NANYANG TECHNOLOGICAL UNIVERSITY

CE2101/ CZ2101: Algorithm Design and Analysis

Part 2

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Topics

- Analysis Techniques (3 hours)
- Dynamic Programming (5 hours)
- String Matching (3 hours)
- Introduction to NP Completeness (2 hours)

Lecture Delivery Method

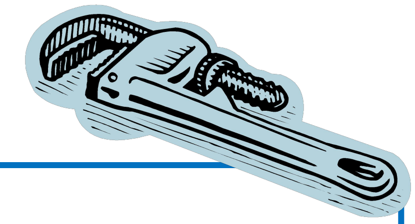
1. Recorded lectures in **Course Media (Media Gallery) /Home**. Videos of one chapter in one PlayList.

2. Weekly review lectures / Q & As in **Zoom** on **Thursdays 2:30pm – 3:30pm**, Week 8 to Week 13. No review lecture on Week 7.

Schedule

Review lectures are from Week 8 to Week 13

Week	Lecture materials to be studied by end of the week	Tutorials	Example classes
7	Analysis techniques	Graphs	Graphs
8	Analysis techniques		
9	Dynamic programming	Analysis techniques	
10	Dynamic programming	DP	Quiz
11	String matching	DP	Quiz
12	NP completeness	String matching	DP
13		NP completeness	DP



Analysis Techniques

Liu Ziwei

Reference: Computer Algorithms: Introduction to Design and Analysis, 3rd Ed, by Sara Basse and Allen Van Gelder.

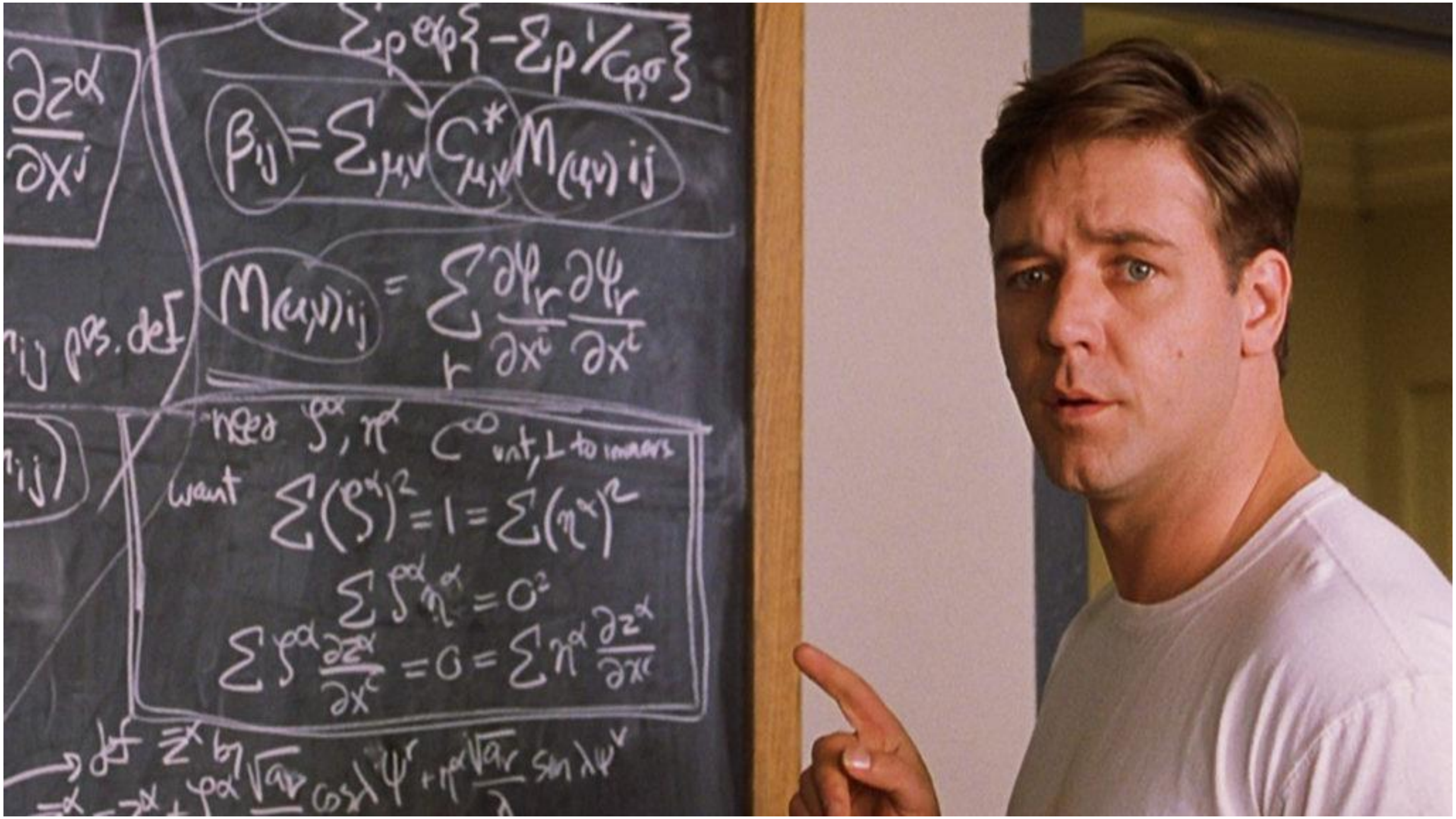
Outline

- Review of the big oh, big omega, big theta

- Solving recurrences (1)
  1. The substitution method
  2. The iteration method
  3. The master method

- Solving recurrences (2)
  1. Solving linear homogeneous recurrences with constant coefficients

Movie “A Beautiful Mind”



Computational Complexity in 1950s

So ~~the~~ a logical way to classify enciphering processes is by the way in which the computation ~~length~~ length for the computation of the key increases with increasing ~~length~~ length of the key. This is ~~at~~ best exponential

and at worst probably a relatively small power of r, ar^2 or ar^3, as in substitution ciphers. 4

Now my general conjecture is as follows: For almost all sufficiently complex types of enciphering, especially ~~the~~ where the ~~information~~ instructions given by different portions of the key

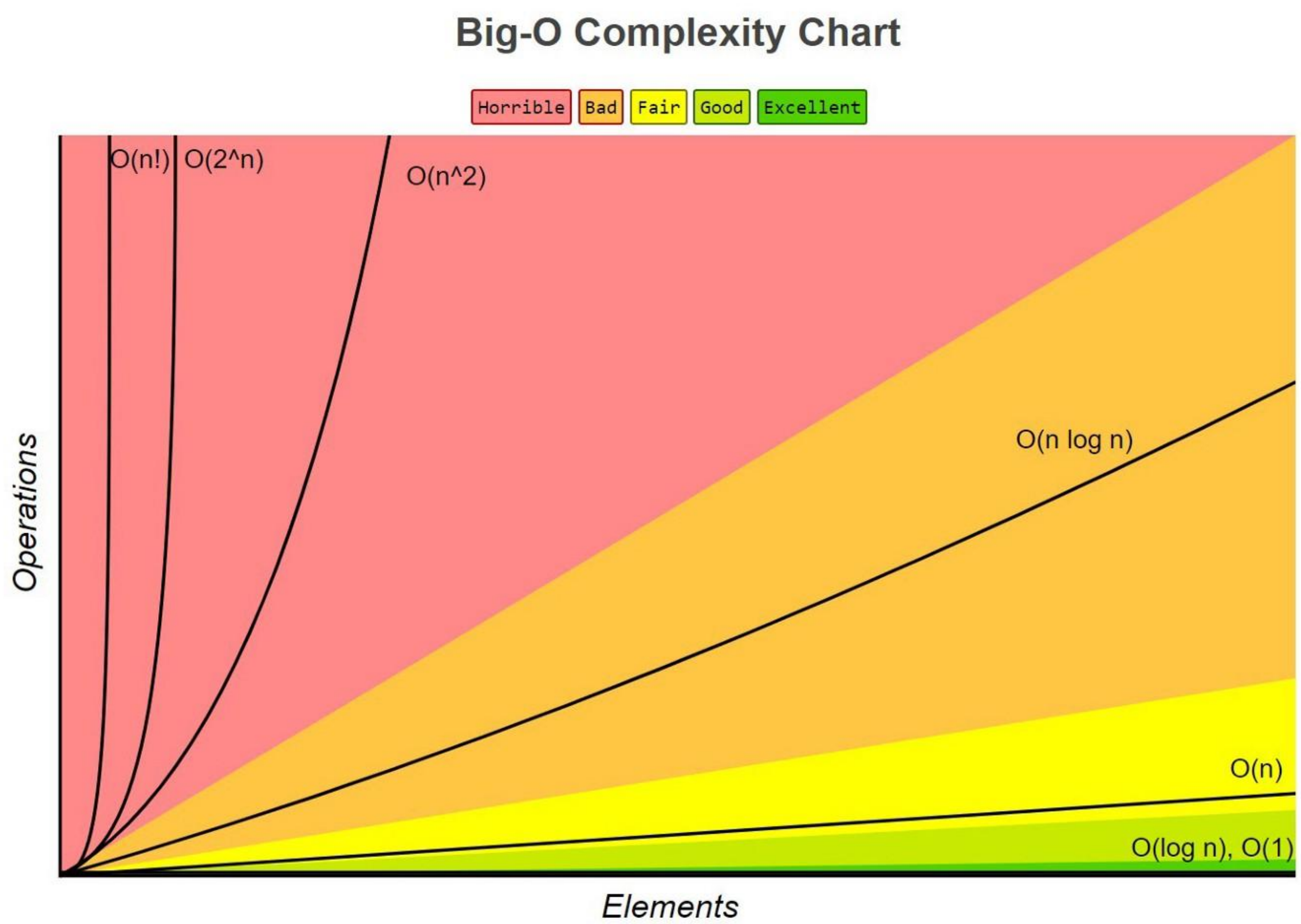
Correspondence between Nash and NSA

John F. Nash (1928–2015)

National Security Agency (est. 1952)

Correspondence between Nash and NSA, 1955
`https://www.nsa.gov/Portals/70/documents/news-features/declassified-documents/nash-letters/nash_letters1.pdf`

Big-O Complexity (Average Case vs. Worst Case)



Ramifications of Big-O Differences

Some numbers:

If we have an algorithm that has 1000 elements, and the O(log n) version runs in 10 nanoseconds…

constant	logarithmic	linear	n log n	quadratic	polynomial (n^3)	exponential (a==2)
$O(1)$	$O(\log n)$	$O(n)$	$O(n \log n)$	$O(n^2)$	$O(n^k)$ $(k \geq 1)$	$O(a^n)$ $(a>1)$
1ns	10ns	1microsec	10microsec	1millisec	1 sec	10^{292} years

Review of the big oh, big omega, big theta

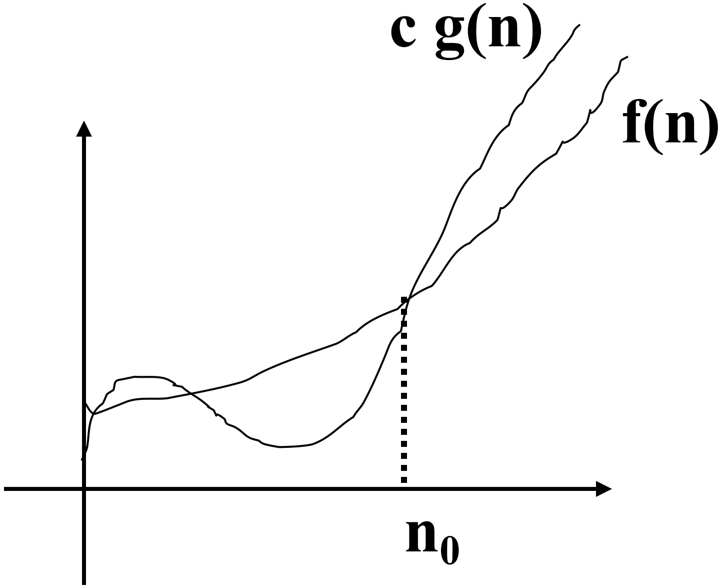
The Big-oh notation:

Definition: Let f and g be 2 functions such that

$f(n) : N \to R^+$ and $g(n) : N \to R^+$, if there exists positive constants c and n_0 such that

$f(n) <= c * g(n)$ for all $n > n_0$

then $f(n) = O(g(n))$.



Alternative definition: if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c < \infty$$

then $f(n) = O(g(n))$.

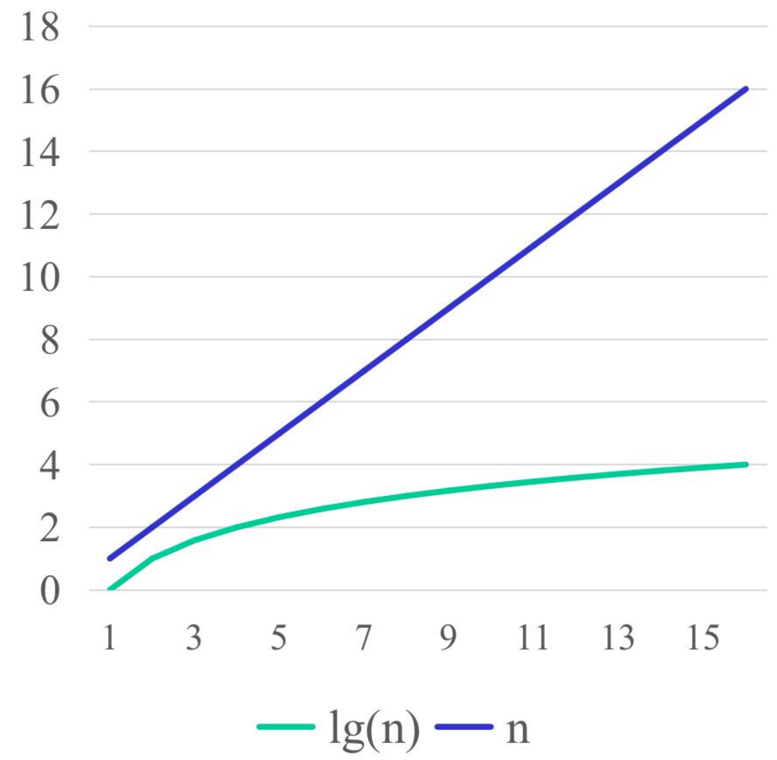
Review of the big oh, big omega, big theta

Example: f(n)=lg(n), g(n)= n,

Let c=1, n_0 =1, then for all n>1

lg(n) <= n, i.e., f(n) <= g(n)

so f(n) = O(g(n)).



Another way: Since

$$\lim_{n\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{\lg(n)}{n} = 0 < \infty$$

so f(n) = O(g(n)).

g(n) gives the asymptotic upper bound for f(n).

Review of the big oh, big omega, big theta

The big Omega notation

Definition: Let f and g be 2 functions such that

$f(n) : N \to R^+$ and $g(n) : N \to R^+$, if there exists positive

constants c and n_0 such that

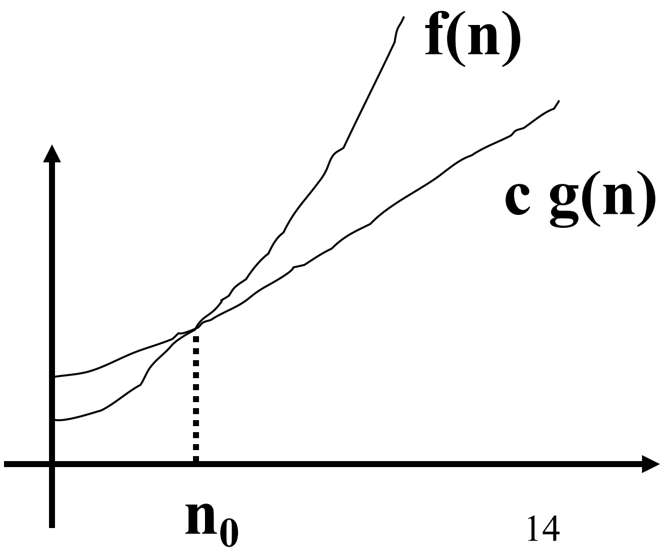
$f(n) >= c * g(n)$ for all $n > n_0$

then $f(n) = \Omega(g(n))$.

Alternative definition: if

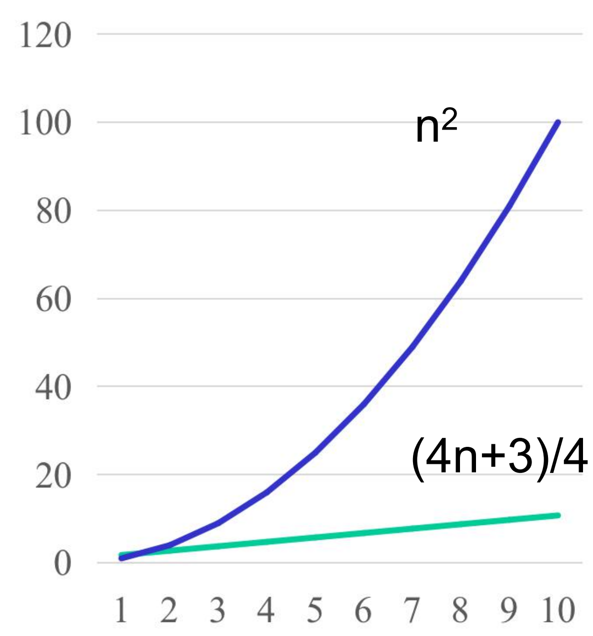
$$\lim_{n \to \infty} \frac{f(n)}{g(n)} > 0$$

then $f(n) = \Omega(g(n))$.



Review of the big oh, big omega, big theta

Example: $f(n)= n^2$, $g(n)=4n+3$



Let $c=1/4$, $n_0 =1$, then for all $n>1$

$n^2 >=(4n+3)/4$ i.e., $f(n) >=(1/4)g(n)$

so $f(n) = \Omega(g(n))$.

Another way: Since

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2}{4n+3} = \lim_{n\to\infty} \frac{n}{4+\frac{3}{n}} = \infty > 0$$

so $f(n) = \Omega(g(n))$.

g(n) gives the asymptotic lower bound for f(n).

Review of the big oh, big omega, big theta

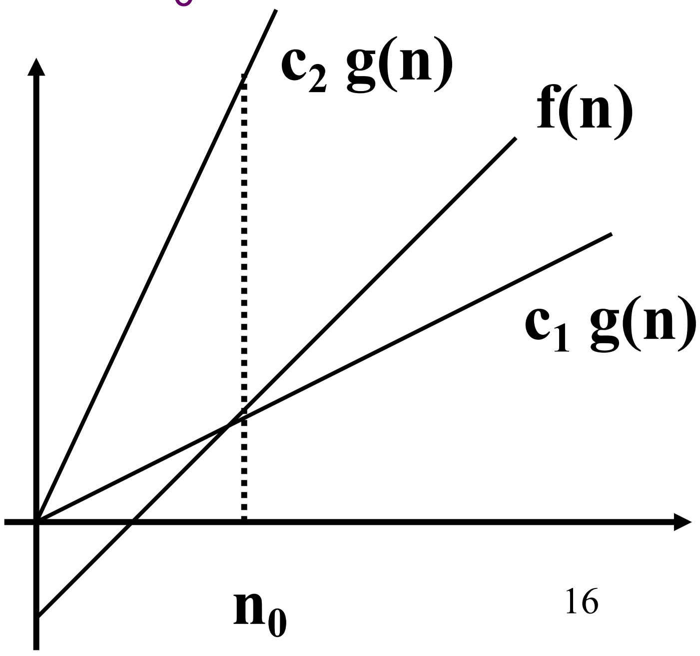
The big Theta notation

Definition: Let f and g be 2 functions such that

$f(n) : N \to R^+$ and $g(n) : N \to R^+$, if there exists positive constants c_1, c_2 and n_0 such that

$c_1 * g(n) <= f(n) <= c_2 * g(n)$ for all $n > n_0$

then $f(n) = \theta(g(n))$.



Alternative definition: if

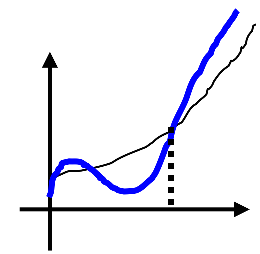
$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c \quad (0 < c < \infty)$$

then $f(n) = \theta(g(n))$.

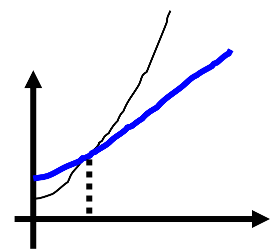
Review of the big oh, big omega, big theta

- The idea of the O, Ω and θ definitions is to establish a relative order among functions.
- We compare the **<u>relative rates of growth</u>**.

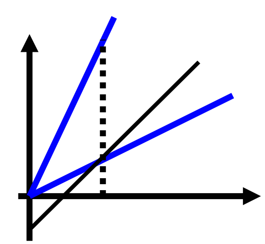
  - If $f(n) = O(g(n))$, $g(n)$ gives the asymptotic upper bound



  - If $f(n) = \Omega(g(n))$, $g(n)$ gives the asymptotic lower bound



  - If $f(n) = \theta(g(n))$, $g(n)$ gives the asymptotic tight bound



Recursive algorithms and Recurrence relations

- Many problems have a recursive solution
- A common way of analysis for such solution algorithms will involve a recurrence relation that needs to be solved
- A recurrence is an equation or inequality that describe a function in terms of its value on smaller inputs, e.g.

$$M(n) = 2M(n-1) + 1$$

Example 1: Towers of Hanoi

Move all disks from the first pole to the third pole subject to the condition that only one disk can be moved at a time and that no disk is ever placed on top of a smaller one.


Numberphile

Pseudo Code !!!

最左 最右 中间

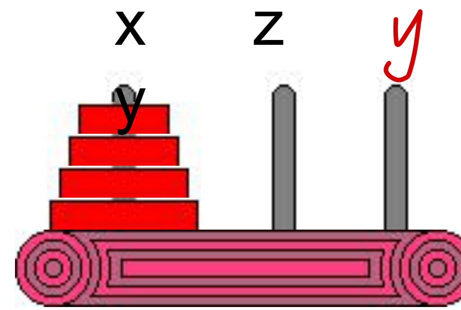
```
void TowersOfHanoi(int n, int x, int y, int z)
{ // Let M(n) be the total no. of disk moves
  if (n == 1)
      cout << "Move  disk from  " << x << " to  " << y << endl;
      // this has one disk move
  else {
     TowersOfHanoi(n-1, x, z, y);
     // this involves M(n-1) disk moves
```

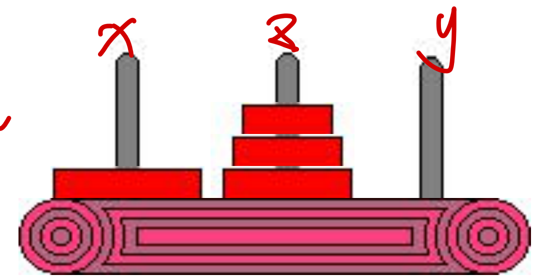
move from x to z with help of y

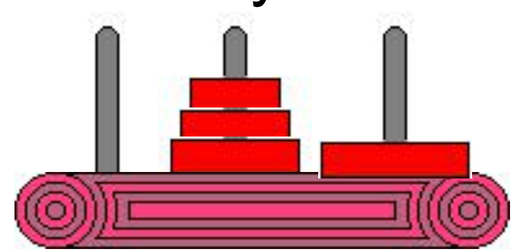
```
     cout << "Move  disk from  " << x << " to  " << y << endl;
    // one disk move
```

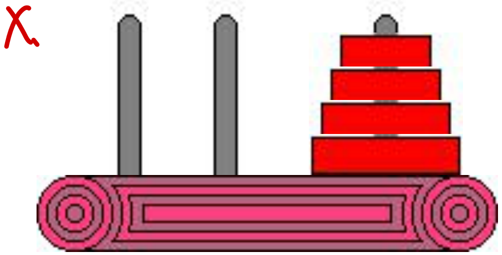
move from z to y with help of x

```
     TowersOfHanoi(n-1, z, y, x);
     // another M(n-1) disk moves
  }
}
```









The number of disk moves: $M(1) = 1;$

$$M(n) = 2M(n-1) + 1$$

Example 2: Merge sort

```
void mergesort(int l, int m)
{
   int mid = (l+m)/2;
   if (m-l > 1) {
      mergesort(l, mid);
      mergesort(mid+1, m);
   }
   merge(l, m);
}
```

Let $M(n)$ be the total no. of comparisons between array elements.

$M(2) = 1;$

$M(n) = 2M(n/2) + n - 1$

Solving recurrences (1)

- We want to solve recurrences of the form

  $$\mathbf{W}(n) = a\mathbf{W}(n/b) + \mathbf{f}(n)$$

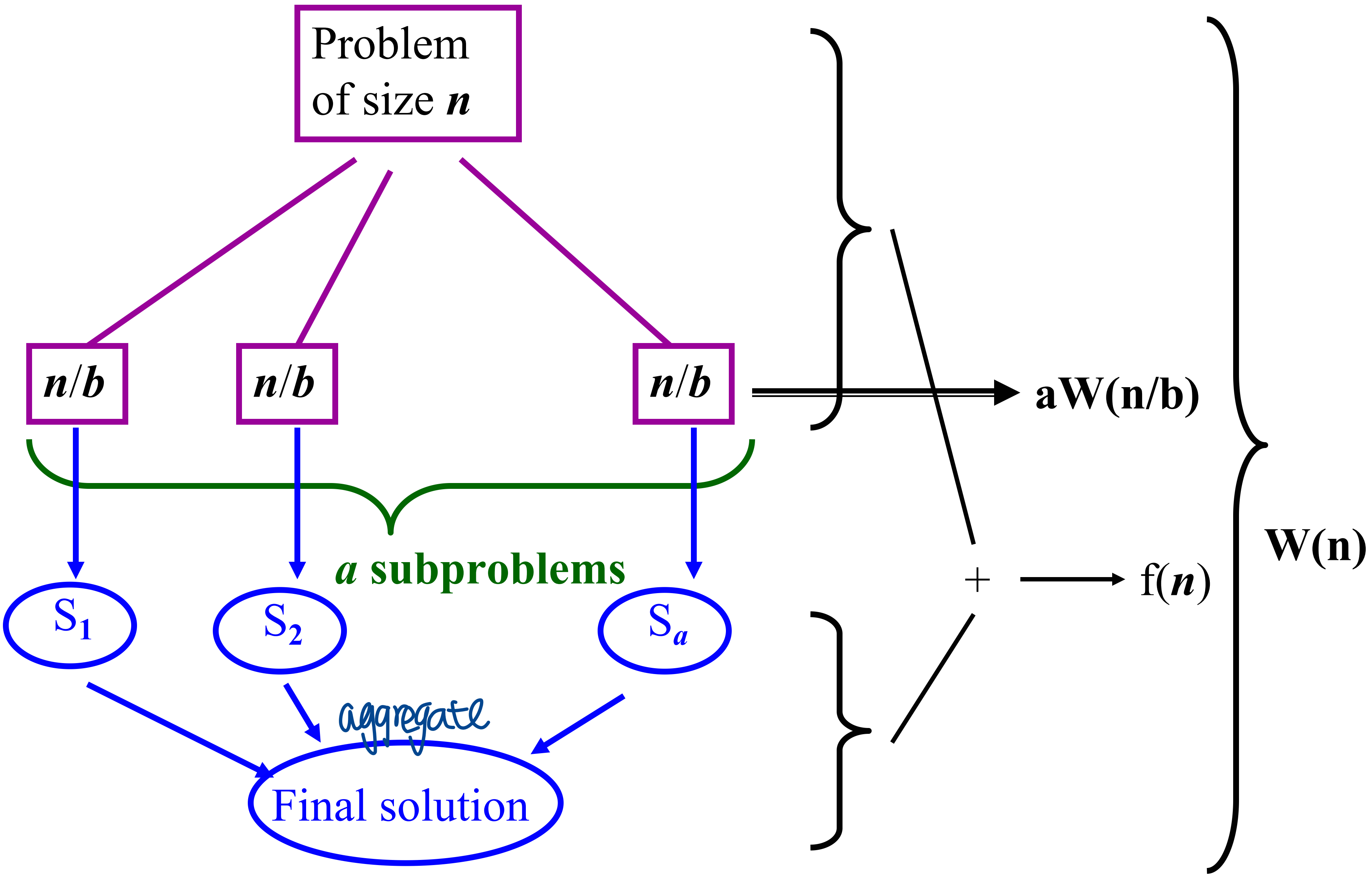
  where $a \geq 1$ and $b > 1$ are constants, f(n) is a function of n.

  factor used to reduce the size of problem

- The recurrence describes the computational cost of an algorithm that uses the "divide-and-conquer" approach.

- f(n) is the cost of dividing the problem and combining the results of the subproblems.

- Usually the problem of size n is divided into subproblems of sizes either $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$. However it does not change the asymptotic behaviour of the recurrence.



Problem of size *n*
n/*b*
n/*b*
n/*b*
a subproblems
S_1
S_2
S_a
aggregate
Final solution
aW(n/b)
+
f(*n*)
W(n)

Solving recurrences (1)

- Examples

cpr 2 times

$W(n) = 2W(n/2) + 2$	Finding the max and min from a sequence
$W(n) = W(n/2) + 2$	Binary search
$W(n) = 3W(n/2) + cn$	Multiplying two 2n-bits integers
$W(n) = 2W(n/2) + n - 1$	Merge sort
$W(n) = 7W(n/2) + 15n^2/4$	Multiplying two nxn matrices

Solving recurrences (1)

We describe three methods:

1) The substitution method
2) The iteration method
3) The master method.

1. The substitution method

- It is a “guess and check” strategy. First guess the form of the solution and then use mathematical induction to prove it.
- A powerful method because often it is easier to prove that a certain bound (in the form of the *O* notation) is valid than to compute the bound.

- but the method is only useful when it is easy to guess the form of the solution.
- **Mathematical Induction**: If $p(a)$ is true and, for some integer $k \geq a$, $p(k+1)$ is true whenever $p(k)$ is true, then $p(n)$ is true for all $n \geq a$.
- Example: The worst case for merge sort ($n = 2^k$)

Base case → Assumption

$$W(2) = 1$$

$$W(n) = \underbrace{2\,W(n/2)}_{\text{mergeSort}} + \underbrace{n - 1}_{\text{merge}}$$

Guess $W(n) = O(f(n))$ then prove it.

Show (i) $W(2) <= f(2)$ (ii) for some integer $k \geq 2$, assume $W(n) = O(f(n))$ for $n \leq 2^k$, prove $W(2n) <= f(2n)$ then $W(n) = O(f(n))$ for all $n \geq 2$.

$W(2^{k+1}) = 2W(2^k) + 2^{k+1} - 1$

$\leq 2(2^k)^2 + 2^{k+1} - 1$

$W(2^{k+1}) \leq 2^{2k+1} + 2^{k+1} - 1$

$\leq 2^{2k+1}$

$\therefore W(2^{k+1}) < 2^{2k+2}$

First guess: W(n) = O(n²)

Proof by mathematical induction that $W(n) \leq cn^2$:

(1) Base case: $W(2) = 1 \leq 2^2$;

(2) Inductive step: assume that $W(n) = O(n^2)$ for $n \leq 2^k$.
Now consider $n = 2^{k+1}$

$$W(2^{k+1}) = 2W(2^k) + 2^{k+1} - 1$$
$$\leq 2 * (2^k)^2 + 2^{k+1} - 1$$
$$= 2 * (2^k)^2 + 2 * 2^k - 1$$
$$\leq 4 * (2^k)^2$$
$$= (2^{k+1})^2$$

i.e. $W(2^{k+1}) \leq (2^{k+1})^2$

A lot is added from step 3 to step 4

Thus $W(n) = O(n^2)$. But is this the best guess?

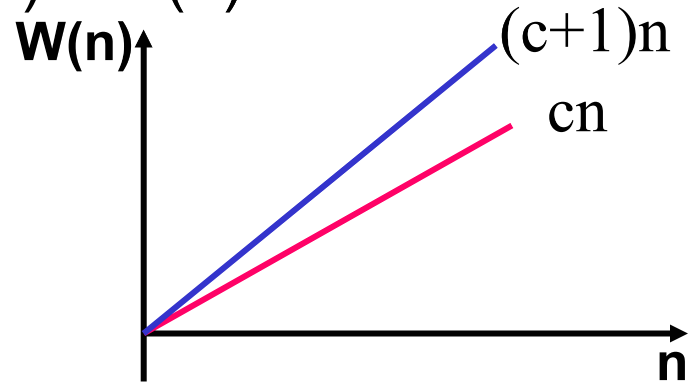
Second guess: **$W(n) = O(n)$, i.e. $W(n) \leq c * n$**

Proof by mathematical induction:

(1) Base case: $W(2) = 1 \leq 2c$;

(2) Inductive step: assume that $W(n) = O(n)$ for $n \leq 2^k$.
Now consider $n = 2^{k+1}$

$$W(2^{k+1}) = 2W(2^k) + 2^{k+1} - 1$$
$$\leq 2 * c * 2^k + 2^{k+1} - 1$$
$$= c * 2^{k+1} + 2^{k+1} - 1$$



Thus $W(2^{k+1}) \leq (c+1) * 2^{k+1} - 1$ but we cannot say
$W(2^{k+1}) \leq c * 2^{k+1}$

Thus $W(n) \neq O(n)$.

Third guess: **W(n) = O(nlgn)**

$W(2^{k+1}) = 2W(2^k) + 2^{k+1} - 1$
$\leq 2^k \cdot \lg 2^k \times 2 + 2^{k+1} - 1$
$= k \cdot 2^{k+1} + 2^{k+1} - 1$
$= (k+1) \times 2^{k+1} - 1$
$< (k+1) \times 2^{k+1}$
$\therefore W(2^{k+1}) < 2^{k+1} \lg 2^{k+1}$
$\therefore$ hold

Proof by mathematical induction:

(1) Base case: $W(2) = 1 \leq 2\lg 2$;

(2) Inductive step: assume that $W(n) \leq n\lg n$ for $n \leq 2^k$.
Now consider $n = 2^{k+1}$

$$W(2^{k+1}) = 2W(2^k) + 2^{k+1} - 1$$
$$\leq 2 * k * 2^k + 2^{k+1} - 1$$
$$= k * 2^{k+1} + 2^{k+1} - 1$$
$$\leq (k + 1) * 2^{k+1}$$

Thus W(n) = O(nlgn) is a very close upper bound.

What if the base condition does not hold?

Consider the recurrence ($n = 2^k$) :

$W(1) = 1$

$W(n) = 2\,W(n/2) + n - 1$

Prove that $W(n) = O(n\lg n)$:

(1) Base case: $W(1) = 1 > c\lg 1$;

(2) Recall the big-O notation: for $f(n) = O(g(n))$, we need $f(n) \le c * g(n)$ for all $n > n_0$. → No doesn't need to be the first element

(3) Thus to prove $W(n) = O(n\lg n)$, we may use another base case.

- We have $W(2) = 3 < c*2*\lg 2$ for any $c > 1$.
- We can assume that $W(n) \le cn\lg n$ for $n \le 2^k$ then prove $W(2^{k+1}) \le c*(k + 1) * 2^{k+1}$

Then $W(n) = O(n\lg n)$.

What can we say about the general case of n?

The worst case for merge sort :

$$W(2) = 1$$

$$W(n) = W(\lceil n/2 \rceil) + W(\lfloor n/2 \rfloor) + n - 1$$

Proof [+]:

单向递增

(1) W(n) is a monotonically increasing function. So when n is not a power of 2, that is, $2^k < n < 2^{k+1}$, then $W(2^k) \leq W(n) \leq W(2^{k+1})$.

(2) We have proved that W(n) = O(nlgn) for powers of 2, so, $W(2^{k+1}) \leq c * (k+1) * 2^{k+1}$.

(3) For any $n < 2^{k+1}$ for some k, $W(n) \leq W(2^{k+1})$. Therefore

$$W(n) \leq c * (k+1) * 2^{k+1} < c * \lg(2n) * (2*n) < 4cn\lg n$$

Therefore W(n) = O(nlgn).

[+] **See *The design and analysis of Algorithms*** by Anany Levitin (pp481-483) about **Smoothness Rule.**

2. The iteration method ⇒ 较常用

- The idea is to expand (iterate) the recurrence and express it as a summation of terms depending only on n and the initial condition.
- Techniques for evaluating summations can then be used to provide bounds on the solution.
- Example:

$W(1) = 1, W(2) = 1, W(3) = 1,$

$W(n) = 3W(\lfloor \frac{n}{4} \rfloor) + n$

we expand (iterate) it:

$$W(n) = 3W(\lfloor \frac{n}{4} \rfloor) + n$$
$$= 3(3W(\lfloor \frac{n}{4^2} \rfloor) + \lfloor \frac{n}{4} \rfloor) + n$$

$$= 3^2\, W(\lfloor \frac{n}{4^2} \rfloor) + 3\lfloor \frac{n}{4} \rfloor + n$$

$$= 3^2(3W(\lfloor \frac{n}{4^3} \rfloor) + \lfloor \frac{n}{4^2} \rfloor) + 3\lfloor \frac{n}{4} \rfloor + n$$

$$= 3^3\, W(\lfloor \frac{n}{4^3} \rfloor) + 3^2\lfloor \frac{n}{4^2} \rfloor + 3\lfloor \frac{n}{4} \rfloor + n$$

we need to iterate until we reach one of the boundary conditions, i.e $\lfloor \frac{n}{4^i} \rfloor = 1, 2$ or 3.

E.g. n=64, $4^3 \leq 64 < 4^4$ and $\lfloor \frac{64}{4^3} \rfloor = 1$;

n=255, $4^3 \leq 255 < 4^4$ and $\lfloor \frac{255}{4^3} \rfloor = 3$;

This means if $4^i \leq n < 4^{i+1}$ then $i = \lfloor \log_4 n \rfloor$. So

$$W(n) = 3^i\, W(a) + 3^{i-1}\lfloor \frac{n}{4^{i-1}} \rfloor + \ldots + 3^2\lfloor \frac{n}{4^2} \rfloor + 3\lfloor \frac{n}{4} \rfloor + n$$

a = 1,2 or 3

$$W(n) = 3^i W(a) + 3^{i-1} \left\lfloor \frac{n}{4^{i-1}} \right\rfloor + \ldots + 3^2 \left\lfloor \frac{n}{4^2} \right\rfloor + 3 \left\lfloor \frac{n}{4} \right\rfloor + n$$

$$\leq 3^{\log_4 n} W(a) + 3^{i-1} \frac{n}{4^{i-1}} + \ldots + 3^2 \frac{n}{4^2} + 3 \frac{n}{4} + n$$

Let $x = 3^{\log_4 n}$ then

$\log_4 x = \log_4 n \log_4 3$ then

$4^{\log_4 x} = 4^{\log_4 n \log_4 3}$ then

$x = n^{\log_4 3}$, i.e. $3^{\log_4 n} = n^{\log_4 3}$

$$\sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i = 4$$

$$W(n) \leq n^{\log_4 3} + n \sum_{i=0}^{\infty} \left(\tfrac{3}{4}\right)^i + = O(n)$$

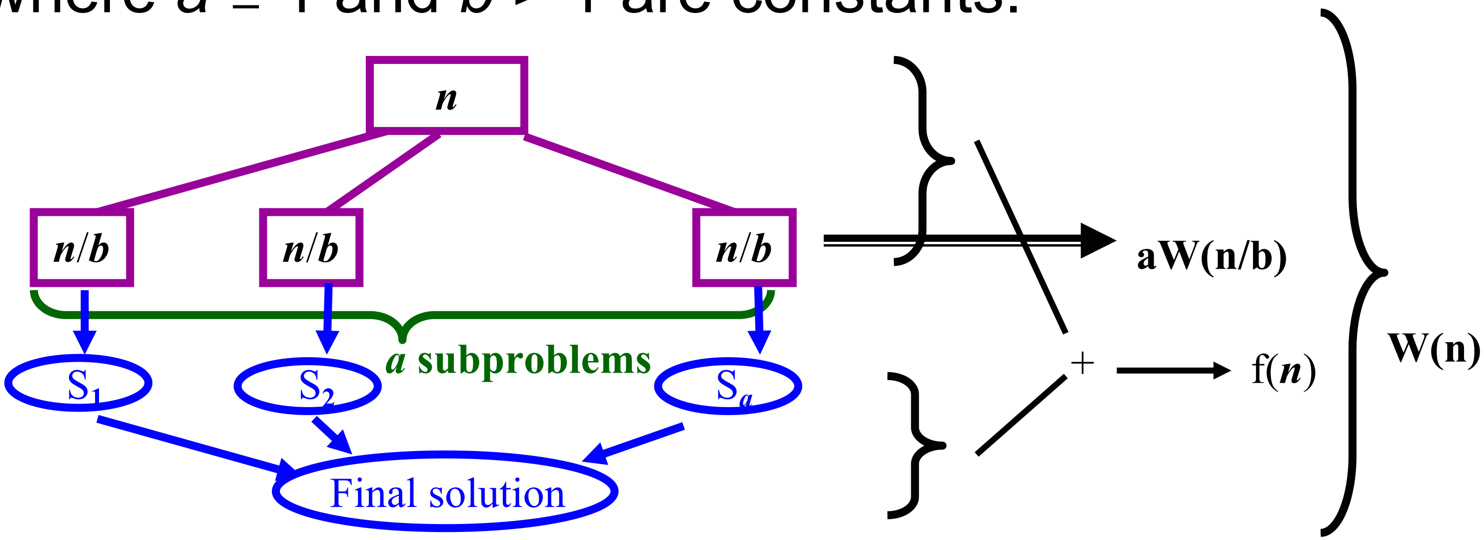
- The iteration method usually leads to lots of algebra.
- We should focus on how many times the recurrence needs to be iterated to reach the boundary condition.

3. The master method

- The master method provides a "manual" for solving recurrences of the form

$$W(n) = aW(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants.



- We are able to determine the asymptotic tight bound in the following three cases

The master theorem ⇒

1. Figure out f(n)
2. Look up the function in 3 scenarios
3. NO NEED FOR CALCULATION!!!

For **W(*n*) = *a*W(*n*/*b*) + f(*n*)** $a \geq 1$ and $b > 1$

The manual:

1. If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $W(n) = \Theta(n^{\log_b a})$. Find a upper bound

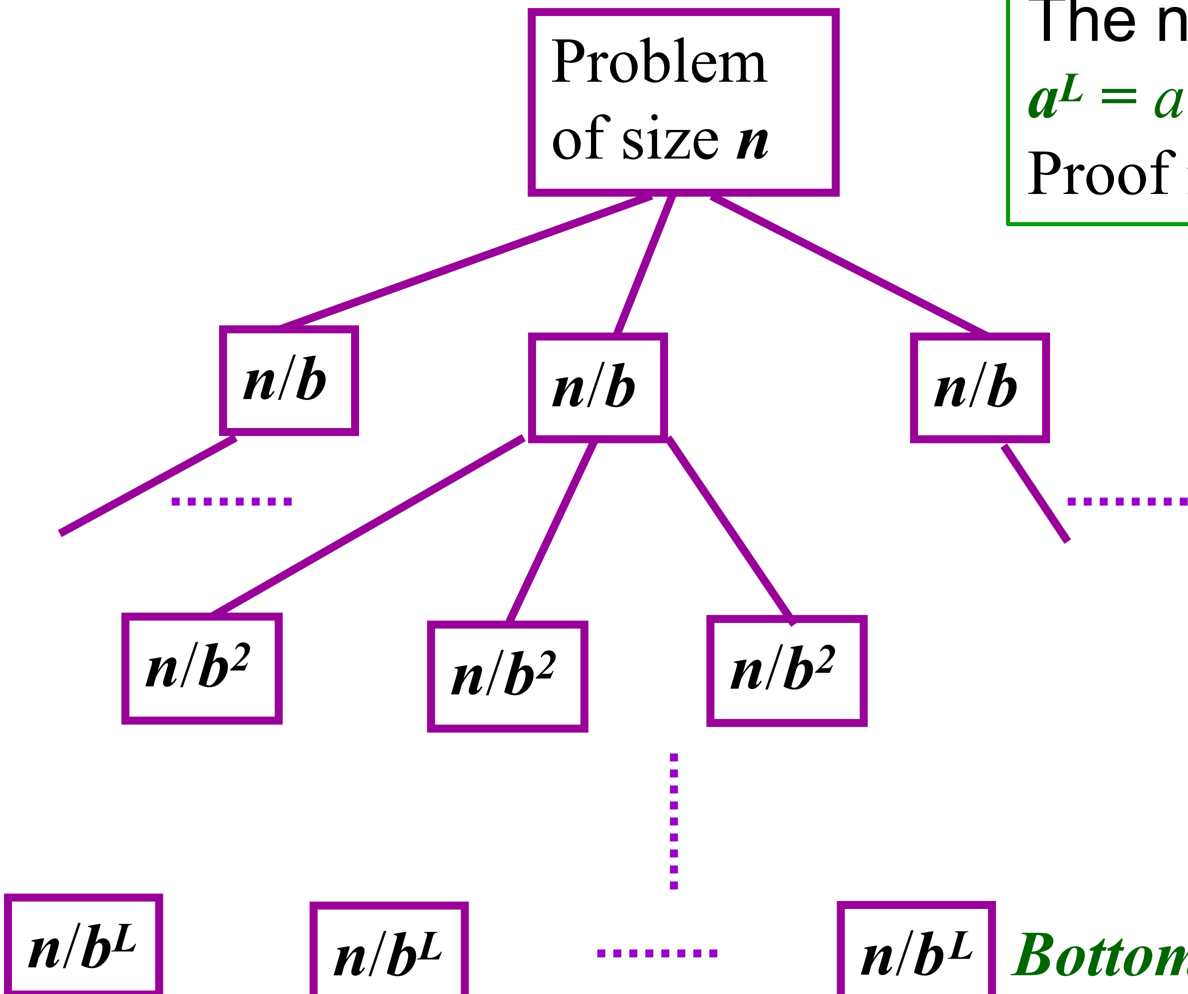
2. If $f(n) = \Theta(n^{\log_b a})$, then $W(n) = \Theta(n^{\log_b a} \log n)$.

   If $f(n) = \Theta(n^{\log_b a} \log^k n)$, $k \geq 0$, → tight boung
   then $W(n) = \Theta(n^{\log_b a} \log^{k+1} n)$

3. If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if $a\, f(n/b) \leq c\, f(n)$ for some constant $c < 1$ and all sufficiently large n, then $W(n) = \Theta(f(n))$. lower bound

What is $n^{\log_b a}$?

The depth of the tree
$L = \log_b n$
The number of leaves is
$a^L = a^{\log_b n} = n^{\log_b a}$
Proof in slide 54



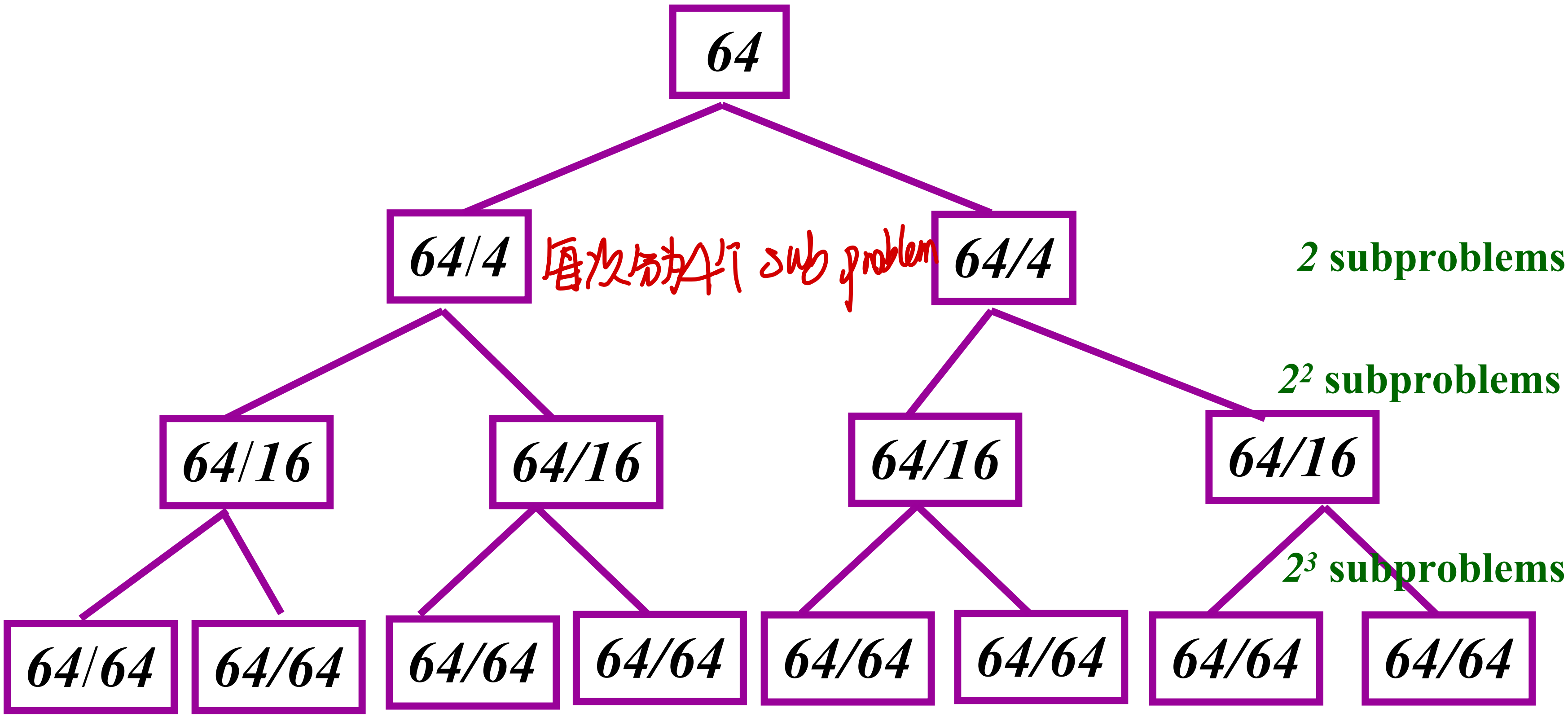
a subproblems

a^2 subproblems

Depth of the tree

Bottom level: a^L **subproblems**

E.g. $\boldsymbol{n} = 64, \quad \boldsymbol{a} = 2, \boldsymbol{b} = 4$



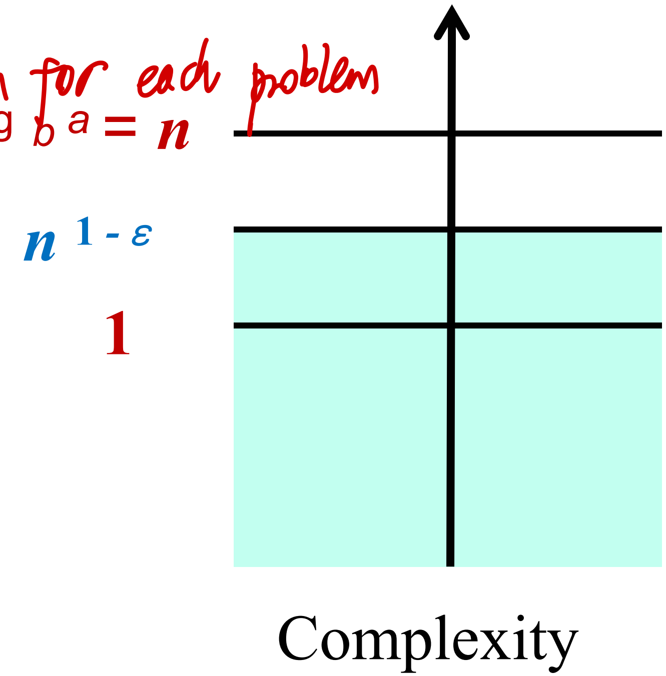
Depth of tree L = $\log_4 64$, Number of leaves = 8 = $2^{\log_4 64} = 64^{\log_4 2}$

$$(a^{\log_b n} = n^{\log_b a})$$

a: no. of problems each level

$\frac{n}{b}$: size of problem for each problem

Examples



1) $W(n) = 3W(n/3) + 2$,

so $a = 3$, $b = 3$,

$n^{\log_b a} = n^1$

$f(n) = 2 = \theta(1) = O(n^1)$

We may let $\varepsilon = 0.5$ then we confirm $2 = O(n^{1-0.5})$,

i.e. $f(n) = O(n^{1-\varepsilon})$

$\Rightarrow f(n) = O(n^{\log_b a - \varepsilon})$ (case 1)

thus $W(n) = \theta(n^{\log_b a})$

$W(n) = \theta(n)$.

Examples

$\quad\quad\quad\quad a \quad\quad\quad\quad b \quad\quad\quad f(n)$

2) $W(n) = 4W(n/4) + n - 1$,

so a = 4, b = 4,

$n^{\log_b a} = n^1$

$f(n) = n - 1$

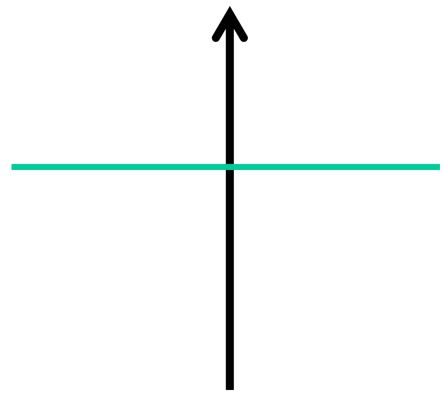
$n^{\log_b a} = n$

$\therefore f(n) = \Theta(n)$

$\Downarrow$

Case 2

$n^{\log_b a} = n$



Complexity

We have

$f(n) = n - 1$

$= \Theta(n^1)$,

$= \Theta(n^{\log_b a})$, (case 2)

thus

$W(n) = \Theta(n^{\log_b a} \log n)$

$= \Theta(n \log n)$

Examples

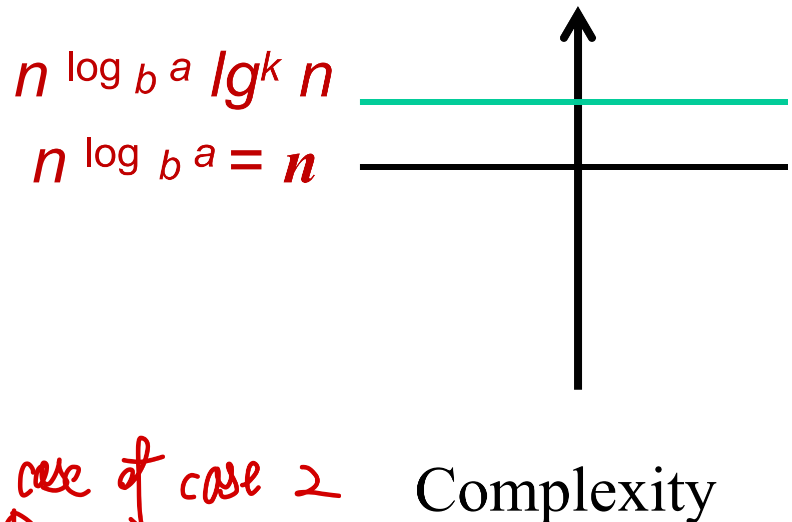
$a=b=2$

3) $W(n) = 2W(n/2) + n \lg n$,

so a = 2, b = 2,

$f(n) = n \lg n$

$n^{\log_b a} = n^1$



We have

$f(n) = \Theta(n^1 \lg n)$,

$= \Theta(n^{\log_b a} lg^k n)$,   (case 2: $k = 1$)

Second case of case 2

thus

$W(n) = \Theta(n^{\log_b a} lg^2 n)$

$= \Theta(n (lg\ n)^2)$

Examples

a=2 b=4 f(n).

4) $W(n) = 2W(n/4) + n$,

so a = 2, b = 4,

$n^{\log_b a} = n^{\log_4 2} = n^{0.5}$

$f(n) = n = \Theta(n)$

Find a lower bound => case 3

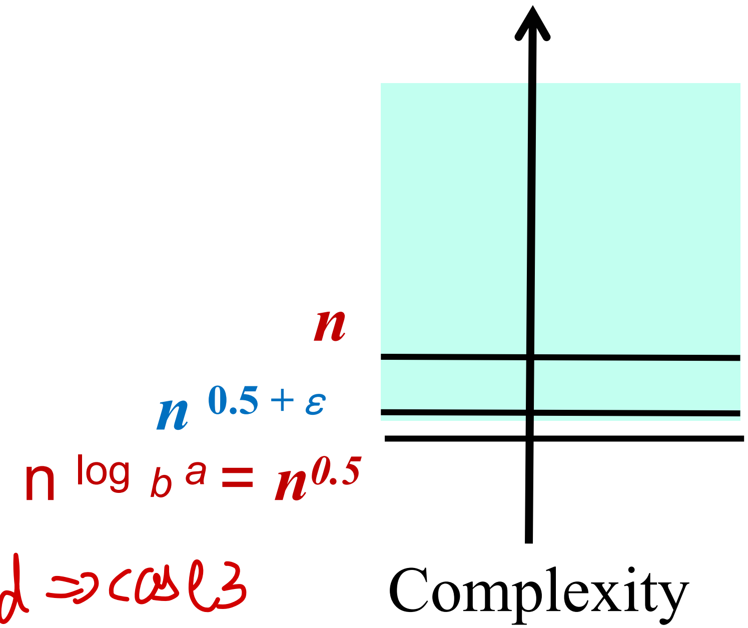
We may let $\varepsilon = 0.1$ then we have $n = \Omega(n^{0.6})$

i.e. $f(n) = \Omega(n^{\log_b a + \varepsilon})$, and

for all sufficiently large n, we can find a value for c, say, $c = ¾$, to show that $a\, f(n/b) \le c\, f(n)$. (case 3)

$a*f(n/b) = 2*f(n/4) = n/2 \le c*n$

thus $W(n) = \Theta(n)$.



Sometimes the master method cannot apply

Example 1: W(n) = 3W(n/3) + n/lgn, $n^{\log_b a} = n^1$

$f(n) = n/\lg n = O(n^1)$ because $\lim_{n\to\infty} \frac{n/\lg n}{n^1} = \lim_{n\to\infty} \frac{1}{\lg n} = 0$

Have an upperbound

$f(n) = O(n^{1-\varepsilon})$? (L'Hôpital's rule, slide 55)

i.e. $n/\lg n = O(n^{1-\varepsilon})$?

No, because asymptotically, $n/\lg n > n^{1-\varepsilon}$ for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{n/\lg n}{n^{1-\varepsilon}} = \lim_{n\to\infty} \frac{n^{\varepsilon}}{\lg n} = \infty$$

This recurrence falls into the gap between case 2 and case 3.

So the Master Theorem cannot apply.

Sometimes the master method cannot apply

Example 2: W(n) = W(n/3) + f(n)

where $f(n) = \begin{cases} 3n + 2^{3n}, & \text{for } n = 2^i \\ 3n & \text{otherwise} \end{cases}$ piecewise func.

so a = 1, b = 3 then $n^{\log_b a} = n^0$ constant.

let $\varepsilon = 1$ then $f(n) = \Omega(n^{0+1})$, case 3?

$a\, f(n/b) \le c\, f(n)$ for all sufficiently large n?

When $n = 3 * 2^i$, $a\, f(n/b) = f(2^i) = n + 2^n$, but $cf(n) = c(3n)$

i.e. $a\, f(n/b) > c\, f(n)$. E.g. for n = 6 or greater

So the Master Theorem cannot apply.

- Notice that when we want to find the order of a recurrence, the initial conditions are not important. This is because the running costs of the terminating conditions are small constants that do not affect the order.

Fast to find complexity for recurrent method.

Solving recurrences (2)

- <u>Definition</u>: A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

  $$a_n = c_1 a_{n-l} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

  where $c_1, c_2, \ldots, c_k$ are real constants and $c_k \neq 0$.

The two different notations: A(n) and a_n

- When using A(n), we mean the function value with parameter n
- When using a_n, we mean the nth term in a sequence $a_1, a_2, \ldots, a_n$.
- If we list A(1), A(2), …, A(n) in a sequence, we can write them as $a_1, a_2, \ldots, a_n$. They are equivalent.

Solving recurrences (2)

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  where $c_1, c_2, \dots, c_k$ are real constants and $c_k \neq 0$.

  - **Linear**: $a_{n-1}, a_{n-2}, \dots, a_{n-k}$ appear in separate terms and to the first power (can't have $a^2 / a^3 / a^4 \dots$)
  - **Homogeneous**: the total degree of each term is the same, e.g. no constant term
  - **Constant coefficients**: $c_1, c_2, \dots, c_k$ are fixed real constants that do not depend on n
  - ***Degree k***: the expression for a_n contains the previous k terms $a_{n-1}, a_{n-2}, \dots, a_{n-k}$, ($c_k \neq 0$)

- Examples
  - A *linear homogeneous recurrence relation of degree 2:* $a_n = a_{n-1} + a_{n-2}$
  - A *linear homogeneous recurrence relation of degree 1:* $a_n = 1.04a_{n-1}$
  - A *linear homogeneous recurrence relation of degree 3 :* $a_n = a_{n-3}$

similar to the concept of step.

- Non-examples

constant terms X

  - $a_n = a_{n-1} + a_{n-2} + 1$: non-homogeneous
  - $a_n = a_{n-1} a_{n-2}$: not linear
  - $a_n = na_{n-1}$: coefficient not constant

- A linear homogeneous recurrence relation of degree *k* can be systematically solved, i.e. find the explicit expression for a_n

- The basic approach is to look for solutions of the form $a_n = t^n$ where t is a constant

- If $a_n = t^n$ is a solution for

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

Then

$$t^n = c_1 t^{n-1} + c_2 t^{n-2} + \ldots + c_k t^{n-k}$$

$\Rightarrow t^k = c_1 t^{k-1} + c_2 t^{k-2} + \ldots + c_k$ (divide both side by t^{n-k})

$\Rightarrow t^k - c_1 t^{k-1} - c_2 t^{k-2} - \ldots - c_k = 0$

- This means if we can solve the equation

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - \ldots - c_k = 0$$

to find t, then $a_n = t^n$ is a solution for

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

We call

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - \ldots - c_k = 0$$

the **characteristic equation** of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

The solutions to the characteristic equation are called the **characteristic roots**

- We consider a linear homogeneous recurrence relation of degree 2

  $a_n = Aa_{n-l} + Ba_{n-2}$ *for all* $n \geq 2$

  where A and B are real constants

- The **characteristic equation**

  $$t^2 - At - B = 0$$

  may have

  1) two distinct roots

  2) a single root

- **Theorem 1 (Distinct Roots Theorem)**

  Suppose a sequence $a_0, a_1, a_2, \ldots.$ satisfies a recurrence relation

  $$a_n = Aa_{n-1} + Ba_{n-2} \text{ for all } n \geq 2$$

  where A and B are real constants and B ≠ 0. If the characteristic equation

  $$t^2 - At - B = 0$$

  has two distinct roots r and s, then $a_0, a_1, a_2, \ldots.$ is given by the explicit formula

  $$\boldsymbol{a_n = Cr^n + Ds^n}$$ (前人证明的 theorem, 直接用就好)

  where C and D are determined by the values of a_0 and a_1.

- Example 1:

  $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$, and $F_0 = F_1 = 1$

  The characteristic equation is

  $t^2 - t - 1 = 0$

  The roots are

For $ax2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{1 \pm \sqrt{1-4(-1)}}{2} = \begin{cases} \frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{cases}$$

$$F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$$

- To find C and D, we have

$$F_0 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^0 + D\left(\frac{1-\sqrt{5}}{2}\right)^0 = C\cdot 1 + D\cdot 1 = C + D$$

$$F_1 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^1 + D\left(\frac{1-\sqrt{5}}{2}\right)^1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right)$$

To solve this system of 2 equations with 2 unknowns, from

$$C + D = 1$$

$$\Rightarrow \left(\frac{1+\sqrt{5}}{2}\right)C + \left(\frac{1+\sqrt{5}}{2}\right)D = \left(\frac{1+\sqrt{5}}{2}\right)$$

Then

$$D\left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right) = \left(\frac{1+\sqrt{5}}{2}\right) - 1$$

$$\Rightarrow \quad D\sqrt{5} = \left(\frac{1+\sqrt{5}}{2}\right) - 1$$

$$\Rightarrow D = \left(\frac{-1+\sqrt{5}}{2\sqrt{5}}\right)$$

Then $C = 1 - D = 1 - \left(\frac{-1+\sqrt{5}}{2\sqrt{5}}\right)$

$$\Rightarrow C = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

We can write

$$D = \left(\frac{-(1-\sqrt{5})}{2\sqrt{5}}\right)$$

- So

$$F_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{-(1-\sqrt{5})}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

After simplifying it, we get

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

for all $n \geq 0$.

- Example 2: → Degree = 2.

$$a_n = 5a_{n-1} - 6a_{n-2}\ ,\ a_0 = 9,\ a_1 = 20$$

The characteristic equation is

$$t^2 - 5t + 6 = 0$$

$\Rightarrow \quad (t-2)(t-3) = 0 \quad \Rightarrow$ two roots: $t = 2,\ t = 3$

$a_n = C2^n + D3^n$ for all $n \geq 0$.

To find C and D:

$9 = C + D, \ \Rightarrow 18 = 2C + 2D$

$20 = 2C + 3D$

Thus $D = 2,\ C = 7$ So $a_n = 7*2^n + 2*3^n$ for all $n \geq 0$

$t^2 - 5t + 6 = 0$

$r = 2 \quad s = 3.$

$a_n = C \cdot s^n + D \cdot r^n$

$\begin{cases} a_0 = C \cdot 3^0 + D \cdot 2^0 = 9 \\ a_1 = C \cdot 3^1 + D \cdot 2^1 = 20 \end{cases}$

$\begin{cases} C + D = 9 \\ 3C + 2D = 20 \end{cases}$

$\Rightarrow \begin{cases} C = 2 \\ D = 7 \end{cases}$

$\therefore a_n = 2 \cdot 3^n + 7 \cdot 2^n$

- **Theorem 2 (Single-Root Theorem)**

  Suppose a sequence $a_0, a_1, a_2, \ldots.$ satisfies a recurrence relation

  $$a_n = Aa_{n-l} + Ba_{n-2} \text{ for all } n \geq 2$$

  where A and B are real constants and $B \neq 0$. If the characteristic equation

  $$t^2 - At - B = 0$$

  $A^2+4B=0.$

  has a single (real) root, then $a_0, a_1, a_2, \ldots.$ is given by the explicit formula

  $$\boldsymbol{a_n = Cr^n + Dnr^n}$$

  where C and D are determined by the values of a_0 and any other known value of the sequence.

- Example

$$b_n = 4b_{n-l} - 4b_{n-2} \text{ for all } n \geq 2$$

with $b_0 = 1$, $b_1 = 3$.

The characteristic equation is

$$t^2 - 4t + 4 = 0$$

$\Rightarrow (t-2)^2 = 0 \quad \Rightarrow$ single root $t = 2$

The explicit formula is

$$b_n = C2^n + Dn2^n$$

where C and D are determined by the values of b_0 and b_1.

We have $1 = C$ and $3 = 2C + 2D$, so $D = ½$ and $C = 1$.

- Therefore

$$b_n = 2^n + (½)n2^n = (1 + n/2)\, 2^n$$

Theorem 1 can be generalised to the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

with characteristic equation

$$t^k - c_1\, t^{k-1} - c_2\, t^{k-2} - \ldots - c_k = 0$$

having *k* distinct roots.

Theorem 2 can be generalised to less than *k* distinct roots.

Proof of $a^{\log_b n} = n^{\log_b a}$

- Let $\mathrm{L} = \log_b n$, i.e. $b^{\mathrm{L}} = n$

$$\Rightarrow (b^{\mathrm{L}})^{\log_{\mathrm{b}} a} = n^{\log_b a}$$

$$\Rightarrow (b^{\log_b a})^{\mathrm{L}} = n^{\log_b a}$$

$$\Rightarrow a^{\mathrm{L}} = n^{\log_b a}$$

$$\Rightarrow a^{\log_b n} = \mathrm{n}^{\log_b a}$$

L'Hôpital's rule

L'Hôpital's rule states that for functions $f(x)$ and $g(x)$, if:

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} g(x) = \pm\infty$$

then:

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \lim_{n \to \infty} \frac{f'(n)}{g'(n)}$$

where the prime (') denotes the derivative.