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MATH 395 HW 2
5.26, 5.32, 5.39
Self Test: TE 5.8, TE 5.9, TE 5.12

5.26 Fair)

If the coin is fair, the conclusion is false when we have probability of head 0.5, and the number of heads tossed is greater than OR EQUAL TO 525.

That is, for number of heads on a fair coin $X_{fair}, P(X_{fair} \geq 525)$

We know that X_{fair} is a binomial random variable with $p = 0.5, n = 1000, E[X_{fair}] = np = 500, Var(X_{fair}) = np(1-p) = 250$

So we want $P(X_{fair} > 525)$.

This can be approximated with a normal distribution

$P(\frac{X_{fair}-E[X_{fair}]}{\sqrt{Var(X_{fair})}} \geq \frac{524.5-E[X_{fair}]}{\sqrt{Var(X_{fair})}}$

Replace with Z, for Z score.

We get $P(Z \geq 1.5495)$

So $P(X_{fair} \geq 525) \approx 1 - \Phi(1.55)$

Using the table for normal distribution, we find that this is approximately equal to 0.0606

Biased)

If the coin is biased, the conclusion is false when the number of heads tossed is STRICTLY LESS than 525.

That is, for number of heads on a biased coin $X_b, P(X_b < 525)$

We know that X_b is a binomial random variable with $p = 0.55, n = 1000, E[X_b] = np = 550, Var(X_b) = np(1-p) = 247.5$

We want $P(X_b < 525)$

This can be approximated with a normal distribution

$P(\frac{X_b-E[X_b]}{\sqrt{Var(X_b)}} < \frac{524.5-E[X_b]}{Var(X_b)}$

Replace with Z, for Z score

We get $P(Z < -1.62)$

So $P(X_b < 525) \approx P(Z < -1.62) \approx 0.052$

5.32 The time in hours required to repair a machine is exponentially distributed random variable with parameter $\lambda = \frac{1}{2}$

a) Find the probability that a repair time exceeds 2 hours

Exponential Random Variable, use formula

$P(X > s) = \int_s^\infty(\lambda e^{-\lambda x})dx$

$P(X > 2) = \int_2^\infty(0.5e^{-0.5x}dx$

$P(X > 2) = -e^{-0.5x}|_2^\infty$

$P(X > 2) = [-e^{-\infty}] - [-e^{-1}] = e^{-1}$

b) Find the CONDITIONAL probability that a repair time is at least 10 hours, given that its duration exceeds 9 hours

We know that for an exponential random variable, it has a memoryless property. That is, $P(X > 10|X > 9) = P(X > 10 - 9)$

So we want $P(X > 1)$

This is $-e^{-0.5x}|_1^\infty = [-e^\infty] - [e^{-0.5}] = e^{-0.5}$

5.39 If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined over $Y = log(x)$

Work with cdf first, $F_Y(y) = P(log(x) \leq y)$

$F_Y(y) = P(x \leq e^y) = F_X(e^y)$

So pdf $f_Y(y)$ is the derivative of the cdf

We get $f_Y(y) = \frac{d}{dx}F_X(e^y)$

$f_Y(y) = f_X(e^y) * e^y$

pdf $f_Y(y) = e^{-e^y}e^y$

TE 5.8 Let X be a random variable that takes on values between 0 and c. That is $P(0 \leq X \leq c) = 1$. Show that $Var(X) \leq \frac{c^2}{4}$

Hint: One approach is to argue that $E[X^2] \leq cE[X]$

And then using the inequality to show that $Var(X) \leq c^2[\alpha(1-\alpha)]$ where $\alpha = \frac{E[X]}{c}$

$E[X^2] = \int_0^c x^2 f(x)dx \leq \int_0^c cxf(x)dx = cE[X]$

And we know $Var(X) = E[X^2] - (E[X])^2$

So $Var(X) \leq cE[X] - (E[X])^2$

$Var(X) \leq c^2(\frac{E[X]}{c} - \frac{(E[X])^2}{c^2})$

So $Var(x) \leq c^2(\alpha - \alpha^2)$ for $\alpha = \frac{E[X]}{c}$

We know that $\alpha - \alpha^2$ on the real line has maximum value at $\alpha = 0.5, \alpha - \alpha^2 = \frac{1}{4}$

So $Var(X) \leq \frac{c^2}{4}$

TE 5.9 Show that if Z is a standard normal random variable; then for $x > 0$

a) Show that $P(Z > x) = P(Z < -x)$

$P(Z > x) = P(-Z < -x)$

But $-Z$ is also a standard normal random variable,

so $P(Z > x) = P(-Z < -x) = P(Z < -x)$

b) Show that $P(|Z| > x) = 2P(Z > x)$

$P(|Z| > x) = P(Z > x) + P(-Z > x)$

Since $-Z$ is another standard normal random variable, then htis is equal to $2P(Z > x)$

c) Show that $P(|Z| < x) = 1 - 2P(Z < x)$

$P(|Z| < x) = 1 - P(|Z| > x)$

Using answer form part b, this is $1 - 2P(Z > x)$

We know that $P(Z > x) = 1 - P(Z < x)$

So $P(|Z| < x) = 1 - 2[1 - P(Z < x)]$

This is $2P(Z < x) - 1$

TE 5.12 The identity from 5.5 tells us that $E[X^n] = \int_0^\infty nx^{n-1}P(X > x)dx$

So for $n = 2$, and an exponential random variable with parameter λ, this is

$E[X^2] = \int_0^\infty 2xP(X > x)dx$

$E[X^2] = \int_0^\infty 2xe^{-\lambda x}dx$

$E[X^2] = 2\int_0^\infty xe^{-\lambda x}dx$

$E[X^2] = 2[(x\frac{e^{-\lambda x}}{-\lambda})|_0^\infty - (\int_0^\infty \frac{e^{-\lambda x}}{-\lambda}dx)]$

First term evaluates to 0

$E[X^2] = \frac{2}{\lambda}\int_0^\infty e^{-\lambda x}dx$

$E[X^2] = \frac{2}{\lambda^2}$