COMP9020 17s1 • Problem Set 6 • 28 April 2017

Solutions

Exercise 1.

(a) True. In general, any bipartite graph can only have cycles of an even length. If V_1, V_2 denote the two groups of nodes in the bipartite graph, then every path starting from, say, a node $v \in V_1$ must alternate its vertices between V_1 and V_2. If it is a cycle (that is, eventually returns to node v), it must have an even number of such alternations, in other words, it contains an even number of edges.

(b) False. For any tree T, $\chi(T) = 2$.

(c) True (in fact, true for all $n \geq 2$), because $\chi(C_n) = 2$ for all even n and $\chi(C_n) = 3$ for all odd n.

(d) True. Remove any two edges which have no common vertex. In the resulting graph there will be no five fully interconnected vertices; in other words it will not contain K_5, and will have a clique number of 4.

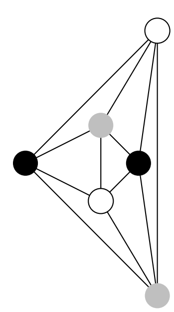
Exercise 2. Let the two connected components have n and m vertices respectively, with $n+m = 20$. The maximum number of edges is achieved by creating two complete graphs K_n and K_m with $\frac{n(n-1)}{2} + \frac{m(m-1)}{2}$ edges overall. This number is maximal for $n = 19$, $m = 1$ which gives 171 edges. The minimum number of edges is obtained when both components are trees with $n - 1$ and $m - 1$ edges respectively, for a total of $n + m - 2 = 18$ edges.

Exercise 3. $K_{i,1}$ and $K_{i,2}$ for all $i \geq 1$. They can be easily presented without self-intersection in a planar drawing. For $i, j \geq 3$ the corresponding bipartite graph would contain $K_{3,3}$, which is known not to be planar.

Exercise 4.

(a) For every 3-partite graph three colours suffice: use a different colour for each of the groups of vertices. Three colours are also necessary: any three vertices selected from three different groups will form a clique.

(b) $K_{4,1,1}$ can be easily drawn without intersections. $K_{3,2,1}$ contains $K_{3,3}$ hence is not planar. A planar drawing of $K_{2,2,2}$ is:



Exercise 5.

(a) 2 edges. To achieve $\chi = 3$, one needs (at least) to avoid having any 4-cliques. Removing one edge leaves a 4-clique (actually two such cliques — including any one but not both of that edge's endpoints, plus the remaining three vertices). Removing two edges suffices — remove any pair of edges which do not share a common vertex; the remaining graph can then be coloured with 3 colours.

(b) 4 edges. A chromatic number of 2 means that the graph is bipartite, with two groups of nodes where each group can be painted with one colour. To minimise the number of *removed* edges, we want to have as many edges as possible in the remaining bipartite graph. We therefore look at *complete bipartite* graphs with a total of 5 vertices. As $K_{1,4}$ has four edges and $K_{2,3}$ has six edges, the latter is the better choice. To reach it we need to remove 4 of the edges in K_5.

(c) 10 edges. A chromatic number of 1 means a fully disconnected graph, with no edges at all. Therefore all 10 edges of the original graph must be removed.