COMP9020 17s1 • Problem Set 10 • 26 May 2017

Solutions

Exercise 1.

(a) Select $\omega_1, \omega_2 \in \Sigma$ for words $\omega_1\omega_2 ace$, $\omega_1 ace\omega_2$ and $ace\omega_1\omega_2$, hence $5^2 \cdot 3 = 75$ words.

(b) The letters can be arranged in 5! different ways, half of which satisfy the additional requirement, hence $\frac{5!}{2} = 60$ words.

(c) Any word of length n in alphabetical order *that ends in a letter* ω can be obtained by

- a word of length $n-1$ in alphabetical order *that ends in*
  - ω *or*
  - any letter that comes before ω in the lexicographic order.

The following table follows the rule $n_{row,col} = n_{1,col-1} + n_{2,col-1} + \ldots + n_{row,col-1}$

word length	1	2	3	4	5
words ending in a	1	1	1	1	1
words ending in c	1	2	3	4	5
words ending in e	1	3	6	10	15
words ending in n	1	4	10	20	35
words ending in s	1	5	15	35	70
sum					126

Exercise 2. Base case: $P(A_1) = P(A_1)$

Inductive step:

$$\begin{aligned} & P((A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}) \\ = \ & P(A_{n+1} \mid A_1 \cap \ldots \cap A_n) \cdot P(A_1 \cap \ldots \cap A_n) && \text{by def. of cond. probab.} \\ = \ & P(A_{n+1} \mid A_1 \cap \ldots \cap A_n) \cdot P(A_1) \cdot P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \ldots \cap A_{n-1}) && \text{by ind. hyp.} \end{aligned}$$

Exercise 3. Let Urn_i denote the event that the i-th urn was selected and TR the event of two red marbles being drawn.

$$\begin{array}{lll} P(TR \mid Urn_1) = 0\ ; & P(TR \mid Urn_2) = \frac{3}{7} \cdot \frac{2}{6}\ ; & P(TR \mid Urn_3) = \frac{2}{4} \cdot \frac{1}{3} \\ P(TR \cap Urn_1) = 0\ ; & P(TR \cap Urn_2) = (\frac{3}{7} \cdot \frac{2}{6}) \cdot \frac{1}{3}\ ; & P(TR \cap Urn_3) = (\frac{2}{4} \cdot \frac{1}{3}) \cdot \frac{1}{3} \end{array}$$

$$P(TR) = P(TR \cap Urn_1) + P(TR \cap Urn_2) + P(TR \cap Urn_3) = \frac{13}{126}$$

$$P(Urn_2 \mid TR) = \frac{P(TR \cap Urn_2)}{P(TR)} = \frac{126}{21 \cdot 13} \approx 0.4615$$

Exercise 4.

(a) Let $p = P(HTH$ comes first$)$. Consider the (recursive) tree of possible outcomes:

HHH	loss
$HHTH$	win
$HHTT$	p
HTH	win
HTT	p
T	p

Therefore, $p = \frac{1}{16} + \frac{1}{16}p + \frac{1}{8} + \frac{1}{8}p + \frac{1}{2}p$; hence $p = \frac{3}{5}$

(b) This is obviously symmetric, so the probability must be $\frac{1}{2}$.