CX2101 Algorithm Design and Analysis

Tutorial 3: Analysis Techniques

School of Computer Science and Engineering *Nanyang Technological University*

1. Solve the following recurrences by the iteration method

   1) T(1) = 1, and for $n \geq 2$, $T(n) = 3T(n-1) + 2$

   2) T(1) = 1, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

2. Solve the recurrences in Question 1 by the substitution method.

3. Solve the following recurrences by the master method.

   1) $W(n) = W(n/3) + 5$

   2) $T(n) = 2T(n/2) + n/4$

   3) $W(n) = 2W(n/4) + \sqrt{n^3}$

4. Determine which of the following are linear homogeneous recurrence relations with constant coefficients. Also find the degree of those that are.

   1) $a_n = 4a_{n-2} + 5a_{n-3}$ $x^3 = 4x + 5$

   2) $a_n = 2na_{n-1} + a_{n-2}$ non-homogeneous

   3) $a_n = a_{n-1} + a_{n-4}$ $a^4 = a^3 + 1$

   4) $a_n = a^2_{n-1} + a_{n-2}$ non - homogeneous

   5) $a_n = a_{n-2} + n$ $\Rightarrow$ non-homogeneous

5. Solve the following recurrence relations together with the initial conditions given. (Due to time constraints, we may not cover every part in the tutorial class.)

   1) $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

   2) $a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

   3) $a_n = 2a_{n-1} - a_{n-2}$ for all $n \geq 2$, $a_0 = 1$, $a_1 = 3$

1) $x^2 - 7x + 10 = 0$

$x \quad -2, -5$

$r = 2, \ s = 5$

$a_n = Cr^n + Ds^n$

$\begin{cases} a_0 = C + D = 1 \\ a_1 = 2C + 5D = 0 \end{cases}$

$\begin{cases} D = -\frac{2}{3} \\ C = \frac{5}{3} \end{cases}$

$\therefore a_n = \frac{5}{3} \times 2^n - \frac{2}{3} \times 5^n$

(2) $t^2 = 4$

$r = 2, \ S = -2$

$\begin{cases} a_0 = C + D = 6 \\ a_1 = 2C - 2D = 8 \end{cases}$

$\begin{cases} C = 5 \\ D = 1 \end{cases}$

$\therefore a_n = 5 \times 2^n + (-2)^n$

(3) $t^2 - 2t + 1 = 0$

$r = s = 1$

$a_n = Cr^n + Dnr^n$

$\begin{cases} a_0 = C = 1 \\ a_1 = C \times 1^1 + D \times 1 \times 1^1 = 3 \end{cases}$

$\begin{cases} C = 1 \\ D = 2 \end{cases}$

$\therefore a_n = 1 + 2n$

Solve by iteration

(1) $T(n) = 3T(n-1)+2,\ T(1)=1$

$$T(n) = 3(3T(n-2)+2)+2 = 9T(n-2)+3\times2+2$$
$$= 9(3T(n-3)+2)+3\times2+2 = 27T(n-3)+3^2\times2+3\times2+2$$
$$= \cdots 3^{n-1}T(1)+3^{n-1}-1$$
$$= 2\times3^{n-1}-1 \le 2\times3^{n-1} \le 3^n \quad \therefore O(3^n)$$

Solve by sustitution

(1) Hypothesis: $T(n) \le 3^n - 2$

(2) Base Case: $T(1)=1 \le 3^1-2$

(3) Inductive steps: Assume for all $n \ge 2$, Hypothesis true.

$$T(n) = 3T(n-1)+2$$
$$\le 3\times(3^{n-1}-2)+2$$
$$\le 3^n-2$$
$$\therefore O(3^n)$$

(2) Let $n=2^k$, $T(2^k)=2T(2^{k-1})+6\times2^k$, $T(1)=1$

$$T(2^k) = 2T(2^{k-1})+6\times2^k$$
$$= 2\cdot(2T(2^{k-2})+6\times2^{k-1})+6\times2^k = 2^2T(2^{k-2})+2\times6\times2^k$$
$$= 2^2(2T(2^{k-3})+6\times2^{k-2})+6\times2^{k+1} = 2^3T(2^{k-3})+3\times6\times2^k$$
$$= 2^kT(2^{k-k})+k\times6\times2^k$$
$$= 2^k+k\times6\times2^k$$
$$= n+6n\lg n$$

(1) Hypothesis: $T(n) = O(n\lg n)$

(2) Base Case: $T(1)=1 \le c\times1\times\sqrt{1}$

(3) Inductive Step: Assume for all $n\ge2$. Hy.. true

$$T(2^k) = 2T(2^{k-1})+6\times2^k$$
$$\le 2\times c\times2^{k-1}\times(k-1)+6\times2^k$$
$$= c\times2^k\times(k-1)+6\times2^k$$

if $c \ge 7$.

$$T(2^k) \le 7\times2^k\times(k-1)+6\times2^k$$
$$= (7k-7+6)\times2^k$$
$$= (7k-1)\times2^k$$
$$\le 7k\cdot2^k$$

∴ Hypothesis True

3. Solve by the master method

(1) $W(n) = W(\frac{n}{3})+5$

$a=1,\ b=3,\ f(n)=5$

$\log_b a = \log_3 1 = 0$

$n^{\log_b a} = n^0 = 1$

$f(n)=5=\Theta(1)=\Theta(f(n))$

$\therefore W(n)=\Theta(\lg n)$

(2) $T(n)=2T(\frac{n}{2})+\frac{n}{4}$

$a=2,\ b=2,\ f(n)=\frac{n}{4}$

$n^{\log_b a} = n^{\log_2 2} = n^1$

$\therefore f(n)=\frac{n}{4}=\Theta(n^{\log_b a})$

$W(n)=\Theta(n\lg n)$

(3) $W(n) = 2W(\frac{n}{4})+\sqrt{n^3}$

$a=2,\ b=4,\ f(n)=n^{\frac{3}{2}}$

$n^{\log_b a} = n^{\log_4 2} = n^{\frac{1}{2}}$

① $f(n)=n^{\frac{3}{2}} = \Omega(n^{\frac{1}{2}+1})$ ✓

② $2\times f(\frac{n}{4}) \le f(n)$

if, $c=\frac{3}{4}$.

② $cW(n) = 2cW(\frac{n}{4})+c\sqrt{n^3} = \frac{3}{2}W(\frac{n}{4})+\frac{3\sqrt{n^3}}{4}$

③ $2\times W(\frac{n}{4}) = 4W(\frac{n}{16})+\frac{\sqrt{n^3}}{4}$

② > ③

$\therefore W(n)=\Theta(\sqrt{n^3})$

CX2101 Algorithm Design and Analysis

Tutorial 3
Analysis Techniques

(Week 9)

Question 1 (1)

Solve the following recurrence by the iteration method

$T(1) = 1$, and for $n \geq 2$, $T(n) = 3T(n-1) + 2$

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$T(1) = 1$, and for $n \geq 2$, $T(n) = 3T(n-1) + 2$

$$
\begin{aligned}
T(n) &= 3T(n-1) + 2 \\
&= 3(3T(n-2) + 2) + 2 \\
&= 3^2T(n-2) + 3*2 + 2 \\
&= 3^2(3T(n-3) + 2) + 3*2 + 2 \\
&= 3^3T(n-3) + 3^2*2 + 3*2 + 2 \\
&= 3^{n-1}T(1) + 3^{n-2}*2 + \ldots + 3^2*2 + 3*2 + 2
\end{aligned}
$$

Question 1 (1)

$= 3^{n-1} + 2(3^{n-2} + \ldots + 3^2 + 3 + 1)$

$= 3^{n-1} + 2(3^{n-2} + \ldots + 3^2 + 3 + 1)$

$= 3^{n-1} + 2(3^{n-1} - 1)/(3-1)$

$= 3^{n-1} + 3^{n-1} - 1$

$\leq 2 * 3^{n-1}$

$\leq 3^n$

So $T(n) = O(3^n)$

Question 1 (2)

Solve the following recurrence by the iteration method

$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

Question 1 (2)

Solve the following recurrence by the iteration method

T(1) = 1, and for n ≥ 2, a power of 2, T(n) = 2T(n/2) + 6n

$$\mathrm{T}(n) = 2\mathrm{T}(\tfrac{n}{2}) + 6n$$
$$= 2(2\mathrm{T}(\tfrac{n}{2^2}) + 6(\tfrac{n}{2})) + 6n$$
$$= 2^2\mathrm{T}(\tfrac{n}{2^2}) + 6n + 6n$$
$$= 2^2\,(2\mathrm{T}(\tfrac{n}{2^3}\,) + 6(\tfrac{n}{2^2}\,)) + 6n + 6n$$
$$= 2^3\;\mathrm{T}(\tfrac{n}{2^3}\,) + 6n + 6n + 6n$$
$$= 2^{\mathrm{k}}\mathrm{T}(1) + 6\mathrm{k}n \qquad \text{assume } n = 2^{\mathrm{k}} \qquad \text{so } k = \lg n$$
$$= n + 6n\lg n$$
$$= \mathrm{O}(n\lg n)$$

Question 2 (1)

Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, $T(n) = 3T(n-1) + 2$

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Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, $T(n) = 3T(n-1) + 2$

Guess that $T(n) = O(3^n)$

Proof: We will prove $T(n) \leq 3^n - 2$ for $n \geq 1$

(a) Base case: $T(1) = 1 \leq 3^1 - 2$.

(b) Inductive step: assume that $T(k) \leq 3^k - 2$, prove that $T(k+1) \leq 3^{k+1} - 2$.

$T(k+1) = 3T(k) + 2$

$\leq 3(3^k - 2) + 2$

$\leq 3^{k+1} - 6 + 2$

$\leq 3^{k+1} - 2$ Thus $T(n) = O(3^n - 2) = O(3^n)$

Question 2 (2)

Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

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Solve the following recurrence by the substitution method

$T(1) = 1$, and for $n \geq 2$, a power of 2, $T(n) = 2T(n/2) + 6n$

Guess $T(n) = O(n\lg n)$

Proof: we show that $T(n) \leq 8\, n\lg n$ for any $n \geq 2$

a) Base case:

$T(2) = 2*1+6*2 = 14$, $8n\lg n = 16$,

so $T(n) \leq 8\, n\lg n$ for $n=2$

b) Inductive step: Assume that $T(2^k) \leq 8\, k\, 2^k$

Prove $T(2^{k+1}) \leq 8\,(k+1)\, 2^{k+1}$

Question 2 (2)

$T(2^{k+1}) = 2\ T(2^k) + 6*\ 2^{k+1}$

$\leq 2*\ 8\ k\ 2^k + 6*\ 2^{k+1}$

$= (8\ k + 6\ )*\ 2^{k+1}$

$\leq 8\ (k + 1\ )*\ 2^{k+1}$

$T(n) = O(n \lg n)$

Question 3(1)

Solve the following recurrence by the master method.

$$W(n) = W(n/3) + 5$$

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Solve the following recurrence by the master method.

$$W(n) = W(n/3) + 5$$

$n^{\log_b a} = n^{\log_3 1} = n^0$, $f(n) = 5 = \theta(1) = \theta(n^0)$

So,

$W(n) = \theta(n^0 \lg n)$

Question 3(2)

Solve the following recurrence by the master method.

$$T(n) = 2T(n/2) + n/4$$

Question 3(2)

Solve the following recurrence by the master method.

$$T(n) = 2T(n/2) + n/4$$

$n^{\log_b a} = n^{\log_2 2} = n^1$, $f(n) = n/4 = \theta(n) = \theta(n^1)$

So,

$W(n) = \theta(n\lg n)$

Question 3(3)

Solve the following recurrence by the master method.

$$W(n) = 2W(n/4) + \sqrt{n^3}$$

Question 3(3)

Solve the following recurrence by the master method.

$W(n) = 2W(n/4) + \sqrt{n^3}$

$n^{\log_b a} = n^{\log_4 2} = n^{0.5}$, $f(n) = \sqrt{n^3} = n^{3/2} = \Omega(n^{0.5 + 0.1})$

And $a*f(n/b) = 2f(n/4) = 2*(n/4)^{3/2} = 2*n^{3/2}(1/4)^{3/2}$

$= (1/4)*n^{3/2} \le (1/4)*n^{3/2} = c*f(n)$ $\quad$ $c = 1/4$

So,

$W(n) = \theta(n^{1.5})$

Question 4

Determine which of the following are linear homogeneous recurrence relations with constant coefficients. Also find the degree of those that are.

1) $a_n = 4a_{n-2} + 5a_{n-3}$ Degree 3
2) $a_n = 2na_{n-1} + a_{n-2}$ Not constant coefficient
3) $a_n = a_{n-1} + a_{n-4}$ Degree 4
4) $a_n = a^2_{n-1} + a_{n-2}$ Not linear
5) $a_n = a_{n-2} + n$ Not homogeneous

Question 5 (no need to cover all parts if running out of time)

(1) Solve the following recurrence relation together with the initial conditions given.

$a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

Question 5(1)

Solve the following recurrence relation together with the initial conditions given.

$a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

The characteristic equation:

$$t^2 - 7t + 10 = 0 \Rightarrow (t - 2)(t - 5) = 0$$

Solution: two distinct roots: $t_1 = 2$, $t_2 = 5$

Thus $a_n = 2^n C + 5^n D$

Question 5(1)

Substitute the initial conditions into $a_n = 2^n C + 5^n D$ to find C and D:

$a_0 = 1 = C + D, \quad \Rightarrow \; 2 = 2C + 2D$

$a_1 = 0 = 2C + 5D$

Thus 2 = -3D, i.e. D = -2/3 then C = 5/3

So we have $a_n = (5/3)*2^n – (2/3)*5^n$

Question 5(2)

Solve the following recurrence relation together with the initial conditions given.

$a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

Question 5(2)

Solve the following recurrence relation together with the initial conditions given.

$a_n = 4a_{n-2}$ for $n \geq 2$, $a_0 = 6$, $a_1 = 8$

The characteristic equation:

$t^2 - 4 = 0 \quad \Rightarrow (t)^2 = 4$

Solution: two distinct roots: $t_1 = 2$, $t_2 = -2$

Thus $a_n = 2^n C + (-2)^n D$

Question 5(2)

Substitute the initial conditions into $a_n = 2^n C + (-2)^n D$ to find C and D:

$a_0 = 6 = C + D$,   => $12 = 2C + 2D$

$a_1 = 8 = 2C - 2D$

Thus $20 = 4C$, i.e. $C = 5$ then $D = 1$

So we have $a_n = 5*2^n + (-2)^n$

Question 5(3)

Solve the following recurrence relations together with the initial conditions given.

$$a_n = 2a_{n-1} - a_{n-2} \text{ for all } n \geq 2, \ a_0 = 1, a_1 = 3$$

Question 5(3)

Solve the following recurrence relations together with the initial conditions given.

$a_n = 2a_{n-1} - a_{n-2}$ for all $n \geq 2$, $a_0 = 1$, $a_1 = 3$

The characteristic equation:

$$t^2 - 2t + 1 = 0 \quad \Rightarrow (t-1)^2 = 0$$

Solution: single root: $t = 1$,

Thus $a_n = C + nD$, we have

$1 = C$, $D = 2$

$a_n = 1 + 2n$