7. Prove that the *gamma function*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \qquad z \in \mathbb{C}, \quad \text{Re} z > 0,$$

is the solution of the difference equation $\Gamma(z+1) = z\Gamma(z)$
[*Hint*: integrate by parts.]

<Pf> $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

$\Rightarrow \Gamma(z+1) = \int_0^\infty e^{-t} t^{z} dt$

Let $u = t^z \quad dv = e^{-t} dt$

$\Rightarrow du = z t^{(z-1)} dt, \; v = -e^{-t}$

$\therefore \Gamma(z+1) = -t^z \cdot e^{-t} \Big|_0^\infty - \int_0^\infty -e^{-t} z t^{(z-1)} dt$

$= 0 + z \int_0^\infty e^{-t} \cdot t^{(z-1)} dt$

$= z \cdot \Gamma(z)$ #

9. Consider the following family of one-step methods depending on the real parameter α

$$u_{n+1} = u_n + h[(1 - \frac{\alpha}{2}) f(x_n, u_n) + \frac{\alpha}{2} f(x_{n+1}, u_{n+1})].$$

Study their consistency as a function of α; then, take $\alpha = 1$ and use the corresponding method to solve the Cauchy problem

$$\begin{cases} y'(x) = -10y(x), & x > 0, \\ y(0) = 1. \end{cases}$$

Determine the values of h in correspondance of which the method is absolutely stable.
[*Solution*: the family of methods is consistent for any value of α. The method of highest order (equal to two) is obtained for $\alpha = 1$ and coincides with the Crank-Nicolson method.]

<Sol> $u_{n+1} = u_n + h[(1-\frac{\alpha}{2}) f(x_n, u_n) + \frac{\alpha}{2} f(x_{n+1}, u_{n+1})]$ — (1)

use taylor expanian to y_{n+1}

$\Rightarrow y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y'' + O(h^3)$

$(y'' = \frac{d}{dx} y' = \frac{d}{dx} f(x, y(x)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = f_x + f_y \cdot f)$

$\Rightarrow y_{n+1} = y_n + h f_n + \frac{h^2}{2}(f_x + f_y \cdot f_n) + O(h^3) \Rightarrow y_{n+1} - y_n = h f_n + \frac{h^2}{2}\underbrace{(f_x + f_y \cdot f)}_{\phi_n} + O(h^3)$ (2)

By (1) $\Rightarrow y_{n+1} = y_n + h[(1-\frac{\alpha}{2}) f_n + \frac{\alpha}{2} f_{n+1}] + h\tau_{n+1}$

$\Rightarrow h\tau_{n+1} = y_{n+1} - y_n - h[(1-\frac{\alpha}{2}) f_n + \frac{\alpha}{2} f_{n+1}]$

By (2) $\overset{\text{||}}{=} h f_n + \frac{h^2}{2}\phi_n - h(1-\frac{\alpha}{2}) f_n - \frac{\alpha}{2} h f_{n+1} + O(h^3)$

$\Rightarrow T_{n+1} = f_n + \frac{h}{2}\phi_n - (1-\frac{\alpha}{2}) f_n - \frac{\alpha}{2} f_{n+1} + O(h^2)$

(for $f_{n+1} = f(x_{n+1}, y_{n+1}) = f(x_n + \Delta x + y_n + \Delta y)$ use Taylor expanian

$f(a+h, b+k) = f(a,b) + \nabla f(a,b)\cdot(h,k) + O(\|(h,k)\|^2)$

$\therefore f(x_{n+1}, y_{n+1}) = f(x_n, y_n) + (f_x, f_y)\cdot(\Delta x, \Delta y) + O(\|(\Delta x, \Delta y)\|^2)$

$= f_n + f_x \cdot \Delta x + f_y \Delta y + O(\|\Delta x, \Delta y\|^2)$

$= f_n + h\cdot f_x + f_y (y_{n+1} - y_n)$

$= f_n + h\cdot f_x + f_y (hf + \frac{h^2}{2}(f_x + f_y \cdot f) + O(h^3))$

$= f_n + h\cdot f_x + f_y (h f_n + O(h^2)) + O(h^2))$

$= f_n + h\cdot f_x + h f_y \cdot f_n + O(h^2)$

$= f_n + h\cdot \phi_n + O(h^2)$)

$\therefore T_{n+1} = f_n + \frac{h}{2}\phi_n - (1-\frac{\alpha}{2}) f_n - \frac{\alpha}{2}(f_n + h\cdot\phi_n) + O(h^2)$

$= \cancel{f_n} + \frac{h}{2}\phi_n - \cancel{f_n} + \cancel{\frac{\alpha}{2} f_n} - \cancel{\frac{\alpha}{2} f_n} - \frac{\alpha}{2} h\cdot\phi_n + O(h^2)$

$= (\frac{h}{2} - \frac{\alpha h}{2})\phi_n + O(h^2)$

$= (\frac{(1-\alpha)}{2} h)\phi_n + O(h^2)$

$\therefore \forall \alpha : T_{n+1} \to 0$ as $h \to 0$ Consistency hold

also only for $\alpha = 1$ is order 2 the other is order 1

So for $\alpha = 1$: $u_{n+1} = u_n + \frac{h}{2} f(x_n, u_n) + \frac{h}{2} f(x_{n+1}, u_{n+1})$ is CN method

test $\begin{cases} y'(x) = -10 y(x) \\ y(0) = 1 \end{cases}$ $\therefore f(x,y) = -10\cdot y(x)$

Solve $u_{n+1} = u_n + \frac{h}{2}(-10\cdot u_n) + \frac{h}{2}(-10\cdot u_{n+1})$

$\Rightarrow u_{n+1} = u_n - 5h u_n - 5h u_{n+1}$

$\Rightarrow u_{n+1} + 5h\, u_{n+1} = u_n - 5h\, u_n$

$\Rightarrow (1+5h)\,u_{n+1} = (1-5h)\,u_n$

$$\Rightarrow u_{n+1} = \frac{(1-5h)}{(1+5h)}u_n = \frac{(1-5h)^n}{(1+5h)^n}u_0 = \frac{(1-5h)^n}{(1+5h)^n}$$

$\therefore$ now we want to find h s.t $\left|\frac{1-5h}{1+5h}\right| \leq 1$

$\Rightarrow |1-5h| \leq |1+5h|$

$\Rightarrow \forall h>0$ the inequality hold $\therefore$ $\forall h>0$ it's absolutely stable