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MATH 395 HW 5
Ch 7: 45, 9, 21, 23, 32, 75

7.45 X_1, X_2, X_3, X_4 are pairwise uncorrelated random variables, each having mean 0, var 1, compute the correlations of

a) $X_1 + X_2,\ X_2 + X_3$

Corr$(X_1 + X_2, X_2 + X_3) = \frac{Cov(X_1+X_2, X_2+X_3)}{\sqrt{Var(X_1+X_2)Var(X_2+X_3}}$

Expand numerator, this is $Cov(X_1, X_2) + Cov(X_1, X_3) + Cov(X_2, X_2) + Cov(X_2, X_3)$

But since pairwise uncorrelated, they are all zero, except for $Cov(X_2, X_2)$, which is just $Var(X_2) = 1$

Expand denominator, this is $\sqrt{(Var(X_1) + Var(X_2) + 2Cov(X_1, X_2))(Var(X_2) + Var(X_3) + 2Cov(X_2, X_3))}$

Since pairwise uncorrelated, cov terms go to 0

Simplifies to $\sqrt{4} = 2$

So Corr$(X_1 + X_2, X_2 + X_3) = \frac{1}{2}$

b) $X_1 + X_2,\ X_3 + X_4$

Corr$(X_1 + X_2, X_3 + X_4) = \frac{Cov(X_1+X_2, X_3+X_4)}{\sqrt{Var(X_1+X_2)Var(X_3+X_4}}$

Expand numerator, this is $Cov(X_1, X_3) + Cov(X_1, X_4) + Cov(X_2, X_3) + Cov(X_2, X_4)$

Since pairwise uncorrelated, cov terms go to 0

Expand denominator, this is $\sqrt{(Var(X_1) + Var(X_2) + 2Cov(X_1, X_2))(Var(X_3) + Var(X_4) + 2Cov(X_3, X_4))}$

Cov terms go to 0, denominator simplifies to 2

So Corr = 0

7.9 A total of n balls numbered 1 through n are put into n urns, also numbered 1 through n in such a wy that ball i is equally likely to go into any of the urns.

Hint: Let $X_i = 1$ if urn i is empty, 0 otherwise

a) Find the expected number of urns that are empty

$E[X_i] = (1 - \frac{1}{i})(1 - \frac{1}{i+1})(1 - \frac{1}{1+2})...(1 - \frac{1}{n})$

$E[X] = \sum_{i=1}^{n} E[X_i]$, independent

$E[X] = \frac{(n-1)}{2}$

b) Find the probability that none of the urns is empty

For the urns not to be empty, the n^{th} ball must be dropped into the n^{th} urn.

This is $\frac{1}{n}\frac{1}{n-1}...\frac{1}{1} = \frac{1}{n!}$

7.21 Hint a: Let $X_i = 0$ if 3 people have birthday on day $i, i = 1, ..365$, 0 otherwise

Hint b: Let $X_i = 1$ if no one has a birthday, day i, 0 otherwise

For a group of 100 people, compute

a) The expected number of days of the year that are birthdays of exactly 3 people

3 people have a birthday on day i, call it event X_i

This happens with probability $\binom{100}{3}\frac{1}{365}^3\frac{364}{365}^{97}$

$E[X] = \sum_{i=1}^{365} E[X_i]$

$= 365\binom{100}{3}\frac{1}{365}^3\frac{364}{365}^{97}$

b) the expected number of distinct birthday

If no one has a birthday on day i, call this event X_i, then this is a $\frac{364}{365}^{100}$ chance

Then if someone does have a birthday on day i, this is complement event, has $1 - \frac{364}{365}^{100}$ probability

Then $E[X] = \sum_{i=1}^{100}(1 - E[X_i])$

$E[X] = 365[1 - \frac{364}{365}^{100}]$

7.23 Urn 1 contains 5 white, 6 black

Urn 2 contains 8 white 10 black

Two balls are randomly selected from urn 1 and are put into urn 2. If 3 balls are then randomly selected from urn 2, compute the expected number of white balls in the trio.

Hint: let $X_i = 1$ if i^{th} white ball initially in urn 1 is oen of the three selected, let $X_i = 0$ otherwise

Similarly, let $Y_i = 1$ if ith white ball from urn 2 is one of the three selected, 0 otherwise.

The number of the white balls in the trio can now be written as

$\sum_1^5 X_i + \sum_1^8 Y_i$

Want $E[\sum_1^5 X_i + \sum_1^8 Y_i] = \sum_1^5 E[X_i] + \sum_1^8 E[Y_i]$

$P(X_i = 1) = P(\text{ith white in urn 2}|\text{ ith white in urn 1)P(ith white in urn 1}) = \frac{3}{20}\frac{2}{11}$

$P(Y_i = 1) = \frac{\binom{19}{2}}{\frac{20}{3}} = \frac{3}{20}$

$E[\sum_1^5 X_i + \sum_1^8 Y_i] = \sum_1^5 \frac{3}{20}\frac{2}{11} + \sum_1^5 \frac{3}{20} = \frac{147}{110}$

7.32 $X_i = 1$ if box i empty, 0 else

$E[X_i] = P(X_i = 1) = \prod_{i=j}^n 1 - \frac{1}{i}$

Then $Var(X_i) = E[X_i](1 - E[X_j])$

And ofr $j < k$, $E[X_jX_k] = \prod_{i=j}^{k-1}(1 - 1/i)\prod_{i=k}^n(1 - 2/i)$

So $Cov(X_j, X_k) = \prod_{i=j}^{k-1}(1 - 1/i)\prod_{i=k}^n(1 - 2/i) - \prod_{i=j}^n(1 - 1/i)\prod_{i=k}^n(1 - 1/i)$

$Var(X) = \sum_{i=1}^n E[X_i](1 - E[X_j]) + 2Cov(X_j, X_k)$

7.75 Hint: Identify the distributions from the mgf, and the njust use hteir probabilities $P(X = 2)$, etc

The moment generating function of X is given by $M_x(t) = e^{2e^t-2}$ and that of Y by $M_y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$

X is poisson distribution with $\lambda = 2$

Y is binomial with parameters $10, \frac{3}{4}$

If X, Y independent, what are

a) $P(X+Y=2)$?

Discrete, sums to 2

$P(X=0)P(Y=2)+P(X=1)P(Y=1)+P(X=2)P(Y=0)$

$=e^{-2}\binom{10}{2}\frac{3}{4}^{2}\frac{1}{4}^{8}+2e^{-2}\binom{10}{1}\frac{3}{4}\frac{1}{4}^{9}+2e^{-2}\frac{1}{4}^{10}$

b) $P(XY=0)$?

Need either 1 or both of X, Y to be 0

This is $P(X=0)+P(Y=0)-P(X=0\cap Y=0)$

$=e^{-2}+\frac{1}{4}^{10}-e^{-2}\frac{1}{4}^{10}$

c) $E[XY]$?

Independent, linear

$E[XY]=E[X]E[Y]=2*10*\frac{3}{4}=15$