10.10 Solve $y^3 - 7y + 6 = 0$

1. Show that Cardano's formula yields the solution

$$y = \sqrt[3]{-3 + \frac{10}{9}\sqrt{-3}} + \sqrt[3]{-3 - \frac{10}{9}\sqrt{-3}}$$

Using Cardano's formula, with $p = -7, q = 6$, we have

$y = \sqrt[3]{-\frac{6}{2} + \sqrt{(\frac{-7}{3})^3 + (\frac{6}{2})^2}} + \sqrt[3]{-\frac{6}{2} + \sqrt{(\frac{-7}{3})^3 - (\frac{6}{2})^2}}$

This is equal to $y = \sqrt[3]{-3 + \frac{10}{9}\sqrt{-3}} + \sqrt[3]{-3 - \frac{10}{9}\sqrt{-3}}$

2. Again, the solutions are not complicated; what is complicated is the cube root calculation that Cardano's formula requires. Check that

$$(1 + \frac{2}{3}\sqrt{-3})^3 = -3 + \frac{10}{9}\sqrt{-3}$$

and

$$(1 - \frac{2}{3}\sqrt{-3})^3 = -3 - \frac{10}{9}\sqrt{-3}$$

Expanding, the first term, we get $(1 + \frac{4}{3}\sqrt{-3} + \frac{4}{9} * (-3))(1 + \frac{2}{3}\sqrt{-3})$

This is equal to $-3 + \frac{10}{9}\sqrt{-3}$

Expanding the second term, we get $(1 - \frac{4}{3}\sqrt{-3} + \frac{4}{9} * (-3))(1 - \frac{2}{3}\sqrt{-3})$

This is equal to $-3 - \frac{10}{9}\sqrt{-3}$

So $y = 1 + \frac{2}{3}\sqrt{-3} + 1 - \frac{2}{3}\sqrt{-3}$

So $y = 2$ is a solution

Then by theorem 9.7, $y - 2$ divides $y^3 - 7y + 6$

By long division, we obtain $\frac{y^3-7y+6}{y-2} = y^2 + 2y - 3$

Which factors as $(y - 1)(y + 3)$

So $y^3 - 7y + 6 = (y - 2)(y - 1)(y + 3)$

With roots $y = 2, y = 1, y = -3$