COMP9020 17s1 • Problem Set 1 • 3 March 2017

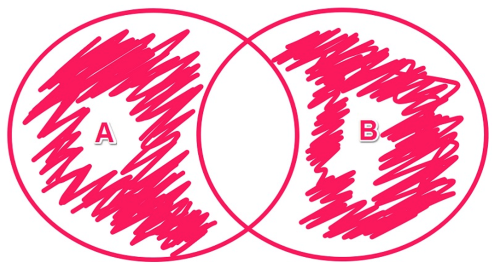
Solutions

Exercise 1. Using the formula $\lfloor \frac{m}{k} \rfloor - \lfloor \frac{n-1}{k} \rfloor$:

- $\lfloor \frac{1000}{3} \rfloor - \lfloor \frac{99}{3} \rfloor = 300$ numbers divisible by 3 $(102, 105, \ldots, 999)$;
- $\lfloor \frac{1000}{5} \rfloor - \lfloor \frac{99}{5} \rfloor = 181$ numbers divisible by 5 $(100, 105, \ldots, 1000)$;
- $\lfloor \frac{1000}{15} \rfloor - \lfloor \frac{99}{15} \rfloor = 60$ numbers divisible by 15 $(105, 120, \ldots, 990)$.

Exercise 2.

(a) Drawing both $(A \setminus B) \cup (B \setminus A)$ and $(A \cup B) \setminus (A \cap B)$ result in the same diagram:



(b) We show in both directions that if an element belongs to $(A \setminus B) \cup (B \setminus A)$ then it also belongs to $(A \cup B) \setminus (A \cap B)$ and vice versa:

- Suppose that an element $x \in (A \setminus B) \cup (B \setminus A)$. Therefore, either $x \in A \setminus B$ or $x \in B \setminus A$. In either case, we conclude that $x \in A \cup B$ and (by the definition of set difference) $x \notin A \cap B$. Therefore, $x \in (A \cup B) \setminus (A \cap B)$.
- Suppose than $x \in (A \cup B) \setminus (A \cap B)$. This means that $x \in A \cup B$ (and, therefore, either $x \in A$ or $x \in B$), but $x \notin A \cap B$. If $x \in A$ and $x \notin A \cap B$, then $x \in A \setminus B$; alternatively, if $x \in B$ and $x \notin A \cap B$, then $x \in B \setminus A$. In either case, we conclude that $x \in (A \setminus B) \cup (B \setminus A)$.

Exercise 3.

(a) $\Sigma^2 = \{aa, ab, ac, ba, \ldots, cc\}$, hence $|\Sigma^2| = 3 \cdot 3 = 9$.

(b) $\Sigma^2 \setminus \Phi^* = \{ab, ba, bb, bc, cb\}$, that is, all words in Σ^2 with the letter b.

(c) No; for example, $ab \in \Sigma^*$ and $ab \notin \Phi^*$, hence $ab \in \Sigma^* \setminus \Phi^*$; but $\Sigma \setminus \Phi = \{b\}$, hence $ab \notin (\Sigma \setminus \Phi)^*$.

Exercise 4. The original Euclid's algorithm for the gcd of two positive numbers works because if d, a, b are integers and d is a common divisor of a and b, then d is a common divisor of $a - b$ and b (or of $b - a$ and a) as well.

For the second ("potentially faster") version, if $a > b$ and $r = a \bmod b$, then $r = a - m \cdot b$ for some integer m. Therefore, again, if d is a common divisor of a and b, then d is a divisor of r as well. Similarly for the case that $b > a$ and $r = b \bmod a$.

In either version, every recursive call operates on a smaller pair of numbers than the previous iteration while retaining all their common divisors. Therefore, eventually, the process must terminate once the greatest common divisor is reached, except that it can happen faster in the second version (the reduction of the pair of numbers in a single iteration can indeed be more dramatic).