COMP9020 17s1 • Problem Set 3 • 17 March 2017

Solutions

Exercise 1. $\neg q \Rightarrow ((r \Rightarrow p) \wedge (p \Rightarrow (q \vee \neg r))) \;\equiv\; q \vee ((\neg r \vee p) \wedge (\neg p \vee q \vee \neg r))$

$$\begin{array}{rll} & q + ((\overline{r}+p)\cdot(\overline{p}+q+\overline{r})) & \\ = & q + \overline{r}\,\overline{p} + \overline{r}\,q + \overline{r}\,\overline{r} + p\overline{p} + pq + p\overline{r} & \text{(by } \textit{distribution}\text{)} \\ = & q + \overline{r}\,\overline{p} + \overline{r}\,q + \overline{r} + pq + p\overline{r} & \text{(since } \overline{r}\cdot\overline{r} = \overline{r} \text{ and omitting } p\cdot\overline{p}=0\text{)} \\ = & q + \overline{r} & \text{(by } \textit{absorption}\text{)} \end{array}$$

Exercise 2. It suffices to show that the three basic operations can be expressed via the **nand** operation:

- $\overline{A} = \overline{A\cdot A} = A$ **nand** A
- $A\cdot B = \overline{\overline{A\cdot B}} = \overline{\overline{A\cdot B}\cdot\overline{A\cdot B}} = (A$ **nand** $B)$ **nand** $(A$ **nand** $B)$
- $A + B = \overline{\overline{A}\cdot\overline{B}} = \overline{\overline{A\cdot A}\cdot\overline{B\cdot B}} = (A$ **nand** $A)$ **nand** $(B$ **nand** $B)$

Therefore, any Boolean expression can be expressed via just the **nand** operation.

Exercise 3.

(a) $\textsc{bool}(1)$ contains $2^{2^1} = 4$ elements:

$$\begin{array}{lcccl} f_1: & p & \mapsto & 0 & \text{, i.e. } f_1(0)=0,\ f_1(1)=0 \\ f_2: & p & \mapsto & p & \text{, i.e. } f_2(0)=0,\ f_2(1)=1 \\ f_3: & p & \mapsto & \overline{p} & \text{, i.e. } f_3(0)=1,\ f_3(1)=0 \\ f_4: & p & \mapsto & 1 & \text{, i.e. } f_4(0)=1,\ f_4(1)=1 \end{array}$$

(b) If $f \in \textsc{bool}(n)$ then f is a function over n boolean variables, that is, $\mathsf{Dom}(f) = \times_{i=1}^{n}\{0,1\}$, hence $|\mathsf{Dom}(f)| = 2^n$. The result of a boolean function is either 0 or 1, that is, $\mathsf{Codom}(f) = \{0,1\}$, hence $|\mathsf{Codom}(f)| = 2$. For each element in the domain, a function can chose any of the elements from the codomain as the function value, hence there are $|\mathsf{Codom}(f)|^{|\mathsf{Dom}(f)|} = 2^{2^n}$ different functions.

Exercise 4. We use B, S, L, J to denote that the corresponding child is a truar, and b, s, l, j to denote that the corresponding child dropped the vase. The information from the parents translates to:

$(B\cdot\overline{S}\cdot\overline{L}\cdot\overline{J}) + (\overline{B}\cdot S\cdot\overline{L}\cdot\overline{J}) + (\overline{B}\cdot\overline{S}\cdot L\cdot\overline{J}) + (\overline{B}\cdot\overline{S}\cdot\overline{L}\cdot J)$

Also, we know that just one child broke the vase, which means:

$(b\cdot\overline{s}\cdot\overline{l}\cdot\overline{j}) + (\overline{b}\cdot s\cdot\overline{l}\cdot\overline{j}) + (\overline{b}\cdot\overline{s}\cdot l\cdot\overline{j}) + (\overline{b}\cdot\overline{s}\cdot\overline{l}\cdot j)$

The children's claims translate to:

$B \Leftrightarrow s$

$S \Leftrightarrow j$

$L \Leftrightarrow \overline{l}$

$J \Leftrightarrow \overline{S}$ (or $J \Leftrightarrow \overline{S}\cdot\overline{j}$)

Now, use a truth table to find the only possible truth assignment. Alternatively, reason as follows: from the last claim, we conclude that either Steve or John must be a truar and the other one must be a liar. This means that the other two children must both be liars. Since Laura is a liar, her claim is false, so Laura broke the vase. (We can now also conclude that the truar is John.)