COMP2521 25T1

Balancing Binary Search Trees

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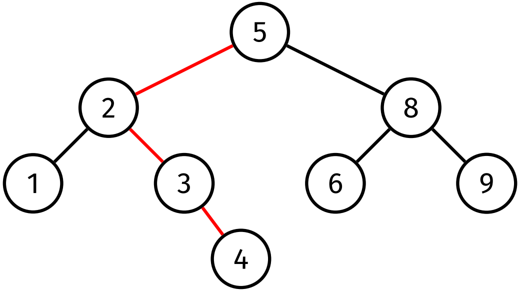
balancing operations
balancing methods

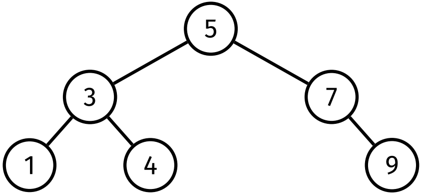
Recap

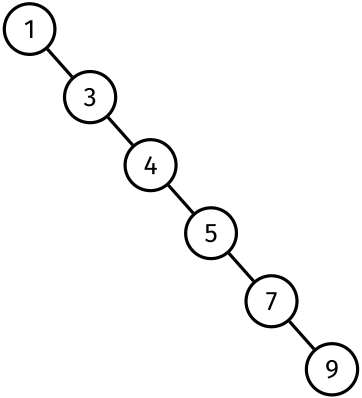
Height of a tree: Maximum path length from the root node to a leaf

- The height of an empty tree is considered to be -1
- The height of the following tree is 3



The structure, height, and hence
performance
of a binary search tree
depends on the order of insertion.



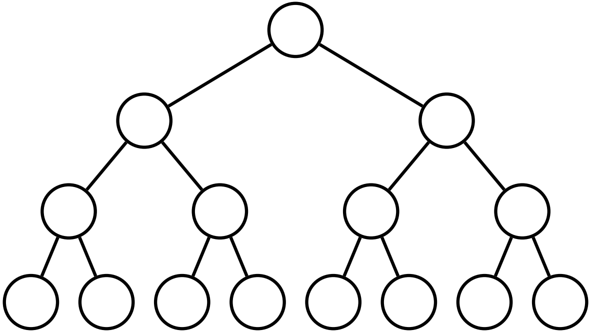


Binary Search Trees

The Best Case

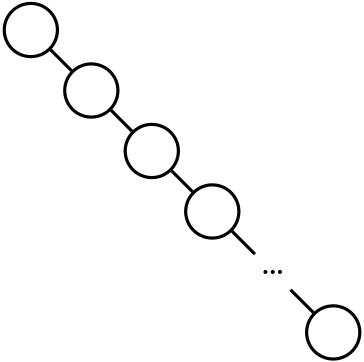
Best case

Items are inserted evenly on the left and right throughout the tree
Height of tree will be $O(\log n)$



Worst case

Items are inserted in ascending or descending order
such that tree consists of a single branch
Height of tree will be $O(n)$



Binary Search Trees

A binary tree of n nodes is said to be
balanced if its height is minimal (or close to minimal) ($O(\log n)$), and
degenerate if it its height is maximal (or close to maximal) ($O(n)$).

Types of Balance

SIZE-BALANCED

a *size-balanced* tree has,
for every node,

$$|\text{SIZE}(l) - \text{SIZE}(r)| \leq 1$$

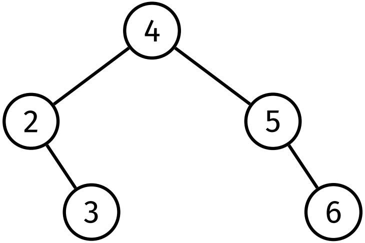
HEIGHT-BALANCED

a *height-balanced* tree has,
for every node,

$$|\text{HEIGHT}(l) - \text{HEIGHT}(r)| \leq 1$$

Types of Balance

Example

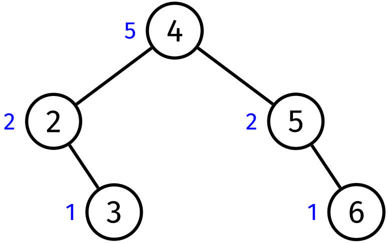


Size-balanced?

Height-balanced?

Types of Balance

Example

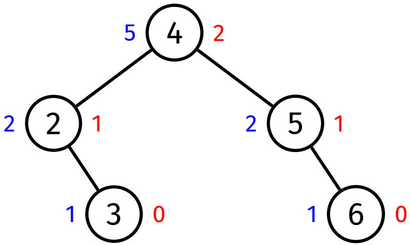


Size-balanced?
Yes

For every node,
$$|\text{SIZE}(l) - \text{SIZE}(r)| \le 1$$

Height-balanced?

Types of Balance
Example



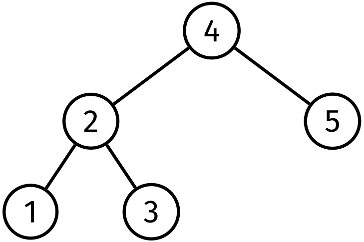
Size-balanced?
Yes

For every node,
$$|\text{SIZE}(l) - \text{SIZE}(r)| \leq 1$$

Height-balanced?
Yes

For every node,
$$|\text{HEIGHT}(l) - \text{HEIGHT}(r)| \leq 1$$

Types of Balance
Example

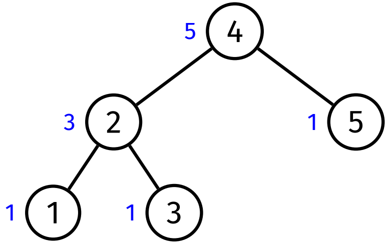


Size-balanced?

Height-balanced?

Types of Balance

Example



Size-balanced?

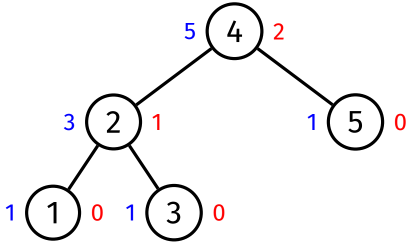
No

At node 4,

$$|\text{SIZE}(l) - \text{SIZE}(r)|$$
$$= |3 - 1| = 2 > 1$$

Height-balanced?

Types of Balance
Example



Size-balanced?
No

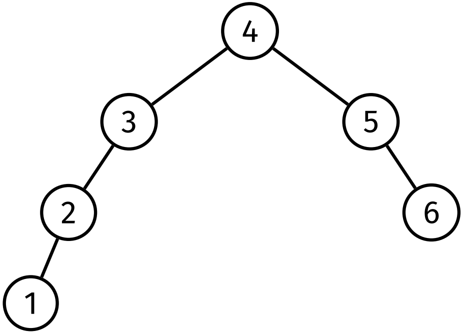
At node 4,
$$|\text{SIZE}(l) - \text{SIZE}(r)|$$
$$= |3 - 1| = 2 > 1$$

Height-balanced?
Yes

For every node,
$$|\text{HEIGHT}(l) - \text{HEIGHT}(r)| \leq 1$$

Types of Balance

Example

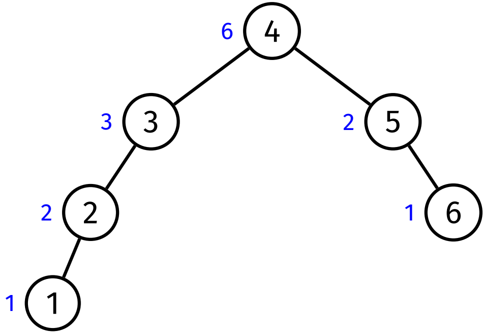


Size-balanced?

Height-balanced?

Types of Balance

Example



Size-balanced?

No

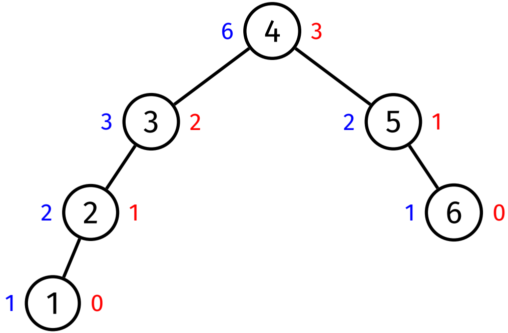
At node 3,

$$|\text{SIZE}(l) - \text{SIZE}(r)|$$
$$= |2 - 0| = 2 > 1$$

Height-balanced?

Types of Balance

Example



Size-balanced?
No

At node 3,
$$|\text{SIZE}(l) - \text{SIZE}(r)|$$
$$= |2 - 0| = 2 > 1$$

Height-balanced?
No

At node 3,
$$|\text{HEIGHT}(l) - \text{HEIGHT}(r)|$$
$$= |1 - (-1)| = 2 > 1$$

Balancing Operations

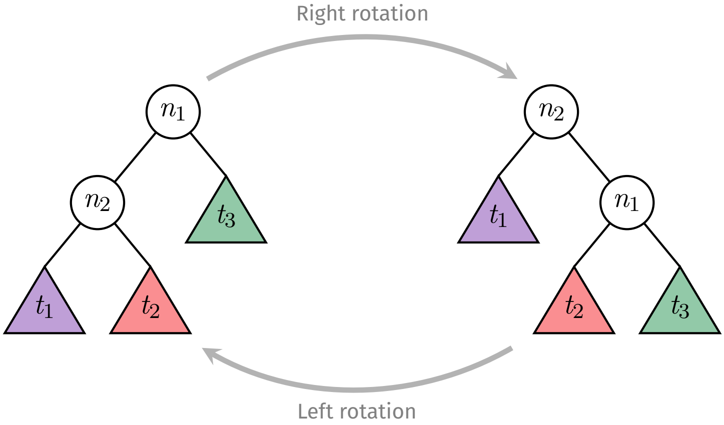
Rotation

- Left rotation
- Right rotation

Partition

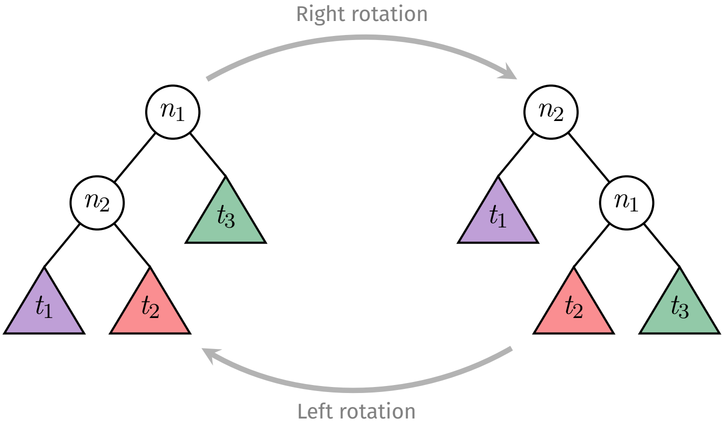
- Rearrange tree around a specified node by rotating it up to the root

LEFT ROTATION and **RIGHT ROTATION**:
a pair of operations
that change the balance of a tree



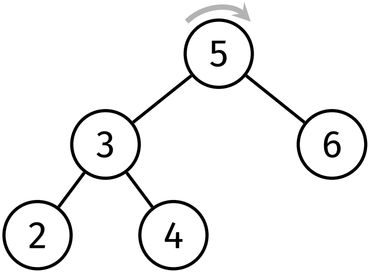
Rotations

Rotations maintain the order of a search tree:

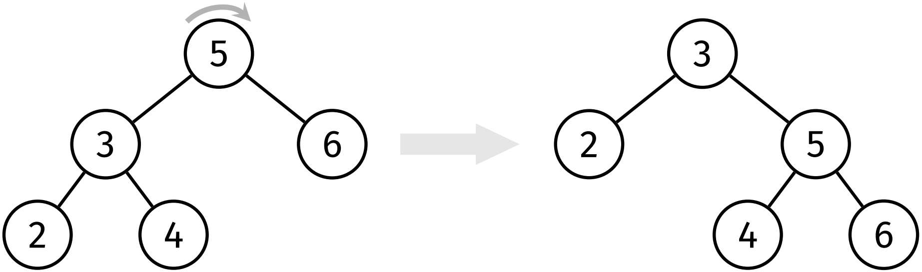


(all values in t_1) $< n_2 <$ (all values in t_2) $< n_1 <$ (all values in t_3)

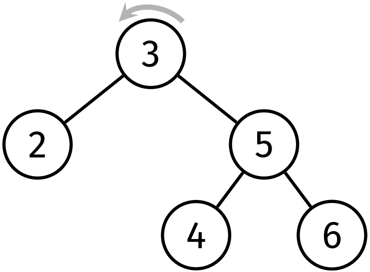
Rotate right at 5



Rotate right at 5



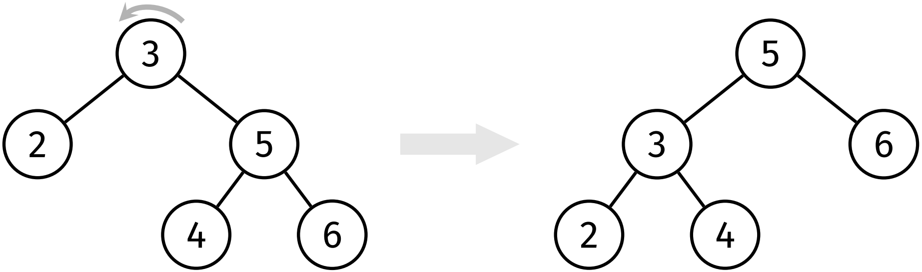
Rotate left at 3



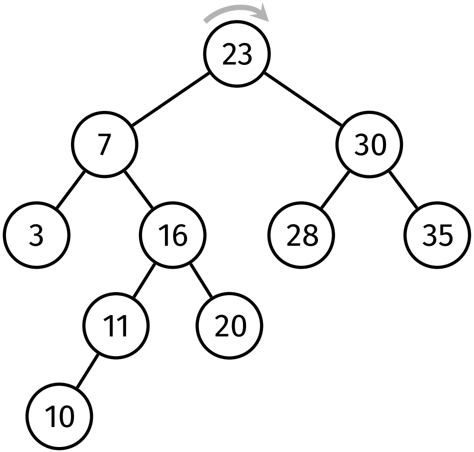
Rotations

Example

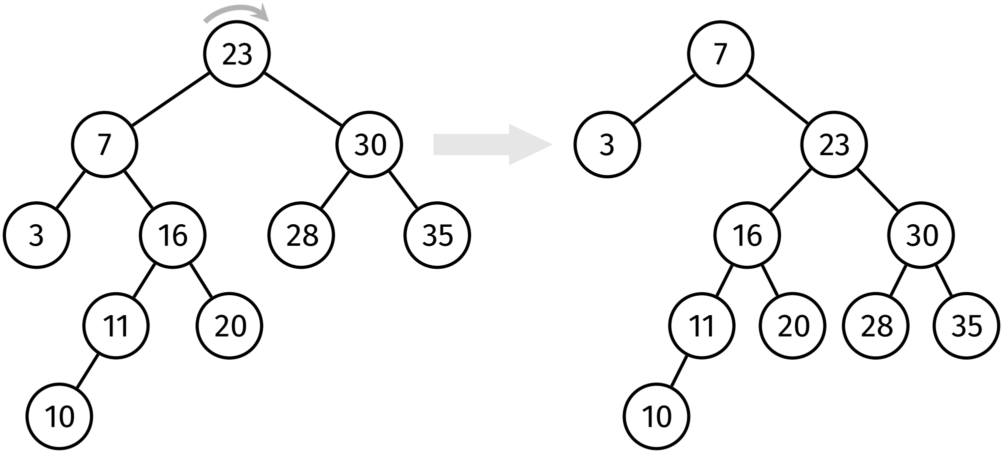
Rotate left at 3



Rotate right at 23



Rotate right at 23



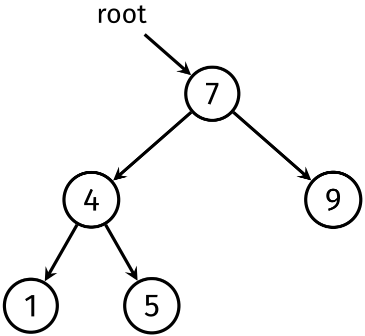
```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;
    root->left = newRoot->right;
    newRoot->right = root;
    return newRoot;
}
```

```
struct node *rotateLeft(struct node *root) {
    if (root == NULL || root->right == NULL) return root;
    struct node *newRoot = root->right;
    root->right = newRoot->left;
    newRoot->left = root;
    return newRoot;
}
```

Rotations

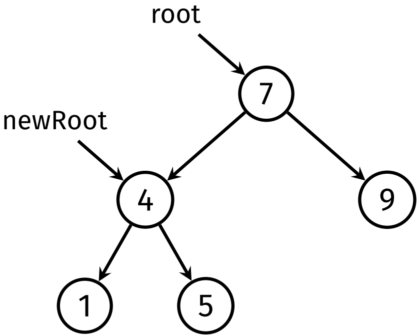
Implementation

```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;



}
```



Rotations
Implementation

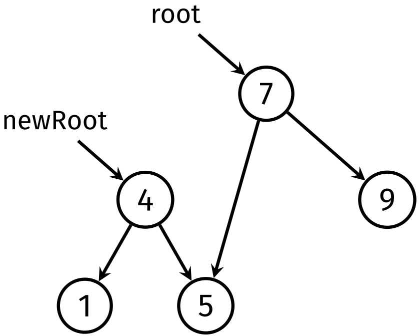
```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;



}
```



Rotations

Implementation

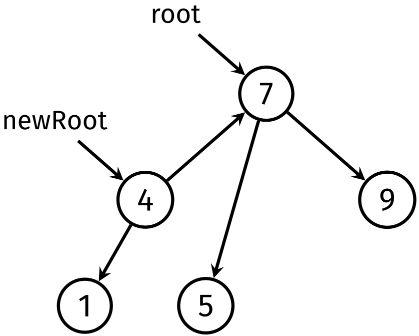
```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;
    root->left = newRoot->right;


}
```



Rotations

Implementation

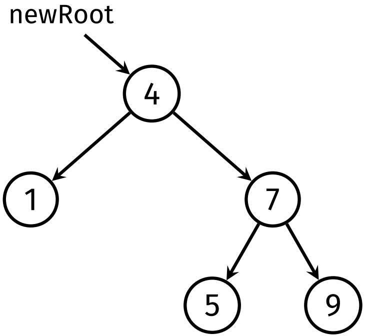
```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;
    root->left = newRoot->right;
    newRoot->right = root;

}
```



Rotations

Implementation

```
struct node *rotateRight(struct node *root) {
    if (root == NULL || root->left == NULL) return root;
    struct node *newRoot = root->left;
    root->left = newRoot->right;
    newRoot->right = root;
    return newRoot;
}
```



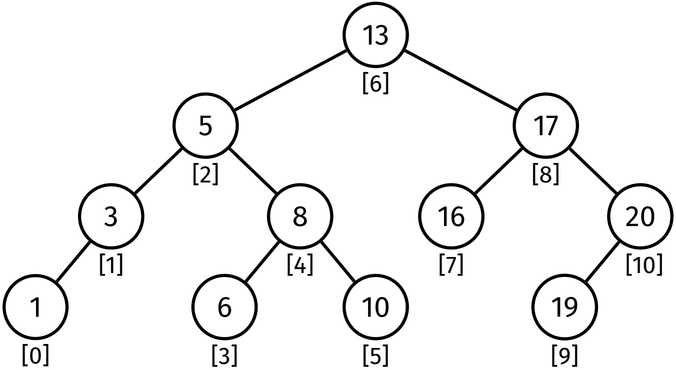
Time complexity: $O(1)$

- Rotation requires only a few localised pointer re-arrangements

Partition

`partition(tree, i)`

Rearrange the tree so that the element with index i becomes the root

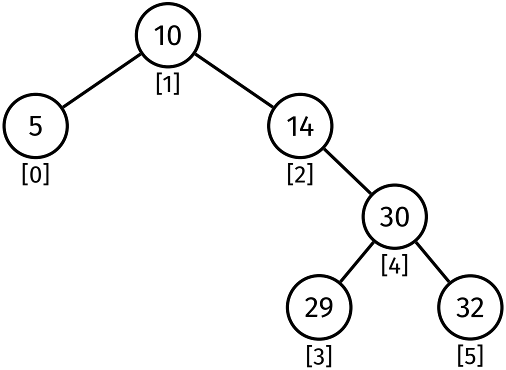


Partition

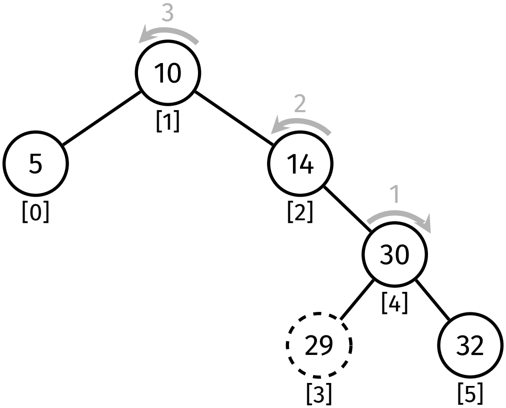
Method:

- Find element with index i
- Perform rotations to lift it to the root
  - If it is the left child of its parent, perform right rotation at its parent
  - If it is the right child of its parent, perform left rotation at its parent
  - Repeat until it is at the root of the tree

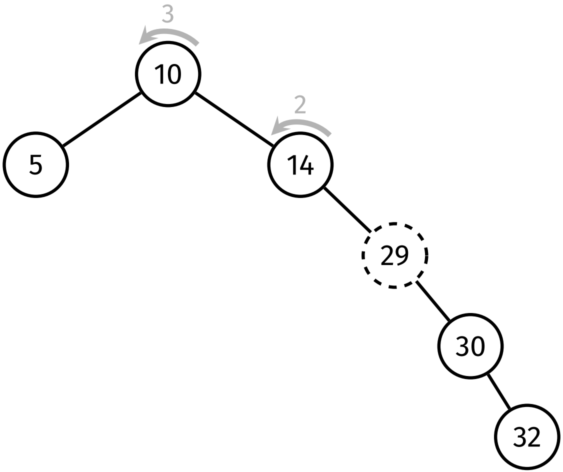
Partition this tree around index 3:



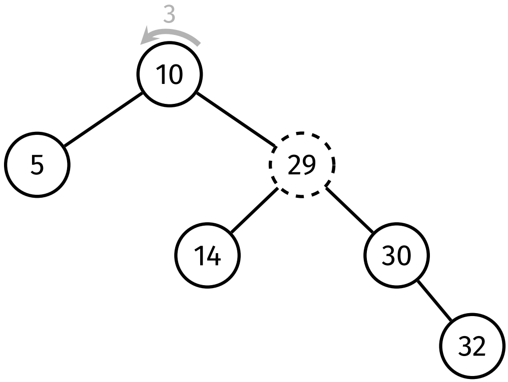
Partition this tree around index 3:



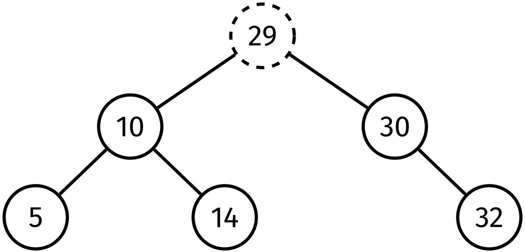
After right rotation at 30:



After left rotation at 14:

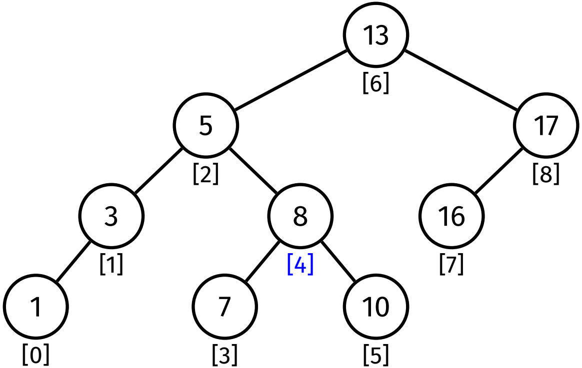


After left rotation at 10:



```
partition(t, i):
    Input:  tree t, index i
    Output: tree with i-th item moved to root

    leftSize = size(t->left)

    if i < leftSize:
        t->left = partition(t->left, i)
        t = rotateRight(t)
    else if i > leftSize:
        t->right = partition(t->right, i - leftSize - 1)
        t = rotateLeft(t)

    return t
```

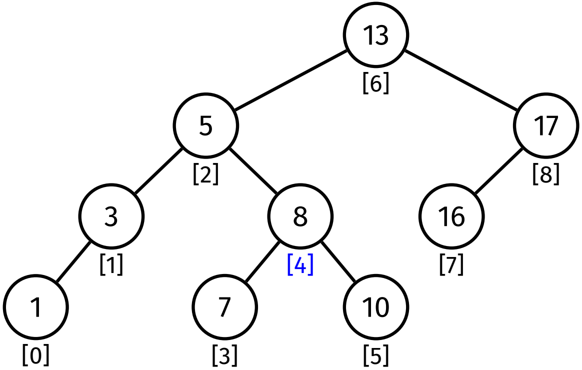
Partition this tree around index 4



Partition

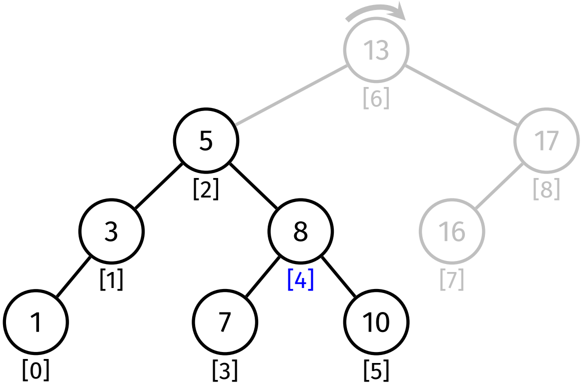
Pseudocode - Example

Size of left subtree is 6, and 4 < 6...

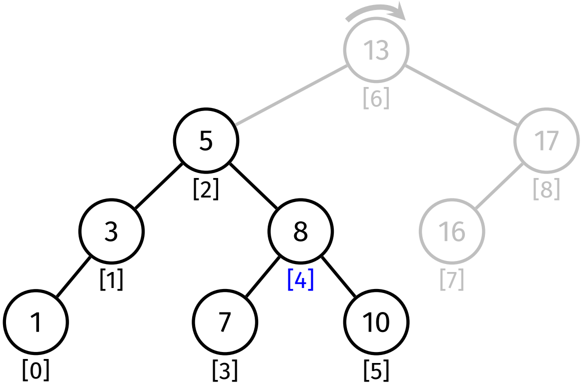


Partition
Pseudocode - Example

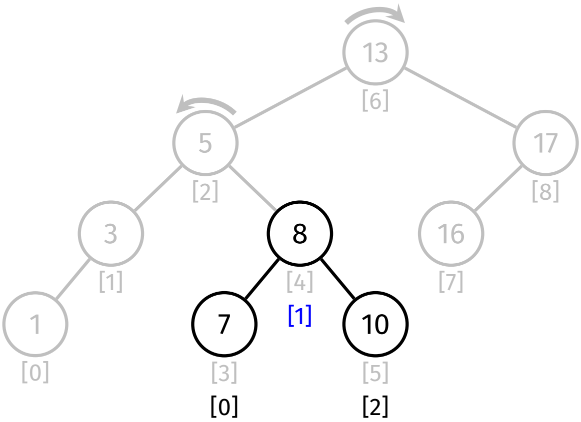
Size of left subtree is 6, and 4 < 6...
so partition left subtree around index 4
and then rotate right at 13



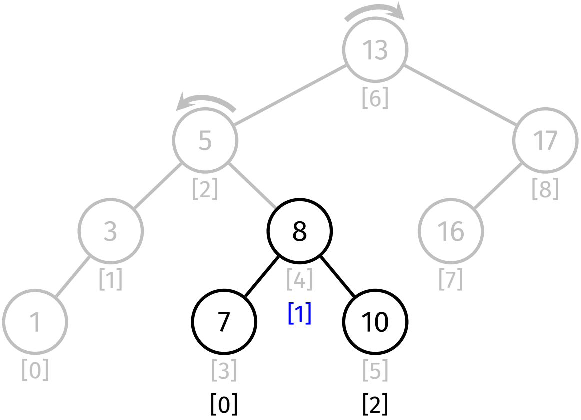
Size of left subtree is 2, and 4 > 2...



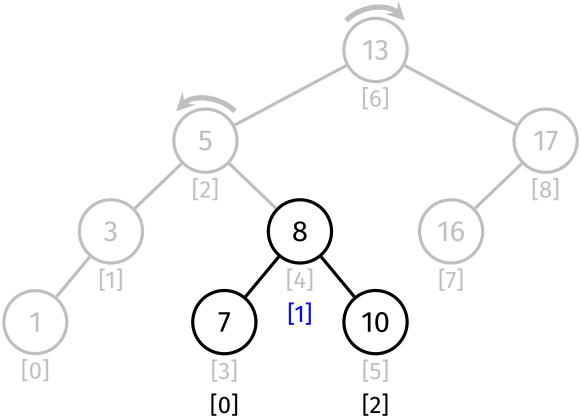
Size of left subtree is 2, and 4 > 2...
so partition right subtree around index (4 - 2 - 1 = 1)
and then rotate left at 5



Size of left subtree is 1, and 1 = 1...

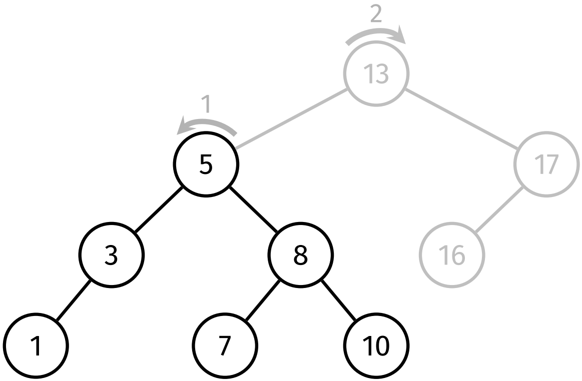


Size of left subtree is 1, and 1 = 1...
so we have found the desired node

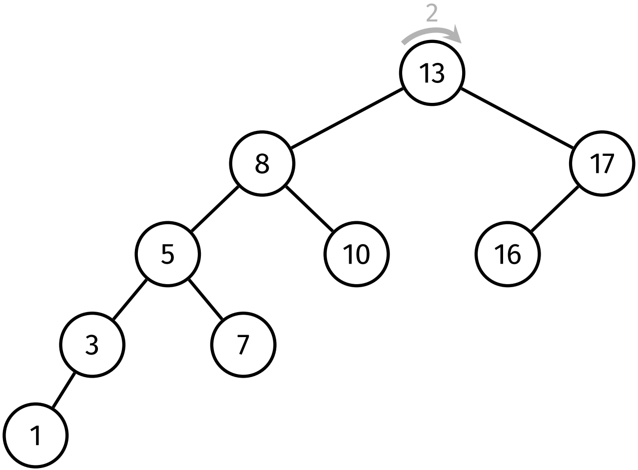


Partition
Pseudocode - Example

Unwinding...
Rotate left at 5



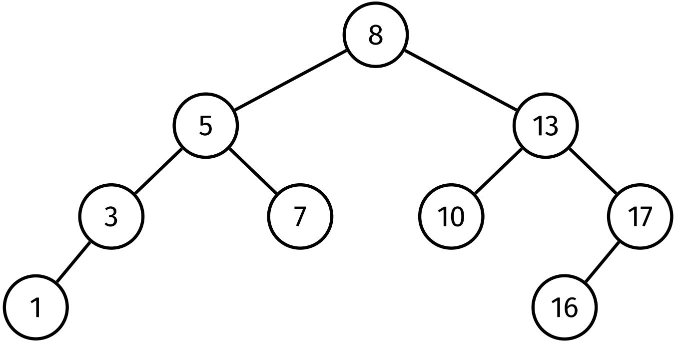
Unwinding...
Rotate right at 13



Partition

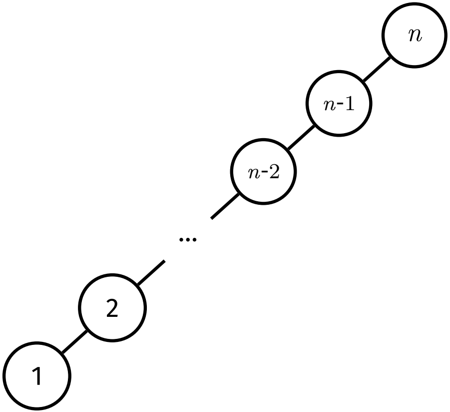
Pseudocode - Example

Done



Analysis:

- size() operation is expensive
- Can cause partition to be $O(n^2)$ in the worst case
  - For example, in the following tree:



Analysis (continued):

- To improve efficiency, can change node structure so that each node stores the size of its subtree in the node itself
    - However, this will require extra work in other functions to maintain

```
struct node {
    int item;
    struct node *left;
    struct node *right;
    int size;
};
```

Balancing Methods

Two categories:

GLOBAL REBALANCING

visit every node and balance its subtree;
$\Rightarrow$ perfectly balanced tree — at cost.

LOCAL REBALANCING

perform small, efficient, localised operations
to try to improve the overall balance of the tree
... at the cost of imperfect balance

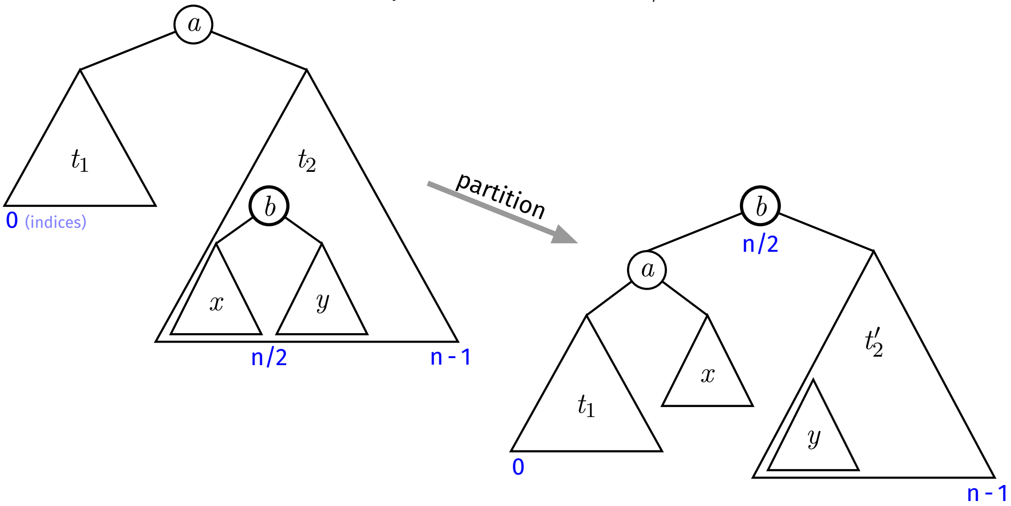
Global Rebalancing

Idea:
Completely rebalance whole tree so it is size-balanced

Method:
Lift the median node to the root
by partitioning on index $\textsc{size}(t)/2$,
then rebalance both subtrees (recursively)

Global Rebalancing

First, partition on index $n/2$...



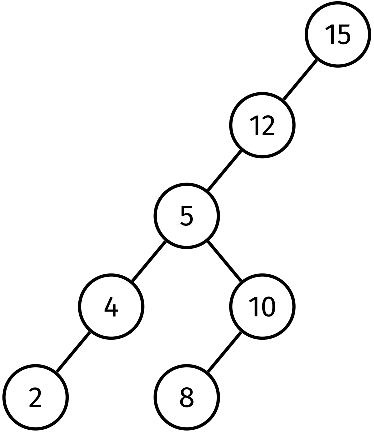
...then rebalance both subtrees

```
rebalance(t):
    Input:  tree t
    Output: rebalanced t

    if size(t) < 3:
        return t

    t = partition(t, size(t) / 2)
    t->left = rebalance(t->left)
    t->right = rebalance(t->right)
    return t
```

Global Rebalancing

Example

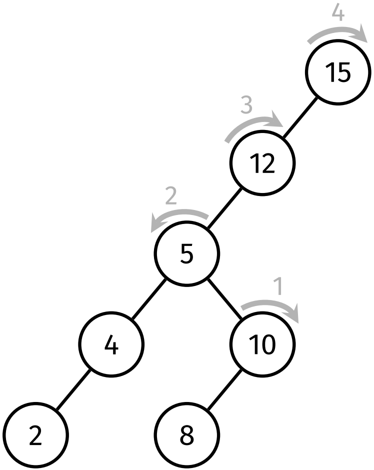
Rebalance the following tree:



Global Rebalancing

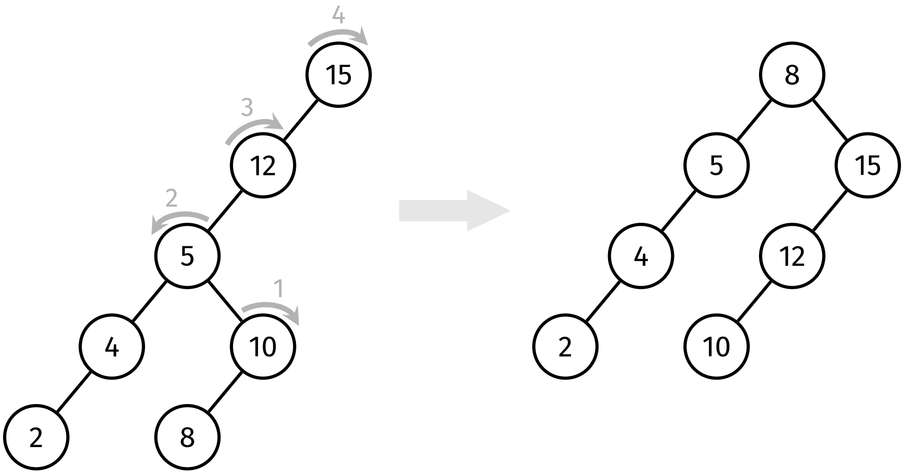
Example

First, partition the tree on index $7/2 = 3$ (node 8)



Global Rebalancing
Example

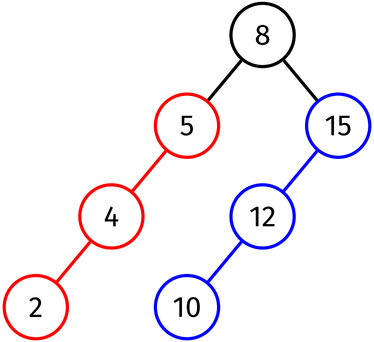
First, partition the tree on index $7/2 = 3$ (node 8)



Global Rebalancing

Example

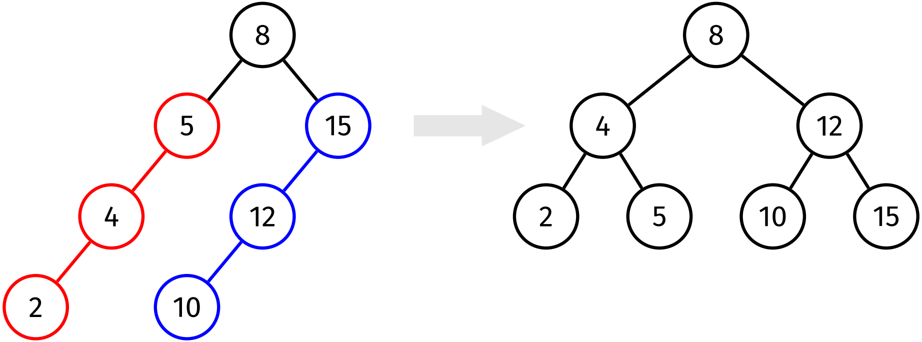
Then, recursively rebalance subtrees



Global Rebalancing

Example

Then, recursively rebalance subtrees



Global Rebalancing

Analysis

Worst-case time complexity: $O(n \log n)$

- Assume nodes store the size of their subtrees
- First step: partition entire tree on index $n/2$
  - This takes at most n recursive calls, n rotations $\Rightarrow$ n steps
  - Result is two subtrees of size $\approx n/2$
- Then partition both subtrees
  - Partitioning these subtrees takes $n/2$ steps each $\Rightarrow$ n steps in total
  - Result is four subtrees of size $\approx n/4$
- ...and so on...
- About $\log_2 n$ levels of partitioning in total, each requiring n steps $\Rightarrow$ $O(n \log n)$

Global Rebalancing

Problems

What if we insert more items?

- Options:
  - Rebalance on every insertion
    - Not feasible
  - Rebalance every k insertions; what k is good?
  - Rebalance when imbalance exceeds threshold.
- It's a tradeoff...
  - We either have more costly insertions
  - Or we have degraded performance for periods of time

bstInsert(t, v):
  Input: tree t, value v
  Output: t with v inserted

  t = insertAtLeaf(t, v)

  if size(t) mod $k = 0$:
    t = rebalance(t)

  return t

Periodic Rebalancing

Remarks

- Good if tree is not modified very often
- Otherwise...
  - Insertion will be slow occasionally due to rebalancing
  - Performance will gradually degrade until next rebalance

Local Rebalancing

Perform small, efficient, localised operations
in an attempt to improve the overall balance of the tree

1. root insertion
2. randomised insertion

Root Insertion

Idea:

Rotations change the structure of a tree

If we perform some rotations every time we insert, that may restructure the tree randomly enough such that it is more balanced

One systematic way to perform these rotations:
Insert new values at the root

Root Insertion

Method:

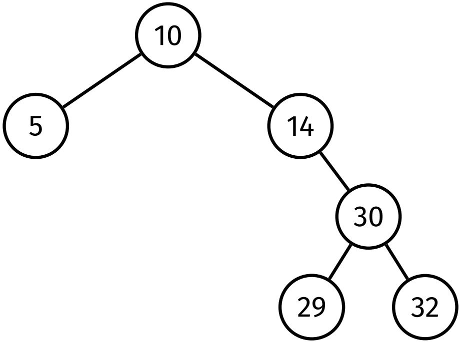
Insert new value normally (at the leaf) ...

... and then rotate the new node up to the root.

Root Insertion

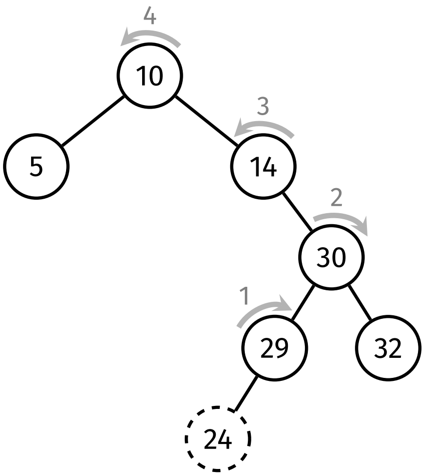
Example

Insert 24 at the root of this tree:



Root Insertion
Example

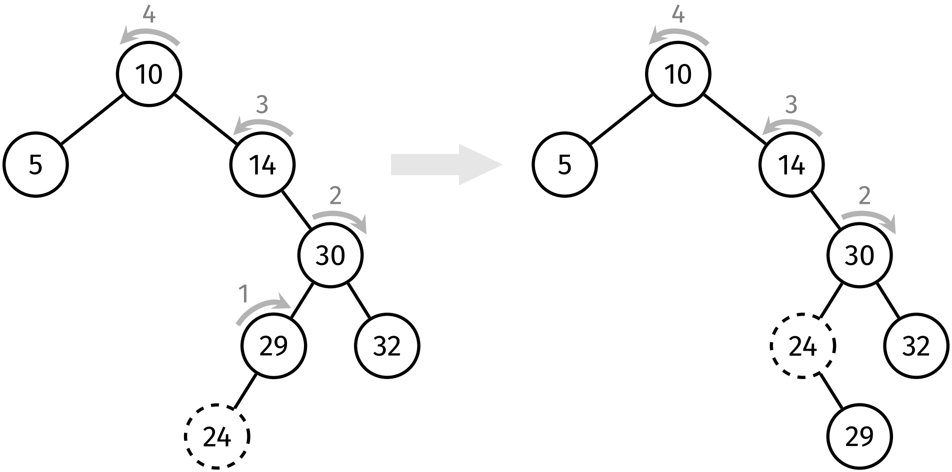
Insert 24 at the root of this tree:



Root Insertion

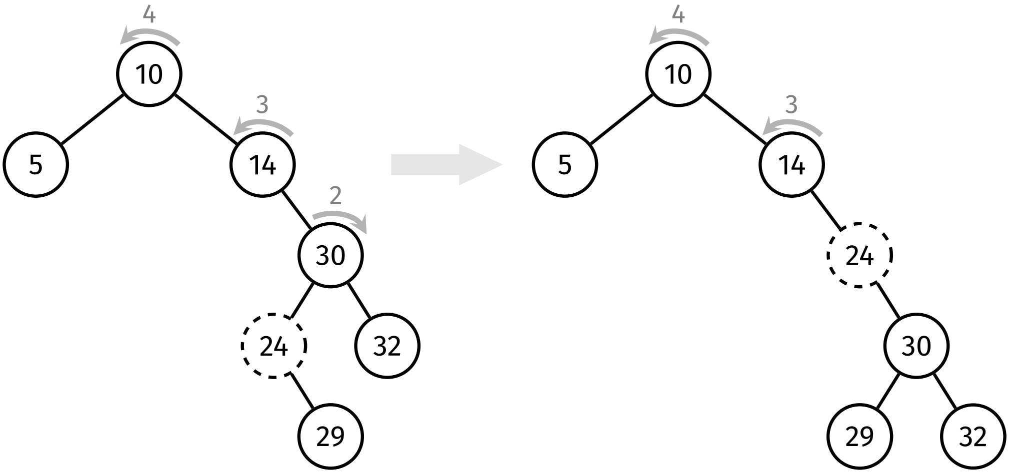
Example

Rotate right at 29



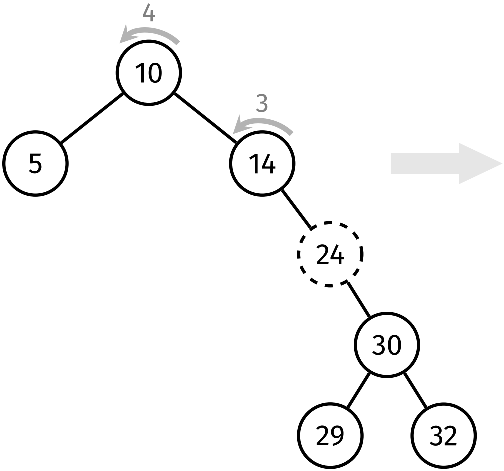
Root Insertion
Example

Rotate right at 30



Root Insertion
Example

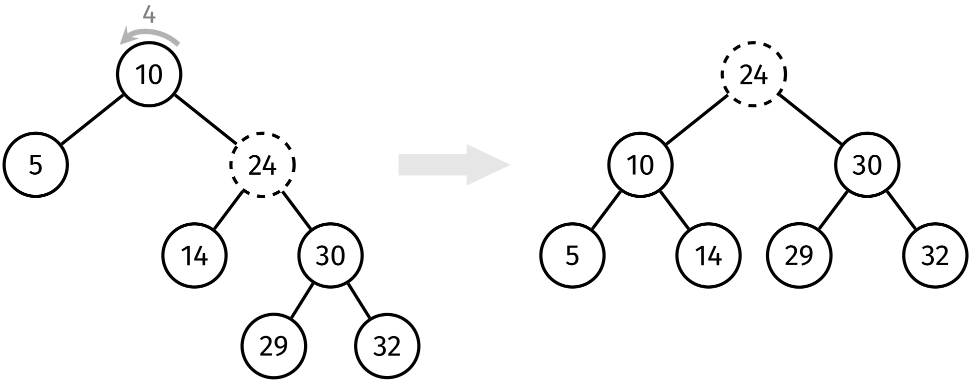
Rotate left at 14



Root Insertion

Example

Rotate left at 10



Root Insertion

Pseudocode


```
insertAtRoot(t, v):
    Input:  tree t, value v
    Output: t with v inserted at the root

    if t is empty:
        return new node containing v
    else if v < t->item:
        t->left = insertAtRoot(t->left, v)
        t = rotateRight(t)
    else if v > t->item:
        t->right = insertAtRoot(t->right, v)
        t = rotateLeft(t)

    return t
```

Analysis:

- Same time complexity as normal insertion: $O(h)$
- Tree is more likely to be balanced, but no guarantee
- Root insertion ensures recently inserted items are close to the root
  - Useful for applications where recently added items are more likely to be searched
- Major problem: ascending-ordered and descending-ordered data is still a worst case for root insertion

Randomised Insertion

BSTs don't have control over insertion order.
Worst cases — (partially) ordered data — are common.

Idea:
Introduce some randomness into insertion algorithm:
Randomly choose whether to insert normally or insert at root

Randomised Insertion

Pseudocode

```
insertRandom(t, v):
    Input:  tree t, value v
    Output: t with v inserted

    if t is empty:
        return new node containing v

    // p/q chance of inserting at root
    if random() mod q < p:
        return insertAtRoot(t, v)
    else:
        return insertAtLeaf(t, v)
```

Note: `random()` is a pseudo-random number generator
30% chance of root insertion ⇒ choose p = 3, q = 10

Randomised Insertion

Remarks

Randomised insertion creates similar results to inserting items in random order.

Tree is more likely to be balanced (but no guarantee)

Summary

	Advantages	Disadvantages
Global rebalancing	Guarantees a balanced tree	Inefficient ($O(n \log n)$ per rebalance), or periods of degraded performance
Local rebalancing	Efficient (adds only a constant factor overhead to insertion)	Not guaranteed to produce a balanced tree

Feedback

https://forms.office.com/r/2BW7BasQ77