10.13 Use Cardano's formula, as clarified in Exercise 10.12, to obtain all three solutions to the cubic equation $y^3 - 7y + 6 = 0$

1. Write down the solution given by the formula as a sum of cuberoots. Observe that it involves the cube roots of $-3 + \frac{10}{9}\sqrt{-3}$ and $-3 - \frac{10}{9}\sqrt{-3}$

The solution is $y = \sqrt[3]{-3 + \sqrt{10}9\sqrt{-3}} + \sqrt[3]{-3 - \frac{10}{9}\sqrt{-3}}$

2. Using the numbers ω, ω^2, and the earlier determination of one cube root of $-3 + \frac{10}{9}\sqrt{-3}$, write expressiosn for the three complex numbers that are cube roots of $-3 + \frac{10}{9}\sqrt{-3}$. Also write down the three complex numbers that are cube roots of $-3 - \frac{10}{9}\sqrt{-3}$

We know from problem 10.10 that a cuberoot of $-3 + \frac{10}{9}\sqrt{-3}$ is $1 + \frac{2}{3}\sqrt{-3}$

So using $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, the cuberoots of $-3 + \frac{10}{9}\sqrt{-3}$ are

$1 + \frac{2}{3}\sqrt{-3}$

$[1 + \frac{2}{3}\sqrt{-3}] * \omega = [1 + \frac{2}{3}\sqrt{-3}] * (-\frac{1}{2} + \frac{\sqrt{-3}}{2}) = \frac{-3}{2} + \frac{\sqrt{-3}}{6}$

$[1 + \frac{2}{3}\sqrt{-3}] * \omega^2 = \frac{1}{2} - \frac{5}{6}\sqrt{-3}$

Cuberoots of $-3 - \frac{10}{9}\sqrt{-3}$ are

$1 - \frac{2}{3}\sqrt{-3}$

$\frac{1}{2} + \frac{5}{6}\sqrt{-3}$

$\frac{-3}{2} - \frac{1}{6}\sqrt{-3}$

3. Pair the cube roots of $-3 + \frac{10}{9}\sqrt{-3}$ and $-3 - \frac{10}{9}\sqrt{-3}$ as specified in exercise 10.12 to get three pairs such that the product of the complex numbers in each pair equals $\frac{7}{3}$

For $A = 1 + \frac{2}{3}\sqrt{-3}, B = 1 - \frac{2}{3}\sqrt{-3}, \omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, the pairs are

$AB, \omega A * \omega^2 B, \omega^2 A * \omega B$

These are:

$[1 + \frac{2}{3}\sqrt{-3}] * [1 - \frac{2}{3}\sqrt{-3}]$

$[-\frac{3}{2} + \frac{\sqrt{-3}}{6}] * [-\frac{-3}{2} - \frac{1}{6}\sqrt{-3}]$

$[\frac{1}{2} - \frac{5}{6}\sqrt{-3}] * [\frac{1}{2} + \frac{5}{6}\sqrt{-3}]$

4. Add together the complex numbers in each pair to obtain all three solutions of $y^3 - 7y + 6 = 0$.

The roots are $r_1 = A + B, r_2 = \omega A + \omega^2 B, r_3 = \omega^2 A + \omega B$

$r_1 = [1 + \frac{2}{3}\sqrt{-3}] + [1 - \frac{2}{3}\sqrt{-3}] = 2$

$r_2 = [-\frac{3}{2} + \frac{\sqrt{-3}}{6}] + [-\frac{-3}{2} - \frac{1}{6}\sqrt{-3}] = -3$

$r_3 = [\frac{1}{2} - \frac{5}{6}\sqrt{-3}] + [\frac{1}{2} + \frac{5}{6}\sqrt{-3}] = 1$