

Note to readers:  
Please ignore these  
sidenotes; they're just  
hints to myself for  
preparing the index,  
and they're often flaky!

KNUTH

# THE ART OF COMPUTER PROGRAMMING

VOLUME 4    PRE-FASCICLE 5B

## INTRODUCTION TO BACKTRACKING

DONALD E. KNUTH *Stanford University*

ADDISON-WESLEY



November 12, 2016

Internet  
Stanford GraphBase  
MMIX

Internet page <http://www-cs-faculty.stanford.edu/~knuth/taocp.html> contains current information about this book and related books.

See also <http://www-cs-faculty.stanford.edu/~knuth/sgb.html> for information about *The Stanford GraphBase*, including downloadable software for dealing with the graphs used in many of the examples in Chapter 7.

See also <http://www-cs-faculty.stanford.edu/~knuth/mmixtureware.html> for downloadable software to simulate the MMIX computer.

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## PREFACE

*Begin at the beginning, and do not allow yourself to gratify  
a mere idle curiosity by dipping into the book, here and there.  
This would very likely lead to your throwing it aside,  
with the remark "This is much too hard for me!"  
and thus losing the chance of adding a very large item  
to your stock of mental delights.*

— LEWIS CARROLL, in *Symbolic Logic* (1896)

THIS BOOKLET contains draft material that I'm circulating to experts in the field, in hopes that they can help remove its most egregious errors before too many other people see it. I am also, however, posting it on the Internet for courageous and/or random readers who don't mind the risk of reading a few pages that have not yet reached a very mature state. *Beware:* This material has not yet been proofread as thoroughly as the manuscripts of Volumes 1, 2, 3, and 4A were at the time of their first printings. And alas, those carefully-checked volumes were subsequently found to contain thousands of mistakes.

Given this caveat, I hope that my errors this time will not be so numerous and/or obtrusive that you will be discouraged from reading the material carefully. I did try to make the text both interesting and authoritative, as far as it goes. But the field is vast; I cannot hope to have surrounded it enough to corral it completely. So I beg you to let me know about any deficiencies that you discover.

To put the material in context, this portion of fascicle 5 previews the opening pages of Section 7.2.2 of *The Art of Computer Programming*, entitled "Backtrack programming." The preceding section, 7.2.1, was about "Generating basic combinatorial patterns" —namely tuples, permutations, combinations, partitions, and trees. Now it's time to consider the non-basic patterns, the ones that have a much less uniform structure. For these we generally need to make tentative choices and then we need to back up when those choices need revision. Several subsections (7.2.2.1, 7.2.2.2, etc.) will follow this introductory material.

\* \* \*

The explosion of research in combinatorial algorithms since the 1970s has meant that I cannot hope to be aware of all the important ideas in this field. I've tried my best to get the story right, yet I fear that in many respects I'm woefully ignorant. So I beg expert readers to steer me in appropriate directions.

Please look, for example, at the exercises that I’ve classed as research problems (rated with difficulty level 46 or higher), namely exercises 14, . . . ; I’ve also implicitly mentioned or posed additional unsolved questions in the answers to exercises 6, 8, 42, 45, . . . . Are those problems still open? Please inform me if you know of a solution to any of these intriguing questions. And of course if no solution is known today but you do make progress on any of them in the future, I hope you’ll let me know.

stamping  
Knuth

I urgently need your help also with respect to some exercises that I made up as I was preparing this material. I certainly don’t like to receive credit for things that have already been published by others, and most of these results are quite natural “fruits” that were just waiting to be “plucked.” Therefore please tell me if you know who deserves to be credited, with respect to the ideas found in exercises 31(b), 33, 44, 50, 51, 62, 66, 67, 75, 100, . . . . Furthermore I’ve credited exercises . . . to unpublished work of . . . . Have any of those results ever appeared in print, to your knowledge?

I’ve got a historical question too: Have you any idea who originated the idea of “stamping” in data structures? (See 7.2.2–(26). This concept is quite different from the so-called time stamps in persistent data structures, and quite different from the so-called time stamps in depth-first search algorithms, and quite different from the so-called time stamps in cryptology, although many programmers do use the name “time stamp” for those kinds of stamp.) It’s a technique that I’ve seen often, in programs that have come to my attention during recent decades, but I wonder if it ever appeared in a book or paper that was published before, say, 1980.

\* \* \*

Special thanks are due to . . . for their detailed comments on my early attempts at exposition, as well as to numerous other correspondents who have contributed crucial corrections.

\* \* \*

I happily offer a “finder’s fee” of \$2.56 for each error in this draft when it is first reported to me, whether that error be typographical, technical, or historical. The same reward holds for items that I forgot to put in the index. And valuable suggestions for improvements to the text are worth 32¢ each. (Furthermore, if you find a better solution to an exercise, I’ll actually do my best to give you immortal glory, by publishing your name in the eventual book:—)

Cross references to yet-unwritten material sometimes appear as ‘00’; this impossible value is a placeholder for the actual numbers to be supplied later.

Happy reading!

*Stanford, California*  
*99 Umbruary 2016*

D. E. K.

## Part of the Preface to Volume 4B

During the years that I've been preparing Volume 4, I've often run across basic techniques of probability theory that I would have put into Section 1.2 of Volume 1 if I'd been clairvoyant enough to anticipate them in the 1960s. Finally I realized that I ought to collect most of them together in one place, near the beginning of Volume 4B, because the story of these developments is too interesting to be broken up into little pieces scattered here and there.

Therefore this volume begins with a special section entitled “Mathematical Preliminaries Redux,” and future sections use the abbreviation ‘MPR’ to refer to its equations and its exercises.

## MATHEMATICAL PRELIMINARIES REDUX

MANY PARTS of this book deal with *discrete probabilities*, namely with a finite or countably infinite set  $\Omega$  of atomic events  $\omega$ , each of which has a given probability  $\Pr(\omega)$ , where

$$0 \leq \Pr(\omega) \leq 1 \quad \text{and} \quad \sum_{\omega \in \Omega} \Pr(\omega) = 1. \quad (1)$$

...



*For the complete text of the special MPR section, please see Pre-Fascicle 5a. Incidentally, Section 7.2.2 intentionally begins on a left-hand page, and its illustrations are numbered beginning with Fig. 68, because Section 7.2.1 ended on a right-hand page and its final illustration was Fig. 67. The editor has decided to treat Chapter 7 as a single unit, even though it will be split across several physical volumes.*

<p><i>Nowhere to go but out, Nowhere to come but back.</i></p> <p>— BEN KING, in <i>The Sum of Life</i> (c. 1893)</p> <p><i>When you come to one legal road that's blocked, you back up and try another.</i></p> <p>— PERRY MASON, in <i>The Case of the Black-Eyed Blonde</i> (1944)</p> <p><i>No one I think is in my tree.</i></p> <p>— JOHN LENNON, in <i>Strawberry Fields Forever</i> (1967)</p>	<p>KING MASON Gardner ES LENNON backtrack Walker domain cutoff properties: logical propositions (relations) <math>P_0()</math> lexicographically</p>
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### 7.2.2. Backtrack Programming

Now that we know how to generate simple combinatorial patterns such as tuples, permutations, combinations, partitions, and trees, we're ready to tackle more exotic patterns that have subtler and less uniform structure. Instances of almost *any* desired pattern can be generated systematically, at least in principle, if we organize the search carefully. Such a method was christened “backtrack” by R. J. Walker in the 1950s, because it is basically a way to examine all fruitful possibilities while exiting gracefully from situations that have been fully explored.

Most of the patterns we shall deal with can be cast in a simple, general framework: We seek all sequences  $x_1x_2 \dots x_n$  for which some property  $P_n(x_1, x_2, \dots, x_n)$  holds, where each item  $x_k$  belongs to some given domain  $D_k$  of integers. The backtrack method, in its most elementary form, consists of inventing intermediate “cutoff” properties  $P_l(x_1, \dots, x_l)$  for  $1 \leq l < n$ , such that

$P_l(x_1, \dots, x_l)$  is true whenever  $P_{l+1}(x_1, \dots, x_{l+1})$  is true; (1)

$P_l(x_1, \dots, x_l)$  is fairly easy to test, if  $P_{l-1}(x_1, \dots, x_{l-1})$  holds. (2)

(We assume that  $P_0()$  is always true. Exercise 1 shows that all of the basic patterns studied in Section 7.2.1 can easily be formulated in terms of domains  $D_k$  and cutoff properties  $P_l$ .) Then we can proceed lexicographically as follows:

**Algorithm B** (*Basic backtrack*). Given domains  $D_k$  and properties  $P_l$  as above, this algorithm visits all sequences  $x_1x_2 \dots x_n$  that satisfy  $P_n(x_1, x_2, \dots, x_n)$ .

**B1.** [Initialize.] Set  $l \leftarrow 1$ , and initialize the data structures needed later.

**B2.** [Enter level  $l$ .] (Now  $P_{l-1}(x_1, \dots, x_{l-1})$  holds.) If  $l > n$ , visit  $x_1x_2 \dots x_n$  and go to B5. Otherwise set  $x_l \leftarrow \min D_l$ , the smallest element of  $D_l$ .

**B3.** [Try  $x_l$ .] If  $P_l(x_1, \dots, x_l)$  holds, update the data structures to facilitate testing  $P_{l+1}$ , set  $l \leftarrow l + 1$ , and go to B2.

**B4.** [Try again.] If  $x_l \neq \max D_l$ , set  $x_l$  to the next larger element of  $D_l$  and return to B3.

**B5.** [Backtrack.] Set  $l \leftarrow l - 1$ . If  $l > 0$ , downdate the data structures by undoing the changes recently made in step B3, and return to B4. (Otherwise stop.) ■

The main point is that if  $P_l(x_1, \dots, x_l)$  is false in step B3, we needn't waste time trying to append any further values  $x_{l+1} \dots x_n$ . Thus we can often rule out huge regions of the space of all potential solutions. A second important point is that very little memory is needed, although there may be many, many solutions.

For example, let's consider the classic *problem of  $n$  queens*: In how many ways can  $n$  queens be placed on an  $n \times n$  board so that no two are in the same row, column, or diagonal? We can suppose that one queen is in each row, and that the queen in row  $k$  is in column  $x_k$ , for  $1 \leq k \leq n$ . Then each domain  $D_k$  is  $\{1, 2, \dots, n\}$ ; and  $P_n(x_1, \dots, x_n)$  is the condition that

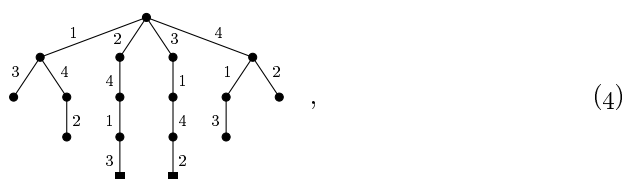
$$x_j \neq x_k \quad \text{and} \quad |x_k - x_j| \neq k - j, \quad \text{for } 1 \leq j < k \leq n. \quad (3)$$

(If  $x_j = x_k$  and  $j < k$ , two queens are in the same column; if  $|x_k - x_j| = k - j$ , they're in the same diagonal.)

This problem is easy to set up for Algorithm B, because we can let property  $P_l(x_1, \dots, x_l)$  be the same as (3) but restricted to  $1 \leq j < k \leq l$ . Condition (1) is clear; and so is condition (2), because  $P_l$  requires testing (3) only for  $k = l$  when  $P_{l-1}$  is known. Notice that  $P_1(x_1)$  is always true in this example.

One of the best ways to learn about backtracking is to execute Algorithm B by hand in the special case  $n = 4$  of the  $n$  queens problem: First we set  $x_1 \leftarrow 1$ . Then when  $l = 2$  we find  $P_2(1, 1)$  and  $P_2(1, 2)$  false; hence we don't get to  $l = 3$  until trying  $x_2 \leftarrow 3$ . Then, however, we're stuck, because  $P_3(1, 3, x)$  is false for  $1 \leq x \leq 4$ . Backtracking to level 2, we now try  $x_2 \leftarrow 4$ ; and this allows us to set  $x_3 \leftarrow 2$ . However, we're stuck again, at level 4; and this time we must back up all the way to level 1, because there are no further valid choices at levels 3 and 2. The next choice  $x_1 \leftarrow 2$  does, happily, lead to a solution without much further ado, namely  $x_1 x_2 x_3 x_4 = 2413$ . And one more solution (3142) turns up before the algorithm terminates.

The behavior of Algorithm B is nicely visualized as a tree structure, called a search tree or *backtrack tree*. For example, the backtrack tree for the four queens problem has just 17 nodes,

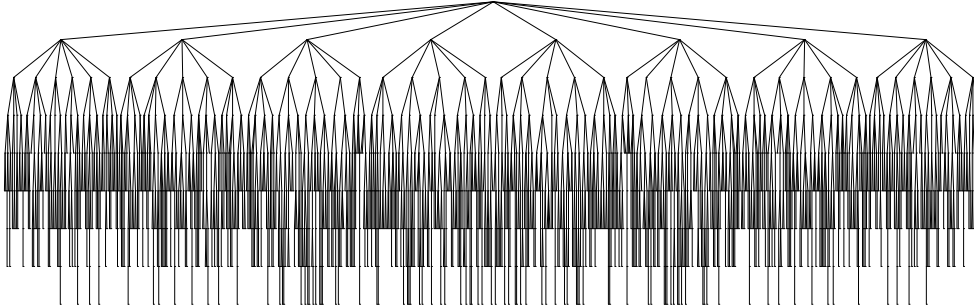


corresponding to the 17 times step B2 is performed. Here  $x_l$  is shown as the label of an edge from level  $l - 1$  to level  $l$  of the tree. (Level  $l$  of the algorithm actually corresponds to the tree's level  $l - 1$ , because we've chosen to represent patterns using subscripts from 1 to  $n$  instead of from 0 to  $n - 1$  in this discussion.) The *profile*  $(p_0, p_1, \dots, p_n)$  of this particular tree—the number of nodes at each level—is  $(1, 4, 6, 4, 2)$ ; and we see that the number of solutions,  $p_n = p_4$ , is 2.

Figure 68 shows the corresponding tree when  $n = 8$ . This tree has 2057 nodes, distributed according to the profile  $(1, 8, 42, 140, 344, 568, 550, 312, 92)$ . Thus the early cutoffs facilitated by backtracking have allowed us to find all 92 solutions by examining only 0.01% of the  $8^8 = 16,777,216$  possible sequences  $x_1 \dots x_8$ . (And  $8^8$  is only 0.38% of the  $\binom{64}{8} = 4,426,165,368$  ways to put eight queens on the board.)

$n$  queens-  
diagonal  
backtrack tree  
profile





data structures—  
mems  
downdating vs updating+  
undoes

**Fig. 68.** The problem of placing eight nonattacking queens has this backtrack tree.

Notice that, in this case, Algorithm B spends most of its time in the vicinity of level 5. Such behavior is typical: The backtrack tree for  $n = 16$  queens has 1,141,190,303 nodes, and its profile is (1, 16, 210, 2236, 19688, 141812, 838816, 3998456, 15324708, 46358876, 108478966, 193892860, 260303408, 253897632, 171158018, 72002088, 14772512), concentrated near level 12.

**Data structures.** Backtrack programming is often used when a *huge* tree of possibilities needs to be examined. Thus we want to be able to test property  $P_l$  as quickly as possible in step B3.

One way to implement Algorithm B for the  $n$  queens problem is to avoid auxiliary data structures and simply to make a bunch of sequential comparisons in that step: “Is  $x_l - x_j \in \{j - l, 0, l - j\}$  for some  $j < l$ ?” Assuming that we access memory whenever referring to  $x_j$ , given a trial value  $x_l$  in a register, such an implementation performs approximately 112 billion memory accesses when  $n = 16$ ; that’s about 98 mems per node.

We can do better by introducing three simple arrays. Property  $P_l$  in (3) says essentially that the numbers  $x_k$  are distinct, and so are the numbers  $x_k + k$ , and so are the numbers  $x_k - k$ . Therefore we can use auxiliary Boolean arrays  $a_1 \dots a_n$ ,  $b_1 \dots b_{2n-1}$ , and  $c_1 \dots c_{2n-1}$ , where  $a_j$  means ‘some  $x_k = j$ ’,  $b_j$  means ‘some  $x_k + k - 1 = j$ ’, and  $c_j$  means ‘some  $x_k - k + n = j$ ’. Those arrays are readily updated and downdated if we customize Algorithm B as follows:

- B1\*.** [Initialize.] Set  $a_1 \dots a_n \leftarrow 0 \dots 0$ ,  $b_1 \dots b_{2n-1} \leftarrow 0 \dots 0$ ,  $c_1 \dots c_{2n-1} \leftarrow 0 \dots 0$ , and  $l \leftarrow 1$ .
- B2\*.** [Enter level  $l$ .] (Now  $P_{l-1}(x_1, \dots, x_{l-1})$  holds.) If  $l > n$ , visit  $x_1 x_2 \dots x_n$  and go to B5\*. Otherwise set  $t \leftarrow 1$ .
- B3\*.** [Try  $t$ .] If  $a_t = 1$  or  $b_{t+l-1} = 1$  or  $c_{t-l+n} = 1$ , go to B4\*. Otherwise set  $a_t \leftarrow 1$ ,  $b_{t+l-1} \leftarrow 1$ ,  $c_{t-l+n} \leftarrow 1$ ,  $x_l \leftarrow t$ ,  $l \leftarrow l + 1$ , and go to B2\*.
- B4\*.** [Try again.] If  $t < n$ , set  $t \leftarrow t + 1$  and return to B3\*.
- B5\*.** [Backtrack.] Set  $l \leftarrow l - 1$ . If  $l > 0$ , set  $t \leftarrow x_l$ ,  $c_{t-l+n} \leftarrow 0$ ,  $b_{t+l-1} \leftarrow 0$ ,  $a_t \leftarrow 0$ , and return to B4\*. (Otherwise stop.) ■

Notice how step B5\* neatly undoes the updates that step B3\* had made, in the reverse order. Reverse order for downdating is typical of backtrack algorithms,

although there is some flexibility; we could, for example, have restored  $a_t$  before  $b_{t+l-1}$  and  $c_{t-l+n}$ , because those arrays are independent.

The auxiliary arrays  $a, b, c$  make it easy to test property  $P_l$  at the beginning of step B3\*, but we must also access memory when we update them and downdate them. Does that cost us more than it saves? Fortunately, no: The running time for  $n = 16$  goes down to about 34 billion mems, roughly 30 mems per node.

Furthermore we could keep the bit vectors  $a, b, c$  entirely in registers, on a machine with 64-bit registers, assuming that  $n \leq 32$ . Then there would be just two memory accesses per node, namely to store  $x_l \leftarrow t$  and later to fetch  $t \leftarrow x_l$ ; however, quite a lot of in-register computation would become necessary.

**Walker's method.** The 1950s-era programs of R. J. Walker organized backtracking in a somewhat different way. Instead of letting  $x_l$  run through all elements of  $D_l$ , he calculated and stored the set

$$S_l \leftarrow \{x \in D_l \mid P_l(x_1, \dots, x_{l-1}, x) \text{ holds}\} \quad (5)$$

upon entry to each node at level  $l$ . This computation can often be done efficiently all at once, instead of piecemeal, because some cutoff properties make it possible to combine steps that would otherwise have to be repeated for each  $x \in D_l$ . In essence, he used the following variant of Algorithm B:

**Algorithm W** (*Walker's backtrack*). Given domains  $D_k$  and cutoffs  $P_l$  as above, this algorithm visits all sequences  $x_1 x_2 \dots x_n$  that satisfy  $P_n(x_1, x_2, \dots, x_n)$ .

**W1.** [Initialize.] Set  $l \leftarrow 1$ , and initialize the data structures needed later.

**W2.** [Enter level  $l$ .] (Now  $P_{l-1}(x_1, \dots, x_{l-1})$  holds.) If  $l > n$ , visit  $x_1 x_2 \dots x_n$  and go to W4. Otherwise determine the set  $S_l$  as in (5).

**W3.** [Try to advance.] If  $S_l$  is nonempty, set  $x_l \leftarrow \min S_l$ , update the data structures to facilitate computing  $S_{l+1}$ , set  $l \leftarrow l + 1$ , and go to W2.

**W4.** [Backtrack.] Set  $l \leftarrow l - 1$ . If  $l > 0$ , downdate the data structures by undoing changes made in step W3, set  $S_l \leftarrow S_l \setminus x_l$ , and retreat to W3. ■

Walker applied this method to the  $n$  queens problem by computing  $S_l = U \setminus A_l \setminus B_l \setminus C_l$ , where  $U = D_l = \{1, \dots, n\}$  and

$$A_l = \{x_j \mid 1 \leq j < l\}, B_l = \{x_j + j - l \mid 1 \leq j < l\}, C_l = \{x_j - j + l \mid 1 \leq j < l\}. \quad (6)$$

He represented these auxiliary sets by bit vectors  $a, b, c$ , analogous to (but different from) the bit vectors of Algorithm B\* above. Exercise 9 shows that the updating in step W3 is easy, using bitwise operations on  $n$ -bit numbers; furthermore, no downdating is needed in step W4. The corresponding run time when  $n = 16$  turns out to be just 9.1 gigamems, or 8 mems per node.

Let  $Q(n)$  be the number of solutions to the  $n$  queens problem. Then we have

$$\begin{array}{cccccccccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ Q(n) & 1 & 1 & 0 & 0 & 2 & 10 & 4 & 40 & 92 & 352 & 724 & 2680 & 14200 & 73712 & 365596 & 2279184 & 14772512 \end{array}$$

and the values for  $n \leq 11$  were computed independently by several people during the nineteenth century. Small cases were relatively easy; but when T. B. Sprague

registers  
Walker  
cutoff properties  
visits  
bitwise operations  
historical notes+  
Sprague

had finished computing  $Q(11)$  he remarked that “This was a very heavy piece of work, and occupied most of my leisure time for several months. . . . It will, I imagine, be scarcely possible to obtain results for larger boards, unless a number of persons co-operate in the work.” [See *Proc. Edinburgh Math. Soc.* **17** (1899), 43–68; Sprague was the leading actuary of his day.] Nevertheless, H. Onnen went on to evaluate  $Q(12) = 14,200$ —an astonishing feat of hand calculation—in 1910. [See W. Ahrens, *Math. Unterhaltungen und Spiele* **2**, second edition (1918), 344.]

All of these hard-won results were confirmed in 1960 by R. J. Walker, using the SWAC computer at UCLA and the method of exercise 9. Walker also computed  $Q(13)$ ; but he couldn’t go any further with the machine available to him at the time. The next step,  $Q(14)$ , was computed by Michael D. Kennedy at the University of Tennessee in 1963, commandeering an IBM 1620 for 120 hours. S. R. Bunch evaluated  $Q(15)$  in 1974 at the University of Illinois, using about two hours on an IBM System 360-75; then J. R. Bitner found  $Q(16)$  after about three hours on the same computer, but with an improved method.

Computers and algorithms have continued to get better, of course, and such results are now obtained almost instantly. Hence larger and larger values of  $n$  lie at the frontier. The whopping value  $Q(27) = 234,907,967,154,122,528$ , found in 2016 by Thomas B. Preußner and Matthias R. Engelhardt, probably won’t be exceeded for awhile! [See *J. Signal Processing Systems* **89** (2017), to appear. This distributed computation occupied a dynamic cluster of diverse FPGA devices for 383 days; those devices provided a total peak of more than 7000 custom-designed hardware solvers to handle 2,024,110,796 independent subproblems.]

**Permutations and Langford pairs.** Every solution  $x_1 \dots x_n$  to the  $n$  queens problem is a permutation of  $\{1, \dots, n\}$ , and many other problems are permutation-based. Indeed, we’ve already seen Algorithm 7.2.1.2X, which is an elegant backtrack procedure specifically designed for special kinds of permutations. When that algorithm begins to choose the value of  $x_l$ , it makes all of the appropriate elements  $\{1, 2, \dots, n\} \setminus \{x_1, \dots, x_{l-1}\}$  conveniently accessible in a linked list.

We can get further insight into such data structures by returning to the problem of Langford pairs, which was discussed at the very beginning of Chapter 7. That problem can be reformulated as the task of finding all permutations of  $\{1, 2, \dots, n\} \cup \{-1, -2, \dots, -n\}$  with the property that

$$x_j = k \text{ implies } x_{j+k+1} = -k, \quad \text{for } 1 \leq j \leq 2n \text{ and } 1 \leq k \leq n. \quad (7)$$

For example, when  $n = 4$  there are two solutions, namely  $234\bar{2}1\bar{3}\bar{1}\bar{4}$  and  $413\bar{1}2\bar{4}\bar{3}\bar{2}$ . (As usual we find it convenient to write  $\bar{1}$  for  $-1$ ,  $\bar{2}$  for  $-2$ , etc.) Notice that if  $x = x_1 x_2 \dots x_{2n}$  is a solution, so is its “dual”  $-x^R = (-x_{2n}) \dots (-x_2)(-x_1)$ .

Here’s a Langford-inspired adaptation of Algorithm 7.2.1.2X, with the former notation modified slightly to match Algorithms B and W: We want to maintain pointers  $p_0 p_1 \dots p_n$  such that, if the positive integers not already present in  $x_1 \dots x_{l-1}$  are  $k_1 < k_2 < \dots < k_t$  when we’re choosing  $x_l$ , we have the linked list

$$p_0 = k_1, p_{k_1} = k_2, \dots, p_{k_{t-1}} = k_t, p_{k_t} = 0. \quad (8)$$

Such a condition turns out to be easy to maintain.

Sprague  
Onnen  
Ahrens  
Walker  
SWAC computer  
UCLA  
Kennedy  
University of Tennessee  
IBM 1620  
Bunch  
University of Illinois  
IBM System 360-75  
Bitner  
Preußner  
Engelhardt  
distributed computation  
FPGA  
permutation  
data structures  
Langford pairs  
dual  
linked list

**Algorithm L** (*Langford pairs*). This algorithm visits all solutions  $x_1 \dots x_{2n}$  to (7) in lexicographic order, using pointers  $p_0 p_1 \dots p_n$  that satisfy (8), and also using an auxiliary array  $y_1 \dots y_{2n}$  for backtracking.

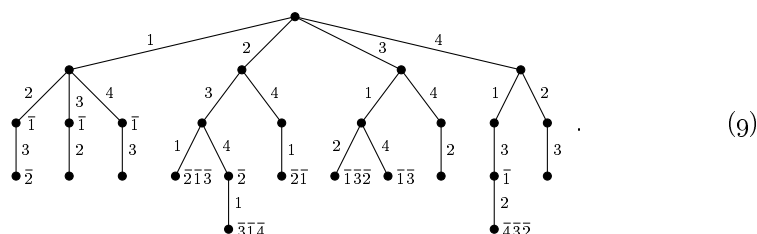
- L1.** [Initialize.] Set  $x_1 \dots x_{2n} \leftarrow 0 \dots 0$ ,  $p_k \leftarrow k+1$  for  $0 \leq k < n$ ,  $p_n \leftarrow 0$ ,  $l \leftarrow 1$ .
- L2.** [Enter level  $l$ .] Set  $k \leftarrow p_0$ . If  $k = 0$ , visit  $x_1 x_2 \dots x_{2n}$  and go to L5. Otherwise set  $j \leftarrow 0$ , and while  $x_l < 0$  set  $l \leftarrow l+1$ .
- L3.** [Try  $x_l = k$ .] (At this point we have  $k = p_j$ .) If  $l+k+1 > 2n$ , go to L5. Otherwise, if  $x_{l+k+1} = 0$ , set  $x_l \leftarrow k$ ,  $x_{l+k+1} \leftarrow -k$ ,  $y_l \leftarrow j$ ,  $p_j \leftarrow p_k$ ,  $l \leftarrow l+1$ , and return to L2.
- L4.** [Try again.] (We've found all solutions that begin with  $x_1 \dots x_{l-1} k$  or something smaller.) Set  $j \leftarrow k$  and  $k \leftarrow p_j$ , then go to L3 if  $k \neq 0$ .
- L5.** [Backtrack.] Set  $l \leftarrow l-1$ . If  $l > 0$  do the following: While  $x_l < 0$ , set  $l \leftarrow l-1$ . Then set  $k \leftarrow x_l$ ,  $x_l \leftarrow 0$ ,  $x_{l+k+1} \leftarrow 0$ ,  $j \leftarrow y_l$ ,  $p_j \leftarrow k$ , and go back to L4. Otherwise terminate the algorithm. ■

undoes  
deletion operation  
dancing links  
search tree  
cutoff principle

Careful study of these steps will reveal how everything fits together nicely. Notice that, for example, step L3 removes  $k$  from the linked list (8) by simply setting  $p_j \leftarrow p_k$ . That step also sets  $x_{l+k+1} \leftarrow -k$ , in accordance with (7), so that we can skip over position  $l+k+1$  when we encounter it later in step L2.

The main point of Algorithm L is the somewhat subtle way in which step L5 undoes the deletion operation by setting  $p_j \leftarrow k$ . The pointer  $p_k$  still retains the appropriate link to the next element in the list, *because  $p_k$  has not been changed by any of the intervening updates*. (Think about it.) This is the germ of an idea called “dancing links” that we will explore in Section 7.2.2.1.

To draw the search tree corresponding to a run of Algorithm L, we can label the edges with the positive choices of  $x_l$  as we did in (4), while labeling the nodes with any previously set negative values that are passed over in step L2. For instance the tree for  $n = 4$  is



Solutions appear at depth  $n$  in this tree, even though they involve  $2n$  values  $x_1 x_2 \dots x_{2n}$ .

Algorithm L sometimes makes false starts and doesn't realize the problem until probing further than necessary. Notice that the value  $x_l = k$  can appear only when  $l+k+1 \leq 2n$ ; hence if we haven't seen  $k$  by the time  $l$  reaches  $2n-k-1$ , we're *forced* to choose  $x_l = k$ . For example, the branch  $12\bar{1}$  in (9) needn't be pursued, because 4 must appear in  $\{x_1, x_2, x_3\}$ . Exercise 20 explains how to incorporate this cutoff principle into Algorithm L. When  $n = 17$ , it reduces the number of nodes in the search tree from 1.29 trillion to 330 billion,

and reduces the running time from 25.0 teramems to 8.1 teramems. (The amount of work has gone from 19.4 mems per node to 24.4 mems per node, because of the extra tests for cutoffs, yet there’s a significant overall reduction.)

Furthermore, we can “break the symmetry” by ensuring that we don’t consider both a solution and its dual. This idea, exploited in exercise 21, reduces the search tree to just 160 billion nodes and costs just 3.94 teramems—that’s 24.6 mems per node.

break the symmetry  
dual  
word rectangles—  
footnote doesn’t show up here  
*n*-letter words  
WORDS  
5-letter words  
Stanford GraphBase  
trie

**Word rectangles.** Let’s look next at a problem where the search domains  $D_l$  are much larger. An  $m \times n$  *word rectangle* is an array of  $n$ -letter words\* whose columns are  $m$ -letter words. For example,

status	
lowest	
utopia	
making	(10)
sledge	

is a  $5 \times 6$  word rectangle whose columns all belong to WORDS(5757), the collection of 5-letter words in the Stanford GraphBase. To find such patterns, we can suppose that column  $l$  contains the  $x_l$ th most common 5-letter word, where  $1 \leq x_l \leq 5757$  for  $1 \leq l \leq 6$ ; hence there are  $5757^6 = 36,406,369,848,837,732,146,649$  ways to choose the columns. In (10) we have  $x_1 \dots x_6 = 1446\ 185\ 1021\ 2537\ 66\ 255$ . Of course very few of those choices will yield suitable rows; but backtracking will hopefully help us to find all solutions in a reasonable amount of time.

We can set this problem up for Algorithm B by storing the  $n$ -letter words in a trie (see Section 6.3), with one trie node of size 26 for each  $l$ -letter prefix of a legitimate word,  $0 \leq l \leq n$ .

For example, such a trie for  $n = 6$  represents 15727 words with 23667 nodes. The prefix *st* corresponds to node number 260, whose 26 entries are

(484,0,0,0,1589,0,0,0,2609,0,0,0,0,1280,0,0,251,0,0,563,0,0,0,1621,0); (11)

this means that *sta* is node 484, *ste* is node 1589, ..., *sty* is node 1621, and there are no 6-letter words beginning with *stb*, *stc*, ..., *stx*, *stz*. A slightly different convention is used for prefixes of length  $n - 1$ ; for example, the entries for node 580, ‘*corne*’, are

(3879,0,0,3878,0,0,0,0,0,0,9602,0,0,0,0,0,171,0,5013,0,0,0,0,0,0), (12)

meaning that *cornea*, *corned*, *cornel*, *corner*, and *cornet* are ranked 3879, 3878, 9602, 171, and 5013 in the list of 6-letter words.

---

\* Whenever five-letter words are used in the examples of this book, they’re taken from the 5757 Stanford GraphBase words as explained at the beginning of Chapter 7. Words of other lengths are taken from the *The Official SCRABBLE® Players Dictionary*, fourth edition (Hasbro, 2005), because those words have been incorporated into many widely available computer games. Such words have been ranked according to the British National Corpus of 2007—where ‘*the*’ occurs 5,405,633 times and the next-most common word, ‘*of*’, occurs roughly half as often (3,021,525). The OSPD4 list includes respectively (101, 1004, 4002, 8887, 15727, 23958, 29718, 29130, 22314, 16161, 11412) words of lengths (2, 3, ..., 12), of which (97, 771, 2451, 4474, 6910, 8852, 9205, 8225, 6626, 4642, 3061) occur at least six times in the British National Corpus.

Suppose  $x_1$  and  $x_2$  specify the 5-letter column-words **slums** and **total** as in (10). Then the trie tells us that the next column-word  $x_3$  must have the form  $c_1c_2c_3c_4c_5$  where  $c_1 \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}, \mathbf{r}, \mathbf{u}, \mathbf{y}\}$ ,  $c_2 \notin \{\mathbf{e}, \mathbf{h}, \mathbf{j}, \mathbf{k}, \mathbf{y}, \mathbf{z}\}$ ,  $c_3 \in \{\mathbf{e}, \mathbf{m}, \mathbf{o}, \mathbf{t}\}$ ,  $c_4 \notin \{\mathbf{a}, \mathbf{b}, \mathbf{o}\}$ , and  $c_5 \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}, \mathbf{u}, \mathbf{y}\}$ . (There are 221 such words.)

Let  $a_{l1} \dots a_{lm}$  be the trie nodes corresponding to the prefixes of the first  $l$  columns of a partial solution to the word rectangle problem. This auxiliary array enables Algorithm B to find all solutions, as explained in exercise 24. It turns out that there are exactly 625,415 valid  $5 \times 6$  word rectangles, according to our conventions; and the method of exercise 24 needs about 19 teramems of computation to find them all. In fact, the profile of the search tree is

$$(1, 5757, 2458830, 360728099, 579940198, 29621728, 625415), \quad (13)$$

indicating for example that just 360,728,099 of the  $5757^3 = 190,804,533,093$  choices for  $x_1x_2x_3$  will lead to valid prefixes of 6-letter words.

With care, exercise 24's running time can be significantly decreased, once we realize that every node of the search tree for  $1 \leq l \leq n$  requires testing 5757 possibilities for  $x_l$  in step B3. If we build a more elaborate data structure for the 5-letter words, so that it becomes easy to run through all words that have a specific letter in a specific position, we can refine the algorithm so that the average number of possibilities per level that need to be investigated becomes only

$$(5757.0, 1697.9, 844.1, 273.5, 153.5, 100.8); \quad (14)$$

the total running time then drops to 1.15 teramems. Exercise 25 has the details. And exercise 28 discusses a method that's faster yet.

**Commafree codes.** Our next example deals entirely with *four*-letter words. But it's not obscene; it's an intriguing question of coding theory. The problem is to find a set of four-letter words that can be decoded even if we don't put spaces or other delimiters between them. If we take any message that's formed from words of the set by simply concatenating them together, **likethis**, and if we look at any seven consecutive letters  $\dots x_1x_2x_3x_4x_5x_6x_7 \dots$ , exactly one of the four-letter substrings  $x_1x_2x_3x_4$ ,  $x_2x_3x_4x_5$ ,  $x_3x_4x_5x_6$ ,  $x_4x_5x_6x_7$  will be a codeword. Equivalently, if  $x_1x_2x_3x_4$  and  $x_5x_6x_7x_8$  are codewords, then  $x_2x_3x_4x_5$  and  $x_3x_4x_5x_6$  and  $x_4x_5x_6x_7$  aren't. (For example, **iket** isn't.) Such a set is called a "commafree code" or a "self-synchronizing block code" of length four.

Commafree codes were introduced by F. H. C. Crick, J. S. Griffith, and L. E. Orgel [*Proc. National Acad. Sci.* **43** (1957), 416–421], and studied further by S. W. Golomb, B. Gordon, and L. R. Welch [*Canadian Journal of Mathematics* **10** (1958), 202–209], who considered the general case of  $m$ -letter alphabets and  $n$ -letter words. They constructed optimum commafree codes for all  $m$  when  $n = 2, 3, 5, 7, 9, 11, 13$ , and 15; and optimum codes for all  $m$  were subsequently found also for  $n = 17, 19, 21, \dots$  (see exercise 32). We will focus our attention on the four-letter case here ( $n = 4$ ), partly because that case is still very far from being resolved, but mostly because the task of finding such codes is especially instructive. Indeed, our discussion will lead us naturally to an understanding of several significant techniques that are important for backtrack programming in general.

teramems  
profile  
search tree  
data structure  
commafree codes—  
4-letter codewords  
four-letter words  
coding theory  
concatenating  
codewords  
self-synchronizing block code  
block code  
Crick  
Griffith  
Orgel  
Golomb  
Gordon  
Welch

To begin, we can see immediately that a commafree codeword cannot be “periodic,” like `dodo` or `gaga`. Such a word already appears within two adjacent copies of itself. Thus we’re restricted to *aperiodic* words like `item`, of which there are  $m^4 - m^2$ . Notice further that if `item` has been chosen, we aren’t allowed to include any of its cyclic shifts `temi`, `emit`, or `mite`, because they all appear within `itemitem`. Hence the maximum number of codewords in our commafree code cannot exceed  $(m^4 - m^2)/4$ .

For example, consider the binary case,  $m = 2$ , when this maximum is 3. Can we choose three four-bit “words,” one from each of the cyclic classes

$$\begin{aligned} [0001] &= \{0001, 0010, 0100, 1000\}, \\ [0011] &= \{0011, 0110, 1100, 1001\}, \\ [0111] &= \{0111, 1110, 1101, 1011\}, \end{aligned} \tag{15}$$

so that the resulting code is commafree? Yes: One solution in this case is simply to choose the smallest word in each class, namely 0001, 0011, and 0111. (Alert readers will recall that we studied the smallest word in the cyclic class of *any* aperiodic string in Section 7.2.1.1, where such words were called *prime strings* and where some of the remarkable properties of prime strings were proved.)

That trick doesn’t work when  $m = 3$ , however, when there are  $(81 - 9)/4 = 18$  cyclic classes. Then we cannot include 1112 after we’ve chosen 0001 and 0011. Indeed, a code that contains 0001 and 1112 can’t contain either 0011 or 0111.

We could systematically backtrack through 18 levels, choosing  $x_1$  in [0001] and  $x_2$  in [0011], etc., and rejecting each  $x_l$  as in Algorithm B whenever we discover that  $\{x_1, x_2, \dots, x_l\}$  isn’t commafree. For example, if  $x_1 = 0010$  and we try  $x_2 = 1001$ , this approach would backtrack because  $x_1$  occurs inside  $x_2x_1$ .

But a naïve strategy of that kind, which recognizes failure only after a bad choice has been made, can be vastly improved. If we had been clever enough, we could have looked a little bit ahead, and never even considered the choice  $x_2 = 1001$  in the first place. Indeed, after choosing  $x_1 = 0010$ , we can automatically exclude *all* further words of the form  $*001$ , such as 2001 when  $m \geq 3$  and 3001 when  $m \geq 4$ .

Even better pruning occurs if, for example, we’ve chosen  $x_1 = 0001$  and  $x_2 = 0011$ . Then we can immediately rule out all words of the forms  $1***$  or  $***0$ , because  $x_11***$  includes  $x_2$  and  $***0x_2$  includes  $x_1$ . Already we could then deduce, in the case  $m \geq 3$ , that classes [0002], [0021], [0111], [0211], and [1112] *must* be represented by 0002, 0021, 0111, 0211, and 2111, respectively; each of the other three possibilities in those classes has been wiped out!

Thus we see the desirability of a lookahead mechanism.

**Dynamic ordering of choices.** Furthermore, we can see from this example that it’s not always good to choose  $x_1$ , then  $x_2$ , then  $x_3$ , and so on when trying to satisfy a general property  $P_n(x_1, x_2, \dots, x_n)$  in the setting of Algorithm B. Maybe the search tree will be much smaller if we first choose  $x_5$ , say, and then turn next to some other  $x_j$ , depending on the particular value of  $x_5$  that was selected. Some orderings might have much better cutoff properties than others, and every branch of the tree is free to choose its variables in any desired order.

periodic  
aperiodic  
cyclic shifts  
prime strings  
lookahead  
dynamic ordering—  
search rearrangement, see dynamic ordering  
cutoff properties

Indeed, our comma-free coding problem for ternary 4-tuples doesn't dictate any particular ordering of the 18 classes that would be likely to keep the search tree small. Therefore, instead of calling those choices  $x_1, x_2, \dots, x_{18}$ , it's better to identify them by the various class names, namely  $x_{0001}, x_{0002}, x_{0011}, x_{0012}, x_{0021}, x_{0022}, x_{0102}, x_{0111}, x_{0112}, x_{0121}, x_{0122}, x_{0211}, x_{0212}, x_{0221}, x_{0222}, x_{1112}, x_{1122}, x_{1222}$ . (Algorithm 7.2.1.1F is a good way to generate those names.) At every node of the search tree we then can choose a convenient variable on which to branch, based on previous choices. After beginning with  $x_{0001} \leftarrow 0001$  at level 1 we might decide to try  $x_{0011} \leftarrow 0011$  at level 2; and then, as we've seen, the choices  $x_{0002} \leftarrow 0002, x_{0021} \leftarrow 0021, x_{0111} \leftarrow 0111, x_{0211} \leftarrow 0211$ , and  $x_{1112} \leftarrow 2111$  are forced, so we should make them at levels 3 through 7.

Furthermore, after those forced moves are made, it turns out that they don't force any others. But only two choices for  $x_{0012}$  will remain, while  $x_{0122}$  will have three. Therefore it will probably be wiser to branch on  $x_{0012}$  rather than on  $x_{0122}$  at level 8. (Incidentally, it also turns out that there *is* no comma-free code with  $x_{0001} = 0001$  and  $x_{0011} = 0011$ , *except* when  $m = 2$ .)

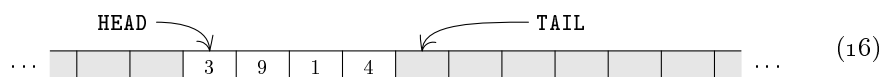
It's easy to adapt Algorithms B and W to allow dynamic ordering. Every node of the search tree can be given a "frame" in which we record the variable being set and the choice that was made. This choice of variable and value can be called a "move" made by the backtrack procedure.

Dynamic ordering can be helpful also after backtracking has taken place. If we continue the example above, where  $x_{0001} = 0001$  and we've explored all cases in which  $x_{0011} = 0011$ , we aren't obliged to continue by trying another value for  $x_{0011}$ . We do want to remember that 0011 should no longer be considered legal, until  $x_{0001}$  changes; but we could decide to explore next a case such as  $x_{0002} = 2000$  at level 2. In fact,  $x_{0002} = 2000$  is quickly seen to be impossible in the presence of 0001 (see exercise 34). An even more efficient choice at level 2, however, is  $x_{0012} = 0012$ , because that branch immediately forces  $x_{0002} = 0002, x_{0022} = 0022, x_{0122} = 0122, x_{0222} = 0222, x_{1222} = 1222$ , and  $x_{0011} = 1001$ .

**Sequential allocation redux.** The choice of a variable and value on which to branch is a delicate tradeoff. We don't want to devote more time to planning than we'll save by having a good plan.

If we're going to benefit from dynamic ordering, we'll need efficient data structures that will lead to good decisions without much deliberation. On the other hand, elaborate data structures need to be updated whenever we branch to a new level, and they need to be downdated whenever we return from that level. Algorithm L illustrates an efficient mechanism based on linked lists; but sequentially allocated lists are often even more appealing, because they are cache-friendly and they involve fewer accesses to memory.

Assume then that we wish to represent a set of items as an unordered sequential list. The list begins in a cell of memory pointed to by HEAD, and TAIL points just beyond the end of the list. For example,



frame  
move  
sequential lists—  
downdating vs updating  
cache-friendly  
unordered sequential list  
cells of memory  
MEM, an array of “cells”–



is one way to represent the set  $\{1, 3, 4, 9\}$ . The number of items currently in the set is  $\text{TAIL} - \text{HEAD}$ ; thus  $\text{TAIL} = \text{HEAD}$  if and only if the list is empty. If we wish to insert a new item  $x$ , knowing that  $x$  isn't already present, we simply set

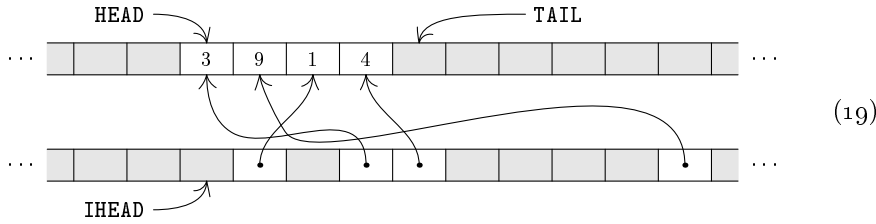
$$\text{MEM}[\text{TAIL}] \leftarrow x, \quad \text{TAIL} \leftarrow \text{TAIL} + 1. \quad (17)$$

Conversely, if  $\text{HEAD} \leq P < \text{TAIL}$ , we can easily delete  $\text{MEM}[P]$ :

$$\text{TAIL} \leftarrow \text{TAIL} - 1; \quad \text{if } P \neq \text{TAIL}, \text{ set } \text{MEM}[P] \leftarrow \text{MEM}[\text{TAIL}]. \quad (18)$$

(We've tacitly assumed in (17) that  $\text{MEM}[\text{TAIL}]$  is available for use whenever a new item is inserted. Otherwise we would have had to test for memory overflow.)

We can't delete an item from a list without knowing its MEM location. Thus we will often want to maintain an "inverse list," assuming that all items  $x$  lie in the range  $0 \leq x < M$ . For example, (16) becomes the following, if  $M = 10$ :



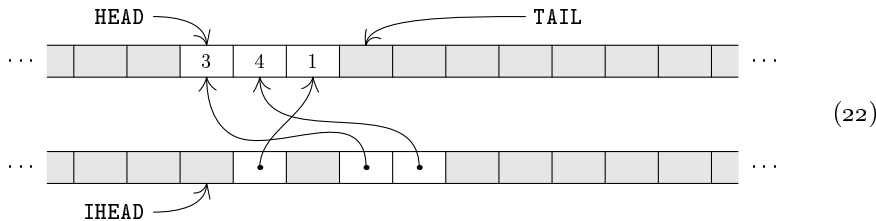
(Shaded cells have undefined contents.) With this setup, insertion (17) becomes

$$\text{MEM}[\text{TAIL}] \leftarrow x, \quad \text{MEM}[\text{IHEAD} + x] \leftarrow \text{TAIL}, \quad \text{TAIL} \leftarrow \text{TAIL} + 1, \quad (20)$$

and  $\text{TAIL}$  will never exceed  $\text{HEAD} + M$ . Similarly, deletion of  $x$  becomes

$$\begin{aligned} P &\leftarrow \text{MEM}[\text{IHEAD} + x], \quad \text{TAIL} \leftarrow \text{TAIL} - 1; \\ \text{if } P \neq \text{TAIL}, \text{ set } y &\leftarrow \text{MEM}[\text{TAIL}], \text{ MEM}[P] \leftarrow y, \text{ MEM}[\text{IHEAD} + y] \leftarrow P. \end{aligned} \quad (21)$$

For example, after deleting '9' from (19) we would obtain this:



In more elaborate situations we also want to test whether or not a given item  $x$  is present. If so, we can keep more information in the inverse list. A particularly useful variation arises when the list that begins at  $\text{IHEAD}$  contains a *complete* permutation of the values  $\{\text{HEAD}, \text{HEAD} + 1, \dots, \text{HEAD} + M - 1\}$ , and the memory cells beginning at  $\text{HEAD}$  contain the inverse permutation—although only the first  $\text{TAIL} - \text{HEAD}$  elements of that list are considered to be "active."

For example, in our comma-free code problem with  $m = 3$ , we can begin by putting items representing the  $M = 18$  cycle classes  $[0001]$ ,  $[0002]$ ,  $\dots$ ,  $[1222]$  into memory cells  $\text{HEAD}$  through  $\text{HEAD} + 17$ . Initially they're all active, with

empty  
insert  
overflow  
inverse list  
inverse permutation  
active

TAIL = HEAD + 18 and MEM[IHEAD +  $c$ ] = HEAD +  $c$  for  $0 \leq c < 18$ . Then whenever we decide to choose a codeword for class  $c$ , we delete  $c$  from the active list by using a souped-up version of (21) that maintains full permutations:

delete  
data structures  
periodic  
radix  $m$

```
P ← MEM[IHEAD + c],  TAIL ← TAIL - 1;
  if P ≠ TAIL, set y ← MEM[TAIL],  MEM[TAIL] ← c,  MEM[P] ← y,
    MEM[IHEAD + c] ← TAIL,  MEM[IHEAD + y] ← P.  (23)
```

Later on, after backtracking to a state where we once again want  $c$  to be considered active, we simply set TAIL ← TAIL + 1, because  $c$  will already be in place!

**Lists for the commafree problem.** The task of finding all four-letter comma-free codes is not difficult when  $m = 3$  and only 18 cycle classes are involved. But it already becomes challenging when  $m = 4$ , because we must then deal with  $(4^4 - 4^2)/4 = 60$  classes. Therefore we'll want to give it some careful thought as we try to set it up for backtracking.

The example scenarios for  $m = 3$  considered above suggest that we'll repeatedly want to know the answers to questions such as, "How many words of the form 02\*\* are still available for selection as codewords?" Redundant data structures, oriented to queries of that kind, appear to be needed. Fortunately, we shall see that there's a nice way to provide them, using sequential lists as in (19)–(23).

In Algorithm C below, each of the  $m^4$  four-letter words is given one of three possible states during the search for commafree codes. A word is *green* if it's part of the current set of tentative codewords. It is *red* if it's not currently a candidate for such status, either because it is incompatible with the existing green words or because the algorithm has already examined all scenarios in which it is green in their presence. Every other word is *blue*, and sort of in limbo; the algorithm might or might not decide to make it red or green. All words are initially blue—except for the  $m^2$  periodic words, which are permanently red.

We'll use the Greek letter  $\alpha$  to stand for the integer value of a four-letter word  $x$  in radix  $m$ . For example, if  $m = 3$  and if  $x$  is the word 0102, then  $\alpha = (0102)_3 = 11$ . The current state of word  $x$  is kept in MEM[ $\alpha$ ], using one of the arbitrary internal codes 2 (GREEN), 0 (RED), or 1 (BLUE).

The most important feature of the algorithm is that every blue word  $x = x_1x_2x_3x_4$  is potentially present in seven different lists, called P1( $x$ ), P2( $x$ ), P3( $x$ ), S1( $x$ ), S2( $x$ ), S3( $x$ ), and CL( $x$ ), where

- P1( $x$ ), P2( $x$ ), P3( $x$ ) are the blue words matching  $x_1***$ ,  $x_1x_2**$ ,  $x_1x_2x_3*$ ;
- S1( $x$ ), S2( $x$ ), S3( $x$ ) are the blue words matching  $***x_4$ ,  $**x_3x_4$ ,  $*x_2x_3x_4$ ;
- CL( $x$ ) hosts the blue words in  $\{x_1x_2x_3x_4, x_2x_3x_4x_1, x_3x_4x_1x_2, x_4x_1x_2x_3\}$ .

These seven lists begin respectively in MEM locations P1OFF +  $p_1(\alpha)$ , P2OFF +  $p_2(\alpha)$ , P3OFF +  $p_3(\alpha)$ , S1OFF +  $s_1(\alpha)$ , S2OFF +  $s_2(\alpha)$ , S3OFF +  $s_3(\alpha)$ , and CLOFF +  $4cl(\alpha)$ ; here (P1OFF, P2OFF, P3OFF, S1OFF, S2OFF, S3OFF, CLOFF) are respectively  $(2m^4, 5m^4, 8m^4, 11m^4, 14m^4, 17m^4, 20m^4)$ . We define  $p_1((x_1x_2x_3x_4)_m) = (x_1)_m$ ,  $p_2((x_1x_2x_3x_4)_m) = (x_1x_2)_m$ ,  $p_3((x_1x_2x_3x_4)_m) = (x_1x_2x_3)_m$ ,  $s_1((x_1x_2x_3x_4)_m) = (x_4)_m$ ,  $s_2((x_1x_2x_3x_4)_m) = (x_3x_4)_m$ ,  $s_3((x_1x_2x_3x_4)_m) = (x_2x_3x_4)_m$ ; and finally  $cl((x_1x_2x_3x_4)_m)$  is an internal number between 0 and  $(m^4 - m^2)/4 - 1$  assigned

**Table 1**LISTS USED BY ALGORITHM C ( $m = 2$ ), ENTERING LEVEL 1reflection  
symmetry breaking  
closed

	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	
	RED	BLUE	BLUE	BLUE	RED	RED	BLUE	BLUE	RED	BLUE	RED	BLUE	BLUE	BLUE	BLUE	RED	
0		20	21	22			23	24		29		2c	28	2b	2a		P1
10	0001	0010	0011	0110	0111				1100	1001	1110	1101	1011				
20	25								2d								
30		50	51	52			54	55		58		59	5c	5e	5d		P2
40	0001	0010	0011		0110	0111			1001	1011			1100	1110	1101		
50	53				56				5a				5f				
60		80	82	83			86	87		88		8a	8c	8d	8e		P3
70	0001		0010	0011			0110	0111	1001		1011		1100	1101	1110		
80	81		84		84		88		89		8b		8e		8f		
90		b8	b0	b9			b1	bb		ba		bd	b2	bc	b3		S1
a0	0010	0110	1100	1110					0001	0011	1001	0111	1101	1011			
b0	b4							be									
c0		e4	e8	ec			e9	ed		e5		ee	e0	e6	ea		S2
d0	1100				0001	1001	1101		0010	0110	1110		0011	0111	1011		
e0	e1				e7			eb					ef				
f0		112	114	116			11c	11e		113		117	118	11a	11d		S3
100			0001	1001	0010		0011	1011	1100		1101		0110	1110	0111		
110	110		114		115		118		119		11b		11e		11f		
120		140	141	144			145	148		147		14b	146	14a	149		CL
130	0001	0010			0011	0110	1100	1001	0111	1110	1101	1011					
140	142				148				14c								
150																	

This table shows MEM locations 0000 through 150f, using hexadecimal notation. (For example, MEM[40d] = 5e; see exercise 36.) Blank entries are unused by the algorithm.

to each class. The seven MEM locations where  $x$  appears in these seven lists are respectively kept in inverse lists that begin in MEM locations  $P10FF - m^4 + \alpha$ ,  $P20FF - m^4 + \alpha, \dots, CLOFF - m^4 + \alpha$ . And the TAIL pointers, which indicate the current list sizes as in (19)–(23), are respectively kept in MEM locations  $P10FF + m^4 + \alpha$ ,  $P20FF + m^4 + \alpha, \dots, CLOFF + m^4 + \alpha$ . (Whew; got that?)

This vast apparatus, which occupies  $22m^4$  cells of MEM, is illustrated in Table 1, at the beginning of the computation for the case  $m = 2$ . Fortunately it's not really as complicated as it may seem at first. Nor is it especially vast: After all,  $22m^4$  is only 13,750 when  $m = 5$ .

(A close inspection of Table 1 reveals incidentally that the words 0100 and 1000 have been colored red, not blue. That's because we can assume without loss of generality that class [0001] is represented either by 0001 or by 0010. The other two cases are covered by left-right reflection of all codewords.)

Algorithm C finds these lists invaluable when it is deciding where next to branch. But it has no further use for a list in which one of the items has become green. Therefore it declares such lists “closed”; and it saves most of the work of list maintenance by updating *only* the lists that remain open. A closed list is represented internally by setting its TAIL pointer to HEAD – 1.

For example, Table 2 shows how the lists in MEM will have changed just after  $x = 0010$  has been chosen to be a tentative codeword. The elements  $\{0001, 0010, 0011, 0110, 0111\}$  of  $P1(x)$  are effectively hidden, because the tail pointer  $\text{MEM}[30] = 1f = 20 - 1$  marks that list as closed. (Those list elements ac-

**Table 2**LISTS USED BY ALGORITHM C ( $m = 2$ ), ENTERING LEVEL 2undoing-  
Floyd  
compiler

	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	
	RED	RED	GREEN	BLUE	RED	RED	BLUE	BLUE	RED	RED	RED	BLUE	BLUE	BLUE	BLUE	RED	
0												29	28	2b	2a		P1
10									1100	1011	1110	1101					
20	1f								2c								
30							54	55				58	5c	5e	5d		P2
40					0110	0111			1011				1100	1110	1101		
50	4f				56				59				5f				
60							86	87				8a	8c	8d	8e		P3
70							0110	0111			1011		1100	1101	1110		
80	80		81		84		88		88		8b		8e		8f		
90				b9				bb				b8		ba			S1
a0									1011	0011	1101	0111					
b0	af								bc								
c0				ec				ed				ee	e0	e4			S2
d0	1100				1101								0011	0111	1011		
e0	e1				e5				e7				ef				
f0				116			11c	11e				117	118	11a	11d		S3
100							0011	1011	1100		1101		0110	1110	0111		
110	110		112		113		118		119		11b		11e		11f		
120				144			145	148				14b	146	14a	149		CL
130					0011	0110	1100		0111	1110	1101	1011					
140	13f				147				14c								
150																	

The word 0010 has become green, thus closing its seven lists and making 0001 red. The logic of Algorithm C has also made 1001 red. Hence 0001 and 1001 have been deleted from the open lists in which they formerly appeared (see exercise 37).

tually do still appear in MEM locations 200 through 204, just as they did in Table 1. But there's no need to look at that list while any word of the form 0\*\*\* is green.)

**A general mechanism for doing and undoing.** We're almost ready to finalize the details of Algorithm C and to get on with the search for commafree codes, but a big problem still remains: The state of computation at every level of the search involves all of the marvelous lists that we've just specified, and those lists aren't tiny. They occupy more than 5000 cells of MEM when  $m = 4$ , and they can change substantially from level to level.

We could make a new copy of the entire state, whenever we advance to a new node of the search tree. But that's a bad idea, because we don't want to perform thousands of memory accesses per node. A much better strategy would be to stick with a single instance of MEM, and to update and downdate the lists as the search progresses, if we could only think of a simple way to do that.

And we're in luck: There *is* such a way, first formulated by R. W. Floyd in his classic paper "Nondeterministic algorithms" [*JACM* **14** (1967), 636–644]. Floyd's original idea, which required a special compiler to generate forward and backward versions of every program step, can in fact be greatly simplified when all of the changes in state are confined to a single MEM array. All we need to do is to replace every assignment operation of the form 'MEM[ $a$ ]  $\leftarrow v$ ' by the

slightly more cumbersome operation

$\text{store}(a, v) :$  Set  $\text{UNDO}[u] \leftarrow (a, \text{MEM}[a])$ ,  $\text{MEM}[a] \leftarrow v$ , and  $u \leftarrow u + 1$ . (24)

Here **UNDO** is a sequential stack that holds (address, value) pairs; in our application we could say ‘ $\text{UNDO}[u] \leftarrow (a \ll 16) + \text{MEM}[a]$ ’, because the cell addresses and values never exceed 16 bits. Of course we’ll also need to check that the stack pointer  $u$  doesn’t get too large, if the number of assignments has no a priori limit.

Later on, when we want to undo all changes to **MEM** since the time when  $u$  had reached a particular value  $u_0$ , we simply do this:

$\text{unstore}(u_0) :$  While  $u > u_0$ , set  $u \leftarrow u - 1$ ,  
 $(a, v) \leftarrow \text{UNDO}[u]$ , and  $\text{MEM}[a] \leftarrow v$ . (25)

In our application the unstacking operation ‘ $(a, v) \leftarrow \text{UNDO}[u]$ ’ here could be implemented by saying ‘ $a \leftarrow \text{UNDO}[u] \gg 16$ ,  $v \leftarrow \text{UNDO}[u] \& \#fff$ ’.

A useful refinement of this reversible-memory technique is often advantageous, based on the idea of “stamping” that is part of the folklore of programming. It puts only one item on the **UNDO** stack when the same memory address is updated more than once in the same round.

$\text{store}(a, v) :$  If  $\text{STAMP}[a] \neq \sigma$ , set  $\text{STAMP}[a] \leftarrow \sigma$ ,  
 $\text{UNDO}[u] \leftarrow (a, \text{MEM}[a])$ , and  $u \leftarrow u + 1$ .  
 Then set  $\text{MEM}[a] \leftarrow v$ . (26)

Here **STAMP** is an array with one entry for each address in **MEM**. It’s initially all zero, and  $\sigma$  is initially 1. Whenever we come to a fallback point, where the current stack pointer will be remembered as the value  $u_0$  for some future undoing, we “bump” the current stamp by setting  $\sigma \leftarrow \sigma + 1$ . Then (26) will continue to do the right thing. (In programs that run for a long time, we must be careful when integer overflow causes  $\sigma$  to be bumped to zero; see exercise 38.)

Notice that the combination of (24) and (25) will perform five memory accesses for each assignment and its undoing. The combination of (26) and (25) will cost seven mems for the first assignment to  $\text{MEM}[a]$ , but only two mems for every subsequent assignment to the same address. So (26) wins, if multiple assignments exceed one-time-only assignments.

**Backtracking through commafree codes.** OK, we’re now equipped with enough basic knowhow to write a pretty good backtrack program for the problem of generating all commafree four-letter codes.

Algorithm C below incorporates one more key idea, which is a lookahead mechanism that is specific to commafree backtracking; we’ll call it the “poison list.” Every item on the poison list is a pair, consisting of a suffix and a prefix that the commafree rule forbids from occurring together. Every green word  $x_1x_2x_3x_4$ —that is, every word that will be a final codeword in the current branch of our backtrack search—contributes three items to the poison list, namely

$$(*x_1x_2x_3, x_4***), \quad (**x_1x_2, x_3x_4**), \quad \text{and} \quad (**x_1, x_2x_3x_4*). \quad (27)$$

**UNDO**  
 stack  
 reversible-memory  
 stamping  
 fallback point  
 bump  
 overflow  
 lookahead  
 poison list

If there's a green word on both sides of a poison list entry, we're dead: The comma-free condition fails, and we must backtrack. If there's a green word on one side but not the other, we can kill off all blue words on the other side by making them red. And if either side of a poison list entry corresponds to an *empty* list, we can remove this entry from the poison list because it will never affect the outcome. (Blue words become red or green, but red words stay red.)

For example, consider the transition from Table 1 to Table 2. When word 0010 becomes green, the poison list receives its first three items:

(\*001, 0\*\*\*),    (\*\*00, 10\*\*),    (\*\*0, 010\*).

The first of these kills off the \*001 list, because 0\*\*\* contains the green word 0010. That makes 1001 red. The last of these, similarly, kills off the 010\* list; but that list is empty when  $m = 2$ . The poison list now reduces to a single item, (\*\*00, 10\*\*), which remains poisonous because list \*\*00 contains the blue word 1100 and 10\*\* contains the blue word 1011.

We'll maintain the poison list at the end of MEM, following the CL lists. It obviously will contain at most  $3(m^4 - m^2)/4$  entries, and in fact it usually turns out to be quite small. No inverse list is required; so we shall adopt the simple method of (17) and (18), but with two cells per entry so that TAIL will change by  $\pm 2$  instead of by  $\pm 1$ . The value of TAIL will be stored in MEM at key times so that temporary changes to it can be undone.

The case  $m = 4$ , in which each codeword consists of four *quaternary* digits  $\{0, 1, 2, 3\}$ , is particularly interesting, because an early backtrack program by Lee Laxdal found that no such comma-free code can make use of all 60 of the cycle classes [0001], [0002],  $\dots$ , [2333]. [See B. H. Jiggs, *Canadian Journal of Math.* **15** (1963), 178–187.] Laxdal's program also reportedly showed that at least three of those classes must be omitted; and it found several valid 57-word sets. Further details were never published, because the proof that 58 codewords are impossible depended on what Jiggs called a “quite time-consuming” computation.

Because size 60 is impossible, our algorithm cannot simply assume that a move such as 1001 is forced when the other words 0011, 0110, 1100 of its class have been ruled out. We must also consider the possibility that class [0011] is entirely absent from the code. Such considerations add an interesting further twist to the problem, and Algorithm C describes one way to cope with it.

**Algorithm C** (*Four-letter comma-free codes*). Given an alphabet size  $m \leq 7$  and a goal  $g$  in the range  $L - m(m - 1) \leq g \leq L$ , where  $L = (m^4 - m^2)/4$ , this algorithm finds all sets of  $g$  four-letter words that are comma-free and include either 0001 or 0010. It uses an array MEM of  $M = \lfloor 23.5m^4 \rfloor$  16-bit numbers, as well as several more auxiliary arrays: ALF of size  $16^3m$ ; STAMP of size  $M$ ; X, C, S, and U of size  $L + 1$ ; FREE and IFREE of size  $L$ ; and a sufficiently large array called UNDO whose maximum size is difficult to guess.

**C1.** [Initialize.] Set  $\text{ALF}[(abcd)_{16}] \leftarrow (abcd)_m$  for  $0 \leq a, b, c, d < m$ . Set  $\text{STAMP}[k] \leftarrow 0$  for  $0 \leq k < M$  and  $\sigma \leftarrow 0$ . Put the initial prefix, suffix, and class lists into MEM, as in Table 1. Also create an empty poison list by

inverse list  
Laxdal  
Jiggs  
stamping++

setting  $\text{MEM}[\text{PP}] \leftarrow \text{POISON}$ , where  $\text{POISON} = 22m^4$  and  $\text{PP} = \text{POISON} - 1$ . Set  $\text{FREE}[k] \leftarrow \text{IFREE}[k] \leftarrow k$  for  $0 \leq k < L$ . Then set  $l \leftarrow 1$ ,  $x \leftarrow \#0001$ ,  $c \leftarrow 0$ ,  $s \leftarrow L - g$ ,  $f \leftarrow L$ ,  $u \leftarrow 0$ , and go to step C3. (Variable  $l$  is the level,  $x$  is a trial word,  $c$  is its class,  $s$  is the “slack,”  $f$  is the number of free classes, and  $u$  is the size of the UNDO stack.)

- C2.** [Enter level  $l$ .] If  $l > L$ , visit the solution  $x_1 \dots x_L$  and go to C6. Otherwise choose a candidate word  $x$  and class  $c$  as described in exercise 39.
- C3.** [Try the candidate.] Set  $\text{U}[l] \leftarrow u$  and  $\sigma \leftarrow \sigma + 1$ . If  $x < 0$ , go to C6 if  $s = 0$  or  $l = 1$ , otherwise set  $s \leftarrow s - 1$ . If  $x \geq 0$ , update the data structures to make  $x$  green, as described in exercise 40, escaping to C5 if trouble arises.
- C4.** [Make the move.] Set  $\text{X}[l] \leftarrow x$ ,  $\text{C}[l] \leftarrow c$ ,  $\text{S}[l] \leftarrow s$ ,  $p \leftarrow \text{IFREE}[c]$ ,  $f \leftarrow f - 1$ . If  $p \neq f$ , set  $y \leftarrow \text{FREE}[f]$ ,  $\text{FREE}[p] \leftarrow y$ ,  $\text{IFREE}[y] \leftarrow p$ ,  $\text{FREE}[f] \leftarrow c$ ,  $\text{IFREE}[c] \leftarrow f$ . (This is (23).) Then set  $l \leftarrow l + 1$  and go to C2.
- C5.** [Try again.] While  $u > \text{U}[l]$ , set  $u \leftarrow u - 1$  and  $\text{MEM}[\text{UNDO}[u] \gg 16] \leftarrow \text{UNDO}[u] \& \#ffff$ . (Those operations restore the previous state, as in (25).) Then  $\sigma \leftarrow \sigma + 1$  and redden  $x$  (see exercise 40). Go to C2.
- C6.** [Backtrack.] Set  $l \leftarrow l - 1$ , and terminate if  $l = 0$ . Otherwise set  $x \leftarrow \text{X}[l]$ ,  $c \leftarrow \text{C}[l]$ ,  $f \leftarrow f - 1$ . If  $x < 0$ , repeat this step (class  $c$  was omitted from the code). Otherwise set  $s \leftarrow \text{S}[l]$  and go back to C5. ■

Exercises 39 and 40 provide the instructive details that flesh out this skeleton.

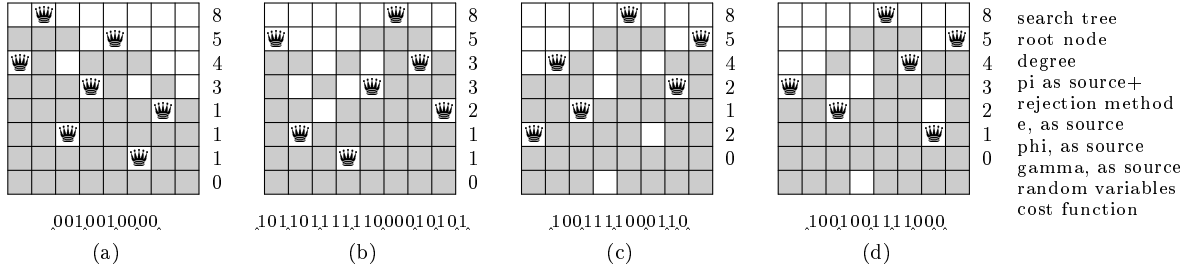
Algorithm C needs just 13, 177, and 2380 megamems to prove that no solutions exist for  $m = 4$  when  $g$  is 60, 59, and 58. It needs about 22800 megamems to find the 1152 solutions for  $g = 57$ ; see exercise 44. There are roughly (14, 240, 3700, 38000) thousand nodes in the respective search trees, with most of the activity taking place on levels  $30 \pm 10$ . The height of the UNDO stack never exceeds 2804, and the poison list never contains more than 12 entries at a time.

**Running time estimates.** Backtrack programs are full of surprises. Sometimes they produce instant answers to a supposedly difficult problem. But sometimes they spin their wheels endlessly, trying to traverse an astronomically large search tree. And sometimes they deliver results just about as fast as we might expect.

Fortunately, we needn’t sit in the dark. There’s a simple Monte Carlo algorithm by which we can often tell in advance whether or not a given backtrack strategy will be feasible. This method, based on random sampling, can actually be worked out by hand *before* writing a program, in order to help decide whether to invest further time while following a particular approach. In fact, the very act of carrying out this pleasant pencil-and-paper method often suggests useful cutoff strategies and/or data structures that will be valuable later when a program is being written. For example, the author developed Algorithm C above after first doing some armchair experiments with random choices of potential comma-free codewords, and noticing that a family of lists such as those in Tables 1 and 2 would be quite helpful when making further choices.

To illustrate the method, let’s consider the  $n$  queens problem again, as represented in Algorithm B\* above. When  $n = 8$ , we can obtain a decent “ballpark

Running time  
estimates of run time—  
Monte Carlo algorithm  
random sampling  
pencil-and-paper method  
cutoff strategies  
data structures  
author



**Fig. 69.** Four random attempts to solve the 8 queens problem. Such experiments help to estimate the size of the backtrack tree in Fig. 68. The branching degrees are shown at the right of each diagram, while the random bits used for sampling appear below. Cells have been shaded in gray if they are attacked by one or more queens in earlier rows.

estimate” of the size of Fig. 68 by examining only a few random paths in that search tree. We start by writing down the number  $D_1 \leftarrow 8$ , because there are eight ways to place the queen in row 1. (In other words, the root node of the search tree has degree 8.) Then we use a source of random numbers—say the binary digits of  $\pi \bmod 1 = (.001001000011\dots)_2$ —to select one of those placements. Eight choices are possible, so we look at three of those bits; we shall set  $X_1 \leftarrow 2$ , because 001 is the second of the eight possibilities (000, 001,  $\dots$ , 111).

Given  $X_1 = 2$ , the queen in row 2 can’t go into columns 1, 2, or 3. Hence five possibilities remain for  $X_2$ , and we write down  $D_2 \leftarrow 5$ . The next three bits of  $\pi$  lead us to set  $X_2 \leftarrow 5$ , since 5 is the second of the available columns (4, 5, 6, 7, 8) and 001 is the second value of (000, 001,  $\dots$ , 100). If  $\pi$  had continued with 101 or 110 or 111 instead of 001, we would incidentally have used the “rejection method” of Section 3.4.1 and moved to the next three bits; see exercise 47.

Continuing in this way leads to  $D_3 \leftarrow 4$ ,  $X_3 \leftarrow 1$ ; then  $D_4 \leftarrow 3$ ,  $X_4 \leftarrow 4$ . (Here we used the two bits 00 to select  $X_3$ , and the next two bits 00 to select  $X_4$ .) The remaining branches are forced:  $D_5 \leftarrow 1$ ,  $X_5 \leftarrow 7$ ;  $D_6 \leftarrow 1$ ,  $X_6 \leftarrow 3$ ;  $D_7 \leftarrow 1$ ,  $X_7 \leftarrow 6$ ; and we’re stuck when we reach level 8 and find  $D_8 \leftarrow 0$ .

These sequential random choices are depicted in Fig. 69(a), where we’ve used them to place each queen successively into an unshaded cell. Parts (b), (c), and (d) of Fig. 69 correspond in the same way to choices based on the binary digits of  $e \bmod 1$ ,  $\phi \bmod 1$ , and  $\gamma \bmod 1$ . Exactly 10 bits of  $\pi$ , 20 bits of  $e$ , 13 bits of  $\phi$ , and 13 bits of  $\gamma$  were used to generate these examples.

In this discussion the notation  $D_k$  stands for a branching degree, not for a domain of values. We’ve used uppercase letters for the numbers  $D_1$ ,  $X_1$ ,  $D_2$ , etc., because those quantities are random variables. Once we’ve reached  $D_l = 0$  at some level, we’re ready to estimate the overall cost, by implicitly assuming that the path we’ve taken is representative of *all* root-to-leaf paths in the tree.

The cost of a backtrack program can be assessed by summing the individual amounts of time spent at each node of the search tree. Notice that every node on level  $l$  of that tree can be labeled uniquely by a sequence  $x_1 \dots x_{l-1}$ , which defines the path from the root to that node. Thus our goal is to estimate the sum of all  $c(x_1 \dots x_{l-1})$ , where  $c(x_1 \dots x_{l-1})$  is the cost associated with node  $x_1 \dots x_{l-1}$ .



For example, the four queens problem is represented by the search tree (4), and its cost is the sum of 17 individual costs

$$c() + c(1) + c(13) + c(14) + c(142) + c(2) + c(24) + \cdots + c(413) + c(42). \quad (28)$$

If  $C(x_1 \dots x_l)$  denotes the total cost of the subtree rooted at  $x_1 \dots x_l$ , then

$$C(x_1 \dots x_l) = c(x_1 \dots x_l) + C(x_1 \dots x_l x_{l+1}^{(1)}) + \cdots + C(x_1 \dots x_l x_{l+1}^{(d)}) \quad (29)$$

when the choices for  $x_{l+1}$  at node  $x_1 \dots x_l$  are  $\{x_{l+1}^{(1)}, \dots, x_{l+1}^{(d)}\}$ . For instance in (4) we have  $C(1) = c(1) + C(13) + C(14)$ ;  $C(13) = c(13)$ ; and  $C() = c() + C(1) + C(2) + C(3) + C(4)$  is the overall cost (28).

In these terms a Monte Carlo estimate for  $C()$  is extremely easy to compute:

**Theorem E.** *Given  $D_1, X_1, D_2, X_2, \dots$  as above, the cost of backtracking is*

$$C() = E(c() + D_1(c(X_1) + D_2(c(X_1 X_2) + D_3(c(X_1 X_2 X_3) + \cdots))). \quad (30)$$

*Proof.* Node  $x_1 \dots x_l$ , with branch degrees  $d_1, \dots, d_l$  above it, is reached with probability  $1/d_1 \dots d_l$ ; so it contributes  $d_1 \dots d_l c(x_1 \dots x_l) / d_1 \dots d_l = c(x_1 \dots x_l)$  to the expected value in this formula. ■

For example, the tree (4) has six root-to-leaf paths, and they occur with respective probabilities  $1/8, 1/8, 1/4, 1/4, 1/8, 1/8$ . The first one contributes  $1/8$  times  $c() + 4(c(1) + 2(c(13)))$ , namely  $c()/8 + c(1)/2 + c(13)$ , to the expected value. The second contributes  $c()/8 + c(1)/2 + c(14) + c(142)$ ; and so on.

A special case of Theorem E, with all  $c(x_1 \dots x_l) = 1$ , tells us how to estimate the total size of the tree, which is often a crucial quantity:

**Corollary E.** *The number of nodes in the search tree, given  $D_1, D_2, \dots$ , is*

$$E(1 + D_1 + D_1 D_2 + \cdots) = E(1 + D_1(1 + D_2(1 + D_3(1 + \cdots))). \quad (31)$$

For example, Fig. 69 gives us four estimates for the size of the tree in Fig. 68, using the numbers  $D_j$  at the right of each  $8 \times 8$  diagram. The estimate from Fig. 69(a) is  $1 + 8(1 + 5(1 + 4(1 + 3(1 + 1(1 + 1(1 + 1)))))) = 2129$ ; and the other three are respectively 2689, 1489, 2609. None of them is extremely far from the true number, 2057, although we can't expect to be so lucky all the time.

The detailed study in exercise 51 shows that the estimate (31) in the case of 8 queens turns out to be quite well behaved:

$$(\min 489, \text{ ave } 2057, \text{ max } 7409, \text{ dev } \sqrt{1146640} \approx 1071). \quad (32)$$

The analogous problem for 16 queens has a much less homogeneous search tree:

$$(\min 2597105, \text{ ave } 1141190303, \text{ max } 131048318769, \text{ dev } \approx 1234000000). \quad (33)$$

Still, this standard deviation is roughly the same as the mean, so we'll usually guess the correct order of magnitude. (For example, ten independent experiments predicted .632, .866, .237, 1.027, 4.006, .982, .143, .140, 3.402, and .510 billion nodes, respectively. The mean of these is 1.195.) A thousand trials with  $n = 64$  suggest that the problem of 64 queens will have about  $3 \times 10^{65}$  nodes in its tree.

subtree  
size of the tree  
standard deviation

Let's formulate this estimation procedure precisely, so that it can be performed conveniently by machine as well as by hand:

binomial tree  
estimating solutions

**Algorithm E** (*Estimated cost of backtrack*). Given domains  $D_k$  and properties  $P_l$  as in Algorithm B, together with node costs  $c(x_1 \dots x_l)$  as above, this algorithm computes the quantity  $S$  whose expected value is the total cost  $C()$  in (30). It uses an auxiliary array  $y_1 y_2 \dots$  whose size should be  $\geq \max(|D_1|, \dots, |D_n|)$ .

- E1.** [Initialize.] Set  $l \leftarrow D \leftarrow 1$ ,  $S \leftarrow 0$ , and initialize any data structures needed.
- E2.** [Enter level  $l$ .] (At this point  $P_{l-1}(X_1, \dots, X_{l-1})$  holds.) Set  $S \leftarrow S + D \cdot c(X_1 \dots X_{l-1})$ . If  $l > n$ , terminate the algorithm. Otherwise set  $d \leftarrow 0$  and set  $x \leftarrow \min D_l$ , the smallest element of  $D_l$ .
- E3.** [Test  $x$ .] If  $P_l(X_1, \dots, X_{l-1}, x)$  holds, set  $y_d \leftarrow x$  and  $d \leftarrow d + 1$ .
- E4.** [Try again.] If  $x \neq \max D_l$ , set  $x$  to the next larger element of  $D_l$  and return to step E3.
- E5.** [Choose and try.] If  $d = 0$ , terminate. Otherwise set  $D \leftarrow D \cdot d$  and  $X_l \leftarrow y_I$ , where  $I$  is a uniformly random integer in  $\{0, \dots, d - 1\}$ . Update the data structures to facilitate testing  $P_{l+1}$ , set  $l \leftarrow l + 1$ , and go back to E2. ■

Although Algorithm E looks rather like Algorithm B, it never backtracks.

Of course we can't expect this algorithm to give decent estimates in cases where the backtrack tree is wildly erratic. The *expected* value of  $S$ , namely  $ES$ , is indeed the true cost; but the *probable* values of  $S$  might be quite different.

An extreme example of bad behavior occurs if property  $P_l$  is the simple condition ' $x_1 > \dots > x_l$ ' and all domains are  $\{1, \dots, n\}$ . Then there's only one solution,  $x_1 \dots x_n = n \dots 1$ ; and backtracking is a particularly stupid way to find it!

The search tree for this somewhat ridiculous problem is, nevertheless, quite interesting. It is none other than the binomial tree  $T_n$  of Eq. 7.2.1.3–(21), which has  $\binom{n}{l}$  nodes on level  $l + 1$  and  $2^n$  nodes in total. If we set all costs to 1, the expected value of  $S$  is therefore  $2^n = e^{n \ln 2}$ . But exercise 50 proves that  $S$  will almost always be much smaller, less than  $e^{(\ln n)^2 \ln \ln n}$ . Furthermore the average value of  $l$  when Algorithm E terminates with respect to  $T_n$  is only  $H_n + 1$ . When  $n = 100$ , for example, the probability that  $l \geq 20$  on termination is only 0.0000000027, while the vast majority of the nodes are near level 51.

Many refinements of Algorithm E are possible. For example, exercise 52 shows that the choices in step E5 need not be uniform. We shall discuss improved estimation techniques in Section 7.2.2.9, after having seen numerous examples of backtracking in practice.

**\*Estimating the number of solutions.** Sometimes we know that a problem has more solutions than we could ever hope to generate, yet we still want to know roughly how many there are. Algorithm E will tell us the approximate number, in cases where the backtrack process never reaches a dead end—that is, if it never terminates with  $d = 0$  in step E5. There may be another criterion for successful termination in step E2 even though  $l$  might still be  $\leq n$ . The expected final value of  $D$  is exactly the total number of solutions, because every solution  $X_1 \dots X_l$  constructed by the algorithm is obtained with probability  $1/D$ .

For example, suppose we want to know the number of different paths by which a king can go from one corner of a chessboard to the opposite corner, without revisiting any square. One such path, chosen at random using the bits of  $\pi$  for guidance as we did in Fig. 69(a), is shown here. Starting in the upper left corner, we have 3 choices for the first move. Then, after moving to the right, there are 4 choices for the second move. And so on. We never make a move that would disconnect us from the goal; in particular, two of the moves are actually forced. (Exercise 58 explains one way to avoid fatal mistakes.)

The probability of obtaining this particular path is exactly  $\frac{1}{3} \frac{1}{4} \frac{1}{6} \frac{1}{6} \frac{1}{2} \frac{1}{6} \frac{1}{7} \dots \frac{1}{2} = 1/D$ , where  $D = 3 \times 4 \times 6 \times 6 \times 2 \times 6 \times 7 \times \dots \times 2 = 1^2 \cdot 2^4 \cdot 3^4 \cdot 4^{10} \cdot 5^9 \cdot 6^6 \cdot 7^1 \approx 8.7 \times 10^{20}$ . Thus we can reasonably guess, at least tentatively, that there are  $10^{21}$  such paths, more or less.

Of course that guess, based on a single random sample, rests on very shaky grounds.

But we know that the average value  $M_N = (D^{(1)} + \dots + D^{(N)})/N$  of  $N$  guesses, in  $N$  independent experiments, will almost surely approach the correct number.

How large should  $N$  be, before we can have any confidence in the results? The actual values of  $D$  obtained from random king paths tend to vary all over the map. Figure 70 plots typical results, as  $N$  varies from 1 to 10000. For each value of  $N$  we can follow the advice of statistics textbooks and calculate the sample variance  $V_N = S_N/(N-1)$  as in Eq. 4.2.2–(16); then  $M_N \pm \sqrt{V_N/N}$  is the textbook estimate. The top diagram in Fig. 70 shows these “error bars” in gray, surrounding black dots for  $M_N$ . This sequence  $M_N$  does appear to settle down after  $N$  reaches 3000 or so, and to approach a value near  $5 \times 10^{25}$ . That’s much higher than our first guess, but it has lots of evidence to back it up.

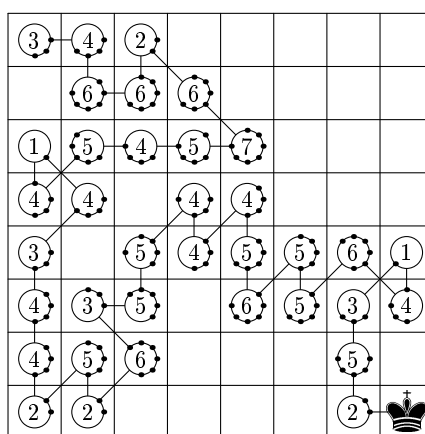
On the other hand, the bottom chart in Fig. 70 shows the distribution of the *logarithms* of the 10000 values of  $D$  that were used to make the top chart. Almost half of those values were totally negligible—less than  $10^{20}$ . About 75% of them were less than  $10^{24}$ . But some of them\* exceeded  $10^{28}$ . Can we really rely on a result that’s based on such chaotic behavior? Is it really right to throw away most of our data and to trust almost entirely on observations that were obtained from comparatively few rare events?

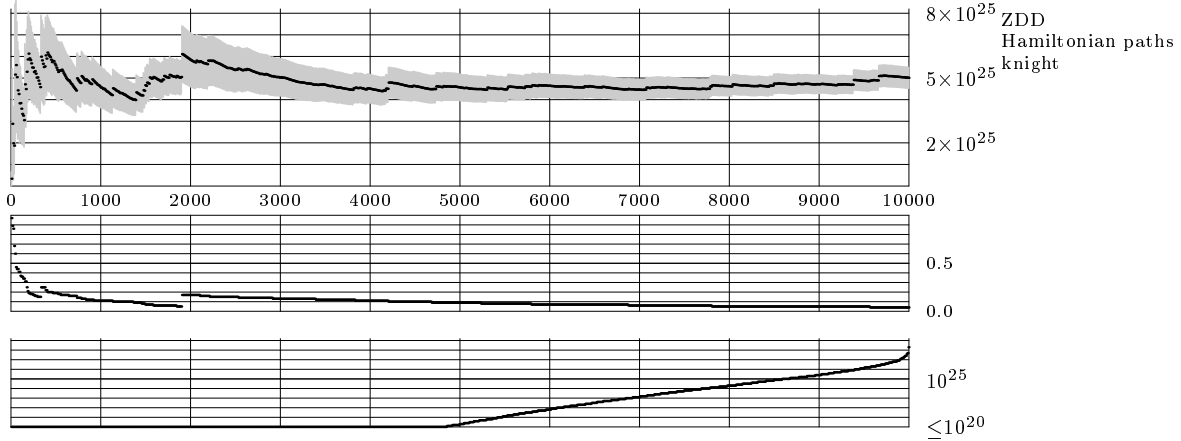
Yes, we’re okay! Some of the justification appears in exercise MPR–124, which is based on theoretical work by P. Diaconis and S. Chatterjee. In the paper cited with that exercise, they defend a simple measure of quality,

$$Q_N = \max(D^{(1)}, \dots, D^{(N)}) / (NM_N) = \frac{\max(D^{(1)}, \dots, D^{(N)})}{D^{(1)} + \dots + D^{(N)}}, \quad (34)$$

\* Four of the actual values that led to Fig. 70 were larger than  $10^{28}$ ; the largest,  $\approx 2.1 \times 10^{28}$ , came from a path of length 57. The smallest estimate, 19361664, came from a path of length 10.

king  
chessboard  
simple paths  
paths, simple  
 $\pi$  for guidance  
statistics  
sample variance  
variance  
error bars  
discarded data  
Diaconis  
Chatterjee





**Fig. 70.** Estimates of the number of king paths, based on up to 10000 random trials. The middle graph shows the corresponding quality measures of Eq. (34). The lower graph shows the *logarithms* of the individual estimates  $D^{(k)}$ , after they've been sorted.

arguing that a reasonable policy in most experiments such as these is to stop sampling when  $Q_N$  gets small. (Values of this statistic  $Q_N$  have been plotted in the middle of Fig. 70.)

Furthermore we can estimate other properties of the solutions to a backtrack problem, instead of merely counting those solutions. For example, the expected value of  $lD$  on termination of the random king's path algorithm is the total *length* of such paths. The data underlying Fig. 70 suggests that this total is  $(2.66 \pm .14) \times 10^{27}$ ; hence the average path length appears to be about 53. The samples also indicate that about 34% of the paths pass through the center; about 46% touch the upper right corner; about 22% touch both corners; and about 7% pass through the center and both corners.

For this particular problem we don't actually need to rely on estimates, because the ZDD technology of Section 7.1.4 allows us to compute the *true* values. (See exercise 59.) The total number of simple corner-to-corner king paths on a chessboard is exactly 50,819,542,770,311,581,606,906,543; this value lies almost within the error bars of Fig. 70 for all  $N \geq 250$ , except for a brief interval near  $N = 1400$ . And the total length of all these paths turns out to be exactly 2,700,911,171,651,251,701,712,099,831, which is a little higher than our estimate. The true average length is therefore  $\approx 53.15$ . The true probabilities of hitting the center, a given corner, both corners, and all three of those spots are respectively about 38.96%, 50.32%, 25.32%, and 9.86%.

The total number of corner-to-corner king paths of the maximum length, 63, is 2,811,002,302,704,446,996,926. This is a number that can *not* be estimated well by a method such as Algorithm E without additional heuristics.

The analogous problem for corner-to-corner *knight* paths, of any length, lies a bit beyond ZDD technology because many more ZDD nodes are needed. Using Algorithm E we can estimate that there are about  $(8.6 \pm 1.2) \times 10^{19}$  such paths.

**Factoring the problem.** Imagine an instance of backtracking that is equivalent to solving two *independent* subproblems. For example, we might be looking for all sequences  $x = x_1 x_2 \dots x_n$  that satisfy  $P_n(x_1, x_2, \dots, x_n) = F(x_1, x_2, \dots, x_n)$ , where

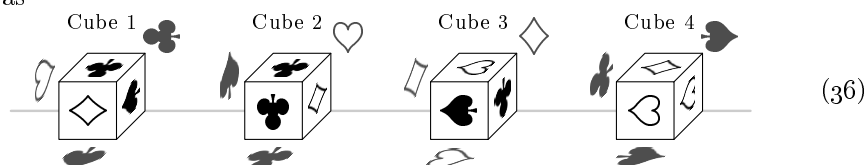
$$F(x_1, x_2, \dots, x_n) = G(x_1, \dots, x_k) \wedge H(x_{k+1}, \dots, x_n). \quad (35)$$

Then the size of the backtrack tree is essentially the *product* of the tree sizes for  $G$  and for  $H$ , even if we use dynamic ordering. Hence it's obviously foolish to apply the general setup of (1) and (2). We can do much better by finding all solutions to  $G$  first, then finding all solutions to  $H$ , thereby reducing the amount of computation to the *sum* of the tree sizes. Again we've divided and conquered, by factoring the compound problem (35) into separate subproblems.

We discussed a less obvious application of problem factorization near the beginning of Chapter 7, in connection with latin squares: Recall that E. T. Parker sped up the solution of 7-(6) by more than a dozen orders of magnitude, when he discovered 7-(7) by essentially factoring 7-(6) into ten subproblems whose solutions could readily be combined.

In general, each solution  $x$  to some problem  $F$  often implies the existence of solutions  $x^{(p)} = \phi_p(x)$  to various simpler problems  $F_p$  that are “homomorphic images” of  $F$ . And if we're lucky, the solutions to those simpler problems can be combined and “lifted” to a solution of the overall problem. Thus it pays to be on the lookout for such simplifications.

Let's look at another example. F. A. Schossow invented a tantalizing puzzle [U.S. Patent 646463 (3 April 1900)] that “went viral” in 1967 when a marketing genius decided to rename it *Instant Insanity*®. The problem is to take four cubes such as



where each face has been marked in one of four ways, and to arrange them in a row so that all four markings appear on the top, bottom, front, and back sides. The placement in (36) is incorrect, because there are two ♣s (and no ♠) on top. But we get a solution if we rotate each cube by 90°.

There are 24 ways to place each cube, because any of the six faces can be on top and we can rotate four ways while keeping the top unchanged. So the total number of placements is  $24^4 = 331776$ . But this problem can be factored in an ingenious way, so that all solutions can be found quickly by hand! [See F. de Carteblanche, *Eureka* 9 (1947), 9–11.] The idea is that any solution to the puzzle gives us two each of {♣, ♦, ♥, ♠}, if we look only at the top and bottom or only at the front and back. That's a much easier problem to solve.

For this purpose a cube can be characterized by its three pairs of markings on opposite faces; in (36) these face-pairs are respectively

$$\{\clubsuit\heartsuit, \clubsuit\spadesuit, \heartsuit\diamondsuit\}, \quad \{\clubsuit\clubsuit, \clubsuit\heartsuit, \spadesuit\diamondsuit\}, \quad \{\heartsuit\heartsuit, \clubsuit\diamondsuit, \spadesuit\diamondsuit\}, \quad \{\clubsuit\heartsuit, \spadesuit\heartsuit, \spadesuit\diamondsuit\}. \quad (37)$$

factorization of problems–  
*independent* subproblems  
backtrack tree  
tree sizes  
dynamic ordering  
divide and conquer paradigm  
latin squares  
Parker  
homomorphic images  
lifted  
Schossow  
Patent  
Tantalizer, see Instant InsanityTM  
Armbruster  
Instant InsanityTM  
Carteblanche

Which of the  $3^4 = 81$  ways to choose one face-pair from each cube will give us  $\{\clubsuit, \clubsuit, \diamond, \diamond, \heartsuit, \heartsuit, \spadesuit, \spadesuit\}$ ? They can all be discovered in a minute or two, by listing the nine possibilities for cubes (1, 2) and the nine for (3, 4). We get just three,

$$(\clubsuit\diamond, \clubsuit\heartsuit, \spadesuit\diamond, \spadesuit\heartsuit), \quad (\spadesuit\heartsuit, \clubsuit\heartsuit, \clubsuit\diamond, \spadesuit\diamond), \quad (\spadesuit\heartsuit, \spadesuit\diamond, \clubsuit\diamond, \clubsuit\heartsuit). \quad (38)$$

Notice furthermore that each solution can be “halved” so that one each of  $\{\clubsuit, \diamond, \heartsuit, \spadesuit\}$  appears on both sides, by swapping face-pairs; we can change (38) to


$$(\diamond\clubsuit, \clubsuit\heartsuit, \spadesuit\diamond, \heartsuit\spadesuit), \quad (\heartsuit\spadesuit, \clubsuit\heartsuit, \diamond\clubsuit, \spadesuit\diamond), \quad (\heartsuit\spadesuit, \spadesuit\diamond, \diamond\clubsuit, \clubsuit\heartsuit). \quad (39)$$

Each of these solutions to the opposite-face subproblem can be regarded as a 2-regular graph, because every vertex of the multigraph whose edges are (say)  $\diamond — \clubsuit, \clubsuit — \heartsuit, \spadesuit — \diamond, \heartsuit — \spadesuit$  has exactly two neighbors.

A solution to Instant Insanity<sup>©</sup> will give us *two* such 2-regular factors, one for top-and-bottom and one for front-and-back. Furthermore those two factors will have disjoint edges: We can’t use the same face-pair in both. Therefore problem (36) can be solved only by using the first and third factor in (39).

Conversely, whenever we have two disjoint 2-regular graphs, we can always use them to position the cubes as desired, thus “lifting” the factors to a solution of the full problem.

Exercise 75 illustrates another kind of problem factorization.

 I may decide to insert a (small?) amount of additional material here, as I prepare sections 7.2.2.1–7.2.2.9.

**Historical notes.** The origins of backtrack programming are obscure. Equivalent ideas must have occurred to many people, yet there was hardly any reason to write them down until computers existed. We can be reasonably sure that James Bernoulli used such principles in the 17th century, when he successfully solved the “Tot tibi sunt dotes” problem that had eluded so many others (see Section 7.2.1.7), because traces of the method exist in his exhaustive list of solutions.

Backtrack programs typically traverse the tree of possibilities by using what is now called depth-first search, a general graph exploration procedure that Édouard Lucas credited to a student named Trémaux [*Récréations Mathématiques* **1** (Paris: Gauthier-Villars, 1882), 47–50].

The eight queens problem was first proposed by Max Bezzel [*Schachzeitung* **3** (1848), 363; **4** (1849), 40] and by Franz Nauck [*Illustrierte Zeitung* **14**, 361 (1 June 1850), 352; **15**, 377 (21 September 1850), 182], perhaps independently. C. F. Gauss saw the latter publication, and wrote several letters about it to his friend H. C. Schumacher. Gauss’s letter of 27 September 1850 is especially interesting, because it explained how to find all the solutions by backtracking—which he called ‘Tatonniren’, from a French term meaning “to feel one’s way.” He also listed the lexicographically first solutions of each equivalence class under reflection and rotation: 15863724, 16837425, 24683175, 25713864, 25741863, 26174835, 26831475, 27368514, 27581463, 35281746, 35841726, and 36258174.

2-regular graph  
disjoint  
lifting  
Bernoulli  
Tot tibi  
depth-first search  
Lucas  
Trémaux  
eight queens problem  
Bezzel  
Nauck  
Gauss  
Schumacher  
lexicographically

Computers arrived a hundred years later, and people began to use them for combinatorial problems. The time was therefore ripe for backtracking to be described as a general technique, and Robert J. Walker rose to the occasion [*Proc. Symposia in Applied Math.* **10** (1960), 91–94]. His brief note introduced Algorithm W in machine-oriented form, and mentioned that the procedure could readily be extended to find variable-length patterns  $x_1 \dots x_n$  where  $n$  is not fixed.

The next milestone was a paper by Solomon W. Golomb and Leonard D. Baumert [*JACM* **12** (1965), 516–524], who formulated the general problem carefully and presented a variety of examples. In particular, they discussed the search for maximum comma-free codes, and noted that backtracking can be used to find successively better and better solutions to combinatorial optimization problems. They introduced certain kinds of lookahead, as well as the important idea of dynamic ordering by branching on variables with the fewest remaining choices.

Other noteworthy early discussions of backtrack programming appear in Mark Wells’s book *Elements of Combinatorial Computing* (1971), Chapter 4; in a survey by J. R. Bitner and E. M. Reingold, *CACM* **18** (1975), 651–656; and in the Ph.D. thesis of John Gaschnig [Report CMU-CS-79-124 (Carnegie Mellon University, 1979), Chapter 4]. Gaschnig introduced techniques of “backmarking” and “backjumping” that we shall discuss later.

Monte Carlo estimates of the cost of backtracking were first described briefly by M. Hall, Jr., and D. E. Knuth in *Computers and Computing*, *AMM* **72**, 2, part 2, Slaughter Memorial Papers No. 10 (February 1965), 21–28. Knuth gave a much more detailed exposition a decade later, in *Math. Comp.* **29** (1975), 121–136. Such methods can be considered as special cases of so-called “importance sampling”; see J. M. Hammersley and D. C. Handscomb, *Monte Carlo Methods* (London: Methuen, 1964), 57–59. Studies of random self-avoiding walks such as the king paths discussed above were inaugurated by M. N. Rosenbluth and A. W. Rosenbluth, *J. Chemical Physics* **23** (1955), 356–359.

Backtrack applications are nicely adaptable to parallel programming, because different parts of the search tree are often completely independent of each other; thus disjoint subtrees can be explored on different machines, with a minimum of interprocess communication. Already in 1964, D. H. Lehmer explained how to subdivide a problem so that two computers of different speeds could work on it simultaneously and finish at the same time. The problem that he considered had a search tree of known shape (see Theorem 7.2.1.3L); but we can do essentially similar load balancing even in much more complicated situations, by using Monte Carlo estimates of the subtree sizes. Although many ideas for parallelizing combinatorial searches have been developed over the years, such techniques are beyond the scope of this book. Readers can find a nice introduction to a fairly general approach in the paper by R. Finkel and U. Manber, *ACM Transactions on Programming Languages and Systems* **9** (1987), 235–256.

M. Alekhovich, A. Borodin, J. Buresh-Oppenheimer, R. Impagliazzo, A. Magen, and T. Pitassi have defined *priority branching trees*, a general model of computation with which they were able to prove rigorous bounds on what backtrack programs can do, in *Computational Complexity* **20** (2011), 679–740.

Walker  
Golomb  
Baumert  
comma-free codes  
optimization  
lookahead  
dynamic ordering  
minimum remaining values  
Wells  
Bitner  
Reingold  
Gaschnig  
backmarking  
backjumping  
Monte Carlo estimates  
Hall  
Knuth  
importance sampling  
Hammersley  
Handscomb  
self-avoiding walks  
king paths  
Rosenbluth  
Rosenbluth  
parallel programming  
search tree  
Lehmer  
load balancing  
Finkel  
Manber  
Alekhovich  
Borodin  
Buresh-Oppenheimer  
Impagliazzo  
Magen  
Pitassi  
priority branching trees

## EXERCISES

- 1. [22] Explain how the tasks of generating (i)  $n$ -tuples, (ii) permutations of distinct items, (iii) combinations, (iv) integer partitions, (v) set partitions, and (vi) nested parentheses can all be regarded as special cases of backtrack programming, by presenting suitable domains  $D_k$  and cutoff properties  $P_l(x_1, \dots, x_l)$  that satisfy (1) and (2).

2. [10] True or false: We can choose  $D_1$  so that  $P_1(x_1)$  is always true.

3. [16] Using a chessboard and eight coins to represent queens, one can follow the steps of Algorithm B and essentially traverse the tree of Fig. 68 by hand in about three hours. Invent a trick to save half of the work.

- 4. [20] Reformulate Algorithm B as a *recursive* procedure called *try*( $l$ ), having global variables  $n$  and  $x_1 \dots x_n$ , to be invoked by saying ‘*try*(1)’. Can you imagine why the author of this book decided *not* to present the algorithm in such a recursive form?

5. [20] Given  $r$ , with  $1 \leq r \leq n$ , in how many ways can 7 nonattacking queens be placed on an  $8 \times 8$  chessboard, if no queen is placed in row  $r$ ?

6. [20] (T. B. Sprague, 1890.) Are there any values  $n > 5$  for which the  $n$  queens problem has a “framed” solution with  $x_1 = 2$ ,  $x_2 = n$ ,  $x_{n-1} = 1$ , and  $x_n = n - 1$ ?

7. [20] Are there two 8-queen placements with the same  $x_1 x_2 x_3 x_4 x_5 x_6$ ?

8. [21] Can a  $4m$ -queen placement have  $3m$  queens on “white” squares?

- 9. [22] Adapt Algorithm W to the  $n$  queens problem, using bitwise operations on  $n$ -bit numbers as suggested in the text.

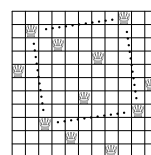
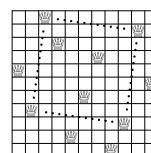
10. [M25] (W. Ahrens, 1910.) Both solutions of the  $n$  queens problem when  $n = 4$  have *chiral symmetry*: Rotation by  $90^\circ$  leaves them unchanged, but reflection doesn’t.

a) Can the  $n$  queens problem have a solution with reflection symmetry?

b) Show that chiral symmetry is impossible when  $n \bmod 4 \in \{2, 3\}$ .

c) Sometimes the solution to an  $n$  queens problem contains four queens that form the corners of a tilted square, as shown here. Prove that we can always get another solution by tilting the square the other way (but leaving the other  $n - 4$  queens in place).

d) Let  $C_n$  be the number of chirally symmetric solutions, and suppose  $c_n$  of them have  $x_k > k$  for  $1 \leq k \leq n/2$ . Prove that  $C_n = 2^{\lfloor n/4 \rfloor} c_n$ .



11. [M28] (*Wraparound queens*.) Replace (3) by the stronger conditions ‘ $x_j \neq x_k$ ,  $(x_k - x_j) \bmod n \neq k - j$ ,  $(x_j - x_k) \bmod n \neq k - j$ ’. (The  $n \times n$  grid becomes a torus.) Prove that the resulting problem is solvable if and only if  $n$  is not divisible by 2 or 3.

12. [M30] For which  $n \geq 0$  does the  $n$  queens problem have at least one solution?

13. [M25] If exercise 11 has  $T(n)$  toroidal solutions, show that  $Q(mn) \geq Q(m)^n T(n)$ .

14. [HM47] Does  $(\ln Q(n))/(n \ln n)$  approach a positive constant as  $n \rightarrow \infty$ ?

15. [21] Let  $H(n)$  be the number of ways that  $n$  queen bees can occupy an  $n \times n$  honeycomb so that no two are in the same line. (For example, one of the  $H(4) = 7$  ways is shown here.) Compute  $H(n)$  for small  $n$ .



16. [15] J. H. Quick (a student) noticed that the loop in step L2 of Algorithm L can be changed from ‘while  $x_l < 0$ ’ to ‘while  $x_l \neq 0$ ’, because  $x_l$  cannot be positive at that point of the algorithm. So he decided to eliminate the minus signs and just set  $x_{l+k+1} \leftarrow k$  in step L3. Was it a good idea?

$n$ -tuples  
tuples  
permutations  
combinations  
integer partitions  
partitions  
set partitions  
nested parentheses  
parentheses  
domains  
cutoff  
properties  
recursive  
global variables  
author  
recursion versus iteration  
Sprague  
bitwise operations  
Ahrens  
chiral symmetry  
Rotation by  $90^\circ$   
Wraparound queens  
broken diagonal, see wraparound  
torus  
 $n$  queens problem  
queen bees  
bees  
honeycomb  
hexagons  
Quick  
Langford pairs+



17. [17] Suppose that  $n = 4$  and Algorithm L has reached step L2 with  $l = 4$  and  $x_1x_2x_3 = 241$ . What are the current values of  $x_4x_5x_6x_7x_8$ ,  $p_0p_1p_2p_3p_4$ , and  $y_1y_2y_3$ ?
19. [M19] What are the domains  $D_l$  in Langford's problem (7)?
- 20. [21] Extend Algorithm L so that it forces  $x_l \leftarrow k$  whenever  $k \notin \{x_1, \dots, x_{l-1}\}$ .
- 21. [M25] If  $x = x_1x_2 \dots x_{2n}$ , let  $x^D = (-x_{2n}) \dots (-x_2)(-x_1) = -x^R$  be its dual.
- Show that if  $n$  is odd and  $x$  solves Langford's problem (7), we have  $x_k = n$  for some  $k \leq \lfloor n/2 \rfloor$  if and only if  $x_k^D = n$  for some  $k > \lfloor n/2 \rfloor$ .
  - Find a similar rule that distinguishes  $x$  from  $x^D$  when  $n$  is even.
  - Consequently the algorithm of exercise 20 can be modified so that exactly one of each dual pair of solutions  $\{x, x^D\}$  is visited.
22. [M26] Explore "loose Langford pairs": Replace ' $j + k + 1$ ' in (7) by ' $j + \lfloor 3k/2 \rfloor$ '.
23. [17] We can often obtain one word rectangle from another by changing only a letter or two. Can you think of any  $5 \times 6$  rectangles that almost match (10)?
24. [20] Customize Algorithm B so that it will find all  $5 \times 6$  word rectangles.
- 25. [25] Explain how to use *orthogonal lists*, as in Fig. 13 of Section 2.2.6, so that it's easy to visit all 5-letter words whose  $k$ th character is  $c$ , given  $1 \leq k \leq 5$  and  $a \leq c \leq z$ . Use those sublists to speed up the algorithm of exercise 24.
26. [21] Can you find nice word rectangles of sizes  $5 \times 7$ ,  $5 \times 8$ ,  $5 \times 9$ ,  $5 \times 10$ ?
27. [22] What profile and average node costs replace (13) and (14) when we ask the algorithm of exercise 25 for  $6 \times 5$  word rectangles instead of  $5 \times 6$ ?
- 28. [23] The method of exercises 24 and 25 does  $n$  levels of backtracking to fill the cells of an  $m \times n$  rectangle one column at a time, using a trie to detect illegal prefixes in the rows. Devise a method that does  $mn$  levels of backtracking and fills just *one* cell per level, using tries for *both* rows and columns.
29. [15] What's the largest commafree subset of the following words?

aced babe bade bead beef cafe cede dada dead deaf face fade feed

- 30. [22] Let  $w_1, w_2, \dots, w_m$  be four-letter words on an  $m$ -letter alphabet. Design an algorithm that accepts or rejects each  $w_j$ , according as  $w_j$  is commafree or not with respect to the accepted words of  $\{w_1, \dots, w_{j-1}\}$ .
31. [M22] A two-letter block code on an  $m$ -letter alphabet can be represented as a digraph  $D$  on  $m$  vertices, with  $a \rightarrow b$  if and only if  $ab$  is a codeword.
- Prove that the code is commafree  $\iff D$  has no oriented paths of length 3.
  - How many arcs can be in a digraph with no oriented paths of length  $r$ ?
- 32. [M30] (W. L. Eastman, 1965.) The following elegant construction yields a commafree code of maximum size for any *odd* block length  $n$ , over any alphabet. Given a sequence of  $x = x_0x_1 \dots x_{n-1}$  of nonnegative integers, where  $x$  differs from each of its other cyclic shifts  $x_k \dots x_{n-1}x_0 \dots x_{k-1}$  for  $0 < k < n$ , the procedure outputs a cyclic shift  $\sigma x$  with the property that the set of all such  $\sigma x$  is commafree.

We regard  $x$  as an infinite periodic sequence  $\langle x_n \rangle$  with  $x_k = x_{k-n}$  for all  $k \geq n$ . Each cyclic shift then has the form  $x_kx_{k+1} \dots x_{k+n-1}$ . The simplest nontrivial example occurs when  $n = 3$ , where  $x = x_0x_1x_2x_0x_1x_2x_0 \dots$  and we don't have  $x_0 = x_1 = x_2$ . In this case the algorithm outputs  $x_kx_{k+1}x_{k+2}$  where  $x_k > x_{k+1} \leq x_{k+2}$ ; and the set of all such triples clearly satisfies the commafree condition.

domains  
dual  
loose Langford pairs  
word rectangle  
orthogonal lists  
trie  
commafree  
two-letter block code  
digraph  
commafree  
Eastman  
aperiodic words  
periodic sequence

One key idea is to think of  $x$  as partitioned into  $t$  substrings by boundary markers  $b_j$ , where  $0 \leq b_0 < b_1 < \dots < b_{t-1} < n$  and  $b_j = b_{j-t} + n$  for  $j \geq t$ . Then substring  $y_j$  is  $x_{b_j} x_{b_{j+1}} \dots x_{b_{j+1}-1}$ . The number  $t$  of substrings is always odd. Initially  $t = n$  and  $b_j = j$  for all  $j$ ; ultimately  $t = 1$ , and  $\sigma x = y_0$  is the desired output.

Eastman's algorithm is based on comparison of adjacent substrings  $y_{j-1}$  and  $y_j$ . If those substrings have the same length, we use lexicographic comparison; otherwise we declare that the longer substring is bigger.

The second key idea is the notion of “dips,” which are substrings of the form  $z = z_1 \dots z_k$  where  $k \geq 2$  and  $z_1 \geq \dots \geq z_{k-1} < z_k$ . It's easy to see that any string  $y = y_0 y_1 \dots$  in which we have  $y_i < y_{i+1}$  for infinitely many  $i$  can be factored into a sequence of dips,  $y = z^{(0)} z^{(1)} \dots$ , and this factorization is unique. For example,

3141592653589793238462643383 ... = 314 15 926 535 89 79 323 846 26 4338 3 ...

Furthermore, if  $y$  is a periodic sequence, its factorization into dips is also ultimately periodic, although some of the initial factors may not occur in the period. For example,

123443550123443550123443550 ... = 12 34 435 501 23 4435 501 23 4435 ...

Given a periodic, nonconstant sequence  $y$  described by boundary markers as above, where the period length  $t$  is odd, its periodic factorization will contain an odd number of odd-length dips. Each round of Eastman's algorithm simply retains the boundary points at the left of those odd-length dips. Then  $t$  is reset to the number of retained boundary points, and another round begins if  $t > 1$ .

- Play through the algorithm by hand when  $n = 19$  and  $x = 3141592653589793238$ .
- Show that the number of rounds is at most  $\lfloor \log_3 n \rfloor$ .
- Exhibit a binary  $x$  that achieves this worst-case bound when  $n = 3^e$ .
- Implement the algorithm with full details. (It's surprisingly short!)
- Explain why the algorithm yields a commafree code.

**33.** [HM28] What is the probability that Eastman's algorithm finishes in one round? (Assume that  $x$  is a random  $m$ -ary string of odd length  $n > 1$ , unequal to any of its other cyclic shifts. Use a generating function to express the answer.)

**34.** [18] Why can't a commafree code of length  $(m^4 - m^2)/4$  contain 0001 and 2000?

► **35.** [15] Why do you think sequential data structures such as (16)–(23) weren't featured in Section 2.2.2 of this series of books (entitled “Sequential Allocation”)?

**36.** [17] What's the significance of (a) MEM[40d] = 5e and (b) MEM[904] = 84 in Table 1?

**37.** [18] Why is (a) MEM[f8] = e7 and (b) MEM[a0d] = ba in Table 2?

**38.** [20] Suppose you're using the undoing scheme (26) and the operation  $\sigma \leftarrow \sigma + 1$  has just bumped the current stamp  $\sigma$  to zero. What should you do?

► **39.** [25] Spell out the low-level implementation details of the candidate selection process in step C2 of Algorithm C. Use the routine store( $a, v$ ) of (26) whenever changing the contents of MEM, and use the following selection strategy:

- Find a class  $c$  with the least number  $r$  of blue words.
- If  $r = 0$ , set  $x \leftarrow -1$ ; otherwise set  $x$  to a word in class  $c$ .
- If  $r > 1$ , use the poison list to find an  $x$  that maximizes the number of blue words that could be killed on the other side of the prefix or suffix list that contains  $x$ .

► **40.** [28] Continuing exercise 39, spell out the details of step C3 when  $x \geq 0$ .

- What updates should be done to MEM when a blue word  $x$  becomes red?
- What updates should be done to MEM when a blue word  $x$  becomes green?

substrings  
boundary markers  
lexicographic comparison  
dips  
pi, as “random” data  
worst-case bound  
analysis of algs  
Eastman  
generating function  
analysis of algs  
data structures  
Sequential Allocation  
undoing scheme  
bumped  
stamp  
poison list

- c) Step C3 begins its job by making  $x$  green as in part (b). Explain how it should finish its job by updating the poison list.
42. [M30] Is there a *binary* ( $m = 2$ ) commafree code with one codeword in each of the  $(\sum_{d \mid n} \mu(d) 2^{n/d})/n$  cycle classes, for every word length  $n$ ?
44. [HM29] A commafree code on  $m$  letters is equivalent to  $2m!$  such codes if we permute the letters and/or replace each codeword by its left-right reflection. Determine all of the nonisomorphic commafree codes of length 4 on  $m$  letters when  $m$  is (a) 2 (b) 3 (c) 4 and there are (a) 3 (b) 18 (c) 57 codewords.
45. [M42] Find a maximum-size commafree code of length 4 on  $m = 5$  letters.
47. [20] Explain how the choices in Fig. 69 were determined from the “random” bits that are displayed. For instance, why was  $X_2$  set to 1 in Fig. 69(b)?
48. [M15] Interpret the value  $E(D_1 \dots D_l)$ , in the text’s Monte Carlo algorithm.
49. [M22] What’s a simple martingale that corresponds to Theorem E?
- 50. [HM25] Elmo uses Algorithm E with  $D_k = \{1, \dots, n\}$ ,  $P_l = [x_1 > \dots > x_l]$ ,  $c = 1$ .
- Alice flips  $n$  coins independently, where coin  $k$  yields “heads” with probability  $1/k$ . True or false: She obtains exactly  $l$  heads with probability  $\binom{n}{l}/n!$ .
  - Let  $Y_1, Y_2, \dots, Y_l$  be the numbers on the coins that come up heads. (Thus  $Y_1 = 1$ , and  $Y_2 = 2$  with probability  $1/2$ .) Show that  $\Pr(\text{Alice obtains } Y_1, Y_2, \dots, Y_l) = \Pr(\text{Elmo obtains } X_1 = Y_l, X_2 = Y_{l-1}, \dots, X_l = Y_1)$ .
  - Prove that Alice q.s. obtains at most  $(\ln n)(\ln \ln n)$  heads.
  - Consequently Elmo’s  $S$  is q.s. less than  $\exp((\ln n)^2(\ln \ln n))$ .
- 51. [M30] Extend Algorithm B so that it also computes the minimum, maximum, mean, and variance of the Monte Carlo estimates  $S$  produced by Algorithm E.
52. [M21] Instead of choosing each  $y_i$  in step E5 with probability  $1/d$ , we could use a biased distribution where  $\Pr(I = i | X_1, \dots, X_{l-1}) = p_{X_1 \dots X_{l-1}}(y_i) > 0$ . How should the estimate  $S$  be modified so that its expected value in this general scheme is still  $C()$ ?
53. [M20] If all costs  $c(x_1, \dots, x_l)$  are positive, show that the biased probabilities of exercise 52 can be chosen in such a way that the estimate  $S$  is always exact.
- 55. [M25] The commafree code search procedure in Algorithm C doesn’t actually fit the mold of Algorithm E, because it incorporates lookahead, dynamic ordering, reversible memory, and other enhancements to the basic backtrack paradigms. How could its running time be reliably estimated with Monte Carlo methods?
57. [M20] Algorithm E can potentially follow  $M$  different paths  $X_1 \dots X_{l-1}$  before it terminates, where  $M$  is the number of leaves of the backtrack tree. Suppose the final values of  $D$  at those leaves are  $D^{(1)}, \dots, D^{(M)}$ . Prove that  $(D^{(1)} \dots D^{(M)})^{1/M} \geq M$ .
58. [27] The text’s king path problem is a special case of the general problem of counting simple paths from vertex  $s$  to vertex  $t$  in a given graph.
- We can generate such paths by random walks from  $s$  that don’t get stuck, if we maintain a table of values  $\text{DIST}(v)$  for all vertices  $v$  not yet in the path, representing the shortest distance from  $v$  to  $t$  through unused vertices. For with such a table we can simply move at each step to a vertex for which  $\text{DIST}(v) < \infty$ .
- Devise a way to update the  $\text{DIST}$  table dynamically without unnecessary work.
59. [26] A ZDD with 3,174,197 nodes can be constructed for the family of all simple corner-to-corner king paths on a chessboard, using the method of exercise 7.1.4–225. Explain how to use this ZDD to compute (a) the total length of all paths; (b) the number of paths that touch any given subset of the center and/or corner points.

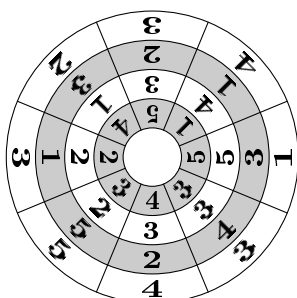
nonisomorphic  
symmetries  
“random” bits  
Monte Carlo  
martingale  
coins  
Stirling cycle numbers  
q.s.  
variance  
biased distribution  
commafree code  
lookahead  
dynamic ordering  
reversible memory  
simple paths  
random walks  
dynamic shortest distances  
shortest distances, dynamic  
ZDD  
corner-to-corner  
king paths  
chessboard

- **60.** [20] Experiment with biased random walks (see exercise 52), weighting each non-dead-end king move to a new vertex  $v$  by  $1 + \text{DIST}(v)^2$  instead of choosing every such move with the same probability. Does this strategy improve on Fig. 70?
- 61.** [HM26] Let  $P_n$  be the number of integer sequences  $x_1 \dots x_n$  such that  $x_1 = 1$  and  $1 \leq x_{k+1} \leq 2x_k$  for  $1 \leq k < n$ . (The first few values are 1, 2, 6, 26, 166, 1626, ...; this sequence was introduced by A. Cayley in *Philosophical Magazine* (4) **13** (1857), 245–248, who showed that  $P_n$  enumerates the partitions of  $2^n - 1$  into powers of 2.)
- Show that  $P_n$  is the number of different profiles that are possible for a binary tree of height  $n$ .
  - Find an efficient way to compute  $P_n$  for large  $n$ . *Hint:* Consider the more general sequence  $P_n^{(m)}$ , defined similarly but with  $x_1 = m$ .
  - Use the estimation procedure of Theorem E to prove that  $P_n \geq 2^{\binom{n}{2}}/(n-1)!$ .
- **66.** [22] When the faces of four cubes are colored randomly with four colors, estimate the probability that the corresponding “Instant Insanity” puzzle has a unique solution. How many 2-regular graphs tend to appear during the “factored” solution process?
- 67.** [20] Find *five* cubes, each of whose faces has one of *five* colors, and where every color occurs at least five times, such that the corresponding puzzle has a unique solution.
- 70.** [24] Assemble five cubes with uppercase letters on each face, using the patterns

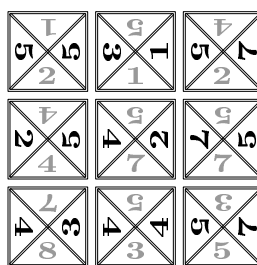


By extending the principles of Instant Insanity, show that these cubes can be placed in a row so that four 5-letter words are visible. (Each word's letters should have a consistent orientation. The letters C and U, H and I, N and Z are related by  $90^\circ$  rotation.)

- **73.** [23] (*The Fool's Disk*.) “Rotate the four disks of the lefthand illustration below so that the four numbers on each ray sum to 12.” (The current sums are  $4 + 3 + 2 + 4 = 13$ , etc.) Show that this problem factors nicely, so that it can be solved readily by hand.



The Fool's Disk



The Royal Aquarium Thirteen Puzzle

- **75.** [26] (*The Royal Aquarium Thirteen Puzzle*.) “Rearrange the nine cards of the righthand illustration above, optionally rotating some of them by  $180^\circ$ , so that the six horizontal sums of gray letters and the six vertical sums of black letters all equal 13.” (The current sums are  $1 + 5 + 4 = 10$ , ...,  $7 + 5 + 7 = 19$ .) The author of *Hoffmann's Puzzles Old and New* (1893) stated that “There is no royal road to the solution. The proper order must be arrived at by successive transpositions until the conditions are fulfilled.” Prove that he was wrong: “Factor” this problem and solve it by hand.

biased random walks  
Cayley  
binary partitions  
partitions  
profiles  
height  $n$   
Instant Insanity  
factored  
speedy schizophrenia  
cubes  
uppercase letters  
alphabet  
Instant Insanity  
5-letter words  
Fool's Disk  
Royal Aquarium Thirteen Puzzle  
Le Nombre Treize, see Royal Aquarium Thirteen Puzzle  
Hoffmann  
Factor

**Table 666**

TWENTY QUESTIONS (SEE EXERCISE 90)

Woods  
questionnaire  
paradoxical

- 
1. The first question whose answer is A is:  
(A) 1            (B) 2            (C) 3            (D) 4            (E) 5
  2. The next question with the same answer as this one is:  
(A) 4            (B) 6            (C) 8            (D) 10           (E) 12
  3. The only two consecutive questions with identical answers are questions:  
(A) 15 and 16   (B) 16 and 17   (C) 17 and 18   (D) 18 and 19   (E) 19 and 20
  4. The answer to this question is the same as the answers to questions:  
(A) 10 and 13   (B) 14 and 16   (C) 7 and 20      (D) 1 and 15      (E) 8 and 12
  5. The answer to question 14 is:  
(A) B            (B) E            (C) C            (D) A            (E) D
  6. The answer to this question is:  
(A) A            (B) B            (C) C            (D) D            (E) none of those
  7. An answer that appears most often is:  
(A) A            (B) B            (C) C            (D) D            (E) E
  8. Ignoring answers that appear equally often, the least common answer is:  
(A) A            (B) B            (C) C            (D) D            (E) E
  9. The sum of all question numbers whose answers are correct and the same as this one is:  
(A)  $\in [59 \dots 62]$    (B)  $\in [52 \dots 55]$    (C)  $\in [44 \dots 49]$    (D)  $\in [61 \dots 67]$    (E)  $\in [44 \dots 53]$
  10. The answer to question 17 is:  
(A) D            (B) B            (C) A            (D) E            (E) wrong
  11. The number of questions whose answer is D is:  
(A) 2            (B) 3            (C) 4            (D) 5            (E) 6
  12. The number of *other* questions with the same answer as this one is the same as the number of questions with answer:  
(A) B            (B) C            (C) D            (D) E            (E) none of those
  13. The number of questions whose answer is E is:  
(A) 5            (B) 4            (C) 3            (D) 2            (E) 1
  14. No answer appears exactly this many times:  
(A) 2            (B) 3            (C) 4            (D) 5            (E) none of those
  15. The set of odd-numbered questions with answer A is:  
(A) {7}           (B) {9}           (C) not {11}      (D) {13}           (E) {15}
  16. The answer to question 8 is the same as the answer to question:  
(A) 3            (B) 2            (C) 13           (D) 18           (E) 20
  17. The answer to question 10 is:  
(A) C            (B) D            (C) B            (D) A            (E) correct
  18. The number of prime-numbered questions whose answers are vowels is:  
(A) prime        (B) square       (C) odd           (D) even           (E) zero
  19. The last question whose answer is B is:  
(A) 14           (B) 15           (C) 16           (D) 17           (E) 18
  20. The maximum score that can be achieved on this test is:  
(A) 18           (B) 19           (C) 20           (D) indeterminate  
(E) achievable only by getting this question wrong
- 

► 90. [M29] (Donald R. Woods, 2000.) Find all ways to maximize the number of correct answers to the questionnaire in Table 666. Each question must be answered with a letter from A to E. *Hint:* Begin by clarifying the exact meaning of this exercise. What answers are best for the following two-question, two-letter “warmup problem”?

1. (A) Answer 2 is B.                      (B) Answer 1 is A.
2. (A) Answer 1 is correct.                (B) Either answer 2 is wrong or answer 1 is A, but not both.

91. [HM28] Show that exercise 90 has a surprising, somewhat paradoxical answer if two changes are made to Table 666: 9(E) becomes ‘ $\in [39 \dots 43]$ ’; 15(C) becomes ‘{11}’.

- **95.** [30] (*A clueless anacrostic.*) The letters of 29 five-letter words

$\overline{1\ 2\ 3\ 4\ 5}, \overline{6\ 7\ 8\ 9\ 10}, \overline{11\ 12\ 13\ 14\ 15}, \overline{16\ 17\ 18\ 19\ 20}, \dots, \overline{141\ 142\ 143\ 144\ 145},$

all belonging to WORDS(1000), have been shuffled to form the following mystery text:

$\overline{30\ 29\ 9\ 140\ 12\ 13\ 145\ 90\ 45\ 99\ 26\ 107\ 47\ 84\ 53\ 51\ 27\ 133\ 39\ 137\ 139\ 66\ 112\ 69\ 14\ 8\ 20\ 91\ 129\ 70}$   
 $\overline{16\ 7\ 93\ 19\ 85\ 101\ 76\ 78\ 44\ 10\ 106\ 60\ 118\ 119\ 24\ 25\ 100\ 1\ 5\ 64\ 11\ 71\ 42\ 122\ 123}$   
 $\overline{103\ 104\ 63\ 49\ 31\ 121\ 98\ 79\ 80\ 46\ 48\ 134\ 135\ 131\ 143\ 96\ 142\ 120\ 50\ 132\ 33\ 43\ 34\ 40\ 32\ 35\ 117\ 116\ 23\ 52}$   
 $\overline{111\ 97\ 113\ 105\ 38\ 102\ 62\ 65\ 114\ 74\ 82\ 81\ 83\ 136\ 37\ 21\ 61\ 88\ 86\ 55\ 56\ 17\ 18\ 94\ 67\ 128\ 15\ 57\ 58\ 89\ 87\ 109\ 2\ 4\ 6\ 28\ 95\ 3\ 126\ 77\ 144\ 54\ 41\ 68\ 115}$   
 $\overline{75\ 138\ 73\ 124\ 36\ 130\ 127\ 141\ 22\ 92\ 72\ 59\ 108\ 125\ 110}$

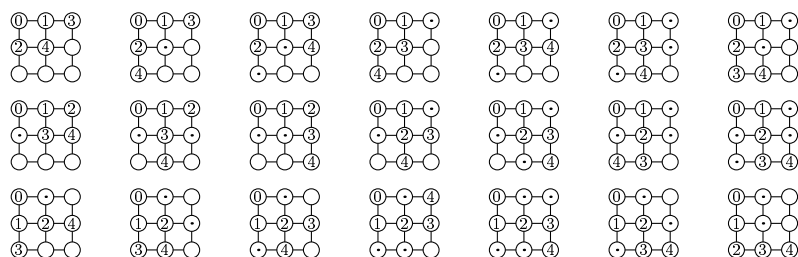
clueless  
 anacrostic  
 five-letter words  
 WORDS(n)  
 mystery text  
 English words  
 factorization  
 Connected subsets  
 canonical  
 grid  
 pentominoes  
 spanning tree  
 lexicographically smallest  
 SGB format  
 ARCS  
 TIP  
 NEXT  
 polyominoes

Furthermore, their initial letters  $\overline{1}, \overline{6}, \overline{11}, \overline{16}, \dots, \overline{141}$  identify the source of that quotation, which consists entirely of common English words. What does it say?

- 96.** [21] The fifteenth mystery word in exercise 95 is ' $\overline{134\ 135\ 131}$ '. Why does its special form lead to a partial *factorization* of that problem?

- **100.** [30] (*Connected subsets.*) Let  $v$  be a vertex of some graph  $G$ , and let  $H$  be a connected subset of  $G$  that contains  $v$ . The vertices of  $H$  can be listed in a canonical way by starting with  $v_0 \leftarrow v$  and then letting  $v_1, v_2, \dots$  be the neighbors of  $v_0$  that lie in  $H$ , followed by the neighbors of  $v_1$  that haven't yet been listed, and so on. (We assume that the neighbors of each vertex are listed in some fixed order.)

For example, if  $G$  is the  $3 \times 3$  grid  $P_3 \square P_3$ , exactly 21 of its connected five-element subsets contain the upper left corner element  $v$ . Their canonical orderings are



if we order the vertices from top to bottom and left to right when listing a vertex's neighbors. (Vertices labeled ①, ②, ③, ④ indicate  $v_0, v_1, v_2, v_3, v_4$ . Other vertices are not in  $H$ .) Notice that in each case the numbered vertices are implicitly connected by a spanning tree whose edges each have the form  $v_i - v_j$  for some  $i < j$ .

The canonical ordering corresponds to  $H$ 's lexicographically smallest spanning tree; for example, the spanning tree in the first solution is  $v_0 - v_1, v_0 - v_2, v_1 - v_3, v_1 - v_4$ . Furthermore, the 21 solutions appear here in lexicographic order of their respective spanning trees.

Design a backtrack algorithm to generate all of the  $n$ -element connected subsets that contain a specified vertex  $v$ , given a graph that is represented in SGB format (which has ARCS, TIP, and NEXT fields, as described near the beginning of Chapter 7).

- 101.** [23] Use the algorithm of exercise 100 to generate *all* of the connected  $n$ -element subsets of a given graph  $G$ . How many such subsets does  $P_n \square P_n$  have, for  $1 \leq n \leq 9$ ?

**102.** [M22] A  $v$ -reachable subset of a directed graph  $G$  is a nonempty set of vertices  $H$  with the property that every  $u \in H$  can be reached from  $v$  by at least one oriented path in  $G|H$ . (In particular,  $v$  itself must be in  $H$ .)

- a) The digraph  $P_3^{\rightarrow} \square P_3^{\rightarrow}$  is like  $P_3 \square P_3$ , except that all arcs between vertices are directed downward or to the right. Which of the 21 connected subsets in exercise 100 are also  $v$ -reachable from the upper left corner element  $v$  of  $P_3^{\rightarrow} \square P_3^{\rightarrow}$ ?
- b) True or false:  $H$  is  $v$ -reachable if and only if  $G|H$  contains a dual oriented spanning tree rooted at  $v$ . (An oriented tree has arcs  $u \rightarrow p_u$ , where  $p_u$  is the parent of the nonroot node  $u$ ; in a *dual* oriented tree, the arcs are reversed:  $p_u \rightarrow u$ .)
- c) True or false: If  $G$  is undirected, so that  $w \rightarrow u$  whenever  $u \rightarrow w$ , its  $v$ -reachable subsets are the same as the connected subsets that contain  $v$ .
- d) Modify the algorithm of exercise 100 so that it generates all of the  $n$ -element  $v$ -reachable subsets of a digraph  $G$ , given  $n$ ,  $v$ , and  $G$ .

**999.** [M00] this is a temporary exercise (for dummies)

$v$ -reachable subset  
 reachable subsets  
 grid, oriented  
 oriented grid  
 dual oriented spanning tree  
 oriented tree  
 parent  
 directed graph versus undirected  
 undirected graph versus directed

## SECTION 7.2.2

1. Although many formulations are possible, the following may be the nicest: (i)  $D_k$  is arbitrary (but hopefully finite), and  $P_l$  is always true. (ii)  $D_k = \{1, 2, \dots, n\}$  and  $P_l = 'x_j \neq x_k \text{ for } 1 \leq j < k \leq l'$ . (iii) For combinations of  $n$  things from  $N$ ,  $D_k = \{1, \dots, N+1-k\}$  and  $P_l = 'x_1 > \dots > x_l'$ . (iv)  $D_k = \{0, 1, \dots, \lfloor n/k \rfloor\}$ ;  $P_l = 'x_1 \geq \dots \geq x_l \text{ and } n - (n-l)x_l \leq x_1 + \dots + x_l \leq n'$ . (v) For restricted growth strings,  $D_k = \{0, \dots, k-1\}$  and  $P_l = 'x_{j+1} \leq 1 + \max(x_1, \dots, x_j) \text{ for } 1 \leq j < l'$ . (vi) For indices of left parentheses (see 7.2.1.6–(8)),  $D_k = \{1, \dots, 2k-1\}$  and  $P_l = 'x_1 < \dots < x_l'$ .

2. True. (If not, set  $D_1 \leftarrow D_1 \cap \{x \mid P_1(x)\}$ .)

3. We can restrict  $D_1$  to  $\{1, 2, 3, 4\}$ , because the reflection  $(9-x_1)\dots(9-x_8)$  of every solution  $x_1 \dots x_8$  is also a solution. (H. C. Schumacher made this observation in a letter to Gauss, 24 September 1850.) Notice that Fig. 68 is left-right symmetric.

4.  $try(l) =$  “If  $l > n$ , visit  $x_1 \dots x_n$ . Otherwise, for  $x_l \leftarrow \min D_l, \min D_l + 1, \dots, \max D_l$ , if  $P_l(x_1, \dots, x_l)$  call  $try(l+1)$ .”

This formulation is elegant, and fine for simple problems. But it doesn't give any clue about why the method is called “backtrack”! Nor does it yield efficient code for important problems whose inner loop is performed billions of times. We will see that the key to efficient backtracking is to provide good ways to update and downgrade the data structures that speed up the testing of property  $P_l$ . The overhead of recursion can get in the way, and the actual iterative structure of Algorithm B isn't difficult to grasp.

5. Excluding cases with  $j = r$  or  $k = r$  from (3) yields respectively (312, 396, 430, 458, 458, 430, 396, 312) solutions. (With column  $r$  also omitted there are just (40, 46, 42, 80, 80, 42, 46, 40).)

6. Yes, almost surely for all  $n > 16$ . One such is  $x_1 x_2 \dots x_{17} = 2 \ 17 \ 12 \ 10 \ 7 \ 14 \ 3 \ 5 \ 9 \ 13 \ 15 \ 4 \ 11 \ 8 \ 6 \ 1 \ 16$ . [See *Proc. Edinburgh Math. Soc.* **8** (1890), 43 and Fig. 52.] Preußner and Engelhardt found 34,651,355,392 solutions when  $n = 27$ .

7. Yes: (42736815, 42736851); also therefore (57263148, 57263184).

8. Yes, at least when  $m = 4$ ; e.g.,  $x_1 \dots x_{16} = 5 \ 8 \ 13 \ 16 \ 3 \ 7 \ 15 \ 11 \ 6 \ 2 \ 10 \ 14 \ 1 \ 4 \ 9 \ 12$ . There are no solutions when  $m = 5$ , but  $7 \ 10 \ 13 \ 20 \ 17 \ 24 \ 3 \ 6 \ 23 \ 11 \ 16 \ 21 \ 4 \ 9 \ 14 \ 2 \ 19 \ 22 \ 1 \ 8 \ 5 \ 12 \ 15 \ 18$  works for  $m = 6$ . (Are there solutions for all even  $m \geq 4$ ?) C. F. de Jaenisch, *Traité des applications de l'analyse mathématique au jeu des échecs* **2** (1862), 132–133, noted that all 8-queen solutions have four of each color. He proved that the number of white queens must be even, because  $\sum_{k=1}^{4m} (x_k + k)$  is even.)

9. Let bit vectors  $a_l, b_l, c_l$  represent the “useful” elements of the sets in (6), with  $a_l = \sum \{2^{x-1} \mid x \in A_l\}$ ,  $b_l = \sum \{2^{x-1} \mid x \in B_l \cap [1..n]\}$ ,  $c_l = \sum \{2^{x-1} \mid x \in C_l \cap [1..n]\}$ . Then step W2 sets  $s_l \leftarrow \mu \ \& \ \bar{a}_l \ \& \ \bar{b}_l \ \& \ \bar{c}_l$ , where  $\mu$  is the mask  $2^n - 1$ .

In step W3 we can set  $t \leftarrow s_l \ \& \ (-s_l)$ ,  $a_l \leftarrow a_{l-1} + t$ ,  $b_l \leftarrow (b_{l-1} + t) \gg 1$ ,  $c_l \leftarrow ((c_{l-1} + t) \ll 1) \ \& \ \mu$ ; and it's also convenient to set  $s_l \leftarrow s_l - t$  at this time, instead of deferring this change to step W4.

(There's no need to store  $x_l$  in memory, or even to compute  $x_l$  in step W3 as an integer in  $[1..n]$ , because  $x_l$  can be deduced from  $a_l - a_{l-1}$  when a solution is found.)

10. (a) Only when  $n = 1$ , because reflected queens can capture each other.

(b) Queens not in the center must appear in groups of four.

(c) The four queens occupy the same rows, columns, and diagonals in both cases.

(d) In each solution counted by  $c_n$  we can independently tilt (or not) each of the  $\lfloor n/4 \rfloor$  groups of four. [*Mathematische Unterhaltungen und Spiele* **1**, second edition (Leipzig: Teubner, 1910), 249–258.]

restricted growth strings  
reflection  
Schumacher  
Gauss  
symmetric  
inner loop  
backtracking, efficient  
iteration versus recursion  
Preußner  
Engelhardt  
de Jaenisch  
mask



**11.** Suppose the  $x_k$  are distinct. Then  $\sum_{k=1}^n (x_k + k) = 2 \binom{n+1}{2} \equiv 0 \pmod{n}$ . If the numbers  $(x_k + k) \bmod n$  are also distinct, we have also  $\sum_{k=1}^n k \equiv \binom{n+1}{2}$ . But that is impossible when  $n$  is even.

Now suppose further that the numbers  $(x_k - k) \bmod n$  are distinct. Then we have  $\sum_{k=1}^n (x_k + k)^2 \equiv \sum_{k=1}^n (x_k - k)^2 \equiv \sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$ . And we also have  $\sum_{k=1}^n (x_k + k)^2 + \sum_{k=1}^n (x_k - k)^2 = 4n(n+1)(2n+1)/6 \equiv 2n/3$ , which is impossible when  $n$  is a multiple of 3. [See W. Ahrens, *Mathematische Unterhaltungen und Spiele* **2**, second edition (1918), 364–366, where G. Pólya cites a more general result of A. Hurwitz that applies to wraparound diagonals of other slopes.]

Conversely, if  $n$  isn't divisible by 2 or 3, we can let  $x_n = n$  and  $x_k = (2k) \bmod n$  for  $1 \leq k < n$ . (The rule  $x_k = (3k) \bmod n$  also works. See Édouard Lucas, *Récréations Mathématiques* **1** (1882), 84–86.)

**12.** The  $(n+1)$  queens problem clearly has a solution with a queen in a corner if and only if the  $n$  queens problem has a solution with a queen-free main diagonal. Hence by the previous answer there's always a solution when  $n \bmod 6 \in \{0, 1, 4, 5\}$ .

Another nice solution was found by J. Franel [*L'Intermédiaire des Mathématiciens* **1** (1894), 140–141] when  $n \bmod 6 \in \{2, 4\}$ : Let  $x_k = (n/2 + 2k - 3[2k \leq n]) \bmod n + 1$ , for  $1 \leq k \leq n$ . With this setup we find that  $x_k - x_j = \pm(k - j)$  and  $1 \leq j < k \leq n$  implies  $(1 \text{ or } 3)(k - j) + (0 \text{ or } 3) \equiv 0 \pmod{n}$ ; hence  $k - j = n - (1 \text{ or } 3)$ . But the values of  $x_1, x_2, x_3, x_{n-2}, x_{n-1}, x_n$  give no attacking queens except when  $n = 2$ .

Franel's solution has empty diagonals, so it provides solutions also for  $n \bmod 6 \in \{3, 5\}$ . We conclude that only  $n = 2$  and  $n = 3$  are impossible.

[A more complicated construction for all  $n > 3$  had been given earlier by E. Pauls, in *Deutsche Schachzeitung* **29** (1874), 129–134, 257–267. Pauls also explained how to find all solutions, in principle, by building the tree level by level (*not* backtracking).]

**13.** For  $1 \leq j \leq n$ , let  $x_1^{(j)} \dots x_m^{(j)}$  be a solution for  $m$  queens, and let  $y_1 \dots y_n$  be a solution for  $n$  toroidal queens. Then  $X_{(i-1)n+j} = (x_i^{(j)} - 1)n + y_j$  (for  $1 \leq i \leq m$  and  $1 \leq j \leq n$ ) is a solution for  $mn$  queens. [I. Rivin, I. Vardi, and P. Zimmermann, *AMM* **101** (1994), 629–639, Theorem 2.]

**14.** [Rivin, Vardi, and Zimmermann, in the paper just cited, observe that in fact the sequence  $(\ln Q(n))/(n \ln n)$  appears to be *increasing*.]

**15.** Let the queen in row  $k$  be in cell  $k$ . Then we have a “relaxation” of the  $n$  queens problem, with  $|x_k - x_j|$  becoming just  $x_k - x_j$  in (3); so we can ignore the  $b$  vector in Algorithm B\* or in exercise 9. We get

$$\begin{array}{cccccccccccccccc} n = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ H(n) = & 1 & 1 & 1 & 3 & 7 & 23 & 83 & 405 & 2113 & 12657 & 82297 & 596483 & 4698655 & 40071743 & 367854835 \end{array}$$

[N. J. Cavenagh and I. M. Wanless, *Discr. Appl. Math.* **158** (2010), 136–146, Table 2.]

**16.** It fails spectacularly in step L5. The minus signs, which mark decisions that were previously forced, are crucial tags for backtracking.

**17.**  $x_4 \dots x_8 = \bar{2}10\bar{4}0$ ,  $p_0 \dots p_4 = 33300$ , and  $y_1 y_2 y_3 = 130$ . (If  $x_i \leq 0$  the algorithm will never look at  $y_i$ ; hence the current state of  $y_4 \dots y_8$  is irrelevant. But  $y_4 y_5$  happens to be 20, because of past history;  $y_6, y_7$ , and  $y_8$  haven't yet been touched.)

**19.** We could say  $D_l$  is  $\{-n, \dots, -2, -1, 1, 2, \dots, n\}$ , or  $\{k \mid k \neq 0 \text{ and } 2 - l \leq k \leq 2n - l - 1\}$ , or anything in between. (But this observation isn't very useful.)

Ahrens  
Pólya  
Hurwitz  
Lucas  
Franel  
Pauls  
tree  
breadth first search  
Rivin  
Vardi  
Zimmermann  
semi-queens  
Cavenagh  
Wanless

**20.** First we add a Boolean array  $a_1 \dots a_n$ , where  $a_k$  means “ $k$  has appeared,” as in Algorithm B\*. It’s  $0 \dots 0$  in step L1; we set  $a_k \leftarrow 1$  in step L3,  $a_k \leftarrow 0$  in step L5.

The loop in step L2 becomes “while  $x_l < 0$ , go to L5 if  $l \geq n-1$  and  $a_{2n-l-1} = 0$ , otherwise set  $l \leftarrow l+1$ .” After finding  $l+k+1 \leq 2n$  in L3, and before testing  $x_{l+k+1}$  for 0, insert this: “If  $l \geq n-1$  and  $a_{2n-l-1} = 0$ , while  $l+k+1 \neq 2n$  set  $j \leftarrow k$ ,  $k \leftarrow p_k$ .”

**21.** (a) In any solution  $x_k = n \iff x_{k+n+1} = -n \iff x_{n-k}^D = n$ .

(b)  $x_k = n-1$  for some  $k \leq n/2$  if and only if  $x_k^D = n-1$  for some  $k > n/2$ .

(c) Let  $n' = n - [n \text{ is even}]$ . Change ‘ $l \geq n-1$  and  $a_{2n-l-1} = 0$ ’ in the modified step L2 to ‘ $l = \lfloor n/2 \rfloor$  and  $a_{n'} = 0$ ’ or ‘ $l \geq n-1$  and  $a_{2n-l-1} = 0$ ’. Insert the following before the other insertion into step L3: “If  $l = \lfloor n/2 \rfloor$  and  $a_{n'} = 0$ , while  $k \neq n'$  set  $j \leftarrow k$ ,  $k \leftarrow p_k$ .” And in step L5—this subtle detail is needed when  $n$  is even—go to L5 instead of L4 if  $l = \lfloor n/2 \rfloor$  and  $k = n'$ .

**22.** The solutions  $1\bar{1}$  and  $21\bar{1}\bar{2}$  for  $n=1$  and  $n=2$  are self-dual; the solutions for  $n=4$  and  $n=5$  are  $431\bar{1}\bar{2}\bar{3}\bar{4}\bar{2}$ ,  $245\bar{2}31\bar{1}\bar{4}\bar{3}\bar{5}$ ,  $451\bar{1}\bar{2}3\bar{4}\bar{2}\bar{5}\bar{3}$ , and their duals. The total number of solutions for  $n=1, 2, \dots$  is  $1, 1, 0, 2, 4, 20, 0, 156, 516, 2008, 0, 52536, 297800, 1767792, 0, 75678864, \dots$ ; there are none when  $n \bmod 4 = 3$ , by a parity argument.

Algorithm L needs only obvious changes. To compute solutions by a streamlined method like exercise 21, use  $n' = n - (0, 1, 2, 0)$  and substitute ‘ $l = \lfloor n/4 \rfloor + (0, 1, 2, 1)$ ’ for ‘ $l = \lfloor n/2 \rfloor$ ’, when  $n \bmod 4 = (0, 1, 2, 3)$ ; also replace ‘ $l \geq n-1$  and  $a_{2n-l-1} = 0$ ’ by ‘ $l \geq \lfloor n/2 \rfloor$  and  $a_{\lfloor (4n+2-2l)/3 \rfloor} = 0$ ’. The case  $n=15$  is proved impossible with 397 million nodes and 9.93 gigamems.

**23.**  $\text{slums} \rightarrow \text{sluff, slump, slurs, slurp, or sluts}; (\text{slums, total}) \rightarrow (\text{slams, tonal})$ .

**24.** Build the list of 5-letter words and the trie of 6-letter words in step B1; also set  $a_{01}a_{02}a_{03}a_{04}a_{05} \leftarrow 00000$ . Use  $\min D_l = 1$  in step B2 and  $\max D_l = 5757$  in step B4. Testing  $P_l$  in step B3, if word  $x_3$  is  $c_1c_2c_3c_4c_5$ , consists of forming  $a_{l1} \dots a_{l5}$ , where  $a_{lk} = \text{trie}[a_{(l-1)k}, c_k]$  for  $1 \leq k \leq 5$ ; but jump to B4 if any  $a_{lk}$  is zero.

**25.** There are  $5 \times 26$  singly linked lists, accessed from pointers  $h_{kc}$ , all initially zero. The  $x$ th word  $c_{x1}c_{x2}c_{x3}c_{x4}c_{x5}$ , for  $1 \leq x \leq 5757$ , belongs to 5 lists and has five pointers  $l_{x1}l_{x2}l_{x3}l_{x4}l_{x5}$ . To insert it, set  $l_{xk} \leftarrow h_{kc_{xk}}$ ,  $h_{kc_{xk}} \leftarrow x$ , and  $s_{kc_{xk}} \leftarrow s_{kc_{xk}} + 1$ , for  $1 \leq k \leq 5$ . (Thus  $s_{kc}$  will be the length of the list accessed from  $h_{kc}$ .)

We can store a “signature”  $\sum_{c=1}^{26} 2^{c-1} [\text{trie}[a, c] \neq 0]$  with each node  $a$  of the trie. For example, the signature for node 260 is  $2^0 + 2^4 + 2^8 + 2^{14} + 2^{17} + 2^{20} + 2^{24} = \#1124111$ , according to (11); here  $a \leftrightarrow 1, \dots, z \leftrightarrow 26$ .

The process of running through all  $x$  that match a given signature  $y$  with respect to position  $z$ , as needed in steps B2 and B4, now takes the following form: (i) Set  $i \leftarrow 0$ . (ii) While  $2^i \& y = 0$ , set  $i \leftarrow i+1$ . (iii) Set  $x \leftarrow h_{z(i+1)}$ ; go to (vi) if  $x = 0$ . (iv) Visit  $x$ . (v) Set  $x \leftarrow l_{xz}$ ; go to (iv) if  $x \neq 0$ . (vi) Set  $i \leftarrow i+1$ ; go to (ii) if  $2^i \leq y$ .

Let  $\text{trie}[a, 0]$  be the signature of node  $a$ . We choose  $z$  and  $y = \text{trie}[a_{(l-1)z}, 0]$  in step B2 so that the number of nodes to visit,  $\sum_{c=1}^{26} s_{zc} [2^{c-1} \& y \neq 0]$ , is minimum for  $1 \leq z \leq 5$ . For example, when  $l=3$ ,  $x_1=1446$ , and  $x_2=185$  as in (10), that sum for  $z=1$  is  $s_{11}+s_{15}+s_{19}+s_{1(15)}+s_{1(18)}+s_{1(21)}+s_{1(25)} = 296+129+74+108+268+75+47 = 997$ ; and the sums for  $z=2, 3, 4, 5$  are 4722, 1370, 5057, and 1646. Hence we choose  $z=1$  and  $y = \#1124111$ ; only 997 words, not 5757, need be tested for  $x_3$ .

The values  $y_l$  and  $z_l$  are maintained for use in backtracking. (In practice we keep  $x, y$ , and  $z$  in registers during most of the computation. Then we set  $x_l \leftarrow x$ ,  $y_l \leftarrow y$ ,  $z_l \leftarrow z$  before increasing  $l \leftarrow l+1$  in step B3; and we set  $x \leftarrow x_l$ ,  $y \leftarrow y_l$ ,  $z \leftarrow z_l$  in

parity argument  
trie  
signature  
trie  
bitwise AND

step B5. We also keep  $i$  in a register, while traversing the sublists as above; this value is restored in step B5 by setting it to the  $z$ th letter of word  $x$ , decreased by 'a'.)

**26.** Here are the author's favorite  $5 \times 7$  and  $5 \times 8$ , and the *only*  $5 \times 9$ 's:

smashes	grandest	pastelist	varistors
partial	renounce	accidence	agentival
immense	episodes	mortgagor	coelomate
emerged	basement	proreform	undeleated
sadness	eyesores	andesytes	oysterers

author  
Gattegno  
compressed trie  
trie, compressed  
saturating ternary addition

No  $5 \times 10$  word rectangles exist, according to our ground rules.

**27.** (1, 15727, 8072679, 630967290 90962081, 625415) and (15727.0, 4321.6, 1749.7, 450.4, 286.0). Total time  $\approx 18.3$  teramems. (In Section 7.2.2.1 we'll study a method that is symmetrical between rows and columns.)

**28.** Build a separate trie for the  $m$ -letter words; but instead of having trie nodes of size 26 as in (11), it's better to convert this trie into a *compressed* representation that omits the zeros. For example, the compressed representation of the node for prefix 'corne' in (12) consists of five consecutively stored pairs of entries ('a', 3879), ('d', 3878), ('l', 9602), ('r', 171), ('t', 5013), followed by (0, 0). Similarly, each shorter prefix with  $c$  descendants is represented by  $c$  consecutive pairs (character, link), followed by (0, 0) to mark the end of the node. Steps B3 and B4 are now very convenient.

Level  $l$  corresponds to row  $i_l = 1 + (l-1) \bmod m$  and column  $j_l = 1 + \lfloor (l-1)/m \rfloor$ . For backtracking we store the  $n$ -trie pointer  $a_{i_l, j_l}$  as before, together with an index  $x_l$  into the compressed  $m$ -trie.

This method was suggested by Bernard Gattegno in 1996 (unpublished). It finds all  $5 \times 6$  word rectangles in just 400 gigamems; and its running time for "transposed"  $6 \times 5$  rectangles turns out to be slightly less (380 gigamems). Notice that only one mem is needed to access each (character, link) pair in the compressed trie.

**29.** Leave out *face* and (of course) *dada*; the remaining eleven are fine.

**30.** Keep tables  $p_i, p'_{ij}, p''_{ijk}, s_i, s'_{ij}, s''_{ijk}$ , for  $0 \leq i, j, k < m$ , each capable of storing a ternary digit. Also keep a table  $x_0, x_1, \dots$  of tentatively accepted words. Begin with  $g \leftarrow 0$ . Then for each input  $w_j = abcd$ , where  $0 \leq a, b, c, d < m$ , set  $x_g \leftarrow abcd$  and also do the following: Set  $p_a \leftarrow p_a + 1, p'_{ab} \leftarrow p'_{ab} + 1, p''_{abc} \leftarrow p''_{abc} + 1, s_d \leftarrow s_d + 1, s'_{cd} \leftarrow s'_{cd} + 1, s''_{bcd} \leftarrow s''_{bcd} + 1$ , where  $x + y = \min(2, x + y)$  denotes saturating ternary addition. Then if  $s_a p'_{b'c'd'} + s'_{a'b'} p'_{c'd'} + s''_{a'b'c'} p'_{d'} = 0$  for all  $x_k = a'b'c'd'$ , where  $0 \leq k \leq g$ , set  $g \leftarrow g + 1$ . Otherwise reject  $w_j$  and set  $p_a \leftarrow p_a - 1, p'_{ab} \leftarrow p'_{ab} - 1, p''_{abc} \leftarrow p''_{abc} - 1, s_d \leftarrow s_d - 1, s'_{cd} \leftarrow s'_{cd} - 1, s''_{bcd} \leftarrow s''_{bcd} - 1$ .

**31.** (a) The word  $bc$  appears in message  $abcd$  if and only if  $a \rightarrow b, b \rightarrow c$ , and  $c \rightarrow d$ .

(b) For  $0 \leq k < r$ , put vertex  $v$  into class  $k$  if the longest path from  $v$  has length  $k$ . Given any such partition, we can include all arcs from class  $k$  to class  $j < k$  without increasing the path lengths. So it's a question of finding the maximum of  $\sum_{0 \leq j < k < r} p_j p_k$  subject to  $p_0 + p_1 + \dots + p_{r-1} = m$ . The values  $p_j = \lfloor (m+j)/r \rfloor$  achieve this (see exercise 7.2.1.4–68(a)). When  $r = 3$  the maximum simplifies to  $\lfloor m^2/3 \rfloor$ .

**32.** (a) The factors of the period, 15 926 535 89 79 323 8314, begin at the respective boundary points 3, 5, 8, 11, 13, 15, 18 (and then  $3 + 19 = 22$ , etc.). Thus round 1 retains boundaries 5, 8, and 15. The second-round substrings  $y_0 = 926, y_1 = 5358979, y_2 = 323831415$  have different lengths, so lexicographic comparison is unnecessary; the answer is  $y_2 y_0 y_1 = x_{15} \dots x_{33}$ .

(b) Each substring consists of at least three substrings of the previous round.

(c) Let  $a_0 = 0$ ,  $b_0 = 1$ ,  $a_{e+1} = a_e a_e b_e$ ,  $b_{e+1} = a_e b_e b_e$ ; use  $a_e$  or  $b_e$  when  $n = 3^e$ .  
 (d) We use an auxiliary subroutine 'less( $i$ )', which returns  $[y_{i-1} < y_i]$ , given  $i > 0$ :  
 If  $b_i - b_{i-1} \neq b_{i+1} - b_i$ , return  $[b_i - b_{i-1} < b_{i+1} - b_i]$ . Otherwise, for  $j = 0, 1, \dots$ , while  $b_i + j < b_{i+1}$ , if  $x_{b_{i-1}+j} \neq x_{b_i+j}$  return  $[x_{b_{i-1}+j} < x_{b_i+j}]$ . Otherwise return 0.

superdip  
 Eastman  
 Scholtz  
 aperiodic

The tricky part of the algorithm is to discard initial factors that aren't periodic. The secret is to let  $i_0$  be the smallest index such that  $y_{i-3} \geq y_{i-2} < y_{i-1}$ ; then we can be sure that a factor begins with  $y_i$ .

- O1.** [Initialize.] Set  $x_j \leftarrow x_{j-n}$  for  $n \leq j < 2n$ ,  $b_j \leftarrow j$  for  $0 \leq j < 2n$ , and  $t \leftarrow n$ .
- O2.** [Begin a round.] Set  $t' \leftarrow 0$ . Find the smallest  $i > 0$  such that  $\text{less}(i) = 0$ . Then find the smallest  $j \geq i + 2$  such that  $\text{less}(j - 1) = 1$  and  $j \leq t + 2$ . (If no such  $j$  exists, report an error: The input  $x$  was equal to one of its cyclic shifts.) Set  $i \leftarrow i_0 \leftarrow j \bmod t$ . (Now a dip of the period begins at  $i_0$ .)
- O3.** [Find the next factor.] Find the smallest  $j \geq i + 2$  such that  $\text{less}(j - 1) = 1$ . If  $j - i$  is even, go to O5.
- O4.** [Retain a boundary.] If  $j < t$ , set  $b'_{t'} \leftarrow b_j$ ; otherwise set  $b'_k \leftarrow b'_{k-1}$  for  $t' \geq k > 0$  and  $b'_0 \leftarrow b_{j-t}$ . Finally set  $t' \leftarrow t' + 1$ .
- O5.** [Done with round?] If  $j < i_0 + t$ , set  $i \leftarrow j$  and return to O3. Otherwise, if  $t' = 1$ , terminate;  $\sigma x$  begins at item  $x_{b'_0}$ . Otherwise set  $t \leftarrow t'$ ,  $b_k \leftarrow b'_k$  for  $0 \leq k < t$ , and  $b_k \leftarrow b_{k-t} + n$  for  $k \geq t$  while  $b_{k-t} < n$ . Return to O2. ■

(e) Say that a "superdip" is a dip of odd length followed by zero or more dips of even length. Any infinite sequence  $y$  that begins with an odd-length dip has a unique factorization into superdips. Those superdips can, in turn, be regarded as atomic elements of a higher-level string that can be factored into dips. The result  $\sigma x$  of Algorithm O is an infinite periodic sequence that allows repeated factorization into infinite periodic sequences of superdips at higher and higher levels, until becoming constant.

Notice that the first dip of  $\sigma x$  ends at position  $i_0$  in the algorithm, because its length isn't 2. Therefore we can prove the comma-free property by observing that, if codeword  $\sigma x''$  appears within the concatenation  $\sigma x \sigma x'$  of two codewords, its superdip factors are also superdip factors of those codewords. This yields a contradiction if any of  $\sigma x$ ,  $\sigma x'$ , or  $\sigma x''$  is a superdip. Otherwise the same observation applies to the superdip factors at the next level. [Eastman's original algorithm was essentially the same, but presented in a more complicated way; see *IEEE Trans. IT-11* (1965), 263–267. R. A. Scholtz subsequently discovered an interesting and totally different way to define the set of codewords produced by Algorithm O, in *IEEE Trans. IT-15* (1969), 300–306.]

**33.** Let  $f_k(m)$  be the number of dips of length  $k$  for which  $m > z_1$  and  $z_k < m$ . The number of such sequences with  $z_2 = j$  is  $(m - j - 1) \binom{m-j+k-3}{k-2} = (k-1) \binom{m-j+k-3}{k-1}$ ; summing for  $0 \leq j < m$  gives  $f_k(m) = (k-1) \binom{m+k-2}{k}$ . Thus  $F_m(z) = \sum_{k=0}^{\infty} f_k(m) z^k = (mz-1)/(1-z)^m$ . (The fact that  $f_0(m) = -1$  in these formulas turns out to be useful!)

Algorithm O finishes in one round if and only if some cyclic shift of  $x$  is a superdip. The number of aperiodic  $x$  that finish in one round is therefore  $n[z^n] G_m(z)$ , where

$$G_m(z) = \frac{F_m(-z) - F_m(z)}{F_m(-z) + F_m(z)} = \frac{(1+mz)(1-z)^m - (1-mz)(1+z)^m}{(1+mz)(1-z)^m + (1-mz)(1+z)^m}.$$

To get the stated probability, divide by  $\sum_{d \in \mathbb{N}} \mu(d) m^{n/d}$ , the number of aperiodic  $x$ . (See Eq. 7.2.1.1–(60). For  $n = 3, 5, 7, 9$  these probabilities are 1, 1, 1, and  $1 - 3/\binom{m^3-1}{3}$ .)

**34.** If so, it couldn't have 0011, 0110, 1100, or 1001.

**35.** That section considered such representations of stacks and queues, but not of unordered sets, because large blocks of sequential memory were either nonexistent or ultra-expensive in olden days. Linked lists were the only decent option for families of variable-size sets, because they could more readily fit in a limited high-speed memory.

stacks  
queues  
memory constraints, historic  
unordered sets  
Linked lists  
deletion

**36.** (a) The blue word  $x$  with  $\alpha = d$  (namely 1101) appears in its P2 list at location 5e.  
(b) The P3 list for words of the form 010\* is empty. (Both 0100 and 0101 are red.)

**37.** (a) The S2 list of 0010 has become closed (hence 0110 and 1110 are hidden).  
(b) Word 1101 moved to the former position of 1001 in its S1 list, when 1001 became red. (Previously 1011 had moved to the former position of 0001.)

**38.** In this case, which of course happens rarely, it's safe to set all elements of STAMP to zero and set  $\sigma \leftarrow 1$ . (Do *not* be tempted to save one line of code by setting all STAMP elements to  $-1$  and leaving  $\sigma = 0$ . That might fail when  $\sigma$  reaches the value  $-1$ !)

**39.** (a) Set  $r \leftarrow m + 1$ . Then for  $k \leftarrow 0, 1, \dots, f - 1$ , set  $t \leftarrow \text{FREE}[k]$ ,  $j \leftarrow \text{MEM}[\text{CLOFF} + 4t + m^4] - (\text{CLOFF} + 4t)$ , and if  $j < r$  set  $r \leftarrow j$ ,  $c \leftarrow t$ ; break out of the loop if  $r = 0$ .

(b) If  $r > 0$  set  $x \leftarrow \text{MEM}[\text{CLOFF} + 4cl(\text{ALF}[x])]$ .

(c) If  $r > 1$  set  $q \leftarrow 0$ ,  $p' \leftarrow \text{MEM}[\text{PP}]$ , and  $p \leftarrow \text{POISON}$ . While  $p < p'$  do the following steps: Set  $y \leftarrow \text{MEM}[p]$ ,  $z \leftarrow \text{MEM}[p + 1]$ ,  $y' \leftarrow \text{MEM}[y + m^4]$ , and  $z' \leftarrow \text{MEM}[z + m^4]$ . (Here  $y$  and  $z$  point to the heads of prefix or suffix lists;  $y'$  and  $z'$  point to the tails.) If  $y = y'$  or  $z = z'$ , delete entry  $p$  from the poison list; this means, as in (18), to set  $p' \leftarrow p' - 2$ , and if  $p \neq p'$  to store( $p, \text{MEM}[p']$ ) and store( $p + 1, \text{MEM}[p' + 1]$ ). Otherwise set  $p \leftarrow p + 2$ ; if  $y' - y \geq z' - z$  and  $y' - y > q$ , set  $q \leftarrow y' - y$  and  $x \leftarrow \text{MEM}[z]$ ; if  $y' - y < z' - z$  and  $z' - z > q$ , set  $q \leftarrow z' - z$  and  $x \leftarrow \text{MEM}[y]$ . Finally, after  $p$  has become equal to  $p'$ , store( $\text{PP}, p'$ ) and set  $c \leftarrow cl(\text{ALF}[x])$ . (Experiments show that this “max kill” strategy for  $r > 1$  slightly outperforms a selection strategy based on  $r$  alone.)

**40.** (a) First there's a routine  $\text{rem}(\alpha, \delta, o)$  that removes an item from a list, following the protocol (21): Set  $p \leftarrow \delta + o$  and  $q \leftarrow \text{MEM}[p + m^4] - 1$ . If  $q \geq p$  (meaning that list  $p$  isn't closed or being killed), store( $p + m^4, q$ ), set  $t \leftarrow \text{MEM}[\alpha + o - m^4]$ ; and if  $t \neq q$  also set  $y \leftarrow \text{MEM}[q]$ , store( $t, y$ ), and store( $\text{ALF}[y] + o - m^4, t$ ).

Now, to redden  $x$  we set  $\alpha \leftarrow \text{ALF}[x]$ , store( $\alpha, \text{RED}$ ); then  $\text{rem}(\alpha, p_1(\alpha), \text{P10FF})$ ,  $\text{rem}(\alpha, p_2(\alpha), \text{P20FF})$ ,  $\dots$ ,  $\text{rem}(\alpha, s_3(\alpha), \text{S30FF})$ , and  $\text{rem}(\alpha, 4cl(\alpha), \text{CLOFF})$ .

(b) A simple routine  $\text{close}(\delta, o)$  closes list  $\delta + o$ : Set  $p \leftarrow \delta + o$  and  $q \leftarrow \text{MEM}[p + m^4]$ ; if  $q \neq p - 1$ , store( $p + m^4, p - 1$ ).

Now, to green  $x$  we set  $\alpha \leftarrow \text{ALF}[x]$ , store( $\alpha, \text{GREEN}$ ); then  $\text{close}(p_1(\alpha), \text{P10FF})$ ,  $\text{close}(p_2(\alpha), \text{P20FF})$ ,  $\dots$ ,  $\text{close}(s_3(\alpha), \text{S30FF})$ , and  $\text{close}(4cl(\alpha), \text{CLOFF})$ . Finally, for  $p \leq r < q$  (using the  $p$  and  $q$  that were just set within ‘close’), if  $\text{MEM}[r] \neq x$  redden  $\text{MEM}[r]$ .

(c) First set  $p' \leftarrow \text{MEM}[\text{PP}] + 6$ , and store( $p' - 6, p_1(\alpha) + \text{S10FF}$ ), store( $p' - 5, s_3(\alpha) + \text{P30FF}$ ), store( $p' - 4, p_2(\alpha) + \text{S20FF}$ ), store( $p' - 3, s_2(\alpha) + \text{P20FF}$ ), store( $p' - 2, p_3(\alpha) + \text{S30FF}$ ), store( $p' - 1, s_1(\alpha) + \text{P10FF}$ ); this adds the three poison items (27).

Then set  $p \leftarrow \text{POISON}$  and do the following while  $p < p'$ : Set  $y, z, y', z'$  as in answer 39(c), and delete poison entry  $p$  if  $y = y'$  or  $z = z'$ . Otherwise if  $y' < y$  and  $z' < z$ , go to C6 (a poisoned suffix-prefix pair is present). Otherwise if  $y' > y$  and  $z' > z$ , set  $p \leftarrow p + 2$ . Otherwise if  $y' < y$  and  $z' > z$ , store( $z + m^4, z$ ), redden  $\text{MEM}[r]$  for  $z \leq r < z'$ , and delete poison entry  $p$ . Otherwise (namely if  $y' > y$  and  $z' < z$ ), store( $y + m^4, y$ ), redden  $\text{MEM}[r]$  for  $y \leq r < y'$ , and delete poison entry  $p$ .

Finally, after  $p$  has become equal to  $p'$ , store( $\text{PP}, p'$ ).

**42.** Exercise 32 exhibits such codes explicitly for all odd  $n$ . The earliest papers on the subject gave solutions for  $n = 2, 4, 6, 8$ . Yoji Niho subsequently found a code for  $n = 10$  but was unable to resolve the case  $n = 12$  [*IEEE Trans. IT-19* (1973), 580–581].

This problem can readily be encoded in CNF and given to a SAT solver. The case  $n = 10$  involves 990 variables and 8.6 million clauses, and is solved by Algorithm 7.2.2.2C in 10.5 gigamems. The case  $n = 12$  involves 4020 variables and 175 million clauses. After being split into seven independent subproblems (by appending mutually exclusive unit clauses), it was proved *unsatisfiable* by that algorithm after about 86 teramems of computation.

So the answer is “No.” The maximum-size code for  $n = 12$  remains unknown.

**44.** (a) There are 28 comma-free binary codes of size 3 and length 4; Algorithm C produces half of them, because it assumes that cycle class [0001] is represented by 0001 or 0010. They form eight equivalence classes, two of which are symmetric under the operation of complementation-and-reflection; representatives are {0001, 0011, 0111} and {0010, 0011, 1011}. The other six are represented by {0001, 0110, 0111 or 1110}, {0001, 1001, 1011 or 1101}, {0001, 1100, 1101}, {0010, 0011, 1101}.

(b) Algorithm C produces half of the 144 solutions, which form twelve equivalence classes. Eight are represented by {0001, 0002, 1001, 1002, 2201, 2001, 2002, 2011, 2012, 2102, 2112, 2122 or 2212} and ({0102, 1011, 1012} or {1020, 1101, 2101}) and ({1202, 2202, 2111} or {2021, 2022, 1112}); four are represented by {0001, 0020, 0021, 0022, 1001, 1020, 1021, 1022, 1121 or 1211, 1201, 1202, 1221, 2001, 2201, 2202} and ({1011, 1012, 2221} or {1101, 2101, 1222}).

(c) Algorithm C yields half of the 2304 solutions, which form 48 equivalence classes. Twelve classes have unique representatives that omit cycle classes [0123], [0103], [1213], one such being the code {0010, 0020, 0030, 0110, 0112, 0113, 0120, 0121, 0122, 0130, 0131, 0132, 0133, 0210, 0212, 0213, 0220, 0222, 0230, 0310, 0312, 0313, 0320, 0322, 0330, 0332, 0333, 1110, 1112, 1113, 2010, 2030, 2110, 2112, 2113, 2210, 2212, 2213, 2230, 2310, 2312, 2313, 2320, 2322, 2330, 2332, 2333, 3110, 3112, 3113, 3210, 3212, 3213, 3230, 3310, 3312, 3313}. The others each have two representatives that omit classes [0123], [0103], [0121], one such being the code {0001, 0002, 0003, 0201, 0203, 1001, 1002, 1003, 1011, 1013, 1021, 1022, 1023, 1031, 1032, 1033, 1201, 1203, 1211, 1213, 1221, 1223, 1231, 1232, 1233, 1311, 1321, 1323, 1331, 2001, 2002, 2003, 2021, 2022, 2023, 2201, 2203, 2221, 2223, 3001, 3002, 3003, 3011, 3013, 3021, 3022, 3023, 3031, 3032, 3033, 3201, 3203, 3221, 3223, 3321, 3323, 3331} and its isomorphic image under reflection and (01)(23).

**45.** (The maximum size of such a code is currently unknown. Algorithm C isn't fast enough to solve this problem on a single computer, but a sufficiently large cluster of machines and/or an improved algorithm should be able to discover the answer. The case  $m = 3$  and  $n = 6$  is also currently unsolved; a SAT solver shows quickly that a full set of  $(3^6 - 3^3 - 3^2 + 3^1)/6 = 116$  codewords cannot be achieved.)

**47.** The 3-bit sequences 101, 111, 110 were rejected before seeing 000. In general, to make a uniformly random choice from  $q$  possibilities, the text suggests looking at the next  $t = \lceil \lg q \rceil$  bits  $b_1 \dots b_t$ . If  $(b_1 \dots b_t)_2 < q$ , we use choice  $(b_1 \dots b_t)_2 + 1$ ; otherwise we reject  $b_1 \dots b_t$  and try again. [This simple method is optimum when  $q \leq 4$ , and the best possible running time for other values of  $q$  uses more than half as many bits. But a better scheme is available for  $q = 5$ , using only  $3\frac{1}{5}$  bits per choice instead of  $4\frac{4}{5}$ ; and for  $q = 6$ , one random bit reduces to the case  $q = 3$ . See D. E. Knuth and A. C. Yao, *Algorithms and Complexity*, edited by J. F. Traub (Academic Press, 1976), 357–428, §2.]

Niho  
CNF  
SAT solver  
unit clauses  
uniformly random  
reject  
Knuth  
Yao  
complexity of calculation  
computational complexity  
Traub

**48.** It's the number of nodes on level  $l+1$  (depth  $l$ ) of the search tree. (Hence we can estimate the profile. Notice that  $D = D_1 \dots D_{l-1}$  in step E2 of Algorithm E.)

**49.**  $Z_0 = C()$ ,  $Z_{l+1} = c() + D_1 c(X_1) + D_1 D_2 c(X_1 X_2) + \dots + D_1 \dots D_l c(X_1 \dots X_l) + D_1 \dots D_{l+1} C(X_1 \dots X_{l+1})$ .

**50.** (a) True: The generating function is  $z(z+1)\dots(z+n-1)/n!$ ; see Eq. 1.2.10-(g).

(b) For instance, suppose  $Y_1 Y_2 \dots Y_l = 1457$  and  $n = 9$ . Alice's probability is  $\frac{1}{1} \frac{1}{2} \frac{2}{3} \frac{1}{4} \frac{1}{5} \frac{5}{6} \frac{1}{7} \frac{7}{8} \frac{8}{9} = \frac{1}{3} \frac{1}{4} \frac{1}{6} \frac{1}{9}$ . Elmo obtains  $X_1 X_2 \dots X_l = 7541$  with probability  $\frac{1}{9} \frac{1}{6} \frac{1}{4} \frac{1}{3}$ .

(c) The upper tail inequality (see exercise 1.2.10-22 with  $\mu = H_n$ ) tells us that  $\Pr(l \geq (\ln n)(\ln \ln n)) \leq \exp(-(\ln n)(\ln \ln n)(\ln \ln \ln n) + O(\ln n)(\ln \ln n))$ .

(d) If  $k \leq n/3$  we have  $\sum_{j=0}^k \binom{n}{j} \leq 2 \binom{n}{k}$ . By exercise 1.2.6-67, the number of nodes on the first  $(\ln n)(\ln \ln n)$  levels is therefore at most  $2(ne/((\ln n)(\ln \ln n)))^{(\ln n)(\ln \ln n)}$ .

**51.** The key idea is to introduce recursive formulas analogous to (29):

$$\begin{aligned} m(x_1 \dots x_l) &= c(x_1 \dots x_l) + \min(m(x_1 \dots x_l x_{l+1}^{(1)}), \dots, m(x_1 \dots x_l x_{l+1}^{(d)})); \\ M(x_1 \dots x_l) &= c(x_1 \dots x_l) + \max(M(x_1 \dots x_l x_{l+1}^{(1)}), \dots, M(x_1 \dots x_l x_{l+1}^{(d)})); \\ \hat{C}(x_1 \dots x_l) &= c(x_1 \dots x_l)^2 + \sum_{i=1}^d (\hat{C}(x_1 \dots x_l x_{l+1}^{(i)})d + 2c(x_1 \dots x_l)C(x_1 \dots x_l x_{l+1}^{(i)})). \end{aligned}$$

They can be computed via auxiliary arrays MIN, MAX, KIDS, COST, and CHAT as follows:

At the beginning of step B2, set  $\text{MIN}[l] \leftarrow \infty$ ,  $\text{MAX}[l] \leftarrow \text{KIDS}[l] \leftarrow \text{COST}[l] \leftarrow \text{CHAT}[l] \leftarrow 0$ . Set  $\text{KIDS}[l] \leftarrow \text{KIDS}[l] + 1$  just before  $l \leftarrow l+1$  in step B3.

At the beginning of step B5, set  $m \leftarrow c(x_1 \dots x_{l-1}) + \text{KIDS}[l] \times \text{MIN}[l]$ ,  $M \leftarrow c(x_1 \dots x_{l-1}) + \text{KIDS}[l] \times \text{MAX}[l]$ ,  $C \leftarrow c(x_1 \dots x_{l-1}) + \text{COST}[l]$ ,  $\hat{C} \leftarrow c(x_1 \dots x_{l-1})^2 + \text{KIDS}[l] \times \text{CHAT}[l] + 2 \times \text{COST}[l]$ . Then, after  $l \leftarrow l-1$  is positive, set  $\text{MIN}[l] \leftarrow \min(m, \text{MIN}[l])$ ,  $\text{MAX}[l] \leftarrow \max(M, \text{MAX}[l])$ ,  $\text{COST}[l] \leftarrow \text{COST}[l] + C$ ,  $\text{CHAT}[l] \leftarrow \text{CHAT}[l] + \hat{C}$ . But when  $l$  reaches zero in step B5, return the values  $m, M, C, \hat{C} - C^2$ .

**52.** Let  $p(i) = p_{X_1 \dots X_{l-1}}(y_i)$ , and simply set  $D \leftarrow D/p(I)$  instead of  $D \leftarrow Dd$ . Then node  $x_1 \dots x_l$  is reached with probability  $\Pi(x_1 \dots x_l) = p(x_1)p_{x_1}(x_2)\dots p_{x_1 \dots x_l}(x_l)$ , and  $c(x_1 \dots x_l)$  has weight  $1/\Pi(x_1 \dots x_l)$  in  $S$ ; the proof of Theorem E goes through as before. Notice that  $p(I)$  is the *a posteriori* probability of having taken branch  $I$ .

(The formulas of answer 51 should now use ' $p(i)$ ' instead of ' $d$ '; and that algorithm should be modified appropriately, no longer needing the KIDS array.)

**53.** Let  $p_{X_1 \dots X_{l-1}}(y_i) = C(x_1 \dots x_{l-1} y_i) / (C(x_1 \dots x_{l-1}) - c(x_1 \dots x_{l-1}))$ . (Of course we generally need to know the cost of the tree before we know the exact values of these ideal probabilities, so we cannot achieve zero variance in practice. But the form of this solution shows what kinds of bias are likely to reduce the variance.)

**55.** The effects of lookahead, dynamic ordering, and reversible memory are all captured easily by a well-designed cost function at each node. But there's a fundamental difference in step C2, because different codeword classes can be selected for branching at the same node (that is, with the same ancestors  $x_1 \dots x_{l-1}$ ) after C5 has undone the effects of a prior choice. The level  $l$  never surpasses  $L+1$ , but in fact the search tree involves hidden levels of branching that are implicitly combined into single nodes.

Thus it's best to view Algorithm C's search tree as a sequence of *binary* branches: Should  $x$  be one of the codewords or not? (At least this is true when the "max kill" strategy of answer 39 has selected the branching variable  $x$ . But if  $r > 1$  and the poison list is empty, an  $r$ -way branch is reasonable (or an  $(r+1)$ -way branch when the slack is positive), because  $r$  will be reduced by 1 and the same class  $c$  will be chosen after  $x$  has been explored.)

profile  
generating function  
tail inequality  
cumulative binomial distribution  
recursive formulas  
a posteriori  
cost function  
search tree  
poison list  
slack

If  $x$  has been selected because it kills many other potential codewords, we probably should bias the branch probability as in exercise 52, giving smaller weight to the “yes” branch because the branch that includes  $x$  is less likely to lead to a large subtree.

**57.** Let  $p_k = 1/D^{(k)}$  be the probability that Algorithm E terminates at the  $k$ th leaf. Then  $\sum_{k=1}^M (1/M) \lg(1/(Mp_k))$  is the Kullback–Leibler divergence  $D(q||p)$ , where  $q$  is the uniform distribution (see exercise MPR-121). Hence  $\frac{1}{M} \sum_{k=1}^M \lg D^{(k)} \geq \lg M$ . (The result of this exercise is essentially true in *any* probability distribution.)

**58.** Let  $\infty$  be any convenient value  $\geq n$ . When vertex  $v$  becomes part of the path we will perform a two-phase algorithm. The first phase identifies all “tarnished” vertices, whose DIST must change; these are the vertices  $u$  from which every path to  $t$  passes through  $v$ . It also forms a queue of “resource” vertices, which are untarnished but adjacent to tarnished ones. The second phase updates the DISTs of all tarnished vertices that are still connected to  $t$ . Each vertex has LINK and STAMP fields in addition to DIST.

For the first phase, set  $d \leftarrow \text{DIST}(v)$ ,  $\text{DIST}(v) \leftarrow \infty + 1$ ,  $R \leftarrow \Lambda$ ,  $T \leftarrow v$ ,  $\text{LINK}(v) \leftarrow \Lambda$ , then do the following while  $T \neq \Lambda$ : (\*) Set  $u \leftarrow T$ ,  $T \leftarrow S \leftarrow \Lambda$ . For each  $w \rightarrow u$ , if  $\text{DIST}(w) < d$  do nothing (this happens only when  $u = v$ ); if  $\text{DIST}(w) \geq \infty$  do nothing ( $w$  is gone or already known to be tarnished); if  $\text{DIST}(w) = d$ , make  $w$  a resource (see below); otherwise  $\text{DIST}(w) = d + 1$ . If  $w$  has no neighbor at distance  $d$ ,  $w$  is tarnished: Set  $\text{LINK}(w) \leftarrow T$ ,  $\text{DIST}(w) \leftarrow \infty$ ,  $T \leftarrow w$ . Otherwise make  $w$  a resource (see below). Then set  $u \leftarrow \text{LINK}(u)$ , and return to (\*) if  $u \neq \Lambda$ .

The queue of resources will start at  $R$ . We will stamp each resource with  $v$  so that nothing is added twice to that queue. To make  $w$  a resource when  $\text{DIST}(w) = d$ , do the following (unless  $u = v$  or  $\text{STAMP}(w) = v$ ): Set  $\text{STAMP}(w) \leftarrow v$ ; if  $R = \Lambda$ , set  $R \leftarrow \text{RT} \leftarrow w$ ; otherwise set  $\text{LINK}(\text{RT}) \leftarrow w$  and  $\text{RT} \leftarrow w$ . To make  $w$  a resource when  $\text{DIST}(w) = d + 1$  and  $u \neq v$  and  $\text{STAMP}(w) \neq v$ , put it first on stack  $S$  as follows: Set  $\text{STAMP}(w) \leftarrow v$ ; if  $S = \Lambda$ , set  $S \leftarrow \text{SB} \leftarrow w$ ; otherwise set  $\text{LINK}(w) \leftarrow S$ ,  $S \leftarrow w$ .

Finally, when  $u = \Lambda$ , we append  $S$  to  $R$ : Nothing needs to be done if  $S = \Lambda$ . Otherwise, if  $R = \Lambda$ , set  $R \leftarrow S$  and  $\text{RT} \leftarrow \text{SB}$ ; but if  $R \neq \Lambda$ , set  $\text{LINK}(\text{RT}) \leftarrow S$  and  $\text{RT} \leftarrow \text{SB}$ . (These shenanigans keep the resource queue in order by DIST.)

Phase 2 operates as follows: Nothing needs to be done if  $R = \Lambda$ . Otherwise we set  $\text{LINK}(\text{RT}) \leftarrow \Lambda$ ,  $S \leftarrow \Lambda$ , and do the following while  $R \neq \Lambda$  or  $S \neq \Lambda$ : (i) If  $S = \Lambda$ , set  $d \leftarrow \text{DIST}(R)$ . Otherwise set  $u \leftarrow S$ ,  $d \leftarrow \text{DIST}(u)$ ,  $S \leftarrow \Lambda$ ; while  $u \neq \Lambda$ , update the neighbors of  $u$  and set  $u \leftarrow \text{LINK}(u)$ . (ii) While  $R \neq \Lambda$  and  $\text{DIST}(R) = d$ , set  $u \leftarrow R$ ,  $R \leftarrow \text{LINK}(u)$ , and update the neighbors of  $u$ . In both cases “update the neighbors of  $u$ ” means to look at all  $w \rightarrow u$ , and if  $\text{DIST}(w) = \infty$  to set  $\text{DIST}(w) \leftarrow d + 1$ ,  $\text{STAMP}(w) \leftarrow v$ ,  $\text{LINK}(w) \leftarrow S$ , and  $S \leftarrow w$ . (It works!)

**59.** (a) Compute the generating function  $g(z)$  (see exercise 7.1.4–209) and then  $g'(1)$ .

(b) Let  $(A, B, C)$  denote paths that touch (center, NE corner, SW corner). Recursively compute eight counts  $(c_0, \dots, c_7)$  at each node, where  $c_j$  counts paths  $\pi$  with  $j = 4[\pi \in A] + 2[\pi \in B] + [\pi \in C]$ . At the sink node  $\square$  we have  $c_0 = 1$ ,  $c_1 = \dots = c_7 = 0$ . Other nodes have the form  $x = (\bar{e}^? x_l: x_h)$  where  $e$  is an edge. Two edges go across the center and affect  $A$ ; three edges affect each of  $B$  and  $C$ . Say that those edges have types 4, 2, 1, respectively; other edges have type 0. Suppose the counts for  $x_l$  and  $x_h$  are  $(c'_0, \dots, c'_7)$  and  $(c''_0, \dots, c''_7)$ , and  $e$  has type  $t$ . Then count  $c_j$  for node  $x$  is  $c'_j + [t=0]c''_j + [t \neq 0][t \& j \neq 0](c''_j + c'_{j-t})$ .

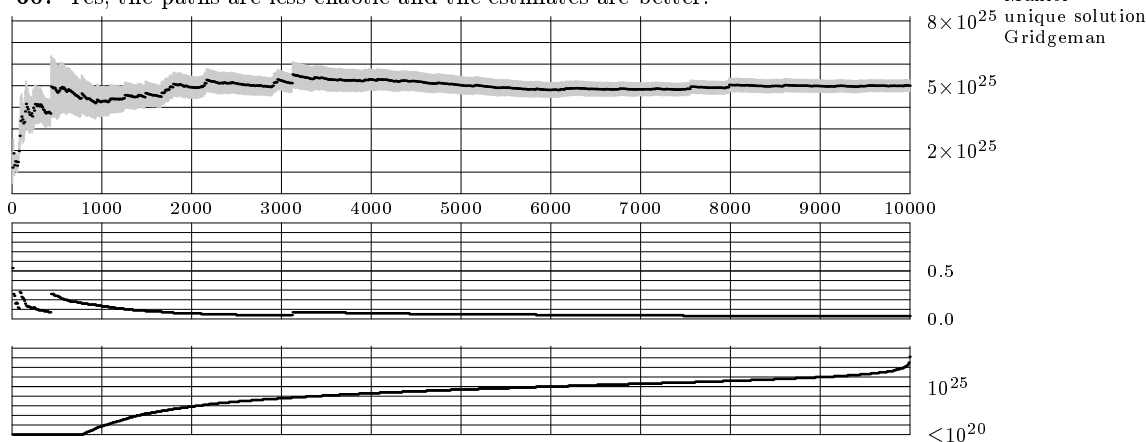
(This procedure yields the following exact “Venn diagram” set counts at the root:  $c_0 = |\bar{A} \cap \bar{B} \cap \bar{C}| = 7653685384889019648091604$ ;  $c_1 = c_2 = |\bar{A} \cap \bar{B} \cap C| = |\bar{A} \cap B \cap \bar{C}| = 7755019053779199171839134$ ;  $c_3 = |\bar{A} \cap B \cap C| = 7857706970503366819944024$ ;

bias  
Kullback  
Leibler  
divergence  
generating function  
Venn diagram



$c_4 = |A \cap \overline{B} \cap \overline{C}| = 4888524166534573765995071$ ;  $c_5 = c_6 = |A \cap \overline{B} \cap C| = |A \cap B \cap \overline{C}| = 4949318991771252110605148$ ;  $c_7 = |A \cap B \cap C| = 5010950157283718807987280$ .)

**60.** Yes, the paths are less chaotic and the estimates are better:



**61.** (a) Let  $x_k$  be the number of nodes at distance  $k - 1$  from the root.

(b) Let  $Q_n^{(m)} = P_n^{(1)} + \dots + P_n^{(m)}$ . Then we have the joint recurrence  $P_1^{(m)} = 1$ ,  $P_{n+1}^{(m)} = Q_n^{(2m)}$ ; in particular,  $Q_1^{(m)} = m$ . And for  $n \geq 2$ , we have  $Q_n^{(m)} = \sum_{k=1}^n a_{nk} \binom{m}{k}$  for certain constants  $a_{nk}$  that can be computed as follows: Set  $t_k \leftarrow P_n^{(k)}$  for  $1 \leq k \leq n$ . Then for  $k = 2, \dots, n$  set  $t_n \leftarrow t_n - t_{n-1}$ ,  $\dots$ ,  $t_k \leftarrow t_k - t_{k-1}$ . Finally  $a_{nk} \leftarrow t_k$  for  $1 \leq k \leq n$ . For example,  $a_{21} = a_{22} = 2$ ;  $a_{31} = 6$ ,  $a_{32} = 14$ ,  $a_{33} = 8$ . The numbers  $P_n^{(m)}$  have  $O(n^2 + n \log m)$  bits, so this method needs  $O(n^5)$  bit operations to compute  $P_n$ .

(c)  $P_n^{(m)}$  corresponds to random paths with  $X_1 = m$ ,  $D_k = 2X_k$ ,  $X_{k+1} = \lceil 2U_k X_k \rceil$ , where each  $U_k$  is an independent uniform deviate. Therefore  $P_n^{(m)} = E(D_1 \dots D_{n-1})$  is the number of nodes on level  $n$  of an infinite tree. We have  $X_{k+1} \geq 2^k U_1 \dots U_k m$ , by induction; hence  $P_n^{(m)} \geq E(2^{\binom{n}{2}} U_1^{n-2} U_2^{n-3} \dots U_{n-2}^1 m^{n-1}) = 2^{\binom{n}{2}} m^{n-1} / (n-1)!$ .

[M. Cook and M. Kleber have discussed similar sequences in *Electronic Journal of Combinatorics* **7** (2000), #R44. See also K. Mahler's asymptotic formula for binary partitions, in *J. London Math. Society* **15** (1940), 115–123, which shows that  $\lg P_n = \binom{n}{2} - \lg(n-1)! + \binom{\lg n}{2} + O(1)$ .]

**66.** Random trials indicate that the expected number of 2-regular graphs is  $\approx 3.115$ , and that the number of disjoint pairs is (0, 1,  $\dots$ , 9, and  $\geq 10$ ) approximately (74.4, 4.3, 8.7, 1.3, 6.2, 0.2, 1.5, 0.1, 2.0, 0.0, and 12.2) percent of the time. If the cubes are restricted to cases where each color occurs at least five times, these numbers change to  $\approx 4.89$  and (37.3, 6.6, 17.5, 4.1, 16.3, 0.9, 5.3, 0.3, 6.7, 0.2, 5.0).

However, the concept of “unique solution” is tricky, because a 2-regular graph with  $k$  cycles yields  $2^k$  ways to position the cubes. Let's say that a set of cubes has a *strongly unique* solution if (i) it has a unique disjoint pair of 2-regular graphs, and furthermore (ii) both elements of that pair are  $n$ -cycles. Such sets occur with probability only about 0.3% in the first case, and 0.4% in the second.

[N. T. Gridgeman, in *Mathematics Magazine* **44** (1971), 243–252, showed that puzzles with four cubes and four colors have exactly 434 “types” of solutions.]

**67.** It's easy to find such examples at random, as in the second part of the previous answer, since strongly unique sets occur about 0.5% of the time (and weakly unique

sets occur with probability  $\approx 8.4\%$ ). For example, the pairs of opposite faces might be (12, 13, 34), (02, 03, 14), (01, 14, 24), (04, 13, 23), (01, 12, 34).

(Incidentally, if we require each color to occur exactly *six* times, every set of cubes that has at least one solution will have at least *three* solutions, because the “hidden” pairs can be chosen in three ways.)

**70.** Each of these cubes can be placed in 16 different ways that contribute legitimate letters to all four of the visible words. (A cube whose faces contain only letters in  $\{C, H, I, N, O, U, X, Z\}$  can be placed in 24 ways. A cube with a pattern like  $\begin{bmatrix} B & D \\ D & D \end{bmatrix}$  cannot be placed at all.) We can restrict the first cube to just two placements; thus there are  $2 \cdot 16 \cdot 16 \cdot 16 = 131072$  ways to place those cubes without changing their order. Of these, only 6144 are “compatible,” in the sense that no rightside-up-only letter appears together with an upside-down-only letter in the same word.

The 6144 compatible placements can then each be reordered in  $5! = 120$  ways. One of them, whose words before reordering are GRHTI, NCICY, OΘRMN, INNNO, leads to the unique solution. (There’s a partial solution with three words out of four. There also are 39 ways to get two valid words, including one that has UNTIL adjacent to HOURS, and several with SYRUP opposite ECHOS.)

**73.** Call the rays N, NE, E, SE, S, SW, W, NW; call the disks 1, 2, 3, 4 from inside to outside. We can keep disk 1 fixed. The sum of rays N, S, E, W must be 48. It is 16 (on disk 1) plus 13 or 10 (on disk 2) plus 8 or 13 (on disk 3) plus 11 or 14. So it is attained either as shown, or after rotating disks 2 and 4 clockwise by  $45^\circ$ . (Or we could rotate any disk by a multiple of  $90^\circ$ , since that keeps the desired sum invariant.)

Next, with optional  $90^\circ$  rotations, we must make the sum of rays N + S equal to 24. In the first solution above it is 9 plus (6 or 7) plus (4 or 4) plus (7 or 4), hence never 24. But in the other solution it’s 9 plus (4 or 6) plus (4 or 4) plus (5 or 9); hence we must rotate disk 2 clockwise by  $90^\circ$ , and possibly also disk 3. However,  $90^\circ$  rotation of disk 3 would make the NE + SW sum equal to 25, so we musn’t move it.

Finally, to get NE’s sum to be 12, via optional rotations by  $180^\circ$ , we have 1 plus (2 or 5) plus (1 or 5) plus (3 or 4); we must shift disks 3 and 4. Hurray: That makes all eight rays correct. Factoring twice has reduced  $8^3$  trials to  $2^3 + 2^3 + 2^3$ .

[See George W. Ernst and Michael M. Goldstein, *JACM* **29** (1982), 1–23. Such puzzles go back to the 1800s; three early examples are illustrated on pages 28 of Slocum and Botermans’s *New Book of Puzzles* (1992). One of them, with six rings and six rays, factors from  $6^5$  trials to  $2^5 + 3^5$ . A five-ray puzzle would have defeated factorization.]

**75.** Call the cards 1525, 5113, ..., 3755. The key observation is that all 12 sums must be odd, so we can first solve the problem mod 2. For this purpose we may call the cards 1101, 1111, ..., 1111; only three cards now change under rotation, namely 1101, 0100, and 1100 (which are the mod 2 images of 1525, 4542, and 7384).

A second observation is that each solution gives  $6 \times 6 \times 2$  others, by permuting rows and/or columns and/or by rotating all nine cards. Hence we can assume that the upper left card is 0011 (8473). Then 0100 (4542) must be in the first column, possibly rotated to 0001 (4245), to preserve parity in the left two black sums. We can assume that it’s in row 2. In fact, after retreating from 13 mod 2 to 13, we see that it *must* be rotated. Hence the bottom left card must be either 4725, 7755, or 3755.

Similarly we see that 1101 (1525) must be in the first row, possibly rotated to 0111 (2515); we can put it in column 2. It must be rotated, and the top right card must be 3454 or 3755. This leaves just six scenarios to consider, and we soon obtain the solution: 8473, 2515, 3454, 4245, 2547, 7452; 7755, 1351, 5537.

invariant

Ernst

Goldstein

Slocum

Botermans

Washington Monument Puzzle, see Fool’s Disk

Rotating Century Puzzle, see Fool’s Disk

Safe Combination Puzzle, see Fool’s Disk

**90.** Suppose there are  $n$  questions, whose answers each lie in a given set  $S$ . A *student* supplies an answer list  $\alpha = a_1 \dots a_n$ , with each  $a_j \in S$ ; a *grader* supplies a Boolean vector  $\beta = x_1 \dots x_n$ . There is a Boolean function  $f_{js}(\alpha, \beta)$  for each  $j \in \{1, \dots, n\}$  and each  $s \in S$ . A graded answer list  $(\alpha, \beta)$  is *valid* if and only if  $F(\alpha, \beta)$  is true, where

$$F(\alpha, \beta) = F(a_1 \dots a_n, x_1 \dots x_n) = \bigwedge_{j=1}^n \bigwedge_{s \in S} ([a_j = s] \Rightarrow x_j \equiv f_{js}(\alpha, \beta)).$$

The *maximum score* is the largest value of  $x_1 + \dots + x_n$  over all graded answer lists  $(\alpha, \beta)$  that are valid. A *perfect score* is achieved if and only if  $F(\alpha, 1 \dots 1)$  holds.

Thus, in the warmup problem we have  $n = 2$ ,  $S = \{A, B\}$ ;  $f_{1A} = [a_2 = B]$ ;  $f_{1B} = [a_1 = A]$ ;  $f_{2A} = x_1$ ;  $f_{2B} = \bar{x}_2 \oplus [a_1 = A]$ . The four possible answer lists are:

$$\begin{aligned} \text{AA: } F &= (x_1 \equiv [A=B]) \wedge (x_2 \equiv x_1) \\ \text{AB: } F &= (x_1 \equiv [B=B]) \wedge (x_2 \equiv \bar{x}_2 \oplus [A=A]) \\ \text{BA: } F &= (x_1 \equiv [B=A]) \wedge (x_2 \equiv x_1) \\ \text{BB: } F &= (x_1 \equiv [B=A]) \wedge (x_2 \equiv \bar{x}_2 \oplus [B=A]) \end{aligned}$$

Thus AA and BA must be graded 00; AB can be graded either 10 or 11; and BB has no valid grading. Only AB can achieve the maximum score, 2; but 2 isn't guaranteed.

In Table 666 we have, for example,  $f_{1C} = [a_2 \neq A] \wedge [a_3 = A]$ ;  $f_{4D} = [a_1 = D] \wedge [a_{15} = D]$ ;  $f_{12A} = [\Sigma_A - 1 = \Sigma_B]$ , where  $\Sigma_s = \sum_{1 \leq j \leq 20} [a_j = s]$ . It's amusing to note that  $f_{14E} = [\{\Sigma_A, \dots, \Sigma_E\} = \{2, 3, 4, 5, 6\}]$ .

The other cases are similar (although often more complicated) Boolean functions — except for 20D and 20E, which are discussed further in exercise 91.

Notice that an answer list that contains both 10E and 17E must be discarded: It can't be graded, because 10E says ' $x_{10} \equiv \bar{x}_{17}$ ' while 17E says ' $x_{17} \equiv x_{10}$ '.

By suitable backtrack programming, we can prove first that no perfect score is possible. Indeed, if we consider the answers in the order (3, 15, 20, 19, 2, 1, 17, 10, 5, 4, 16, 11, 13, 14, 7, 18, 6, 8, 12, 9), many cases can quickly be ruled out. For example, suppose  $a_3 = C$ . Then we must have  $a_1 \neq a_2 \neq \dots \neq a_{16} \neq a_{17} = a_{18} \neq a_{19} \neq a_{20}$ , and early cutoffs are often possible. (We might reach a node where the remaining choices for answers 5, 6, 7, 8, 9 are respectively  $\{C, D\}$ ,  $\{A, C\}$ ,  $\{B, D\}$ ,  $\{A, B, E\}$ ,  $\{B, C, D\}$ , say. Then if answer 8 is forced to be B, answer 7 can only be D; hence answer 6 is also forced to be A. Also answer 9 can no longer be B.) An instructive little propagation algorithm will make such deductions nicely at every node of the search tree. On the other hand, difficult questions like 7, 8, 9, are best *not* handled with complicated mechanisms; it's better just to wait until all twenty answers have been tentatively selected, and to check such hard cases only when the checking is easy and fast. In this way the author's program showed the impossibility of a perfect score by exploring just 52859 nodes, after only 3.4 megamems of computation.

The next task was to try for score 19 by asserting that only  $x_j$  is false. This turned out to be impossible for  $1 \leq j \leq 18$ , based on very little computation whatsoever (especially, of course, when  $j = 6$ ). The hardest case,  $j = 15$ , needed just 56 nodes and fewer than 5 kilomems. But then, ta da, three solutions were found: One for  $j = 19$  (185 kilonodes, 11 megamems) and two for  $j = 20$  (131 kilonodes, 8 megamems), namely

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
D	C	E	A	B	E	B	C	E	A	B	E	A	E	D	B	D	A	b	B	(i)
A	E	D	C	A	B	C	D	C	A	C	E	D	B	C	A	D	A	A	c	(ii)
D	C	E	A	B	A	D	C	D	A	E	D	A	E	D	B	D	B	E	e	(iii)

student  
grader  
valid  
dynamic ordering  
propagation algorithm  
search tree  
author

(The incorrect answers are shown here as lowercase letters. The first two solutions establish the truth of 20B and the falsity of 20E.)

**91.** Now there's only *one* list of answers with score  $\geq 19$ , namely (iii). But that is paradoxical — because it claims 20E is false; hence the maximum score *cannot* be 19!

Paradoxical situations are indeed possible when the global function  $F$  of answer 90 is used recursively within one or more of the local functions  $f_{js}$ . Let's explore a bit of recursive territory by considering the following two-question, two-letter example:

1. (A) Answer 1 is incorrect. (B) Answer 2 is incorrect.
2. (A) Some answers can't be graded consistently. (B) No answers achieve a perfect score.

Here we have  $f_{1A} = \bar{x}_1$ ;  $f_{1B} = \bar{x}_2$ ;  $f_{2A} = \exists a_1 \exists a_2 \forall x_1 \forall x_2 \neg F(a_1 a_2, x_1 x_2)$ ;  $f_{2B} = \forall a_1 \forall a_2 \neg F(a_1 a_2, 11)$ . (Formulas quantified by  $\exists a$  or  $\forall a$  expand into  $|S|$  terms, while  $\exists x$  or  $\forall x$  expand into two; for example,  $\exists a \forall x g(a, x) = (g(A, 0) \wedge g(A, 1)) \vee (g(B, 0) \wedge g(B, 1))$  when  $S = \{A, B\}$ .) Sometimes the expansion is undefined, because it has more than one "fixed point"; but in this case there's no problem because  $f_{2A}$  is true: Answer AA can't be graded, since 1A implies  $x_1 \equiv \bar{x}_1$ . Also  $f_{2B}$  is true, because both BA and BB imply  $x_1 \equiv \bar{x}_2$ . Thus we get the maximum score 1 with either BA or BB and grades 01.

On the other hand the simple one-question, one-letter questionnaire '1. (A) The maximum score is 1' has an *indeterminate* maximum score. For in this case  $f_{1A} = F(A, 1)$ . We find that if  $F(A, 1) = 0$ , only (A, 0) is a valid grading, so the only possible score is 0; similarly, if  $F(A, 1) = 1$ , the only possible score is 1.

OK, suppose that the maximum score for the modified Table 666 is  $m$ . We know that  $m < 19$ ; hence (iii) isn't a valid grading. It follows that 20E is true, which means that every *valid graded list of score  $m$  has  $x_{20}$  false*. And we can conclude that  $m = 18$ , because of the following two solutions (which are the only possibilities with 20C false):

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
B	A	d	A	B	E	D	C	D	A	E	D	A	E	D	E	D	B	E	c
A	E	D	C	A	B	C	D	C	A	C	E	D	B	a	C	D	A	A	c

But wait: If  $m = 18$ , we can score 18 with 20A true and two errors, using (say)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
D	e	D	A	B	E	D	e	C	A	E	D	A	E	D	B	D	C	C	A

or 47 other answer lists. This *contradicts*  $m = 18$ , because  $x_{20}$  is true.

End of story? No. This argument has implicitly been predicated on the assumption that 20D is false. What if  $m$  is indeterminate? Then a new solution arises

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
D	C	E	A	B	E	D	C	E	A	E	B	A	E	D	B	D	A	d	D

of score 19. With (iii) it yields  $m = 19$ ! If  $m$  is determinate, we've shown that  $m$  cannot actually be defined consistently; but if  $m$  is indeterminate, it's definitely 19.

*Question 20 was designed to create difficulties. [:-)]*

— DONALD R. WOODS (2001)

**95.** The 29 words **spark, often, lucky, other, month, ought, names, water, games, offer, lying, opens, magic, brick, lamps, empty, organ, noise, after, raise, drink, draft, backs, among, under, match, earth, roots, topic** yield this: "The success or failure of backtrack often depends on the skill and ingenuity of the programmer. ... Backtrack programming (as many other types of programming) is somewhat of an art." — Solomon W. Golomb, Leonard D. Baumert.

recursively  
quantified  
fixed point  
indeterminate  
WOODS  
Golomb  
Baumert

That solution can be found interactively, using inspired guesses based on a knowledge of English and its common two-letter and three-letter words. But could a computer that knows common English words discover it without understanding their meanings?

We can formulate that question as follows: Let  $w_1, \dots, w_{29}$  be the unknown words from WORDS(1000), and let  $q_1, \dots, q_{29}$  be the unknown words of the quotation. (By coincidence there happen to be just 29 of each.) We can restrict the  $q$ 's to words that appear, say, 32 times or more in the British National Corpus. That gives respectively (85, 562, 1863, 3199, 4650, 5631, 5417, 4724, 3657, 2448) choices for words of (2, 3, ..., 11) letters; in particular, we allow 3199 possibilities for the five-letter words  $q_7, q_{11}, q_{21}, q_{22}$ , because they aren't required to lie in WORDS(1000). Is there a unique combination of words  $w_i$  and  $q_j$  that meets the given anacrostic constraints?

This is a challenging problem, which we shall discuss in later sections. The answer turns out (surprisingly?) to be *no*; in fact, here is the first solution found by the author's machine(!): "The success or failure of backtrack often depends on roe skill and ingenuity at the programmer. ... Backtrack programming (as lacy offal types of programming) as somewhat al an art." (The OSPD4 includes 'al' as the name of the Indian mulberry tree; the BNC has 'al' 3515 times, mostly in unsuitable contexts, but that corpus is a blunt instrument.) Altogether 720 solutions satisfy the stated constraints; they differ from the "truth" only in words of at most five letters.

Anacrostic puzzles, which are also known by other names such as double-crostics, were invented in 1933 by E. S. Kingsley. See E. S. Spiegelthal, *Proceedings of the Eastern Joint Computer Conference* **18** (1960), 39–56, for an interesting early attempt to solve them—*without* backtracking—on an IBM 704 computer.

**96.** Instead of considering 1000 possibilities for  $\frac{131}{137} \frac{132}{118} \frac{133}{46} \frac{134}{48} \frac{135}{32}$ , it suffices to consider the 43 pairs  $xy$  such that  $cxyab$  is in WORDS(1000) and  $abc$  is a common three-letter word. (Of these pairs **ab**, **ag**, ..., **ve**, only **ar** leads to a solution. And indeed, the 720 solutions factor into three sets of 240, corresponding to choosing **earth**, **harsh**, or **large** as  $\frac{131}{137} \frac{132}{118} \frac{133}{46} \frac{134}{48} \frac{135}{32}$ .) Similar reductions, but not so dramatic, occur with respect to  $\frac{131}{137} \frac{132}{118} \frac{133}{46} \frac{134}{48} \frac{135}{32}$ , and  $\frac{131}{137} \frac{132}{118} \frac{133}{46} \frac{134}{48} \frac{135}{32}$ .

**100.** The following algorithm uses an integer utility field  $\text{TAG}(u)$  in the representation of each vertex  $u$ , representing the number of times  $u$  has been "tagged." The operations "tag  $u$ " and "untag  $u$ " stand respectively for  $\text{TAG}(u) \leftarrow \text{TAG}(u) + 1$  and  $\text{TAG}(u) \leftarrow \text{TAG}(u) - 1$ . Vertices shown as  $\odot$  in the 21 examples have a nonzero TAG field, indicating that the algorithm has decided not to include them in this particular  $H$ .

State variables  $v_l$  (a vertex),  $i_l$  (an index), and  $a_l$  (an arc) are used at level  $l$  for  $0 \leq l < n$ . We assume that  $n > 1$ .

- R1.** [Initialize.] Set  $\text{TAG}(u) \leftarrow 0$  for all vertices  $u$ . Then set  $v_0 \leftarrow v$ ,  $i \leftarrow i_0 \leftarrow 0$ ,  $a \leftarrow a_0 \leftarrow \text{ARCS}(v)$ ,  $\text{TAG}(v) \leftarrow 1$ ,  $l \leftarrow 1$ , and go to R4.
- R2.** [Enter level  $l$ .] (At this point  $i = i_{l-1}$ ,  $v = v_i$ , and  $a = a_{l-1}$  is an arc from  $v$  to  $v_{l-1}$ .) If  $l = n$ , visit the solution  $v_0 v_1 \dots v_{n-1}$  and set  $l \leftarrow n - 1$ .
- R3.** [Advance  $a$ .] Set  $a \leftarrow \text{NEXT}(a)$ , the next neighbor of  $v$ .
- R4.** [Done with level?] If  $a \neq \Lambda$ , go to R5. Otherwise if  $i = l - 1$ , go to R6. Otherwise set  $i \leftarrow i + 1$ ,  $v \leftarrow v_i$ ,  $a \leftarrow \text{ARCS}(v)$ .
- R5.** [Try  $a$ .] Set  $u \leftarrow \text{TIP}(a)$  and tag  $u$ . If  $\text{TAG}(u) > 1$ , return to R3. Otherwise set  $i_l \leftarrow i$ ,  $a_l \leftarrow a$ ,  $v_l \leftarrow u$ ,  $l \leftarrow l + 1$ , and go to R2.

two-letter  
three-letter  
British National Corpus  
constraints  
author  
OSPD4  
double-crostics  
Kingsley  
Spiegelthal  
IBM 704  
utility field  
tagged  
 $\Lambda$  the null pointer

**R6.** [Backtrack.] Set  $l \leftarrow l - 1$ , and stop if  $l = 0$ . Otherwise set  $i \leftarrow i_l$ . Untag all neighbors of  $v_k$ , for  $l \geq k > i$ . Then set  $a \leftarrow \text{NEXT}(a_l)$ ; while  $a \neq \Lambda$ , untag  $\text{TIP}(a)$  and set  $a \leftarrow \text{NEXT}(a)$ . Finally set  $a \leftarrow a_l$  and return to R3. ■

backtracking, variant structure  
 $n$ -omino placement

This instructive algorithm differs subtly from the conventional structure of Algorithm B. Notice in particular that  $\text{TIP}(a_l)$  is not untagged in step R6; that vertex won't be untagged and chosen again until some previous decision has been reconsidered.

**101.** Let  $G$  have  $N$  vertices. For  $1 \leq k \leq N$ , perform Algorithm R on the  $k$ th vertex  $v$  of  $G$ , except that step R1 should tag the first  $k - 1$  vertices so that they are excluded. (A tricky shortcut can be used: If we untag all neighbors of  $v = v_0$  after Algorithm R stops, the net effect will be to tag only  $v$ .)

The  $n$ -omino placement counts 1, 4, 22, 113, 571, 2816, 13616, 64678, 302574 are computed almost instantly, for small  $n$ . (Larger  $n$  are discussed in Section 7.2.3.)

- 102.** (a) All but the 13th and 18th, which require an upward or leftward step.  
 (b) True. If  $u \in H$  and  $u \neq v$ , let  $p_u$  be any node of  $H$  that's one step closer to  $v$ .  
 (c) Again true: The oriented spanning trees are also ordinary spanning trees.  
 (d) The same algorithm works, except that step R4 must return to itself after setting  $a \leftarrow \text{ARCS}(v)$ . (We can no longer be sure that  $\text{ARCS}(v) \neq \Lambda$ .)

**999.** ...

# INDEX AND GLOSSARY

GOLDSMITH

*He writes indexes to perfection.*

— OLIVER GOLDSMITH, *Citizen of the World* (1762)

When an index entry refers to a page containing a relevant exercise, see also the *answer* to that exercise for further information. An answer page is not indexed here unless it refers to a topic not included in the statement of the exercise.

- 2-letter block codes, 28.
- 2-letter words of English, 8, 48.
- 2-regular graphs, 25, 31.
- 3-letter words of English, 8, 48.
- 4-letter codewords, 9–18, 28–30.
- 4-letter words of English, 8, 48.
- 5-letter words of English, 8–9, 28, 31, 33.
- 6-letter and  $k$ -letter words of English, 8–9, 48.
- $\gamma$  (Euler’s constant), as source of “random” data, 19.
- $\Lambda$  (the null link), 43, 48.
- $\pi$  (circle ratio), as source of “random” data, 19–20, 22, 29.
- $\phi$  (golden ratio), as source of “random” data, 19.
- A posteriori probability, 42.
- Active elements of a list, 12.
- Ahrens, Joachim Heinrich Lüdecke, 6, 27, 36.
- Alekhnovich, Michael (Misha) Valentinovich (Алехнович, Михаил (Миша) Валентинович), 26.
- Alphabet, 31.
- Anacrostic puzzle, 33.
- Analysis of algorithms, 29, 30.
- Aperiodic words, 10, 28, 39.
- ARCS( $v$ ) (first arc of vertex  $v$ ), 33, 48.
- Armbruster, Franz Owen, 24.
- Backjumping, 26.
- Backmarking, 26.
- Backtrack programming, 2– $\infty$ .
  - efficiency of, 35.
  - history of, 2, 5–6, 25–26.
  - introduction to, 2–32.
  - variant structure, 48–49.
- Backtrack trees, 3, 4, 7, 9–11, 18–20, 24, 26, 41, 42, 44–46.
  - estimating the size of, 20, 41.
- Baumert, Leonard Daniel, 26, 47, 51.
- Bees, queen, 27.
- Bernoulli, Jacques (= Jakob = James), 25.
- Bezzel, Max Friedrich Wilhelm, 25.
- Biased random walks, 30–31, 43.
- Binary partitions, 31.
- Binomial trees, 21.
- Bitner, James Richard, 6, 26.
- Bitwise operations, 5, 27, 37.
- Block codes, 9, 28.
- Borodin, Allan Bertram, 26.
- Botermans, Jacobus (= Jack) Petrus Hermana, 45.
- Boundary markers, 29.
- Breadth-first search, 36.
- Breaking symmetry, 8, 14.
- British National Corpus, 8, 48.
- Broken diagonals, *see* Wraparound.
- Bumping the current stamp, 16, 29.
- Bunch, Steve Raymond, 6.
- Buresh-Oppenheim, Joshua, 26.
- Cache-friendly data structures, 11.
- Canonical labeling, 33.
- Carroll, Lewis (= Dodgson, Charles Lutwidge), iii.
- Carteblanche, Filet de (pseudonym, most likely of C. A. B. Smith), 24.
- Cavenagh, Nicholas John, 36.
- Cayley, Arthur, 31.
- Cells of memory, 11.
- Chatterjee, Sourav (সৌরভ চ্যাটার্জী), 22.
- Chessboard, 2–6, 22–26, 27, 31.
- Chiral symmetry, 27.
- Closed lists, 14.
- Clueless anacrostic, 33.
- CNF: Conjunctive normal form, 41.
- Codewords, comma-free, 9–18, 28–30.
- Coin flipping, 30.
- Combinations, 27.
- Comma-free codes, 9–18, 26, 28–30.
- Compilers, 15.
- Complexity of calculation, 41.
- Compressed tries, 38.
- Computational complexity, 41.
- Concatenation, 9.
- Connected subsets, 33.
- Constraints, 48.
- Cook, Matthew Makonnen, 44.
- Corner-to-corner paths, 22–23, 30–31.
- Cost function, 19, 42.
- Crick, Francis Harry Compton, 9.
- Cubes, 24–25, 31.
- Cumulative binomial distribution, 42.
- Cutoff principle, 7.
- Cutoff properties, 2, 5, 10, 18, 27.
- Cyclic shifts, 10, 28.

- Dancing links, 7.
- Data structures, 4–6, 9, 11–14, 18, 29.
- de Carteblanche, Filet (pseudonym, most likely of C. A. B. Smith), 24.
- de Jaenisch, Carl Friedrich Andreevitch (Янишъ, Карлъ Андреевичъ), 35.
- Degree of a node, 19.
- Deletion operation, 7, 12–13, 40.
- Depth-first search, 25.
- Diaconis, Persi Warren, 22.
- Diagonal lines (slope  $\pm 1$ ), 3, 36, *see also* Wraparound.
- Digraphs, 28, 34.
- DIMACS: DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, inaugurated in 1990.
- Dips, 29.
- Directed graphs versus undirected graphs, 34.
- Discarded data, 22.
- Discrete probabilities, 1.
- Disjoint sets, 25, 44.
- Distributed computations, 6.
- Divergence, Kullback–Leibler, 43.
- Divide and conquer paradigm, 24.
- Dodgson, Charles Lutwidge, iii.
- Domains, 2, 27, 28.
- Double-crostics, 48.
- Downdating versus updating, 4–5, 11, 15.
- Dual oriented spanning tree, 34.
- Dual solutions, 6, 8, 28.
- Dynamic ordering, 10–11, 24, 26, 30, 46.
- Dynamic shortest distances, 30.
  
- $e$ , as source of “random” data, 19.
- Eastman, Willard Lawrence, 28, 29, 39.
- Eight queens problem, 3–4, 19–20, 25–26.
- Empty lists, 12, 17.
- Engelhardt, Matthias Rüdiger, 6, 35.
- English words, 8–9, 28, 33.
- Ernst, George Werner, 45.
- Error bars, 22.
- Estimates of run time, 18–21.
- Estimating the number of solutions, 21–23.
  
- Factorization of problems, 24–25, 31, 33.
- Fallback points, 16.
- Finkel, Raphael Ari, 26.
- Five-letter words, 8–9, 28.
- Fixed point of recursive formula, 47.
- Floyd, Robert W, 15.
- Fool’s Disk, 31.
- Four-letter codewords, 9–18, 28–30.
- FPGA devices: Field-programmable gate arrays, 6.
- Frames, 11.
- Franel, Jérôme, 36.
  
- Gardner, Erle Stanley, 2.
- Gaschnig, John Gary, 26.
- Gattegno, Bernard, 38.
- Gauß (= Gauss), Johann Friderich Carl (= Carl Friedrich), 25, 35.
- Generating functions, 29, 42, 43.
- Gigamem ( $G\mu$ ): One billion memory accesses, 5.
- Global variables, 27.
- Goldsmith, Oliver, 50.
- Goldstein, Michael Milan, 45.
- Golomb, Solomon Wolf, 9, 26, 47, 51.
- Gordon, Basil, 9, 51.
- Graders, 46.
- Grid graphs, 27, 33.  
oriented, 34.
- Gridgeman, Norman Theodore, 44.
- Griffith, John Stanley, 9.
  
- Hales, Alfred Washington, 51.
- Hall, Marshall, Jr., 26.
- Hamilton, William Rowan, paths, 23.
- Hammersley, John Michael, 26.
- Handscorn, David Christopher, 26.
- Height of binary trees, 31.
- Hexagons, 27.
- Historical notes, 2, 5–6, 25–26.
- Hoffmann, Louis (pen name of Angelo John Lewis), 24.
- Homomorphic images, 24.
- Honeycombs, 27.
- Hurwitz, Adolf, 36.
  
- IBM 704 computer, 48.
- IBM 1620 computer, 6.
- IBM System 360-75 computer, 6.
- Impagliazzo, Russell Graham, 26.
- Importance sampling, 26.
- Independent subproblems, 24.
- Indeterminate statements, 47.
- Inner loops, 35.
- Insertion operation, 12.
- Instant Insanity, 24–25, 31.
- Integer partitions, 27, 31.
- Internet, ii, iii.
- Invariant relations, 45.
- Inverse lists, 12–15, 17.
- Inverse permutations, 12–13.
- Iteration versus recursion, 27, 35.
  
- Jaenisch, Carl Friedrich Andreevitch de (Янишъ, Карлъ Андреевичъ), 35.
- Jewett, Robert Israel, 51.
- Jiggs, B. H. (pen name of Baumert, Hales, Jewett, Imaginary, Golomb, Gordon, and Selfridge), 17.
  
- Kennedy, Michael David, 6.
- Kilomem ( $K\mu$ ): One thousand memory accesses, 44.
- King, Benjamin Franklin, Jr., 2.



- King paths, 22–23, 26, 30–31.  
 Kingsley, Hannah Elizabeth Seelman, 48.  
 Kleber, Michael Steven, 44.  
 Knight moves, 23.  
 Knuth, Donald Ervin (高德纳), i, iv, 18, 26, 27, 38, 41, 46, 48.  
 Kullback, Solomon, 43.
- Langford, Charles Dudley, pairs, 6–8, 27–28.  
 Latin squares, 24.  
 Laxdal, Albert Lee, 17.  
 Le Nombre Treize, *see* Royal Aquarium Thirteen Puzzle.  
 Lehmer, Derrick Henry, 26.  
 Leibler, Richard Arthur, 43.  
 Lennon, John Winston Ono, 2.  
 Lewis, Angelo John, 51.  
 Lexicographic order, 2, 7, 25, 29, 33.  
 Lifting, 24–25.  
 Linked lists, 6–7, 40.  
 Load balancing, 26.  
 Lookahead, 10, 16, 26, 30.  
 Loose Langford pairs, 28.  
 Lucas, François Edouard Anatole, 25, 36.
- Magen, Avner (אבר מן), 26.  
 Mahler, Kurt, 44.  
 Manber, Udi (עדי מנבר) [not אודי!], 26.  
 Martingales, 30.  
 Masks, 35.  
 Mason, Perry, 2.  
 Megamem ( $M\mu$ ): One million memory accesses, 18.  
 MEM, an array of “cells,” 11–18, 29.  
 Memory constraints, historic, 40.  
 Mem ( $\mu$ ): One 64-bit memory access, 4.  
 Minimum remaining values heuristic, 26.  
 MMIX computer, ii.  
 Monte Carlo estimates, 18–23, 26, 30.  
 Moves, 11.  
 MPR: Mathematical Preliminaries Redux, v, 1.  
 Mystery text, 33.
- $n$ -letter words of English, 8.  
 $n$ -omino placement, 49.  
 $n$  queens problem, 3–6, 18–20, 25–27.  
 $n$ -tuples, 27.  
 Nauck, Franz, 25.  
 Nested parentheses, 27.  
 NEXT( $a$ ) (the next arc with the same initial vertex as  $a$ ), 33, 48.  
 Niho, Yoji Goff (仁保洋二), 41.  
 Nonisomorphic solutions, 30.
- Onnen, Hendrick, Sr., 6.  
 Optimization, 26.  
 Orgel, Leslie Eleazer, 9.  
 Oriented grids, 34.  
 Oriented trees, 34.  
 Orthogonal lists, 28.  
 OSPD4: *Official SCRABBLE® Players Dictionary*, 8, 48.  
 Overflow of memory, 12, 16.
- $P_0()$ , 2.  
 Paradox, 32.  
 Parallel programming, 46.  
 Parent in a tree, 34.  
 Parentheses, 27.  
 Parity argument, 37, 45.  
 Parker, Ernest Tilden, 24.  
 Partitions, 27, 31.  
 Patents, 24.  
 Paths, simple, 22, 26, 30.  
 Pauls, Emil, 36.  
 Pencil-and-paper method, 18–20.  
 Pentominoes, 33.  
 Periodic sequences, 28.  
 Periodic words, 10, 13.  
 Permutations, 6, 27.  
 Phi ( $\phi$ ), as source of “random” data, 19.  
 Pi ( $\pi$ ), as source of “random” data, 19–20, 22, 29.  
 Pitassi, Toniann, 26.  
 Poison list, 16–17, 29–30, 42.  
 Pólya, György (= George), 36.  
 Polyominoes, 33.  
 Preußner, Thomas Bernd, 6, 35.  
 Prime strings, 10.  
 Priority branching trees, 26.  
 Probabilities, 1.  
 Profile of a tree, 3, 9, 28, 31, 42.  
 Propagation algorithm, 46.  
 Properties: Logical propositions (relations), 2, 27.
- q.s., 30.  
 Quantified Boolean formulas, 47.  
 Queen bees, 27.  
 Queens, *see*  $n$  queens problem.  
 Questionnaires, 31.  
 Queues, 40, 43.  
 Quick, Jonathan Horatio, 27.
- Radix  $m$  representation, 13.  
 Random bits, 30.  
 Random sampling, 18.  
 Random variables, 19.  
 Random walks, 18–23, 30–31.  
 Reachable subsets, 34.  
 Recurrence relations, 42, 44.  
   in a Boolean equation, 47.  
 Recursion versus iteration, 27, 35.  
 Recursive algorithms, 27, 42.  
 Reflection symmetry, 14, 27, 35, *see also* Dual solutions.  
 Registers, 5, 37–38.

- Reingold, Edward Martin (ריינגולד, יצחק משה בן חיים), 26.
- Rejection method, 19, 41.
- Restricted growth strings, 35.
- Reversible memory technique, 16, 30.
- Rivin, Igor (Ривин, Игорь Евгеньевич), 36.
- Root node, 19.
- Rosenbluth, Arianna Wright, 26.
- Rosenbluth, Marshall Nicholas, 26.
- Rotating Century Puzzle, *see* Fool's Disk.
- Rotation by  $90^\circ$ , 27.
- Royal Aquarium Thirteen Puzzle, 31.
- Running time estimates, 18–21.
- Safe Combination Puzzle, *see* Fool's Disk.
- Sample variance, 22.
- SAT solvers, 41.
- Saturating ternary addition, 38.
- Scholtz, Robert Arno, 39.
- Schossow, Frederick Alvin, 24.
- Schumacher, Heinrich Christian, 25, 35.
- Search rearrangement, *see* Dynamic ordering.
- Search trees, 3, 4, 7, 9–11, 18–20, 24, 26, 36, 41, 42, 44–46.
  - estimating the size, 20, 41.
- Self-avoiding walks, 22, 26, 30.
- Self-reference, 32, 53.
- Self-synchronizing block codes, 9.
- Selfridge, John Lewis, 51.
- Semi-queens, 36.
- Sequential allocation, 29.
- Sequential lists, 11–15.
- Set partitions, 27.
- SGB, *see* Stanford GraphBase.
- Shortest distances, dynamic, 30.
- Signature of a trie node, 37.
- Simple paths, 22, 26, 30.
- Slack, 18, 42.
- Slocum, Gerald Kenneth (= Jerry), 45.
- Smith, Cedric Austen Bardell, 51.
- Spanning trees, 33–34.
- Speedy Schizophrenia, 31.
- Spiegelthal, Edwin Simeon, 48.
- Sprague, Thomas Bond, 5, 6, 27.
- Stacks, 16, 40.
- Stamping, iv, 16–19, 29.
- Standard deviation, 20, 22, 30.
- Stanford GraphBase, ii, 8.
  - format for digraphs and graphs, 33, 48.
- Statistics, 22.
- Stirling, James, cycle numbers, 30.
- Students, 46.
- Substrings, 29.
- Subtrees, 20, 26.
- Superdips, 39.
- SWAC computer, 6.
- Symmetries, 30, 35.
  - breaking, 8, 14.
- November 12, 2016
- Tagged vertices, 48–49.
- Tail inequality, 42.
- Tantalizer, *see* Instant Insanity.
- Teramem ( $T\mu$ ): One trillion memory accesses, 9.
- TIP( $a$ ) (final vertex of arc  $a$ ), 33, 48.
- Torus, 27.
- Tot tibi ..., 25.
- Traub, Joseph Frederick, 41.
- Trémaux, Charles Pierre, 25.
- Tries, 8–9, 28, 37.
  - compressed, 38.
- Tuples, 27.
- Twenty Questions, 32.
- Two-letter block codes, 28.
- UCLA: The University of California at Los Angeles, 6.
- Undirected graphs versus directed graphs, 34.
- UNDO stack, 16.
- Undoing, 4–5, 7, 15–16, 29.
- Uniformly random numbers, 41.
- Unique solutions, 44–45.
- Unit clauses, 41.
- University of California, 6.
- University of Dresden, 6.
- University of Illinois, 6.
- University of Tennessee, 6.
- Unordered sequential lists, 11.
- Unordered sets, 40.
- Uppercase letters, 31.
- Utility fields in SGB format, 47.
- $v$ -reachable subsets, 34.
- Valid gradings, 46.
- Vardi, Ilan, 36.
- Variance of a random variable, 22, 30.
- Venn, John, diagram, 43.
- Visiting an object, 2, 4, 5, 7, 18.
- Walker, Robert John, 2, 5, 6, 26.
- Wanless, Ian Murray, 36.
- Washington Monument Puzzle, *see* Fool's Disk.
- Welch, Lloyd Richard, 9.
- Wells, Mark Brimhall, 26.
- White squares, 27.
- Woods, Donald Roy, 32, 47.
- Word rectangles, 8–9, 28.
- WORDS( $n$ ), the  $n$  most common five-letter words of English, 8, 33.
- Worst-case bounds, 29.
- Wraparound, 27.
- Yao, Andrew Chi-Chih (姚期智), 41.
- ZDD: A zero-suppressed decision diagram, 23, 30.
- Zimmermann, Paul Vincent Marie, 36.