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MATH 444 HW 4
3 H, J, K, L

3H Prove that every circle contains infinitely many points [hint: use exercise 3F and 3G]

Given any circle, it has a center $O$ and radius $r$

The circle is described by $\mathscr{C}(O,r) = \{P : OP = r\}$

We know by exercise 3F that the point $O$ lies on infinitely many distinct lines.

Given any one of those distinct lines, call it $\ell$, there exists a coordinate function $f : \ell \to \mathbb{R}$ such that $f(O) = 0$

$f$ is bijective, take $P = f^{-1}(r)$

Then $OP = |f(P) - f(O)| = |r - 0| = r$, so $P$ is on the circle

Claim: For any point $P$ on the circle, there is only one line connecting $O$ and $P$

Assume two lines, $\ell_1, \ell_2$ connect $O$ and $P$

Then this violates that distinct lines must not contain 2 of the same points.

So for any point $P$ on the circle, there must be only one unique line connecting $O$ and $P$

Since there are infinitely many distinct lines through $O$, there are infinitely distinct points on the circle.

3J Prove Theorem 3.35 (the segment construcion theorem) [Hint: use an adapted coordinate function. Look ath the proof of theorem 3.27 for inspiration.]

Theorem 3.35: Suppose $\overrightarrow{a}$ is a ray starting at the point $A$ and $r$ is a positive real number. Then there exists a unique point $C$ in the interior of $\overrightarrow{a}$ such that $AC = r$

$\overrightarrow{a}$ is a ray, it is part of a line, call it $\ell$

$\ell$ has a coordinate function $f : \ell \to \mathbb{R}$, and $f(A) = 0$

And $\overrightarrow{a}$ is described as $\{P \in \ell : f(P) > 0\}$

Let $C = f^{-1}(r)$

Then $f(C) > 0$, so $C$ is on the ray

And $AC = |f(C) - f(A)| = |r - 0| = r$

So there exists a point $C$ on the ray such that the distance $AC = r$

Show that this point $C$ is unique

Assume that $C$ is not unique, then there must be another point, call it $D$ such that it is on the ray and $AD = r$

Then $AD = |f(D) - f(A)| = r$

Then $|f(D)| = r$

Then $f(D) = \pm r$

Case: $f(D) = +r$

Then since $f$ is bijective, $C = D$, contradicts that $D$ is not $C$

Case: $f(D) = -r$

Then $f(D) < 0$

Then $D$ is not on the ray, contradiction

So $C$ must be unique.

3K Prove Corollary 3.37 (Euclid's segment cutoff theorem)

Corollary 3.37: If $\overline{AB}$ and $\overline{CD}$ are segments with $CD > AB$, there is a unique point $E$ in the interior of $\overline{CD}$ such that $\overline{CE} \cong \overline{AB}$

Let $r = AB$

$\overline{CD}$ is a segment on the line $\ell$

$\ell$ has a coordinate function $f : \ell \to \mathbb{R}$ such that $f(C) = 0, f(D) > 0$

Then all points on $\overline{CD}$ are $\{P \in \ell : 0 \leq f(P) \leq f(D)\}$

We know $AB < CD$, so $|f(D) - f(C)| = |f(D)| = f(D) > r$

Then let $E = f^{-1}(r)$

Then $f(E) = r, 0 < r < f(D)$, so $E$ is on the segment

And $CE = |f(E) - f(C)| = |r - 0| = AB$

So there exists a point $E$ on the segment $\overline{CD}$ such that $CE = AB$

Show that $E$ is a unique point

Assume that $E$ is not unique, then there exists another point $F$ such that $CF = AB$, and $F$ is on the segment $CD$

If $CF = AB$, then $|f(F) - f(C)| = r = AB$

$|f(F)| = r$

Then $f(F) = \pm r$

Case 1: $f(F) = r$

Then since $f$ is bijective, $F = E$, contradicts that $F \neq E$

Case 2: $f(F) = -r$

Then $f(F) = -r < 0$, then $F$ is not on the segment $\overline{CD}$, contradiction

So $E$ must be unique.

3L Prove Theorem 3.42 (on intersections and unions of rays).

Theorem 3.42: Suppose $A, B$ are two distinct points. Then the following set of equalities hold

a) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$

Show that $\overrightarrow{AB} \cap \overrightarrow{BA} \subseteq \overline{AB}$

Rays $\overrightarrow{AB}, \overrightarrow{BA}$ exist on the line $\ell$ containing $A, B$

Then there exists a coordinate function $f : \ell \to \mathbb{R}, f(A) = 0, f(B) > 0$

So $\overline{AB} = \{P \in \ell : 0 \leq f(P) \leq f(B)\}$

Points in $\overrightarrow{AB}$ satisfy $\{P \in \ell : f(P) \geq 0\}$

And $\overrightarrow{BA}$ satisfy $\{P \in \ell : f(P) < f(B)\}$

So points in $\overrightarrow{AB} \cap \overrightarrow{BA}$ satify $P \in \ell : 0 \leq f(P) \leq f(B)$

So $\overrightarrow{AB} \cap \overrightarrow{BA} \subseteq \overline{AB}$

Show that $\overline{AB} \subseteq \overrightarrow{AB} \cap \overrightarrow{BA}$

Points in the segment $\overline{AB}$ are $\{P \in \ell : 0 \leq f(P) \leq f(B)\}$

And we know points in $\overrightarrow{AB} \cap \overrightarrow{BA}$ satify $P \in \ell : 0 \leq f(P) \leq f(B)$

So $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$

b) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$

Show that $\overrightarrow{AB} \cup \overrightarrow{BA} \subseteq \overleftrightarrow{AB}$

Points in $\overrightarrow{AB}$ satisfy $\{P \in \ell : f(P) \geq 0\}$

And $\overrightarrow{BA}$ satisfy $\{P \in \ell : f(P) < f(B)\}$

So $\overrightarrow{AB} \cup \overrightarrow{BA} = \{P \in \ell : f(P) \geq 0\} \cup \{P \in \ell : f(P < f(B)\}$

So $\overrightarrow{AB} \cup \overrightarrow{BA} \subseteq \ell$

Show $\ell \subseteq \overrightarrow{AB} \cup \overrightarrow{BA}$

Given a point $P$ in $\ell$, then show that $P \in \overrightarrow{AB} \cup \overrightarrow{BA}\}$

We know $\overrightarrow{AB} \cup \overrightarrow{BA} = \{P \in \ell : f(P) \geq 0\} \cup \{P \in \ell : f(P < f(B)\}$

So $P \in \ell$, which is true.

So $\overrightarrow{AB} \cup \overrightarrow{BA} \subseteq \overleftrightarrow{AB}$