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MATH 395 HW 6
6[2a, 4a], 7[3a, 36]
7.48, 7.51, 7.40, 7.50

6[2a,4a]

6.2) Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the ith ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of

a) X_1, X_2

Call $X_1 = X, X_2 = Y$

$P(X = 1, Y = 1) = \frac{5}{13}\frac{4}{12}$

$P(X = 1, Y = 0) = \frac{5}{13}\frac{8}{12}$

$P(X = 0, Y = 1) = \frac{8}{13}\frac{5}{12}$

$P(X = 0, Y = 0) = \frac{8}{13}\frac{7}{12}$

6.4) Repeat problem 6.2 when the ball is selected and replaced before the next selection

a) Call $X_1 = X, X_2 = Y$

$P(X = 1, Y = 1) = \frac{5}{13}\frac{5}{13}$

$P(X = 1, Y = 0) = \frac{5}{13}\frac{8}{13}$

$P(X = 0, Y = 1) = \frac{8}{13}\frac{5}{13}$

$P(X = 0, Y = 0) = \frac{8}{13}\frac{8}{13}$

6[3a, 36]

6.3) In problem 6.2, suppose that the white balls are numbered, and let Y_i equal 1 if the ith white ball is selected and 0 otherwise. Find hte joint probability mass function of

a) Y_1, Y_2

3 balls are chosen, equally likely

Want 1st white ball and 2nd white ball

We picked 3, so probability of first is $\frac{3}{13}$

Then the second white ball would have probability $\frac{2}{12}$

$P(Y_1 = 1, Y_2 = 1) = \frac{3}{13}\frac{2}{12}$

Similarly

$P(Y_1 = 1, Y_2 = 0) = \frac{3}{13}\frac{10}{12}$

$P(Y_1 = 0, Y_2 = 0) = \frac{10}{13}\frac{2}{12}$

$P(Y_1 = 0, Y_2 = 0) = \frac{10}{13}\frac{9}{12}$

6.36) In problem 6.3, calculate the conditional probability mass function of Y_1 given that

a) $Y_2 = 1$

$P(Y_1 = 1|Y_2 = 1) = \frac{\frac{3*2}{13*12}}{\frac{3}{13}} = \frac{1}{6}$

$P(Y_1 = 0|Y_2 = 1) = \frac{\frac{3*10}{13*12}}{\frac{3}{13}} = \frac{5}{6}$

b) $Y_2 = 0$

$P(Y_1 = 1|Y_2 = 0) = \frac{\frac{3*10}{13*12}}{\frac{10}{13}} = \frac{1}{4}$

$P(Y_1 = 0|Y_2 = 0) = \frac{\frac{10*9}{13*12}}{\frac{10}{13}} = \frac{3}{4}$

7.48 A fair die is successively rolled. Let X, Y denote respectively, the number of rolls to obtain a 6 and a 5. Find

a) $E[X]$

$E[X] = \sum_{x=1}^{\infty} xP(X = x) = \sum_{x=1}^{\infty} x(\frac{5}{6}^{x-1}\frac{1}{6} = 6$

$E[X] = 6$

b) $E[X|Y = 1]$

$E[X|Y = 1] = \sum_{x=1}^{\infty} xP(X = x|Y = 1) = \sum_{x=1}^{\infty} x\frac{5}{6}^{x-2}\frac{1}{6}$

$= \frac{6}{5}\sum_{x=1} x\frac{5}{6}^{x-1}\frac{1}{6}$

$= \frac{1}{5}\sum_{x=2} x\frac{5}{6}^{x-1}$

$= \frac{-1}{5} + \frac{1}{5} * 36$

$= 7$

c) $E[X|Y = 5]$

$= \sum_{x=1}^{\infty} xP(X = x, Y = 5)$

$= \sum_{x=1}^{\infty} \frac{P(X=x,Y=5}{P(Y=5)}$

$P(Y = 5) = \frac{5}{6}^4\frac{1}{6}$

If $x \leq 4$, then $P(X = x, Y = 5) = (\frac{4}{6})^{x-1}\frac{1}{6}\frac{5}{6}^{4-x}\frac{1}{6}$

If $x = 5$, then $P(X = x, Y = 5) = \frac{4}{6}^4\frac{1}{6}\frac{5}{6}^{x-6}\frac{1}{6}$

Then $E[X|Y = 5] = \frac{1}{(\frac{5}{6})^4(\frac{1}{6})}(\sum_{x=1}^{4} x4^{x-1}5^{4-x}6^{-5} + \sum_{x=6}^{\infty} x4^4 5^{-x}6^6{}^{-x})$

$= \frac{3636}{625}$

7.51 The joint density of X, Y is given by $f(x, y) = \frac{e^{-y}}{y}, \quad 0 < x < y, 0 < y < \infty$

Compute $E[X^3|Y = y]$

$f_{X|Y}(x|y) = \frac{\frac{e^- y}{y}}{\int_0^y e^{-y}/ydx}, \quad 0 < x < y$

$f_{X|Y}(x|y) = \frac{1}{y}, \quad 0 < x < y$

$E[X^3|Y = y] = \int_0^y x^3\frac{1}{y}dx = \frac{y^3}{4}$

7.40 The joint density function of X, Y is given by $f(x, y) = \frac{1}{y}e^{-(y+\frac{x}{y})}, \quad x > 0, y > 0$

Find $E[X], E[Y]$, and show that $Cov(X, Y) = 1$

$Cov(X, Y) = E[XY] - E[X]E[Y]$

$f_Y(y) = e^{-y}\int_0^{\infty} \frac{1}{y}e^{-x/y}dx = e^{-y}$

$E[Y] = \int_0^\infty e^{-y} = 1$

Then $E[X] = E[E[X|Y]] = E[Y] = 1$

$E[XY] = E[E[XY|Y] = E[YE[X|Y]] = E[Y^2] = 2$

Then $Cov(X, Y) = 2 - 1 * 1 = 1$

7.50 The joint density of $X, Y, f(x, y) = \frac{e^{\frac{-x}{y}} e^{-y}}{y}, \quad 0 < x < \infty, 0 < y < \infty$

$f_{X|Y}(x|y) = \frac{e^{-x/y} e^{-y}/y}{\int_0^\infty e^{-x/y} e^{-y}/y dy}$

$f_{X|Y}(x|y) = \frac{1}{y} e^{-x/y}, \quad 0 < x < \infty$

$E[X^2|Y = y] = \int_0^\infty x^2 \frac{1}{y} e^{-x/y} dx$

$E[X^2|Y = y] = 2y^2$