RRY025- SOLUTIONS TO PROBLEMS
PROBLEM SET B - FOURIER TRANSFORMS

1)a) $\delta(x-1, y-2) = 0$ unless $x = 1$ and $y = 2$, hence the product $f(x,y)\delta(x-1,y-2)$ is also zero unless both $x = 1$ and $y = 2$. The product is therefore also a delta function at the same position. However the size of the delta function is multiplied by the value of $f(x,y) = (x+y)^3$ at $x = 1$,$y = 2$. Hence the final answer is $27\delta(x-1, y-2)$; this means that the area under the delta function is now 27.

b) To convolve $f(x,y)$ with $\delta(x-1, y-2)$ rotate the second function by 180 degrees to give $\delta(-x'-1, -y'-2)$ make different shifts x,y., multiply by $f(x', y')$ and integrate.

$$f'(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\delta(x - x' - 1, y - y' - 2)dx'dy';$$

the contents of the integral are zero unless $x - x' - 1 = 0$ and $y - y' - 2 = 0$. Therefore $f'(x,y)$ equals the value of $f(x', y')$ at $x' = x - 1$ and $y' = y - 2$,

$$f'(x,y) = (x - 1 + y - 2)^3 = (x + y - 3)^3$$

so the effect of convolving with a delta function is to shift the function in position.

2) Let $g(x,y) = f(ax, by)$, what is the effect of doing two forward FTs in succession?
The definition of the forward FT is

$$G(u,v) = FT(g(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)exp(-2\pi i(ux + vy))dxdy$$

while the inverse transform is defined as

$$g(x,y) = IFT(G(u,v)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u,v)exp(+2\pi i(ux + vy))dudv$$

take the above equation and replace x by -x and y by -y then

$$g(-x,-y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u,v)exp(-2\pi i(ux + vy))dudv$$

but this fits the definition of a forward FT of the forward transform $(G(u,v))$. So two forward transforms one after the other convert $g(x,y)$ to $g(-x,-y)$, i.e. cause a reflection through the origin or equivalently a rotation of 180 degrees.

Taking into account that $g(x,y)$ is a scaled version of $f(x,y)$ the final output image is a scaled and inverted (rotated 180° about origin) version of $f(x,y)$.

3).

a). The continuous FT of the circular pillbox is circularly symmetric, the FT as a function of radius is a Bessel function of the first kind divided by r i.e. $J_1(r)/r$.

b) Can consider this input image as the product of two 'top-hat' functions in x and y respectively $\sqcap(x/a) \sqcap (y/b)$ where $\sqcap(x/a)$ equals 1 unless $|(x/a)| > 1$ in which case the functions equal 0. The input 2D image is a separable function, hence the 2D continuous FT is the product of the two 1D FTs i.e. $sin(au)sin(bv)/abuv$.

c) Since convolution of an input image with a delta function shifts that image (see question 1 above), consider input

$$I(x,y) = \delta(x - x_o, y) * C(x,y) + \delta(x + x_o, y) * C(x,y)$$

where $C(x,y)$ is the circle function in part a). Via the convolution theorem the FT of I(x,y) is the product of the FT of C(x,y) and the FT of $\delta(x - x_o, y) + \delta(x + x_o, y)$ or $2cos(2\pi u x_o)$.

4) a) $f_{even}(x) = 0.5f(x) + 0.5f(-x)$, is clearly true since $f_{even}(x) = f_{even}(-x)$, likewise $f_{odd}(x) = 0.5f(x) - 0.5f(-x)$, is true since $f_{odd}(x) = -f_{odd}(-x)$. Note that any function can be written $f(x) = f_{even}(x) + f_{odd}(x)$

b) The Fourier transform of $f_{even}(x)$ is;

$$\int_{-\infty}^{\infty} f_{even}(x) exp(-2\pi ixu)dx$$

$$= \int_{-\infty}^{\infty} 0.5f(x)exp(-2\pi ixu)dx + \int_{-\infty}^{\infty} 0.5f(-x)exp(-2\pi ixu)dx$$

let $x' = -x$

$$= \int_{-\infty}^{\infty} 0.5f(x)exp(-2\pi ixu)dx + \int_{-\infty}^{\infty} 0.5f(x')exp(2\pi ix'u)dx'$$

$$= \int_{-\infty}^{\infty} f(x)0.5(exp(-2\pi ixu) + exp(2\pi ixu))dx = \int_{-\infty}^{\infty} f(x)cos(2\pi ixu)dx = Re(F(u))$$

and a similar calculation for f_{odd}.

5) i) Having imaginary part zero requires the image to be symmetric in the origin so that $I(x, y) = I(-x, -y)$, this is true for the following images **A, B, C, F, H**

ii) The real part of the FT will be positive if we can write $FT(I(x, y)) = |FT(I'(x, y))|^2$ where $I'(x, y)$ is any function.
the above can be written $FT(I(x, y)) = FT(I'(x, y)) \times [FT(I'(x, y))]^*$.
Using the convolution theorem this implies that $I(x, y)$ is the autocorrelation of some function $I'(x, y)$ with itself. This condition is only true for image **B**.

iii) $F(0, 0) = \int\int f(x, y)dxdy$, so $F(0, 0) = 0$ for **E,H**

iv) $F(u, v)$ has circular symmetry if $f(x, y)$ has circular symmetry, which occurs only for **F**.

v) The real part will be zero for any antisymmetric image such that $f(x, y) = -f(-x, -y)$, which is only true for **E**.

6)a)

$$f(x, y) = 2cos[2\pi(4x + 6y)] = exp(-2\pi i(4x + 6y)) + exp(2\pi i(4x + 6y))$$

From inspection the 2D continuous FT is

$$F(u, v) = \delta(u - 4, v - 6) + \delta(u + 4, v + 6)$$

and is shown on the left of Fig 1.
The sampled image is multiplied by the sampling function $III(x, y)$ so that $f'(x, y) = III(x, y)f(x, y)$ where

$$III(x, y) = \sum_n \sum_m \delta(x - n\Delta x, y - m\Delta y)$$

where $\Delta x = 0.1$ and $\Delta y = 0.2$

From the convolution theoreom $F'(u, v) = FT(f'(x, y)) = FT(f(x, y)) * FT(III(x, y))$ or $F'(u, v) = F(u, v) * FT(III(x, y))$. Where the FT of the sampling function

$$FT(III(x, y)) = III(u, v) = \sum_n \sum_m \delta(u - n\Delta u, v - m\Delta v)$$

where $\Delta u = 1/\Delta x = 10$, $\Delta v = 1/\Delta y = 5$ and is shown in Fig 1, right. The result of convolving the spectrum of the continuous function F(u,v) with the FT of the sampling function is shown in Fig 2.

b) The ideal lowpass filter has cutoffs at half the sampling frequencies. The x and y sampling frequencies are $\Delta u = 1/\Delta x = 10$ and $\Delta v = 1/\Delta y = 5$. Hence the low pass restoration filter

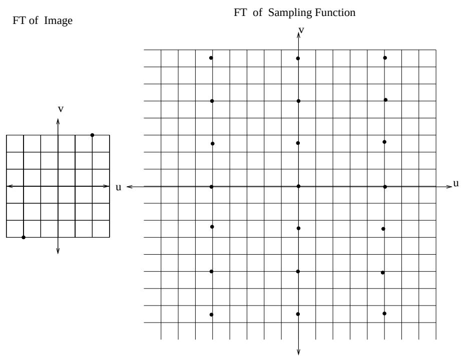

Figure 1: Left - FT of the continuous image. Right- FT of the sampling function.Small solid dots represent Delta functions. Each grid spacing is 2 units in u and v.

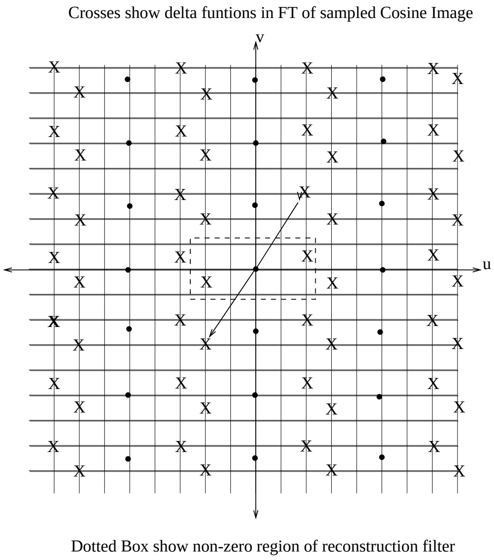

Figure 2: FT of the sampled image indicated by the crosses, each of which is a delta function.

is a rectangle which has value 1 between u=-5 and u=+5 and v=-2.5 and v=+2.5 and zero elsewhere (see dashed box in Fig 2). After multiplying by the lowpass filter only two (aliased) delta functions remain, i.e.

$$\delta(u-4, v-1) + \delta(u+4, v+1)$$

c) Hence after doing the inverse FT the reconstructed image is $2cos(2\pi(4x + y))$

d) Using horizontal and vertical axes the bandwidth of the continuous image are in the u -direction $w_u = 4$ and in the v-direction $w_v = 6$. To prevent aliasing we need Nyquist sampling, i.e. $\Delta x < 1/2w_u = 1/8$ and $\Delta y < 1/2w_v = 1/12$.

For the reconstruction filter minimum bandwidths are w_u and w_v respectively. The maximum bandwidths depend on Δx and Δy and are $w_{max,u} = 1/\Delta x - w_u$ and $w_{max,v} = 1/\Delta y - w_v$ respectively,

7)a) Linear convolutions, rotate second function by 180° about origin, shift, multiply each element (if there is no element assume zero), sum the results [check results using 'conv2' in MATLAB].

$$x(m,n) * A(m,n) = \begin{matrix} -2 & -8 & -2 \\ -1 & 2 & -3 \\ 7 & 19 & 10 \\ 2 & 5 & 3 \end{matrix}$$

$$x(m,n) * B(m,n) = \begin{matrix} 1 & 6 & 12 & 14 & 3 \\ 3 & 13 & 20 & 21 & 9 \\ 2 & 5 & 3 & 0 & 0 \end{matrix}$$

b) Cross-correlation. Like convolution but don't rotate second function [check results using 'filter2' in MATLAB using option 'full'].

$$x(m,n) \text{ correlated with } A(m,n) = \begin{matrix} 1 & 4 & 1 \\ 5 & 17 & 6 \\ 4 & 7 & 7 \\ -4 & -10 & -6 \end{matrix}$$

c) Circular convolution. The quantity $x(m,n) ** B(m,n)$ is a periodic function in which the 2x3 matrix below repeats an infinite number of times

$$\begin{matrix} 17 & 14 & 15 \\ 24 & 22 & 20 \end{matrix}$$

if we wished to linearly convolve x and B via the DFT method we must zero pad both until they are the size of the linearly convolved image i.e. 5 columns by 3 rows.

8)

1-C - This is the hardest FT to match. The FT C is broad in u direction corresponding to closely spaced vertical lines in figure (these are unfortunatley a little hard to see in image 1 unless you magnify the image).

2-F - Diagonal spikes in F perpendicular to sharp ring structure.

3-B - Roughly circular FT but strong vertical spike due to brightness difference between top and bottom edges of image.

4-D - Both image and FT consist of vertical and horizontal stripes

5-A - Dice is circles convolved with 5 delta functions. from convolution theoroem expect FT to be FT of circles times FT of delta functions. Figure A looks like this.

6-E Ordinary image, no circular symmetry of FT, vertical and horizontal spikes due to difference between top and bottom and left and right edges of image.