10.1 1. For a real number a, verify that $(x^2 + a) = x^2 + 2a + a^2$

Done.

2. For a real number b, conclude that the polynomial $x^2 + bx + \frac{b}{4}$ is the square of a degree one polynomial

$x^2 + bx + \frac{b}{4} = (x + \frac{b}{2})^2$

3. For real numbers b, c, rewrite $x^2 + bx + c$ by adding and subtracting $\frac{b^2}{4}$ and find that solving the equation $x^2 + bx + c = 0$ is equivalent to solving an equation of the form $(x + \frac{b}{2})^2 = \frac{d}{4}$

$x^2 + bx + c = 0 \leftrightarrow (x + \frac{b}{2})^2 - \frac{b^2}{4} + c = 0$

$(x + \frac{b}{2})^2 = \frac{b^2}{4} - c$

So solving for $x^2 + bx + c = 0$ is the same as solving for $(x + \frac{b}{2})^2 = \frac{d}{4}$ for $d = b^2 - 4c$

4. Deduce that if $d = 0$, then $x^2 + bx + c$ factors as $(x + \frac{b}{2})$, and one solution to $x^2 + bx + c = 0$ is $x = -\frac{b}{2}$

We know that solving $x^2 + bx + c = 0$ is equivalent to solving for $(x + \frac{b}{2})^2 = \frac{d}{4}$. So when $d = 0$, $(x + \frac{b}{2})^2 = 0$, so it factors as $(x + \frac{b}{2})(x + \frac{b}{2})$, then one of the roots is $x = -\frac{b}{2}$

5. Deduce that if d is negative, then there is no real solution in $\mathbb{R}$ to the equation $x^2 + bx + c = 0$, and is irreducible in $\mathbb{R}[x]$

$(x + \frac{b}{2})^2 = \frac{d}{4}$, for d negative

So $(x + \frac{b}{2}) = \pm\sqrt{\frac{-d}{4}}i$.

So there are no real solutions, so it is irreducible in $\mathbb{R}[x]$

6. Deduce that if d is positive, then there are 2 real solutions to $x^2 + bx + c = 0$. Write them out explicitly in terms of b, c

$(x + \frac{b}{2})^2 = \frac{d}{4}$

$(x + \frac{b}{2}) = \pm\sqrt{\frac{d}{4}}$

$x = -\frac{b}{2} \pm \frac{\sqrt{d}}{2}$

So $x = -\frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$

10.4 Consider the quadratic polynomial $x^2 + bx + c$, where b, c real with non negative discriminant, so that the roots r_1, r_2 real

1. Recall that $c = r_1, r_2$, $b = -(r_1 + r_2)$

2. If $c = 0$, the nature of the roots are easy to determine. Explain why

This is because the equation is now $x^2 + bx$, which factors as $x(x + b)$, so we have roots $r_1, r_2 = x, -b$

3. Assume that $c \neq 0$, show that if the roots r_1, r_2 have the same sign, then $c > 0$, and if the roots have the opposite sign, $c < 0$

r_1, r_2 have same signs:

Case 1: both positive

Then $c = r_1 * r_2$ is positive * positive = positive, $c > 0$

Case 2: both negative

Then $c = r_1 * r_2$ is negative * negative = positive, $c > 0$

r_1, r_2 have opposite signs:

Without loss of generality, assume r_1 negative, r_2 positive, then the product $c = r_1 * r_2$ is the product of a negative and a positive, so c is negative, so $c < 0$

4. Conclude that there is an odd number of positive roots when c is negative, and an even number of positive roots when c is positive

This is because when c is positive, r_1, r_2 either both positive or both negative, so there are 0 or 2 positive roots. And if exactly one of the roots is negative and the other positive, then there is an odd number of positive roots for c negative.

5. Assume that c is positive, show that the roots are positive precisely when $b < 0$, and negative when $b > 0$

$c > 0$, so $r_1 * r_2 > 0$, and r_1, r_2 must have the same sign

And $b = -(r_1 + r_2)$

Case 1: $b < 0$

Then the sum $-(r_1 + r_2)$ must be positive

So r_1, r_2 must both be positive

Case 2: $b > 0$

Then the sum $-(r_1 + r_2)$ must be negative

Then r_1, r_2 must both be negative

6. Conclude that you can use b, c to determine the signs of the roots. Describe exactly how you would do so

For nonnegative discriminant, we can use b, c to determine the signs of the roots. If $c = 0$, finding roots is trivial, they are $x, -b$. First, if c nonzero, we can determine whether the roots are the same sign (if $c > 0$) or oppposite sign ($c < 0$). And if they are the same sign, we can use b to determine whether both are positive ($b < 0$), or both negative ($b > 0$).

10.7 We begin with the cubic polynomial $y^3 + py + q$. We can assume that p is nonzero, for if $p = 0$, the equation is $y^3 = -q$, and the solution is easily obtained as the cube root of $-q$. Introduce a new variable satisfying

$$y = z - \frac{p}{3z}$$

1. Substitute $z - \frac{p}{3z}$ for y in the equation, expand, and simplify, to obtain

$$z^3 - \frac{p^3}{27z^3} + q = 0$$

For $y = z - \frac{p}{3z}$, we have $(z - \frac{p}{3z})^3 + p(z - \frac{p}{3z}) + q = 0$

$(z^2 - \frac{2p}{3} + \frac{p^2}{9z^2})(z - \frac{p}{3z}) + pz - \frac{p^2}{3z} + q = 0$

$[z^3 - \frac{2pz}{3} + \frac{p^2}{9z}] - [\frac{pz}{3} - \frac{2p^2}{9z} + \frac{p^3}{27z^3}] + pz - \frac{p^2}{3z} + q = 0$

$z^3 - pz + \frac{p^2}{3z} - \frac{p^3}{27z^3} + pz - \frac{p^2}{3z} + q = 0$

$z^3 - \frac{p^3}{27z^3} + q = 0$

2. Multiply by z^3 to clear the value in the denominator to obtain $z^6 - \frac{p^3}{27} + qz^3 = 0$

3. Observe that this is the quadratic equation in z^3. Use the quadratic formula to obtain $z^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2 + \frac{4p^3}{27}}{4}}$

If we let x to be z^3, then we can apply the quadratic formula to solve for $x^2 + qx - \frac{p^3}{27} = 0$

The roots we obtain are $x = \frac{-q}{2} \pm \frac{\sqrt{q^2 - \frac{4p^3}{27}}}{2}$

Plugging back in $z^3 = x$, we get $z^3 = \frac{-q}{2} \pm \frac{\sqrt{q^2 - \frac{4p^3}{27}}}{2}$

Which is equal to $z^3 = \frac{-q}{2} \pm \sqrt{\frac{q^2 - \frac{4p^3}{27}}{4}}$

4. Introduce R as an abbreviation for $(\frac{p}{3})^3 + (\frac{q}{2})^2$ and rewrite the last equality as $z^3 = -\frac{q}{2} \pm \sqrt{R}$

The value inside the square root term is equal to $\frac{q^2}{4} - \frac{p^3}{27}$, which is equal to $(\frac{p}{3})^3 + (\frac{q}{2})^2$

So we can rewrite the equality as $z^3 = -\frac{q}{2} \pm \sqrt{R}$

5. There are two possible values for z^3, namely, $-\frac{q}{2} + \sqrt{R}, -\frac{q}{2} - \sqrt{R}$

Multiply these two values together and simplify. Show that you get

$$(-\frac{q}{2} + \sqrt{R})(-\frac{q}{2} - \sqrt{R}) = (-\frac{p}{3})^3$$

Multiplying, we get $(\frac{q^2}{4} - R^2)$

Plugging back in $R = (\frac{p}{3})^3 + (\frac{q}{2})^2$, we get

$(\frac{q^2}{4} - (\frac{p}{3})^3 + (\frac{q}{2})^2$

Which simplifies to $(-\frac{p}{3})^3$

6. Take the cube root of both sides above and deduce that the two values of z have a product satisfying

$$\sqrt[3]{-\frac{q}{2} + \sqrt{R}} * \sqrt[3]{-\frac{q}{2} - \sqrt{R}} = -\frac{p}{3}$$

7. Observe that this means that if you choose z to be the cube root of $\frac{q}{2} + \sqrt{R}$, then $-\frac{p}{3z}$ is the cube root of $-\frac{q}{2} - \sqrt{R}$

This is true, since if it is the cuberoot, then $z * \sqrt[3]{-\frac{q}{2} - \sqrt{R}} = -\frac{p}{3}$

Which is equal to $\sqrt[3]{-\frac{q}{2} - \sqrt{R}} = -\frac{p}{3z}$

8. Recall that z was introduced to satisfy $z - \frac{p}{3z}$. You have shown that the two terms on the right of this equation, $z, -\frac{p}{3z}$ are the cube roots of $-\frac{q}{2} + \sqrt{R}$ and $-\frac{q}{2} - \sqrt{R}$ respectively.

9. Conclude that y is the sum of these two cube roots

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt[3]{-\frac{q}{2} - \sqrt{R}}$$

Since $y = z - \frac{p}{3z}$, we can plug in $z = \sqrt[3]{-\frac{q}{2} + \sqrt{R}}, \frac{p}{3z} = \sqrt[3]{-\frac{q}{2} - \sqrt{R}}$

We obtain $y = \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt[3]{-\frac{q}{2} - \sqrt{R}}$

10.10 Solve $y^3 - 7y + 6 = 0$

1. Show that Cardano's formula yields the solution

$$y = \sqrt[3]{-3 + \frac{10}{9}\sqrt{-3}} + \sqrt[3]{-3 - \frac{10}{9}\sqrt{-3}}$$

Using Cardano's formula, with $p = -7, q = 6$, we have

$y = \sqrt[3]{-\frac{6}{2} + \sqrt{(\frac{-7}{3})^3 + (\frac{6}{2})^2}} + \sqrt[3]{-\frac{6}{2} + \sqrt{(\frac{-7}{3})^3 - (\frac{6}{2})^2}}$

This is equal to $y = \sqrt[3]{-3 + \frac{10}{9}\sqrt{-3}} + \sqrt[3]{-3 - \frac{10}{9}\sqrt{-3}}$

2. Again, the solutions are not complicated; what is complicated is the cube root calculation that Cardano's formula requires. Check that

$$(1 + \frac{2}{3}\sqrt{-3})^3 = -3 + \frac{10}{9}\sqrt{-3}$$

and

$$(1 - \frac{2}{3}\sqrt{-3})^3 = -3 - \frac{10}{9}\sqrt{-3}$$

Expanding, the first term, we get $(1 + \frac{4}{3}\sqrt{-3} + \frac{4}{9} * (-3))(1 + \frac{2}{3}\sqrt{-3})$

This is equal to $-3 + \frac{10}{9}\sqrt{-3}$

Expanding the second term, we get $(1 - \frac{4}{3}\sqrt{-3} + \frac{4}{9} * (-3))(1 - \frac{2}{3}\sqrt{-3})$

This is equal to $-3 - \frac{10}{9}\sqrt{-3}$

So $y = 1 + \frac{2}{3}\sqrt{-3} + 1 - \frac{2}{3}\sqrt{-3}$

So $y = 2$ is a solution

Then by theorem 9.7, $y - 2$ divides $y^3 - 7y + 6$

By long division, we obtain $\frac{y^3-7y+6}{y-2} = y^2 + 2y - 3$

Which factors as $(y-1)(y+3)$

So $y^3 - 7y + 6 = (y-2)(y-1)(y+3)$

With roots $y = 2, y = 1, y = -3$

10.13 Use Cardano's formula, as clarified in Exercise 10.12, to obtain all three solutions to the cubic equation $y^3 - 7y + 6 = 0$

1. Write down the solution given by the formula as a sum of cuberoots. Observe that it involves the cube roots of $-3 + \frac{10}{9}\sqrt{-3}$ and $-3 - \frac{10}{9}\sqrt{-3}$

The solution is $y = \sqrt[3]{-3 + \sqrt{109}\sqrt{-3}} + \sqrt[3]{-3 - \frac{10}{9}\sqrt{-3}}$

2. Using the numbers ω, ω^2, and the earlier determination of one cube root of $-3 + \frac{10}{9}\sqrt{-3}$, write expressiosn for the three complex numbers that are cube roots of $-3 + \frac{10}{9}\sqrt{-3}$. Also write down the three complex numbers that are cube roots of $-3 - \frac{10}{9}\sqrt{-3}$

We know from problem 10.10 that a cuberoot of $-3 + \frac{10}{9}\sqrt{-3}$ is $1 + \frac{2}{3}\sqrt{-3}$

So using $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, the cuberoots of $-3 + \frac{10}{9}\sqrt{-3}$ are

$1 + \frac{2}{3}\sqrt{-3}$

$[1 + \frac{2}{3}\sqrt{-3}] * \omega = [1 + \frac{2}{3}\sqrt{-3}] * (-\frac{1}{2} + \frac{\sqrt{-3}}{2}) = \frac{-3}{2} + \frac{\sqrt{-3}}{6}$

$[1 + \frac{2}{3}\sqrt{-3}] * \omega^2 = \frac{1}{2} - \frac{5}{6}\sqrt{-3}$

Cuberoots of $-3 - \frac{10}{9}\sqrt{-3}$ are

$1 - \frac{2}{3}\sqrt{-3}$

$\frac{1}{2} + \frac{5}{6}\sqrt{-3}$

$\frac{-3}{2} - \frac{1}{6}\sqrt{-3}$

3. Pair the cube roots of $-3 + \frac{10}{9}\sqrt{-3}$ and $-3 - \frac{10}{9}\sqrt{-3}$ as specified in exercise 10.12 to get three pairs such that the product of the complex numbers in each pair equals $\frac{7}{3}$

For $A = 1 + \frac{2}{3}\sqrt{-3}, B = 1 - \frac{2}{3}\sqrt{-3}, \omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, the pairs are

$AB, \omega A * \omega^2 B, \omega^2 A * \omega B$

These are:

$[1 + \frac{2}{3}\sqrt{-3}] * [1 - \frac{2}{3}\sqrt{-3}]$

$[-\frac{3}{2} + \frac{\sqrt{-3}}{6}] * [-\frac{-3}{2} - \frac{1}{6}\sqrt{-3}]$

$[\frac{1}{2} - \frac{5}{6}\sqrt{-3}] * [\frac{1}{2} + \frac{5}{6}\sqrt{-3}]$

4. Add together the complex numbers in each pair to obtain all three solutions of $y^3 - 7y + 6 = 0$.

The roots are $r_1 = A + B, r_2 = \omega A + \omega^2 B, r_3 = \omega^2 A + \omega B$

$r_1 = [1 + \frac{2}{3}\sqrt{-3}] + [1 - \frac{2}{3}\sqrt{-3}] = 2$

$r_2 = [-\frac{3}{2} + \frac{\sqrt{-3}}{6}] + [-\frac{-3}{2} - \frac{1}{6}\sqrt{-3}] = -3$

$r_3 = [\frac{1}{2} - \frac{5}{6}\sqrt{-3}] + [\frac{1}{2} + \frac{5}{6}\sqrt{-3}] = 1$

10.24 We can immediately dispose of one special case, that in which $r = 0$. Consider the quartic polynomial $z^4 + qz^2 + s$

1. Factor z^4+qz^2+s (which is a quadratic polynomial in z^2) in the form $(z^2-r_1)(z^2-r_2)$ for two real or possibily complex numbers r_1, r_2.

Substitute $x = z^2, x^2 + xq + s = 0$

$x = -\frac{q}{2} \pm \sqrt{q^2 - 4s}$

Plugging back in $z^2 = x$,

So equation is $(z^2 - [-\frac{q}{2} + \frac{\sqrt{q^2-4s}}{2}])(z^2 - [-\frac{q}{2} - \frac{\sqrt{q^2-4s}}{2}])$

So there are two real or possibly complex numbers r_1, r_2

2. If r_1, r_2 real, you have obtained the desired factorization of $z^4 + qz^2 + s$ in $\mathbb{R}[x]$

Yes.

3. Alternatively, if r_1, r_2 are nonreal, observe that they are complex conjugates of each other. Using their square roots, factor $z^4 + qz^2 + s$ as a product of degree one polynomial in $\mathbb{C}[x]$. Show that the degree-one terms can be regrouped and combined in pairs to obtain a factorization of $z^4 + qz^2 + s$ as a product of quadratic polynomials in $\mathbb{R}[x]$

r_1, r_2 are complex conjugates of each other, since they are in the forms $a + bi, a - bi$ respectively

We can rewrite complex numbers r_1, r_2 as $a + bi$ and $a - bi$

If we take the square root of $(z^2 - (a + bi))$ and $(z^2 - (a - bi))$, we get roots of $z^4 + qz^2 + s = 0$ are $(z + \sqrt{a - bi}), (z - \sqrt{a - bi}), (z + \sqrt{a + bi}), (z - \sqrt{a + bi})$

Then $z^4 + qz^2 + s = 0$ is the product of 4 degree one polynomials in $\mathbb{C}[x]$

And these 4 degree one polynomials may be regrouped as a product of quadratic polynomials in $\mathbb{R}[x]$

10.37 Prove Theorem 10.6

Let $f(x)$ be a polynomial in $\mathbb{R}[x]$ of positive degree n

1. Show that $f(x)$ is a product of irreducible polynomials of degree 1 or 2:

By induction (on n)

Base case: If $n = 1$ or $n = 2$, true

Inductive step

Inductive Hypothesis: If $f(x)$ is a polynomial of degree n, $f(x)$ factors in $\mathbb{R}[x]$ as the product of irreducible polynomials of degree 1 or 2

Show that for $f(x)$ of degree $n+1$, $f(x)$ factors in $\mathbb{R}[x]$ as the product of polynomials of degree 1 or 2

$f(x)$ is degree $n+1$, so by theorem 10.5, it is not irreducible, rewrite it as $f(x) = g(x)h(x)$ for $g(x)$ degree 1 or 2, and $h(x)$ degree n or $n-1$

Then by the inductive hypothesis, $h(x)$ is a product of irreducible polynomials of degree 1 or 2

So $f(x)$ is a product of irreducible polynomials of degree 1 or 2

2. Show that $f(x)$ has n roots in $\mathbb{C}$

We know from part 1 that $f(x)$ is of degree n, and is the product of irreducible polynomials whose sum of degree is n, where there are r degree one factors and s degree two factors, such that $r + 2s = n$

If a factor is degree one in the form $(x - \gamma)$, with root $\gamma \in \mathbb{R}$, then it has a root $\gamma \in \mathbb{C}$

If a factor is degree 2 and irreducible in $\mathbb{R}[x]$, then it may be reduced in $\mathbb{C}[x]$ as $(x - r)(x - \bar{r})$. Then it has 2 complex roots, $r, \bar{r} \in \mathbb{C}$

Then the total number of roots in $\mathbb{C}$ is $r + 2s = n$

Then there are n roots of $f(x) \in \mathbb{C}$

3. Show that the polynomial $f(x)$ factors in $\mathbb{C}[x]$ as the product of n degree-one polynomials

We know from part 1 that $f(x)$ is of degree n, and is the product of polynomials whose sum of degree is n, where there are r degree one factors and s degree two factors, such that $r + 2s = n$

Each degree one factor is a polynomial in $\mathbb{R}[x]$, so it is a polynomial in $\mathbb{C}[x]$

Each irreducible degree 2 factor in $\mathbb{R}[x]$ may be reduced in $\mathbb{C}[x]$ as two degree one factors, $(x - \gamma), (x - \bar{\gamma})$

Then $f(x)$ is the product of $r + 2s = n$ degree one factors in $\mathbb{C}[x]$

10.40 Prove theorem 10.9

Suppose that $f(x)$ is a polynomial of positive degree in $\mathbb{R}[x]$, and that r is a root of $f(x)$ in $\mathbb{C}$

1. For the first part, show that $\bar{f}(x) = f(x)$

Since $f(x) \in \mathbb{R}[x]$, then $f(x)$ is in the form $(a_0 + 0i)x^0 + (a_1 + 0i)x^1 + ...(a_n + 0i)x^n$

So its conjugate, $\bar{f}(x)$ is in the form $(a_0 - 0i)x^0 + (a_1 - 0i)x^1 + ...(a_n - 0i)x^n$, which is equal to $f(x)$

2. For the second part, check that the coefficients of $(x - r)(x - \bar{r})$ are real, so that $(x - r)(x - \bar{r})$ lies in $\mathbb{R}[x]$

The polynomial $(x - r)(x - \bar{r})$ is equal to $x^2 + r^2$, with real coefficients.

3. Deduce that if r is a nonreal complex number and $x - r$ divides $f(x)$ in $\mathbb{C}[x]$, then $(x - r)(x - \bar{r})$ divides $f(x)$ in $\mathbb{C}[x]$

We know by theorem 10.8 that since $(x - r)$ divides $f(x)$ in $\mathbb{C}[x]$, then $(x - \bar{r})$ divides $\bar{f}(x)$ in $\mathbb{C}[x]$

Since $\bar{f}(x) = f(x)$, then $(x - \bar{r})$ divides $f(x)$

4. Observe that to prove that $(x - r)(x - \bar{r})$ divides $f(x)$ in $\mathbb{R}[x]$, it suffices to prove the following statement: Suppose $f(x), g(x)$ are nonzero polynomials in $\mathbb{R}[x]$ and $h(x)$ is a polynomial in $\mathbb{C}[x]$ such that $f(x) = g(x)h(x)$. Then $h(x)$ lies in $\mathbb{R}[x]$

Yes

5. Prove this last statement.

We know by theorem 10.8 that since $f(x) \in \mathbb{R}[x]$, and therefore is in $\mathbb{C}[x]$, then for $f(x) = g(x)h(x)$, $\bar{f}(x) = \bar{g}(x)\bar{h}(x)$

Since $f(x), g(x)$ are in $\mathbb{R}[x]$, then the work shown in part 1 shows that $f(x) = \bar{f}(x), g(x) = \bar{g}(x)$

Then $f(x) = g(x)\bar{h}(x)$

Then $\bar{h}(x) = h(x)$

But this is only true when $h(x) \in \mathbb{R}[x]$

So $h(x)$ must be in $\mathbb{R}[x]$