5. Prove that

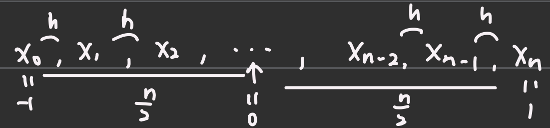
$$(n-1)!h^{n-1}|(x-x_{n-1})(x-x_n)| \leq |\omega_{n+1}(x)| \leq n!h^{n-1}|(x-x_{n-1})(x-x_n)|,$$

where n is even, $-1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$, $x \in (x_{n-1}, x_n)$ and $h = 2/n$.
[*Hint* : let $N = n/2$ and show first that

$$\omega_{n+1}(x) = (x+Nh)(x+(N-1)h)\ldots(x+h)x \\ (x-h)\ldots(x-(N-1)h)(x-Nh). \tag{8.74}$$

Then, take $x = rh$ with $N-1 < r < N$.]

<P.f> Let $h := \frac{2}{n}$ and $x_0 = -1$, $x_1 = -1+h$, $x_2 = -1+2h$, $\cdots$, $x_k = -1+kh$



$\Rightarrow$ Let $N := \frac{n}{2}$

$$\Rightarrow W_{n+1} = \prod_{i=0}^{n}(x-x_i) = (x-(-Nh))(x-(-(N-1)h)\cdots(x+h)x(x-h)\cdots(x-Nh)$$

$$= \prod_{m=-N}^{N}(x-mh)$$

$$= (x-x_{n-1})(x-x_n)\prod_{m=-N}^{N-2}(x-mh)$$

Let $x = rh$, $r \in (N-1, N)$

$$\Rightarrow W_{n+1}(x) = (x-x_{n-1})(x-x_n)\prod_{m=-N}^{N-2}(rh-mh)$$

$$= (x-x_{n-1})(x-x_n)h^{n-1}\prod_{m=-N}^{N-2}(r-m)$$

focus on $\prod_{m=-N}^{N-2}(r-m)$

since $r \in (N-1, N)$ for $m = -N, \cdots, N-2$

if $m = -N \Rightarrow r-m = r+N \therefore |r-m| \in (2N-1, 2N)$

$m = -N+1 \Rightarrow r-m = r+N-1 \therefore |r-m| \in (2N-2, 2N-1)$

$\vdots$

$m = N-2 \Rightarrow r-m = r-(N-2) \therefore |r-m| \in (1, 2)$

$$\Rightarrow \prod_{m=-N}^{N-2}|r-m| = |(r+N)(r+N-1)\cdots(r-N-2)| \in (n-1!, n!)$$

$$\Rightarrow |W_{n+1}(x)| = |(x-x_{n-1})(x-x_n)|h^{n-1}\prod_{m=-N}^{N-2}|r-m|$$

$$\Rightarrow (n-1)!h^{n-1}|(x-x_{n-1})(x-x_n)| \leq |W_{n+1}(x)| \leq n!h^{n-1}|(x-x_{n-1})(x-x_n)|$$

6. Under the assumptions of Exercise 5, show that $|\omega_{n+1}|$ is maximum if $x \in (x_{n-1}, x_n)$ (notice that $|\omega_{n+1}|$ is an even function).
[*Hint* : use (8.74) to prove that $|\omega_{n+1}(x+h)/\omega_{n+1}(x)| > 1$ for any $x \in (0, x_{n-1})$ with x not coinciding with any interpolation node.]

<P.f> define $R(x) = \dfrac{W_{n+1}(x+h)}{W_{n+1}(x)}$

first we prove $|R(x)| > 1 \quad \forall x \in (0, x_{n-1})$

$$W_{n+1}(x) = \prod_{m=-N}^{N} (x - mh)$$

$$W_{n+1}(x+h) = \prod_{m=-N}^{N} (x+h-mh)$$

$(-N-1, -\cancel{N}, \cdots, \cancel{N-1})$

$$\Rightarrow \frac{W_{n+1}(x+h)}{W_{n+1}(x)} = \frac{\prod_{m=-N}^{N}(x+h-mh)}{\prod_{m=-N}^{N}(x-mh)} = \frac{\prod_{m=-N}^{N}(x-(m-1)h)}{\prod_{m=-N}^{N}(x-mh)}$$

$(-\cancel{N}, \cdots, N)$

$$= \frac{(x-(-N-1)h)}{(x-Nh)} = \frac{x+(N+1)h}{x-Nh}$$

(note: $x_k = (-1 + kh)$ if let $N = \frac{n}{2}$ then $x_n = Nh \Rightarrow x_{n-1} = (N-1)h$)

$\because \forall x \in (0, x_{n-1}) = (0, (N-1)h)$

$\Rightarrow \quad x - Nh < (N-1)h - Nh < -h < 0$

$x + (N+1)h > (N+1)h > 0$

$\because x > 0$

$$\therefore |R(x)| = \left|\frac{W_{n+1}(x+h)}{W_{n+1}(x)}\right| = \left|\frac{x+(N+1)h}{x-Nh}\right| = \frac{x+(N+1)h}{Nh-x} > 1$$

By $\left|\dfrac{W_{n+1}(x+h)}{W_{n+1}(x)}\right| > 1 \Rightarrow |W_{n+1}(x+h)| > |W_{n+1}(x)|$

$\therefore |W_{n+1}|$ is strickly increasing $\forall\ 0 < x < (N-1)h$

Notice that $|W_{n+1}|$ is a even function $\therefore |W_{n+1}(-x)| = |W_{n+1}(x)|$

$\Rightarrow |W_{n+1}|$ is also strickly decreasing $\forall\ (N-1)h < x < 0$

$\therefore$ the maximum of $|W_{n+1}(x)|$ is in $x \in ((N-1)h, Nh) = (x_{n-1}, x_n)$

8. Determine an interpolating polynomial $Hf \in \mathbb{P}_n$ such that

$$(Hf)^{(k)}(x_0) = f^{(k)}(x_0), \qquad k = 0, \ldots, n,$$

and check that

$$Hf(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j,$$

that is, the Hermite interpolating polynomial on one node coincides with the *Taylor polynomial.*

<Pf> Let $T_n[f](x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$ be the j-th taylor polynomial at x_0

$$= \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$\Rightarrow \frac{d^k}{dx^k} T_n[f](x)\Big|_{x=x_0} = \sum_{j=k}^{n} \frac{f^{(j)}(x_0)}{j!} j(j-1)\cdots(j-k+1)(x-x_0)^{j-k}\Big|_{x=x_0} = f^{(k)}(x_0)$$

$\therefore$ $T_n[f](x)$ can be $(Hf)(x)$

claim : $T_n[f](x) = (Hf)(x)$

$\because$ $T_n[f]^{(k)}(x_0) = (Hf)^{(k)}(x_0) = f^{(k)}(x_0)$ $\forall k \leq n$ $\therefore$ Let $R(x) = T_n[f](x) - (Hf)(x)$

$\Rightarrow R^{(k)}(x_0) = T_n^{(k)}[f](x_0) - (Hf)^{(k)}(x_0) = 0$ $\forall k \leq n$

$\therefore$ if we take the n-th taylor polynomial of $R(x)$ at x_0

$\Rightarrow R(x) = \sum_{i=0}^{n+1} \frac{R^{(i)}(x_0)}{i!}(x-x_0)^i$ we will found out the coefficient equal 0

for the tern degree $\leq n$

So $R(x) = S(x)(x-x_0)^{n+1}$ where $S(x)$ is some polynomial

but notice that $\deg(R(x)) \leq n$ $\Rightarrow$ $R(x) \equiv 0$ hence $T_n(f)(x) = (Hf)(x)$ $\#$