COMP9020 17s1 • Problem Set 7 • 5 May 2017

Solutions

Exercise 1.

(a) Base case: $1 \cdot 1! = 1 = (1 + 1!) - 1$

Inductive step:

$$\begin{array}{rcll} 1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! + (n+1) \cdot (n+1)! & = & (n+1)! - 1 + (n+1) \cdot (n+1)! & \text{by ind. hyp.} \\ & = & (1 + (n+1)) \cdot (n+1)! - 1 & \\ & = & (n+2) \cdot (n+1)! - 1 & \\ & = & (n+2)! - 1 & \text{by def. of !} \end{array}$$

(b) Base case: $s_1 = 1 = \frac{1}{1} = \frac{\text{FIB}(1)}{\text{FIB}(2)}$

Inductive step:

$$s_{n+1} = \frac{1}{1 + s_n} = \frac{1}{1 + \frac{\text{FIB}(n)}{\text{FIB}(n+1)}} = \frac{1}{\frac{\text{FIB}(n+1)+\text{FIB}(n)}{\text{FIB}(n+1)}} = \frac{1}{\frac{\text{FIB}(n+2)}{\text{FIB}(n+1)}} = \frac{\text{FIB}(n+1)}{\text{FIB}(n+2)}$$

Exercise 2. For a tree T, let $\ell(T)$ and $r(T)$ denote, respectively, the number of leaves and the number of vertices with a right sibling.

Base case: A tree consisting of just a root has 1 leaf and no vertex with a right sibling:

$$T = \langle v; \rangle \;\Rightarrow\; \ell(T) = 1 = 0 + 1 = r(T) + 1$$

Inductive step: The leaves of a tree $T = \langle r; T_1, T_2, \ldots, T_k \rangle$ are all the leaves of all the subtrees T_i. Moreover, in T the roots of all the subtrees $T_1, \ldots, T_{k-1}$ (but not T_k) are vertices with a right sibling, in addition to all the other vertices in all the subtrees that have a right sibling. Hence,

$$T = \langle r; T_1, T_2, \ldots, T_k \rangle \;\Rightarrow\; \ell(T) = \sum_{i=1}^{k} \ell(T_i) \text{ and } r(T) = (k-1) + \sum_{i=1}^{k} r(T_i)$$

From the induction hypothesis, $\ell(T_i) = r(T_i) + 1$ for all $1 \le i \le k$, it follows that

$$\ell(T) = \sum_{i=1}^{k} \ell(T_i) = \sum_{i=1}^{k} (r(T_i) + 1) = k + \sum_{i=1}^{k} r(T_i) = r(T) + 1$$

Exercise 3. Base case: A graph with $v(G) = 1$ node is connected, has $e(G) = 0$ edges and hence satisfies $e(G) \ge v(G) - 1$.

Inductive step (proof by contradiction): Consider graph G with $v(G) \ge 2$ nodes such that $e(G) < v(G) - 1$. We will show that G is not connected. From the lecture we know that $\sum_{v \in V} deg(v) = 2e(G) < 2v(G) - 2$. It follows that there is at least one vertex $v_0 \in V$ with $deg(v_0) \le 1$. If $deg(v_0) = 0$ then G is not connected and we are done. If $deg(v_0) = 1$, consider the graph G' obtained by removing v_0 and its only connecting edge from G. It follows that $e(G') < v(G') - 1$ since $e(G') = e(G) - 1$, $v(G') = v(G) - 1$ and $e(G) < v(G) - 1$. By the induction hypothesis, G' is not connected. But then neither is G: if v and w are vertices with no path between them in G' then adding v_0 doesn't help.

Exercise 4. Base case: For $i = 1$, we have

$$T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j) = T(n-1) + \sum_{j=0}^{0} g(n-j) = T(n-1) + g(n)$$

which is true by the definition of $T(n)$.

Inductive step: Suppose $T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j)$ is true for $i = k < n-1$. We show that it holds for $i = k+1$. By the definition of $T(\cdot)$ we have $T(n-k) = T(n-k-1) + g(n-k)$, therefore

$$T(n) = T(n-k-1) + g(n-k) + \sum_{j=0}^{k-1} g(n-j) = T(n-k-1) + \sum_{j=0}^{k} g(n-j)$$

which is exactly what needs to be shown for $i = k+1$.