COMP0043 Assessment 3
The Heston Model and Improvements
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Basic Heston Model

Heston model is an famous method for evaluating European Option with the assumption that the volatility of underlying assets follow stochastic distribution or random process, while the volatility is assumed as constant value in Black Scholes Model. The Heston model improves the understanding of the volatility surface and serves as an efficient calibration method.
The formula of basic Heston model is as follows:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_{1t}$$
$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t} dW_{2t}$$

The difference between Heston and BSM is that Vt^½ replaces constant volatility sigma and it is described by Cox–Ingersoll–Ross model. The two brownian motions in dS and dV are also correlated.

Volatility Smilling

When it comes to joint calibration derivative portfolios in two related markets, for instance solving the joint S&P 500/VIX smile calibration problem, the basic Heston model needs to be improved by some approaches. Volatility Smiles, got the name by its U-shape, shows the relationship of change of implied volatility and strike price of a series of options which are written on the same underlying asset and also with expiration date. We will discuss two main approaches to achieve the goal of joint calibration and their advantages and disadvantages respectively.

Multifactor or Multiscale Models

The basic Heston model(1993) is one of the most common models of one-factor stochastic volatility models. In this basic model, volatility is described by one single factor in a square root function, which can be solved by the Fast Fourier Transform method(Carr and Madan (1999)). Extending the basic one, multifactor approach is to add one or even more extra terms, such as jumps or deterministic displacement in the basic Heston model to achieve more accuracy in calibrating complex models. In H++ model for example, a deterministic shift and jumps are added into an affine model to calibrate SPX and VIX derivatives simultaneously.

The advantages are:
- If the added factor is simple, for example by adding deterministic displacement Φ, there is not much extra cost in computation.
- Additional factors bring more flexibility, which perform better fitting on smile-skew effect at long and short maturities.

The disadvantages are:
- Too many parameters cause overfitting. Too many model parameters are introduced to fit the in-the-sample data, which will make the model fit the sample data too much, and reduce the ability to simulate the out-of-sample data. In C. Pacati et al(2018), Heston++ model shows overfitting by out-of-sample pricing. Nevertheless, the Heston++ model also shows better pricing performance compared to the H model even in this circumstance. So this drawback is acceptable.

Rough Heston Model

According to Gatheral et al., the log-volatility time series can be described by fractional Brownian motion, with Hurst parameter of order 0.1. The following formula describe the Rough Heston Model:

$$\frac{dS_u}{S_u} = \sigma_u \left[\rho dW_u^* + \sqrt{1-\rho^2} dW_u^{\perp}\right]$$

$$\sigma_u^2 = \sigma_t^2 + \frac{1}{\Gamma(\alpha)} \int_t^u (u-s)^{\alpha-1} \lambda \left[\theta - \sigma_s^2\right] ds + \frac{\nu}{\Gamma(\alpha)} \int_t^u (u-s)^{\alpha-1} \sigma_s dW_s^*$$

Where we notice that α must be in the range of 0.5 to 1.

The advantages are:
- Rough Heston dynamics provide conditions for short-time option prices to possess similar explosive behaviour of the second-order approximation for implied volatility.
- Quasi-closed formula for characteristic function, which allows fast pricing
- non-markov, so that infinite dimensional markovian

- allows solving by new technology in computer science, such as neural network

The disadvantages are:

- Fractional Terms in the Rough Heston model may cause higher error bound than approximation in the basic Heston model. This drawback limits the usage in real pricing procedure because of its natural inaccuracies.

Calibration

Calibration is an important procedure for using the numerical method to one or more specific dataset. When we focus on calibration of the quadratic rough Heston model, according to J. Gatheral et al. 2020, we first need this function Θ to calibrate this parameter set v=(α,⅄,a,b,c,Z0):

$$\theta_0(t) = \frac{Z_0}{\lambda\Gamma(1-\alpha)} t^{-\alpha}.$$

And the following objective function needs to be minimized using metrics described following to obtain parameters. In the following procedure, we need to obtain v by using Monte-Carlo methods. In this way, Rough Heston would be calibrated on the two types of derivatives: SPX and VIX.

$$F(\nu) = \frac{1}{\#\mathcal{O}^{SPX}} \sum_{o \in \mathcal{O}^{SPX}} (\sigma^{o,mid} - \sigma^{o,\nu})^2 + \frac{1}{\#\mathcal{O}^{VIX}} \sum_{o \in \mathcal{O}^{VIX}} (\sigma^{o,mid} - \sigma^{o,\nu})^2,$$

Fitting performance can be analyzed by mean percentage error(MPE), mean absolute percentage error(MAPE), mean squared error(MSE) and so on.Those metrics serve as different dimensional tools for presenting errors. MSE represents the volatility (standard deviation) of errors, MPE measures direction of errors and MAPE is used for indicating how large the error is(magnitude).

References

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