CX2101 Algorithm Design and Analysis

Tutorial 4
Dynamic Programming
(Weeks 10-11)

Question 1

Find the length of the longest common subsequence and a longest common subsequence of CAGAG and ACTGG by the dynamic programming algorithm in the lecture notes.

	1	2	3	4	5
x	C	A	G	A	G
y	A	C	T	G	G

c		A	C	T	G	G
	0	0	0	0	0	0
C	0	0	1	1	1	1
A	0	1	1	1	1	1
G	0	1	1	1	2	2
A	0	1	1	1	2	2
G	0	1	1	1	2	3

h		A	C	T	G	G
	—	—	—	—	—	—
C	|	|	\	—	—	—
A	|	\	|	|	|	|
G	|	|	|	|	\	\
A	|	\	|	|	|	|
G	|	|	|	|	\	\

```
for i = 1 to n
   for j = 1 to m
      if x[i] == y[j]  {
         c[i][j] = c[i-1][j-1] + 1;
         h[i][j] = '\';   }
      else if c[i-1][j] >= c[i][j-1] {
         c[i][j] = c[i-1][j];
         h[i][j] = '|';   }
      else {
         c[i][j] = c[i][j-1];
         h[i][j] = '—';   }
```

LCS(5,5) = 3

<table>
<tr><th>h</th><th></th><th>A</th><th>C</th><th>T</th><th>G</th><th>G</th></tr>
<tr><td></td><td>—</td><td>—</td><td>—</td><td>—</td><td>—</td><td>—</td></tr>
<tr><td>C</td><td>|</td><td>|</td><td>\</td><td>—</td><td>—</td><td>—</td></tr>
<tr><td>A</td><td>|</td><td>\</td><td>|</td><td>|</td><td>|</td><td>|</td></tr>
<tr><td>G</td><td>|</td><td>|</td><td>|</td><td>|</td><td>\</td><td>\</td></tr>
<tr><td>A</td><td>|</td><td>\</td><td>|</td><td>|</td><td>|</td><td>|</td></tr>
<tr><td>G</td><td>|</td><td>|</td><td>|</td><td>|</td><td>\</td><td>\</td></tr>
</table>

The subsequence: C G G

Question 2

The H-number H(n) is defined as follows:

H(0) =1, and for n>0:

$H(n) = H(n-1) + H(n-3) + H(n-5)+ \ldots.+H(0)$ when n is odd

$H(n) = H(n-2) + H(n-4) + H(n-6)+ \ldots.+H(0)$ when n is even.

a) Give a recursive algorithm to compute H(n) for an arbitrary n as suggested by the recurrence equation given for H(n). Draw the tree that represents the recursive calls made when H(8) is computed.

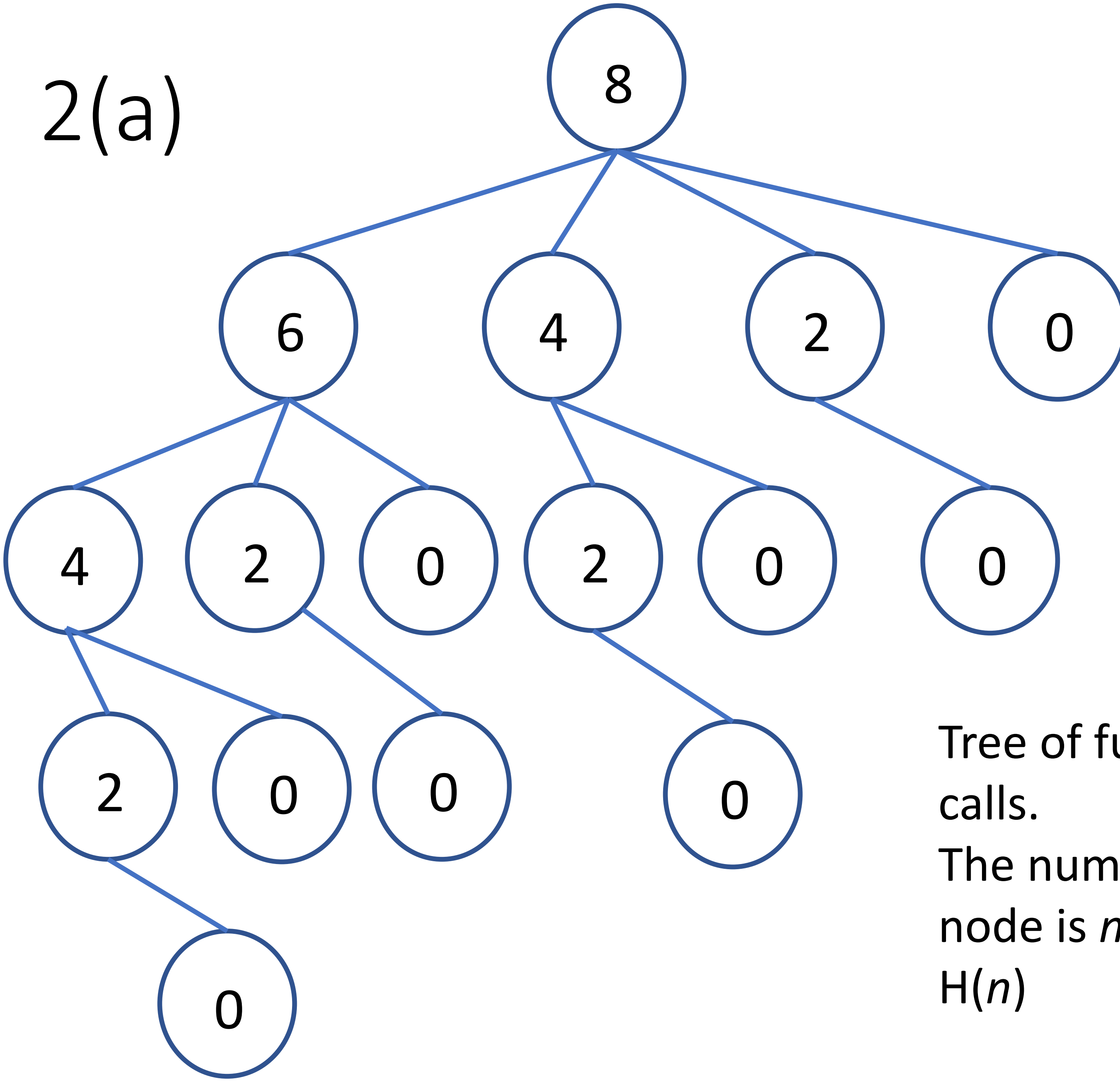
b) Draw the subproblem graph for H(8) and H(9).

c) Write an iterative algorithm using the dynamic programming approach (bottom-up). What are the time and space required?

Question 2(a)

```
int  hn( int n) {
{    if (n == 0)    return 1;
     else {
          S = 0;
          if (n mod 2)     j=n-1;      else  j=n-2;
          for (k = 0; k <= j; k = k+2)
                S += hn(k);
      }
      return S;
}
```

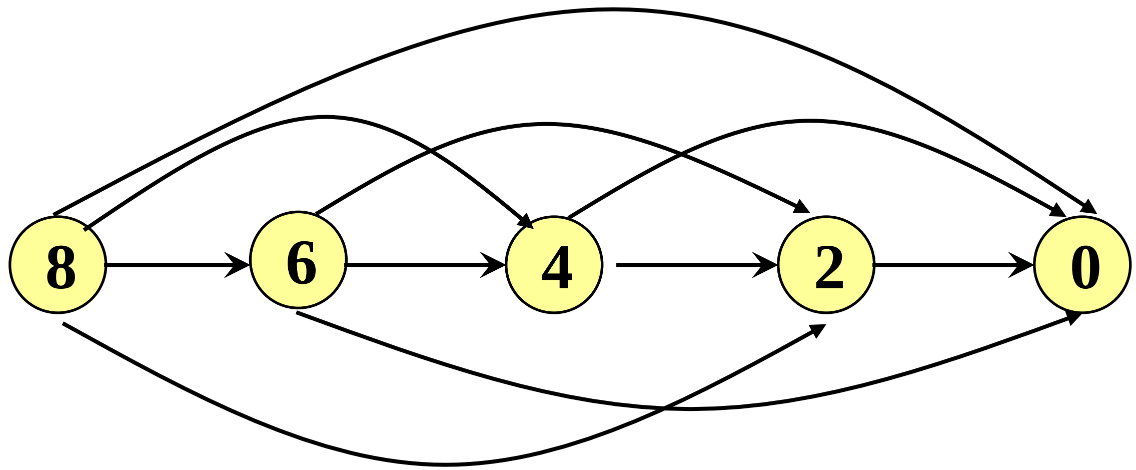
2(a)



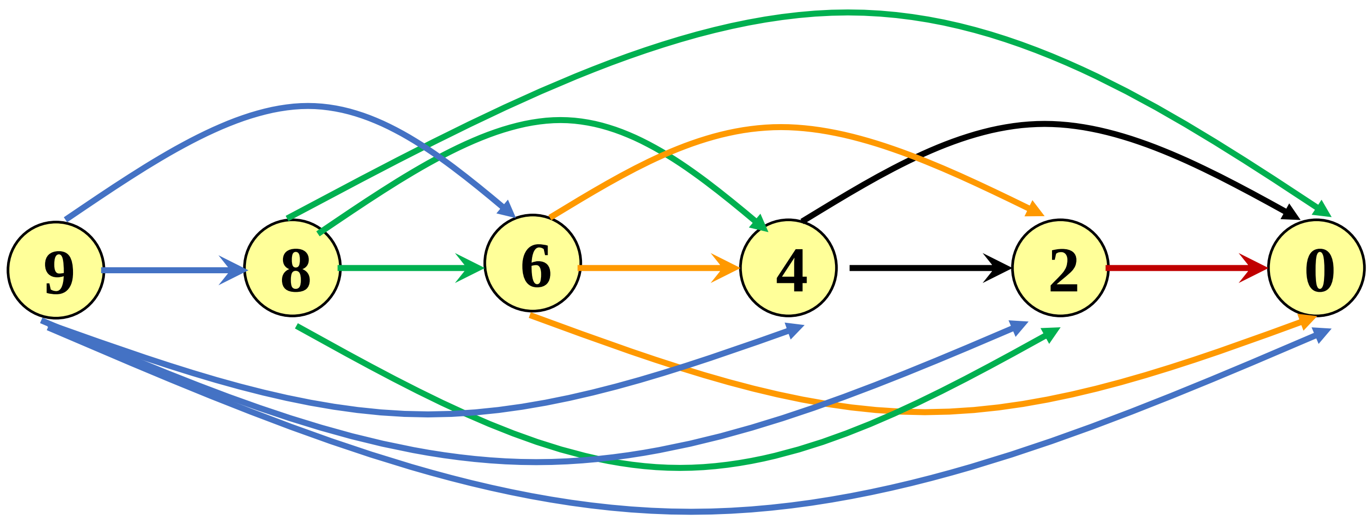
Tree of function calls.
The number in a node is *n* for a call H(*n*)

2(b)

The subproblem graph for H(8)



The subproblem graph for H(9)



2(c)

```
int  hn_DP(int n)
{     // Make use of an array S[0..n]
      S[0]=1;
      for (i = 1; i<=n; i++) {
            S[i] = 0;
            if (i mod 2)    j = i-1;    else  j=i-2;
            while (j>=0) { S[i] += S[j]; j-=2;};
       }
       return S[n];
}
```

Space Complexity: $O(n)$.   Time complexity: $O(n^2)$

Question 3

The binomial coefficients can be defined by the recurrence equation:

$C(n, k) = C(n - 1, k - 1) + C(n - 1, k)$ for $n > 0$ and $k > 0$

$C(n, 0) = 1$ for $n >= 0$

$C(0, k) = 0$ for $k > 0$

$C(n, k)$ is also called "*n* choose *k*". This is the number of ways to choose *k* distinct objects from a set of *n* objects.

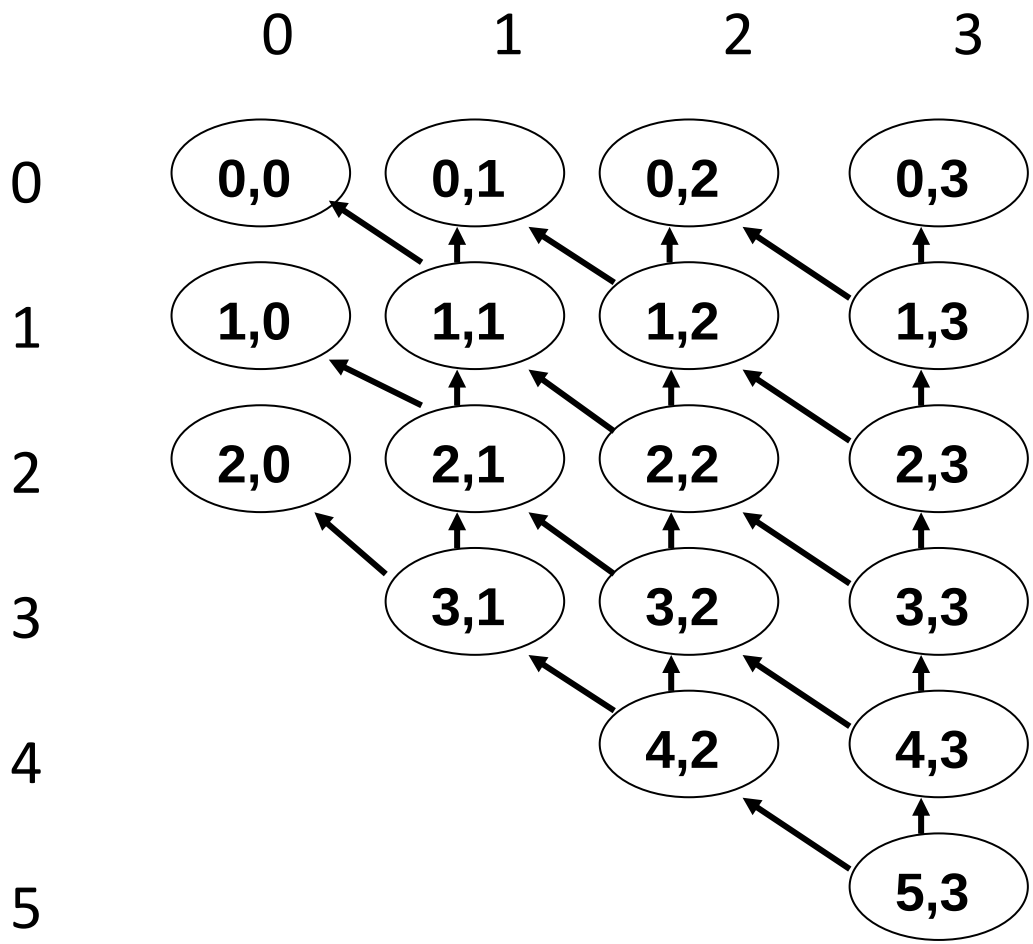
3(a)

Give a recursive algorithm as suggested by the recurrence equation given for C(*n*, *k*).

```
int C(int n, int k)
{
        if (k == 0)   return 1;
        if (n == 0) return 0;

        return C(n – 1, k – 1) + C(n – 1, k);
}
```

3(b) Draw the subproblem graph for C(5, 3).



3(c)

Write a recursive algorithm using the dynamic programming approach (top-down) stating the data structure used for the dictionary.

Use dictionary: int dic[n+1][k+1];
// initialised to –1 in all entries

```
int C(int n, int k, int [] [] dic)
{     int c1, c2;

      if (k == 0)   {
            dic[n][0] = 1;
            return 1;  }
      if (n == 0) {
            dic[0][k] = 0;
            return 0;   }

      if (dic[n – 1][k – 1] == -1)
            c1 = C(n – 1, k – 1);
      else c1 = dic[n – 1][k – 1] ;
      if (dic[n – 1][k] == -1)
            c2 = C(n – 1, k);
      else c2 = dic[n – 1][k] ;

      dic[n][k] = c1 + c2;
      return dic[n][k];
}
```

Time complexity: O(nk)
Space complexity: O(nk)

3(d)

Write an iterative algorithm using the dynamic programming approach (bottom-up).

```
int C(int n, int k, int [] [] dic)
{    int dic[n+1][k+1];

     For (i = 1;  i<= k;  i++)  dic[0][i] = 0;
     For (i = 0;  i<= n;  i++)  dic[i][0] = 1;
     For (i = 1;  i<= n;  i++)
          For (j = 1;  j<= k;  j++)
               dic[i][j] = dic[i-1][j-1] + dic[i-1][j];

    Return dic[n][k];
}
```

Time complexity: O(nk)
Space complexity: O(nk)

```
int C(int n, int k, int [] [] dic)  // more optimized
{     int dic[n+1][k+1];

      For (i = 1;  i<= k;  i++)  dic[0][i] = 0;
      For (i = 0;  i<= n-k;  i++)  dic[i][0] = 1;
      For (i = 1;  i<= n;  i++)
            For (j = max(i-(n-k), 1);  j<= k;  j++)
                  dic[i][j] = dic[i-1][j-1] + dic[i-1][j];

      Return dic[n][k];
}
```

Question 4

Suppose the dimensions of the matrices *A*, *B*, *C*, and *D* are 20x2, 2x15, 15x40, and 40x4, respectively, and we want to know how best to compute *A*x*B*x*C*x*D*. Show the arrays **cost**, **last**, and **multOrder** computed by Algorithms matrixOrder() in the lecture notes.

Array **d**

20	**2**	**15**	**40**	**4**
0	**1**	**2**	**3**	**4**

Cost

	0	1	2	3	4
0		0	600	2800	1680
1			0	1200	1520
2				0	2400
3					0
4					

Last

	0	1	2	3	4
0			1	1	1
1				2	3
2					3
3					
4					

Array **d**

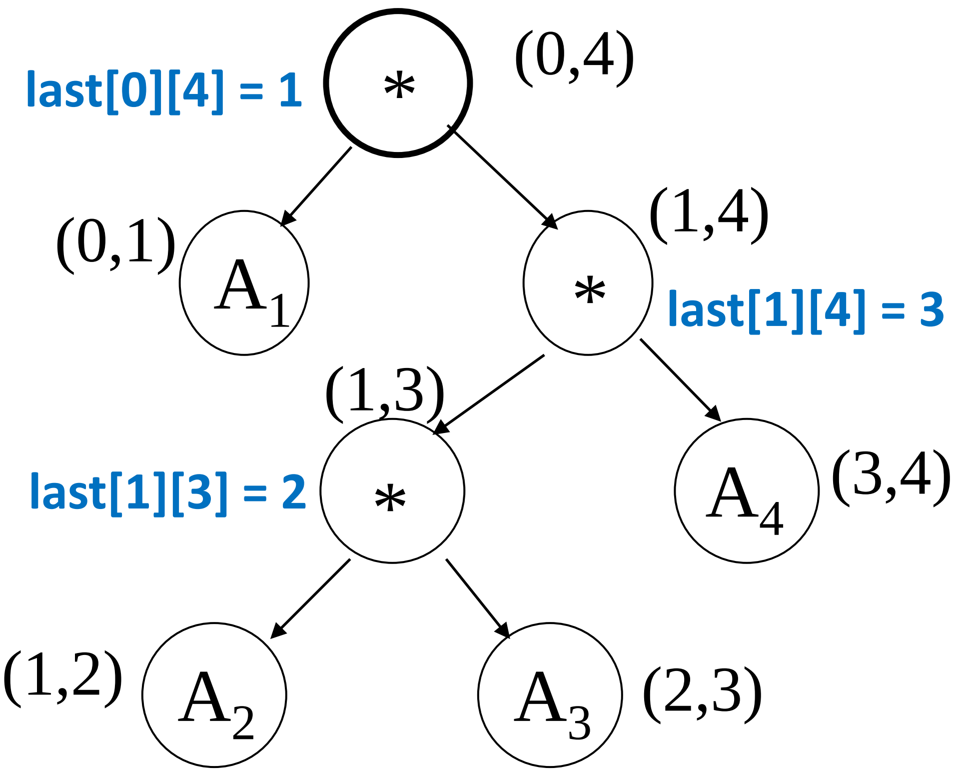
20	2	15	40	4

Cost[0][3] = min(Cost[0][1] +Cost[1][3] + d[0]*d[1]*d[3], Cost[0][2] +Cost[2][3] + d[0]*d[2]*d[3])
=min(1200+1600, 600+12000)
=2800

Cost[1][4] = min(Cost[1][2] +Cost[2][4] + d[1]*d[2]*d[4], Cost[1][3] +Cost[3][4] + d[1]*d[3]*d[4])

Cost[0][4] = min (Cost[0][1] +Cost[1][4] + d[0]*d[1]*d[4], Cost[0][2] +Cost[2][4] + d[0]*d[2]*d[4], Cost[0][3] +Cost[3][4] + d[0]*d[3]*d[4])

Computation of MultOrder is not examinable



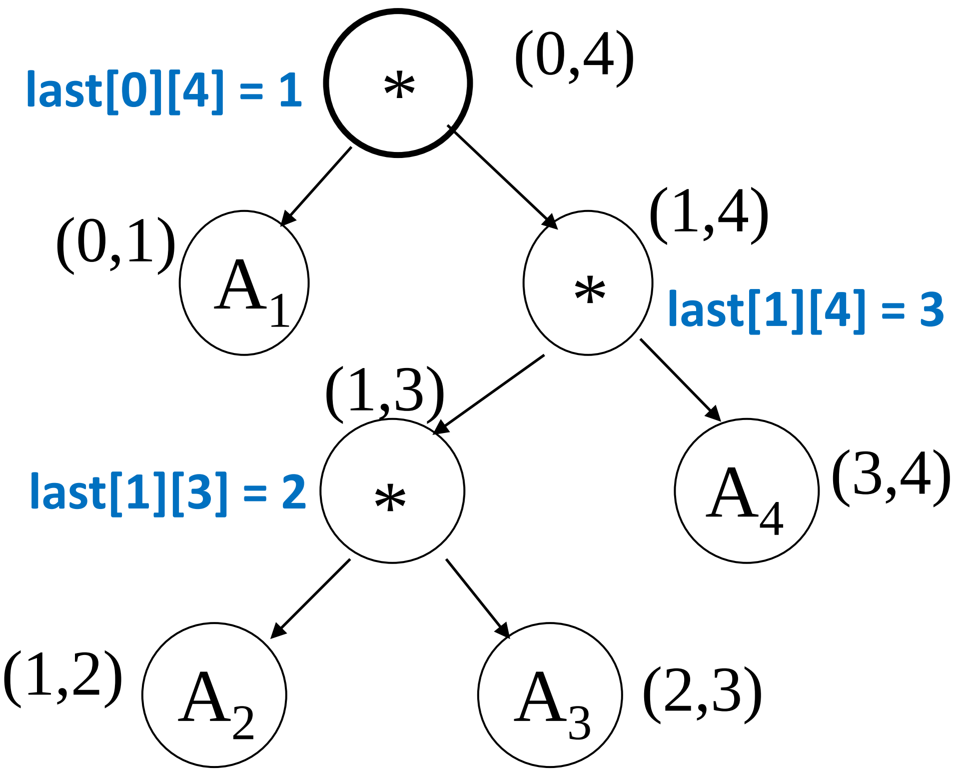
	0	1	2	3	4
0			1	1	**1**
1				**2**	**3**
2					3
3					
4					

last

Starting from last[0][4] = 1
Output to MultOrder;
Continue with last[1][4]
then last[0][1]

multOrder

			1



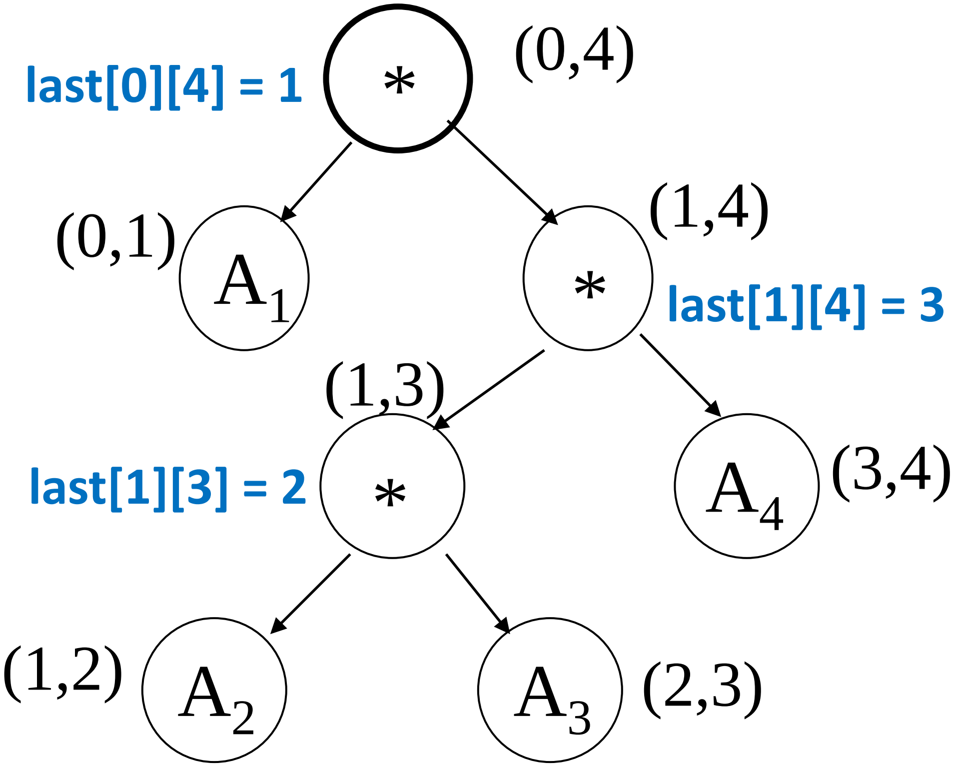
	0	**1**	**2**	**3**	**4**
0			1	1	**1**
1				**2**	**3**
2					3
3					
4					

last

last[1][4] = 3
Output to MultOrder;
Continue with last[3][4]
then last[1][3]

multOrder

		3	1



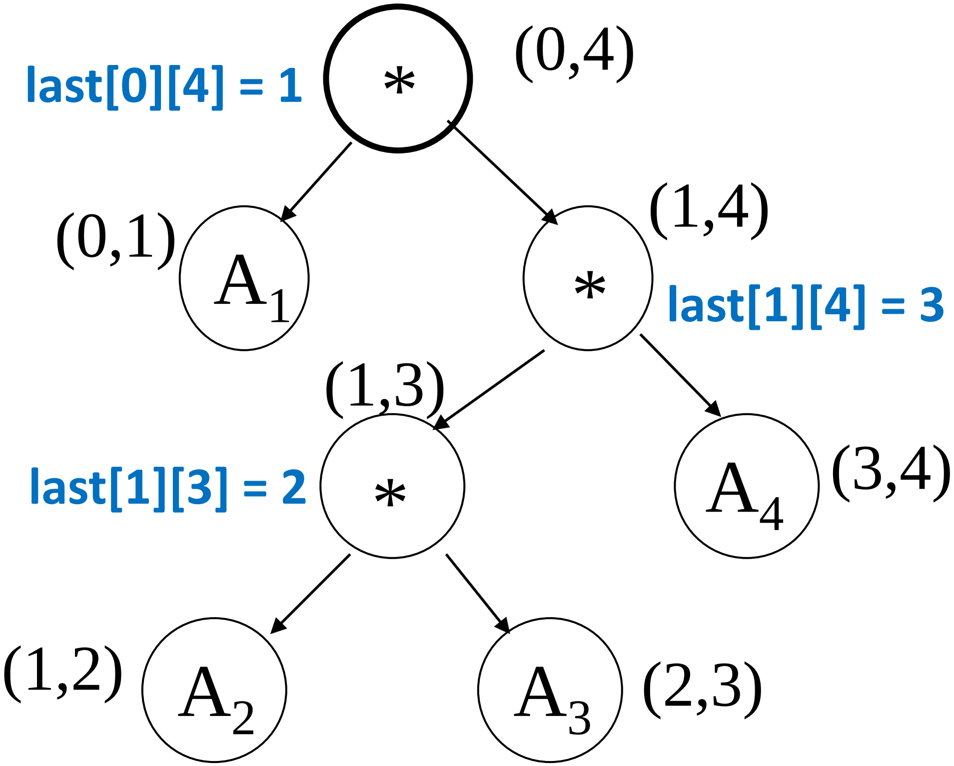
	0	1	2	3	4
0			1	1	**1**
1				**2**	**3**
2					3
3					
4					

last

last[3][4] is a single matrix:
Do nothing

multOrder

		3	1



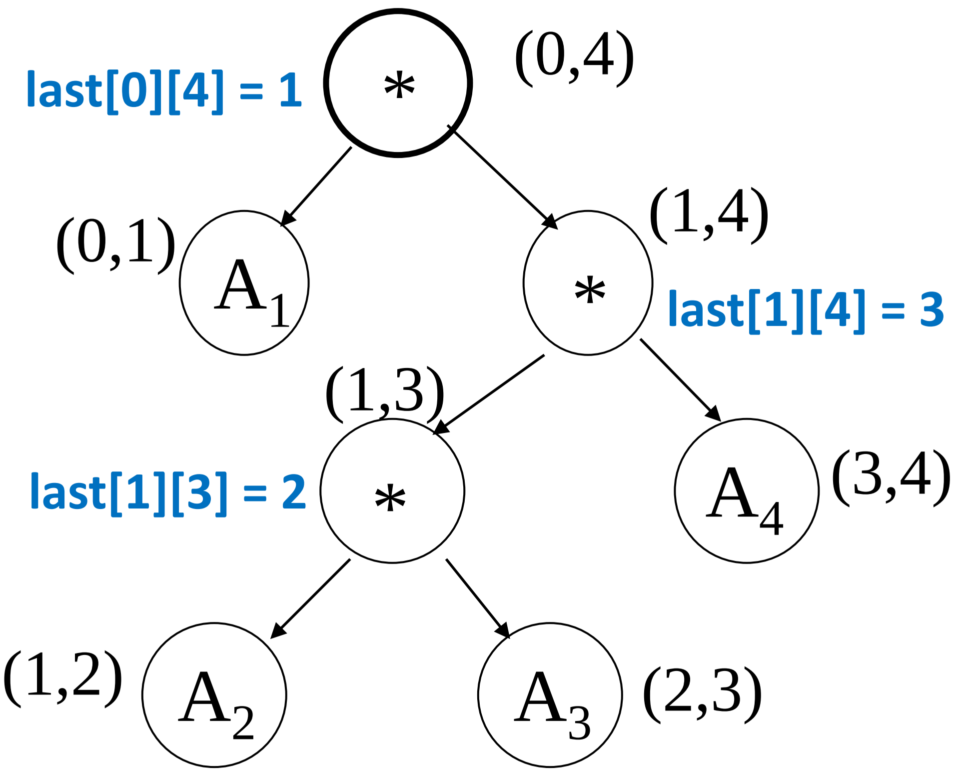
	0	1	2	3	4
0			1	1	**1**
1				**2**	**3**
2					3
3					
4					

last

last[1][3] = 2
Output to MultOrder;
Continue with last[2][3]
then last[1][2]

multOrder

	2	3	1



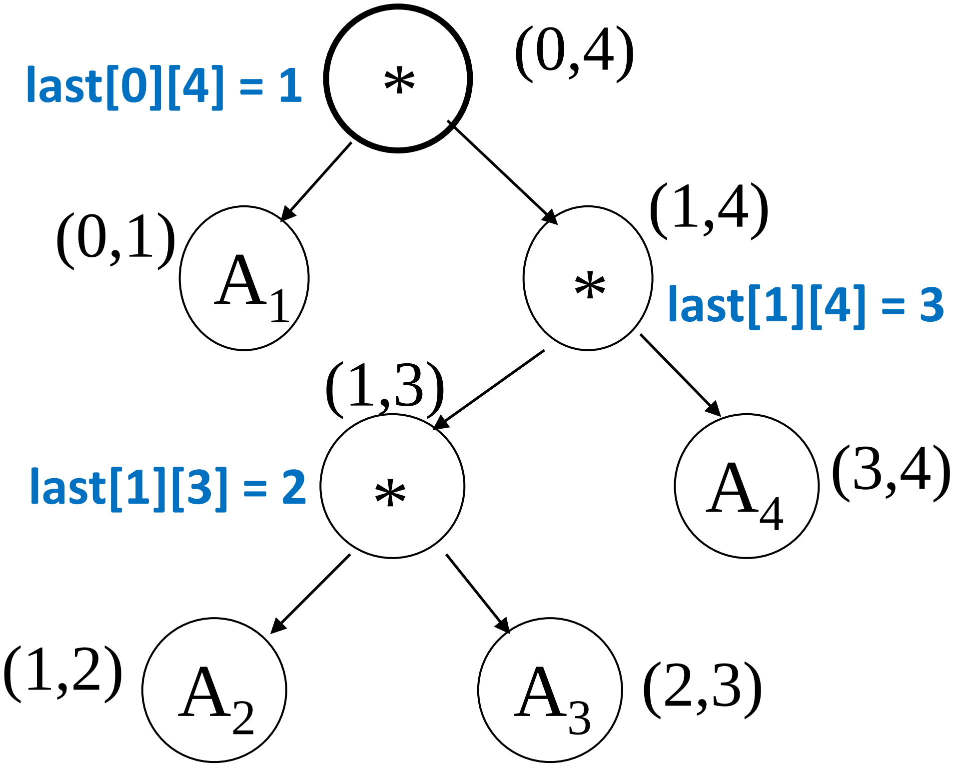
	0	1	2	3	4
0			1	1	**1**
1				**2**	**3**
2					3
3					
4					

last

last[2][3] is a single matrix:
Do nothing
last[1][2] is a single matrix:
Do nothing

multOrder

	2	3	1



	0	1	2	3	4
0			1	1	**1**
1				**2**	**3**
2					3
3					
4					

last

last[0][1] is a single matrix:
Do nothing

So the best sequence is
(A1 x ((A2 x A3) x A4))

multOrder

	2	3	1

Question 5

Construct an example with only three or four matrices where the worst multiplication order does at least 100 times as many element-wise multiplications as the best order.

Let the dimensions of A, B and C be 100x1, 1x100, 100x1 respectively.
Best order: A(BC) – the no. of multiplications is 200
Worst order: (AB)C – the no. of multiplications is 20000

Question 6

We have a knapsack of size 10 and 4 objects. The sizes and the profits of the objects are given by the table below. Find a subset of the objects that fits in the knapsack that maximizes the total profit by the dynamic programming algorithm in the lecture notes.

p	10	40	30	50
s	5	4	6	3

C = 10

p	10	40	30	50
w	5	4	6	3

profit

	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	0	0	0
3	0	0	0	0	50
4	0	0	40	40	50
5	0	10	40	40	50
6	0	10	40	40	50
7	0	10	40	40	90
8	0	10	40	40	90
9	0	10	50	50	90
10	0	10	50	70	90

```
for r = 1 to C
    for c = 1 to n
        profit[r][c] = profit[r][c-1];
        if (w[c] <= r)
            if (profit[r][c] <
        profit[r-w[c]][c-1] + p[c])
                profit[r][c] =
                    profit[r-w[c]][c-1]
                    + p[c]
```