CE2101/ CZ2101: Algorithm Design and Analysis

Week 5: Review Lecture

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Content

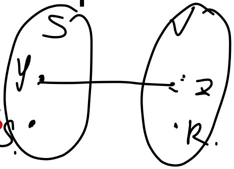
- Dijkstra’s Correctness
- Prim’s Algorithm and Its Correctness

Dijkstra's Correctness

Lemma 1: The sub-path of a shortest path is also shortest. min: $d(y)+W(y,z)$

- Prove by contradiction. $\forall y \in S,\ z \in V-S$

usage: already have a shortest path, extend on it.



Theorem D1: Assume that the shortest distances in S are correctly computed. The greedy step that chooses

→ for any y, any z.

$$\min\{d[y] + W(y, z) \mid \forall y \in S,\ z \in (V - S)\},$$

produces the shortest distance for z.

Dijkstra's Correctness

Proof of Theorem D1 (continued)

Let P be a shortest path from **s** to **y** followed by edge (y, z)

Let W(P) = the distance travelled along P

Let P' = any shortest path different from P, i.e., P' = s, z1, …, zk, …, z

Assume that zk is the first vertex in P' not in set S.

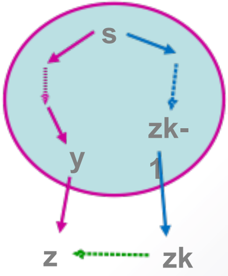
W(P) = d[y] + W(y, z)

W(P') = d[zk–1] + W(zk–1, zk) + distance from zk to z

Note that: d[zk–1] + W(zk–1, zk) □ d[y] + W(y, z)

Since distance from zk to z is non-negative,

therefore, **W(P) <= W(P')**.



Dijkstra's Correctness

Theorem D2: Dijkstra's algorithm computes the shortest distances for every vertex correctly.

- Prove by mathematical induction on the size of S.
- The inductive step applies Theorem D1.

Theorem D2 and Proof

Theorem D2: Given a directed weighted graph G with nonnegative weights and a source vertex s, Dijkstra's algorithm computes the shortest distance from s to each vertex of G that is reachable from s.

Proof (By induction):

We will show by induction that as each vertex v is added into set S, d[v] is the shortest distance from s to v.

Basis: (Base Case)

The algorithm assigns d[s] to zero when the source vertex s is added to S. So d[s] is the shortest distance from s to s when S has the first vertex in it.

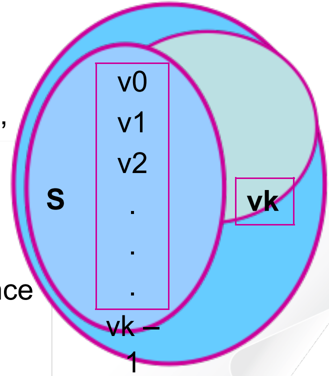
Theorem D2 and Proof (Continued)

Inductive Hypothesis: ✓ level 1: Proof by contradiction

Assume the theorem is true when S has **k** vertices. That is, assume v0, v1, v2, …, vk–1 are added where d[v1], d[v2]… are the shortest distances.

When vk is chosen by Dijkstra's algorithm, it means an edge (vi, vk), where i ∈ {0, 1, 2, …, k – 1}, is chosen to minimise d[vi] + W(vi, vk) among all edges with one vertex in S and one vertex not in S.

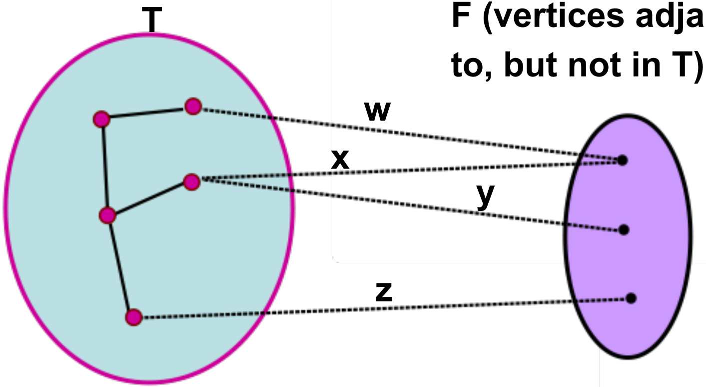
By Theorem D1, d[vk] is the shortest distance from source to vk. So the theorem is true when S has **k + 1** vertices.



Prim's Algorithm

- To compute a minimum spanning tree on an undirected, weighted graph.
- The input graph must be connected.

Main Idea of Prim's Algorithm

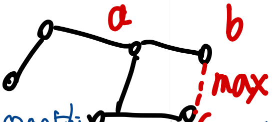


Choose min(w, x, y, z)

Prim's Correctness

Need this to help prove correctness of Prim's Algo.

MST property: If we add any edge that is in G but not in a spanning tree *T*, this edge will create a cycle in *T* and carries the max weight in this cycle.



$(b,c) \in E(G)$

引理: used to prove something more important

Lemma 1: If two spanning trees in G both have the MST property, then they have the same total weight.

- Prove by mathematical induction on the number of differing edges.

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Prim's Correctness

Theorem 1: MST = MST property

- Only if: prove by contradiction. (If MST doesn't have MST property)
- If: prove by "only if" result and Lemma 1.

(For any intermediate step, the sub-tree have MST properties)

Lemma 2: The tree constructed at every step of Prim's has the MST property in induced subgraph.

- Proof not required.

At last step, the tree would have MST property

→ Prove by Lemma 2 and Theorem 1 tree with MST property is MST

Theorem 2: Prim's correctly computes the MST.

Exercise

- Given a directed graph G with non-negative edge weights, Dijkstra's algorithm is used to compute a shortest path P from a source vertex s to a target vertex t. If the weight of every edge in G is increased by 10, does P remain the shortest path from s to t in the modified graph? If your answer is "yes", give a proof. Otherwise, give a counterexample. **[AY1920S1]**

No. Assume there are l edges $(e_1, e_2, e_3, \ldots e_l)$.

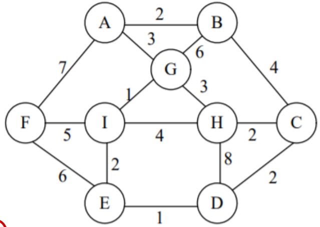
Original G: shortest P: $w(P) = \sum_{i=1}^{l} w(e_i)$

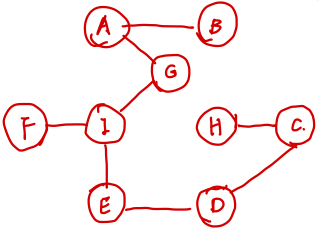
New Graph G': $w(P') = \sum_{i=1}^{l} w(e_i) + 10 = 10l + w(P)$

Counter example:

Exercise

- Execute Prim's algorithm on G in the figure starting at vertex A. Draw the minimum spanning tree obtained. **[AY1920S1]**





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Exercise

- Does Prim's algorithm work if some edges in the input graph have negative weights? If your answer is "yes", give a proof. Otherwise, give a counterexample. **[AY1819S1]**

Yes

By theorem 1 and Lemma 2, can prove the correctness of Prim's ...

In all prove, there's no special mentioning on the sign of weight.

need to point out to get full mark

No need to go into details.