5. Prove the estimate (12.23).

[*Hint*: for each internal node $x_j$, $j = 1, \dots, n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j)$$

$$= -u''(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} u''(t)(x_j - h - t)^2\, dt - \int_{x_j}^{x_j+h} u''(t)(x_j + h - t)^2\, dt\right].$$

Then, pass to the squares and sum $\tau_h(x_j)^2$ for $j = 1, \dots, n-1$. On noting that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, for any real numbers $a, b, c$, and applying the Cauchy-Schwarz inequality yields the desired result.]

<P.f>

$$\because\ \tau_h(x_j) = \underbrace{-u''(x_j)}_{a} - \frac{1}{h^2}\Big(\underbrace{\int_{x_{j-1}}^{x_j} u''(t)(x_j-h-t)^2\,dt}_{b} - \underbrace{\int_{x_j}^{x_{j+1}} u''(t)(x_j+h-t)^2\,dt}_{c}\Big)$$

$$\therefore\ (\tau_h(x_j))^2 \le 3\left(|u''(x_j)|^2 + \frac{b^2}{h^4} + \frac{c^2}{h^4}\right)$$

for $b^2 = \left(\int_{x_{j-1}}^{x_j} u''(t)(x_j-h-t)^2\,dt\right)^2$ by Cauchy-Schwarz

$$b^2 \le \int_{x_{j-1}}^{x_j}(u''(t))^2 dt \int_{x_{j-1}}^{x_j}(x_j-h-t)^4 dt = \int_{x_{j-1}}^{x_j}(u''(t))^2 dt \int_{x_{j-1}}^{x_j}(x_{j-1}-t)^4 dt$$

Let $s = x_{j-1} - t$

$$= \int_{x_{j-1}}^{x_j}(u''(t))^2 dt \cdot \int_0^h s^4\,ds = \int_{x_{j-1}}^{x_j}(u''(t))^2 dt \cdot \frac{s^5}{5}\Big|_0^h = \int_{x_{j-1}}^{x_j}(u''(t))^2 dt \cdot \frac{h^5}{5}$$

$$\therefore\ b^2 \le \int_{x_{j-1}}^{x_j}(u''(t))^2 dt \cdot \frac{h^5}{5}$$

for $c^2 = \int_{x_j}^{x_{j+1}} u''(t)(x_{j+1}-t)^2\,dt \le \int_{x_j}^{x_{j+1}}(u''(t))^2 dt \int_{x_j}^{x_{j+1}}(x_{j+1}-t)^4 dt$

$$= \int_{x_j}^{x_{j+1}}(u''(t))^2 dt \int_0^h s^4\,ds = \int_{x_j}^{x_{j+1}}(u''(t))^2 dt \cdot \frac{h^5}{5}$$

$$\therefore\ \|\tau_h\|_h^2 = h\sum_{j=1}^{n-1}|\tau_h(x_j)|^2 \le 3\left(h\sum_{j=1}^{n-1}|u''(x_j)|^2 + \frac{h}{5}\sum_{j=1}^{n-1}\int_{x_{j-1}}^{x_j}(u''(t))^2 dt + \frac{h}{5}\sum_{j=1}^{n-1}\int_{x_j}^{x_{j+1}}(u''(t))^2 dt\right)$$

$$= 3\left(\|u''\|_h^2 + \frac{h}{5}\int_0^1 (u''(t))^2 dt + \frac{h}{5}\int_0^1 (u''(t))^2 dt\right)$$

$$= 3\left(\|u''\|_h^2 + \frac{2h}{5}\int_0^1 (u''(t))^2 dt\right)$$

$$= 3\left(\|u''\|_h^2 + \frac{2h}{5}\|u\|^2_{L^2(0,1)}\right)$$

$$\le 3\left(\|f\|_h^2 + \|f\|^2_{L^2(0,1)}\right)$$

7. Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2}x_j(1-x_j)$.
[*Solution*: use the definition (12.25) with $g(x_k) = 1$, $k = 1, \ldots, n-1$ and recall that $G^k(x_j) = hG(x_j, x_k)$ from the exercise above. Then

$$T_h g(x_j) = h\left[\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right]$$

from which, after straightforward computations, one gets the desired result.]

<Pf> $T_h g = \sum_{k=1}^{n-1} g(x_k)\, G^k$

Let $g(x_k) \equiv 1$ for $k = 1, \cdots, n-1$

$\Rightarrow T_h g(x_j) = \sum_{k=1}^{n-1} G^k = h\sum_{k=1}^{n-1} G(x_j, x_k)$

$\because G(x,s) = \begin{cases} x(1-s) & x \le s \\ s(1-x) & x \ge s \end{cases}$

$\Rightarrow T_h g(x_j) = h\left(\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right)$

$= h\left((1-x_j)\sum_{k=1}^{j} x_k + x_j\sum_{k=j+1}^{n-1}(1-x_k)\right)$

$= h\left((1-x_j)\sum_{k=1}^{j} kh + x_j\sum_{k=j+1}^{n-1}(1-kh)\right)$

$= h\left((1-x_j)\cdot h\frac{j(j+1)}{2} + x_j\left(\sum_{k=j+1}^{n-1} 1 - \sum_{k=j+1}^{n-1} kh\right)\right)$

$= h\left((1-x_j)\cdot h\cdot\frac{j(j+1)}{2} + x_j\left[(n-1-j) - h\left(\frac{(n-1)n - j(j+1)}{2}\right)\right]\right)$

$= h^2(1-x_j)\cdot\frac{j(j+1)}{2} + hx_j\left[(n-1-j) - h\left(\frac{(n-1)n - j(j+1)}{2}\right)\right]$

$= h^2(1-jh)\cdot\frac{j(j+1)}{2} + jh^2\left[(n-1-j) - h\left(\frac{(n-1)n - j(j+1)}{2}\right)\right]$

$= \frac{h^2 j(j+1)}{2}(1-jh) + jh^2\left[(n-1-j) - h\left(\frac{n(n-1) - j(j+1)}{2}\right)\right]$

$= \frac{h^2 j(j+1)}{2} - \frac{j^2h^3(j+1)}{2} + jh^2(n-1-j) - jh^3\left(\frac{n(n-1) - j(j+1)}{2}\right)$

$jh^3\left(-\frac{j(j+1)}{2} - \frac{n(n-1)}{2} + \frac{j(j+1)}{2}\right) = jh^3\left(-\frac{n(n-1)}{2}\right) = -\frac{h^3 j\, n(n-1)}{2}$

$jh^2\left(\frac{(j+1)}{2} + n-1-j\right) = jh^2\left(n-1-\frac{j+1}{2}\right)$

$= jh^2\left(n-1-\frac{j+1}{2}\right) - \frac{h^3 j\, n(n-1)}{2}$

$= \frac{j}{n^2}\left(n-1-\frac{j+1}{2}\right) - \frac{j}{n^3}\,\frac{n(n-1)}{2}$

$= \frac{j}{n^2}\cdot\frac{2n-2-j-1}{2} = \frac{j(2n-j-3)}{2n^2}$ $\quad \frac{j(n-1)}{2n^2}$

$$= \frac{j(2n-j-3)}{2n^2} - \frac{j(n-1)}{2n^2} = \frac{j(2n-j-3-n+1)}{2n^2} = \frac{j(n-j-2)}{2n^2}$$

$$= \frac{x_j(n-j-2)}{2n} = \frac{x_j}{2} \cdot h(n-j-2) = \frac{x_j}{2}(hn - x_j - 2h) = \frac{x_j}{2}(h(n-2) - x_j)$$

$$= \frac{x_j}{2}\left(1 - \frac{1}{n} - x_j\right) \approx \frac{x_j}{2}(1 - x_j)$$

8. Prove Young's inequality (12.40).

We recall now the following Young's inequality (see Exercise 8)

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \qquad \forall a, b \in \mathbb{R}, \quad \forall \varepsilon > 0.$$

<P.f> Consider $\left(\sqrt{\varepsilon}\, a - \frac{1}{2\sqrt{\varepsilon}} b\right)^2 \ge 0$

$$\Rightarrow \varepsilon a^2 - 2 \cdot \sqrt{\varepsilon} \cdot \frac{1}{2\sqrt{\varepsilon}} ab + \frac{1}{4\varepsilon} b^2 \ge 0$$

$$\Rightarrow \varepsilon a^2 - ab + \frac{1}{4\varepsilon} b^2 \ge 0$$

$$\Rightarrow ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \#$$

9. Show that $\|v_h\|_h \le \|v_h\|_{h,\infty}$ $\forall v_h \in V_h$.

<P.f> By the def

$$\|V_h\|_h^2 = h \sum_{j=1}^{n-1} V_j^2 \quad \text{and}$$

$$\|V_h\|_{h,\infty} = \max_{1 \le j \le n-1} |V_j|$$

Obviously we have $|V_j| \le \|V_h\|_{h,\infty}$, $\forall j$

$$\therefore \sum_{j=1}^{n-1} V_j^2 \le \sum_{j=1}^{n-1} \|V_h\|_{h,\infty} = (n-1)\|V_h\|_{h,\infty}$$

$$\Rightarrow \|V_h\|_h^2 = h \sum_{j=1}^{n-1} V_j^2 \le h(n-1)\|V_h\|_{h,\infty}$$

$$\because h = \frac{1}{n} \Rightarrow h(n-1) = 1 - \frac{1}{n} \le 1$$

$$\Rightarrow h(n-1)\|V_h\|_{h,\infty}^2 \le \|V_h\|_{h\infty}^2$$

$$\therefore \|V_h\|_h^2 \le \|V_h\|_{h\infty}^2 \Rightarrow \|V_h\|_h \le \|V_h\|_{h\infty} \quad \#$$

11. Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.
[*Solution*: apply twice the second order centered finite difference operator $L_h$ defined in (12.9).]

<P.f> $\displaystyle (L_h u)(x_{j-1}) = -\frac{u_j - 2u_{j-1} + u_{j-2}}{h^2}$

$$(L_h u)(x_j) = -\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

$$(L_h u)(x_{j+1}) = -\frac{u_{j+2} - 2u_{j+1} + u_j}{h^2}$$

$$\therefore (L_h u)^2(x_j) = L_h\left(-\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}\right)$$

$$= -\frac{L_h u_{j-1} - 2L_h u_j + L_h u_{j+1}}{h^2}$$

$$= -\frac{1}{h^2}\left(\left(\frac{-1}{h^2}(u_j - 2u_{j-1} + u_{j-2}) + \frac{2}{h^2}(u_{j+1} - 2u_j + u_{j-1}) + \frac{-1}{h^2}(u_{j+2} - 2u_{j+1} + u_j)\right)\right)$$

$$= -\frac{1}{h^4}(-u_j + 2u_{j-1} - u_{j-2} + 2u_{j+1} - 4u_j + 2u_{j-1} - u_{j+2} + 2u_{j+1} - u_j)$$

$$= -\frac{1}{h^4}(-u_{j-2} + 4u_{j-1} - 6u_j + 4u_{j+1} - u_{j+2})$$

$$= \frac{1}{h^4}(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2})$$

$$\therefore u^{(iv)}(x) = -(-u'')''(x) = L_h u\,(L_h u)(x_j) = L_h^2 u(x_j)$$

$$Lu(x) = -u^{(iv)}(x) = -L_h^2 u(x_j) = \frac{-1}{h^4}(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2})$$