10.7 We begin with the cubic polynomial y^3+py+q. We can assume that p is nonzero, for if $p=0$, the equation is $y^3=-q$, and the solution is easily obtained as the cube root of $-q$. Introduce a new variable satisfying

$$y = z - \frac{p}{3z}$$

1. Substitute $z - \frac{p}{3z}$ for y in the equation, expand, and simplify, to obtain

$$z^3 - \frac{p^3}{27z^3} + q = 0$$

For $y = z - \frac{p}{3z}$, we have $(z - \frac{p}{3z})^3 + p(z - \frac{p}{3z}) + q = 0$

$(z^2 - \frac{2p}{3} + \frac{p^2}{9z^2})(z - \frac{p}{3z}) + pz - \frac{p^2}{3z} + q = 0$

$[z^3 - \frac{2pz}{3} + \frac{p^2}{9z}] - [\frac{pz}{3} - \frac{2p^2}{9z} + \frac{p^3}{27z^3}] + pz - \frac{p^2}{3z} + q = 0$

$z^3 - pz + \frac{p^2}{3z} - \frac{p^3}{27z^3} + pz - \frac{p^2}{3z} + q = 0$

$z^3 - \frac{p^3}{27z^3} + q = 0$

2. Multiply by z^3 to clear the value in the denominator to obtain $z^6 - \frac{p^3}{27} + qz^3 = 0$

3. Observe that this is the quadratic equation in z^3. Use the quadratic formula to obtain $z^3 = -\frac{q}{2} \pm \sqrt{\frac{q^2 + \frac{4p^3}{27}}{4}}$

If we let x to be z^3, then we can apply the quadratic formula to solve for $x^2 + qx - \frac{p^3}{27} = 0$

The roots we obtain are $x = \frac{-q}{2} \pm \frac{\sqrt{q^2 - \frac{4p^3}{27}}}{2}$

Plugging back in $z^3 = x$, we get $z^3 = \frac{-q}{2} \pm \frac{\sqrt{q^2 - \frac{4p^3}{27}}}{2}$

Which is equal to $z^3 = \frac{-q}{2} \pm \sqrt{\frac{q^2 - \frac{4p^3}{27}}{4}}$

4. Introduce R as an abbreviation for $(\frac{p}{3})^3 + (\frac{q}{2})^2$ and rewrite the last equality as $z^3 = -\frac{q}{2} \pm \sqrt{R}$

The value inside the square root term is equal to $\frac{q^2}{4} - \frac{p^3}{27}$, which is equal to $(\frac{p}{3})^3 + (\frac{q}{2})^2$

So we can rewrite the equality as $z^3 = -\frac{q}{2} \pm \sqrt{R}$

5. There are two possible values for z^3, namely, $-\frac{q}{2} + \sqrt{R}, -\frac{q}{2} - \sqrt{R}$

Multiply these two values together and simplify. Show that you get

$$(-\frac{q}{2} + \sqrt{R})(-\frac{q}{2} - \sqrt{R}) = (-\frac{p}{3})^3$$

Multiplying, we get $(\frac{q^2}{4} - R^2)$

Plugging back in $R = (\frac{p}{3})^3 + (\frac{q}{2})^2$, we get

$(\frac{q^2}{4} - (\frac{p}{3})^3 + (\frac{q}{2})^2$

Which simplifies to $(-\frac{p}{3})^3$

6. Take the cube root of both sides above and deduce that the two values of z have a product satisfying

$$\sqrt[3]{-\frac{q}{2} + \sqrt{R}} * \sqrt[3]{-\frac{q}{2} - \sqrt{R}} = -\frac{p}{3}$$

7. Observe that this means that if you choose z to be the cube root of $\frac{q}{2} + \sqrt{R}$, then $-\frac{p}{3z}$ is the cube root of $-\frac{q}{2} - \sqrt{R}$

This is true, since if it is the cuberoot, then $z * \sqrt[3]{-\frac{q}{2} - \sqrt{R}} = -\frac{p}{3}$

Which is equal to $\sqrt[3]{-\frac{q}{2} - \sqrt{R}} = -\frac{p}{3z}$

8. Recall that z was introduced to satisfy $z - \frac{p}{3z}$. You have shown that the two terms on the right of this equation, $z, -\frac{p}{3z}$ are the cube roots of $-\frac{q}{2} + \sqrt{R}$ and $-\frac{q}{2} - \sqrt{R}$ respectively.

9. Conclude that y is the sum of these two cube roots

$$y = \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt[3]{-\frac{q}{2} - \sqrt{R}}$$

Since $y = z - \frac{p}{3z}$, we can plug in $z = \sqrt[3]{-\frac{q}{2} + \sqrt{R}}, \frac{p}{3z} = \sqrt[3]{-\frac{q}{2} - \sqrt{R}}$

We obtain $y = \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt[3]{-\frac{q}{2} - \sqrt{R}}$