1. Consider the boundary value problem (12.1)-(12.2) with $f(x) = 1/x$. Using (12.3) prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ but $u(0)$ is not defined and u', u'' do not exist at $x = 0$ ($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

<Pf> by (12.3) the solution of boundary value problem is

$u(x) = \int_0^1 G(x,s) f(s)\, ds$, where $G(x,s)$ is Green's function

Let $f(x) = \frac{1}{x}$

$$
\begin{aligned}
\Rightarrow u(x) &= \int_0^1 G(x,s)\cdot\frac{1}{s}\, ds \\
&= \int_0^x s(1-x) f(s)\, ds + \int_x^1 x(1-s) f(s)\, ds \\
&= \int_0^x s(1-x)\cdot\frac{1}{s}\, ds + \int_x^1 x(1-s)\frac{1}{s}\, ds \\
&= (1-x)\int_0^x 1\, ds + x\int_x^1 \frac{1}{s} - 1\, ds \\
&= (1-x)\cdot s\Big|_0^x + x(\ln s - s)\Big|_x^1 \\
&= (1-x)\cdot x - 0 + x\left[(0-1) - (\ln x - x)\right] \\
&= x - x^2 + x\left[-1 - \ln x + x\right] \\
&= x - x^2 - x - x\ln x + x^2 = -x\ln x ,
\end{aligned}
$$

So $u(0)$ is not defined , also $u'(x) = -\ln x - 1$, $u''(x) = -\frac{1}{x}$

where $u'(0)$, $u''(0)$ is also not defined

6. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4). [*Solution*: we prove the result by verifying that $L_h G = he^k$. Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_h G = 0$ at every node x_l with $l = 0, \ldots, k-1$ and $l = k+1, \ldots, n+1$. Finally, a direct computation shows that $(L_h G)(x_k) = 1/h$ which concludes the proof.]

<p.f> for fixed x_k the function $G(x_k, s) = \begin{cases} s(1-x_k) & s \le x_k \\ x_k(1-s) & s \ge x_k \end{cases}$

is a straigh line in $[0, x_k]$ and $[x_k, 1]$

So $L_h G(x_k, x_j) = \dfrac{G(x_k, x_{j-1}) - 2G(x_k, x_j) + G(x_k, x_{j+1})}{h^2}$

for x_j, x_{j+1}, x_{j-1} in one of the interval $[0, x_k]$ or $[x_k, 1]$

and $j \neq k$ we have

$G(x_k, x_{j-1}) = x_{j-1}(1-x_k) = (x_j - h)(1-x_k)$, $G(x_k, x_{j+1}) = x_{j+1}(1-x_k) = (x_j+h)(1-x_k)$, $G(x_k, x_j) = x_j(1-x_k)$

$\Rightarrow G(x_k, x_{j-1}) - 2G(x_k, x_j) + G(x_k, x_{j+1})$

$= (x_j - h)(1-x_k) - 2x_j(1-x_k) + (x_j+h)(1-x_k)$

$= (1-x_k)[x_j - h - 2x_j + x_j + h] = 0$

if $j = k$

$L_h G(x_k, x_j) = -\dfrac{G(x_k, x_{k-1}) - 2G(x_k, x_k) + G(x_k, x_{k+1})}{h^2}$

$G(x_k, x_{k-1}) = x_{k-1}(1-x_k) = (x_k - h)(1-x_k)$, $G(x_k, x_k) = x_k(1-x_k)$, $G(x_k, x_{k+1}) = x_k(1-x_{k+1}) = x_k(1-(x_k+h))$

$\Rightarrow G(x_k, x_{k-1}) - 2G(x_k, x_k) + G(x_k, x_{k+1})$

$= (x_k - h)(1-x_k) - 2x_k(1-x_k) + x_k(1-x_k-h)$

$= (1-x_k)(-h-x_k) + x_k(1-x_k-h)$

$= -h + \cancel{hx_k} - \cancel{x_k} + \cancel{x_k^2} + \cancel{x_k} - \cancel{x_k^2} - \cancel{x_k h} = -h$

$\therefore L_h G(x_j, x_k) = -\dfrac{-h}{h^2} = \dfrac{1}{h}$ $\Rightarrow L_h G(x_k, x_j) = \begin{cases} \frac{1}{h} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$

define $\tilde{G}^k(x_j) = hG(x_j, x_k) \Rightarrow (L_h \tilde{G}^k)(x_j) = h(L_h G(x_j, x_k) = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases} = e^k$

and $\tilde{G}^k(x_0) = hG(x_0, x_k) = hG(x_n, x_k) = \tilde{G}^k(x_n) = 0$

$\therefore$ $\tilde{G}^k(x_j)$ satisfy the same boundary value problem with G^k

$\therefore$ $\tilde{G}^k(x_j) = G^k(x_j) = hG(x_j, x_k)$.

2. Check for any sequence $\{w_j\}_{j=0}^{n}$ and $\{v_j\}_{j=0}^{n}$ we have

$$\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = w_n v_{n-1} - w_0 v_0 - \sum_{j=1}^{n-1} w_j (v_j - v_{j-1})$$

<P.f> $\sum_{j=0}^{n-1}(w_{j+1}-w_j)v_j = \sum_{j=0}^{n-1} w_{j+1}v_j - \sum_{j=0}^{n-1} w_j v_j$

for $\sum_{j=0}^{n-1} w_{j+1}v_j$, Let $i=j+1$

$$\Rightarrow \sum_{i=1}^{n} w_i v_{i-1} = w_n v_{n-1} + \sum_{i=1}^{n-1} w_i v_{i-1} = w_n v_{n-1} + \sum_{j=1}^{n-1} w_j v_{j-1} \quad -(1)$$

for $\sum_{j=0}^{n-1} w_j v_j = w_0 v_0 + \sum_{j=1}^{n-1} w_j v_j \quad -(2)$

$$\Rightarrow (1)-(2) \Rightarrow w_n v_{n-1} + \sum_{j=1}^{n-1} w_j v_{j-1} - w_0 v_0 - \sum_{j=1}^{n-1} w_j v_j$$

$$= w_n v_{n-1} - w_0 v_0 + \sum_{j=1}^{n-1} w_j (v_{j-1} - v_j)$$

$$= w_n v_{n-1} - w_0 v_0 - \sum_{j=1}^{n-1} w_j (v_j - v_{j-1}) \#$$

we want to show that $(L_h v_h, v_h)_h = h^{-1}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2$

<P.f> By the def of inner product $(u.v)_h = h\sum_{j=1}^{n-1} u_j v_j$

also $L_h v_h = -\dfrac{v_{j-1}-2v_j+v_{j+1}}{h^2} = -\dfrac{(v_{j+1}-v_j)-(v_j-v_{j-1})}{h^2}$

$$\therefore (L_h v_h, v_h)_h = h\sum_{j=1}^{n-1}\left(-\frac{(v_{j+1}-v_j)-(v_j-v_{j-1})}{h^2}\right)v_j$$

$$= -\frac{1}{h}\left[\sum_{j=1}^{n-1}(v_{j+1}-v_j)v_j - \sum_{j=1}^{n-1}(v_j-v_{j-1})v_j\right]$$

Let $a_j = v_{j+1}-v_j$

$$(L_h v_h, v_h)_h = -\frac{1}{h}\left[\underbrace{\sum_{j=1}^{n-1} a_j v_j}_{S_1} - \underbrace{\sum_{j=1}^{n-1} a_{j-1} v_j}_{S_2}\right]$$

$\because v_0 = 0$

for $S_1 = \sum_{j=1}^{n-1} a_j v_j = \sum_{j=0}^{n-1} a_j v_j$

$\because v_n = 0$

for $S_2 = \sum_{j=1}^{n-1} a_{j-1} v_j = \sum_{k=0}^{n-2} a_k v_{k+1} = \sum_{k=0}^{n-1} a_k v_{k+1}$

$$\therefore S_1 - S_2 = \sum_{j=0}^{n-1} a_j v_j - \sum_{j=0}^{n-1} a_j v_{j+1} = \sum_{j=0}^{n-1} a_j (v_j - v_{j+1}) = -\sum_{j=0}^{n-1}(a_j)^2$$

$$\therefore (L_h v_h, v_h)_h = -\frac{1}{h}\sum_{j=0}^{n-1} -(a_j)^2 = \frac{1}{h}\sum_{j=0}^{n-1}(v_{j+1}-v_j)^2 \#$$