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 MATH 445 HW 1
 7: B, D, E, F, G

7B Prove Theorem 7.7 (Existence and uniqueness of perpendicular bisectors)

Theorem 7.7: Every segment has a unique perpendicular bisector

Let \overline{AB} be the segment formed by two distinct points A, B .

We know there exists a midpoint M on \overline{AB}

Let \overleftrightarrow{AB} be the line containing segment \overline{AB}

Then by theorem 4.30, we know there exists a unique line ℓ that is perpendicular to \overleftrightarrow{AB} through point M

7D Prove Theorem 7.10 (Existence and uniqueness of a reflected point)

Theorem 7.10: Let ℓ be a line and let A be a point not on ℓ . Then there is a unique point A' , called the reflection of A across ℓ , such that ℓ is the perpendicular bisector of $\overline{AA'}$.

By theorem 7.1, taking point A not on ℓ , we can construct a line m through point A perpendicular to ℓ

m, ℓ perpendicular, intersect at a point, call it P

m is a line, so there exists a coordinate function f such that $f(A) = 0, f(p) > 0$

Then $AP = |f(p) - f(a)| = f(p)$

Let A' be the point on the opposite side of ℓ such that $A' = f^{-1}(2p)$

Then $A'P = |f(2p) - f(p)| = f(p) = AP$

This A' is unique by bijectivity of coordinate function f

m is perpendicular to ℓ through point P

Then for the segment $\overline{AA'}$, it has midpoint P , and has line ℓ perpendicular to $\overline{AA'}$

7E Prove Lemma 7.12 (Properties of closest points)

Let P be a point and let S be any set of points.

a) If C is a closest point to P in S , then $C' \in S$ is also a closest point to P if and only if $PC' = PC$

b) If C is a point in S such that $PX > PC$ for every point $X \in S$ other than C , then C is the unique closest point to P in S .

prove a:

Forwards: Assume C and C' are both closest points to P in S .

C is a closest point, and C' is another point in S , so we have that $PC \leq PC'$

C' is a closest point, and C is another point in S , so we have that $PC' \leq PC$

Then $PC = PC'$

Backwards: Assume C is a closest point. Assume $PC' = PC$

By definition of closest point, $PC \leq PX$ for all $X \in S$

and $PC' = PC$

So $PC' \leq PX$ for all $X \in S$

So C' is also a closest point to P in S

prove b:

By contradiction

Assume C is a point in S , $PX > PC$ for every point $X \in S$ other than C

Assume C is not the unique closest point to P in S

C is not unique closest, then there exists another closest point, C'

Then by part a, $PC = PC'$

But this contradicts that $PX > PC$ for every point $X \in S$ other than C

In particular, that $PC' > PC$

Then C must be the unique closest point to P in S

7F Prove Theorem 7.13 (The closest point on a line)

Suppose ℓ is a line, P is a point not on ℓ , and F is the foot of the perpendicular from P to ℓ

a) F is the unique closest point to P on ℓ

b) If A, B are points on ℓ such that $F * A * B$, then $PB > PA$

prove a:

ℓ is a line, P not on ℓ , F is the foot of the perpendicular from P to ℓ

Let A be a point on ℓ not equal to F

Then $\triangle PFA$ is a right triangle with measure $\angle PFA = 90$

Then by corollary 5.15, since every triangle must have two acute angles, angles $\angle FAP, \angle FPA$ must both be acute.

So $\angle PFA > \angle FAP$ and $\angle PFA > \angle FPA$

So by Theorem 5.16, scalene inequality, \overline{PA} is the longest side of the triangle.

So for any point A on ℓ not equal to F , the distance $FP < AP$

So F is a closest point to P on ℓ

Show that F is the unique closest point to P on ℓ

$FP < AP$ for all A on ℓ not equal to F

So by Lemma 7.12b, F is the unique closest point to P on ℓ

prove b:

7G Prove theorem 7.14 (The closest point on a segment) [Hint: consider separately the cases in which $P \in \overleftrightarrow{AB}$ and $P \notin \overleftrightarrow{AB}$, and divide the second case into two subcases depending on whether the foot of the perpendicular from P to \overleftrightarrow{AB} does or does not lie in \overline{AB}].

Theorem 7.14: Suppose \overleftrightarrow{AB} is a segment and P is any point. Then there is a unique closest point to P in \overleftrightarrow{AB}