RRY025- SOLUTIONS TO PROBLEMS

PROBLEM SET F - IMAGE RESTORATION

1) a) The distorted image $f(x,y) = g(x,y) * h(x,y) + n(x,y)$

Where * indicates convolution. To restore by the Inverse filter, take FT, multiply by the inverse filter i.e. 1/FT(h(x,y), do the Inverse FT back, the estimated true image $g'(x,y) = g(x,y) + IFT[FT(n(x,y))/FT(h(x,y))]$

so the noise in the restored image $n'(x,y) = IFT[FT(n(x,y))/FT(h(x,y))]$

b) Take FT and multiply by $W(u,v)$ where

$$W(u,v) = \frac{H^*(u,v)S_{gg}(u,v)}{|H(u,v)|^2 S_{gg}(u,v) + S_{nn}(u,v)}$$

Where $H(u,v)$ is the FT of $h(x,y)$, and where $S_{gg}(u,v)$ and $S_{nn}(u,v)$ are Fourier transforms of the image covarience and noise covariance over the ensemble of convolution problems - or equivalently the mean square power of the FTs of image and noise.

Advantages of Wiener filter - stable in presence of nulls in $H(u,v)$ - reduces the effect of additive noise - produces best reconstruction in terms of mean least squares error compared to true image.

Disadvantages of Wiener filter - To get optimum solution must know statistics of image set and of noise - unless the signal to noise is very large will leave some blur in image.

c) In cartesian coordinates the PSF is space variable. Solve by going to polar coordinates where problem becomes one of convolution with a space-invariable psf.

First step, convert the problem to polar coordinates, if the data is recorded on cartesian coordinates first interpolate to r, θ coordinates. Now at each radius, $r = r_o$ problem is a circular convolution.

$$f'(r_o, \theta) = g'(r_o, \theta) ** h'(\theta)$$

where ** indicates circular convolution, and $h'(\theta)$ is a boxcar function, which is zero if $|\theta| > \theta_r/2$ and one if $|\theta| < \theta_r/2$ where θ_r is the angle rotated through during the exposure. If at each radius we carry out a DFT we can convert the above convolution problem to a multiplication problem and recover an estimate of g' using a Wiener filter (note an inverse Filter is not suitable because the DFT of h' has zeros). After doing this at each r_o back interpolate onto a cartesian grid to give the final restored image.

2) i) Linear but space-variant. Due to perspective objects appear to move away from a central point. If the camera has a limited exposure then points will be convolved with a streak whose length and direction depends on where in the image it is.

ii) Linear but space-variant. The jerky up and down motions of the car are OK, they cause space invariant distortions, but there will be small rotations also which produce distortions which depend on how far we are from the image centre.

iii) Linear and space-invariant. Acceleration means that the PSF is not just a box car in the direction of motion. But the PSF is the same for all parts of the image, so its space invariant.

iv) Non-linear, consider a pixel with value 1.5_o and let F be the operator, then clearly $F(1.5I_o) \neq F(0.75I_o) + F(0.75I_o)$. Distortion is space-invariant.

v) Linear, but space-variant. Input delta function in I gives output delta function in I' but multiplied by a number which depends on position.

vi) Linear, but space space-variant. An input delta function at the centre of rotation give an output delta function at the same place, but a delta function elsewhere gives an output delta function shifted.

3) Calculate the continuous F's of the two functions

$$H_1(u,v) = 1 + 2cos(2\pi u) + 2cos(2\pi v)$$

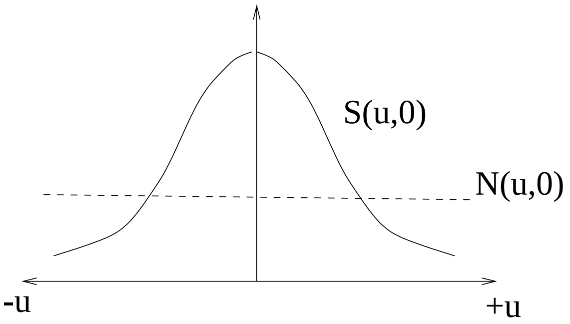
$$H_2(u,v) = 5 + 2cos(2\pi u) + 2cos(2\pi v)$$

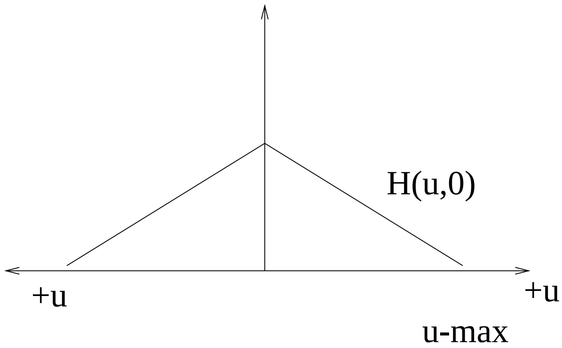
It is clear that $H_1(u,v)$ has zeros but that $H_2(u,v)$ does not. The inverse filter will go to infinity for the first PSF and so is not suitable.

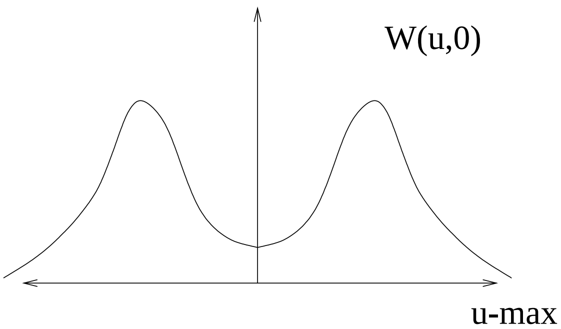
4) The inverse transform uses a filter which is just $R = 1/H$. the Wiener filter is defined to be

$$W(u,v) = \frac{H^*(u,v)S(u,v)}{|H(u,v)|^2 S(u,v) + N(u,v)}$$

We are given in the diagrams the values of H,S, N along the u-axis and so can calculate $W(u,0)$. When $S >> N$ Wiener filter approximates to $1/H$. In other limit $S << N$ approximates to HS/N which goes to zero as H goes to zero.







5) a) The PSF in the x direction is a top hat function which is non-zero from $x = -s\Delta t/2$ to $x = s\Delta t/2$, where it has value $1/(s\Delta t)$. In the y-direction the PSF is a delta function. The OFT $H(u,v)$ is the FT of the PSF

$$H(u,v) = \frac{sin(\pi s\Delta tu)}{\pi s\Delta tu}$$

Not suitable to use inverse filter for restoration since $H(u,v)$ contains zeros.

b) Use formula for Wiener filter

$$W(u,v) = \frac{H^* S_{ff}}{|H|^2 S_{ff} + S_{nn}}$$

inputing the above expression for $H(u,v)$ and the values $S_{ff} = 1/u$, $S_{nn} = s\Delta t$, we get

$$W(u,v) = \frac{(\pi u/u_o)sin(\pi u/u_o)}{sin^2(\pi u/u_o) + (u/u_o)^3\pi^2}$$

where $u_o = 1/(s\Delta t)$, therefore when $u = 1/(s\Delta t) = u_o$, then $W(u,v) = 0$.

c) To work out the PSF consider photons falling on camera film or CCD, observing a point source. Number of photons per sec emitted by object constant, so brightness per ccd pixel of size dx depend on length of time dt object stays in pixel. Since for the accelerating car

$$x(t) = (a/2)t^2$$

$$dx = atdt$$

$$dt = dx/(at) = dx/\sqrt{2ax}$$

so PSF is proportional to $x^{-0.5}$ out to some maximum value of x, $x_{max} = 0.5at_{exp}{}^2$ where t_{exp} is the exposure time.

Differences to constant motion PSF - the acceleration PSF is not symmetric therefore its FT is complex. Because of this it generally has no nulls in its amplitude, because when the real part is zero, the imaginary generally is not and vice versa.

d) In this case objects are at different distances and are blurred different amounts, the PSF is space variable and so Wiener of inverse filter methods are not useful.

6) a) If $\gamma = -1$ then without preprocessing the recorded image is simply the input image convolved with $h(m, n)$. i.e $u(m, n) * h(m, n)$. We can preprocess the input image to create $u'(m, n)$ so that after distortion by the flying spot scanner the recorded image equals the true image $u' * h = u$. If we choose the preprocessing such that $u' = u * h'$ then the output image is $h * u * h'$, and to make this output image equal the true image we must choose h' so that $h(m, n) * h'(m, n) = \delta(m, n)$. In the Fourier domain this means $H'(u, v) = 1/H(u, v)$, or the inverse filter; given the form of $h(m, n)$ this means

$$H'(u, v) = \frac{1}{1 + 0.5cos(2\pi u) + 0.5cos(2\pi v)}$$

and $h'(m, n)$ is the inverse FT of this.

b) If we ignore the spot size, then output image $aw^{-\gamma}$, output equals input u, if we choose

$$w = (v/\alpha)^{-1/\gamma}$$

c) In the hardware recording system, image is first convolved and is then subject to pixel non-linearity. To preprocess to avoid distortion must apply operations in reverse order - first apply inverse non-linear distortion, then convolve the result with a $h'(m, n)$. where the $h'(m, n)$ is the inverse FT of $1/H(u, v)$.

We have in this case $H(u, v) = FT(h(m, n)) = FT(sinc^2(m)sinc^2(n))$ i.e. we take the FT of a seperable function of m,n. so $H(u, v) = FT(sinc^2(m))FT(sinc^2(n))$. Using the convolution theorem the 1D FT of sinc squared is the convolution with itself of the FT of the sinc function - since the FT of sinc is top-hat function, the FT of sinc squared is a top-hat convolved with itself or a triangle function.

Since the triangle function goes to zero at beyond some maximum u,v, we cannot recover the information at these spatial frequencies using the inverse filter. Therefore even after applying our preprocessing, the recorded image lacks the high spatial frequency information of the original.

7) X-ray tomography problem

i)

$$A_x(1) = a + c$$

$$A_x(2) = b + d$$

$$A_y(1) = c + d$$

$$A_y(2) = a + b$$

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| A_{x}(1)| = | 1  0  1  0 |   | a |
| A_{x}(2)| = | 0  1  0  1 |   | b |
| A_{y}(1)| = | 0  0  1  1 |   | c |
| A_{y}(2)| = | 1  1  0  0 |   | d |
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If the matrix B has an inverse there is a unique solution for a,b,c,d, otherwise there is not. To have an inverse the matrix must have a determinant which is non-zero. Can show that B has a zero determinant. Equivalently we can show that the four equations are not independent (each equation can be be derived from a combination of the others).

ii) Given the solution presented the data is fitted no matter what the value of γ, e.g.

$$A_x(1) = (2 + \gamma) + (2 - \gamma) = 4$$

$$A_x(2) = (1 - \gamma) + (\gamma) = 1$$

$$A_y(1) = (2 - \gamma) + (-\gamma) = 2$$

$$A_y(2) = (2 - \gamma) + (\gamma) = 2$$

The absorption can only be positive, so all of a,b,c,d all must be positive so $b = (1 - \gamma) > 0$ and $d = \gamma > 0$, hence $0 < \gamma < 1$.

iii) Find value of γ that minimises total energy $E = a^2 + b^2 + c^2 + d^2$. Substituting values for a,b,c,d

$$E = (2 + \gamma)^2 + (1 - \gamma)^2 + (2 - \gamma)^2 + (\gamma)^2$$

$$= 9 - 2\gamma + 4\gamma^2$$

$$dE/d\gamma = -2 + 8\gamma = 0$$

$$\gamma = 0.25$$