1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12). Prove that $|E_1(f)| \simeq 2|E_0(f)|$.

$E_1(f) = -\frac{h^3}{12} f''(\xi)$, $h = b-a$

$E_0(f) = \frac{h'^3}{3} f''(\xi)$, $h' = \frac{b-a}{2}$

Setting $h = b-a \Rightarrow E_1(f) = -\frac{h^3}{12} f''(\xi_1)$ and $E_0 = \frac{h^3}{24} f''(\xi_0)$

$$\Rightarrow \frac{|E_1(f)|}{2|E_0(f)|} = \frac{\left|-\frac{h^3}{12} f''(\xi_1)\right|}{2\left|\frac{h^3}{24} f''(\xi_0)\right|} = \frac{\frac{h^3}{12}|f''(\xi_1)|}{\frac{h^3}{12}|f''(\xi_0)|} = \frac{|f''(\xi_1)|}{|f''(\xi_0)|}$$

Let $c \in [a,b]$ and $\xi_1, \xi_0 \in [c-h, c+h]$

$$\Rightarrow \lim_{h\to 0} \frac{|E_1(f)|}{2|E_0(f)|} = \lim_{h\to 0} \frac{|f''(\xi_1(h))|}{|f''(\xi_0(h))|} = \frac{|f''(c)|}{|f''(c)|} = 1$$

$\Rightarrow 2|E_0(f)| \simeq |E_1(f)|$ $\square$

3. Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes. Compute the degree of exactness r of the formulae:
(a) $I_2(f) = (2/3)[2f(-1/2) - f(0) + 2f(1/2)]$,
(b) $I_4(f) = (1/4)[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)]$.
Which is the order of infinitesimal p for (a) and (b)?
[*Solution*: $r = 3$ and $p = 5$ for both $I_2(f)$ and $I_4(f)$.]

(a) $I_2(f) = \frac{2}{3}[2f(-\frac{1}{2}) - f(0) + 2f(\frac{1}{2})]$

$\{\alpha_k\} = \{\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\}$, $\{x_k\} = \{-\frac{1}{2}, 0, \frac{1}{2}\}$

for $k=0 \Rightarrow x^0 = 1 \Rightarrow \int_{-1}^{1} 1\,dx = 2$

$\Rightarrow \frac{2}{3}[2\cdot 1 - 1 + 2\cdot 1] = \frac{2}{3}\cdot 3 = 2$ ✓

$k=1 \Rightarrow x \Rightarrow \int_{-1}^{1} x\,dx = 0$

$\Rightarrow \frac{2}{3}[2\cdot(-\frac{1}{2}) - 0 + 2\cdot\frac{1}{2}] = 0$ ✓

$k=2 \Rightarrow x^2 \Rightarrow \int_{-1}^{1} x^2 dx = \frac{2}{3}$

$\Rightarrow \frac{2}{3}[2\cdot(\frac{1}{4}) - 0 + 2\cdot(\frac{1}{4})] = \frac{2}{3}$ ✓

$k=3 \Rightarrow x^3 \Rightarrow \int_{-1}^{1} x^3 dx = 0$

$\Rightarrow \frac{2}{3}[2\cdot(-\frac{1}{8}) - 0 + 2(\frac{1}{8})] = 0$

$k=4 \Rightarrow x^4 \Rightarrow \int_{-1}^{1} x^4\,dx = \frac{2}{5}$

$\Rightarrow \frac{2}{3}[2(\frac{1}{16}) - 0 + 2(\frac{1}{16})] = \frac{1}{6}$ ×

$\therefore r=3$

for $P \Rightarrow \because r=3 \therefore E(f) = \frac{f^{4}(\xi)}{4!}\left(\int_{-1}^{1} x^4\,dx - I_3(f(x^4))\right)$

$\Rightarrow \frac{f^{4}(\xi)}{4!}\left(\frac{2}{5} - \underline{I_3(x^4)}\right) = \frac{f^{(4)}(\xi)}{4!}\left(\frac{2}{5} - \frac{1}{6}\right) = \frac{f^{(4)}(\xi)}{4!}\cdot\frac{7}{30}$

$= \frac{2}{3}[2\cdot(-\frac{1}{2})^4 - 0 + 2\cdot(\frac{1}{2})^4] = \frac{2}{3}[\frac{2}{16} + \frac{2}{16}] = \frac{1}{6}$

$\therefore E(f) = \frac{f^{(4)}(\xi)}{24}\cdot\frac{7}{30} = \frac{7}{720} f^{(4)}(\xi)$, $\xi \in [-1,1]$

change $[-1,1] \to [a,b] \Rightarrow \frac{7}{720}\cdot\frac{h^5}{32} f^{(4)}(\xi)$, $\xi \in [a,b]$ $\therefore P=5$

(b) $I_4(f) = \frac{1}{4}[f(-1) + 3f(-\frac{1}{3}) + 3f(\frac{1}{3}) + f(1)]$

for $k=0 \Rightarrow x^0 = 1 \Rightarrow \frac{1}{4}[1+3+3+1] = 2$ ✓ (we have calculate $\int_{-1}^{1} f(x)\,dx$ for $f(x) = 1, x, x^2, x^3$)

for $k=1 \Rightarrow x \Rightarrow \frac{1}{4}[-1 + 3\cdot(-\frac{1}{3}) + 3\cdot(\frac{1}{3}) + 1] = 0$ ✓

for $k=2 \Rightarrow x^2 \Rightarrow \frac{1}{4}[1 + 3\cdot\frac{1}{9} + 3\cdot\frac{1}{9} + 1] = \frac{2}{3}$ ✓

for $k=3 \Rightarrow x^3 \Rightarrow \frac{1}{4}((-1)^3 + 3\cdot(-\frac{1}{3})^3 + 3\cdot(\frac{1}{3})^3 + 1^3] = 0$ ✓

for $k=4 \Rightarrow x^4 \Rightarrow \frac{1}{4}((-1)^4 + 3\cdot(-\frac{1}{3})^4 + 3(\frac{1}{3})^4 + 1^4] = \frac{14}{27} \neq \frac{2}{5}$

$\therefore r=3$

$\frac{1}{4}[(-1)^4 + 3\cdot(-\frac{1}{3})^4 + 3\cdot(\frac{1}{3})^4 + 1^4] = \frac{28}{54} = \frac{14}{27}$

for $P \Rightarrow \because r=3 \therefore E(f) = \frac{f^{(4)}(\xi)}{4!}\left(\int_{-1}^{1} x^4\,dx - \underline{I_4(x^4)}\right) =$

$= \frac{f^{(4)}(\xi)}{4!}\left(\frac{2}{5} - \frac{14}{27}\right) = -\frac{16}{135}$

$\therefore E(f) = -\frac{2}{405} f^{(4)}(\xi)$ $\xi \in [-1,1]$

change $[-1,1] \Rightarrow [a,b] \Rightarrow E(f) = -\frac{2}{405}\cdot\frac{h^5}{32} f^{(4)}(\xi)$ $\therefore P=5$

5. Let $I_w(f) = \int_0^1 w(x)f(x)dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = af(x_1)$. Find a and x_1 in such a way that Q has maximum degree of exactness r.
[*Solution*: $a = 2/3$, $x_1 = 3/5$ and $r = 1$.]

if $\deg f(x) \le r \Rightarrow Q(f) = I_w(f)$

$\therefore$ for $r = 0 \Rightarrow f(x) = 1 \Rightarrow I_w(1) = \int_0^1 \sqrt{x}\,dx = \frac{2}{3} \cdot x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}$

$\therefore Q(1) = \alpha = \frac{2}{3}$

for $r = 1 \Rightarrow f(x) = x \Rightarrow I_w(x) = \int_0^1 x\sqrt{x}\,dx = \int_0^1 x^{\frac{3}{2}} dx = \frac{2}{5} x^{\frac{5}{2}} \Big|_0^1 = \frac{2}{5}$

$\therefore Q(x) = \alpha \cdot f(x_1) = \frac{2}{3} \cdot x_1 = \frac{2}{5} \Rightarrow x_1 = \frac{3}{5}$

for $r = 2 \Rightarrow f(x) = x^2 \Rightarrow I_w(x) = \int_0^1 x^2\sqrt{x}\,dx = \int_0^1 x^{\frac{5}{2}} dx = \frac{2}{7} x^{\frac{7}{2}} \Big|_0^1 = \frac{2}{7}$

$\therefore Q(x^2) = \alpha f(x_1) = \frac{2}{3} \cdot \left(\frac{3}{5}\right)^2 = \frac{6}{25} \neq \frac{2}{7}$

$\therefore r = 1, \quad \alpha = \frac{2}{3} \quad x_1 = \frac{3}{5}$

6. Let us consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for the approximation of $I(f) = \int_0^1 f(x)dx$, where $f \in C^1([0,1])$. Determine the coefficients α_j, for $j = 1, 2, 3$ in such a way that Q has degree of exactness $r = 2$.
[*Solution*: $\alpha_1 = 2/3$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.]

for $r = 0 \Rightarrow f(x) = 1 \Rightarrow I(f) = \int_0^1 1\,dx = 1$

$\therefore Q(1) = \alpha_1 + \alpha_2 = 1$

for $r = 1 \Rightarrow f(x) = x \Rightarrow I(f) = \int_0^1 x\,dx = \frac{1}{2}$

$Q(x) = \alpha_1 \cdot 0 + \alpha_2 + \alpha_3 = \frac{1}{2} \Rightarrow \alpha_2 + \alpha_3 = \frac{1}{2}$

for $r = 2 \Rightarrow f(x) = x^2 \Rightarrow I(f) = \int_0^1 x^2 dx = \frac{1}{3}$

$Q(x^2) = \frac{1}{2} \cdot 0 + \alpha_2 + 0 = \frac{1}{3} \Rightarrow \alpha_2 = \frac{1}{3}$

$\therefore \alpha_1 = \frac{2}{3}, \quad \alpha_3 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

check $r = 3 \Rightarrow f(x) = x^3 \Rightarrow I(f) = \int_0^1 x^3 dx = \frac{1}{4}$

$Q(x^3) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 1 + \frac{1}{6} \cdot 0 = \frac{1}{3} \neq \frac{1}{4}$

$\therefore r = 2$ is maximum and $Q(f) = \frac{2}{3} f(0) + \frac{1}{3} f(1) + \frac{1}{6} f'(0)$