9. Given the following set of data

$$\{f_0 = f(-1) = 1,\ f_1 = f'(-1) = 1,\ f_2 = f'(1) = 2,\ f_3 = f(2) = 1\},$$

prove that the Hermite-Birkoff interpolating polynomial H_3 does not exist for them.
[*Solution* : letting $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, one must check that the matrix of the linear system $H_3(x_i) = f_i$ for $i = 0, \ldots, 3$ is singular.]

Let $H_3(X) = a_3 X^3 + a_2 X^2 + a_1 X + a_0$

$H_3(X_i) = f_i \quad i = 0, \cdots 3$

$H_3(X_0) = f_0 = f(-1) = -a_3 + a_2 - a_1 + a_0 = 1$

$H_3(X_1) = f_1 = f'(-1) = 1 \; H_3'(-1) = 3a_3X^2 + 2a_2X + a_1$

$= 3a_3 - 2a_2 + a_1 = 1$

$H_3(X_2) = f_2 = f'(1) = 3a_3 + 2a_2 + a_1 = 2$

$H_3(X_3) = f_3 = f(2) = 8a_3 + 4a_2 + 2a_1 + a_0 = 1$

$$\Rightarrow \begin{bmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$\Rightarrow$ Setting $\begin{bmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix} = A \Rightarrow$ then calculate $\det(A)$

$$\Rightarrow \det(A) = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 9 & 3 & 3 & 0 \end{bmatrix} = \begin{vmatrix} 3 & -2 & 1 \\ 3 & 2 & 1 \\ 9 & 3 & 3 \end{vmatrix} = \cancel{18} - \cancel{18} + \cancel{4} - 18 - \cancel{4} + \cancel{18}$$

$$= 0$$

$\Rightarrow A$ is singular

$\therefore$ $a_0\ a_1\ a_2\ a_3$ has no sol

12. Let $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$; then, consider the following rational approximation

$$r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}, \tag{8.75}$$

called the *Padé approximation.* Determine the coefficients of r in such a way that

$$f(x) - r(x) = \gamma_8 x^8 + \gamma_{10}x^{10} + \ldots.$$

[*Solution:* $a_0 = 1$, $a_2 = -7/15$, $a_4 = 1/40$, $b_2 = 1/30$.]

$$r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}$$

according to $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \ldots$

$$\therefore \frac{1}{1+b_2x^2} = \frac{1}{1-(-b_2x^2)} = 1 + (-b_2x^2) + (b_2^2x^4) + (-b_2^3x^6)$$

$$= 1 - b_2x^2 + b_2^2x^4 - b_2^3x^6 + b_2^4x^8 + \ldots$$

$$\therefore r(x) = (a_0 + a_2x^2 + a_4x^4)(1 - b_2x^2 + b_2^2x^4 - b_2^3x^6 + b_2^4x^8 + \ldots)$$

$$= a_0 + (-a_0b_2 + a_2)x^2 + (a_0b_2^2 - a_2b_2 + a_4)x^4 + (-a_0b_2^3 + a_2b_2^2 - a_4b_2)x^6 + \ldots$$

if $f(x) - r(x) = \gamma_8 x^8 + \gamma_{10}x^{10} + \ldots$

then
$$\begin{cases} a_0 = 1 \\ (-a_0b_2 + a_2) = -\frac{1}{2} \\ (a_0b_2^2 - a_2b_2 + a_4) = \frac{1}{24} \\ (-a_0b_2^3 + a_2b_2^2 - a_4b_2) = -\frac{1}{720} \end{cases}$$

$\Rightarrow a_0 = 1 \quad \Rightarrow \begin{cases} (-b_2 + a_2) = -\frac{1}{2} \Rightarrow a_2 = b_2 - \frac{1}{2} \\ (b_2^2 - a_2b_2 + a_4) = \frac{1}{24} = b_2^2 - b_2(b_2 - \frac{1}{2}) + a_4 = -\frac{1}{2}b_2 + a_4 = \frac{1}{24} \Rightarrow a_4 = \frac{1}{24} - \frac{1}{2}b_2 \\ (-b_2^3 + a_2b_2^2 - a_4b_2) = -\frac{1}{720} \end{cases}$

$$\Rightarrow -b_2^3 + (b_2 - \frac{1}{2})b_2^2 - (\frac{1}{24} - \frac{1}{2}b_2)b_2 = -\frac{1}{720}$$

$$= -\cancel{b_2^3} + \cancel{b_2^3} - \frac{1}{2}\cancel{b_2^2} - \frac{1}{24}b_2 + \frac{1}{2}\cancel{b_2^2} = -\frac{1}{720}$$

$$\Rightarrow -\frac{1}{24}b_2 = -\frac{1}{720} \quad \Rightarrow b_2 = \frac{1}{30}$$

$$\Rightarrow a_2 = \frac{1}{30} - \frac{1}{2} = -\frac{7}{15}, \quad a_4 = \frac{1}{24} - \frac{1}{60} = \frac{1}{40} \quad \#$$