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# CE2101/ CZ2101: Algorithm Design and Analysis

## Quicksort

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# Learning Objectives

At the end of this lecture, students should be able to:

- Explain how “Divide and Conquer” approach is used in Quicksort
- Explain the pseudo code of Quicksort
- Manually execute Quicksort on an example input array
- Analyse time complexities of Quicksort in the best, average and worst cases

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# Quicksort

- Fastest general purpose in-memory sorting algorithm in the average case
- Implemented in Unix as qsort() which can be called in a program (see 'man qsort' for details)
- Main steps
  - Select one element in array as pivot
  - Partition list into two sublists with respect to pivot such that all elements in left sublist are less than pivot; all elements in right sublist are greater than or equal to pivot
  - Recursively partition until input list has one or zero element
- No merging is required because the pivot found during partitioning is already at its final position

Since we do it recursively, every element should have a chance to be pivot.

# Quicksort (Pseudo Code)

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# Quicksort (Pseudo Code)

start index

```
void quicksort(int n, int m)
{
    int pivot_pos;
    if (n >= m)
        return;
    pivot_pos = partition(n, m);
    quicksort(n, pivot_pos - 1);
    quicksort(pivot_pos + 1, m);
}
```

end index

(Do all the dirty work).

范围

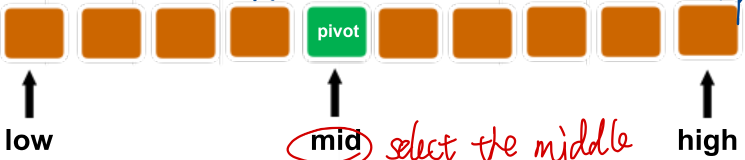
→此函数作用:找到n,m中的pivot position. 小于该pivot的移左,大于的移右. 最后返回pivot position.

excludes pivot itself

# Partition Routine in Quicksort

# Partition Routine in Quicksort

use this version !!! (select middle as pivot)




select the middle as pivot.

**int partition(int low, int high)**

```
{
    int i, last_small, pivot;
    int mid = (low+high)/2;
    swap(low, mid);
    pivot = slot[low];
    last_small = low;
```

traverse index

# define boundry

key value of pivot (not index).

point to the last index of the list containg "smallgup"

# move the pivot to begining of the list.

(swap content, not index)

# Partition Routine in Quicksort

after swap, pivot at first. (Don't need to check for pivot later on)

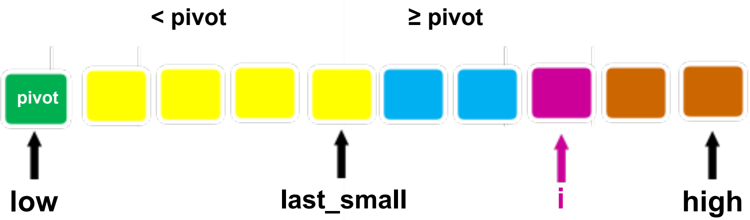

Initial State

```
int partition(int low, int high)
{.......
    for (i = low+1; i <= high; i++)
        if (slot[i] < pivot)
            swap(++last_small, i);
    swap(low, last_small);
    return last_small;
}
```

* Let's look at a general case to see if the code is correct.

# Partition Routine in Quicksort

General Case.




```
int partition(int low, int high)
{.......
        for (i = low+1; i <= high; i++)
          if (slot[i] < pivot)
               swap(++last_small, i);
     swap(low, last_small);
        return last_small;
 }
```

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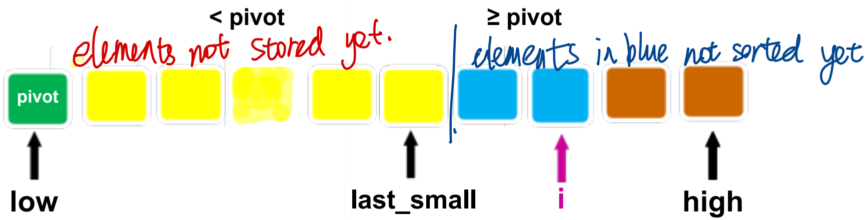
# Partition Routine in Quicksort

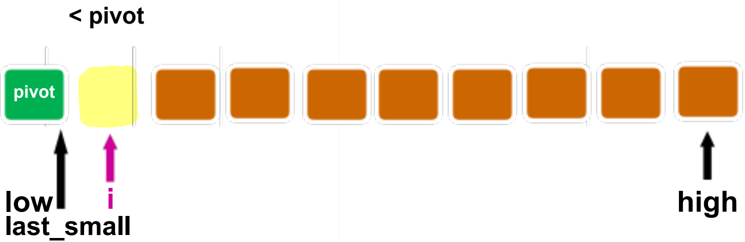



```
int partition(int low, int high)
{.......
      for (i = low+1; i <= high; i++)
        if (slot[i] < pivot)
            swap(++last_small, i);
      swap(low, last_small);
        return last_small;
}
```

→ This make Quicksort unstable.

Might change the initial order of

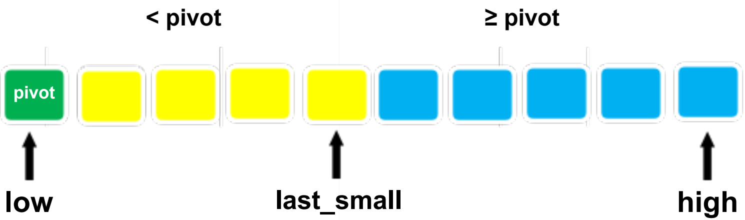
# Partition Routine in Quicksort




```
int partition(int low, int high)
{.......
        for (i = low+1; i <= high; i++)
          if (slot[i] < pivot)
                swap(++last_small, i);
      swap(low, last_small);
          return last_small;
 }
```

如果第一个元素 < pivot.
进行一次 dummy swap. (可接受)

# Partition Routine in Quicksort




```
int partition(int low, int high)
{.......
        for (i = low+1; i <= high; i++)
          if (slot[i] < pivot)
                swap(++last_small, i);
      swap(low, last_small);
        return last_small;
}
```

**Note:**

- Loop terminates when *i* reaches high;
- swap **pivot** from position low to position last_small, to obtain the final position of pivot element.

# Quicksort (Example)

# Quicksort (Example)

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Start** | 77 | 15 | 96 | 89 | 42 | 80 | 35 | 04 | 93 | 06 |
| | | | | | ↑ pivot | | | | | |
| **Swap** | 77 | 15 | 96 | 89 | 42 | 80 | 35 | 04 | 93 | 06 |

**Partition the elements …**

# Quicksort (Example)

## Partitioning…

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 42 | 15 | 96 | 89 | 77 | 80 | 35 | 04 | 93 | 06 |
| last_small | i | | | | | | | | |

last_small

35 04 06

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 42 | 15 | 96 | 89 | 77 | 80 | 35 | 04 | 93 | 06 |
| | last_small | i | | | i | | | | |

96 89 77

最终: 06 15 35 04 42 80 96 89 93 77

pivot

We have done 9 comparisons.

# Quicksort (Example)

**Step 1:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 42 | 15 | 96 | 89 | 77 | 35 | 04 | 93 | 06 | |

**After partitioning…**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 06 | 15 | 35 | 04 | 42 | 80 | 96 | 89 | 93 | 77 |

**9** comparisons

**Step 2:**

**Swap**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 15 | 06 | 35 | 04 | 42 | | | | | |

**Insert**

| 0 | 1 | 2 | 3 | 4 | | | | | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | | | | | |

**3** comparisons

> **Recursively call**
> **Quicksort (low,pivot_pos-1);**
> **Ignore RHS for time being**

# Quicksort (Example)

already in pivot position

**Step 3:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 80 | 96 | 89 | 93 | 77 |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 80 | 96 | 89 | 93 | 77 |

**1 comparison**

**Step 4:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 80 | 96 | 89 | 93 | 77 |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 80 | 96 | 89 | 93 | 77 |

**0 comparison**

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# Quicksort (Example)

**Step 5:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 06 | 15 | 35 | 42 | 80 | 96 | 89 | 93 | 77 | |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 80 | 96 | 89 | 93 | 77 |

**0 comparison**

**Sorting of LHS completed**

# Quicksort (Example)

**Dealing with right half of the array:**

**Step 6:**

| 0 | 1 | [redacted] | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 89 | 96 | 80 | 93 | 77 |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

4 comparisons

**Step 7:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

1 comparison

# Quicksort (Example)

**Dealing with right half of the array:**

Quick Sort is not very efficient in smallsize.

**Step 8:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

0 comparison

**Step 9:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

1 comparison

# Quicksort (Example)

**Step 10:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

0 comparison

**Final outcome:**

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 04 | 06 | 15 | 35 | 42 | 77 | 80 | 89 | 93 | 96 |

# Comments on Quicksort

- **Which element of array should be pivot?** In this implementation, we take the middle element as pivot (other choices possible).
- Use quicksort(0, size − 1) to invoke quick sort; ‘size’ is the number of elements in array slot[ ].
- During partitioning, the middle element (pivot) is moved to the 1st position (i.e. slot[0]).
- A ‘for’ loop goes through the rest of array to split it into two portions.

• Not efficient in sorting small dataset

• Usually used in hybrid mode

large size: QuickSort

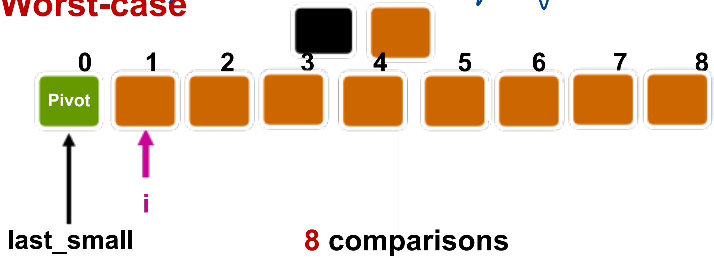
small size: InsertionSort.

# Quicksort’s Performance

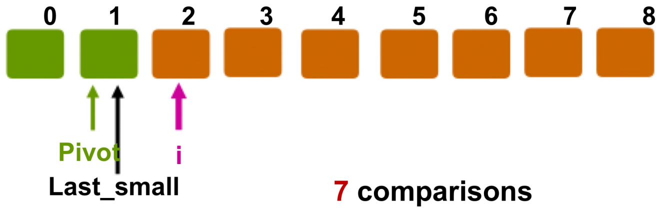
# Quicksort's Performance

## Worst-case

→ pivot is the smallest/largest element in the sublist



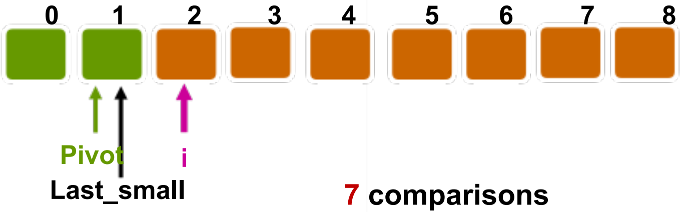

**8 comparisons**




**7 comparisons**

# Quicksort's Performance

## Worst-case




7 comparisons

# Quicksort's Performance

**Worst case** happens when the pivot does a bad job at splitting the array evenly, if pivot is the smallest or the largest key each time, then the total no. of key comparisons is O($n$2).

$$\sum_{k=2}^{n}(k-1) = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$$

# Quicksort's Performance

**Best case** happens when the pivot happens to divide the array into two sub-arrays of equal length, in every partitioning.

For simplicity, let's assume:

- ~~$n = 2k$~~ $n=2^k$, i.e. $k = \lg n$.
- Each step, the pivot divides the array of length $n$ into two sub-arrays each of length approximately $n/2$.

□□□□ … □ $(n-1)$

Recursive function:

$$\begin{cases} T(1)=0 \\ T(n) = 2T(\frac{n}{2}) + c(n-1) \end{cases}$$

$n-1$次 key comparison

解: $T(n) = 2T(\frac{n}{2}) + cn$ 为了更好计算, $n-1$ 简化为 $n$

$= 2\cdot[2T(\frac{n}{2^2}) + c\cdot\frac{n}{2}] + cn = 2^2T(\frac{n}{4}) + 2cn$

$= 4(2T(\frac{n}{8}) + c\cdot\frac{n}{4}) + 2cn = 8T(\frac{n}{8}) + 3cn$

⋮

$= \underbrace{\boxed{2^kT(\frac{n}{2^k})}}_{=0} + kcn.$

$\therefore T(n) = cn\cdot\lg n = n\lg n$

# Quicksort's Performance

The recurrence equati[redacted]:

T(1) = 0,

T(n) = 2T(n/2) + cn, where c is a constant

T(n) = 2 (2T(n/4) + cn/2) + cn

= 22T(n/4) + 2cn

= 23T(n/8) + 3cn

…

= 2k T(n/2k) + kcn

= nT(1) + cnlgn = cnlgn

$$\therefore \; T(n) = \Theta(n \lg n)$$

**Because n $\approx 2^k$, i.e. k = lg n, and T(1) = 0**

# Quicksort's Performance

**Average case:** assume that the keys are distinct and that all permutations of the keys are equally likely.

*k* = no. of elements in the range of the array being sorted,

$A(k)$ = no. of comparisons done for this range, — $T(k)$, $W(k)$

*i* = final position of the pivot, counting from 0,

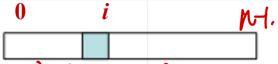
A(n)

A(i) A(n-i-1)

For partition function:

n element → n-1 comparisons.

(compare each value to pivot value).



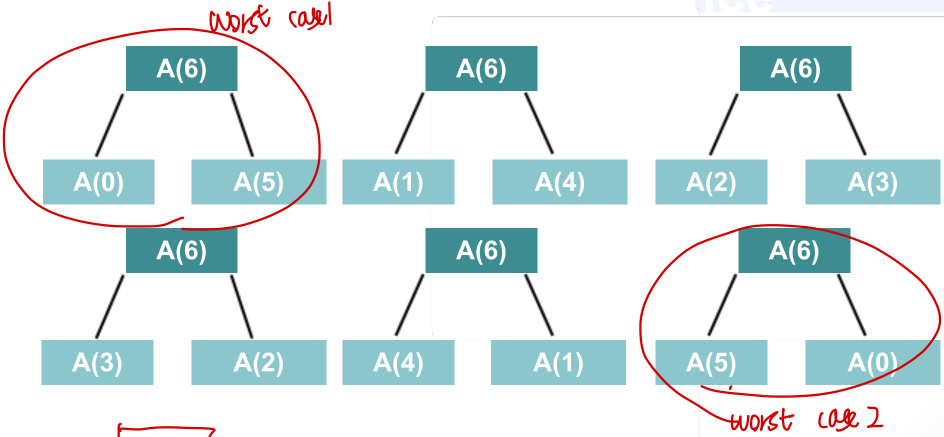

<pivot

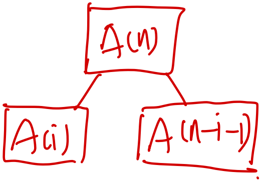
*i* elements

>pivot.

*n* – *i* – 1 elements

# Quicksort's Performance



# Quicksort's Performance

Thus,

cost of partition for n=6.

A(6) = 5 + 1/6( A(0) + A(5) + A(1) + A(4) + A(2) + A(3) + … + A(5) + A(0))

A(5) = 4 + 2[A(0)+A(4)] + 2[A(1)+A(3)] + A(2)+A(2)) × 1/5

$$A(0) = A(1) = 0$$

$$A(n) = n - 1 + \frac{1}{n}\sum_{i=0}^{n-1}\left[A(i) + A(n-i-1)\right] = \Theta(n \lg n)$$

Proof is not required

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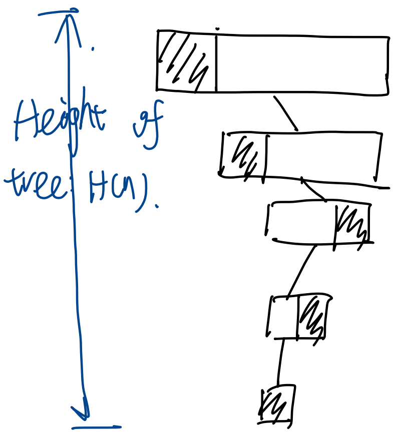
# Quicksort's Performance

QuickSort usually used in hybrid-algo.

- Strengths:
  - Fast on average
    Constant subsumed by big-O notation, $\Theta(n\lg n)$. (Why unique?)
  - No merging required
  - Best case occurs when pivot always splits array into equal halves
- Weaknesses:
  - Poor performance when pivot does not split the array evenly
  - Quicksort also performs badly when the size of list to be sorted is small
  - If more work is done to select pivot carefully, the bad effects can be reduced
    pivot can be sorted more wisely

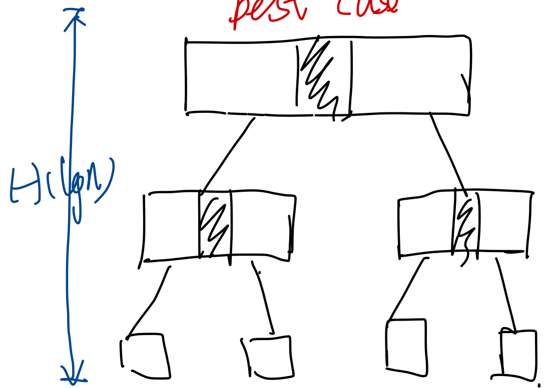
# Recursive Tree

worst case.




$H(n) \cdot O(n) = O(n^2)$.

best case




$H(\lg n) \cdot O(n) = O(n \lg n)$.

# Summary

- Quicksort uses the “Divide and Conquer” approach.
- Partition function splits an input list into two sub-lists by comparing all elements with the pivot:
  - Elements in the left sub-list are < pivot and
  - Elements in the right sub-list are ≥ pivot.
- Quicksort is called recursively on each sub-list.
- The worst-case time complexity of Quicksort is $\Theta(n2)$.
- The best-case and average-case time complexities of Quicksort are both $\Theta(n\lg n)$.