CE2101/ CZ2101: Algorithm Design and Analysis

Mergesort

Ke Yiping, Kelly

Learning Objectives

At the end of this lecture, students should be able to:

- Explain the approach of Divide and Conquer
- Describe how Mergesort works by:
  - Recalling the pseudo code
  - Manually executing the algorithm on a toy input array
- Analyse the time complexity of Mergesort, by using:
  - Recurrence equation
  - Recursion tree

通常用递归 recursive function 来解决

The Divide and Conquer approach

The skeleton of this approach:

```
solve (problem of size n)
{   if (n <= minimum size)
        solve the problem directly;
    else {
            divide the problem into p1, p2, … , pk;
            for each sub-problem ps
                solutions = solve (ps);
            combine all solutions;
    }
}
```

usually, $k=2$

为什么不把k变得更大? 即多分成几份

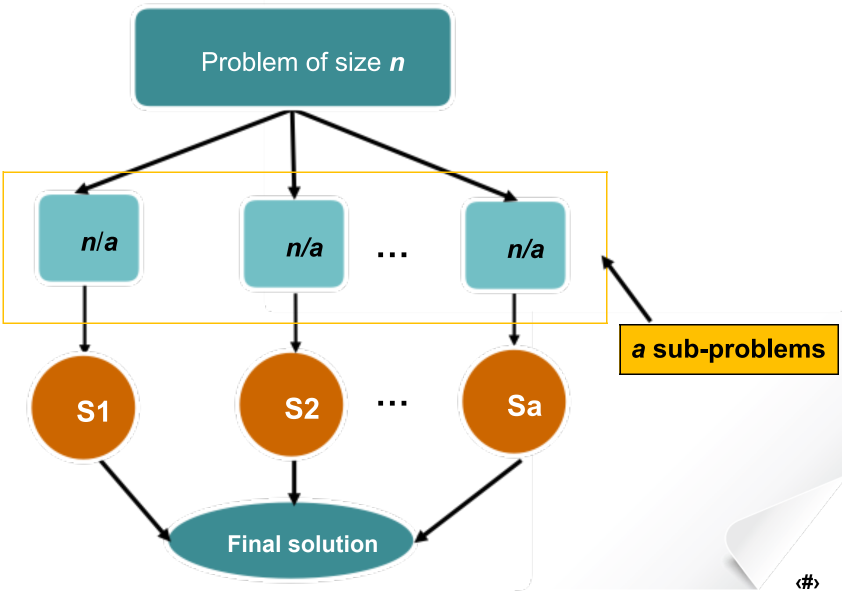
原因: 1. n分2份 → $\log_2 n$

2. n分k份 → $\log_k n$

$$\log_k n = \frac{\log_2 n}{\boxed{\log_k 2}}$$ → constant.

3. 效率变化不明显，而combine的成本明显↑.

Mergesort



a sub-problems

Mergesort (Algorithm)

Mergesort (Algorithm)

排序的基本base case都是这俩

Base case: n=0 or n=1

return

```
mergeSort(list) {
    if (length of list > 1) {
        Partition list into two (approx.) equal sized
            lists, L1 & L2;
        mergeSort (L1);
        mergeSort (L2);
        merge the sorted L1 & L2;
    }
}
```

all dirty work done in the function

Mergesort

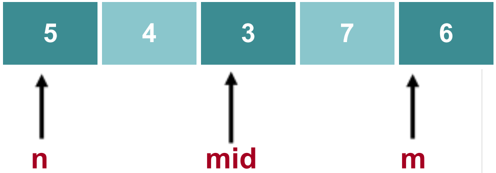
index

```
void mergesort(int n, int m)
{    int mid = (n+m)/2;
   if (m-n <= 0)
          return;
   else if (m-n > 1) {
      mergesort(n, mid);
      mergesort(mid+1, m);
   }
   merge(n, m);
}
```

不需要进入 recursive call

m-n =1 会直接，跳到这一步

5 | 4



if m-n = 0, 5

m = n

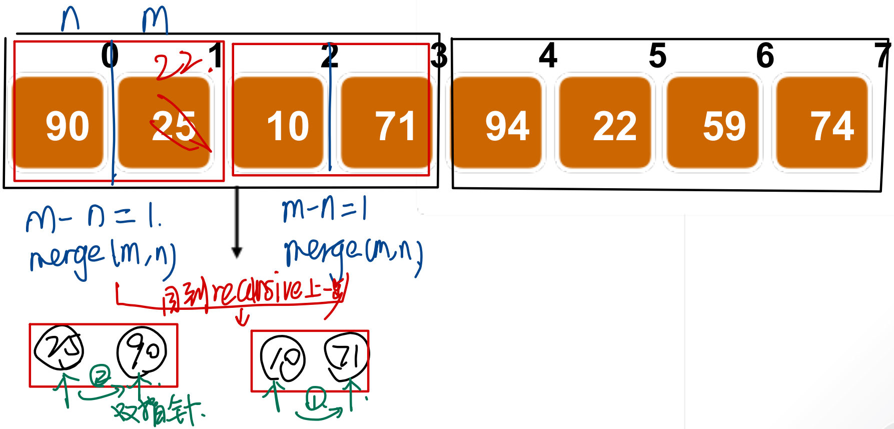
if m-n < 0, Empty array

问：如果列表有重复?

指针同时后移.

Mergesort

Sort in ascending order



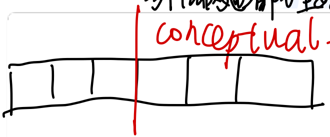
① 25<10? → 10

② 25<71? → 10 25.

Merge (Pseudo Code)

Merge (Pseudo Code)

为什么两边都理好了，还要分?



conceptual → 分隔两边列表

```
void merge(int n, int m) {
  if (m-n <= 0) return;
  divide the list into 2 halves;  // both halves are sorted
     while (both halves are not empty) {
               compare the 1st elements of the 2 halves;  // 1 comparison
     if (1st element of 1st half is smaller)
                    1st element of 1st half joins the end of the merged list;  ?
     else if (1st element of 2nd half is smaller)
                         move the 1st element of 2nd half to the end of the
        merged list;  ?
```

Merge (Pseudo Code)

ctie).

```
else {   // the 1st elements of the 2 halves are equal
              if (they are the last elements)   break;
            1st element of 1st half joins end of the merged list;
            move the 1st element of 2nd half to the end of the
   merged list;
    }
  } // end of while loop;
} // end of merge
```

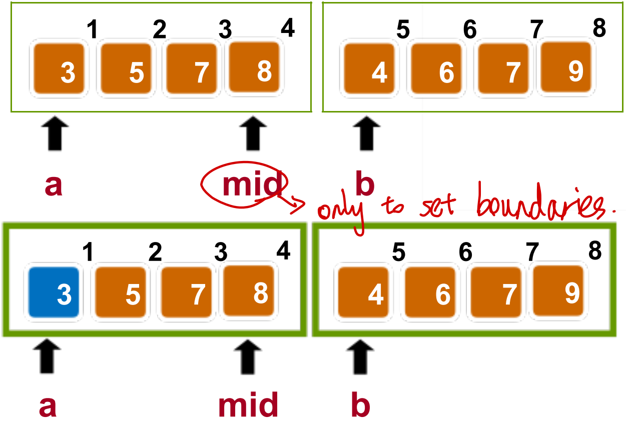
Challenge:
How to do it without auxiliary storage for the merged list?

Merge (Case Scenarios)

(without additional space)

Merge (Case Scenarios)

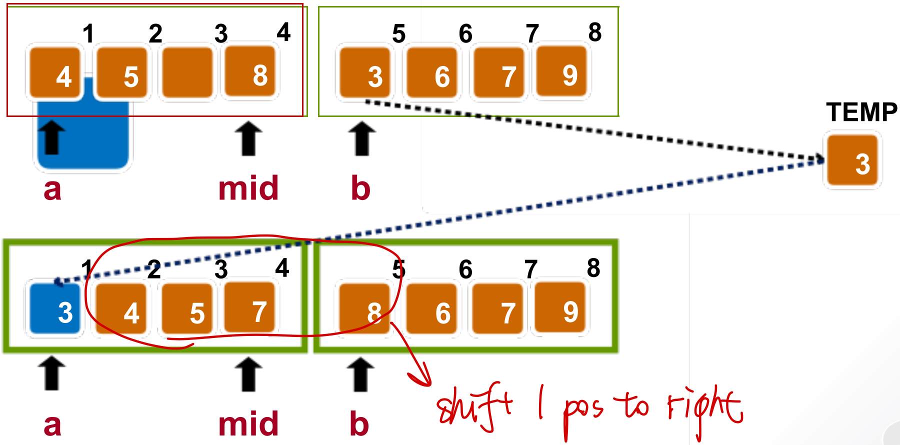
Case 1: 1st element of 1st half is smaller



only to set boundaries.

Merge (Case Scenarios)

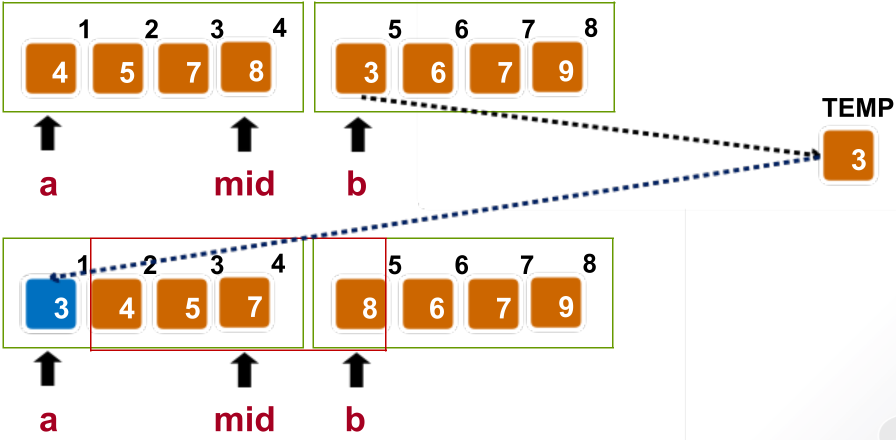
Case 2: 1st element of 2nd half is smaller



shift 1 pos to right

Merge (Case Scenarios)

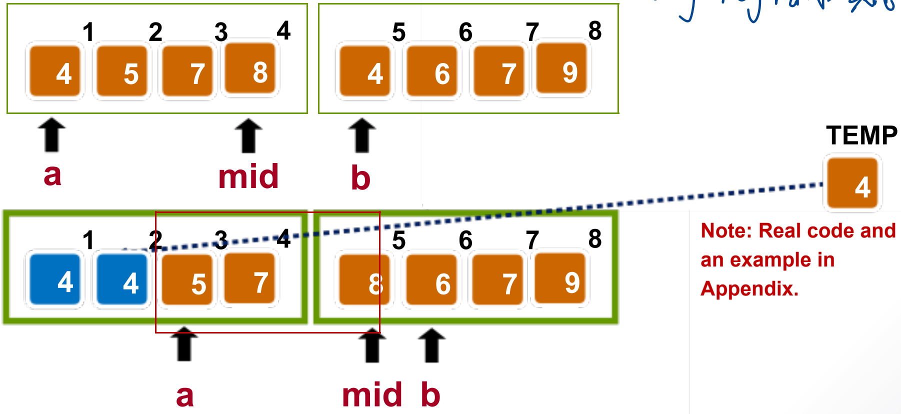
Case 2: 1st element of 2nd half is smaller



Merge (Case Scenarios)

Case 3: **1st element of 2nd half is equal** → cannot move and copy

(only key相同，其他可能会不一样)



Note: Real code and an example in Appendix.

Complexity:

Mergesort Algorithm (Recap)

Mergesort Algorithm (Recap)

- Since merging is performed directly on the original array, swapping and shifting are needed
- mergesort() partitions a contiguous array of elements between index n and m into two subarrays

```
void mergesort(int n, int m)
{   int mid = (n+m)/2;
   if (m-n <= 0)
      return;
   else if (m-n > 1) {
      mergesort(n, mid);
      mergesort(mid+1, m);
   }
   merge(n, m);
}
```

Mergesort Algorithm (Recap)

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- Recursively partitions until m-n<=0, then merge the resulting two subarrays

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void mergesort(int n, int m)
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  …..
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```

Mergesort Algorithm (Recap)

- Since merging is performed directly on the original array, swapping and shifting are needed
- mergesort() partitions a contiguous array of elements between index n and m into two subarrays
- Recursively partitions until m-n<=0, then merge the resulting two subarrays
- merge() function merges two sub-arrays of elements between index n and ‘mid’, and between ‘mid+1’ and m

```
void mergesort(int n, int m)
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   if (m-n <= 0)
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  merge(n, m);
}
```

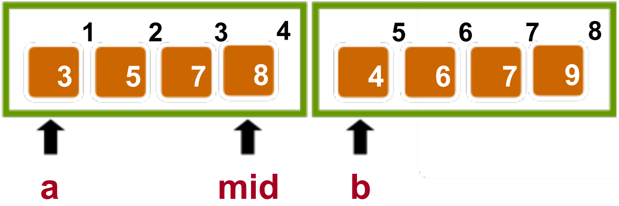
Mergesort Algorithm (Recap)

- Since merging is performed directly on the original array, swapping and shifting are needed
- mergesort() partitions a contiguous array of elements between index n and m into two subarrays
- Recursively partitions until m-n<=0, then merge the resulting two subarrays
- merge() function merges two sub-arrays of elements between index n and 'mid', and between 'mid+1' and m
- During merging, one element from each subarray is compared and the smaller one is inserted into new list

```
void mergesort(int n, int m)
{   .....
   merge(n, m);
}
```

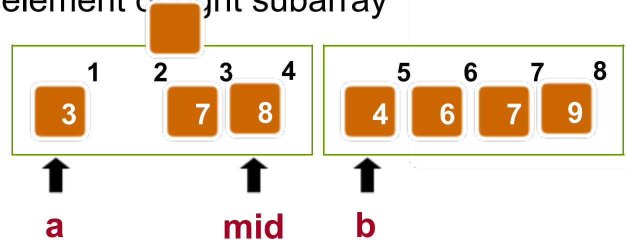
Mergesort Algorithm (Recap)

- Left subarray runs from n to ‘mid’ with a as running index; right subarray runs from mid+1 to m with b as running index



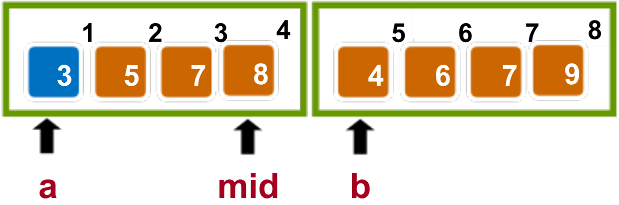
Mergesort Algorithm (Recap)

- Left subarray runs from n to ‘mid’ with a as running index; right subarray runs from mid+1 to m with b as running index
- slot[a] is the head element of left subarray, slot[b] is the head element of right subarray



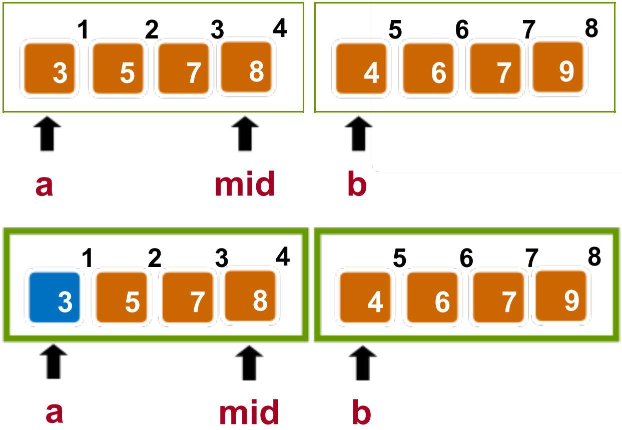
Mergesort Algorithm (Recap)

- Left subarray runs from n to 'mid' with a as running index; right subarray runs from mid+1 to m with b as running index
- slot[a] is the head element of left subarray, slot[b] is the head element of right subarray
- During merging, both left and right subarrays shrink towards the right to make space for the newly merged array



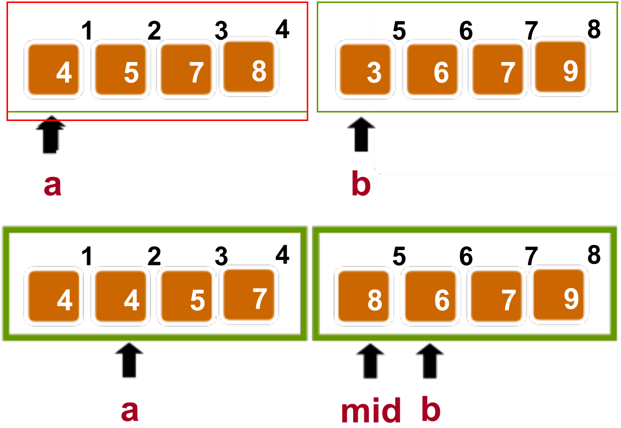
Mergesort Algorithm (Recap)

Case 1: if slot[a] < slot[b], there is nothing much to do since smaller element already in correct position (with regard to the merged array)



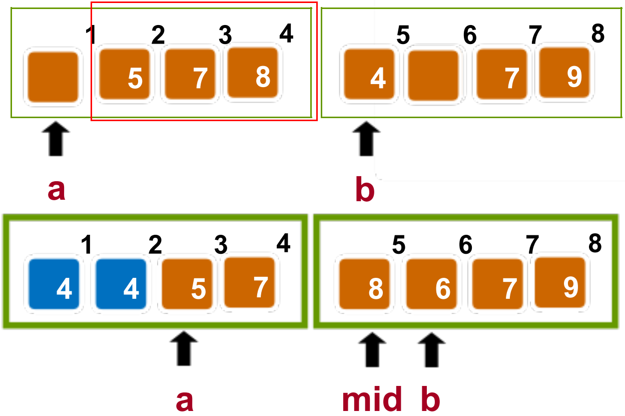
Mergesort Algorithm (Recap)

Case 2: if slot[a] > slot[b], then Right-shift (by one) elements of left subarray from index a to 'mid' and insert element at slot[b] into slot[a]



Mergesort Algorithm (Recap)

Case 3: if slot[a] == slot[b], then slot[a] is in the correct position. So, move slot[b] next to beside slot[a], by Right-shifting and swapping



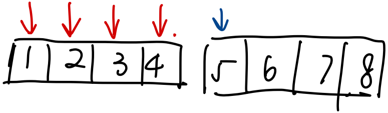
Complexity of Mergesort

Complexity of Mergesort

-. Complexity of Merge Function : worst $(n-1)$ best $(\frac{n}{2})$. $\}$ $O(n)$

- After each comparison of keys from the two sub-lists, at least one element is moved to the new merged list and never compared again
- After the last key comparison, at least two elements will be moved into the merged list
- Thus, to merge two sub-lists of n elements in total, the number of key comparisons needed is at most $n - 1$

at least $= \frac{n}{2}$.



二. Complexity of Mergesort.

```
void mergesort(int s, int e) // s=start, e=end
{   int mid = (s+e)/2;
    if (e-s <= 0) return;
    else if (e-s > 1) {
        mergesort(s, mid);
        mergesort(mid+1, e);
    }
    merge(s, e);
}
```

time complexity,

if (e-s <= 0) return; → $\boxed{W}(1) = 0$

mergesort(s, mid); → W(n/2)

mergesort(mid+1, e); → W(n/2)

W(n)

merge(s, e); → Worst case: n-1

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Complexity of Mergesort

Mergesort performance (assume n = 2k)

k = lg n

Worst case :

W(1) = 0, ↗first half ↗second half merge.

$W(n) = \boxed{W(n/2)} + \boxed{W(n/2)} + \boxed{n-1}$ Or

$W(2k) = 2W(2k-1) + 2k -1$

$= 2(2W(2k-2) + 2k-1 -1) + 2k -1$

$= 22W(2k-2) + 2k -2 + 2k -1$

$= 22(2W(2k-3)+ 2k-2 -1) + 2k -2 + 2k -1$

$= 23W(2k-3)+ 2k -22 + 2k -2 + 2k -1$

…

$= 2kW(2k-k) + k2k – (1 + 2 + 4 + … + 2k-1)$

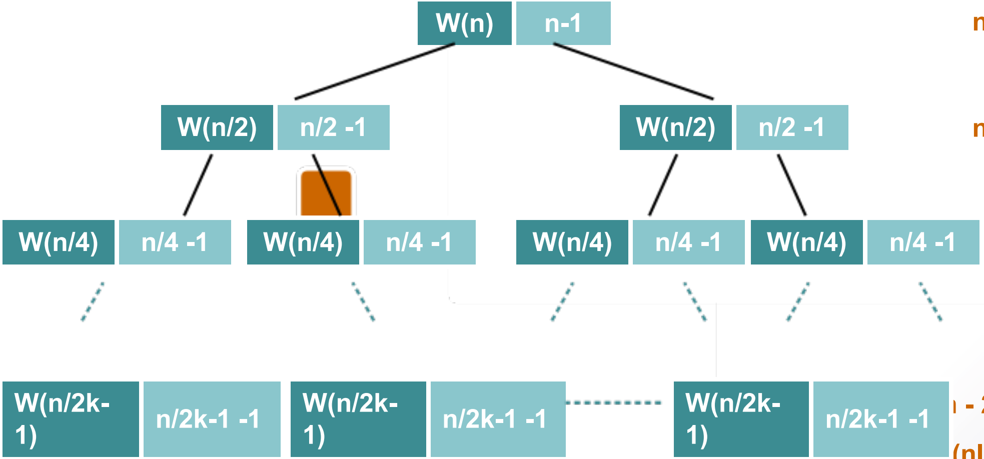
$= k2k – (2k – 1)$

$= n \lg n – (n – 1)$

$= O(n \lg n)$

Geometric series

Visually :Recursion Tree



W(n) n-1

W(n/2) n/2 -1 W(n/2) n/2 -1

W(n/4) n/4 -1 W(n/4) n/4 -1 W(n/4) n/4 -1 W(n/4) n/4 -1

W(n/2k-1) n/2k-1 -1 W(n/2k-1) n/2k-1 -1 ---------- W(n/2k-1) n/2k-1 -1

n-1

n-2

ı-4

ı - 2k-

(nlgn)

W(2) = 2W(1) + 1 = 1

Height of tree is $k = O(\lg_2 n)$

Evaluation of Mergesort

- **Strengths:**
- Simple and good runtime behavior
- Easy to implement when using linked list
- **Weaknesses:**
- Difficult to implement for contiguous data storage such as array without auxiliary storage (requires data movements during merging)

Summary

- Mergesort uses the Divide and Conquer approach.
- It recursively divide a list into two halves of approximately equal sizes, until the sub-list is too small (no more than two elements).
- Then, it recursively merges two sorted sub-lists into one sorted list.
- The worst-case running time for merging two sorted lists of total size n is $n – 1$ key comparisons.
- The running time of Mergesort is $O(n\lg n)$.

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Appendix

(Merge operation in Mergesort)

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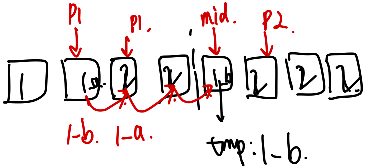
Merge Function

```
void merge(int n, int m)
  {
     int mid = (n+m)/2;
      int a = n, b = mid+1, i, tmp;
      if (m-n <= 0) return;
     while (a <= mid && b <= m) {
      cmp = compare(slot[a], slot[b]);
      if (cmp > 0) { //slot[a] > slot[b]
               tmp = slot[b++];
             for (i = ++mid; i > a; i--)
               slot[i] = slot[i-1];
```



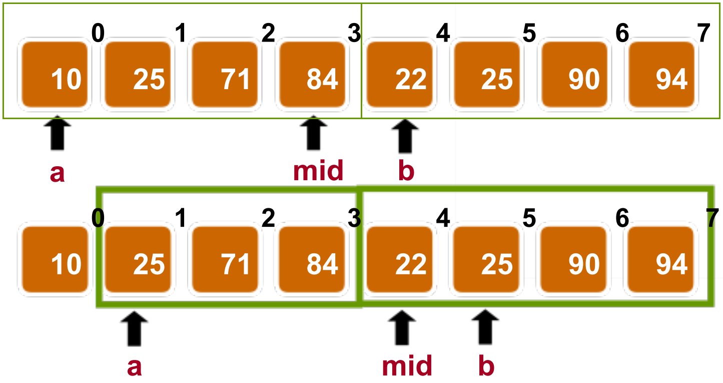
Question:

why is merge sort stable?

Merge Function

```
          slot[a++] = tmp;
  } else if (cmp < 0) //slot[a] < slot[b]
       a++;
   else {   //slot[a] == slot[b]
             if (a == mid && b == m)
          break;
             tmp = slot[b++];
             a++;
             for (i = ++mid; i > a; i--)
          slot[i] = slot[i-1];
             slot[a++] = tmp;
      }
  } // end of while loop;
} // end of merge
```

Merge Operation



a : the 1st element of the 1st half

mid : the last element of the 1st half

b : the 1st element of the 2nd half

Parameters for merge:

n:0, m: 7

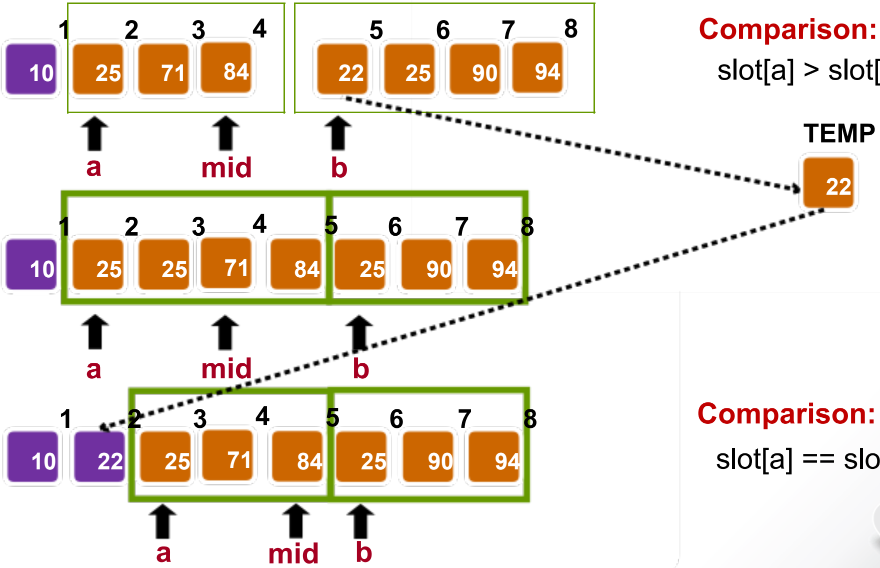
mid = (0+7)/2 = 3;

a = n; **b** = mid+1;

Comparison:

slot[a] < slot[b]

Merge Operation



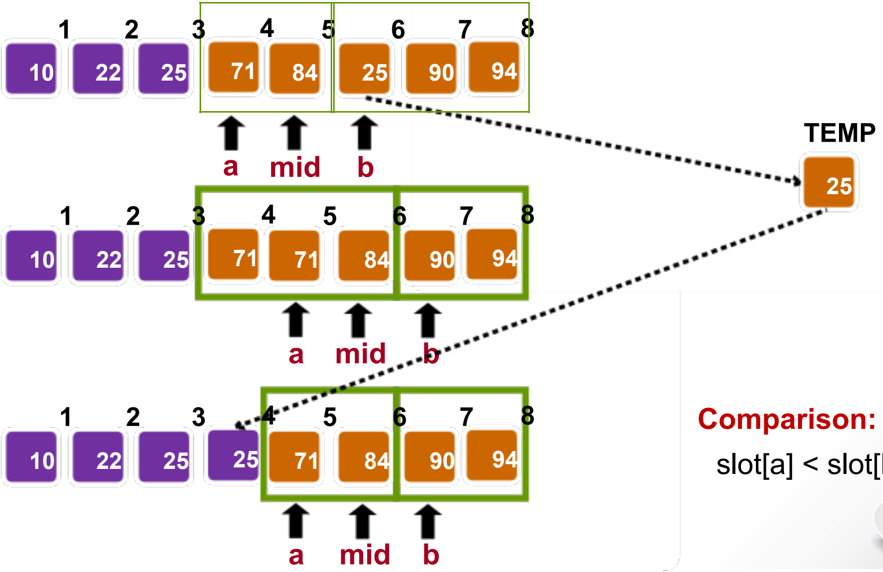
Comparison:

slot[a] > slot[b]

Comparison:

slot[a] == slot[b]

Merge Operation



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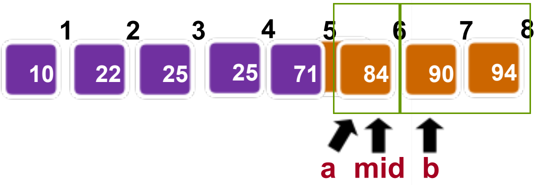
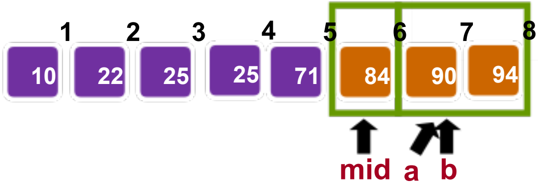
Merge Operation



Comparison:

slot[a] < slot[b]



1st half empty

Merge operation completed