1. $u'' = f, \quad u(0) = 0, \quad u(1) = 0$

$$f(x) = \begin{cases} 1 & 0.4 \le x \le 0.6 \\ 0 & \text{otherwise} \end{cases}$$

for $0 \le x < 0.4$

$u_1'' = 0 \Rightarrow u_1' = a \Rightarrow u_1 = ax + c \Rightarrow u_1(0) = 0 \Rightarrow c = 0 \Rightarrow u_1 = ax$

for $0.4 \le x \le 0.6$

$u_2'' = 1 \Rightarrow u_2' = x + c \Rightarrow u_2 = \frac{1}{2}x^2 + cx + d$

for $0.6 < x \le 1$

$u_3'' = 0 \Rightarrow u_3' = e \Rightarrow u_3 = ex + f \Rightarrow u_3(1) = 0 \Rightarrow e + f = 0$

$\because u_2(0.4) = u_1(0.4) = \frac{1}{2}\left(\frac{2}{5}\right)^2 + \frac{2}{5}c + d = \frac{2}{5}a$

$u_3(0.6) = u_2(0.6) = \frac{1}{2}\left(\frac{3}{5}\right)^2 + \frac{3}{5}c + d = \frac{3}{5}e + f$

$u_2'(0.4) = u_1'(0.4) \Rightarrow \frac{2}{5} + c = a$

$u_3'(0.6) = u_2'(0.6) \Rightarrow \frac{3}{5} + c = e$

$$\begin{cases} e + f = 0 & (1) \\ \frac{1}{2}\left(\frac{2}{5}\right)^2 + \frac{2}{5}c + d = \frac{2}{5}a & (2) \\ \frac{1}{2}\left(\frac{3}{5}\right)^2 + \frac{3}{5}c + d = \frac{3}{5}e + f & (3) \\ \frac{2}{5} + c = a & (4) \\ \frac{3}{5} + c = e & (5) \end{cases}$$

$\frac{2}{25} + \frac{2}{5}c + d = \frac{2}{5}a$

$\frac{9}{50} + \frac{3}{5}c + d = \frac{3}{5}e + f$

$\Rightarrow \frac{5}{50} + \frac{1}{5}c = \frac{3}{5}e - \frac{2}{5}a + f$

$= \frac{3}{5}\left(\frac{3}{5} + c\right) - \frac{2}{5}\left(\frac{2}{5} + c\right) + f$

$= \frac{9}{25} + \frac{3}{5}c - \frac{4}{25} - \frac{2}{5}c + f$

$\frac{5}{50} + \frac{1}{5}c = \frac{5}{25} + \frac{1}{5}c + f \Rightarrow \frac{5}{50} - \frac{10}{50} = f \Rightarrow f = -\frac{5}{50} = -\frac{1}{10}$

$\therefore e = \frac{1}{10} \quad d = \frac{2}{25}$

$a = -\frac{1}{10}$

$c = -\frac{1}{2}$

$\Rightarrow a = -\frac{1}{10}, \quad c = -\frac{1}{2}, \quad d = \frac{2}{25}, \quad e = \frac{1}{10}, \quad f = -\frac{1}{10}$

$$\therefore u(x) = \begin{cases} -\frac{1}{10}x & 0 \le x < 0.4 \\ \frac{1}{2}x^2 - \frac{1}{2}x + \frac{2}{25} & 0.4 \le x \le 0.6 \\ \frac{1}{10}x - \frac{1}{10} & 0.6 < x \le 1 \end{cases}$$

2. $u''-2u'+u=1$, $u(0)=0$. $u'(1)=1$

(a) Let u_1 and u_2 be two sol of this BVP

Let $w=u_1-u_2 \Rightarrow w(0)=u_1(0)-u_2(0)=0$

and $w'=u_1'-u_2' \Rightarrow w'(1)=u_1'(1)-u_2'(1)=0$

$\Rightarrow w''-2w'+w = u_1''-u_2''-2u_1'+2u_2'+u_1-u_2$

$= (u_1''-2u_1'+u_1) - (u_2''-2u_2'+u_2) = 1-1=0$

$\therefore$ w is a sol of a homogenis question of $w''-2w'+w=0$, $w(0)=0$. $w'(1)=0$

$\therefore$ the charastic poly is $r^2-2r+1=0 \Rightarrow (r-1)^2=0 \Rightarrow r=1$

$\therefore W(x)=(c_1+c_2x)e^x \Rightarrow W(0)=c_1e^0=c_1=0$

$\Rightarrow W(x)=c_2x\cdot e^x \Rightarrow W'(1)=c_2e^x+c_2xe^x\big|_1 = c_2+c_2=0 \Rightarrow c_2=0$

$w(x)\equiv 0 \Rightarrow u_1-u_2=0 \Rightarrow u_1=u_2$ #

(b) Let $h=\frac{1}{n}$, $x_j=jh$

$u''(x_j)-2u'(x_j)+u(x_j)=1$

$\because u''(x_j)=\dfrac{u(x_{j+1})-2u(x_j)+u(x_{j-1})}{h^2}$ and $u'(x_j)=\dfrac{u(x_{j+1})-u(x_{j-1})}{2h}$

$\Rightarrow \dfrac{u(x_{j+1})-2u(x_j)+u(x_{j-1})}{h^2}-\dfrac{u(x_{j+1})-u(x_{j-1})}{h}+u(x_j)=1$

$\Rightarrow u(x_{j+1})-2u(x_j)+u(x_{j-1})-hu(x_{j+1})+hu(x_{j-1})+h^2u(x_j)=h^2$

$\Rightarrow (1-h)u(x_{j+1})+(-2+h^2)u(x_j)+(1+h)u(x_{j-1})=h^2$

$\therefore$ for $u'(1)=1$, $u'(1)\approx\dfrac{3u_n-4u_{n-1}+u_{n-2}}{2h}$

for $j=1 \Rightarrow (1+h)u_0+(-2+h^2)u_1+(1-h)u_2=h^2$

$\Rightarrow (-2+h^2)u_1+(1-h)u_2=h^2$

$\therefore$ for $(u_1, \cdots, u_n)^T$

$$A: \begin{pmatrix} -2+h^2 & 1-h & 0 & 0 & \cdots & 0 \\ 1+h & -2+h^2 & 1-h & 0 & \cdots & 0 \\ 0 & 1+h & -2+h^2 & 1-h & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & 1 & 4 & -3 \end{pmatrix} \quad b = \begin{pmatrix} h^2 \\ h^2 \\ h^2 \\ \vdots \\ 2h \end{pmatrix}$$

3. $u'' = \sin(2\pi x)$, $u'(0) = 0$, $u'(1) = 0$

(a) $u'' = \sin(2\pi x) \Rightarrow \int_0^1 u''(x)\,dx = \int_0^1 \sin(2\pi x)\,dx$

$$\Rightarrow u'(1) - u'(0) = \frac{-\cos(2\pi x)}{2\pi}\Big|_0^1 = \frac{-\cos(2\pi) + \cos(0)}{2\pi} = \frac{-1+1}{2\pi} = 0$$

$$\Rightarrow 0 - 0 = 0$$

$\therefore$ the consistency condition hold

(b) $x \in [0,1]$, $h = \frac{1}{n}$ $\quad x_j = jh \;\; j = 0, 1, \cdots, n$

$$u'(0) = \frac{-3u_0 + 4u_1 - u_2}{2h} = 0 \Rightarrow -3u_0 + 4u_1 - u_2 = 0$$

$$u'(1) = \frac{3u_n - 4u_{n-1} + u_{n-2}}{2h} = 0 \Rightarrow 3u_n - 4u_{n-1} + u_{n-2} = 0$$

$$u''(x_j) = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = \sin(2\pi x_j)$$

$$\therefore A = \begin{bmatrix} -3 & 4 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ 0 & 0 & 1 & -2 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \; -4 & 3 \end{bmatrix} \begin{bmatrix} u_0 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ \sin(2\pi x_1) \\ \sin(2\pi x_2) \\ \vdots \\ \sin(2\pi x_{n-1}) \\ 0 \end{bmatrix}$$

4. $u'' = e^{\sin(x)}$, $u'(0)=0$, $u'(1)=\alpha$

(a) $f(x) = e^{\sin(x)}$

$\Rightarrow \int_0^1 e^{\sin(x)} dx = \int_0^1 u'' dx = u'(1) - u'(0) = \alpha$

$\Rightarrow \alpha = \int_0^1 e^{\sin(x)} dx$

$\therefore$ if $\alpha = \int_0^1 e^{\sin(x)} dx$ then it must have solution

(b) $u'' = e^{\sin(x)}$

$\Rightarrow u'(x) = \int e^{\sin(x)} dx = \int_0^x e^{\sin(s)} ds + C_1$

$\Rightarrow u'(0) = 0 \Rightarrow \int_0^0 e^{\sin(s)} ds + C_1 = 0$

$\Rightarrow C_1 = 0$

$u'(1) = \alpha = \int_0^1 e^{\sin(x)} dx$

$\Rightarrow u(x) = \int_0^x \int_0^t e^{\sin(s)} ds\, dt + C_2$

$= \int_0^x (x-s) e^{\sin(s)} ds + C_2$

C_2 is Constant.