COMP9020 17s1 • Problem Set 5 • 31 March 2017

Equivalence Relations, Orderings

Exercise 1. Prove or disprove that $\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$ is an equivalence relation if

(a) $\mathcal{R}$ is defined by $a\mathcal{R}b$ iff $a + b$ is divisible by 3;

(b) $\mathcal{R}$ is defined by $a\mathcal{R}b$ iff $a + 2b$ is divisible by 3.

Exercise 2. Prove that if $m, n \in \mathbb{Z}$ and $m = n \pmod{p}$ then $m^2 = n^2 \pmod{p}$.

Exercise 3. Consider the relation $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}$ defined by $(a, b) \in \mathcal{R}$ iff either $a \leq b - 0.5$ or $a = b$. Show that $\mathcal{R}$ is a partial order, but not a total order.

Exercise 4. Let binary relation $\mathcal{R}$ on $\{1, \ldots, 20\}$ be defined by $a\mathcal{R}b$ iff either $a = b$ or $a - b > 10$.

(a) Show that $\mathcal{R}$ is a partial order.

(b) Is $\langle \{1, \ldots, 20\}, \mathcal{R} \rangle$ a lattice?

$*$**Exercise 5.** Using the set $\{1, \ldots, 10\}$ with the natural total order, define $A = \{1, \ldots, 10\} \times \{1 \ldots, 10\}$ and consider the two orderings over A:

- product $\sqsubseteq_P$
- lexicographic $\sqsubseteq_L$

Find the maximal length of a chain $a_1 \sqsubseteq a_2 \sqsubseteq \ldots \sqsubseteq a_n$ (such that $a_i \neq a_{i+1}$) for each of these orderings.