COMP9020 17s1 • Problem Set 5 • 31 March 2017

Solutions

Exercise 1.

(a) $\mathcal{R}$ is not an equivalence relation since it is neither reflexive nor transitive: $(1,2) \in \mathcal{R}$ and $(2,1) \in \mathcal{R}$, but $(1,1) \notin \mathcal{R}$.

(b) Yes, $\mathcal{R}$ is an equivalence relation: Notice that $a + 2b$ is divisible by 3 whenever $a - b$ is divisible by 3, hence $(a,b) \in \mathcal{R}$ iff $a \bmod 3 = b \bmod 3$, which is reflexive, symmetric and transitive.

Exercise 2. If $m = n \pmod p$ then $m = k \cdot p + r$ and $n = l \cdot p + r$ for some $k, l \in \mathbb{Z}$ and $r \in \{0, \ldots, p-1\}$. Then,

$$\begin{aligned} m^2 &= k^2p^2 + 2kpr + r^2 \\ n^2 &= l^2p^2 + 2lpr + r^2 \end{aligned}$$

Hence, $m^2 \bmod p = r^2 \bmod p = n^2 \bmod p$, so $m^2 = n^2 \pmod p$.

Exercise 3.

- $\mathcal{R}$ is reflexive: for every a, b such that $a = b$, by definition $(a,b) \in \mathcal{R}$.
- $\mathcal{R}$ is antisymmetric: for any $a \neq b$, if $(a,b) \in \mathcal{R}$ then it must be that $a \leq b - 0.5$, therefore $b \geq a + 0.5 > a - 0.5$ so $(b,a) \notin \mathcal{R}$.
- $\mathcal{R}$ is transitive: for any a, b, c, this is trivial if $a = b$ or $b = c$, otherwise if $a \neq b$ and $b \neq c$ then $(a,b) \in \mathcal{R} \wedge (b,c) \in \mathcal{R} \Rightarrow a \leq b - 0.5 \wedge b \leq c - 0.5 \Rightarrow a \leq c - 1 \Rightarrow a \leq c - 0.5 \Rightarrow (a,c) \in \mathcal{R}$.

Therefore $\mathcal{R}$ is a partial order. It is not a total order since any pair a, b where $a < b < a + 0.5$ (for instance, 1.1 and 1.2) are not related in either direction.

Exercise 4.

(a) Check that $\mathcal{R}$ is reflexive, antisymmetric and transitive in the same way as for the relation in Exercise 2.

(b) It is not a lattice; for example, the pair (1,2) do not have a greatest lower bound (or, in fact, any lower bound — both 1 and 2 are minimal elements).

Exercise 5. For the product order the maximum length is 19; for example,

$$(1,1) \sqsubseteq_P (1,2) \sqsubseteq_P \ldots \sqsubseteq_P (1,9) \sqsubseteq_P (1,10) \sqsubseteq_P (2,10) \sqsubseteq_P \ldots \sqsubseteq_P (9,10) \sqsubseteq_P (10,10)$$

For the lexicographic order, since it is a total order, the longest chain contains all of the 100 elements in the set:

$$(1,1) \sqsubseteq_L (1,2) \sqsubseteq_L \ldots \sqsubseteq_L (1,10) \sqsubseteq_L (2,1) \sqsubseteq_L \ldots \sqsubseteq_L (9,10) \sqsubseteq_L (10,1) \sqsubseteq_L \ldots \sqsubseteq_L (10,10)$$