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MATH 445 HW 1
7: B, D, E, F, G

7B Prove Theorem 7.7 (Existence and uniqueness of perpendicular bisectors)

Theorem 7.7: Every segment has a unique perpendicular bisector

Let $\overline{AB}$ be the segment formed by two distinct points A, B.

We know there exists a midpoint M on $\overline{AB}$

Let $\overleftrightarrow{AB}$ be the line containing segment $\overline{AB}$

Then by theorem 4.30, we know there exists a unique line ℓ that is perpenedicular to $\overleftrightarrow{AB}$ through point M

So ℓ goes through the bisector M and is perpendicular to $\overleftrightarrow{AB}$, so ℓ is the unique perpendicular bisector of $\overline{AB}$

7D Prove Theorem 7.10 (Existence and uniqueness of a reflected point)

Theorem 7.10: Let ℓ be a line and let A be a point not on ℓ. Then there is a unique point A', called the reflection of A across ℓ, such that ℓ is the perpendicular bisector of $\overline{AA'}$.

By theorem 7.1, taking point A not on ℓ, we can construct a line m through point A perpendicular to ℓ

m, ℓ perpendicular, intersect at a point, call it P

m is a line, so there exists a coordinate function f such that $f(A) = 0$, $f(p) > 0$

Then $AP = |f(p) - f(a)| = f(p)$

Let A' be the point on the opposite side of ℓ such that $A' = f^{-1}(2p)$

Then $A'P = |f(2p) - f(p)| = f(p) = AP$

This A' is unique by bijectivity of coordinate function f

m is perpendicular to ℓ through point P

Then for the segment $\overline{AA'}$, it has midpoint P, and has line ℓ perpendicular to $\overline{AA'}$ through the midpoint P

Then ℓ is the unique perpendicular bisector of $\overline{AA'}$

7E Prove Lemma 7.12 (Properties of closest points)

Let P be a point and let S be any set of points.

a) If C is a closest point to P in S, then $C' \in S$ is also a closest point to P if and only if $PC' = PC$

b) If C is a point in S such that $PX > PC$ for every point $X \in S$ other than C, then C is the unique closest point to P in S.

prove a:

Forwards: Assume C and C' are both closest points to P in S.

C is a closest point, and C' is another point in S, so we have that $PC \leq PC'$

C' is a closest point, and C is another point in S, so we have that $PC' \leq PC$

Then $PC = PC'$

Backwards: Assume C is a closest point. Assume $PC' = PC$

By definition of closest point, $PC \leq PX$ for all $X \in S$

and $PC' = PC$

So $PC' \leq PX$ for all $X \in S$

So C' is also a closest point to P in S

prove b:

By contradiction

Assume C is a point in S, $PX > PC$ for every point $X \in S$ other than C

Assume C is not the unique closest point to P in S

C is not unique closest, then there exists another closest point, C'

Then by part a, $PC = PC'$

But this contradicts that $PX > PC$ for every point $X \in S$ other than C

In particular, that $PC' > PC$

Then C must be the unique cloest point to P in S

7F Prove Theorem 7.13 (The closest point on a line)

Suppose ℓ is a line, P is a point not on ℓ, and F is the foot of the perpendicular from P to ℓ

a) F is the unique closest point to P on ℓ

b) If A, B are points on ℓ such that $F * A * B$, then $PB > PA$

prove a:

ℓ is a line, P not on ℓ, F is hte foot of the perpendicular form P to ℓ

Let A be a point on ℓ not equal to F

Then ΔPFA is a right triangle with measure $\angle PFA = 90$

Then by corrolary 5.15, since every triangle must have two acute angles, angles $\angle FAP, \angle FPA$ must both be acute.

So $\angle PFA > \angle FAP$ and $\angle PFA > \angle FPA$

So by Theorem 5.16, scalene inequality, $\overline{PA}$ is the longest side of the triangle.

So for any point A on ℓ not equal to F, the distance $FP < AP$

So F is a closest point to P on ℓ

Show that F is the unique cloest point to P on ℓ

$FP < AP$ for all A on ℓ not equal to F

So by Lemma 7.12b, F is the unique closest point to P on ℓ

prove b:

Assume A, B on ℓ such that $F * A * B$

These form right triangles $\Delta PFA, \Delta PFB$

$\angle PFA$ is right, so by corollary 5.15, $\angle FAP$ is acute

So $\angle PAB$ is obtuse

So in the triangle ΔPAB, by corollary 5.15, $\angle APB, \angle ABP$ are acute

So $\angle PAB < \angle APB, \angle PAB < \angle ABP$

So by theorem 5.16 Scalene inequality, $PB > PA$

7G Prove theorem 7.14 (The closest point on a segment) [Hint: consider separately the cases in which $P \in \overleftrightarrow{AB}$ and $P \notin \overleftrightarrow{AB}$, and divide the second case into two subcases depending on whether the foot of the perpendicular from P to $\overleftrightarrow{AB}$ does or does not lie in $\overline{AB}$].

Theorem 7.14: Suppose $\overleftrightarrow{AB}$ is a segment and P is any point. Then there is a unique closest point to P in $\overleftrightarrow{AB}$

$\overline{AB}$ is a segment formed by two distinct points A, B

P is any point.

Case: P on $\overline{AB}$

Then the closest point to P on $\overline{AB}$ is itself

Case: P on $\overleftrightarrow{AB}$, not on $\overline{AB}$

Then either $P * A * B$, or $A * B * P$

If $P * A * B$, then there exists a coordinate function f such that $f(P) = 0, f(A) > 0$

Claim: A is a closest point for P on $\overline{AB}$

For $X \in \overline{AB}$, $A * X * B$.

So $AP = f(A)$, and $AX = f(X)$

Since $A * X * B$, then $f(A) \leq f(X)$ for all $X \in \overline{AB}$, so A is a closest point

Claim: A is the unique closest point.

$A \neq X$, so $f(A) < f(X)$ and $AP < XP$ for all $X \neq A \in \overline{AB}$

So by lemma 7.12b, A is the unique closest point.

Similarly, if $A * B * P$, we can show that B is the unique closest point.

Case: P not on $\overleftrightarrow{AB}$

By theorem 7.1, we can construct a line m perpendicular to $\overleftrightarrow{AB}$ perpendicular to ℓ through point P

Let F be the point of intersection of lines m, ℓ

If $F \in \overline{AB}$, then by theorem 7.13a, F is the unique closest point to P on $\overline{AB}$

If $F \notin \overline{AB}$, then either $F * A * B$ or $A * B * F$. Use the coordinate function f as defined in previous case.

If $F * A * B$, then by theorem 7.13b, for $X \in \overline{AB}, X \neq A$, we have $XP > AP$

Then by lemma 7.12b, A must be the unique closest point to P on $\overline{AB}$

If $A * B * F$, we can show that for $X \neq B$, $XP > BP$, so that B is the unique closest point to P on $\overline{AB}$