CX1107 Data Structures and Algorithms **2020/21 Semester 2**

0: Mathematic Revision for Algorithms

School of Computer Science and Engineering *Nanyang Technological University*

0.1 Overview of Review Topics

1. Sets and Functions
2. Floor and Ceiling Functions
3. Power and Logarithm Functions
4. Series
5. Limits
6. Differentiation of Functions
7. Modular Arithmetic
8. Proof by Induction

0.2 Sets

Set theory is a branch of mathematical logic that studies sets. A set is a collection of objects, called its **members** or **elements**. In set theory, the objects are usually mathematical objects. If A is a set and a is its element, then we write $a \in A$. $\emptyset$ denotes the empty set, that is the set containing no element.
We can describe a set by using a Venn diagram as illustrated in Figure 0.1. We also can define a set by writing $A = \{1, 2, 3\}$. In this case, $1 \in A$ or 1 is a element of A. We also can define sets by their elements' property. For example, we can define the set of even integer by $B = \{x | x \in \mathbb{Z}$ and $x/2$ is an integer$\}$

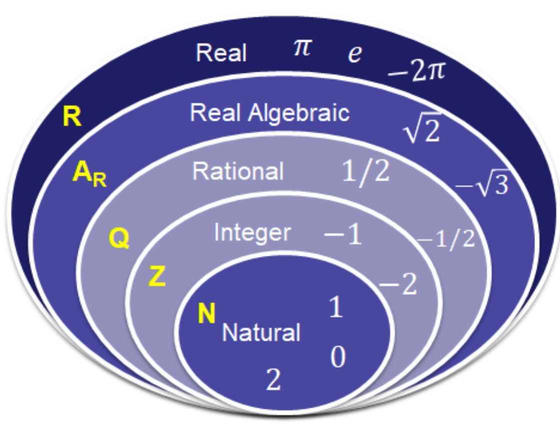


Figure 0.1: The number sets: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{A}_{\mathbb{R}} \subset \mathbb{R}$

0.2.1 Basic notation

$A \subset B$: A is a subset of B
$A \cap B$: The intersection of sets A and B or $\{x|x \in A$ and $x \in B\}$
$A \cup B$: The union of sets A and B or $\{x|x \in A$ or $x \in B\}$

0.3 Functions

Given that $f : \mathbf{D} \rightarrow \mathbf{R}$, $y = f(x)$

- **Function** f : A rule that assigns a unique element in a set **R** to each element in a set **D**.
- Independent variables: The inputs of the function, x.
- Dependent variables: The corresponding outputs, y.
- Image: y is the image of x under f.
- Preimage or inverse image: x is the preimage of y under f.
- **Domain**: The domain of the function, **D**, is the input set of the function.
- **Range**: The range of the function, **R**, is the corresponding output set of the function.
- **Codomain**: The codomain of the function, **C** is the set within which the corresponding output values of the function lie. (The **R** is a subset of **C**, $f(\mathbf{D}) = \mathbf{R} \subset \mathbf{C}$)

0.3.1 Mapping Functions

- **One-to-one function** or **injective function**: $f : \mathbf{D} \rightarrow \mathbf{R}$ is one-to-one function if it maps distinct elements in **D** to distinct elements in **R**. If $f(x_1) = f(x_2)$, then it must be $x_1 = x_2$.
- **Onto function** or **surjective function**: $f : \mathbf{D} \rightarrow \mathbf{R}$ is onto function if there always exists an element in **D** is preimage of the element in **R** for any $y \in \mathbf{R}$.
- **One-to-one onto function** or **bijective function**: $f : \mathbf{D} \rightarrow \mathbf{R}$ is one-to-one onto function if it is both injective and surjective. A function f is invertible iff (*if and only if*) f is an one-to-one onto function.
- **Many-to-one function**: $f : \mathbf{D} \rightarrow \mathbf{R}$ is many-to-one function if any element in **R** of f is the image of more than one element in the domain, **D** of f.

0.4 Function Representations

1. Analytical Method

$$\begin{aligned} A(r) &= \pi r^2 \\ I(V) &= I_S(\exp^{V/nV_T} - 1) \\ Z(x,y) &= x^2 + y^2 \end{aligned}$$

2. Venn Diagram Method

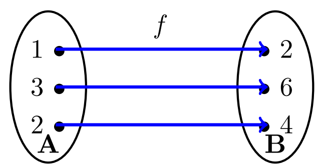


Figure 0.2: $f : A \to B,\ f(x) = 2x$

3. Graphical Method

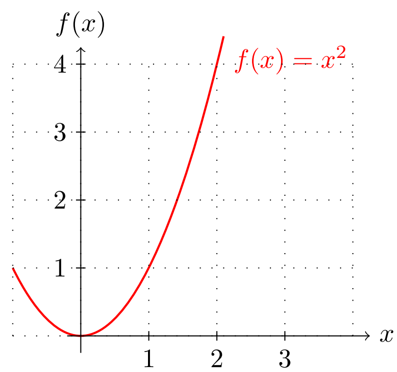


Figure 0.3: $f : \mathbb{R} \to \mathbb{R}\ f(x) = x^2$

4. Tabulation Method

Period	1Q16	4Q15	3Q15	2Q15	1Q15	4Q14	3Q14	2Q14	1Q14	...
Number of units	2847	3199	4159	4104	2655	2760	3061	4211	2815	...

Table 0.1: The number of private residential unit transactions in the whole of Singapore

0.5 Floor and Ceiling Functions

- The floor and ceiling functions map a real number to the largest previous or the smallest following integer, respectively.
- $\lfloor x \rfloor$ (floor of x) $= \max\{m \in \mathbb{Z} | m \leq x\}$ the largest integer, m not greater than x.

- $\lceil x \rceil$ (ceiling of x) $= \min\{n \in \mathbb{Z} | n \geq x\}$ the smallest integer, n not less than x.
- $\lfloor x \rfloor \leq x \leq \lceil x \rceil$ e.g. $\lfloor 5.5 \rfloor = 5$, $\lfloor 5 \rfloor = 5$, $\lceil 5.5 \rceil = 6$

0.5.1 Equivalences of Floor Function

$$\lfloor x \rfloor = m \qquad m \in \mathbb{Z} \tag{0.1}$$

$$m \leq x < m + 1 \tag{0.2}$$
$$x - 1 < m \leq x \tag{0.3}$$
$$\tag{0.4}$$

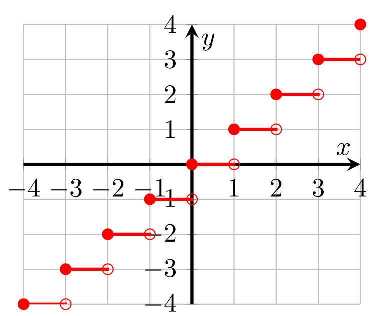
$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1 \tag{0.5}$$

0.5.2 Equivalences of Ceiling Function

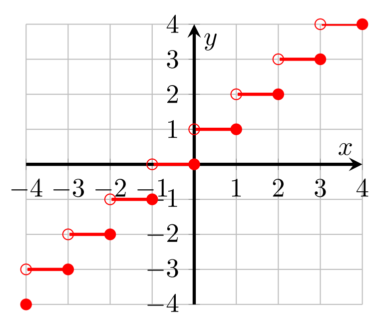
$$\lceil x \rceil = n \qquad n \in \mathbb{Z} \tag{0.6}$$

$$n - 1 < x \leq n \tag{0.7}$$
$$x \leq n \leq x + 1 \tag{0.8}$$
$$\tag{0.9}$$

$$\lceil x \rceil + \lceil y \rceil - 1 \leq \lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil \tag{0.10}$$



(a) Floor Function



(b) Ceiling Function

Figure 0.4: floor function and ceiling function

0.6 Power and Logarithm Functions

0.6.1 Exponentiation

Exponentiation is a mathematical operation, written as a^n, involving two numbers, the base b and the exponent (a.k.a. index or power) n. When n is a positive integer, exponentiation corresponds to repeated multiplication.

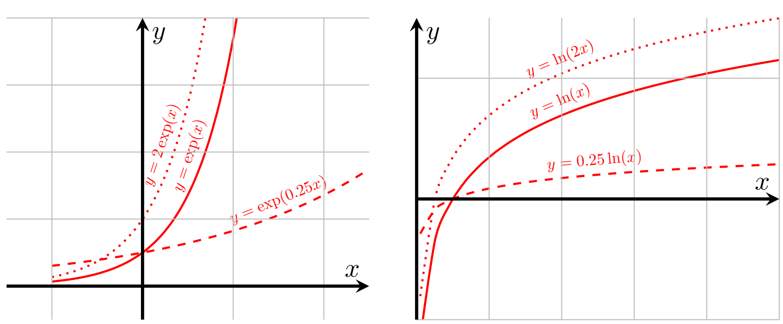
$$
\begin{aligned}
a^{-n} &\equiv \frac{1}{a^n} && (0.11)\\
a^{\frac{1}{n}} &\equiv \sqrt[n]{a} && (0.12)\\
a^n a^m &\equiv a^{n+m} && (0.13)\\
\frac{a^n}{a^m} &\equiv a^{n-m} && (0.14)\\
(a^n)^m &\equiv a^{nm} && (0.15)
\end{aligned}
$$

0.6.2 Logarithm

The logarithm of a number to the base b is the exponent by which the base b has to be raised to produce that number.

$$\log_a b = c \Leftrightarrow b = a^c \tag{0.16}$$

$$
\begin{aligned}
\log_a 1 &= 0 && (0.17)\\
\log_a 0 &= \text{undefined} && (0.18)\\
\log_a x + \log_a y &= \log_a xy && (0.19)\\
\log_a x - \log_a y &= \log_a \frac{x}{y} && (0.20)\\
\log_a x^y &= y \log_a x && (0.21)\\
\log_a c &= \frac{\log_b c}{\log_b a} && (0.22)
\end{aligned}
$$



(a) $y = \exp(x)$ (b) $y = \ln(x)$

Figure 0.5: Logarithms and Exponential Functions

0.7 Series

0.7.1 Geometric Series

$$
\begin{aligned}
G_n &= a + ar + ar^2 + \ldots + ar^{n-1} \\
rG_n &= ar + ar^2 + \ldots + ar^{n-1} + ar^n \\
(1-r)G_n &= a - ar^n \\
G_n &= \frac{a(1-r^n)}{1-r} \\
G_\infty = \frac{a}{1-r} & \qquad |r| < 1
\end{aligned}
$$

0.7.2 Arithmetic Series

$$
\begin{aligned}
S_n &= a + (a+d) + (a+2d) + \ldots + [a + (n-1)d] \\
&= \frac{n}{2}[2a + (n-1)d] \\
&= \frac{n}{2}(a_0 + a_{n-1})
\end{aligned}
$$

where the first term of the series, $a_0 = a$ and the last term of the series, $a_{n-1} = a + (n-1)d$.

0.7.3 Arithmetico-geometric Series

$$1 \cdot r^0 + 2 \cdot r^1 + 3 \cdot r^2 + \cdots + k \cdot r^{k-1}$$

For example, $r = 2$

$$
\begin{aligned}
\sum_{t=1}^{k} t2^{t-1} &= 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 4 + 4 \cdot 8 + \ldots + k \cdot 2^{k-1} \\
2\sum_{t=1}^{k} t2^{t-1} &= \qquad 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 8 + \ldots + (k-1) \cdot 2^{k-1} + k \cdot 2^k \\
(2-1)\sum_{t=1}^{k} t2^{t-1} &= -1 \cdot 1 - 1 \cdot 2 - 1 \cdot 4 - 1 \cdot 8 - \ldots - 1 \cdot 2^{k-1} + k \cdot 2^k \quad \triangleright \text{eq. 2 - eq. 1} \\
\sum_{t=1}^{k} t2^{t-1} &= -2^k + 1 + k \cdot 2^k \quad \triangleright \text{geometric series} \\
&= 2^k(k-1) + 1
\end{aligned}
$$

0.7.4 Faulhaber's formula

The sum of the p^{th} powers of the first n positive integers:

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6} \\ \sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4}\end{aligned}$$

0.7.4.1 The Sum of The Squares of The First n Positive Integers

The binomial expansion of $(k-1)^3$:

$$(k-1)^3 = k^3 - 3k^2 + 3k - 1$$

We can rearrange the terms:

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1$$

Thus, we have k equations from $k = 1$ to n,

$$\begin{aligned}1^3 - 0^3 &= 3(1^2) - 3(1) + 1 \\ 2^3 - 1^3 &= 3(2^2) - 3(2) + 1 \\ \cdots &= \cdots \\ (n-1)^3 - (n-2)^3 &= 3((n-1)^2) - 3(n-1) + 1 \\ n^3 - (n-1)^3 &= 3(n^2) - 3(n) + 1\end{aligned}$$

Summing these equations,

$$\begin{aligned}n^3 &= 3(\sum_{k=1}^{n} k^2) - 3(\sum_{k=1}^{n} k) + (\sum_{k=1}^{n} 1) \\ n^3 &= 3(\sum_{k=1}^{n} k^2) - 3(\frac{n(n+1)}{2}) + n \\ 3(\sum_{k=1}^{n} k^2) &= n^3 + 3(\frac{n(n+1)}{2}) - n\end{aligned}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

0.8 Limits

Informal definition, limit is the value that a function or sequence "approaches" as the input or index approaches some value.

Let $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$, then

1. **Constant Rule**: $\lim_{x\to c} k = k$
2. **Sum and Difference Rule**:
$$\lim_{x\to c}[f(x) \pm g(x)] = L \pm M$$
3. **Constant Multiple Rule**: $\lim_{x\to c} kf(x) = kL$
4. **Product Rule**:
$$\lim_{x\to c}[f(x) \cdot g(x)] = L \cdot M$$
5. **Quotient Rule**:
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M} \qquad M \neq 0$$
6. **Power Rule**:
$$\lim_{x\to c}[f(x)]^n = L^n$$
7. **Root Rule**:
$$\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} \qquad L > 0 \text{ for even } n$$

0.8.1 L'Hôpital's Rule

If

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$
2. f and g are differentiable at the interval and $g'(x) \neq 0$
3. $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists

,then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example 1 :

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} &= \lim_{x\to 2} \frac{2x}{2x+1} \\ &= \frac{4}{5}\end{aligned}$$

Example 2:

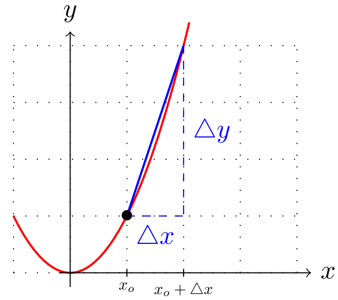
$$\begin{aligned}\lim_{x\to\infty} \frac{x^n}{e^x} &= \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} \\ &\dots \\ &= \lim_{x\to\infty} \frac{n!}{e^x} = 0\end{aligned}$$

0.8.1.1 Exercise

1. $\lim_{x\to 0} \frac{x-\sin x}{x^3}$
2. $\lim_{x\to \infty} \frac{\ln x}{x}$
3. $\lim_{x\to 0} x \ln x$

Answer: 1. $\frac{1}{6}$, 2. 0, 3. 0

0.9 Differentiation



- The instantaneous rate of change of the dependent variable with respect to the independent variable, $\frac{dy}{dx}$
- The **gradient** of the curve at the point.
- The process of finding a derivative is called **differentiation**
- $m = \frac{dy}{dx} = \lim_{\triangle x\to 0} \frac{\triangle y}{\triangle x} = \frac{f(x_o+\triangle x)-f(x_o)}{\triangle x}$
- $f'(a)$ exists iff the limit of $\frac{f(a+\triangle x)-f(a)}{\triangle x}$ exists.

0.9.1 Differentiation Properties

$$\begin{aligned}
\frac{d}{dx}(kf(x)) &= k\frac{df(x)}{dx} \\
\frac{d}{dx}(f(x)+g(x)) &= \frac{df(x)}{dx}+\frac{dg(x)}{dx} \\
\frac{d}{dx}(f(x)-g(x)) &= \frac{df(x)}{dx}-\frac{dg(x)}{dx}
\end{aligned}$$

Example 1:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$
\begin{aligned}
\frac{d}{dx}[\sum_{n=0}^{N} a_n x^n] &= \sum_{n=0}^{N} \frac{d}{dx}[a_n x^n] \\
&= \sum_{n=0}^{N} a_n \frac{d}{dx}[x^n] \\
&= \sum_{n=1}^{N} a_n n[x^{n-1}]
\end{aligned}
$$

0.9.2 Rules of Differentiation

1. Product Rule: $\frac{d}{dx}(f(x)g(x)) = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$
2. Quotient Rule: $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{g^2(x)}$
3. Chain Rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ or $(f \circ g)'(x) = f'(g(x))g'(x)$

Example 1:

$$\frac{x^2 + 2x + 2}{2x^3 + x - 1}$$

Answer:

$$\frac{-2x^4 - 8x^3 - 11x^2 - 2x - 4}{(2x^3 + x - 1)^2}$$

0.9.3 Some Common Use Formula

- $\frac{d}{dx}c = 0$
- $\frac{d}{dx}x = 1$
- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}\ln x = \frac{1}{x}$
- $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$
- $\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)}f'(x)$
- $\frac{d}{dx}2^{f(x)} = 2^{f(x)}f'(x)\ln 2$
- $\frac{d}{dx}a^{f(x)} = a^{f(x)}f'(x)\ln a$
- $\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$
- $\frac{d}{dx}\log_b f(x) = \frac{1}{f(x)\ln b}f'(x)$

0.10 Modular Arithmetic

Given two integers, a and n, we can write a division of a by n as the following

$$\frac{a}{n} = Q \text{ remainder } R$$

where

- a is the dividend
- n is the divisor
- Q is the quotient
- R is the remainder

In modular arithmetic, we are interested at the remainder R. We can rewrite the expression in

$$a \bmod n = R$$

We would say that a modulo n is equal to R. n is referred to as the modulus and $n > 0$. $R = [0, n-1]$.

Example 1: $a \bmod 3 = R$

a	...	-4	-3	-2	-1	0	1	2	3	4	...
R	...	2	0	1	2	0	1	2	0	1	...

With regard to the modulo n arithmetic operations, the following equalities are easily shown to be true:

$$\begin{aligned}
((a \bmod n) + (b \bmod n)) \bmod n &= (a+b) \bmod n \\
((a \bmod n) - (b \bmod n)) \bmod n &= (a-b) \bmod n \\
((a \bmod n) \times (b \bmod n)) \bmod n &= (a \times b) \bmod n
\end{aligned}$$

0.11 Mathematical Induction

1. **Base case** is correct. It is noted that n_1 is not necessary equal 1.
2. **Induction step**: if the statement holds for n, then statement holds for $n+1$

Example 1:
Proof that:

$$S_n = \sum_{m=0}^{n-1} ar^m = \frac{a(1-r^n)}{1-r}$$

1. $n = 1, m = 0$, $S_1 = ar^0 = \frac{a(1-r^1)}{1-r}$

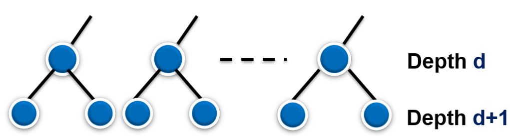
2. If $S_n = \sum_{m=0}^{n-1} ar^m = \frac{a(1-r^n)}{1-r}$ is correct, then

$$\begin{aligned} S_{n+1} &= \frac{a(1-r^n)}{1-r} + ar^n \\ &= \frac{a(1-r^n) + ar^n(1-r)}{1-r} \\ &= \frac{a - ar^n + ar^n - ar^{n+1}}{1-r} \\ &= \frac{a - ar^{n+1}}{1-r} \end{aligned}$$

   By the Method of Induction, the Geometric Series formula holds for any $n \in \mathbb{N}$

Example 2:
Prove that: there are at most 2^d nodes at depth d of a binary tree.

1. By definition of a binary tree, each node has at most 2 children. Let d denote the depth of the tree.
2. Base case: At $d = 0$, there is at most 1 root node, i.e. 2^0 node.
3. Induction Step: We assume that the tree has, for any depth d, at most 2^d nodes at that depth. Prove that at depth $d + 1$, there are at most $2^{(d+1)}$ nodes.
   - By assumption, at depth d, there are at most 2^d nodes.
   - Each of the node at depth d can have at most 2 children, hence there are at most $2 * 2d = 2^{d+1}$ nodes. Thus the result is true for depth $d + 1$



By the Method of Induction, the result is true for all depths of a binary tree.

0.11.1 Exercise

Using mathematic induction to prove the following equations and statement:

1. $1 + r + r^2 + r^3 + \ldots + r^n = \frac{1-r^{n+1}}{1-r}$
2. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$
3. $x + y$ is a factor of $x^{2n} - y^{2n}$ for $n \in \mathbb{N}$