1. Prove that Heun's method has order 2 with respect to h.
[*Hint*: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

and

$$E_2 = \frac{h}{2}\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method.]

<P.f> the truncation error : $y_{n+1} = y_n + hf(t_n.y_n) + h\tau_{n+1}$

$$\Rightarrow h\tau_{n+1} = y_{n+1} - y_n - hf(t_n.y_n)$$

By Heun's method $\quad u_{n+1} = u_n + \frac{h}{2}[f(t_n.u_n) + f(t_{n+1}, u_n + hf(t_n, u_n))]$, $y_{n+1} = y(t_{n+1})$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}[f(t_n.y_n) + f(t_{n+1}, y_n + hf(t_n.y_n))]$$

$$\Rightarrow h\tau_{n+1} = y_{n+1} - y_n - \frac{h}{2}[f(t_n.y_n) + f(t_{n+1}, y_n + hf(t_n,y_n))] \quad -(1)$$

(i) $y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} f(s.y_n)\,ds$

(ii) (1) RHS : $+\frac{h}{2}[f(t_n.y_n) + f(t_{n+1}, y_{n+1})] - \frac{h}{2}[f(t_n.y_n) + f(t_{n+1}, y_{n+1})]$

$$\Rightarrow h\tau_{n+1} = \underbrace{\int_{t_n}^{t_{n+1}} f(s.y_n)\,ds - \frac{h}{2}[f(t_n.y_n) + f(t_{n+1}.y_{n+1})]}_{E_1}$$
$$\underbrace{+\frac{h}{2}[f(t_n.y_n) + f(t_{n+1}.y_{n+1})] - \frac{h}{2}[f(t_n.y_n) + f(t_{n+1}.y_n + hf(t_n.y_n))]}_{E_2}$$

E_1 : $\int_{t_n}^{t_{n+1}} f(s.y_n)\,ds - \frac{h}{2}[f(t_n.y_n) + f(t_{n+1}.y_{n+1})]$

here E_1 is error of numerical interger of trapezoidal rule

Setting $a = t_n$, $b = t_{n+1}$, $h = b-a$ and $g(t) = f(t.y(t))$

then $E_1 = \int_a^b g(t)\,dt - \frac{h}{2}(g(a)+g(b)) = -\frac{h^3}{12}g''(\xi)$, $\xi \in [a.b]$

$\therefore$ E_1 is $O(h^3)$

E_2 : $\frac{h}{2}[f(t_{n+1}.y_{n+1}) - f(t_{n+1}.y_n + hf(t_n.y_n))]$

Suppose f is Lipshiz continuous

$$\Rightarrow |f(t_{n+1}.y_{n+1}) - f(t_{n+1}, y_n + hf(t_n.y_n))| \le L|y_{n+1} - (y_n + hf(t_n.y_n))|$$

$$\therefore |E_2| \le \frac{h}{2}L\,|y_{n+1} - y_n + hf(t_n.y_n)|$$

how define $h = t_{n+1} - t_n$ and use taylor expansion to $y(t)$ in t_n

$\Rightarrow y(t) = y(t_n) + y'(t_n)\cdot(t-t_n) + \frac{1}{2}y''(\xi)(t-t_n)^2$. $\xi \in (t_n, t_n+h)$

take $t = t_{n+1}$ $\Rightarrow y(t_{n+1}) = y(t_n) + y'(t_n)\cdot h + \frac{1}{2}y''(\xi)h^2$

$\therefore y(t_{n+1}) = y(t_n) + h\, f(t_n, y_n) + \frac{h^2}{2}y''(\xi)$

$\Rightarrow y(t_{n+1}) - y(t_n) - h\cdot f(t_n, y_n) = \frac{h^2}{2}y''(\xi)$

$\Rightarrow |y_{n+1} - y_n - hf(t_n, y_n)| = \frac{h^2}{2}|y''(\xi)| = O(h^2)$

$\therefore |E_2| \le \frac{h}{2}L\cdot\frac{h^2}{2}|y''(\xi)| = \frac{h^3}{4}\cdot L|y''(\xi)| = O(h^3)$

$\therefore h\tau_{n+1} = E_1 + E_2 = O(h^3) + O(h^3) = O(h^3)$

$\Rightarrow h\tau_{n+1} = O(h^3) \Rightarrow \tau_{n+1} = O(h^2)$

2. Prove that the Crank-Nicoloson method has order 2 with respect to h.
[*Solution*: using (9.12) we get, for a suitable ξ_n in (t_n, t_{n+1})

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

or, equivalently,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^2}{12} f''(\xi_n, y(\xi_n)). \qquad (11.90)$$

Therefore, relation (11.9) coincides with (11.90) up to an infinitesimal of order 2 with respect to h, provided that $f \in C^2(I)$.]

<P.f> By $y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(\tau_n, y_n)\, d\tau$

$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] + h\tau_{n+1}$

$\Rightarrow h\tau_{n+1} = y_{n+1} - y_n - \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$

$= \int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\, d\tau - \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] \quad (1)$

$\therefore$ By the result of ch9 ((9.12) the quadrature error of trapezoidal rule is $-\frac{h^3}{12} f''(\xi)$)

(1) $\Rightarrow h\tau_{n+1} = \int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\, d\tau - \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] = -\frac{h^3}{12} f''(\xi_n)$, $\xi_n \in [t_n, t_{n+1}]$

$\Rightarrow h\tau_{n+1} = O(h^3) \Rightarrow \tau_{n+1} = O(h^2)$

$\therefore$ CN is order 2