Ch 1 Riemann Integral

Def 1.1 Let $a, b \in \mathbb{R}$, $a < b$. A partition of $[a,b]$ is a set of $\{x_0, \dots, x_n\}$ with $x_i \neq x_j$ if $i \neq j$. and $a = x_0 < x_1 < \dots < x_n = b$.

Def 1.2 Let $f: [a,b] \to \mathbb{R}$ be a bounded function and let P be a partition of $[a,b]$. The lower Riemann sum is

$$L(f, P, [a,b]) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{[x_{i-1}, x_i]} f.$$

the upper Riemann sum is

$$U(f, P, [a,b]) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{[x_{i-1}, x_i]} f.$$

Note: Clearly $L(f, P, [a,b]) \leq U(f, P, [a,b])$

Def 1.3 Let P and P' be two partitions of $[a,b]$. P' is finer than P if $P \subseteq P'$. Note: finer is not smaller, P' has more points.

Prop 1.4 If $P \subseteq P'$, then $L(f, P, [a,b]) \leq L(f, P', [a,b]) \leq U(f, P', [a,b])$ $\leq U(f, P, [a,b])$

Proof. Let $P = \{x_0, \dots, x_n\}$, $P' = \{x_0', \dots, x_N'\}$, $N > n$.
$\forall j = 1, \dots, n$, let $k = 0, 1, \dots, N-1$ s.t. $x_{j-1} = x_k' < x_{k+1}' < \dots < x_{k+m}' = x_j$ for some $m \geq 1$ s.t. $k + m \leq N$.
Then $(x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f = \sum_{i=1}^{m} (x_{k+i}' - x_{k+i-1}') \inf_{[x_{j-1}, x_j]} f \leq \sum_{i=1}^{m} (x_{k+i}' - x_{k+i-1}') \inf_{[x_{k+i-1}', x_{k+i}']} f$.
Same to sup.

Def 1.5 Let $f: [a,b] \to \mathbb{R}$ be a bounded function.
The lower Riemann integral of f is: $L(f, [a,b]) = \sup_P L(f, P, [a,b])$
The upper Riemann integral of f is: $U(f, [a,b]) = \inf_P U(f, P, [a,b])$
Note: Clearly $L(f, [a,b]) \leq U(f, [a,b])$

Def 1.6 Let $f: [a,b] \to \mathbb{R}$ be a bounded function. f is Riemann integrable if $L(f, [a,b]) = U(f, [a,b])$ in which case we define

$$\int_a^b f(x)\,dx = L(f, [a,b]) = U(f, [a,b])$$

Note: Clearly $(b-a) \inf_{[a,b]} f \leq \int_a^b f(x)\,dx \leq (b-a) \sup_{[a,b]} f$.

Prop 1.7 Let $f, g: [a,b] \to \mathbb{R}$ be bounded Riemann integrable functions. Then $\alpha f + \beta g$ is Riemann integrable, $\forall \alpha, \beta \in \mathbb{R}$. And

$$\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

Lemma 1.8 If $f: [a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.

Proof. Since f is continuous on a closed bounded interval, then f is bounded and uniformly continuous.

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Let $n \in \mathbb{N}_+$ s.t. $\frac{b-a}{n} < \delta$. Let P be the equal-space partition.

Then, $U(f,[a,b]) - L(f,[a,b]) \leq U(f,P,[a,b]) - L(f,P,[a,b])$

$= \frac{b-a}{n} \sum_{i=1}^{n} \left(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \right) \leq \frac{b-a}{n} n\varepsilon = (b-a)\varepsilon$.

Let $\varepsilon \searrow 0$, we conclude that $U(f,[a,b]) \leq L(f,[a,b])$

Appendix. Why Riemann integral is not good enough? It cannot deal with ① Discontinuities ② Unbounded functions ③ Limits

Examples ① $f: [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$

$\forall [a,b] \subseteq [0,1]$, $\sup_{[a,b]} f = 1$ and $\inf_{[a,b]} f = 0$ since $\mathbb{Q}$ is dense in $\mathbb{R}$.

② $f: [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$

$\sup_{[0,x_1]} f = \infty$, $\forall x_1 \in (0,1]$. But $\lim_{a \to 0^+} \int_a^1 f(x)\,dx = \lim_{a \to 0^+} 2\sqrt{x}\Big|_a^1 = 2$.

③ Let $\mathbb{Q} \cap [0,1] = \{r_1, r_2, \dots, r_n, \dots\}$, $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \dots, r_n\} \\ 0 & \text{otherwise} \end{cases}$

We know $f_n \to f$ pointwise, where $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$

However, $\int_0^1 f_n(x)\,dx = 0$ but $f(x)$ is not Riemann integrable.

Ch 2 Outer Measures

Def 2.1 Let $\xi \subseteq \mathcal{P}(X)$ be a family of elementary sets, with $\emptyset \in \xi$, and $X = \bigcup_{n=1}^{\infty} X_n$ for some $X_n \in \xi$.
Let $\rho : \xi \to [0, \infty]$ be the elementary measure s.t. $\rho(\emptyset) = 0$.
Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by
$$\mu^*(E) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : E_n \in \xi, \forall n \in \mathbb{N}_+, E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}$$

Def 2.2 If $I \subseteq \mathbb{R}$ is an interval, define $l(I) = \sup(I) - \inf(I)$, so that $l(I) = l(\mathring{I}) = l(\bar{I})$. A rectangle is a set of form $R = I_1 \times \dots \times I_n$ with $l(R) = \prod_{i=1}^{n} l(I_i)$. $\forall \alpha \in \mathbb{R}^N$, $l(R) = l(\alpha + R)$, $\forall R \subseteq \mathbb{R}^N$.
Note: with rectangles and $l(\cdot)$, we have Lebesgue outer measure.

Def 2.3 Let X be a nonempty set. A mapping $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure if ① $\mu^*(\emptyset) = 0$ ② $E \subseteq F \Rightarrow \mu^*(E) \leq \mu^*(F)$
③ $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$, $\forall E_n \in \mathcal{P}(X)$

Prop. 2.4 μ^* defined in Def 2.1 is an outer measure and $\mu^*(E) \leq \rho(E)$ $\forall E \in \xi$.

Proof. ① Since $\emptyset \in \xi$ and $\rho(\emptyset) = 0$, then $\mu^*(\emptyset) = 0$
② Let $E \subseteq F \subseteq \bigcup_{n=1}^{\infty} F_n$, $F_n \in \xi$. Then $E \subseteq \bigcup_{n=1}^{\infty} F_n$, so $\mu^*(E) \leq \sum_{n=1}^{\infty} \rho(F_n)$. Taking inf on both sides, we have $\mu^*(E) \leq \mu^*(F)$
③ Let $E_n \subseteq X$, $n \in \mathbb{N}_+$.
$\forall \varepsilon > 0$, $\forall n \in \mathbb{N}_+$, find $\{E_{n,k}\}_{k \in \mathbb{N}_+} \subseteq \xi$ s.t. $\sum_{k=1}^{\infty} \rho(E_{n,k}) \leq \mu^*(E_n) + \frac{\varepsilon}{2^n}$, $E_n \subseteq \bigcup_{k=1}^{\infty} E_{n,k}$. Then $\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{n,k}$, hence $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho(E_{n,k}) \leq \sum_{n=1}^{\infty} [\mu^*(E_n) + \frac{\varepsilon}{2^n}] = \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon$. Let $\varepsilon \downarrow 0$.

Prop 2.5 $\mu^*(E) = \rho(E)$, $E \in \xi$ iff $\rho(E) \leq \sum_{i=1}^{\infty} \rho(E_i)$, $E_i \in \xi$, $E \subseteq \bigcup_{i=1}^{\infty} E_i$

Proof. $\Leftarrow$ Let $E \subseteq \bigcup_{i=1}^{\infty} E_i$, $i \in \mathbb{N}_+$. Then $\rho(E) \leq \sum_{i=1}^{\infty} \rho(E_i)$. Taking inf on both sides, we get $\rho(E) \leq \mu^*(E)$. However, $\mu^*(E) \leq \rho(E)$ by def.
$\Rightarrow$ Assume that $\mu^*(E) = \rho(E)$, $\forall E \in \xi$. Let $E, E_n \in \xi$, $n \in \mathbb{N}_+$, $E \subseteq \bigcup_{n=1}^{\infty} E_n$. Then $\rho(E) = \mu^*(E) \leq \sum_{n=1}^{\infty} \rho(E_n)$
Note: in the Lebesgue case, we have $(\mathcal{L}^N)^*(R) = l(R)$.

Remark 2.6 The Lebesgue outer measure is translation invariant, i.e. if $A \subseteq \mathbb{R}^N$ and $x_0 \in \mathbb{R}^N$, then $(\mathcal{L}^N)^*(A) = (\mathcal{L}^N)^*(A + x_0)$.

Proof. If $A \subseteq \bigcup_{n=1}^{\infty} R_n$, then $A + x_0 \subseteq \bigcup_{n=1}^{\infty} (R_n + x_0)$ and $l(x_0 + R_n) = l(R_n)$ because $l(\alpha + I) = l(\alpha + (a, b)) = l((a + \alpha, b + \alpha)) = b - a = l(I)$.

Remark 2.7 It's possible to construct $A, B \subseteq \mathbb{R}$ s.t. $A \cap B = \emptyset$ and $(\mathcal{L}')^*(A \cup B) < (\mathcal{L}')^*(A) + (\mathcal{L}')^*(B)$.

Proof. In $[-1,1]$, $a \sim b$ if $a-b \in \mathbb{Q}$. Since $\forall a \in [-1,1]$, $[-1,1] = \bigcup_{a \in [-1,1]} [a]$.
Define $E = \{[a] : a \in [-1,1]\}$ and V be the set containing exactly one element from each equivalence class. $|V \cap [a]| = 1$.
Define $[-2,2] \cap \mathbb{Q} = \{r_1, r_2, \dots\}$. If $a \in [-1,1]$, let $V \cap [a] = [\tilde{a}]$.
Then $a - \tilde{a} \in \mathbb{Q} \Rightarrow a = \tilde{a} + r_k$ for some $k \in \mathbb{N}_+$. Thus $[-1,1] \subseteq \bigcup_{k=1}^{\infty} (r_k + V)$
Thus $2 = l([-1,1]) \le \sum_{n=1}^{\infty} \mathcal{L}'(r_k + V) = \sum_{n=1}^{\infty} \mathcal{L}'(V)$. Hence $\mathcal{L}'(V) > 0$.

If $\exists\, i, j$ with $i \neq j$, such that $t \in (r_i + V) \cap (r_j + V)$, then $t = r_i + v_i = r_j + v_j$, implying $v_i - v_j \in \mathbb{Q}$. $\Rightarrow [v_i] = [v_j]$. Contradiction!
Therefore, if $i \neq j$, $(r_i + V) \cap (r_j + V) = \emptyset$.
Now $V \subseteq [-1,1]$, $r_k \in [-2,2]$, then $\bigcup_{k=1}^{n} (r_k + V) \subseteq [-3,3]$. Therefore $\mathcal{L}'(\bigcup_{k=1}^{n} (r_k + V)) \le l([-3,3]) = 6$.
However, by translation invariant, $\sum_{k=1}^{n} \mathcal{L}'(r_k + V) = n\mathcal{L}'(V)$.
Since $\mathcal{L}'(V) > 0$, $\exists\, n_0$ s.t. $n_0 \mathcal{L}'(V) > 6$. then $6 \ge \mathcal{L}'(\bigcup_{k=1}^{n} (r_k + V))$ and $\sum_{i=1}^{n} \mathcal{L}'(r_k + V) > 6$, contradiction!

Remark 2.8 Indeed, there does not exist a function $\mu : \mathcal{P}(X) \to [0, \infty]$ s.t.
① $\mu(I) = l(I)$ ② $\mu(A) = \mu(\alpha + A)$ ③ $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

Ch3 Sigma Algebra

Def 3.1 Let X be a nonempty set and let $\mu^*: \mathcal{P}(X) \to [0,\infty]$ be an outer measure. A set E is μ^*-measurable if $\forall F \subseteq X$, $\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \setminus E)$.

Remark 3.2 If $B \subseteq X$ is μ^*-measurable, then $\forall A \subseteq X$ s.t. $A \cap B = \emptyset$, it holds $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$

Proof. $\mu^*(A \cup B) = \mu^*((A \cup B) \setminus B) + \mu^*((A \cup B) \cap B) = \mu^*(A) + \mu^*(B)$

Def 3.3 Let X be a non-empty set. A collection $\mathcal{M} \subseteq \mathcal{P}(X)$ is a σ-algebra if ① $\phi \in \mathcal{M}$ ② if $E \in \mathcal{M}$, then $X \setminus E \in \mathcal{M}$
③ if $E_n \in \mathcal{M}$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$.
The pair $(X, \mathcal{M})$ is called a measurable space.

Prop 3.4 Let $(X, \mathcal{M})$ be a measurable space. Then ① $X \in \mathcal{M}$,
② if $E, F \in \mathcal{M}$, then $E \cup F$, $E \cap F$, $E \setminus F \in \mathcal{M}$
③ if $E_n \in \mathcal{M}$, then $\bigcap_{n=1}^{\infty} E_n \in \mathcal{M}$.

Proof. Trivial to prove, Use set opperation and apply De Morgan Law

Def 3.5 Let X be a non-empty set. $\mu^*: \mathcal{P}(X) \to [0,\infty]$ is regular if $\forall E \subseteq X$, $\exists F$ μ^*-measurable s.t. $\mu^*(E) = \mu^*(F)$.

Prop. 3.6 The intersection of σ-algebras is a σ-algebra. But the union may not be.

Def 3.7 Let X be a non-empty set and $\mathcal{E} \subseteq \mathcal{P}(X)$. The σ-algebra generated by $\mathcal{E}$ is the intersection of all σ-algebras containing $\mathcal{E}$. For instance, the σ-algebra generated by all open sets is called Borel σ-algebra, denoted by $\mathcal{B}(X)$.

Def 3.8 Let (X, d) be a metric space. $\mu^*: \mathcal{P}(X) \to [0,\infty]$ is called a metric outer measure if $\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$ for all sets $E, F \subseteq X$ s.t. $\mathrm{dist}(E,F) := \inf\{d(x,y) : x \in E, y \in F\} > 0$.

Prop. 3.9 If $\mu^*: \mathcal{P}(X) \to [0,\infty]$ is a metric outer measure, then every Borel set is μ^*-measurable.

Proof. Enough to show every closed set is μ^*-measurable.

Let $C \subseteq X$ be closed and $F \subseteq X$ be such that $\mu^*(F) < \infty$.

Claim: $\mu^*(F \cap C) + \mu^*(F \setminus C) \leq \mu^*(F)$.

For $n \in \mathbb{N}_+$, define $E_0 = \{x \in F \setminus C : \text{dist}(x, C) \geq 1\}$

$E_n = \{x \in F \setminus C : \frac{1}{n+1} \leq \text{dist}(x, C) < \frac{1}{n}\}$, disjoint. Since C is closed,

$F \setminus C = \bigcup_{n=0}^{\infty} E_n$.

Let $x \in E_{2n}$, $y \in E_{2n+2}$ and $\frac{1}{2n+1} \leq \text{dist}(x, C) \leq d(x,y) + \text{dist}(y, C)$

$< d(x,y) + \frac{1}{2n+2}$. Thus $d(x,y) > 0$. i.e. $\text{dist}(E_{2n}, E_{2n+2}) > 0$.

By definition, we have $\sum_{n=0}^{k} \mu^*(E_{2n}) = \mu^*(\bigcup_{n=0}^{k} E_{2n}) \leq \mu^*(F) < \infty$.

Similarly, $\sum_{n=0}^{k} \mu^*(E_{2n+1}) < \infty$. Therefore $\sum_{n=0}^{\infty} \mu^*(E_n)$ converges.

$\forall n \in \mathbb{N}_+$, if $x \in F \cap C$ and $y \in \bigcup_{n=0}^{k} E_n$, then $\frac{1}{k+1} \leq \text{dist}(y, C) \leq d(x,y)$

Since μ^* is metric outer measure, we have $\mu^*(F \setminus C) + \mu^*(F \cap C)$

$= \mu^*(F \cap C) + \mu^*(\bigcup_{n=0}^{\infty} E_n) \leq \mu^*(F \cap C) + \mu^*(\bigcup_{n=0}^{k} E_n) + \mu^*(\bigcup_{n=k+1}^{\infty} E_n) \leq$

$\mu^*((F \cap C) \cup \bigcup_{n=0}^{k} E_n) + \sum_{n=k+1}^{\infty} \mu^*(E_n) \leq \mu^*(F) + \sum_{n=k+1}^{\infty} \mu^*(E_n)$.

Letting $k \to \infty$, we have $\mu^*(F \setminus C) + \mu^*(F \cap C) \leq \mu^*(F)$

Def 3.10 The class of all $(\mathcal{L}^N)^*$-measurable sets of $\mathbb{R}^N$ is called the σ-algebra of Lebesgue measurable sets.

Prop. 3.11 $(\mathcal{L}^N)^*$ is a metric outer measure. In particular, every Borel set of $\mathbb{R}^N$ is $(\mathcal{L}^N)^*$-measurable.

Proof. Let $E, F \subseteq \mathbb{R}^N$ be such that $d := \text{dist}(E, F) > 0$.

Consider a sequence of rectangles R_n s.t. $E \cup F \subseteq \bigcup_{n=1}^{\infty} R_n$. We can assume that $\text{diam}(R_n) < \frac{d}{2}$, $\forall n \in \mathbb{N}_+$.

Hence, if $R_n \cap F \neq \emptyset$, then $R_n \cap E = \emptyset$; if $R_n \cap E \neq \emptyset$, then $R_n \cap F = \emptyset$. Therefore, $\sum_{n=1}^{\infty} l(R_n) = \sum_{R_n \cap E \neq \emptyset} l(R_n) + \sum_{R_n \cap F \neq \emptyset} l(R_n) \geq$ $(\mathcal{L}^N)^*(E) + (\mathcal{L}^N)^*(F)$.

Taking inf over all sequences of R_n: $(\mathcal{L}^N)^*(E \cup F) \geq (\mathcal{L}^N)^*(E) + (\mathcal{L}^N)^*(F)$

Note: $\mathcal{B}(\mathbb{R}^N) \subseteq \mathcal{L}(\mathbb{R}^N) \subseteq \mathcal{P}(\mathbb{R}^N)$

Ch4 Measures

Def 4.1 Let $(X, \mathcal{M})$ be a measurable space. A measure on $(X, \mathcal{M})$ is a function $\mu: \mathcal{M} \to [0, \infty]$ s.t. ① $\mu(\phi) = 0$; ② $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ for pairwise disjoint E_n.

Prop 4.2 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E, F \in \mathcal{M}$, $E \subseteq F$.
① $\mu(E) \le \mu(F)$, ② $\mu(F \setminus E) = \mu(F) - \mu(E)$ if $\mu(E) < \infty$

Proof. ① $F = E \cup (F \setminus E)$ and $\mu(F \setminus E) \ge 0$. ② Trivial

Prop 4.3 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E, F \in \mathcal{M}$ with $\mu(E \cap F) < \infty$. Then $\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$.

Proof. $E \cup F = E \setminus (E \cap F) \cup F \setminus (E \cap F) \cup E \cap F$. Then apply Prop 4.2.

Prop 4.4 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E_n \in \mathcal{M}$. Then $\mu(\bigcup_{n=1}^{\infty} E_n) \le \sum_{n=1}^{\infty} \mu(E_n)$

Proof. Set $E_k' = E_n \setminus \bigcup_{1 \le k < n} E_k$ and $E_1' = E_1$. Hence $\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcup_{n=1}^{\infty} E_n')$
$= \sum_{n=1}^{\infty} \mu(E_n') \le \sum_{n=1}^{\infty} \mu(E_n)$

Lemma 4.5 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $E_n \in \mathcal{M}$ s.t. $E_n \subseteq E_{n+1}$.
Then $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \to \infty} \mu(E_n)$.

Proof. If $\exists n_0 \in \mathbb{N}_+$ s.t. $\mu(E_{n_0}) = \infty$, Then by monotonicity it's true.
Assume $\mu(E_n) < \infty$, $\forall n \in \mathbb{N}_+$. Then $\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcup_{n=1}^{\infty} (E_n \setminus E_{n-1}))$ $E_0 = \phi$
$= \sum_{n=1}^{\infty} \mu(E_n \setminus E_{n-1}) = \lim_{k \to \infty} \sum_{n=1}^{k} (\mu(E_n) - \mu(E_{n-1})) = \lim_{k \to \infty} \mu(E_k)$.
Note: if $E_n \supseteq E_{n+1}$, then we require $\mu(E_1) < \infty$. Prove by De Morgan $E_1 \setminus \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E_1 \setminus E_n)$ and $E_1 \setminus E_1 \subseteq E_1 \setminus E_2 \subseteq \dots \subseteq E_1 \setminus E_n$

Def 4.6 Given a measure space $(X, \mathcal{M}, \mu)$, the measure μ is Complete if $\forall E \in \mathcal{M}$, $\mu(E) = 0$, $\forall F \subseteq E$, we have $F \in \mathcal{M}$.

Thm 4.7 Borel - Cantelli Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $E_n \in \mathcal{M}$ s.t. $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then a.e. $x \in X$ belongs to finitely many E_n.
$\mu(\{x \in X : x \in E_k \text{ i.o.}\}) = 0$

Proof. $\{x \in X : x \in E_k \text{ i.o.}\} = \limsup_{n \to \infty} E_n = \bigcap_{m=1}^{\infty} \bigcup_{n \ge m} E_n =: E$. Suppose $\mu(E) = \alpha > 0$
Then $0 < \alpha = \mu(E) \le \mu(\bigcup_{n \ge m} E_n) \le \sum_{n \ge m}^{\infty} \mu(E_n)$, $\forall m \in \mathbb{N}_+$.
However, since $\sum_{n=1}^{\infty} \mu(E_n) < \infty$, $\exists m_0$ s.t. $\sum_{n=m_0}^{\infty} \mu(E_n) < \frac{\alpha}{2}$.
Contradiction!

Thm 4.8 Caratheodory Theorem

Let X be a non-empty set and $\mu^*: \mathcal{P}(X) \to [0, \infty]$ be an outer measure. Then $\mathcal{M}^* = \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\}$ is a σ-algebra and $\mu^*|_{\mathcal{M}^*}$ is a complete measure. Note: Proof is not required.

Fact 4.9 ① If $A \subseteq \mathbb{R}$ is Lebesgue integrable and $\mathcal{L}^1(A) > 0$. Then there exists a non-measurable subset of A.

② In general, the image of a Borel set under a continuous function is not a Borel set.

③ The image of a Lebesgue set under a continuous function may not be measurable.

④ Let X be a complete, seperable metric space and let $f: X \to \mathbb{R}$ be continuous. If $B \subseteq X$ is a Borel set, then $f(B) \subseteq \mathbb{R}$ is Lebesgue measurable. Not necessarily Borel measurable.

⑤ The inverse image of a Borel set under a continuous func. is a Borel set.

⑥ $\mathcal{B}(\mathbb{R}^N) \subsetneq \{\text{Lebesgue measurable sets}\} \subsetneq \mathcal{P}(X)$.

Note: "Fact" means I don't know how to prove but my professor said it's true.

Remark 4.10 $\mathcal{L}(\mathbb{R}^N) := \{\text{Lebesgue measurable sets}\} = \mathcal{B}(\mathbb{R}^N) + \mathcal{N}$ where $\mathcal{N} := \{A \subseteq \mathbb{R}^N : A \in \mathcal{L}(\mathbb{R}^N) \text{ and } \mathcal{L}^N(A) = 0\}$

Proof. Later in Ch 6.

Ch 5 Uniqueness

Def 5.1 We say that $\Pi \subseteq \mathcal{P}(X)$ is a Π-system if
① $\Pi \neq \emptyset$; ② $A, B \in \Pi \Rightarrow A \cap B \in \Pi$

Def 5.2 We say that $\Lambda \subseteq \mathcal{P}(X)$ is a λ-system if ① $X \in \Lambda$;
② $A, B \in \Lambda$, $A \subseteq B \Rightarrow B \setminus A \in \Lambda$ ③ $A_n \in \Lambda$, $A_n \subseteq A_{n+1} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \Lambda$
Note: σ-algebra is both a Π-system and a λ-system

Thm 5.3 Dynkin π-λ Theorem
If Π is a Π-system and Λ is a λ-system, then
$\Pi \subseteq \Lambda \Rightarrow \sigma(\Pi) \subseteq \Lambda$

Proof. Not required. Also there are other versions of this theorem.

Prop 5.4 If μ and ν are finite measures agree on Π, and $X \in \Pi$
then $\mu = \nu$ on $\sigma(\Pi)$

Proof. Since $X \in \Pi$, $\Lambda = \{A \in \mathcal{M} : \nu(A) = \mu(A)\}$ is a λ-system.
Since $\Pi \subseteq \Lambda$, by Thm 5.3, we have $\sigma(\Pi) \subseteq \Lambda$.

Remark 5.5 ① The intersection of λ-systems is a λ-system.
Given $\mathcal{E} \subseteq \mathcal{P}(X)$. $\lambda(\mathcal{E})$ is the λ-system generated by $\mathcal{E}$.
② If $\mathcal{F}$ is both a Π-system and a λ-system, then it is a σ-algebra.
③ If μ is a measure on $(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N))$, for every rectangle R, $\mu(R) = l(R)$, then $\forall A \in \mathcal{L}(\mathbb{R}^N)$, $\mu(A) = (\mathcal{L}^N)(A)$.

Proof. ③ Step 1: prove $\mu \leq \mathcal{L}^N$
Let $A \in \mathcal{L}(\mathbb{R}^N)$ and $A \subseteq \bigcup_{n=1}^{\infty} R_n$, R_n are rectangles. Then $\mu(A) \leq \sum_{n=1}^{\infty} \mu(R_n) = \sum_{n=1}^{\infty} l(R_n)$. Taking inf, $\mu(A) \leq (\mathcal{L}^N)^*(A) = (\mathcal{L}^N)(A)$

Step 2: prove $\mathcal{L}^N \leq \mu$
Assume first $A \in \mathcal{L}(\mathbb{R}^N)$ is bounded, $A \subseteq R$, Then $\mu(R \setminus A) \leq (\mathcal{L}^N)(R \setminus A)$ which implies $(\mathcal{L}^N)(A) \leq \mu(A)$. For arbitrary $A \in \mathcal{L}(\mathbb{R}^N)$, write $A = \bigcup_{n=1}^{\infty} A \cap (-n,n)^N$. We have $\mu(A) = \lim_{n\to\infty} \mu(A \cap (-n,n)^N) \geq$ it's bounded $\lim_{n\to\infty} (\mathcal{L}^N)(A \cap (-n,n)^N) = \mathcal{L}^N(A)$.

Ch6 Regularity

Def 6.1 An outer measure $\mu^* : \mathcal{P}(X) \to [0,\infty]$ is regular if $\forall E \subseteq X$, $\exists$ a μ^*-measurable $F \subseteq X$ s.t. $E \subseteq F$, $\mu^*(E) = \mu^*(F)$.

Prop 6.2 Let μ^* be a regular outer measure. If $E_n \in \mathcal{P}(X)$ is increasing, then $\mu^*(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu^*(E_n)$

Proof. Since μ^* is regular, $\forall n \in \mathbb{N}_+$, $\exists F_n$ μ^*-measurable s.t. $E_n \subseteq F_n$, $\mu^*(E_n) = \mu^*(F_n)$. If $\exists n_0$ s.t. $\mu^*(E_{n_0}) = \infty$, then we're done, $\infty = \infty$. Otherwise, By Caratheodory, $G_n := \bigcap_{k=n}^{\infty} F_k$ are μ^*-measurable, with $E_n \subseteq G_n \subseteq F_n$ and $\mu^*(E_n) \leq \mu^*(G_n) \leq \mu^*(F_n) = \mu^*(E_n)$. Set $\mathcal{M}^*$ as the family of μ^*-measurable sets, we know $\mu^* \llcorner_{\mathcal{M}^*}$ is a complete measure. Since $G_n \subseteq G_{n+1}$, we have $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \mu^*(\bigcup_{n=1}^{\infty} G_n) = \lim_{n\to\infty} \mu^*(G_n) \leq \liminf_{n\to\infty} \mu^*(F_n) \leq \limsup_{n\to\infty} \mu^*(F_n) \leq \mu^*(\bigcup_{n=1}^{\infty} E_n)$ and the result follows.

Remark 6.3 Given an outer measure $\mu^* : \mathcal{P}(X) \to [0,\infty]$, we can contruct a regular outer measure $\nu^*(E) = \inf\{\mu^*(F) : E \subseteq F, F \text{ is } \mu^*\text{-measurable}\}$. Then ① if E is μ^*-measurable, then E is ν^*-measurable, $\mu^*(E) = \nu^*(E)$ ② if E is ν^*-measurable and $\nu^*(E) < \infty$, then E is μ^*-measurable.

Proof. See HW4 Q2.

Def 6.4 Let X be a topological space and $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be an outer measure. A set $E \subseteq X$ is inner regular if $\mu^*(E) = \sup\{\mu^*(K) : K \subseteq E, K \text{ is compact}\}$ and is outer regular if $\mu^*(E) = \inf\{\mu^*(U) : E \subseteq U, U \text{ is open}\}$. E is regular if it is both.

Def 6.5 Let X be a topological space and $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be an outer measure. μ^* is a Borel outer measure if $\mathcal{B}(X) \subseteq \mathcal{M}^*$ and is a Borel regular outer measure if $\forall E \subseteq X$, $\exists$ Borel set $B \subseteq X$ s.t. $E \subseteq B$ and $\mu^*(E) = \mu^*(B)$

Note: $\mathcal{B}(X) \subseteq \mathcal{M}^*$ means every Borel set is μ^*-measurable.

Def 6.6 Let X be a topological space and $\mu^* : \mathcal{P}(X) \to [0,\infty]$ be an outer measure. μ^* is a Radon outer measure if

① μ^* is a Borel outer measure.

② $\forall K \subseteq X$ compact, $\mu^*(K) < \infty$,

③ $\forall U \subseteq X$ open, U is inner regular,

④ $\forall E \subseteq X$ is outer regular.

Remark 6.7 A Radon outer measure is Borel regular.

Proof. $\forall E \subseteq X$. If $\mu(E)=\infty$, then X is open, $E \subseteq X$, $\mu^*(E)=\mu^*(X)$.
If $\mu(E)<\infty$, since it's outer regular, $\exists$ open sets U_n s.t. $E \subseteq U_n$ and $\mu^*(E) \geq \mu^*(U_n)-\frac{1}{n}$. Then $B := \bigcap_{n=1}^{\infty} U_n$ is Borel and $E \subseteq B$, $\mu^*(E)=\mu^*(B)$.

Note: the converse is false in general.

Prop 6.8 Let X be a topological space and let μ^* be a Radon measure. Then every σ-finite μ^*-measurable set is inner regular.

Proof. Let $E \subseteq X$ be μ^*-measurable.
Step 1: Assume $\mu^*(E)<\infty$.
Since E is outer regular, $\exists$ open $V \supseteq E$ s.t. $\mu^*(V)<\mu^*(E)+\varepsilon$.
Since V is inner regular, $\exists$ compact $K \subseteq V$ s.t. $\mu^*(V)<\mu^*(K)+\varepsilon$.
Since E is μ^*-measurable, we have $\mu^*(E)+\mu^*(V \setminus E)=\mu^*(V)$, so $\mu^*(V\setminus E)<\varepsilon$.
Since $V \setminus E$ is outer regular, $\exists$ open $U \supseteq V\setminus E$ s.t. $\mu^*(U)<\varepsilon$.
The compact set $C := K \setminus U$ is contained in E. Since U is μ^*-meas.
$\mu^*(C)=\mu^*(K\setminus U)=\mu^*(K)-\mu^*(K\cap U) > \mu^*(V)-\varepsilon-\mu^*(K\cap U) \geq \mu^*(V)-\mu^*(U)-\varepsilon > \mu^*(E)-2\varepsilon$. Hence E is inner regular.
Step 2: if $\mu^*(E)=\infty$, then $E=\bigcup_{n=1}^{\infty} E_n$ with $\mu^*(E_n)<\infty$. Since μ^* is Borel regular, $\exists F_n \in \mathcal{B}(X)$ s.t. $E_n \subseteq F_n$ and $\mu^*(E_n)=\mu^*(F_n)$.
Apply Step 1 to $F_n \cap E$!

Prop 6.9 If $E \subseteq \mathbb{R}^N$, then $(\mathcal{L}^N)^*(E)=\inf\{\mathcal{L}^N(U): E \subseteq U, U \text{ open}\}$ and if E is Lebesgue measurable, $\mathcal{L}^N(E)=\sup\{\mathcal{L}^N(K): K \subseteq E, K \text{ compact}\}$

Proof. This lemma is useful, but the proof is not required.

Prop 6.10 A set $E \subseteq \mathbb{R}^N$ is Lebesgue measurable iff $\forall \varepsilon>0$, $\exists$ open set $U \supseteq E$ s.t. $(\mathcal{L}^N)^*(U\setminus E)<\varepsilon$.

Proof. $\Rightarrow$ Assume that E is Lebesgue measurable:
If $\mathcal{L}^N(E)<\infty$, fix $\varepsilon>0$, by Prop 6.9 we have open set U, $E \subseteq U$ and $\mathcal{L}^N(U)<\mathcal{L}^N(E)+\varepsilon$. And $\mathcal{L}^N(U)=\mathcal{L}^N(E)+\mathcal{L}^N(U\setminus E)$. Thus we have $\mathcal{L}^N(U\setminus E)<\varepsilon$.
If $\mathcal{L}^N(E)=\infty$, then $E_k := E \cap B(0,k)$ measurable and $\varepsilon>0$.
Since $\mathcal{L}^N(E_k)<\infty$, then $\exists U_n$ open s.t. $E_n \subseteq U_n$ and $\mathcal{L}^N(U_n\setminus E_n)<\frac{\varepsilon}{2^n}$.
Then $U := \bigcup_{n=1}^{\infty} U_n$ is an open set containing E. $\mathcal{L}^N(U\setminus E)=$

$\mathcal{L}^N(\bigcup_{n=1}^{\infty}(U_n \setminus E)) \le \sum_{n=1}^{\infty} \mathcal{L}^N(U_n \setminus E_n) < \varepsilon.$

$\Leftarrow$ Assume that $(\mathcal{L}^N)^*(U \setminus E) < \varepsilon$:
Now $A \setminus E = (A \setminus U) \cup (A \cap (U \setminus E))$ so $\mathcal{L}^{N*}(A \cap E) + \mathcal{L}^N(A \setminus E) \le \mathcal{L}^{N*}(A \cap E) +$
$\mathcal{L}^{N*}(A \setminus U) + \mathcal{L}^{N*}(A \cap (U \setminus E)) \le \mathcal{L}^{N*}(A \cap U) + \mathcal{L}^{N*}(A \setminus U) + \mathcal{L}^{N*}(U \setminus E) \le \mathcal{L}^{N*}(A) + \varepsilon$

Prop 6.11 A set $E \subseteq \mathbb{R}^N$ is Lebesgue measurable iff $\forall \varepsilon > 0$, $\exists$ open set $U \supseteq E$ and closed set $F \subseteq E$ s.t. $(\mathcal{L}^N)^*(U \setminus F) < \varepsilon$.
If $(\mathcal{L}^N)(E) < \infty$, then F can be compact.

Proof. $\Rightarrow$ By Prop 6.10, $\exists$ open sets $U \supseteq E$ and $V \supseteq E^c$ s.t. $\mathcal{L}^N(U \setminus E) < \frac{\varepsilon}{2}$ and $\mathcal{L}^N(V \setminus E^c) < \frac{\varepsilon}{2}$. Set $F := V^c$, then F is closed, $F \subseteq E \subseteq U$.
and $\mathcal{L}^N(U \setminus F) \le \mathcal{L}^N(U \setminus E) + \mathcal{L}^N(E \setminus F) < \varepsilon$.
$\Leftarrow$ By monotonicity, $\mathcal{L}^N(U \setminus E) \le \mathcal{L}^N(U \setminus F) < \varepsilon$. Then apply Prop 6.10.

If $\mathcal{L}^N(E) < \infty$, then by Prop 6.9, $\exists$ compact set $K \subseteq E$ s.t.
$\mathcal{L}^N(E) < \mathcal{L}^N(K) + \frac{\varepsilon}{2}$, so that $\mathcal{L}^N(E \setminus K) = \mathcal{L}^N(E) - \mathcal{L}^N(K) < \frac{\varepsilon}{2}$.
Also, $\exists$ open set $U \supseteq E$ s.t. $\mathcal{L}^N(U) < \mathcal{L}^N(E) + \frac{\varepsilon}{2}$. Then $K \subseteq E \subseteq U$
and $\mathcal{L}^N(U \setminus K) \le \mathcal{L}^N(U \setminus E) + \mathcal{L}^N(E \setminus K) < \varepsilon$.

Def 6.12 A G_δ is a countable intersection of open sets.
A F_σ is a countable union of closed sets.

Prop 6.13 A set E is Lebesgue measurable iff $\exists$ G_δ and F_σ s.t.
$F_\sigma \subseteq E \subseteq G_\delta$ and $\mathcal{L}^N(G_\delta \setminus F_\sigma) = 0$.

Proof. $\Rightarrow$ By Prop 6.11, find U_k, F_k s.t. $F_k \subseteq E \subseteq U_k$, $\mathcal{L}^N(U_k \setminus F_k) \le \frac{1}{k}$,
$\forall k \in \mathbb{N}_+$. Set $F_\sigma := \bigcup_{k=1}^{\infty} F_k$ and $G_\delta := \bigcap_{k=1}^{\infty} U_k$. Since $\mathcal{L}^N(G_\delta \setminus F_\sigma) \le$
$\mathcal{L}^N(U_k \setminus F_k) \le \frac{1}{k}$, $\forall k \in \mathbb{N}_+$. Let $k \to \infty$, $\mathcal{L}^N(G_\delta \setminus F_\sigma) = 0$
$\Leftarrow$ Since $\mathcal{L}^N$ is complete, then $E \setminus F_\sigma$ is measurable. And
$E = (E \setminus F_\sigma) \cup F_\sigma$. From Ch4.

Prop 6.14 A set E is Lebesgue measurable iff $\exists B \in \mathcal{B}(\mathbb{R}^N)$, $A \in \mathcal{L}(\mathbb{R}^N)$ with
$E = B \cup A$ and $\mathcal{L}^N(A) = 0$ with $A \subseteq H \in \mathcal{B}(\mathbb{R}^N)$ with $\mathcal{L}^N(H) = 0$.

Proof. $\Leftarrow$ Obvious. If $B \in \mathcal{B}(\mathbb{R}^N)$, then $B \in \mathcal{L}(\mathbb{R}^N)$.
$\Rightarrow$ We have G_δ and F_σ so that $A := G_\delta \setminus F_\sigma$ has measure 0.
Then we have G_δ' and F_σ' s.t. $F_\sigma' \subseteq A \subseteq G_\delta'$. Define $H := G_\delta'$.
Finally, we have $B := F_\sigma$.

Prop 6.15 The Lebesgue outer measure $(\mathcal{L}^n)^*$ is Radon.

Proof. It follows from Def 6.6, Prop 3.11, and Prop 6.9.

Prop. 6.16 Let (X, d) be a metric space and let μ be finite Borel measure. Then $\forall B \in \mathcal{B}(X)$ and $\forall \varepsilon > 0$, $\exists$ closed set C and open set U s.t. $C \subseteq B \subseteq U$ and $\mu(U \setminus C) < \varepsilon$.

Proof. Not required.

Ch7 Completion

Def 7.1 Let $(X, \mathcal{M}, \mu)$ be a measure space and let
$\mathcal{N} := \{A \subseteq X : \exists H \in \mathcal{M}, \mu(H) = 0, A \subseteq H\}$
The completion $\mathcal{M}_\mu$ of $\mathcal{M}$ w.r.t. μ is $\mathcal{M}_\mu : \{A \cup B : A \in \mathcal{N}, B \in \mathcal{M}\}$

Prop 7.2 $\mathcal{L}(\mathbb{R}^N)$ is the completion of $\mathcal{B}(\mathbb{R}^N)$ w.r.t. $\mathcal{L}^N$.

Proof. Direct result from Prop. 6.14.

Lemma 7.3 $E \in \mathcal{M}_\mu$ iff $\exists F, G \in \mathcal{M}$ s.t. $F \subseteq E \subseteq G$ and $\mu(G \setminus F) = 0$.

Proof. $\Rightarrow$ Let $E \in \mathcal{M}_\mu$, then by definition, $E = A \cup B$, $A \in \mathcal{N}$, $B \in \mathcal{M}$.
Let $H \in \mathcal{M}$ s.t. $A \subseteq H$, $\mu(H) = 0$. Set $F = B$ and $G = B \cup H$.
$\Leftarrow$ Given F, G. Write $E = F \cup (E \setminus F)$. We have $E \setminus F \subseteq G \setminus F$.
Since $F \in \mathcal{M}$, $E \setminus F \subseteq G \setminus F$, $\mu(G \setminus F) = 0$, $E \in \mathcal{M}_\mu$.

Lemma 7.4 Let $(X, \mathcal{M}, \mu)$ be a measure space, and $(X, \mathcal{M}_\mu, \bar{\mu})$ be its completion.
where $\bar{\mu}(E) = \mu(B)$ Note: $E \in \mathcal{M}_\mu$ and $E = A \cup B$, $B \in \mathcal{M}$.
Then ① $\mathcal{M}_\mu$ is a σ-algebra;
② $\bar{\mu}$ is a measure on $\mathcal{M}_\mu$ and $\bar{\mu} \llcorner \mathcal{M} = \mu$;
③ $\mathcal{M}_\mu$ is complete w.r.t. $\bar{\mu}$.

Proof. ① 1. Clearly $\emptyset \in \mathcal{M}_\mu$
2. If $E \in \mathcal{M}_\mu$, then apply Lemma 7.3 to get F and G.
Then $X \setminus G \subseteq X \setminus E \subseteq X \setminus F$ and $\mu((X \setminus F) \setminus (X \setminus G)) = 0$. Thus $X \setminus E \in \mathcal{M}_\mu$
3. If $E_n \in \mathcal{M}_\mu$, let $E_n = A_n \cup B_n$. Set $A = \bigcup_{n=1}^\infty A_n$ and $B = \bigcup_{n=1}^\infty B_n$, we have
$\bigcup_{n=1}^\infty E_n = A \cup B \in \mathcal{M}_\mu$

② Claim: $\bar{\mu}$ is well defined.
Assume $E \in \mathcal{M}_\mu$ and $E = A_1 \cup B_1 = A_2 \cup B_2$ Note: corresponding H_i.
$\mu(B_1) \le \mu(B_2 \cup H_2) \le \mu(B_2) + \mu(H_2) = \mu(B_2)$.
Similarly, $\mu(B_2) \le \mu(B_1)$. Thus $\mu(B_1) = \mu(B_2)$

Clearly $\bar{\mu} \ge 0$ and $\bar{\mu}(\emptyset) = 0$. Note: regular setting
Let $E_n \in \mathcal{M}_\mu$ be disjoint and $E_n = A_n \cup B_n$. Set $A = \bigcup_{n=1}^\infty A_n$, $B = \bigcup_{n=1}^\infty B_n$
then $\bigcup_{n=1}^\infty E_n = A \cup B$. Since B_n are disjoint, we have
$\bar{\mu}(\bigcup_{n=1}^\infty E_n) = \mu(B) = \sum_{n=1}^\infty \mu(B_n) = \sum_{n=1}^\infty \bar{\mu}(E_n)$

Finally, $\bar{\mu}(E) = \mu(E)$ because $E = E \cup \emptyset$.

③ $\forall E \in \mathcal{M}_\mu$ s.t. $\bar{\mu}(E) = 0$, By Lemma 7.3, get our F and G.
Now $E = A \cup B$ where $A \in \mathcal{N}$, $B \in \mathcal{M}$. Since $\bar{\mu}(E) = 0$, $\mu(B) = 0$
and $A \subseteq H$, $\mu(H) = 0$. Hence $\mu(G) = \mu(F) \leq \mu(B \cup H) \leq \mu(B) + \mu(H)$
$= 0$. If a set $D \subseteq E$, then $\phi \subseteq D \subseteq G$ with $\mu(G \setminus \phi) = 0$.
By Lemma 7.3, we have $D \in \mathcal{M}_\mu$.

Remark 7.5 Taking the completion doesn't amount to adding all zero measure sets. We only pick up necessary elements.
For example, $\mu = \mathcal{L}^N$, $X = \mathbb{R}^N$, $\mathcal{M} = \{\phi, X\}$. Then $\mathcal{M}_\mu = \mathcal{M}$.

Ch 8 Measurable Functions

Def. 8.1 Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $f: X \to Y$ is measurable if $f^{-1}(F) \in \mathcal{M}$ for every set $F \in \mathcal{N}$.

Lemma. 8.2 Let $(X, \mathcal{M})$ be a measurable space, and let Y be a nonempty set. Let $f: X \to Y$. Then $\mathcal{N}' := \{F \subseteq Y : f^{-1}(F) \in \mathcal{M}\}$ is a σ-algebra.

Proof. ① Clearly $f^{-1}(\phi) = \phi$. Thus $\phi \in \mathcal{N}'$.

② If $F \in \mathcal{N}'$, then $f^{-1}(Y \setminus F) = f^{-1}(Y) \setminus f^{-1}(F) = X \setminus f^{-1}(F) \in \mathcal{M}$. Thus $Y \setminus F \in \mathcal{N}'$.

③ If $\{F_n\}_{n \in \mathbb{N}_+} \in \mathcal{N}'$, then $f^{-1}(\bigcup_{n=1}^{\infty} F_n) = \bigcup_{n=1}^{\infty} f^{-1}(F_n) \in \mathcal{M}$. Thus $\bigcup_{n=1}^{\infty} F_n \in \mathcal{N}'$.

Lemma. 8.3 Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. Let $\mathcal{N} := \sigma(\xi)$ where $\xi \subseteq \mathcal{P}(Y)$. Then $f: X \to Y$ is measurable iff $f^{-1}(F) \in \mathcal{M}$ for every $F \in \xi$.

Proof. $\Rightarrow$ Clearly $\xi \subseteq \mathcal{N}$. It follows from Def. 8.1.

$\Leftarrow$ Define $\mathcal{N}' := \{F \subseteq Y : f^{-1}(F) \in \mathcal{M}\}$. Since $\xi \subseteq \mathcal{N}'$, by lemma 8.2, $\mathcal{N} := \sigma(\xi) \subseteq \mathcal{N}'$.

Remark 8.4. ① Unless stated, otherwise the σ-algebra of Y is $\mathcal{B}(Y)$.

② For $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$, $f: X \to Y$ is called a Borel function.

③ For $(\mathbb{R}^N, \mathcal{L}^N)$ and $(Y, \mathcal{N})$, $f: \mathbb{R}^N \to Y$ is Lebesgue measurable.

④ If $f: X \to Y$ is continuous, then f is Borel measurable.

Remark 8.5. ① Let $(X, \mathcal{M}), (Y, \mathcal{N}), (Z, \mathcal{O})$ be measurable spaces. Let $f: X \to Y$ and $g: Y \to Z$ be measurable functions. Then $g \circ f$ is measurable.

Proof. If $F \in \mathcal{O}$, then $g^{-1}(F) \in \mathcal{N}$, leading to $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F)) \in \mathcal{M}$.

② $\tilde{g}$ is Lebesgue measurable, i.e. $\tilde{g}: (\mathbb{R}^N, \mathcal{L}^N) \to (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$ and $\tilde{f}$ is Borel measurable, i.e. $\tilde{f}: (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N)) \to (\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N))$.

Then $\tilde{g} \circ \tilde{f}$ may fail to be measurable. See Ex. 4.12.

③ Let $(X, \mathcal{M})$ be a measurable space. $f, g: X \to [-\infty, \infty]$ be arbitrary. f, g are measurable iff $(f, g): X \to [-\infty, \infty]^2$ is measurable.

Proof. $\Rightarrow$ If $U, V \subseteq [-\infty, \infty]$ are open, then $(f, g)^{-1}(U \times V) = f^{-1}(U) \cap g^{-1}(V) \in \mathcal{M}$.

$\Leftarrow$ Let $\pi_1: (x, y) \mapsto x$ and $\pi_2: (x, y) \mapsto y$ be continuous projections. Continuity implies measurability. Then $f = \pi_1 \circ (f, g)$ and $g = \pi_2 \circ (f, g)$ are measurable functions.

④ Let $(X, \mathcal{M})$ be a measurable space, $f, g : X \to \overline{\mathbb{R}}$ are measurable.
Then so are f^2, $|f|$, f^+, f^-, αf $(\alpha \in \mathbb{R})$, $f \pm g$, $f \cdot g$, $f \wedge g$, $f \vee g$

Proof. Directly apply ③.
For instance, define $\theta : (x, y) \mapsto x + y$ and $f + g = \theta \circ (f, g)$.

⑤ Let $(X, \mathcal{M})$ be a measurable space and let $f_n : X \to \overline{\mathbb{R}}$ be measurable.
Then so are $\sup_n f_n$, $\inf_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$.
In particular, if $f_n \to f$ pointwise, then f is measurable.

Proof. $\forall a \in \mathbb{R}$, $(\sup_n f_n)^{-1}((a, +\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((a, +\infty]) \in \mathcal{M}$.
Likewise, $(\inf_n f_n)^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f_n^{-1}([-\infty, a)) \in \mathcal{M}$.
Now, $\liminf_n f_n = \sup_n (\inf_{k>n} f_k)$ and $\limsup_n f_n = \inf_n (\sup_{k>n} f_k)$.

Def. 8.6. Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that a property P holds almost everywhere on a set $A \subseteq X$ if $\exists E \in \mathcal{M}$ s.t. $\mu(E) = 0$, P holds on $A \setminus E$.

Ex. 8.7. ① Irrational numbers are almost everywhere on $\mathbb{R}$. $E = \mathbb{Q}$
② $f(x) = |x|$ is differentiable almost everywhere on $\mathbb{R}$. $E = \{0\}$

Prop. 8.8 Let $(X, \mathcal{M}, \mu)$ be a measure space with a complete measure μ.
Let $(Y, \mathcal{N})$ be a measurable space. Let $f : X \to Y$ be measurable and $g : X \to Y$ s.t. $f = g$ a.e. Then g is measurable.

Proof. Let $E_0 = \{x \in X : f(x) \neq g(x)\}$. Since $f = g$ a.e., $\exists E \in \mathcal{M}$, $\mu(E) = 0$ s.t. $f(x) = g(x)$ if $x \in X \setminus E$. Hence $X \setminus E \subseteq X \setminus E_0$, leading to $E_0 \subseteq E$.
Since μ is complete, we have $\mu(E_0) = 0$ and $E_0 \in \mathcal{M}$.
$\forall F \in \mathcal{M}$, we have $g^{-1}(F) = \{x \in X \setminus E_0 : g(x) \in F\} \cup \{x \in E_0 : g(x) \in F\}$
$= \{x \in X \setminus E_0 : f(x) \in F\} \cup \{x \in E_0 : g(x) \in F\} = (f^{-1}(F) \setminus E_0) \cup \{x \in E_0 : g(x) \in F\}$
Now $f^{-1}(F) \setminus E_0 \in \mathcal{M}$ and $\{x \in E_0 : g(x) \in F\} \subseteq E_0$, thus it is measurable.

Prop. 8.9 Let $(X, \mathcal{M}, \mu)$ be a measure space with a complete measure μ.
Let $f_n : X \to [-\infty, \infty]$ be measurable. If $\exists \lim_{n\to\infty} f_n(x)$ for μ-a.e. $x \in X$, then $\lim_{n\to\infty} f_n(x)$ is measurable.

Proof. $\exists E \in \mathcal{M}$, $\mu(E) = 0$ s.t. $\lim_{n\to\infty} f_n(x)$ exists for all $x \in X \setminus E$.
By 8.5. ⑤, $\overline{\lim_{n\to\infty}} f_n$ and $\underline{\lim_{n\to\infty}} f_n$ are measurable.
By 8.8, define $g(x) = \begin{cases} \lim_{n\to\infty} f_n & x \in X \setminus E \\ \text{anything} & \text{otherwise} \end{cases}$ would be measurable.

Def 8.10. Let $(X, \mathcal{M})$ be a measurable space. We say that $s: X \to R$ is a simple function if it is measureb and has finite range, i.e. $s = \sum_{i=1}^{n} a_i I_{E_i}$, $n \in \mathbb{N}_+$, $a_i \neq a_j$ if $i \neq j$, $E_i := s^{-1}(\{a_i\}) \in \mathcal{M}$.

Thm 8.11. Approximation of measurable functions by simple functions

Let $(X, \mathcal{M})$ be a measurable space and let $f: X \to [-\infty, \infty]$ be a measurable function. Then $\exists$ simple functions $S_n: X \to R$ s.t.

① $|S_n(x)| \le |S_{n+1}(x)| \le |f(x)|$, $\forall n \in \mathbb{N}_+, x \in X$;

② $\lim_{n\to\infty} S_n(x) = f(x)$, $\forall x \in X$; ③ if f is bounded, then $S_n \to f$ uniformly.

Proof. Define, for $n \in \mathbb{N}_+$,
$$S_n(x) = \begin{cases} \frac{m}{2^n} & \text{if } f(x) \in [\frac{m}{2^n}, \frac{m+1}{2^n}),\ m = 0, \dots, n2^n - 1 \\ \frac{m+1}{2^n} & \text{if } f(x) \in [\frac{m}{2^n}, \frac{m+1}{2^n}),\ m = -n2^n, \dots, -1 \\ n & \text{if } f(x) \ge n \\ -n & \text{if } f(x) < -n \end{cases}$$

Prop 8.12 Let $(X, \mathcal{M}, \mu)$ be a measurable space with its completion $(X, \mathcal{M}_\mu, \bar{\mu})$. Then $f: X \to \mathbb{R}$ is $\mathcal{M}_\mu$-measurable iff $\exists\, g: X \to \mathbb{R}$ is $\mathcal{M}$-measurable and $f = g$ μ-a.e.

Proof. $\Leftarrow$ Let $\mathcal{U} \subseteq \mathbb{R}$ be open and let $E = \{x \in X : f(x) \neq g(x)\}$.

In the proof of Prop 8.8, we've shown that $E \in \mathcal{M}$ and $\mu(E) = 0$.

Thus, $f^{-1}(\mathcal{U}) = \{x \in X \setminus E : f(x) \in \mathcal{U}\} \cup \{x \in E : f(x) \in \mathcal{U}\}$
$= \{x \in X \setminus E : g(x) \in \mathcal{U}\} \cup \{x \in E : f(x) \in \mathcal{U}\}$
$= (g^{-1}(\mathcal{U}) \setminus E) \cup \{x \in E : f(x) \in \mathcal{U}\}$

Now, $g^{-1}(\mathcal{U}) \setminus E \in \mathcal{M} \subseteq \mathcal{M}_\mu$. $\{x \in E : f(x) \in \mathcal{U}\} \subseteq E \subseteq \mathcal{M}$ with $\mu(E) = 0$, i.e. $\{x \in E : f(x) \in \mathcal{U}\} \in \mathcal{N}$. Thus $f^{-1}(\mathcal{U}) \in \mathcal{M}_\mu$.

$\Rightarrow$ Tedious. Proved by constructing simple functions.

Thm 8.13. Egoroff's Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $f_n: X \to \mathbb{R}$ be measurable functions, $n \in \mathbb{N}_+$, s.t. $f_n \to f$ pointwise. $(f: X \to \mathbb{R})$ Then, $\forall \varepsilon > 0$, $\exists E \in \mathcal{M}$ s.t. $\mu(X \setminus E) < \varepsilon$ and $f_n \to f$ uniformly on E.

Proof. Since $f_n \to f$ pointwise, we have $X = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in X : |f_n(x) - f(x)| < \frac{1}{k}\}$, $\forall k \in \mathbb{N}_+$. Now set $A_{m,k} = \bigcap_{n=m}^{\infty} \{x \in X : |f_n(x) - f(x)| < \frac{1}{k}\} \in \mathcal{M}$.

Note: f_n is measurable, then f is measurable. And $(-\frac{1}{k}, \frac{1}{k})$ is open.

Since $A_{i,k} \subseteq A_{i+1,k}$, $\forall k \in \mathbb{N}_+$, we have $\mu(X) = \lim_{m\to\infty} \mu(A_{m,k}) < \infty$.

$\forall \varepsilon > 0$, $\exists m_k \in \mathbb{N}_+$ s.t. $\mu(A_{m_k,k}) > \mu(X) - \frac{\varepsilon}{2^k}$.

Define $E = \bigcap_{k=1}^{\infty} A_{m_k,k} \in \mathcal{M}$. Then $\mu(X \setminus E) = \mu(X \setminus \bigcap_{k=1}^{\infty} A_{m_k,k}) = \mu(\bigcup_{k=1}^{\infty} (X \setminus A_{m_k,k}))$
$\leq \sum_{k=1}^{\infty} \mu(X \setminus A_{m_k,k}) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon$.

Fix $\delta > 0$. Let $k \in \mathbb{N}_+$ s.t. $\frac{1}{k} < \delta$. Then $E \subseteq A_{m_k,k} = \bigcap_{n \geq m_k} \{x \in X : |f_n(x) - f(x)| < \frac{1}{k}\}$.
which means $\sup_{x \in E} |f_n(x) - f(x)| \leq \frac{1}{k} < \delta$, $\forall n \geq m_k$. i.e. $f_n \to f$ uniformly.

Thm 8.14 **Tietze Extension Theorem**

Let X be a metric space. Let $C \subseteq X$ be a closed set and let $f : C \to \mathbb{R}$ be a continuous function. Then there exists $F : X \to \mathbb{R}$ continuous s.t. $F|_C = f$.

Proof. Define
$$F(x) = \begin{cases} f(x) & \text{if } x \in C \\ \inf_{z \in C} \left\{ f(z) + \frac{d(x,z)}{d(x,C)} - 1 \right\} & \text{if } x \notin C \end{cases}$$

Lemma 8.15 Let X be a metric space and let μ be a finite measure on X s.t. every Borel set is inner regular. Let $f : X \to \mathbb{R}$ be measurable. Then $\forall \varepsilon > 0$, $\exists$ a closed set $C \subseteq X$ s.t. $\mu(X \setminus C) < \varepsilon$ and $f|_C$ is continuous.

Proof. ① Assume f is bounded. WLOG, $f : X \to [0,1]$.

Let $A_{k,n} := f^{-1}([\frac{k}{2^n}, \frac{k+1}{2^n}))$, $0 \leq k \leq 2^n$. Note: $[\frac{k}{2^n}, \frac{k+1}{2^n})$ is Borel.

By inner regularity, $\forall \varepsilon > 0$, $\forall k, n \in \mathbb{N}_+$, $\exists$ compact set $K_{k,n} \subseteq A_{k,n}$ s.t. $\mu(A_{k,n}) - \mu(K_{k,n}) < \frac{\varepsilon}{2^n(2^{n+1})}$. Since $\mu(X) < \infty$, $\mu(A_{k,n}) - \mu(K_{k,n}) = \mu(A_{k,n} \setminus K_{k,n})$

Set $C_n := \bigcup_{k=0}^{2^n} K_{k,n}$. We know C_n are closed and $\mu(X \setminus C_n) \leq \sum_{k=0}^{2^n} \mu(A_{k,n} \setminus K_{k,n}) < \frac{\varepsilon}{2^n}$.
Set $C := \bigcap_{n=1}^{\infty} C_n$, which is closed. Then $\mu(X \setminus C) = \mu(\bigcup_{n=1}^{\infty} (X \setminus C_n)) \leq \sum_{n=1}^{\infty} \mu(X \setminus C_n) < \varepsilon$.
Set $S_n := \sum_{k=0}^{2^n} \frac{k}{2^n} \mathbb{1}_{\underset{\text{disjoint}}{A_{k,n}}}$. Clearly $S_n|_{A_{k,n}}$ is constant, thus $S_n|_{K_{k,n}}$ is constant, implying $S_n|_{C_n} = S_n|_{\bigcup_{k=0}^{2^n} K_{k,n}}$ is continuous. Note: disjoint restriction! powerful
Since $C \subseteq C_n$, we conclude that $S_n : C \to \mathbb{R}$ is continuous. Further, since $\sup_{x \in C} |f(x) - S_n(x)| \leq \frac{1}{2^n}$, we have $S_n \to f$ uniformly on C. Thus $f|_C$ is continuous.

② If f is not bounded. Define $h : \mathbb{R} \to (-1,1)$ as a homeomorphism.
Consider $\bar{f} : h \circ f : X \to (-1,1)$.
By ①, we know $\bar{f}|_C$ is continuous. Hence, $f = h^{-1} \circ \bar{f}$ is also continuous on C.

Thm 8.16 **Lusin's Theorem**

Let X be a metric space and let μ be a finite measure on X s.t. every Borel set is inner regular. If $f : X \to \mathbb{R}$ is measurable, then $\forall \varepsilon > 0$, $\exists g : X \to \mathbb{R}$ continuous s.t. $\mu(\{x \in X : f(x) \neq g(x)\}) < \varepsilon$.

Proof. Fix $\varepsilon > 0$. Apply 8.15, $\exists\, C \subseteq X$ closed, $f|_C$ continuous, $\mu(X \setminus C) < \varepsilon$.
Then apply 8.14, $\exists\, g : X \to \mathbb{R}$ continuous s.t. $g|_C = f$. Thus $\{f \neq g\} \subseteq X \setminus C$.

Remark 8.17. Lusin's Theorem still holds with $X = \mathbb{R}^N$ and $\mu = \mathcal{L}^N$, in spite the fact that $\mathcal{L}^N(\mathbb{R}^N) = +\infty$. Note: in the proof, we use $\mu(X) < \infty$ to show $\mu(A_{k,n} \setminus K_{k,n}) < \frac{\varepsilon}{2^n 2^{n+1}}$. However, we don't need $K_{k,n}$ be compact here. Being closed suffices, so here we can use another theorem 6.15:
"If $E \subseteq \mathbb{R}^N$ is Lebesgue measurable, $\forall \varepsilon > 0$, $\exists$ a closed set $F \subseteq E$ s.t. $\mathcal{L}^N(E \setminus F) < \varepsilon$"

Ch9 Integration

Def 9.1 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $s: X \to [0, \infty)$ be a simple function, i.e. $s = \sum_{i=1}^{n} a_i I_{E_i}$ with disjoint $E_i \in \mathcal{M}$, $a_i \neq a_j$ if $i \neq j$, $n \in \mathbb{N}_+$.
The Lebesgue integral of s w.r.t. μ is defined as Note: $0 \cdot \infty = 0$

Note: $\int s\,d\mu =$ $\int_X s(x)\,d\mu(x) := \sum_{i=1}^{n} a_i \mu(E_i)$

If $A \in \mathcal{M}$, then $\int_A s\,d\mu = \sum_{i=1}^{n} a_i \mu(E_i \cap A) = \int_X s I_A\,d\mu$.

Remark 9.2 ① Let s, t be simple functions, $\alpha, \beta \in \mathbb{R}^+$.
Then $\int (\alpha s + \beta t)\,d\mu = \alpha \int s\,d\mu + \beta \int t\,d\mu$

② If $A, B \in \mathcal{M}$, $A \cap B = \phi$, then $\int_{A \cup B} s\,d\mu = \int_A s\,d\mu + \int_B s\,d\mu$.

③ Let $E_1, \dots, E_n, F_1, \dots, F_m \in \mathcal{M}$ and $a_1, \dots, a_n, b_1, \dots, b_m \in \mathbb{R}^+$ s.t.
$\sum_{i=1}^{n} a_i I_{E_i} = \sum_{j=1}^{m} b_j I_{F_j}$. Then $\sum_{i=1}^{n} a_i \mu(E_i) = \sum_{j=1}^{m} b_j \mu(F_j)$.

Proof. Simple but tedious. Do algebra on different simple functions.

Def 9.3 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f: X \to [0, +\infty]$ be a measurable function. Then $\int_X f\,d\mu := \sup_{0 \leq s \leq f} \int_X s\,d\mu$.

Prop 9.4 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f, g: X \to [0, +\infty]$ be measurable functions s.t. $f \leq g$ μ-a.e. Then $\int_X f\,d\mu \leq \int_X g\,d\mu$.

Proof. By definition, $\exists E \in \mathcal{M}$ s.t. $\mu(E) = 0$ and $f(x) \leq g(x)$, $\forall x \in X \setminus E$.
Let $0 \leq s \leq f$ be simple, i.e. $s = \sum_{i=1}^{n} a_i I_{E_i}$. Define $t(x) = \begin{cases} a_i & \text{if } x \in E_i \setminus E \\ 0 & \text{if } x \in E_i \cap E \end{cases}$
Then $t: X \to [0, \infty]$ is simple and $t \leq g$.
We have $\int_X s\,d\mu = \sum_{i=1}^{n} a_i \mu(E_i \setminus E) + \sum_{i=1}^{n} a_i \underbrace{\mu(E_i \cap E)}_{=0} = \sum_{i=1}^{n} a_i \mu(E_i \setminus E)$
$= \int_X t\,d\mu \leq \int_X g\,d\mu$. Taking sup on both sides: $\int_X f\,d\mu \leq \int_X g\,d\mu$.

Thm 9.5 Monotone Convergence Theorem ☆

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n: X \to [0, \infty]$ be an increasing sequence of measurable functions, $f_n \leq f_{n+1}$, $\forall n \in \mathbb{N}_+$.
Let $f = \lim_{n \to \infty} f_n$. Then f is measurable and $\int_X f\,d\mu = \lim_{n \to \infty} \int_X f_n\,d\mu$.

Proof. By Remark 8.5. ⑤, f is measurable.
By Prop. 9.4. $\lim_{n \to \infty} \int_X f_n\,d\mu \leq \int_X f\,d\mu$.

Let $s: X \to (0, \infty]$ be simple, i.e. $s = \sum_{i=1}^{m} a_i I_{E_i}$ and $f(x) \geq s(x)$, $\forall x \in X$.
Fix $t \in (0, 1)$ and set $A_n := \{x \in X : f_n(x) \geq t\,s(x)\}$, $\forall n \in \mathbb{N}_+$.
Then $A_1 \subseteq A_2 \subseteq \dots$ and $X = \bigcup_{n=1}^{\infty} A_n$. Thus $\mu(F) = \lim_{n \to \infty} \mu(A_n \cap F)$, $\forall F \in \mathcal{M}$.
Note: If not, then $\exists x_0 \in X$ s.t. $x_0 \notin A_n$, $\forall n \in \mathbb{N}_+$, implying $f_n(x_0) < t\,s(x_0)$.
Since $s(x) > 0$, then $f(x_0) = \lim_{n \to \infty} f_n(x_0) \leq t\,s(x_0) < s(x_0)$, Contradiction.

Note that $f_n(x) \geq I_{A_n} \cdot [t \cdot \sum_{i=1}^{m} a_i I_{E_i}] = \sum_{i=1}^{m} t a_i I_{A_n \cap E_i}$. Then by Remark 9.2 ③ $\int_X f_n d\mu \geq \sum_{i=1}^{m} t a_i \mu(A_n \cap E_i)$, leading to $\underset{\sim\sim\sim}{\lim_{n\to\infty}} \int_X f_n d\mu \geq \sum_{i=1}^{m} t a_i \mu(E_i) = t \int_X s\, d\mu$.

Note: We have $\mu(F) = \lim_{n\to\infty} \mu(A_n \cap F)$. Now replace F by E_i.

Letting $t \uparrow 1^-$, we have $\lim_{n\to\infty} \int_X f_n d\mu \geq \int_X s\, d\mu$. Taking sup over s on both sides.

Remark 9.6 MCT doesn't hold for decreasing sequences.

Counter example: $f_n(x) = n I_{(0, \frac{1}{n}]}$ in $[0,1]$. $f := \lim_{n\to\infty} f_n = 0$.

Clearly, $\lim_{n\to\infty} \int_{[0,1]} f_n dx = 1$ and $\int_{[0,1]} f dx = 0$.

Thm 9.7 **Beppo Levi Theorem**

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n : X \to [0,\infty]$ be measurable functions. Then $\sum_{n=1}^{\infty} f_n$ is measurable and $\sum_{n=1}^{\infty} \int_X f_n d\mu = \int_X \sum_{n=1}^{\infty} f_n d\mu$.

Proof. Let $g_k = \sum_{n=1}^{k} f_n$, $\forall k \in \mathbb{N}_+$. Then $g_k \leq g_{k+1}$ and $\lim_{k\to\infty} g_k = \sum_{n=1}^{\infty} f_n$ pointwise.

By MCT, we have $\int_X \sum_{n=1}^{\infty} f_n d\mu = \lim_{k\to\infty} \underset{\sim\sim\sim}{\sum_{n=1}^{k} \int_X f_n d\mu} = \sum_{n=1}^{\infty} \int_X f_n d\mu$.

Note: we use Remark 9.8 in advance.

Remark 9.8 Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f, g : X \to [0,\infty]$ be measurable functions and $\alpha, \beta \in \mathbb{R}^+$. Then $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g\, d\mu$.

Proof. By Thm 8.11, we have simple functions $f_n \leq f_{n+1}$ and $g_n \leq g_{n+1}$, $n \in \mathbb{N}_+$ s.t. $f(x) = \lim_{n\to\infty} f_n(x)$ and $g(x) = \lim_{n\to\infty} g_n(x)$. Then $\alpha f_n + \beta g_n \nearrow \alpha f + \beta g$.

By MCT, $\int_X (\alpha f + \beta g) d\mu = \lim_{n\to\infty} \int_X (\alpha f_n + \beta g_n) d\mu = \lim_{n\to\infty} [\alpha \int_X f_n d\mu + \beta \int_X g_n d\mu]$

$= \alpha \int_X f d\mu + \beta \int_X g\, d\mu$.

Example. If $a_{n,k} \geq 0$, $\forall n, k \in \mathbb{N}_+$, then $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{n,k} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} a_{n,k}$.

Proof. Set $X = \mathbb{N}$ and μ = counting measure. Apply Thm 9.7.

Thm 9.9 **Fatou's Lemma**

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n : X \to [0,\infty]$ be measurable functions. Then $\int_X \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$.

Proof. Set $g_k = \inf_{k \geq n} f_n$. Hence $g_n \leq f_n$ and $\liminf_{n\to\infty} f_n = \lim_{k\to\infty} g_k = \sup_k g_k$ Note: $g_k \leq g_{k+1}$

By Remark 8.5 ⑤, $\liminf_{n\to\infty} f_n$ and g_k are measurable.

By Prop 9.4, we have $\int_X g_n d\mu \leq \int_X f_n d\mu$. By Thm 9.5 MCT, we have $\int_X \liminf_{n\to\infty} f_n d\mu = \int_X \lim_{n\to\infty} g_n d\mu = \lim_{n\to\infty} \int_X g_n d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$.

Remark 9.10 Let $(X, \mathcal{M}, \mu)$ be a measure space and $f : X \to [0,\infty]$ be measurable:

① $\int_X f d\mu = 0$ iff $f = 0$ μ-a.e. on X.

② If $A \in \mathcal{M}$, then $\int_A f d\mu = \int_X f I_A d\mu$.

③ If $\mu(A)=0$, then $\int_A f\,d\mu = 0$.

④ If $\int_X f\,d\mu < \infty$, then $f<\infty$ μ-a.e. on X.

Proof. ① $\Rightarrow$ Assume $\int_X f\,d\mu = 0$. Set $A_n = \{x\in X : f(x) \geq \frac{1}{n}\}$. Clearly $f \geq \frac{1}{n}\mathbb{1}_{A_n}$, implying $0 = \int_X f\,d\mu \geq \int_X \frac{1}{n}\mathbb{1}_{A_n}\,d\mu = \frac{1}{n}\mu(A_n)$. Thus $\mu(A_n)=0$, $\forall n\in\mathbb{N}_+$. Therefore $\mu(\bigcup_{n=1}^{\infty} A_n) = 0$ where $\bigcup_{n=1}^{\infty} A_n = \{x\in X : f(x)>0\}$, i.e. $f=0$ μ-a.e.

$\Leftarrow$ Assume $f=0$ μ-a.e., i.e. $\exists E\in\mathcal{M}$ s.t. $\mu(E)=0$, $f(x)=0$, $\forall x\in X\setminus E$.

Then $\int_X s\,d\mu = \sum_{i=1}^{n} a_i\,\mu(E_i) = 0 + \sum_{i=1}^{n} a_i\,\mu(E_i\cap E) = 0$. Note: $\mu(E_i\cap E)\leq\mu(E)$

② Simple but tedious manipulating simple functions.

Consider $\bar{s}(x) = \begin{cases} s(x) & x\in A \\ 0 & x\in X\setminus A. \end{cases}$

③ If $\mu(A)=0$, then $\int_A f\,d\mu \underset{\text{by ②}}{=} \int_X f\,\mathbb{1}_A\,d\mu \underset{\text{by ①}}{=} 0$ since $f\,\mathbb{1}_A = 0$ μ-a.e.

Note:

④ Assume $\int_X f\,d\mu < \infty$. Set $A = \{x\in X : f(x) = +\infty\}$. Let $s = n\,\mathbb{1}_A$, which is simple and $0\leq s\leq f$. Thus $n\mu(A) = \int_X s\,d\mu \leq \int_X f\,d\mu < \infty$, $\forall n\in\mathbb{N}_+$, implying $\mu(A)=0$.

Def 9.11 Let $(X,\mathcal{M},\mu)$ be a measure space and $f: X\to[-\infty,\infty]$ be measurable. If at least one of $\int_X f^+\,d\mu$ and $\int_X f^-\,d\mu$ is finite, then $\int_X f\,d\mu$ is **well-defined**, i.e. $\int_X f\,d\mu := \int_X f^+\,d\mu - \int_X f^-\,d\mu$.

If both are finite, then f is **integrable**.

And if f is integrable over every compact set $K\subseteq X$, then f is **locally integrable**.

Prop 9.12 Let $(X,\mathcal{M},\mu)$ be a measure space. A measurable function f is integrable iff $\int_X |f|\,d\mu < \infty$.

Proof. $\Rightarrow$ $|f| = f^+ + f^-$. By Remark 9.8, $\int_X |f|\,d\mu = \int_X f^+\,d\mu + \int_X f^-\,d\mu < \infty$.

$\Leftarrow$ $f^\pm \leq |f|$. By Prop 9.4. $\int_X f^\pm\,d\mu \leq \int_X |f|\,d\mu < \infty$.

Thm 9.13 **Riemann–Lebesgue Theorem**

Let $a<b$, $f:[a,b]\to\mathbb{R}$ is bounded. Then f is Riemann integrable iff $\mathcal{L}^1(\{x\in[a,b] : f \text{ is not continuous on } x\}) = 0$. Furthermore, if f is Riemann integrable, then f is Lebesgue integrable and $\int_a^b f(x)\,dx = \int_{[a,b]} f\,d\mathcal{L}^1$.

Note: Recall f continuous $\Rightarrow$ f Riemann integrable.

Remark 9.14 Note that MCT may fail if we don't require $f_n \geq 0$. Even $f_n \geq -1$.

Counter example: $f_n = -\frac{1}{n}$, $X = \mathbb{R}$. Then $\int_X f_n d\mu = -\infty$ and $\int_X \lim_{n\to\infty} f_n \, d\mu = 0$.

Lemma 9.15 Let $f: \mathbb{R}^N \to \mathbb{R}$ be integrable. $\forall \varepsilon > 0$, $\exists$ continuous function $g: \mathbb{R}^N \to \mathbb{R}$ s.t. $\| f-g \|_{L^1} := \int_X |f-g| \, d\mu < \infty$.

Proof. Since f is integrable, we have $\int_X f^{\pm} \, d\mu < \infty$.

Let $S_1, S_2 : \mathbb{R}^N \to \mathbb{R}^+$ be simple, $S_1 = \sum_{k=1}^{n_1} a_k I_{A_k}$ and $S_2 = \sum_{j=1}^{n_2} b_j I_{B_j}$ s.t.

$\int_{\mathbb{R}^N} f^+ d\mu \leq \int_{\mathbb{R}^N} S_1 d\mu + \frac{\varepsilon}{4}$ and $\int_{\mathbb{R}^N} f^- d\mu \leq \int_{\mathbb{R}^N} S_2 d\mu + \frac{\varepsilon}{4}$.

Since $\sum_{k=1}^{n_1} a_k \mu(A_k) \leq \int_{\mathbb{R}^N} f^+ d\mu < \infty$, we have $\mu(A_k) < \infty$, $\forall k \in \mathbb{N}_+$.

Since A_k is measurable, $\exists\, \mathcal{U}_k^{(1)}$ open and $F_k^{(1)}$ compact s.t. $F_k^{(1)} \subseteq A_k \subseteq \mathcal{U}_k^{(1)}$ and $\mu(\mathcal{U}_k^{(1)} \setminus F_k^{(1)}) < \frac{\varepsilon}{4n_1} \cdot \frac{1}{2\max a_k}$. By Urysohn's lemma, we have continuous $\varphi_k^{(1)}: \mathbb{R}^N \to [0,1]$ s.t. $\varphi_k^{(1)}|_{F_k^{(1)}} \equiv 1$ and $\varphi_k^{(1)}|_{\mathbb{R}^N \setminus \mathcal{U}_k^{(1)}} \equiv 0$.

Similarly, we have $\mathcal{U}_j^{(2)}, F_j^{(2)}, \varphi_j^{(2)}$ and $\mu(\mathcal{U}_j^{(2)} \setminus F_j^{(2)}) < \frac{\varepsilon}{4n_2} \cdot \frac{1}{2\max b_j}$.

Note: this is continuous, which means g.

$$\text{Then } \Big\| f - \underbrace{\Big(\sum_{k=1}^{n_1} a_k \varphi_k^{(1)} - \sum_{j=1}^{n_2} b_j \varphi_j^{(2)} \Big)}_{} \Big\|_{L^1} \leq \Big\| f^+ - \sum_{k=1}^{n_1} a_k \varphi_k^{(1)} \Big\|_{L^1} + \Big\| f^- - \sum_{j=1}^{n_2} b_j \varphi_j^{(2)} \Big\|_{L^1}$$

$$\leq \| f^+ - S_1 \|_{L^1} + \Big\| \sum_{k=1}^{n_1} a_k (I_{A_k} - \varphi_k^{(1)}) \Big\|_{L^1} + \| f^- - S_2 \|_{L^1} + \Big\| \sum_{j=1}^{n_2} b_j (I_{B_j} - \varphi_j^{(2)}) \Big\|_{L^1}$$

$$\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \sum_{k=1}^{n_1} a_k \int_{\mathbb{R}^N} |I_{A_k} - \varphi_k^{(1)}| \, d\mu + \sum_{j=1}^{n_2} b_j \int_{\mathbb{R}^N} |I_{B_j} - \varphi_j^{(2)}| \, d\mu$$

$$\leq \frac{\varepsilon}{2} + \sum_{k=1}^{n_1} a_k \int_{\mathcal{U}_k^{(1)} \setminus F_k^{(1)}} |I_{A_k} - \varphi_k^{(1)}| \, d\mu + \sum_{j=1}^{n_2} b_j \int_{\mathcal{U}_j^{(2)} \setminus F_j^{(2)}} |I_{B_j} - \varphi_j^{(2)}| \, d\mu$$

$$\leq \frac{\varepsilon}{2} + 2 \Big(\sum_{k=1}^{n_1} a_k \cdot \mu(\mathcal{U}_k^{(1)} \setminus F_k^{(1)}) + \sum_{j=1}^{n_2} b_j \cdot \mu(\mathcal{U}_j^{(2)} \setminus F_j^{(1)}) \Big) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Prop. 9.16 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n : X \to \mathbb{R}$ be measurable s.t. $\int_X f_1 \, d\mu < \infty$ and $f_n \geq f_{n+1}$ μ-a.e., $\forall n \in \mathbb{N}_+$.

Then $f := \lim_{n\to\infty} f_n$ is measurable and $\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$.

Proof. If $\int_X f_1 \, d\mu = -\infty$, then it's trivial.

If $\int_X f_1 \, d\mu \in \mathbb{R}$, then we have $f_n^+ \leq f_1^+$ μ-a.e. leading to $\int_X f_n^+ d\mu < +\infty$, which means that $\int_X f_n \, d\mu$ exists.

Now $f_1 - f_n$ is an increasing sequence of nonnegative measurable functions. Apply MCT, we have $\int_X f_1 - f \, d\mu = \lim_{n\to\infty} \int_X f_1 - f_n \, d\mu$, i.e. $-\int_X f \, d\mu = \lim_{n\to\infty} -\int_X f_n \, d\mu$.

Thm 9.17 Dominated Convergence Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n : X \to \overline{\mathbb{R}}$ be measurable s.t. $f := \lim_{n\to\infty} f_n$ μ-a.e. Suppose that $\exists\, g: X \to [0, \infty]$ integrable s.t. $|f_n| \leq g$ μ-a.e. $\forall n \in \mathbb{N}_+$, then f is integrable and $\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$.

In particular, $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

Proof. To deal with integral, WLOG, modify f_n and g on zero measure sets.
We have $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall x \in X$ and $|f_n(x)| \le g(x)$, $\forall x \in X$, $\forall n \in \mathbb{N}_+$.
Then f is measurable. By Fatou's lemma, we have $\int_X |f| d\mu \le \liminf_{n\to\infty} \int_X |f_n| d\mu \le \int_X g\, d\mu < \infty$. Thus f is integrable.

Note: $g+f \ge 0$. Apply Fatou!

Now $\int_X g\, d\mu + \int_X f d\mu = \int_X (g+f)\, d\mu \le \liminf_{n\to\infty} \int_X (g+f_n)\, d\mu = \int_X g\, d\mu + \liminf_{n\to\infty} \int_X f_n\, d\mu$.
Since $\int_X g\, d\mu \in \mathbb{R}$, we have $\int_X f d\mu \le \liminf_{n\to\infty} \int_X f_n\, d\mu$.
Similarly, we have $-\int_X f d\mu \le \liminf_{n\to\infty} -\int_X f_n d\mu$, which means $\int_X f d\mu \ge \limsup_{n\to\infty} \int_X f_n d\mu$. Therefore, $\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$.

Note: we use $|f_n - f| \le 2g$.

Further, by definition, $|f_n - f| \to 0$ pointwise. Thus $\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0$

Remark 9.18 The integrable upper bound g is important!
Counterexample: $([0,1], \mathcal{L}(\mathbb{R})|_{[0,1]}, \mathcal{L}^1 \llcorner_{[0,1]})$, $f_n = n\, \mathbb{1}_{[0,\frac{1}{n})}$
Then $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$ and $\int_0^1 f(x)\, dx = 0$.

Def 9.19 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f: X \to Y$ where Y is a set. Define $\mathcal{M}' \subseteq \mathcal{P}(Y)$ by $\mathcal{M}' = \{A \subseteq Y : f^{-1}(A) \in \mathcal{M}\}$. Define the pushforward measure $\nu : \mathcal{M}' \to [0,\infty]$ by $\nu(A) = \mu(f^{-1}(A))$.

Prop 9.20 $\mathcal{M}'$ is a σ-algebra and ν is a measure on $(Y, \mathcal{M}')$.
If $g: Y \to [-\infty, \infty]$ is integrable, then $\int_Y g\, d\nu = \int_X (g\circ f)\, d\mu$

Proof. First part is trivial and tedious.
Second part: Clearly $g\circ f$ is measurable.
$\forall A \in \mathcal{M}'$, $\int_Y \mathbb{1}_A\, d\nu = \nu(A) = \mu(f^{-1}(A)) = \int_X \mathbb{1}_{f^{-1}(A)}\, d\mu = \int_X (\mathbb{1}_A \circ f)\, d\mu$
Here, $\mathbb{1}_A$ is simple. Similarly, we extend to nonnegative and arbitrary.

Prop. 9.21 Let $h: \mathbb{R}^N \to \mathbb{R}$ be integrable and $x_0 \in \mathbb{R}^N$. Then $\int_{\mathbb{R}^N} h(x)dx = \int_{\mathbb{R}^N} h(x+x_0)\, dx$

Proof. Consider $(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N), \mathcal{L}^N)$ and $f: \mathbb{R}^N \to \mathbb{R}^N$ with $f(x) = x + x_0$
$\mathcal{M}' = \{A \subseteq \mathbb{R}^N : f^{-1}(A) \in \mathcal{L}(\mathbb{R}^N)\} = \{A \subseteq \mathbb{R}^N : A - x_0 \in \mathcal{L}(\mathbb{R}^N)\} = \mathcal{L}(\mathbb{R}^N)$
and $\nu(A) = \mathcal{L}^N(f^{-1}(A)) = \mathcal{L}^N(A - x_0) = \mathcal{L}^N(A)$ Note: $\nu = \mathcal{L}^N$.
By Prop. 9.20, $\int_{\mathbb{R}^N} h\, d\nu = \int_{\mathbb{R}^N} h\circ f\, d\mu$, i.e. $\int_{\mathbb{R}^N} h(x)dx = \int_{\mathbb{R}^N} h(x+x_0)\, dx$.

Ch 10 Convergence

Def 10.1 Let $(X, \mathcal{M}, \mu)$ be a measure space and $f_n, f: X \to \mathbb{R}$ be measurable:

① $f_n \to f$ μ-almost everywhere:
$\exists E \in \mathcal{M}$, $\mu(E) = 0$ s.t. $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall x \in X \setminus E$.

② $f_n \to f$ almost uniformly:
$\forall \varepsilon > 0$, $\exists E \in \mathcal{M}$ s.t. $\mu(E) < \varepsilon$, $\lim_{n\to\infty} \sup_{x\in X\setminus E} |f_n(x) - f(x)| = 0$.

③ $f_n \to f$ in measure: $\forall \varepsilon > 0$, $\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) = 0$

④ $f_n \to f$ in L^p: $\lim_{n\to\infty} \int_X |f_n - f|^p \, d\mu = 0$

⑤ $f_n \to f$ weakly in L^1: Note: denoted by $f_n \rightharpoonup f$.
$\lim_{n\to\infty} \int_X f_n g \, d\mu = \int_X f g \, d\mu$, for every bounded measurable $g: X \to \mathbb{R}$.

Examples ① $f_n(x) = x^n$ converges to $f(x) = 0$ almost uniformly on $[0,1]$,
but not uniformly.

Proof. $\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & x = 1 \\ 0 & 0 \le x < 1 \end{cases}$ but uniform convergence of continuous functions is continuous.

Note: $|f(x) - f(y)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$

However, fix $\varepsilon > 0$, let $E = [1-\varepsilon, 1]$. Then $\mu(E) = \varepsilon$.
$\sup_{x\in[0,1-\varepsilon)} |f_n(x) - 0| = \sup_{x\in[0,1-\varepsilon)} x^n = (1-\varepsilon)^n$. Then $\lim_{n\to\infty} (1-\varepsilon)^n = 0$.

② $f_n \to f$ μ-a.e. $\not\Rightarrow$ in measure / in L^1 / almost uniformly

Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{L}^1)$, $f_n(x) := 1_{[n,\infty)}(x)$.

Proof. Clearly, $f_n \to f \equiv 0$ pointwise. $\forall x \in \mathbb{R}$, $f_n(x) = 0$ if $n > x$.

Claim: $f_n \not\to 0$ in measure.

Fix $\varepsilon \in (0,1)$, $\mathcal{L}^1(\{x \in \mathbb{R} : |f_n(x) - 0| > \varepsilon\}) = \mathcal{L}^1([n,\infty)) = \infty$

Claim: $f_n \not\to 0$ in L^1.

Later we'll prove convergence in L^1 implies convergence in measure.

Claim: $f_n \not\to 0$ almost uniformly.

$\forall \varepsilon > 0$, let $E \in \mathcal{B}(\mathbb{R})$ s.t. $\mathcal{L}^1(E) < \varepsilon$. We have $\mathcal{L}^1([n,\infty) \setminus E) = \infty$.

Thus $\forall n \in \mathbb{N}_+$, $\exists x_n \in [n,\infty) \setminus E$,

Therefore, $\sup_{x\in[n,\infty)\setminus E} |f_n(x) - 0| \ge f_n(x_n) = 1$

③ $f_n \to f$ in measure / in L^1 $\not\Rightarrow$ μ-a.e.

Consider $([0,1), \mathcal{B}([0,1)), \mathcal{L}^1 \llcorner_{[0,1)})$, intervals $[0,\frac{1}{2})$, $[\frac{1}{2}, 1)$,
$[0,\frac{1}{3})$, $[\frac{1}{3}, \frac{2}{3})$, ... f_n is the indicator function of n-th interval.

Proof. Clearly, we have $\int_{[0,1)} |f_n| \, d\mathcal{L}^1 = \int_{[0,1)} 1_{I_n} \, d\mathcal{L}^1 = \mathcal{L}^1(I_n) \to 0$ and
$\mathcal{L}^1(\{x \in [0,1) : |f_n| > \varepsilon\}) = \mathcal{L}^1(I_n) \to 0$.

However, $\lim_{n\to\infty} f_n(x)$ doesn't exist for every $x \in [0,1)$.

Lemma 10.2 Let $(X, \mathcal{M}, \mu)$ be a measurable space and $f_n, f : X \to \mathbb{R}$ be measurable

① If $\mu(X) < +\infty$ and $f_n \to f$ μ-a.e. then $f_n \to f$ in measure.

② If $f_n \to f$ almost uniformly, then $f_n \to f$ in measure / μ-a.e.

③ If $f_n \to f$ in measure, then $\exists$ subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ almost uniformly.

④ If $f_n \to f$ in L^1, then $f_n \to f$ in measure and $f_n \rightharpoonup f$.

Moreover, $\exists g \in L^1(X)$ s.t. $f_{n_k} \to f$ almost uniformly and $|f_{n_k}| \le g$.

Proof. ① Since $f_n \to f$ μ-a.e., $\forall \varepsilon > 0$, $I_{\{|f_n - f| > \varepsilon\}} \to 0$ μ-a.e.

Since $\mu(X) < \infty$, $g(x) \equiv 1$ is integrable. By DCT, we have

$\lim_{n\to\infty} \mu(|f_n - f| > \varepsilon) = \lim_{n\to\infty} \int_X I_{\{|f_n - f| > \varepsilon\}} \, d\mu = 0$

② Claim 1: $f_n \to f$ in measure

Fix $\varepsilon, \delta > 0$. Let $E_\delta \in \mathcal{M}$ s.t. $\mu(E_\delta) < \delta$ and $\sup_{x \in X \setminus E_\delta} |f_n - f| \to 0$.

which means $\exists n_0$ s.t. $\forall n \ge n_0$, $\sup_{x \in X \setminus E_\delta} |f_n - f| < \varepsilon$.

Thus $\forall n \ge n_0$, $\mu(\{x \in X : |f_n - f| \ge \varepsilon\}) \le \mu(E_\delta) < \delta$.

Claim 2: $f_n \to f$ μ-a.e.

$\forall k \in \mathbb{N}_+$, let $E_k \in \mathcal{M}$ s.t. $\mu(E_k) < \frac{1}{k}$ and $f_n \to f$ uniformly on $X \setminus E_k$.

Set $\tilde{E}_{k+1} = \tilde{E}_k \cap E_{k+1}$ and $\tilde{E}_1 = E_1$, then $\tilde{E}_{k+1} \subseteq \tilde{E}_k$, $\mu(\tilde{E}_k) < \frac{1}{k}$ and

$\lim_{n\to\infty} \sup_{x \in X \setminus \tilde{E}_k} |f_n(x) - f(x)| = 0$, $\forall k \in \mathbb{N}_+$. Define $E_0 = \bigcap_{k=1}^{\infty} \tilde{E}_k$.

Since $\mu(E_1) < \infty$, we have $\mu(E_0) = \lim_{k\to\infty} \mu(\tilde{E}_k) = 0$.

Now, if $x \in X \setminus E_0$, then $\exists k$ s.t. $x \notin \tilde{E}_k$, thus $f_n(x) \to f(x)$.

Hence $f_n \to f$ μ-a.e. Note: $f_n \to f$ uniformly on $X \setminus \tilde{E}_k$.

③ Assume that $f_n \to f$ in measure. We know that $\forall \varepsilon > 0$,

$\lim_{n\to\infty} \mu(\{x \in X : |f_n - f| > \varepsilon\}) = 0$.

With $\varepsilon = \frac{1}{2^k}$, find $n_k \in \mathbb{N}_+$, $\forall n \ge n_k$, $\mu(|f_n - f| > \frac{1}{2^k}) \le \frac{1}{2^{k+1}}$.

We choose the sequence $\{n_k\}$ to be strictly increasing.

Set $E_k := \{x \in X : |f_{n_k} - f| > \frac{1}{2^k}\}$, $F_k = \bigcup_{j=k}^{\infty} E_j$. Then we have

$\mu(F_k) \le \sum_{j=k}^{\infty} \mu(E_j) \le \sum_{j=k}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^k}$.

Fix $\varepsilon > 0$, choose k_0 s.t. $\frac{1}{2^{k_0}} < \varepsilon$. Let $E := F_{k_0}$ s.t. $\mu(E) < \varepsilon$.

$\forall \delta > 0$, $\exists m \ge k_0$ s.t. $\frac{1}{2^m} < \delta$. If $x \in X \setminus E$, then $x \notin F_{k_0}$, hence

$x \notin E_j$ where $j \ge m$. Thus $|f_{n_j} - f| \le \frac{1}{2^j} \le \frac{1}{2^m} < \delta$.

④ Assume that $f_n \to f$ in L^1.

Claim 1: $f_n \to f$ in measure. Note: Use Chebyshev's inequality!

$\forall \varepsilon > 0$, $\mu(|f_n - f| > \varepsilon) \le \frac{1}{\varepsilon} \int_X |f_n - f| \, d\mu \to 0$ as $n \to \infty$.

Claim 2: $f_n \rightharpoonup f$.

Let $g: X \to \mathbb{R}$ be a bounded measurable function. Then

$\left|\int_X (f_n - f) g \, d\mu\right| \le \int_X |f_n - f| \cdot |g| \, d\mu \le \sup |g| \cdot \int_X |f_n - f| \, d\mu \to 0$

Claim 3: $\exists \{f_{n_k}\} \subseteq \{f_n\}$, $g \in L^1$ s.t. $f_{n_k} \to f$ almost uniformly and

$|f_{n_k}(x)| \le g(x)$ μ-a.e. $x \in X$.

$\forall n \in \mathbb{N}_+$, find $n_k \in \mathbb{N}_+$ Note: increasing w.r.t. k. s.t. $\int_X |f_{n_k} - f| \, d\mu \le \frac{1}{2^k}$

Set $h(x) := \sum_{k=1}^{\infty} |f_{n_k}(x) - f(x)|$. Clearly $h(x)$ is measurable.

By Beppo-Levi, $\int_X h(x) \, d\mu = \sum_{k=1}^{\infty} \int_X |f_{n_k} - f| \, d\mu \le 1$. Thus $h(x) < \infty$ μ-a.e.

and $|f_{n_k}(x) - f(x)| \to 0$ as $k \to \infty$.

Note that $|f_{n_k}(x)| \le |f_{n_k}(x) - f(x)| + |f(x)| \le \underbrace{h(x) + |f(x)|}_{\text{our required function } g(x)}$

Thm 10.3 Chebyshev Inequality

Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^1(X, \mu)$.

$\forall c > 0$, $\mu(\{x \in X : |f(x)| \ge c\}) \le \frac{1}{c} \int_X |f| \, d\mu$.

Proof. $\mu(\{x \in X : |f(x)| \ge c\}) = \frac{1}{c} \cdot c \cdot \int_X \mathbb{1}_{\{x \in X : |f(x)| \ge c\}} \, d\mu$ Note: use Prop. 9.4

$\le \frac{1}{c} \int_{\{x \in X : |f(x)| \ge c\}} |f(x)| \, d\mu \le \frac{1}{c} \int_X |f| \, d\mu$.

Ch 11 L^p Space

Def 11.1 Let V be a vector space. Then $\|\cdot\|: V \to [0,\infty)$ is a norm if:
① $\|v\| = 0$ iff $v = 0$; ② $\forall \alpha \in \mathbb{R}$, $v \in V$, $\|\alpha v\| = |\alpha| \cdot \|v\|$
③ $\forall u, v \in V$, $\|u+v\| \le \|u\| + \|v\|$.

Given a normed space $(V, \|\cdot\|)$, we can define a metric $d: V \times V \to [0,\infty)$ by $d(u,v) = \|u-v\|$. $\forall u, v \in V$.
We say V is a Banach space if it's a complete metric space, i.e. every Cauchy sequence in V converges to some element of V.

Def 11.2. Let $(X, \mathcal{M}, \mu)$ be a measure space. If $1 \le p < \infty$, we define $\mathcal{M}^p(E) := \{f: X \to [-\infty,\infty] : f \text{ is measurable}, \|f\|_{\mathcal{M}^p(E)} < \infty\}$, where $\|f\|_{\mathcal{M}^p(E)} = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}$.
If $p = \infty$, then $\mathcal{M}^\infty(E) := \{f: X \to \mathbb{R} : f \text{ is measurable}, \|f\|_{\mathcal{M}^\infty(E)} < \infty\}$, where $\|f\|_{\mathcal{M}^\infty(E)} = \sup_{x \in X} |f(x)|$. Note: $\|\cdot\|_{\mathcal{M}^p(E)}$ for $1 \le p \le \infty$ is almost a norm.

Def 11.3 Let $1 \le p \le \infty$. The conjugate exponent of p is $q = \begin{cases} \frac{p}{p-1} & 1 < p < \infty \\ \infty & p = 1 \\ 1 & p = \infty \end{cases}$
Note that $\frac{1}{p} + \frac{1}{q} = 1$.

Thm 11.4 Hölder Inequality
Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \le p \le \infty$. If $f, g: X \to \overline{\mathbb{R}}$ are measurable, then $\|fg\|_{\mathcal{M}^1(X)} \le \|f\|_{\mathcal{M}^p(X)} \cdot \|g\|_{\mathcal{M}^q(X)}$
In particular, if $f \in \mathcal{M}^p(X)$ and $g \in \mathcal{M}^q(X)$, then $f \cdot g \in \mathcal{M}^1(X)$.

Proof. If $\|f\|_{\mathcal{M}^p(X)} = 0$ or $\|g\|_{\mathcal{M}^q(X)} = 0$, then $f = 0$ or $g = 0$ μ-a.e. implying $f \cdot g = 0$ μ-a.e. Trivial to prove.
If $\|f\|_{\mathcal{M}^p(X)} = \infty$ or $\|g\|_{\mathcal{M}^q(X)} = \infty$, then trivial to prove.
Hence, consider $\|f\|_{\mathcal{M}^p(X)}, \|g\|_{\mathcal{M}^q(X)} \in (0,\infty)$.

Assume first $p \in (1,\infty)$. Recall Young Inequality: $ab \le \frac{1}{p}a^p + \frac{1}{q}b^q$.
we have $|f(x)g(x)| \le \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$. which means
$\int_X |fg| \, d\mu \le \frac{1}{p}\int_X |f|^p d\mu + \frac{1}{q}\int_X |g|^q d\mu$.
Let's involve $t > 0$: $t\int_X |fg| \, d\mu \le \frac{1}{p}\int_X |tf|^p d\mu + \frac{1}{q}\int_X |g|^q d\mu$, implying
$\int_X |fg| \, d\mu \le \frac{t^{p-1}}{p}\int_X |f|^p d\mu + \frac{1}{tq}\int_X |g|^q d\mu =: h(t)$.
With some algebra, we have $h'(t) = 0$ iff $t = \dfrac{\|g\|_{\mathcal{M}^q(X)}^{\frac{q}{p}}}{\|f\|_{\mathcal{M}^p(X)}}$.

Plug this t into $h(t)$, we have $\|fg\|_{\mathcal{M}^1(X)} \le \|f\|_{\mathcal{M}^p(X)} \cdot \|g\|_{\mathcal{M}^q(X)}$

If $p=1, q=\infty$, then $\int_X |fg|\,d\mu \le \int_X |f| \sup_X |g|\,d\mu = \sup_X |g| \cdot \int_X |f|\,d\mu$.
Same for the case $p=\infty, q=1$.

Thm 11.5 **Minkowski Inequality**

Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \le p \le \infty$. Let $f, g : X \to \overline{\mathbb{R}}$ be measurable s.t. $f+g$ is well-defined. Then
$\|f+g\|_{\mathcal{M}^p(X)} \le \|f\|_{\mathcal{M}^p(X)} + \|g\|_{\mathcal{M}^p(X)}$.

Def 11.6 Let's introduce an **equivalence relation** on $\mathcal{M}^p(X)$:
$f \sim g$ if $f = g$ μ-a.e. Then $f \sim g \Rightarrow \|f\|_{\mathcal{M}^p(X)} = \|g\|_{\mathcal{M}^p(X)}$, $1 \le p < \infty$.

Then we define **L^p norm**: $\|f\|_{L^p(X)}$. And $L^p(X)$ or $L^p(X,\mu) :=$
$\{[f] : f : X \to [-\infty, \infty]$ measurable, and $\|f\|_{\mathcal{M}^p(X)} < \infty\}$ where
$[f] := \{g : X \to [-\infty, \infty]$ measurable and $f \sim g\}$ Note: equivalence class.
$\|f\|_{L^p(X)} := \|[f]\|_{\mathcal{M}^p(X)}$.

Note: if $p = \infty$, then $f \sim g \not\Rightarrow \|f\|_{\mathcal{M}^p(X)} = \|g\|_{\mathcal{M}^p(X)}$. Hence we need another notion to modify zero measure sets.

Define the **essential supremum** of f as:
$$\operatorname{ess\,sup} f := \inf\{t \in \mathbb{R} : \mu(\{x \in X : f(x) > t\}) = 0\}$$
Then $L^\infty(X)$ or $L^\infty(X,\mu) := \{[f] : f : X \to \overline{\mathbb{R}}$ is measurable and $\|f\|_{L^\infty} < \infty\}$
where $\|f\|_{L^\infty} = \operatorname{ess\,sup} f$. Note: easy to verify it's a norm.

Lemma 11.7 Let $1 \le p < \infty$, $f \in L^p(X)$. Then $\|f\|_{L^p} = \sup_{g \in L^q \setminus \{0\}} \frac{1}{\|g\|_{L^q}} \int_X |fg|\,d\mu$.

Proof. By Holder inequality, $\int_X |fg|\,d\mu \le \|f\|_{L^p} \cdot \|g\|_{L^q}$. Thus RHS $\le$ LHS.
Conversely, if $f \equiv 0$, then trivial to prove. WLOG, assume $\|f\|_{L^p} > 0$, define $F = \frac{f}{\|f\|_{L^p}}$. WTS $1 \le \sup \cdots \int_X |F \cdot g|\,d\mu$.

Set $g = |F|^{p-1} \cdot \operatorname{sign}(F)$. Then $\int_X |g|^q\,d\mu = \int_X |F|^p\,d\mu = 1$. Therefore,
RHS $\ge \int_X |F \cdot |F|^{p-1} \operatorname{sign}(F)|\,d\mu = \int_X |F|^p\,d\mu = 1$.

Lemma 11.8 Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L^\infty(X)$.
If $f \in L^p(X)$, then $\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$.

Proof. Step 1. Assume $0 < \mu(X) < \infty$.
$\|f\|_{L^p} = (\int_X |f|^p\,d\mu)^{\frac{1}{p}} \le \|f\|_{L^\infty} [\mu(X)]^{\frac{1}{p}}$, so $\limsup_{p\to\infty} \|f\|_{L^p} \le \|f\|_{L^\infty}$.
Conversely, $\forall r < \|f\|_{L^\infty}$, $\exists E := \{x \in X : |f(x)| > r\}$. Since $0 < \mu(E) < \infty$,
$\int_X |f|^p\,d\mu \ge \int_E |f|^p\,d\mu \ge r^p \mu(E)$. Thus $\liminf_{p\to\infty} \|f\|_{L^p} \ge r$. Note: ess sup!

Let $r \uparrow \|f\|_{L^\infty}$. We conclude that $\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$.

Step 2. Assume $\mu(X) = \infty$.

Let $M = \|f\|_{L^\infty}$. If $M = 0$, then $f = 0$ μ-a.e. Trivial.

Now assume $0 < M < \infty$. Let $0 < \varepsilon < M$ and $D = \{x \in X : |f(x)| \geq M - \varepsilon\}$

By definition of $\|\cdot\|_{L^\infty}$, we have $0 < \mu(D) < \infty$. Note: the reason is

$(M-\varepsilon)^p \mu(D) \leq \int_D |f(x)|^p d\mu \leq \int_X |f|^p d\mu < \infty$.

So $\|f\|_{L^p} \geq (M-\varepsilon)[\mu(D)]^{\frac{1}{p}}$, which means $\liminf_{p\to\infty} \|f\|_{L^p} \geq M - \varepsilon$.

Let $\varepsilon \downarrow 0$, we have $\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}$.

Let $\varepsilon > 0$, set $\tilde{f}(x) := \frac{f(x)}{M+\varepsilon}$. Then $0 \leq |\tilde{f}(x)| \leq \frac{M}{M+\varepsilon} < 1$ and we have

$(\int_X |f|^p d\mu)^{\frac{1}{p}} = (M+\varepsilon)(\int_X |\tilde{f}|^p d\mu)^{\frac{1}{p}}$.

Fix $p_0 > 1$, $\forall p \geq p_0$, $\int_X |\tilde{f}|^p d\mu \leq \int_X |\tilde{f}|^{p_0} d\mu \cdot (\frac{M}{M+\varepsilon})^{p-p_0} \to 0$ as $p \to \infty$.

Since $\exists p_0$, $\forall p \geq p_0$, $\int_X |\tilde{f}|^p d\mu \leq 1$, let $\varepsilon \downarrow 0$, then $\|f\|_{L^\infty} \geq \limsup_{p\to\infty} \|f\|_{L^p}$

Lemma 11.9 Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p < q \leq +\infty$. If $\mu(X) < \infty$, then $L^q(X) \subseteq L^p(X)$.

Proof. Assume $q = \infty$, then $\int_X |f|^p d\mu \leq (\text{ess sup } f)^p \cdot \mu(X) < \infty$.

Suppose that $1 \leq q < \infty$. By Holder inequality, $\int_X |f|^p d\mu \leq$

$(\int_X 1^? d\mu)^{\frac{1}{?}} \cdot [\int_X (|f|^p)^{\frac{q}{p}} d\mu]^{\frac{p}{q}} = [\mu(X)]^{\frac{1}{?}} \cdot (\int_X |f|^q d\mu)^{\frac{p}{q}} < \infty$

where $?$ is the conjugate exponent of $\frac{q}{p}$.

Remark 11.10 Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \leq p \leq q < \infty$.

① $L^p(X) \nsubseteq L^q(X)$ iff $\exists$ measurable sets with small positive measure

② $L^q(X) \nsubseteq L^p(X)$ iff $\exists$ measurable sets with large finite measure

Note: Complicated to prove! Note: arbitrarily

Appendix. Some relevant theorems from functional analysis.

① Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 \le p \le \infty$.
Then $L^p(X)$ is a Banach space.

② Let $(V, \|\cdot\|)$ be a normed space. Then V is complete iff whenever $\sum_{n=1}^{\infty} \|v_n\| < \infty$, $v_n \in V$, $\exists v \in V$ s.t. $\lim_{n\to\infty} \|\sum_{k=1}^{n} v_n - v\| = 0$.

③ Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $\mathcal{S} = \{ s : X \to \mathbb{R} :$ s is simple and $\mu(\{s \neq 0\}) < +\infty \}$ is dense in $L^p(X)$, $1 \le p \le \infty$.

④ Let X be a σ-compact metric space. Let μ be finite on compact sets s.t. every Borel set is inner regular.
Then $C_c(X)$ is dense in $L^p(X)$, $1 \le p < \infty$.

Note: $p = \infty$ it fails, because uniform limit of cts function is cts.

Ch12 Signed Measures

Def 12.1 Let $(X, \mathcal{M})$ be a measurable space and let $\mu, \nu : X \to [0,\infty]$ be two measures. ν is absolutely continuous w.r.t. μ, denoted by $\nu \ll \mu$, if $\forall E \in \mathcal{M}$, $\mu(E) = 0 \Rightarrow \nu(E) = 0$.

Remark 12.2 Given $(X, \mathcal{M}, \mu)$ and $f : X \to \mathbb{R}$ measurable. Let $\nu(E) = \int_E f \, d\mu$, $\forall E \in \mathcal{M}$. Then ν is a measure and $\nu \ll \mu$.

Proof. Trivial to verify.

Lemma 12.3 Let $(X, \mathcal{M})$ be a measurable space and $\mu, \nu : \mathcal{M} \to [0,\infty]$ be two measures on $\mathcal{M}$. $\forall E \in \mathcal{M}$, define $\nu_{ac}(E) := \sup_f \{ \int_E f d\mu : f : X \to [0,\infty]$ measurable s.t. $\int_{E'} f \, d\mu \le \nu(E')$, $\forall E' \subseteq E$, $E' \in \mathcal{M} \}$

Then ① ν_{ac} is a measure ② $\nu_{ac} \ll \mu$

③ $\forall E \in \mathcal{M}$, $\exists f_E$ s.t. $\nu_{ac}(E) = \int_E f_E \, d\mu$

④ If ν_{ac} is σ-finite, then f_E is independent of E.

Proof. ① Clearly $\nu_{ac}(\phi) = 0$. (Remark 9.10 ③)

Now let $E_n \in \mathcal{M}$ and $E = \bigcup_{n=1}^{\infty} E_n$. Let $f : X \to [0,\infty]$ be a measurable function s.t. $\int_{E'} f \, d\mu \le \nu(E')$, $\forall E' \subseteq E$, $E' \in \mathcal{M}$. Then $\forall n \in \mathbb{N}_+$, if $E' \subseteq E_n$, we have $\int_{E_n} f d\mu \le \nu_{ac}(E_n)$. In turn, $\int_E f \, d\mu \le \sum_{n=1}^{\infty} \nu_{ac}(E_n)$

Taking sup on both sides, we get $\nu_{ac}(E) \le \sum_{n=1}^{\infty} \nu_{ac}(E_n)$

Conversely, if $\nu_{ac}(E) = \infty$, then trivial to prove.

Assume $\nu_{ac}(E) < \infty$. We have $\nu_{ac}(E_n) \le \nu_{ac}(E) < \infty$ since $E_n \subseteq E$,

$\forall \varepsilon > 0$, $\forall n \in \mathbb{N}_+$, find $f_n : X \to [0,\infty]$ measurable s.t. $\int_{E'} f_n \, d\mu \le \nu(E')$ where $E' \subseteq E_n$, $E' \in \mathcal{M}$ and $\nu_{ac}(E_n) \le \int_{E_n} f_n d\mu + \frac{\varepsilon}{2^n}$.

Define $f = \sum_{n=1}^{\infty} 1_{E_n} f_n$, which is measurable by Beppo-Levi. We have $\int_{E'} f d\mu = \sum_{n=1}^{\infty} \int_{E' \cap E_n} f \, d\mu \le \sum_{n=1}^{\infty} \nu(E' \cap E_n) = \nu(E')$, thus f is admissible for $\nu_{ac}(E)$.

Hence $\sum_{n=1}^{\infty} \nu_{ac}(E_n) \le \sum_{n=1}^{\infty} (\int_{E_n} f_n \, d\mu + \frac{\varepsilon}{2^n}) = \int_E f d\mu + \varepsilon \le \nu_{ac}(E) + \varepsilon$.

Taking $\varepsilon \searrow 0$, we set $\sum_{n=1}^{\infty} \nu_{ac}(E) \le \nu_{ac}(E)$.

② If $\mu(E) = 0$, then $\int_E f \, d\mu = 0$, implying $\nu_{ac}(E) = 0$.

③ If $f, g : X \to [0,\infty]$ are measurable functions s.t. $\int_{E'} f \, d\mu \le \nu(E')$ and $\int_{E'} g \, d\mu \le \nu(E')$, $\forall E' \subseteq E$, $E' \in \mathcal{M}$, then $f \vee g$ is still admissible because $\int_{E'} f \vee g \, d\mu = \int_{E' \cap \{f \ge g\}} f d\mu + \int_{E' \cap \{f < g\}} g \, d\mu \le \nu(E' \cap \{f \ge g\}) + \nu(E' \cap \{f < g\}) = \nu(E')$

Hence, if $f_n : X \to [0,\infty]$ are measurable s.t. $\lim_{n\to\infty} \int_E f_n d\mu = \nu_{ac}(E)$ and f_n are admissible for $\nu_{ac}(E)$. WLOG, assume $f_n \le f_{n+1}$ μ-a.e.

Define $f_E = \lim_{n\to\infty} f_n$ with f_E admissible for $\nu_{ac}(E)$. By MCT, we have $\int_E f_E \, d\mu = \lim_{n\to\infty} \int_E f_n d\mu = \nu_{ac}(E)$

Note: $f := f_X$

④ Assume first that V_{ac} is finite.
Since f_X is admissible for $V_{ac}(X)$, obtained in ③, we have that
$\int_{E'} f_X \, d\mu \le \nu(E')$, $\forall E' \in \mathcal{M}$. For given $E \in \mathcal{M}$, $\int_E f \, d\mu \le V_{ac}(E)$.
If $\int_E f \, d\mu < V_{ac}(E)$, since $V_{ac}(X \setminus E) \ge \int_{X \setminus E} f \, d\mu$, then $V_{ac}(X) = V_{ac}(E)$
$+ V_{ac}(X \setminus E) > \int_E f \, d\mu + \int_{X\setminus E} f \, d\mu = \int_X f \, d\mu = V_{ac}(X)$. Contradiction!
If V_{ac} is σ-finite, then decompose $X = \bigcup_{n=1}^{\infty} X_n$, $V_{ac}(X_n) < \infty$.
$\forall n \in \mathbb{N}_+$, choose f_n s.t. $V_{ac}(X_n) = \int_{X_n} f_n \, d\mu$. Define $f = \sum_{n=1}^{\infty} 1_{X_n} f_n$.

Def 12.4 Let $(X, \mathcal{M})$ be a measurable space. A signed measure is
$\lambda : \mathcal{M} \to (-\infty, \infty]$ or $[-\infty, \infty)$ s.t. ① $\lambda(\emptyset) = 0$, ② $\lambda(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda(E_n)$
if $E_n \in \mathcal{M}$ are pairwise disjoint.

Def 12.5 Let $(X, \mathcal{M})$ be a measurable space and let λ be a signed measure.
We say that $E \in \mathcal{M}$ is positive if $\lambda(F) \ge 0$, $\forall F \subseteq E$, $F \in \mathcal{M}$.
$E \in \mathcal{M}$ is negative if $\lambda(F) \le 0$, $\forall F \subseteq E$, $F \in \mathcal{M}$.
Set $P(\lambda) = \{\lambda\text{-positive sets of } \mathcal{M}\}$, $N(\lambda) = \{\lambda\text{-negative sets of } \mathcal{M}\}$

Remark 12.6 ① If $\lambda(A) \in \mathbb{R}$, then $\forall E \subseteq A$, $E \in \mathcal{M}$, $\lambda(A) = \lambda(E) + \lambda(A \setminus E)$
② $P(\lambda)$ and $N(\lambda)$ are closed w.r.t. countable unions, intersection,
and relative compliment.
③ If $P \in P(\lambda)$, $N \in N(\lambda)$, then P and N are λ-disjoint, i.e.
if $E \subseteq P \cap N$, then $\lambda(E) = 0$.

Note: $\lambda : \mathcal{M} \to (-\infty, +\infty]$

Lemma 12.7 Let $A \in \mathcal{M}$ and λ be a signed measure. Assume that $\lambda(A) \in (-\infty, 0)$.
Then $\exists E \in \mathcal{M}$, $E \subseteq A$, $E \in N(\lambda)$ s.t. $\lambda(E) \le \lambda(A)$.

Proof. Define $\delta_1 = \sup\{\lambda(E) : E \subseteq A, E \in \mathcal{M}\}$. If $\delta_1 = 0$, then $E = A$.
If not, $\delta_1 > 0$, so let $A_1 \subseteq A$, $A_1 \in \mathcal{M}$, $\lambda(A_1) \ge \frac{\delta_1}{2}$.
Define $\delta_2 = \sup\{\lambda(E) : E \subseteq A \setminus A_1, E \in \mathcal{M}\}$. keep doing ...
If this process stops at some $n \in \mathbb{N}_+$, we are done. $E = A \setminus (\bigcup_{k=1}^{n-1} A_k)$
Otherwise, define $E = A \setminus (\bigcup_{n=1}^{\infty} A_n)$ Note: $\sum_{n=1}^{\infty} \lambda(A_n) > 0$, so $\lambda(E) < \lambda(A)$
Since $-\infty < \lambda(E) < \lambda(A) < 0$, we know $\sum_{n=1}^{\infty} \lambda(A_n) < \infty$, implying
$\sum_{n=1}^{\infty} \delta_n < \infty$. Hence $\delta_n \to 0$.
Assume $D \subseteq E$, $D \in \mathcal{M}$ and $\lambda(D) \ge 0$. Then $\forall n \in \mathbb{N}_+$, $D \subseteq A \setminus (\bigcup_{k=1}^{n} A_k)$ so
$\lambda(D) \le \delta_n$ (because D is admissible for δ_n). Since $\delta_n \to 0$, $\lambda(D) = 0$.
Thus $\lambda(D) \le 0$, which means $E \in N(\lambda)$.

Lemma 12.8 Similarly, if $0<\lambda(A)<\infty$, then $\exists E\in\mathcal{M}$, $E\subseteq A$, $\lambda(E)\geq\lambda(A)$ and $E\in P(\lambda)$. Proof. Apply Lemma 12.7 to $-\lambda$.

Thm 12.9 Hahn Decomposition

Let λ be a signed measure on $(X,\mathcal{M})$. Then there exists $P\in P(\lambda)$ and $N\in N(\lambda)$ s.t. $X=P\cup N$. Note: let $P'=P\setminus N$, so $P'\cap N=\varnothing$.

Proof. WLOG, assume that $\lambda:\mathcal{M}\to(-\infty,\infty]$.

Let $L=\inf\{\lambda(A): A\in N(\lambda)\}$. Let $A_n\in N(\lambda)$ s.t. $\lambda(A_n)\to L$.

Let $N=\bigcup_{n=1}^{\infty}A_n$. By 12.6 ②, $N\in N(\lambda)$ so $L\leq\lambda(N)$.

$\forall n\in\mathbb{N}_+$, $\lambda(N)=\lambda(A_n)+\lambda(N\setminus A_n)\leq\lambda(A_n)$. Taking limit, $\lambda(N)\leq L$.

Hence $\lambda(N)=L$. Set $P=X\setminus N$. If $P\notin P(\lambda)$, then $\exists A\subseteq P$, $A\in\mathcal{M}$, $\lambda(A)<0$. By Lemma 12.7, $\exists E\in\mathcal{M}$, $E\in N(\lambda)$, $E\subseteq A$ s.t. $\lambda(E)\leq\lambda(A)$. Thus $\lambda(E\cup N)=\lambda(E)+\lambda(N)\leq\lambda(A)+\lambda(N)<L$.

However $E\cup N\in N(\lambda)$, which leads to a contradiction.

Def 12.10 Let $(X,\mathcal{M})$ be a measurable space, $\mu,\nu:\mathcal{M}\to[0,\infty]$ be two measures, μ and ν are mutually singular, $\mu\perp\nu$, if $\exists X_\mu, X_\nu\in\mathcal{M}$ disjoint s.t. $X=X_\mu\cup X_\nu$ and $\mu(E)=\mu(E\cap X_\mu)$, $\nu(E)=\nu(E\cap X_\nu)$ $\forall E\in\mathcal{M}$.

Lemma 12.11 Let $(X,\mathcal{M})$ be a measurable space, $\mu,\nu:\mathcal{M}\to[0,\infty]$ be two measures, $\forall E\in\mathcal{M}$, define $\nu_s(E)=\sup\{\nu(F): F\subseteq E, F\in\mathcal{M}, \mu(F)=0\}$. Then ① ν_s is a measure ② $\forall E\in\mathcal{M}$, this supremum is attainable. ③ if ν_s is σ-finite, then $\nu_s\perp\mu$.

Proof. ① Clearly $\nu_s(\varnothing)=0$. Let $E_n\in\mathcal{M}$ be mutually disjoint.

Let $F\in\mathcal{M}$, $\mu(F)=0$, $F\subseteq\bigcup_{n=1}^{\infty}E_n$. Then $F\cap E_n\in\mathcal{M}$, $\mu(F\cap E_n)=0$

Since $F\cap E_n\subseteq E_n$, we have $\nu_s(E_n)\geq\nu(F\cap E_n)$.

So $\sum_{n=1}^{\infty}\nu_s(E_n)\geq\sum_{n=1}^{\infty}\nu(F\cap E_n)\overset{\text{disjoint}}{=}\nu\left(\bigcup_{n=1}^{\infty}(F\cap E_n)\right)=\nu(F)$. Taking sup on both sides, we get $\sum_{n=1}^{\infty}\nu_s(E_n)\geq\nu_s\left(\bigcup_{n=1}^{\infty}E_n\right)$.

Conversely, let $\varepsilon>0$, $\forall n\in\mathbb{N}_+$, choose $F_n\in\mathcal{M}$, $\mu(F_n)=0$, $F_n\subseteq E_n$ and $\mu(F_n)\geq\nu_s(E_n)-\frac{\varepsilon}{2^n}$. Then $\bigcup_{n=1}^{\infty}F_n\subseteq\bigcup_{n=1}^{\infty}E_n$, $\bigcup_{n=1}^{\infty}F_n\in\mathcal{M}$ with $\mu\left(\bigcup_{n=1}^{\infty}F_n\right)=0$. Hence, $\nu_s\left(\bigcup_{n=1}^{\infty}E_n\right)\geq\nu\left(\bigcup_{n=1}^{\infty}F_n\right)=\sum_{n=1}^{\infty}\nu(F_n)\geq\sum_{n=1}^{\infty}\nu_s(E_n)-\varepsilon$

Taking $\varepsilon\downarrow 0$, we conclude that $\sum_{n=1}^{\infty}\nu_s(E_n)\leq\nu_s\left(\bigcup_{n=1}^{\infty}E_n\right)$

② If $\nu_s(E)=0$, then set $F=\varnothing$. Now assume $\nu_s(E)>0$.

Consider a sequence $0<t_n<\nu_s(E)$ s.t. $t_n\to\nu_s(E)$. By definition,

$\forall n \in \mathbb{N}_+$, find $F_n \subseteq E$, $F_n \in \mathcal{M}$, $\mu(F_n) = 0$ and $t_n < \nu(F_n) \leq \nu_s(E)$.
Let $E_n := \bigcup_{k=1}^{n} F_k \in \mathcal{M}$, $E_n \subseteq E$, $\mu(E_n) = 0$. So $\nu_s(E) \geq \nu(E_n) \geq \nu(F_n)$
Hence, $\nu_s(E) \geq \lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \lim_{n\to\infty} t_n = \nu_s(E)$. Therefore, with
$F = \bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$, we have $\mu(F) = 0$ and $\nu_s(E) = \nu(F)$.

(3) With $X = \bigcup_{n=1}^{\infty} X_n$, $\nu_s(X_n) < \infty$, X_n pairwise disjoint. If $\nu_s \lfloor_{(\mathcal{M} \lfloor X_n)} \perp$
$\mu \lfloor_{(\mathcal{M} \lfloor X_n)}$, then clearly $\nu_s \perp \mu$. Thus, WLOG, assume $\nu_s(X) < \infty$
From (2), $\exists X_s \in \mathcal{M}$, $\mu(X_s) = 0$, $\nu_s(X) = \nu(X_s)$.
Let $E \in \mathcal{M}$, $\exists E_s \in \mathcal{M}$, $E_s \subseteq E$, $\mu(E_s) = 0$, $\nu_s(E) = \nu(E_s)$.
Then $\nu(E_s \setminus X_s) = 0$ or $E_s \cup X_s$ is admissible for $\nu_s(X)$. With
$\infty > \nu_s(X) \geq \nu(E_s \cup X_s) = \nu(X_s) + \nu(E_s \setminus X_s) > \nu(X_s)$ contradiction!
Hence, $\nu_s(E) = \nu(E_s) = \nu(E_s \cap X_s) + \nu(E_s \setminus X_s) = \nu(E_s \cap X_s) \leq$
$\nu(E \cap X_s)$. And $\mu(E \cap X_s) \leq \mu(X_s) = 0$ so it's admissible for
$\nu_s(E \cap X_s)$. We conclude that $\nu_s(E) = \nu_s(E \cap X_s)$. Then
$\mu(E) = \mu(X_s) + \mu(E \setminus X_s) = \mu(E \setminus X_s) = \mu(E \cap (X \setminus X_s))$. Thus $\nu_s \perp \mu$.

Thm 12.12 **Jordan Decomposition**

Let $(X, \mathcal{M})$ be a measurable space and let λ be a signed measure on $\mathcal{M}$. There exists unique nonnegative measures $\lambda^+, \lambda^- : \mathcal{M} \to [0, \infty]$ s.t. $\lambda = \lambda^+ - \lambda^-$, $\lambda^+ \perp \lambda^-$, at least one of them is finite.

Proof. Let (P, N) be the Hahn Decomposition for λ. Define for $E \in \mathcal{M}$,
$\lambda^+(E) = \lambda(E \cap P)$, $\lambda^-(E) = -\lambda(E \cap N)$. Since we can take $X = P \cup N$
with P, N disjoint, then $\lambda^+ \perp \lambda^-$ and clearly $\lambda(E) = \lambda^+(E) - \lambda^-(E)$.
To addres uniqueness, we show that $\forall A \in \mathcal{M}$, $\lambda^+(A) = \sup\{\lambda(E) :$
$E \subseteq A, E \in \mathcal{M}\}$. If so, then replacing λ by $-\lambda$, we have $\lambda^-(A) =$
$-\inf\{\lambda(E) : E \subseteq A, E \in \mathcal{M}\}$

($\geq$) Let $E \in \mathcal{M}$, $E \subseteq A$. then $\lambda^+(A) \geq \lambda^+(E) = \lambda(E) + \lambda^-(E) \geq \lambda(E)$
($\leq$) With $X^+, X^- \in \mathcal{M}$, $X^+ \cap X^- = \emptyset$ and $\lambda^+(A) = \lambda^+(A \cap X^+)$,
$\lambda^-(A) = \lambda^-(A \cap X^-)$. We have $\lambda^+(A) = \lambda^+(A \cap X^+) = \lambda(A \cap X^+) + \lambda^-(A \cap X^+)$
$= \lambda(A \cap X^+) + \lambda^-(A \cap X^+ \cap X^-) = \lambda(A \cap X^+)$. So set $E = A \cap X^+$.

Finally, if $\lambda : \mathcal{M} \to [-\infty, \infty)$ then $\lambda^+(X) = \lambda(X \cap P) < \infty$ while if
$\lambda : \mathcal{M} \to (-\infty, \infty]$ then $\lambda^-(X) = -\lambda(X \cap N) < \infty$.

Def 12.13 The total variation of λ is a measure $|\lambda| := \lambda^+ + \lambda^-$.
If $|\lambda|(X) < \infty$, then $\|\lambda\| := |\lambda|(X)$ is the total variation norm.

Def 12.14 Let $(X, \mathcal{M})$ be a measurable space and let λ_1, λ_2 be two signed measures. We say that $\lambda_1 << \lambda_2$ if $\forall E \in \mathcal{M}$, $|\lambda_2|(E) = 0 \Rightarrow \lambda_1(E) = 0$.
$\lambda_1 \perp \lambda_2$ if $\exists A \in \mathcal{M}$ s.t. $|\lambda_1|(A) = 0$, $|\lambda_2|(X \setminus A) = 0$.
Note: being mutually singular is symmetric.

Prop. 12.15 Let $(X, \mathcal{M})$ be a measurable space and let $\mu, \nu : \mathcal{M} \to (0, \infty)$ be finite positive measures. Then $\nu << \mu$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $E \in \mathcal{M}$, $\mu(E) < \delta \Rightarrow \nu(E) < \varepsilon$.

Proof. $\Rightarrow$ Suppose not, then $\exists \varepsilon_0$, $\forall \delta > 0$ s.t. $E \in \mathcal{M}$, $\mu(E) < \delta \Rightarrow \nu(E) \geq \varepsilon_0$
Then set $F_n := \bigcup_{k=n}^{\infty} E_k$ and $F_n \supseteq F_{n+1}$, $\mu(F_n) \leq \frac{1}{2^{n-1}}$ and $\nu(F_n) \geq \varepsilon_0$.
We have $\mu(\bigcap_{n=1}^{\infty} F_n) = \lim_{n \to \infty} \mu(F_n) = 0$ while $\nu(\bigcap_{n=1}^{\infty} F_n) \geq \varepsilon_0$. Hence $\nu \not\ll \mu$.
$\Leftarrow$ Suppose $\mu(E) = 0$. Then $\nu(E) < \varepsilon$ for any $\varepsilon > 0$, i.e. $\nu(E) = 0$.

Thm 12.16 Radon–Nikodym Theorem
Let $(X, \mathcal{M})$ be a measurable space and $\mu, \nu : X \to [0, \infty]$ be two measures, with μ σ-finite and $\nu << \mu$. Then there exists a
Note: unique up to μ-meas. 0
unique measurable function $f : X \to [0, \infty]$ s.t. $d\nu = f\,d\mu$, i.e.
$\nu(E) = \int_E f\,d\mu$, $\forall E \in \mathcal{M}$. Note: $f := \frac{d\nu}{d\mu}$ is Radon–Nikodym derivative.

Proof. Too long. It's not required.
Note: if ν is a signed measure λ, then $f \in L^1(X, \mu)$. $\lambda(E) = \int_E f\,d|\mu|$.

Thm 12.17 Lebesgue Decomposition Theorem
Let $(X, \mathcal{M})$ be a measurable space and $\mu, \nu : \mathcal{M} \to [0, \infty]$ be measures with ν σ-finite. Then $\exists \nu_s, \nu_{ac} : \mathcal{M} \to [0, \infty]$ s.t.
1 $\nu = \nu_{ac} + \nu_s$ 2 $\nu_{ac} << \mu$ 3 $\nu_s \perp \mu$. The decomposition is unique.
If μ is σ-finite, then $\nu_{ac}(E) = \int_E f\,d\mu$.
Note: same to signed measures.

Proof. Too long. It's not required.

Ch 13 Product Spaces

Def 13.1 Given two measure spaces, the product σ-algebra of $\mathcal{M}, \mathcal{N}$, denoted by $\mathcal{M} \otimes \mathcal{N}$, is the smallest σ-algebra that
$\mathcal{M} \otimes \mathcal{N} := \sigma(\{E \times F : E \in \mathcal{M}, F \in \mathcal{N}\})$.

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Consider class of elementary sets in $X \times Y$: $\mathcal{E} := \{E \times F : E \in \mathcal{M}, F \in \mathcal{N}\}$ and elementary measure $\rho(E \times F) = \mu(E)\nu(F)$.

Define the product outer measure:
$$(\mu \times \nu)^*(G) := \inf\left\{\sum_{n=1}^{\infty} \mu(E_n)\nu(F_n) : E_n \in \mathcal{M}, F_n \in \mathcal{N}, G \subseteq \bigcup_{n=1}^{\infty} E_n \times F_n\right\}$$

By Caratheodory Thm, $(\mu \times \nu)^*\lfloor : \mathcal{M} \times \mathcal{N} \to [0, \infty]$ is a complete measure, where $\mathcal{M} \times \mathcal{N}$ is the set of $(\mu \times \nu)^*$-measurable sets.

Lemma 13.2 Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces. If $E \in \mathcal{M}$, $F \in \mathcal{N}$, then $E \times F$ is $(\mu \times \nu)^*$-measurable, and $(\mu \times \nu)^*(E \times F) = \mu(E)\nu(F)$. In particular, $\mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{M} \times \mathcal{N}$.

Proof. Observe that if $E_1, E_2 \in \mathcal{M}$, $F_1, F_2 \in \mathcal{N}$, then $(E_1 \times F_1) \cap (E_2 \times F_2) = (E_1 \cap E_2) \times (F_1 \cap F_2)$ and $(E_1 \times F_1) \setminus (E_2 \times F_2) = ((E_1 \setminus E_2) \times F_1) \cup ((E_1 \cap E_2) \times (F_1 \setminus F_2))$.

Step 1: If $E \in \mathcal{M}$, $F \in \mathcal{N}$, then $E \times F$ is $(\mu \times \nu)^*$-measurable
Let $G \subseteq X \times Y$. Let $E_n \in \mathcal{M}$, $F_n \in \mathcal{N}$ s.t. $G \subseteq \bigcup_{n=1}^{\infty} E_n \times F_n$.
Then $G \setminus (E \times F) \subseteq \bigcup_{n=1}^{\infty} (E_n \times F_n) \setminus (E \times F) = \bigcup_{n=1}^{\infty} [((E_n \setminus E) \times F_n) \cup ((E_n \cap E) \times (F_n \setminus F))]$
and $G \cap (E \times F) \subseteq \bigcup_{n=1}^{\infty} (E_n \times F_n) \cap (E \times F) = \bigcup_{n=1}^{\infty} (E_n \cap E) \times (F_n \cap F)$. Therefore,
$(\mu \times \nu)^*(G \setminus (E \times F)) + (\mu \times \nu)^*(G \cap (E \times F)) \leq \sum_{n=1}^{\infty} [\mu(E_n \setminus E)\nu(F_n) + \mu(E_n \cap E)\nu(F_n \setminus F) + \mu(E_n \cap E)\nu(F_n \cap F)] = \sum_{n=1}^{\infty} \mu(E_n)\nu(F_n)$. Taking inf on both sides:
we have $(\mu \times \nu)^*(G \setminus (E \times F)) + (\mu \times \nu)^*(G \cap (E \times F)) \leq (\mu \times \nu)^*(G)$.

Step 2: If $E \in \mathcal{M}$, $F \in \mathcal{N}$, then $(\mu \times \nu)(E \times F) = \mu(E)\nu(F)$
By definition, $(\mu \times \nu)(E \times F) \leq \mu(E)\nu(F)$. WTS: $\ldots \geq \ldots$
Let $E \times F \subseteq \bigcup_{n=1}^{\infty} E_n \times F_n$, $E_n \in \mathcal{M}$, $F_n \in \mathcal{N}$.
Note that $E \times F = \bigcup_{n=1}^{\infty} (E \cap E_n) \times (F \cap F_n)$, so $\mathbb{1}_E(x)\mathbb{1}_F(y) = \mathbb{1}_{E \times F}(x,y) \leq \sum_{n=1}^{\infty} \mathbb{1}_{(E \cap E_n) \times (F \cap F_n)}(x,y) = \sum_{n=1}^{\infty} \mathbb{1}_{E \cap E_n}(x) \cdot \mathbb{1}_{F \cap F_n}(y)$.
$\forall y \in Y$, the function $x \mapsto \mathbb{1}_{E \cap E_n}(x) \cdot \mathbb{1}_{F \cap F_n}(y)$ is measurable, so by Beppo-Levi, $x \mapsto \sum_{n=1}^{\infty} \mathbb{1}_{E \cap E_n}(x) \cdot \mathbb{1}_{F \cap F_n}(y)$ is measurable.

$$\mu(E)\cdot I_F(y) = \int_X I_E(x) I_F(y)\, d\mu(x) = \sum_{n=1}^{\infty} \int_X 1_{E\cap E_n}(x)\cdot 1_{F\cap F_n}(y)\, d\mu(x) = \underbrace{\sum_{n=1}^{\infty} \mu(E\cap E_n)\, I_{F\cap F_n}(y)}$$

For the same reason, $y \mapsto \sum_{n=1}^{\infty} 1_{E\cap E_n}(x)\cdot 1_{F\cap F_n}(y)$ is measurable. Thus

$\mu(E)\nu(F) = \int_Y \mu(E) I_F(y)\, d\nu(y) = \sum_{n=1}^{\infty} \mu(E\cap E_n)\int_Y 1_{F\cap F_n}(y)\, d\nu(y) = \sum_{n=1}^{\infty} \mu(E\cap E_n)\nu(F\cap F_n) \le \sum_{n=1}^{\infty} \mu(E_n)\nu(F_n)$. Finally, take inf on both sides.

Lemma 13.3 Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measurable spaces. Then $\mathcal{M}\otimes\mathcal{N}$ is regular.

Proof. $\forall k\in\mathbb{N}_+$, let $E_n^{(k)}\in\mathcal{M}$ and $F_n^{(k)}\in\mathcal{N}$ s.t. $G\subseteq \bigcup_{n=1}^{\infty} E_n^{(k)}\times F_n^{(k)}$ and $\sum_{n=1}^{\infty}\mu(E_n^{(k)})\nu(F_n^{(k)}) \le (\mu\times\nu)^*(G) + \frac{1}{k}$. Define $\tilde{G} = \bigcap_{k=1}^{\infty}\left(\bigcup_{n=1}^{\infty} E_n^{(k)}\times F_n^{(k)}\right)$.

Then $G\subseteq\tilde{G}$ and $(\mu\times\nu)(\tilde{G}) \le \sum_{n=1}^{\infty}\mu(E_n^{(k)})\nu(F_n^{(k)}) \le (\mu\times\nu)^*(G) + \frac{1}{k}$.

Let $k\to\infty$.

Lemma 13.4 Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measurable spaces. Assume that μ,ν are complete and G has finite $\mu\times\nu$ measure. Then

① $\mu(\{x\in X : \{y\in Y : (x,y)\in G\}\notin\mathcal{N}\}) = 0$.

② $\nu(\{y\in Y : \{x\in X : (x,y)\in G\}\notin\mathcal{M}\}) = 0$.

③ $x\mapsto \nu(\{y\in Y : (x,y)\in G\})$ is measurable

④ $y\mapsto \mu(\{x\in X : (x,y)\in G\})$ is measurable

⑤ $\mu\times\nu(G) = \int_X \nu(\{y\in Y : (x,y)\in G\})\, d\mu(x) = \int_Y \mu(\{x\in X : (x,y)\in G\})\, d\nu(y)$

Note: $\mu\times\nu$ is complete $\not\Rightarrow$ μ and ν are complete.

Proof. Tedious. Four pages of manipulating σ-algebras. Just accept it. ˙◠˙

Thm 13.5 Tonelli Theorem

Let $(X,\mathcal{M},\mu)$ and $(Y,\mathcal{N},\nu)$ be measurable spaces. Assume that μ,ν are complete and σ-finite. Let $f: X\times Y\to[0,\infty]$ be $\mathcal{M}\times\mathcal{N}$-measurable. Then,

① μ-a.e. $x\in X$, $f(x,\cdot) = y\mapsto f(x,y)$ is measurable

② ν-a.e. $y\in Y$, $f(\cdot,y) = x\mapsto f(x,y)$ is measurable

③ $x\mapsto\int_Y f(x,y)\, d\nu(y)$ is measurable

④ $y\mapsto\int_X f(x,y)\, d\mu(x)$ is measurable

⑤ $\int_{X\times Y} f(x,y)\, d(\mu\times\nu)(x,y) = \int_X\left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x) = \int_Y\left(\int_X f(x,y)\, d\mu(x)\right) d\nu(y)$

Proof. If f is simple function, then directly apply Lemma 13.4.

If f is arbitrary, then let $s_n: X\times Y\to[0,\infty]$ be measurable s.t. $0\le s_1(x,y)\le\dots\le s_n(x,y)\le s_{n+1}(x,y)\le\dots\le f(x,y)$.

Apply MCT to s_n twice: $\int_{X\times Y} f(x,y)\, d(\mu\times\nu) = \lim_{n\to\infty}\int_{X\times Y} s_n\, d(\mu\times\nu) = \lim_{n\to\infty}\int_X\left(\int_Y s_n\, d\nu\right) d\mu = \int_X\left(\int_Y \lim_{n\to\infty} s_n\, d\nu\right) d\mu$. Same for another direction.

Thm 13.6 Fubini Theorem

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measurable spaces. Assume that μ, ν are complete. Let $f: X \times Y \to [-\infty, \infty]$ be $\mu \times \nu$-integrable. Then

① μ-a.e. $x \in X$, $f(x, \cdot) = y \mapsto f(x,y)$ is measurable

② ν-a.e. $y \in Y$, $f(\cdot, y) = x \mapsto f(x,y)$ is measurable

③ $x \mapsto \int_Y f(x,y)\, d\nu(y)$ is measurable

④ $y \mapsto \int_X f(x,y)\, d\mu(x)$ is measurable

⑤ $\int_{X \times Y} f(x,y)\, d(\mu \times \nu)(x,y) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X f(x,y)\, d\mu(x) \right) d\nu(y)$

Proof. Define $G_n := \{ (x,y) \in X \times Y : |f(x,y)| \geq \frac{1}{n} \}$. Since $f \in L^1$, $(\mu \times \nu)(G_n) < \infty$.

Now $\bigcup_{n=1}^{\infty} G_n$ is σ-finite, apply Tonelli to f^+ and f^-.

Note: $\int_{X \times Y} f = \int_{\cup G_n} f$.

Remark 13.7 For both Tonelli and Fubini, they don't require f is $\mathcal{M} \otimes \mathcal{N}$-measurable.

Also, σ-finiteness may not be necessary.

Note: eg. $X = \mathbb{N}$, μ be counting measure, ν any complete measure.

In this case, both Tonelli and Fubini hold.

Example: Let $(X, \mathcal{M}, \mu)$ be a measure space, μ is complete, $1 \leq p < \infty$, $f: X \to \mathbb{R}$ is measurable. Then $\int_X |f|^p d\mu = p \int_0^\infty s^{p-1} \mu(|f| > s)\, ds$.

Proof. If $\exists s_0 > 0$ s.t. $\mu(|f| > s_0) = \infty$, then $\forall\, 0 \leq s < s_0$, $\mu(|f| > s) = \infty$.

Hence RHS $= \infty$. By Chebyshev, $\int_X |f|^p d\mu \geq \int_{\{|f| > s_0\}} |s_0|^p d\mu = \infty$.

Now assume that $\mu(|f| > s) < \infty$, $\forall s > 0$. Define $X_0 = \bigcup_{n=1}^{\infty} \{|f| > \frac{1}{n}\}$

$\int_X |f|^p d\mu = \int_{X_0} |f|^p d\mu = \int_{X_0} \left(\int_0^{|f(x)|} p s^{p-1} ds \right) d\mu = p \int_{X_0} \left(\int_0^\infty s^{p-1} 1_{|f| > s}\, ds \right) d\mu$

by Tonelli $= p \int_0^\infty s^{p-1} \left(\int_{X_0} 1_{|f| > s}\, d\mu \right) ds = p \int_0^\infty s^{p-1} \mu(|f| > s)\, ds$.