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MATH 394 HW 4
3.79, 3.84 on p.97 of Chapter 3.
4, 11, 13, 19, 20, 21, 24, 29, 32, 33, 35 and 39 on page 163 of Chapter 4

3.79 Let A = {sum = 7} = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}

Then P(A) = $\frac{6}{36} = \frac{1}{6}$

Let B = {sum = even}

Then P(B) = $\frac{1}{2}$, since half the time, both dice will be both even or both odd.

We want the probability of getting 2 sevens before 6 evens.

So in successive rolls, P(A) = $\frac{\frac{1}{6}}{\frac{1}{6}+\frac{1}{2}} = \frac{1}{4}$

And P(B) = $\frac{\frac{1}{2}}{\frac{1}{6}+\frac{1}{2}} = \frac{3}{4}$

Call the roll a success if the sum is 7

We know that the probability of exactly k successes in m+n-1 trials is $\binom{m+n-1}{k}p^k(1-p)^{m+n-1-k}$

So it follows that the desired probability of n successes before m failures is $P_{n,m} = \sum_{k=n}^{m+n-1}\binom{m+n-1}{k}p^k(1-p)^{m+n-1-k}$

So $P(2 sevens before 6 evens) = \sum_{k=2}^{2+7-1}\binom{2+6-1}{k}p^k(1-p)^{2+6-1-k}$

We know $\sum_{k=0}^{2+7-1}\binom{2+6-1}{k}p^k(1-p)^{2+6-1-k} = 1$

So $P(2 sevens before 6 evens) = 1 - \sum_{k=0}^{1}\binom{7}{k}p^k(1-p)^{7-k}$

This is equal to $1 - [\binom{7}{0}p^0(1-p)^7 + \binom{7}{1}p^1(1-p)^6]$

We call 7 a success, so $p = \frac{1}{4}, (1-p) = \frac{3}{4}$

So the desired probability is $1 - [\binom{7}{0}\frac{1}{4}^0(\frac{3}{4})^7 + \binom{7}{1}\frac{1}{4}^1(\frac{3}{4})^6]$

Which is equal to $\frac{4547}{8192} \approx 0.55505$

3.84 a) If the balls are replaced, the probability of grabbing a white ball is always $\frac{4}{12} = \frac{1}{3}$

P(A) winning = $\sum_{i=0}^{\infty} P(\text{A wins on the } i^{th} \text{ draw})$

For A to win on any given i^{th} draw, A must draw a white with probability $\frac{1}{3}$, and everyone from previous hands must lose with probability $\frac{2}{3}$ each for each previous draw.

This is equal to $\sum_{i=0}^{\infty}\frac{1}{3}(\frac{2}{3})^{3i}$

This is a geometric series, and converges to $\frac{a}{1-r}$

That is $\frac{1}{3}\frac{1}{1-\frac{8}{27}} = \frac{9}{19}$

P(B) winning = $\sum_{i=0}^{\infty} P(\text{B wins on the } i^{th} \text{ draw})$

For B to win on any given i^{th} draw, A must draw a non white ball with probability $\frac{2}{3}$, and B must draw a white ball with probability $\frac{2}{3}$, and all previous draws must be non white with probability $\frac{2}{3}$

This is equal to $\frac{2}{3}\frac{1}{3}\frac{1}{1-\frac{8}{27}} = \frac{6}{19}$

P(C) winning = 1 - P(A) - P(B) = $\frac{4}{19}$

b) If the balls are not replaced

P(A) = Sum P(A wins on i^{th} draw)

This is equal to $\frac{4}{12} + [\frac{8}{12}\frac{7}{11}\frac{6}{10}] * \frac{4}{9} + [\frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}] * \frac{4}{6} = \frac{7}{15} \approx 0.4666$

P(B) = Sum P(B wins on i^{th} draw)

$\frac{8}{12}\frac{4}{11} + [\frac{8}{12}\frac{7}{11}\frac{6}{10}]\frac{5}{9}\frac{4}{8} + [\frac{8}{12}\frac{7}{11}\frac{6}{10}\frac{5}{9}\frac{4}{8}\frac{3}{7}]\frac{2}{6}\frac{4}{5} = \frac{53}{165} \approx 0.3212$

P(C) = 1 - P(A) - P(B) ≈ 0.212

4.4 There are 5 men, 5 women, they are ranked, all unique, all rankings equally likely

Let X=i denote women's highest rank achieved at rank at i

P(X=1) is when the highest ranked woman is rank 1

Out of 5 women, pick 1 to be rank 1. There are 9! remaning permutations for the remaining people

So P(X=1) = $\frac{\binom{5}{1}9!}{10!} = \frac{1}{2}$

P(X=2) when the highest ranked woman is rank 2

The rank 1 person is male. 5 males to choose one to be rank 1

The rank 2 is female, 5 females to choose one to be rank 2

8! permutations for the remaining rankings

P(X=2) = $\frac{5\binom{5}{1}8!}{10!} = \frac{5}{18}$

Rank 1 and 2 are male, 5 choices male rank 1, 4 choices male rank 2

Rank 3 female, 5 females choose 1 for rank 3, 7! permutations for others

P(X=3) = $\frac{5*4\binom{5}{1}7!}{10!} = \frac{5}{36}$

Rank 1,2, and 3 are male. 5 choices for rank 1, 4 for 2, 3 for 3. 5 choices for female rank 4. 6! permutations for remaining people.

P(X=4) = $\frac{5*4*3\binom{5}{1}6!}{10!} = \frac{5}{84}$

Rank 1,2,3, and 4 male. 5 choices female for rank 5. 5! permutations for remaining people.

P(X=5) = $\frac{5*4*3*2\binom{5}{1}5!}{10!} = \frac{5}{252}$

Rank 1,2,3,4, and 5 are male. 5 choices for female rank 6. 4! permutations for remaining.

P(X=6) = $\frac{5!\binom{5}{1}4!}{10!} = \frac{1}{252}$

Rank 1,2,3,4,5, and 6 are male. impossible.

P(X=7) = P(X=8) = P(X=9) = P(X=10) = 0

4.11 a) For 3, $10^3 = 3 * 333 + 1$, so 333 numbers between 1 and 10^3 are divisible by 3

So P(3) = $\frac{333}{1000}$

For 5, $10^3 = 5 * 200$, so 200 numbers between 1 and 1000 are divisible

P(5) = $\frac{200}{1000}$

For 7, $10^3 = 7 * 142 + 6$, so 142 numbers between 1 and 1000 are divisible

P(7) = $\frac{142}{1000}$

For 15, $10^3 = 15 * 66 + 10$, so 66 numbers

P(15) = $\frac{66}{1000}$

For 105, $10^3 = 105 * 9 + 55$, so 9 numbers

P(105) = $\frac{9}{1000}$

For k large enough, $P(a$ divides $N)$ converges to $\frac{1}{a}$

b) Find $P(\mu(N) = 0)$, as $k \to \infty$

$P(\mu(N) = 0)$ = P(N is not divisible by $P_i^2, i \geq 1$)

$= \prod_i^\infty$ P(N is not divisible by p_i^2)

$= \prod_i^\infty$ P(1 - N is divisible by p_i^2)

Know that the probability of a number N being divisible by a number a from 10^k for k large converges to $\frac{1}{a}$

In this case, $a = p_i^2, i \geq 1$

$= \prod_i^\infty (1 - \frac{1}{p_i^2})$

Know that this is equal to $\frac{6}{\pi^2}$

4.13 Let s be a successful sale

Let 1st be the first sale, and 2nd be the second sale

Let dlx be a deluxe sale, and std be a standard sale

$P(s|1st) = 0.3$

$P(s|2nd) = 0.6$

$P(dlx|s) = 0.5$

$P(std|s) = 0.5$

Let X be the value of all sales.

X = {0, 500, 1000, 1500, 2000}

P(X = 0) = $P(s^c|1st) * P(s^c|2nd) = (1 - 0.3)(1 - 0.6) = 0.28$

P(X = 500) = $P(s^c|1st) * P(s|2nd) * P(std|s) + P(s|1st)(std|s) * P(s^c|2nd) = (1-0.3)(0.6)(0.5) + (0.3)(0.5) * (1 - 0.6) = 0.27$

P(X = 1000) =

$P(s|1st) * P(std|s) * P(s|2nd)P(std|s) + P(s|1st)P(dlx|s)P(s^c|2nd) + P(s^c|1st)P(s|2nd)P(dlx|s) = [0.3 * 0.5 * 0.6 * 0.5] + [0.3 * 0.5 * 0.4] + [0.7 * 0.6 * 0.5] = 0.315$

P(X = 1500) = $P(s|1st) * P(dlx|s) * P(s|2nd) * P(std|s) + P(s|1st) * P(std|s) + P(s|2nd) * P(dlx|s) = [0.3 * 0.5 * 0.6 * 0.5] + [0.3 * 0.5 * 0.6 * 0.5] = 0.09$

P(X = 2000) = $P(s|1st) * P(dlx|s) * P(s|2nd) * P(dlx|s) = [0.3 * 0.5 * 0.6 * 0.5] = 0.045$

4.19 $P(b < 0) = 0$

$P(0 \leq b < 1) = \frac{1}{2}$

$P(1 \leq b < 2) = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$

$P(2 \leq b < 3) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$

$P(3 \leq b < 3.5) = \frac{9}{10} - \frac{4}{5} = \frac{1}{10}$

$p(b \geq 3.5) = 1 - \frac{9}{10} = \frac{1}{10}$

4.20 a) $P(X > 0) =$ P(Win first) + P(lose, win, win)

$= \frac{18}{38} + \frac{20}{38}\frac{18}{38}^2 \approx 0.5918$

b) No, this is not a foolproof strategy because it is possible to win 1, or lose 1 or 3.

c) To win 1, player can win first, or lose and win 2nd and 3rd.

To end with 0, impossible.

To lose 1, player loses first, and then plays a second and third round. There are 2 choices to pick 1 to win, the other is a loss.

To lose 2, it is impossible.

To lose 3, the player loses the first, and loses the second and third.

$E[X] = [1 * \frac{18}{38} + \frac{20}{38}\frac{18}{38}^2] + [(-1) * \frac{20}{38}\binom{2}{1}\frac{20}{38}\frac{18}{38}] + [(-3) * \frac{20}{38}^3] \approx -0.10803$

4.21 148 students

a) It is more likely to select a student in a bigger bus than it is to select a student from any bus with equal probability. So $E[X] > E[Y]$

b) $E[X] = 40 * \frac{40}{148} + 33 * \frac{33}{148} + 25 * \frac{25}{148} + 50 * \frac{50}{148} \approx 39.28$

$E[Y] = \frac{(40+33+25+50)}{4} = 37$

4.24 P(b guess 1) = p P(b guess 2) = 1 - p

Loss = 3/4 unit paid

Win = i units received

a) Determine expected if A has written down number 1

$E[X] = p * 1 - (1-p)\frac{3}{4}$

b) Determine expected if A has written down number 2

$E[X] = p * -\frac{3}{4} + (1-p) * 2$

Set the two equations equal to each other to find best p

$p * 1 - (1-p)\frac{3}{4} = p * -\frac{3}{4} + (1-p) * 2$

Solving for p, we get $p = \frac{11}{18}$

So maximum at $p = \frac{11}{18}$, so max $= \frac{23}{72}$

c) P(a writes 1) = q

P(a writes 2) = 1-q

$E[X] = q * 1 - (1-q) * \frac{3}{4}$

d) $E[X] = -\frac{3}{4} * q + (1-q) * 2$

Setting the equations equal and solving for q,

We get $q = \frac{11}{18}$. At this value, A's maximum loss is equal to B's maximum gain.

4.29 There are two possibilities: Machine 1 is broken, with probability p, or Machine 2 is broken with probability 1-p

If we check machine 1 first, cost is C_1, and there is a $(1-p)$ probability that machine 2 is broken instead, and we need to check that for C_2 and pay R_2 for repairs. Otherwise there is a p probability that 1 is the problem so we pay R_1 to fix it

Then the expected value for cost is $C_1 + (1-p)(C_2 + R_2) + (p)(R_1)$

If we check machine 2 first, cost is C_2, and there is a p probability that machine 2 is broken, and we need to check that for C_1 and pay R_1 for repair. Otherwise there is a $(1-p)$ probability that 2 is the problem, so we pay R_2 to fix it

Then the expected value for cost is $C_2 + (p)(C_1 + R_1) + (1-p)(R_2)$

We would want to it in ascending order if the expected value for cost of checking 1 first is less than the expected value of cost for checking 2 first

So we check when $C_1 + (1-p)(C_2 + R_2) + (p)(R_1) \leq C_2 + (p)(C_1 + R_1) + (1-p)(R_2)$

This is equivalent to $C_1 \leq p(C_1 + C_2)$

So when the inequality $C_1 \leq p(C_1 + C_2)$ is not satisfied, it is better to reverse the checking order.

4.32 100 people, 10 groups of 10 people

0.1 probability for disease, each person, independent

Let p = probability disease = 0.1

If someone in the group of 10 is positive, with a probability $(p)^{10}$, then 11 tests must be conducted

If no one in the group of 10 is positive, with a probability $(1-p)^{10}$, then only 1 test is performed

$E[X] = 1[(1-0.1)^{10}] + 11[(0.1)^{10}] \approx 0.3486$

4.33 Purchase for 10

Sell for 15

Demand is a binomial variable, n = 10

Probability of purchase $p = \frac{1}{3}$

Binomial, so we know $E[X] = np = \frac{10}{3} \approx 3.333$

So he can expect to sell 3 papers (can't buy and sell fractions of paper), so he should purchase 3 papers to maximize his profit.

4.35 5 red marbles, 5 blue marbles

Draw 2 randomly

If they are the same color, win 1.10

If different colors, lose 1

a) Find the expected value of win

Two events can happen, either they are both the same color, or not the same

Let p = both same color

Then E[X] = p(1.1) + (1-p)(-1) = 2.1p - 1

There are 5 red marbles, and 5 blue marbles. Then the probability of picking two different colors $= 1 - p = \frac{{5 \atop 1}{5 \atop 1}}{\binom{10}{2}} = \frac{5}{9} \approx 0.555$

So $p = \frac{4}{9} \approx 0.4444$

So $E[X] = 2.1(\frac{4}{9}) - 1 = \frac{-1}{15} \approx -0.0666$

b) Find the variance of the amount you win

$Var(X) = E[X^2] - (E[X])^2$

Find $E[X^2]$

$E[X^2] = (1.1)^2\frac{4}{9} + (-1)^2\frac{5}{9} = \frac{82}{75} \approx 1.09333$

So $Var(X) = \frac{82}{75} - [\frac{-1}{15}]^2 = \frac{49}{45} \approx 1.0888$

4.39 Urn contains 3 white balls, 3 black balls

Draw a ball, with replacement.

What is the probability that of the first 4 balls drawn, exactly 2 are white?

We know $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$

Probability of drawing a black or white ball is 0.5 each

So $n = 4, k = 2, p = 0.5$

So $P(X = 2) = \binom{4}{2}0.5^2(1 - 0.5)^{4-2} = \frac{3}{8} \approx 0.375$