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CE2101/ CZ2101: Algorithm Design and Analysis

Insertion Sort

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复刷的简要slide

Insertion Sort (Pseudo Code)

desired outcome: 1 2 3 4 5 6.

Best Case: input 1 2 3 4 5 6.

$O(n)$ time $= n-1$ (每个项比较一次)

Worst Case: input 6 5 4 3 2 1

$O(n^2)$ time $= \sum_{i=2}^{n}(i-1) = 1+2+3+\dots+n-1$ 每项比较 i-1 次

$= \frac{(1+n-1)\times(n-1)}{2} = \frac{n(n-1)}{2}$

The incremental approach

```
void InsertionSort (ALIST slot[ ], int n)
{ // input slot is an array of n records;
 // assume n > 1;
 for (int i=1; i < n; i++)
     for (int j=i; j > 0; j--) {
       if (slot[j].key < slot[j-1].key)
          swap(slot[j], slot[j-1]);
       else break;
     }
}
```

Average Case: input 3 5 6 2 1 4

$O(n^2)$ 每一项都可能比较 1, 2, … i 次, 比较次数出现的几率都是 $\frac{1}{i}$.

对任意一项, $t = \sum_{j=1}^{i} \frac{1}{i}\times j = \frac{1}{i}\sum_{j=1}^{i} j = \frac{1}{i}\times\frac{(i+1)\times i}{2} = \frac{i+1}{2}$

对全部n项: $t = \sum_{i=1}^{n-1} \frac{i+1}{2} = \frac{(1+\frac{n-1+1}{2})\times n}{2}$ ← $O(n^2)$

Learning Objectives

At the end of this lecture, students should be able to:

- Explain the incremental approach as a strategy of algorithm design
  sort one-by-one
- Describe how Insertion sort algorithm works, by manually running its pseudo code on a toy example
- Analyse the time complexities of Insertion sort in the best case, worst case and average case

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Insertion Sort of a Hand of Cards

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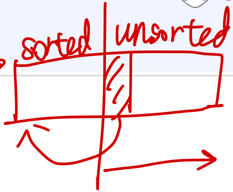
Insertion Sort

The incremental approach

- An intuitive, primitive sorting method
- A form of insertion into an ordered list
- Given an unordered set of objects, repeatedly remove an entry from the set and insert it into a new ordered list
- Ensure that the new list is ordered at all times
- Each insertion requires movements of certain entries in the ordered list



Sorted: grow by 1
unsorted: shrink by 1

Insertion Sort (Pseudo Code)

The incremental approach

array size.

```
void InsertionSort (ALIST slot[ ], int n)
{ // input slot is an array of n records;
 // assume n > 1;
 for (int i=1; i < n; i++)
   for (int j=i; j > 0; j--) {
     if (slot[j].key < slot[j-1].key)
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```

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Insertion Sort (Pseudo Code)

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45	29	06	64	12	16

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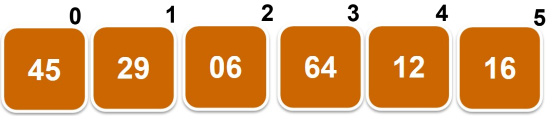

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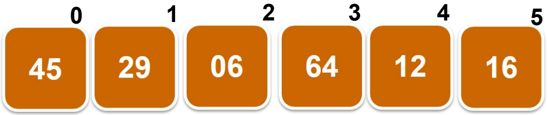

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```

index starts at 0.

Pick up a new item from slot[ ]

Insertion Sort (Pseudo Code)

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}
```

→ inner loop.

往前查找位置

Find the correct position to insert the item.

Insertion Sort (Pseudo Code)

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 for (int i=1; i < n; i++)
    for (int j=i; j > 0; j--) {
      if (slot[j].key < slot[j-1].key)
        swap(slot[j], slot[j-1]);
      else break;
   }
}
```

compare only key values.

swap record.

*该算法 swap了好多次.

Insertion Sort Example

Insertion Sort Example

Index j	0	1	2	3	4	5
	45	29	06	64	12	16

Insertion Sort Example

If (slot[j].key < slot[j-1].key)

(slot[**2**].key < slot[**1**].key)

16 < **12**

Index j	0	1	2	3	4	5
		12	16	29	45	64

Insertion Sort Algorithm
(Recap)

Insertion Sort Algorithm

- Original unsorted set and final sorted list are both in array slot[ ].
- Since sorting is performed directly on original array without any working storage, swapping and shifting are essential.

06	12	29	45	64	16

Insertion Sort Algorithm

- Original unsorted set and final sorted list are both in array slot[ ].

No extra space required

- Since sorting is performed directly on original array without any working storage, swapping and shifting are essential.
- During sorting, slot[ ] contains sorted portion on the ‘left’ and unsorted portion on the ‘right’; sorted portion grows while unsorted portion shrinks.

Sorted				Unsorted	
06	12	29	45	64	16

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Insertion Sort Algorithm

- In the outer 'for' loop, i begins with 1 because the ordered list begins with one element (slot[0]); hence slot[1] is the first element from the unordered list.

```
for (int i=1; i < n; i++)
     for (int j=i; j > 0; j--) {
       if (slot[j].key < slot[j-1].key)
           swap(slot[j], slot[j-1]);
       else break;
}
```

Insertion Sort Algorithm

- At each iteration, number at slot[i] is inserted into the new ordered list.

The inner ‘for’ loop finds the correct position in the ordered list by swapping slot[j] with slot[j-1] as long as the key of slot[j-1] is > the key of slot[j].

```
for (int i=1; i < n; i++)
    for (int j=i; j > 0; j--) {
      if (slot[j].key < slot[j-1].key)
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Insertion Sort Algorithm

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for (int i=1; i < n; i++)
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      else break;
```

Complexity of Insertion Sort

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Complexity of Insertion Sort

Number of key comparisons:

- There are $n - 1$ iterations **(the outer loop)**

已经排好，每个都只需比较、确认一次即可

- **Best case:** 1 key comparison/ iteration, **total: $n - 1$**
- **Already sorted:** [06] [12] [16] [29] [45] [64]
- **Worst case:** i key comparisons for the ith iteration
- **Reversely sorted:** [64] [45] [29] [16] [12] [06]

Total: $$1 + 2 + 3 + \ldots + (n-1) = \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} \in \Theta(n^2)$$

Insertion Sort Performance

Average case: the *i*th iteration may have 1, 2, …, *i* key comparisons, each with 1/*i* chance.

The average no. of comparisons in the *i*th iteration:

$$\frac{1}{i}\sum_{j=1}^{i} j = \frac{1}{i}\left(1+2+\ldots+i\right)$$

Summation for the ***n*-1** iterations:

$$1+\frac{1}{2}\left(1+2\right)+\frac{1}{3}\left(1+2+3\right)+\ldots+\frac{1}{n-1}\left(1+\ldots+n-1\right) = \sum_{i=1}^{n-1}\left(\frac{1}{i}\sum_{j=1}^{i} j\right)$$

$$= \sum_{i=1}^{n-1}\left(\frac{1}{i}\frac{i(i+1)}{2}\right) = \frac{1}{2}\sum_{i=1}^{n-1}(i+1) = \frac{1}{2}\left(\frac{(n-1)(n+2)}{2}\right) = \Theta\left(n^2\right)$$

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Insertion Sort Performance

- **Strengths:**
  - Good when the unordered list is almost sorted.
  - Need minimum time to verify if the list is sorted.
  - Fast with linked storage implementation: no movement of data.

linked list → array is not the preferred data structure for insertion sort

- **Weaknesses:**
  - When an entry is inserted, it may still not be in the final position yet.
  - Every new insertion necessitates movements for some inserted entries in ordered list.
  - When each slot is large (e.g., a slot contains a large record of 10Mb), movement is expensive.
  - Less suitable with contiguous storage implementation.

Summary

- Insertion sort uses the incremental approach.
- **Main idea:** Repeatedly pick up an element *x* to insert into a sorted sub-array on the left side, by comparing *x* with its left neighbour. If they are out of order, swap them; otherwise, insert *x* there.

Index j	**0**	**1**	**2**	**3**	**4**	**5**
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- **Main idea:** Repeatedly pick up an element *x* to insert into a sorted sub-array on the left side, by comparing *x* with its left neighbour. If they are out of order, swap them; otherwise, insert *x* there.
- **Time complexity analysis:**
  - Best case: □(*n*), when input array is already sorted.
  - Worst case: □(*n*2), when input array is reversely sorted.
  - Avera : □(*n*2).