ON abc TRIPLES OF THE FORM $(1, c-1, c)$

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Abstract. By an abc triple, we mean a triple (a, b, c) of relatively prime positive integers a, b, and c such that $a+b=c$ and $\operatorname{rad}(abc)<c$, where $\operatorname{rad}(n)$ denotes the product of the distinct prime factors of n. The necessity of the ϵ in the abc conjecture is demonstrated by the existence of infinitely many abc triples. For instance, $\left(1, 9^k-1, 9^k\right)$ is an abc triple for each positive integer k. In this article, we study abc triples of the form $(1, c-1, c)$ and deduce two general results that allow us to recover existing sequences of abc triples with $a=1$.

1. Introduction

In 1985, Masser and Oesterlé proposed the abc conjecture [Oes88, Mas17], which states:
The abc conjecture. *For every $\epsilon>0$, there are finitely many relatively prime positive integers a, b, and c with $a+b=c$ such that*

$$\operatorname{rad}(abc)^{1+\epsilon}<c,$$

where $\operatorname{rad}(n)$ denotes the product of the distinct prime factors of n.

Due to its profound implications, this simple-to-state conjecture is one of the most important open questions in number theory. For instance, some consequences of the abc Conjecture include an asymptotic version of Fermat's Last Theorem, Faltings's Theorem, Roth's Theorem, and Szpiro's Conjecture [Elk91, Lan90, Oes88].

The statement of the abc conjecture naturally leads us to ask if the ϵ is necessary? This leads us to the "simplistic abc conjecture," which asks if there are finitely many relatively prime positive integers a, b, and c with $a+b=c$ for which $\operatorname{rad}(abc)<c$. We call such triples (a, b, c) an abc triple. The "simplistic abc conjecture" is false, as demonstrated by the triple $\left(1, 3^{2^k}-1, 3^{2^k}\right)$, which is an abc triple for each positive integer k. This infinite sequence of abc triples is one of the first documented counterexamples to the simplistic abc conjecture and was communicated to Lang by Jastrzebowski and Spielman [Lan90]. A theorem of Stewart [Ste84] leads to similar infinite sequences of abc triples such as $\left(1, 8^{7^k}-1, 8^{7^k}\right)$, where k is a positive integer [MM16]. Jastrzebowski and Spielman's counterexample can also be recovered from the following result: for each odd prime p and each positive integer k, $\left(1, p^{(p-1)k}-1, p^{(p-1)k}\right)$ is an abc triple [Bar23]. Another construction, due to Granville and Tucker [GT02], shows that for each odd prime p, $\left(1, 2^{p(p-1)}-1, 2^{p(p-1)}\right)$ is an abc triple.

In this article, we consider abc triples of the form $(1, c-1, c)$ and establish necessary conditions from which we can recover an infinite sequence of abc triples. As a consequence of our results, we recover each of the sequences mentioned above of abc triples. To state our theorems, we first define the *cosocle* of a positive integer m to be $\operatorname{cosocle}(m)=\frac{m}{\operatorname{rad}(m)}$. For instance, $\operatorname{cosocle}(360)=\frac{360}{30}=12$. With this terminology, we prove:

Theorem 1. *Let n, m, l be positive integers with $n>1$ such that m divides n^l-1 and $\operatorname{cosocle}(m)>\operatorname{rad}(n)$. Then $\left(1, n^{lk}-1, n^{lk}\right)$ is an abc triple for each positive integer k*

We prove Theorem 1 in Section 2. While the proof is elementary, the result allows us to recover each of the previously mentioned sequences of *abc* triples. It also leads to new sequences of *abc* triples, such as $\left(1, n^{(n-1)k} - 1, n^{(n-1)k}\right)$ which is an *abc* triple for each positive integer k whenever n is a positive integer that is either odd or even and non-squarefree (see Corollary 3.3). A slight modification of the proof of Theorem 1 leads us to our next result:

Theorem 2. *Let n, m, l be positive integers such that l is odd, m divides $n^l + 1$, and* $\operatorname{cosocle}(m) > \operatorname{rad}(n)$*. Then $\left(1, n^{lk}, n^{lk} + 1\right)$ is an abc triple for each odd positive integer k.*

This is also proven in Section 2. In Section 3, we demonstrate various consequences of Theorems 1 and 2. We note that the results mentioned in this article lead to further sequences of *abc* triples by employing the transfer method: if (a, b, c) is an *abc* triple, then $\left(a^3, b^3, c\left(b^2 - ab + a^2\right)\right)$ and $\left(a^2, c\,(b-a)\,, b^2\right)$ are *abc* triples [MM16].

We conclude the article with Section 4, which is an analysis of the *abc* triples of the form $(1, c-1, c)$ found by the ABC@Home Project. The ABC@Home Project was a network computing project that was started in 2006 by the Mathematics Department of Leiden University in the Netherlands, together with the Dutch Kennislink Science Institute. By 2011, they found that there are exactly 14 482 065 *abc* triples (a, b, c) with $c < 10^{18}$. By the time the project came to a close in 2015, the ABC@Home Project had found a total of 23 827 716 *abc* triples (a, b, c) with $c < 2^{63}$. We note that this list is not exhaustive of all *abc* triples with $c < 2^{63}$. In particular, the ABC@Home project found that there exactly 45 604 *abc* triples of the form $(1, c-1, c)$ with $c < 10^{18}$. We show that the results of this article account for approximately 1.57% of the *abc* triples of the form $(1, c-1, c)$ with $c < 10^{18}$.

2. Main Results

Lemma 2.1. *Let m and n be positive integers. If m divides n, then* $\operatorname{rad}(n) = \operatorname{rad}\left(\frac{n}{\operatorname{cosocle}(m)}\right)$.

Proof. If $m = 1$, there is nothing to show. So suppose that $m > 1$ and let $m = \prod_{i=1}^{r} p_i^{e_i}$ be the unique prime factorization of m, with each p_i denoting a distinct prime. Since m divides n, we have that the unique prime factorization of n is $n = a \cdot \prod_{i=1}^{r} p_i^{f_i}$ where $e_i \leq f_i$ if $1 \leq i \leq r$ and a is coprime to m. Since $\operatorname{cosocle}(m) = \prod_{i=1}^{r} p_i^{e_i - 1}$, we deduce that

$$\frac{n}{\operatorname{cosocle}(m)} = a \cdot \prod_{i=1}^{r} p_i^{f_i - e_i + 1}$$

For $1 \leq i \leq r$, observe that $f_i - e_i + 1 \geq 1$ and thus $\operatorname{rad}\left(\frac{n}{\operatorname{cosocle}(m)}\right) = \operatorname{rad}(n)$. □

With this lemma, we are now ready to prove Theorem 1:

Proof of Theorem 1. Since $n^{lk} - 1 = \left(n^l - 1\right) \sum_{j=0}^{k-1} n^{lj}$, we deduce that $\operatorname{cosocle}(m)$ divides $n^{lk} - 1$ for each positive integer k. By Lemma 2.1, $\operatorname{rad}\left(n^{lk} - 1\right) = \operatorname{rad}\left(\frac{n^{lk}-1}{\operatorname{cosocle}(m)}\right)$. By assumption, $\frac{\operatorname{rad}(n)}{\operatorname{cosocle}(m)} < 1$ and thus

$$\operatorname{rad}\left(n\left(n^{lk} - 1\right)\right) = \operatorname{rad}(n)\operatorname{rad}\left(\frac{n^{lk} - 1}{\operatorname{cosocle}(m)}\right) \leq \frac{\operatorname{rad}(n)}{\operatorname{cosocle}(m)}\left(n^{lk} - 1\right) < n^{lk} - 1.$$

The result now follows since

$$n^{lk} - \operatorname{rad}\left(n\left(n^{lk} - 1\right)\right) > n^{lk} - n^{lk} + 1 = 1.$$

□

We now prove Theorem 2:

Proof of Theorem 2. If lk is an odd positive integer, then $n^{lk}+1 = \left(n^l+1\right)\sum_{j=0}^{k-1}(-1)^j n^{lj}$. It follows that m divides $n^{lk}+1$ for each positive integer k. By Lemma 2.1, $\operatorname{rad}(n^{lk}+1) = \operatorname{rad}\left(\frac{n^{lk}+1}{\operatorname{cosocle}(m)}\right)$. Since $\frac{\operatorname{rad}(n)}{\operatorname{cosocle}(m)} < 1$, we observe that

$$\operatorname{rad}\left(n\left(n^{lk}+1\right)\right) = \operatorname{rad}(n)\operatorname{rad}\left(\frac{n^{lk}+1}{\operatorname{cosocle}(m)}\right) \leq \frac{\operatorname{rad}(n)}{\operatorname{cosocle}(m)}\operatorname{rad}(n^{lk}+1) < n^{lk}+1.$$

Consequently,

$$n^{lk}+1-\operatorname{rad}\left(n\left(n^{lk}+1\right)\right) > n^{lk}+1-n^{lk}-1 = 0. \qquad \square$$

3. Consequences

In this section, we consider various consequences of Theorems 1 and 2. From these consequences, we deduce the sequences of abc triples that were mentioned in the introduction. To state our first consequence, let m be a positive integer. The Carmichael function $\lambda : \mathbb{N} \to \mathbb{N}$ has the property that $\lambda(n)$ is the least positive integer for which $a^{\lambda(m)} \equiv 1 \mod m$ for each integer a that is relatively prime to m. In particular, $\lambda(n)$ divides $\varphi(n)$, where φ denotes the Euler-totient function.

Corollary 3.1. *Let λ denote the Carmichael function and let m and n be relatively prime positive integers such that* $\operatorname{cosocle}(m) > \operatorname{rad}(n) > 1$. *Then $\left(1, n^{\lambda(m)k}-1, n^{\lambda(m)k}\right)$ is an abc triple for each positive integer k.*

Proof. Since $n^{\lambda(m)} \equiv 1 \mod m$, we have and thus m divides $n^{\lambda(m)}-1$. In particular, $\operatorname{cosocle}(m)$ divides $n^{\lambda(m)}-1$. Now let $l = \lambda(m)$. Then n, m, l satisfy the assumptions of Theorem 1 and thus $\left(1, n^{\lambda(m)k}-1, n^{\lambda(m)k}\right)$ is a good abc triple for each positive integer k. $\square$

As an example, choose $n = 7$ and $m = 64$. Then $\operatorname{cosocle}(64) = 32 > 7 = \operatorname{rad}(n)$ and therefore the conditions of Corollary 3.1 are satisfied. As a result, we find $\left(1, 7^{\lambda(64)k}-1, 7^{\lambda(64)k}\right) = (1, 7^{16k}-1, 7^{16k})$ is an infinite sequence of ABC triples.

Corollary 3.2. *Let $n > 1$ be an integer and p be an odd prime such that $p > \operatorname{rad}(n)$. Then for each positive integer k, $\left(1, n^{p(p-1)k}-1, n^{p(p-1)k}\right)$ is an abc triple.*

Proof. By assumption, $\operatorname{cosocle}\left(p^2\right) = p > \operatorname{rad}(n)$. Moreover, $\lambda\left(p^2\right) = p\,(p-1)$ since p is prime. It follows from Corollary 3.1 that $\left(1, n^{\lambda(p^2)k}-1, n^{\lambda(p^2)k}\right) = \left(1, n^{p(p-1)k}-1, n^{p(p-1)k}\right)$ is an abc triple. $\square$

By taking $n = 2$ and $k = 1$ in Corollary 3.2, we obtain that $\left(1, 2^{p(p-1)}-1, 2^{p(p-1)}\right)$ is an abc triple. This result is originally due to Granville and Tucker [GT02].

Corollary 3.3. *Let $n > 1$ be an integer that is either odd or even and non-squarefree. Then $\left(1, n^{(n-1)k}-1, n^{(n-1)k}\right)$ is an abc triple for each integer k.*

Proof. Let $P = \sum_{j=0}^{n-2} n^j$ and observe that $n^{n-1}-1 = (n-1)\,P$. Moreover,

$$P \equiv \sum_{j=0}^{n-2}(1)^j \mod (n-1) = 0 \mod (n-1)\,.$$

In particular, $(n-1)^2$ divides $n^{n-1}-1$ and thus

$$\operatorname{rad}(n^{n-1}-1)=\operatorname{rad}\left(\frac{n^{n-1}-1}{n-1}\right).$$

Now suppose that n is odd. We claim that 4 divides P. If $n \equiv 1 \mod 4$, then this follows since P is divisible by $n-1$. So suppose that $n \equiv 3 \mod 4$. Then P is divisible by 4 since

$$P \equiv \sum_{j=0}^{n-2} (-1)^j \mod (n+1) = 0 \mod (n+1).$$

Consequently,

$$\operatorname{rad}(n^{n-1}-1)=\operatorname{rad}\left(\frac{n^{n-1}-1}{2\,(n-1)}\right) \leq \frac{n^{n-1}-1}{2\,(n-1)}. \tag{3.1}$$

Now observe that by (3.1),

$$\operatorname{cosocle}(n^{n-1}-1)=\frac{n^{n-1}-1}{\operatorname{rad}(n^{n-1}-1)} \geq 2\,(n-1) > \operatorname{rad}(n).$$

The claim now follows by Theorem 1 with $l=n-1$ and $m=n^l-1$.

Lastly, suppose that n is an even non-squarefree positive integer. Then $n=a^2b$ for some positive integers a and b with $a>1$ and b squarefree. Then $\operatorname{rad}(n)=\operatorname{rad}(ab)\leq ab < n-1$. Since $\operatorname{rad}(n^{n-1}-1)=\operatorname{rad}\left(\frac{n^{n-1}-1}{n-1}\right) \leq \frac{n^{n-1}-1}{n-1}$, we deduce that

$$\operatorname{cosocle}(n^{n-1}-1)=\frac{n^{n-1}-1}{\operatorname{rad}(n^{n-1}-1)} \geq n-1 > \operatorname{rad}(n).$$

The result follows by Theorem 1 with $l=n-1$ and $m=n^l-1$. ☐

Consider the smallest ABC triple $(1,8,9)$ and its corresponding infinite sequence, $\left(1,9^k-1,9^k\right)$. Then $\left(1,9^k-1,9^k\right)=\left(1,3^{2k}-1,3^{2k}\right)$ is an infinite sequence of ABC triples by Corollary 3.3. By taking $n=12$ in Corollary 3.3, then $\left(1,12^{11k}-1,12^{11k}\right)$ is an infinite sequence of ABC triples.

Corollary 3.4. *Let $n>1$ be an integer. Then $\left(1,n^{(n+1)k}-1,n^{(n+1)k}\right)$ is an abc triple whenever $(n+1)\,k$ is an even positive integer.*

Proof. Let l be an even integer and let $P=\sum_{j=0}^{l-1}(-1)^{j+1}\,n^j$. Then $n^l-1=(n+1)\,P$. We now proceed by cases.

Case 1. Suppose that n is a positive even integer and let $l=2\,(n+1)$. Then $P\equiv 0 \mod (n+1)$ and thus

$$\operatorname{rad}(n^l-1)=\operatorname{rad}\left(\frac{n^l-1}{n+1}\right)\leq\frac{n^l-1}{n+1}.$$

The claim now holds by Theorem 1 with $m=n^l-1$ since

$$\operatorname{cosocle}(n^l-1)=\frac{n^l-1}{\operatorname{rad}(n^l-1)}\geq n+1>\operatorname{rad}(n).$$

Case 2. Suppose that n is a positive odd integer. Then $l=n+1$ is even and $P\equiv 0 \mod (n+1)$. A similar argument to that of case 1 with $m=n^l-1$ shows that the result holds by Theorem 1. ☐

As an example, choose $n=21$. As a result, $(n+1)k$ is even for every k and by corollary 3.4 $\left(1,21^{22k}-1,21^{22k}\right)$ is an infinite sequence of ABC triples.

Corollary 3.5. *Let n be an even positive integer. Then $\left(1, n^{(n+1)k}, n^{(n+1)k}+1\right)$ is an abc triple for each positive odd integer k.*

Proof. Observe that $n^{n+1}+1 = (n+1)\sum_{j=0}^{n}(-n)^j$. Moreover, $\sum_{j=0}^{n}(-n)^j \equiv 0 \mod(n+1)$ and thus $\mathrm{rad}(n^{n+1}+1) = \mathrm{rad}\left(\frac{n^{n+1}+1}{n+1}\right) \leq \frac{n^{n+1}+1}{n+1}$. Consequently,

$$\mathrm{cosocle}(n^{n+1}+1) = \frac{n^{n+1}+1}{\mathrm{rad}(n^{n+1}+1)} \geq n+1 > \mathrm{rad}(n)\,.$$

The result now follows from Theorem 2 by taking $l = n+1$ and $m = n^l$. Choose $n = 22$, then by corollary 3.5 $\left(1, 22^{23k}, 22^{23k}+1\right)$ is an infinite sequence of ABC triples for each positive odd integer k. □

4. The *ABC*@Home Project

As mentioned prior, the transfer method provides a way to map existing ABC triples to new ABC triples by applying polynomial identities. For our purposes, we consider polynomial identities that preserve $a = 1$. By splitting the binomial formula $(a+b)^n$, we obtain the following family of identities:

$$a^{n-k}\left(\sum_{j=0}^{k}\binom{n}{j}a^{k-j}b^j\right) + b^{k+1}\left(\sum_{j=0}^{n-k-1}\binom{n}{j}a^j b^{n-k-1-j}\right) = c^n$$

Choosing $k = 0$, we obtain a transfer theorem that sends $(1, c-1, c)$ to $(1, c^n-1, c^n)$ for each positive integer n. However, we note that Theorem 1 recovers this result, since given any ABC triple $\left(1, n^l-1, n^l\right)$ that satisfies the conditions of Theorem 1, $\left(1, n^{lk}-1, n^{lk}\right)$ is an ABC triple. Therefore, the two non-trivial polynomials identities that we apply in the next section are the ones stated in the introduction, namely, $(a,b,c) \mapsto \left(a^3, b^3, c\left(b^2-ab+a^2\right)\right)$ and $(a,b,c) \mapsto \left(a^2, c\left(b-a\right), b^2\right)$

Motivated by Theorem 1 and Theorem 2, we compute the quality the ABC triples found by them and compare them to the quality of the ABC triples found by the ABC@Home project. As mentioned prior, the ABC@Home project found 45,603 ABC triples of the form $(1, c-1, c)$. Using the theorems stated above and various transfer methods, our theorems account for approximately 719 or 1.57% of the ABC triples found by the ABC@Home project. We have sorted our triples by quality, the extent to which an ABC triple satisfies the ABC Conjecture.

Definition 4.1. Given an ABC triple (a,b,c), the quality $q(a,b,c)$ of the triple is defined to be

$$q(a,b,c) = \frac{\log c}{\log \mathrm{rad}(abc)}$$

The triple with the highest quality is $(2, 3^{10}\cdot 109, 23^5)$, bearing a quality of 1.6299. The highest quality triple in the data set from our results is $(1, 2400, 2401)$ with a quality of 1.455 and the lowest quality triple has a quality of 1.00005.

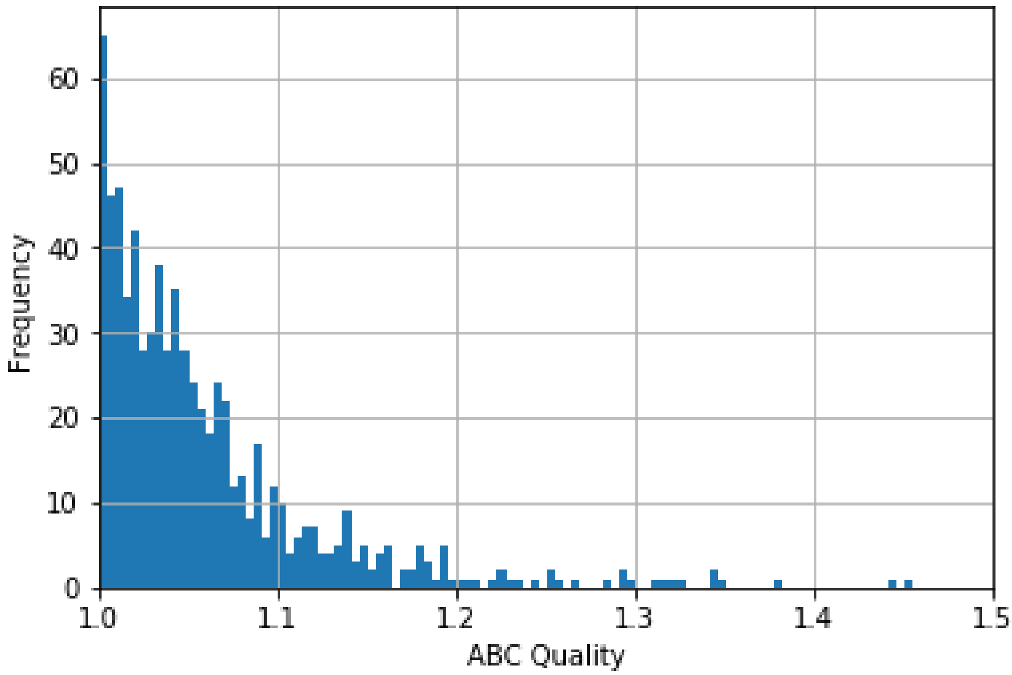

Figure 1. Histogram of ABC Triple Quality

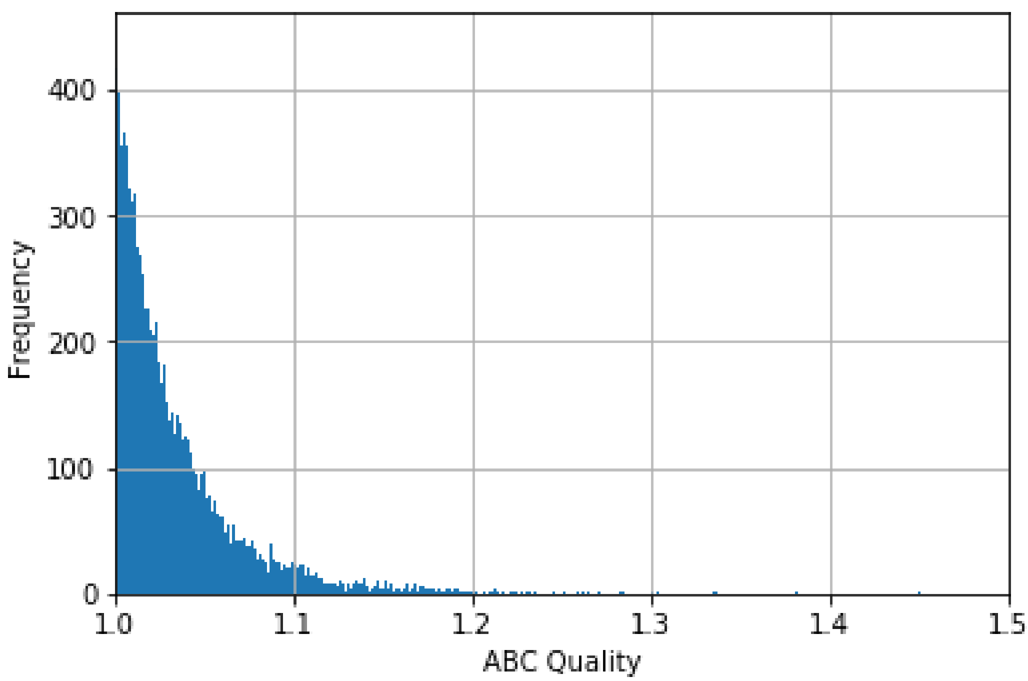

Figure 2. Histogram of Quality of 45,603 ABC Triples

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