NANYANG TECHNOLOGICAL UNIVERSITY

CE2101/ CZ2101: Algorithm Design and Analysis

Week 9

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Outline

- Quiz: Week 10 & Week 11
- Substitution Method
- Linear Homogeneous Recurrence Relation
- Dynamic Programming
- Extended Topics (not covered in the exam)

Please feel free to interrupt me if you have any questions :)

Substitution Method

Solving Recurrences – Substitution method

References for guess:

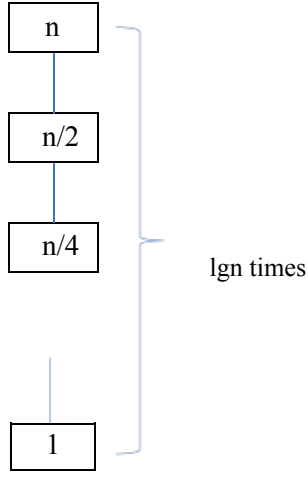
Binary search:

$W(n) = W(n/2) + 1,$

$W(1) = 1$

$W(n) =$

$O(\lg n)$



Solving Recurrences – Substitution method

References for guess:

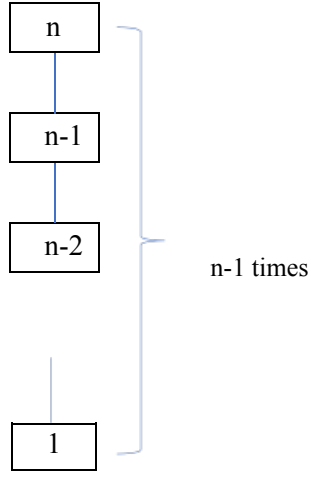
Insertion sort (best case)

$W(n) = W(n-1) + 1,$

$W(1) = 0$

$W(n) =$

$O(n)$



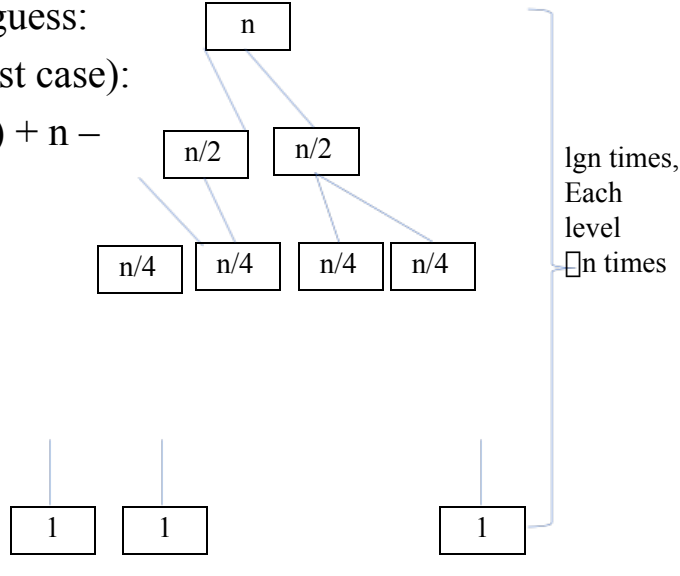
Solving Recurrences – Substitution method

References for guess:

MergeSort (worst case):

$W(n) = 2W(n/2) + n - 1$, $W(1) = 0$

$W(n) = O(n \lg n)$

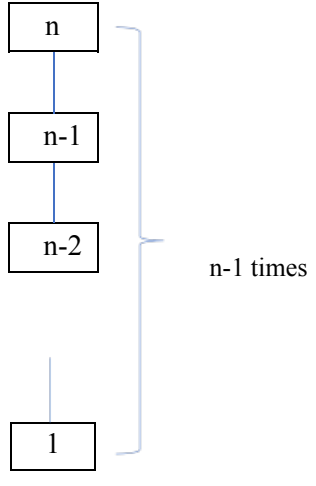


Solving Recurrences – Substitution method

References for guess:

QuickSort (worst case): $W(n) = W(n-1) + n - 1$, $W(1) = 0$

$W(n) =$
$O(n2)$



Solving Recurrences – Substitution method

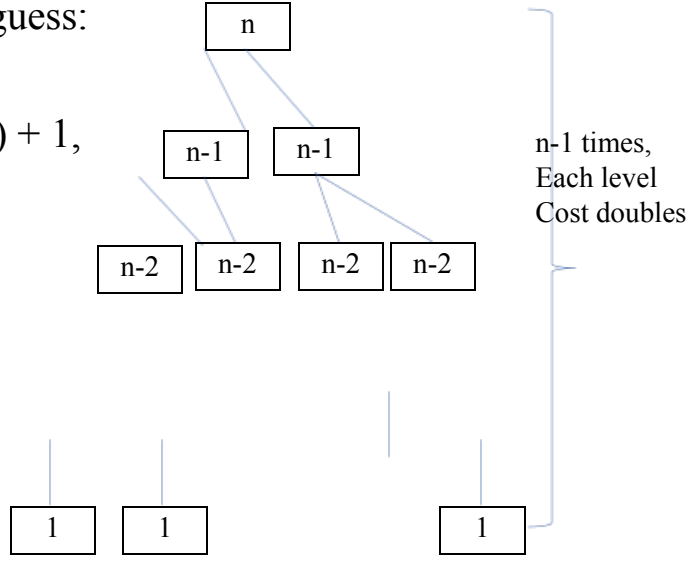
References for guess:

Tower of Hanoi

$W(n) = 2W(n-1) + 1,$

$W(1) = 0$

$W(n) =$
$O(2n)$



Linear Homogeneous Recurrence Relation

We consider a linear homogeneous recurrence relation of degree 2

$a_n = Aa_{n-1} + Ba_{n-2}$ for all $n \geq 2$

where A and B are real constants

The **characteristic equation**

$$t^2 - At - B = 0$$

may have

1) two distinct roots

2) a single root

Theorem 1 (Distinct Roots Theorem)

Suppose a sequence *a0, a1, a2, ….* satisfies a recurrence relation

an = Aan-I + Ban-2 for all n ≥ 2

where A and B are real constants and B ≠ 0. If the characteristic equation

t2 - At – B = 0

has two distinct roots *r* and *s*, then *a0, a1, a2, ….* is given by the explicit formula

an = Crn + Dsn

where C and D are determined by the values of *a0* and *a1*.

Theorem 2 (Single-Root Theorem)

Suppose a sequence *a0, a1, a2, ….* satisfies a recurrence relation

an = Aan-I + Ban-2 for all n ≥ 2

where A and B are real constants and B ≠ 0. If the characteristic equation

t2 - At – B = 0

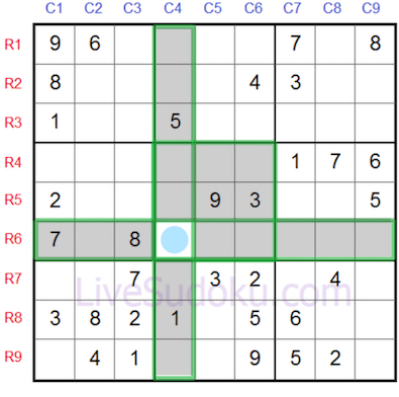
has a single (real) root, then *a0, a1, a2, ….* is given by the explicit formula

an = Crn + Dnrn

where C and D are determined by the values of *a0* and any other known value of the sequence.

Dynamic Programming

Sudoku


C1 C2 C3 C4 C5 C6 C7 C8 C9
R1 9 6 7 8
R2 8 4 3
R3 1 5
R4 1 7 6
R5 2 9 3 5
R6 7 8
R7 7 3 2 4
R8 3 8 2 1 5 6
R9 4 1 9 5 2
LiveSudoku.com

Dynamic Programming

Consider the Virahanka number V(n) defined by the following equations:

V(n) = 1, when n = 0 or 1;

V(n) = V(n-1) + V(⌊n/2⌋), when n>1;

(here ⌊n/2⌋ is the floor function for n/2. E.g. ⌊3/2⌋ =

⌊2/2⌋ =1).

Dynamic programming (Bottom Up)

Subproblem graphs

For a recursive algorithm A, the subproblem graph for A is a directed graph whose vertices are the instances for this problem. The directed edges (I, J) for all pairs that indicate: if A is invoked on problem I, it makes a recursive call directly on instance J.

E.g. the subproblem graph for fib(6):

Dynamic programming (Bottom Up)

1. Formulate the problem P in terms of smaller versions of the problem (recursively), say, Q1, Q2, …
2. Turn this formulation into a recursive function to solve problem P
3. Draw the subproblem graph and find the dependencies among subproblems
4. Use a dictionary to store solutions to subproblems
5. In the iterative function to solve P

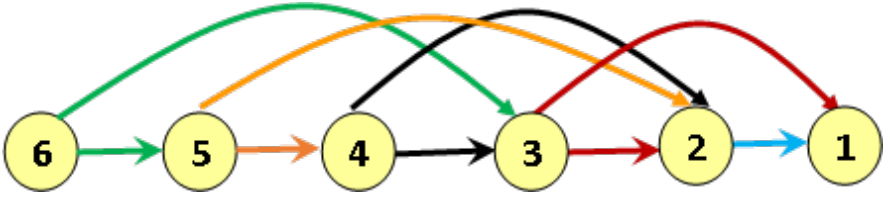
- compute the solutions of subproblems of a problem first
- The solution to P is computed based on the solutions to its subproblems and is stored into the dictionary

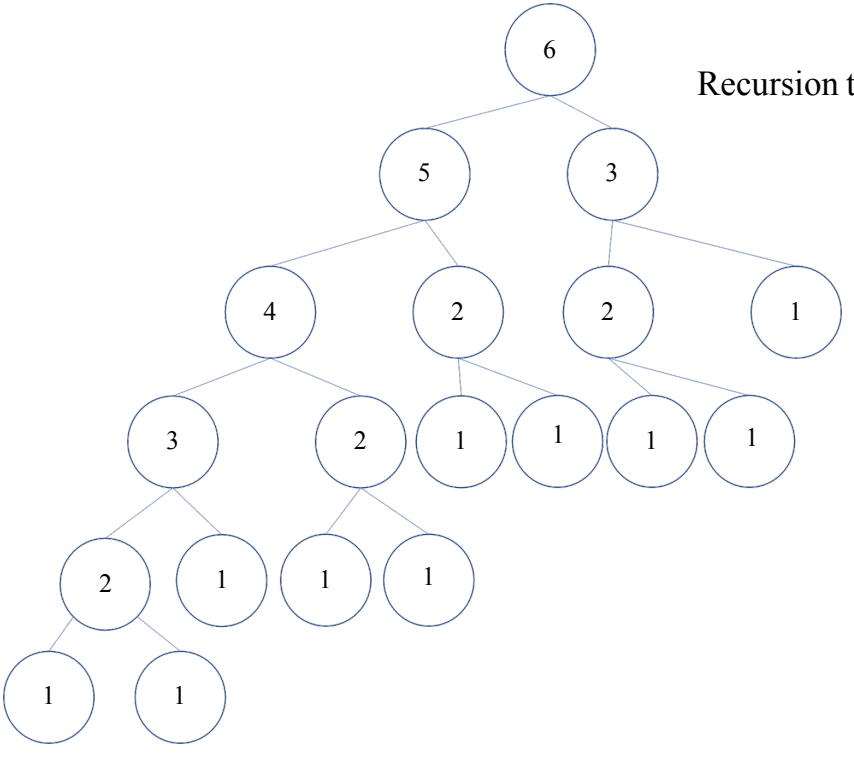
Dynamic Programming

V(n) = 1, when n = 0 or 1;

V(n) = V(n-1) + V(⌊n/2⌋), when n>1;

(i) Draw the subproblem graph for V(6).





Recursion tree

Dynamic Programming

V(n) = 1, when n = 0 or 1;

V(n) = V(n-1) + V(⌊n/2⌋), when n>1;

(ii)Give dynamic programming algorithm to compute V(n) using the bottom up approach.

```
v[0] = v[1] = 1
For (j= 2 to n)
    v[j] = v[j-1] +
v[j/2]; return v[n];
```

Dynamic Programming

$V(n) = 1$, when $n = 0$ or 1;

$V(n) = V(n-1) + V(\lfloor n/2 \rfloor)$, when $n>1$;

(iii) Give dynamic programming algorithm to compute V(n) using the top down approach.

Dynamic programming (Top Down)

1. Formulate the problem P in terms of smaller versions of the problem (recursively), say, Q1, Q2, …
2. Turn this formulation into a recursive function to solve problem P
3. Use a dictionary to store solutions to subproblems
4. In the recursive function to solve P

- Before any recursive call, say on subproblem Qi, check the dictionary to see if a solution for Qi has been stored
- If no solution has been stored, make the recursive call
- Otherwise, retrieve the stored solution
- Just before returning the solution for P, store the solution in the dictionary - memorization

```
// v array initialized to all -1 DP_V(n)
{
If (n == 0 or n == 1) { v[n] = 1; return 1 }
If (v[n-1] == -1) v1 = DP_V(n-1) else v1 = v[n-1]; If (v[n/2] == -1) v2 = DP_V(n/2) else v2 = v[n/2]; v[n] = v1 + v2;
Return v[n];
}
```

Thanks!

Q & A