Brandon Chen
MATH 445 HW 4
Due Wednesday 4/25
12 A,D,E
13 A,C

12A Prove theorem 12.5 (The SSS similarity theorem). [Hint: Use the similar triangle construction theorem to construct triangle $\Delta D'E'F'$ that is similar to ΔABC, but with $\overline{D'E'} \cong \overline{DE}$. Then use the hypothesis and a little algebra to show that $\Delta D'E'F' \cong \Delta DEF$

Theorem 12.5: If ΔABC and ΔDEF are triangles such that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$, then $\Delta ABC \sim \Delta DEF$

By theorem 12.4, we can construct a triangle $\Delta D'E'F'$ similar to ΔABC with side length $D'E' \cong DE$.

Then by construction of similar triangle, $\frac{AB}{D'E'} = \frac{BC}{E'F'} = \frac{BC}{E'F'}$

By transitive property of similarity, $\Delta DEF \sim \Delta D'E'F'$

Then by definition of similar triangle, $\frac{DE}{D'E'} = \frac{D'F'}{D'F'} = \frac{E'F'}{E'F'}$

But we know that $D'E' = DE$

So $DE = D'E', D'F' = D'F', E'F' = E'F'$

Then $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

12D CONVERSE TO THE ANGLE BISECTOR PROPORTION THEOREM: Suppose ΔABC is a triangle and D is a point in the interior of $\overline{BC}$ such that $\frac{BD}{DC} = \frac{AD}{AC}$. Prove that $\overrightarrow{AD}$ is the bisector of $\angle BAC$

Let ΔABC be a triangle, and let D be a point in the interior of $\overline{BC}$ such that $\frac{BD}{DC} = \frac{AD}{AC}$

Assume for contradiction that $\overrightarrow{AD}$ is not the bisector of angle $\angle BAC$

Then E be a point on $\overline{BC}$ such that $\overrightarrow{AE}$ is the bisector of $\angle BAC$

Then by theorem 12.9, $\frac{BE}{EC} = \frac{AB}{BC}$

We know by hypothesis that $\frac{AB}{BC} = \frac{BD}{DC}$

Then $\frac{BE}{EC} = \frac{BD}{DC}$

And $\frac{BD}{BE} = \frac{DC}{EC}$

Both D, E on interior of $\overline{BC}$

Then either $B * D * E * C$

or $B * E * D * C$

Case: $B * D * E * C$

Then $BD < BE$, and $\frac{BD}{BE} < 1$

And $DC > EC$, and $\frac{DC}{EC} > 1$

This contradicts that $\frac{BD}{BE} = \frac{DC}{EC}$

Case: $B * E * D * C$

Then $BD > BE$, and $\frac{BD}{BE} > 1$

And $DC < EC$, and $\frac{DC}{EC} < 1$

This contradicts that $\frac{BD}{BE} = \frac{DC}{EC}$

Then $\overrightarrow{AD}$ must be the angle bisector.

12E Prove that the three midsegments of a triangle yield an admissible decomposition of the triangle into four congruent triangles, each of which is similar to the original one and has one quarter the area.

Construct a triangle ΔABC

Let D be the midpoint of $\overline{AB}$

Let E be the midpoint of $\overline{BC}$

Let F be the midpoint of $\overline{AC}$

$\frac{AD}{AB} = \frac{1}{2} = \frac{AF}{AC}$

For triangle ΔADF, it has angle $\angle BAC \cong \angle DAF$

So by theorem 12.6, SAS similarity, $\Delta ADF \sim \Delta ABC$

Using a similar argument for $\Delta DBE, \Delta EFC$, we find that

$\Delta DBE \sim \Delta ABC$, and $\Delta EFC \sim \Delta ABC$

So $\Delta ADF \sim \Delta DBE \sim \Delta EFC$

So by definition of similar triangles $\frac{AD}{DB} = \frac{AF}{DE} = \frac{DF}{BE}$

Since length $AD = DB$, then $AF = DE$ and $DF = BE$

So by SSS congruence, $\Delta ADF \cong \Delta DBE$

Similarly, we show that $\Delta EFC \cong \Delta ADF \cong DBE$

For triangle ΔDEF, it is formed by shared side lengths with the other triangles.

We get that $DF \cong DF$, $DE \cong DE \cong AF$, and $FE \cong FE \cong AD$

Then $\Delta DEF \cong \Delta ADF \cong DBE \cong EFC$

And $\Delta DEF \sim \Delta ADF \sim \Delta DBE \sim \Delta EFC \sim \Delta ABC$

13A Use the idea suggested by figure 13.13 to give a proof of the pythagorean theorem. [Hint: bacuse the only figure given to you by the hypothesis is an arbitrary right triangle, first you have to explain how a figure like the one in the diagram can be constructed and justify any claims you make about relationships that the diagram suggests.]

Construct a right triangle ΔABC with side lengths $AB = a, BC = b, AC = c$ and let side length $AC = c$ be the hypotenuse.

On $\overleftrightarrow{BC}$, take point D such that $BD = a + b$

Through point D, there exists a line ℓ parallel to $\overline{AB}$, take point E on ℓ such that $DE = b$, E on the same half plane as A

Then $\angle CDE = 90$, is right

And ΔCDE is a triangle with $CD = a = AB, \angle CDE = \angle ABC, DE = b = BC$

So by SAS congruence, $\Delta CDE \cong \Delta ABC$

So $CE = AC = c$

$\angle BCA, \angle ACE, \angle ECD$ form a linear triple, so $\angle BCA + \angle ACE + \angle ECD = 180$

We know that $\angle ECD \cong \angle BAC$

So $\angle BCA + \angle ACE + \angle BAC$

So $\angle ACE \cong \angle ABC$, is right angle.

So for ΔACE, $AC = c = CE$, and $\angle ACE = 90$

$\overline{AB} \parallel \overline{DE}$ by construction,

So points $ABDE$ form a trapezoid.

Let $\overline{AB}, \overline{ED}$ be the bases of the trapezoid.

then $ABDE$ has height $a + b$ and bases with lengths a, b

Then $ABDE$ has area $\frac{1}{2}[a + b] * [a + b] = \frac{a^2+b^2+2ab}{2}$

$\Delta ABC, \Delta CDE, \Delta ACE$ are an admissible decomposition of $ABDE$

So the area $\alpha(ABDE)$ is equal to $\alpha(ABC) + \alpha(CDE) + \alpha(ACE)$

For right triangle ΔABC, let $\overline{BC} = b$ be the base, and the height be $\overline{AB} = a$, then it has area $\frac{1}{2}ab$

And since $\Delta ABC \cong \Delta CDE$, then $\alpha(CDE) = \frac{1}{2}ab$

For right triangle ΔACE, let side $\overline{AC}$ be the base, and let $\overline{CE}$ with s, so it has area $\frac{1}{2}c^2$

Then $\frac{a^2+b^2+2ab}{2} = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$

Then $a^2 + b^2 = c^2$

13C Prove theorem 13.2 (the converse to the pythagorean theorem). [Hint: Construct a right triangle whose legs have lengths a, b and show that it is congruent to ΔABC.]

Theorem 13.2: Suppose ΔABC is a triangle with side lengths a, b, c. If $a^2 + b^2 = c^2$, then ΔABC is a right triangle, and its hypotenuse is the side of length c

By contradiction, assume ΔABC is a triangle with side lengths a, b, c and $a^2 + b^2 = c^2$, but ΔABC not a right triangle

Construct another triangle ΔPQR with $PQ = a, QR = b, \angle Q$ is a right angle

By pythagorean theorem, $(PR)^2 = a^2 + b^2$

We know by hypothesis that $a^2 + b^2 = c^2$

So $PR^2 = c^2$, and $PR = c$

So $PR = c = AC$, so by SSS congruence, $\Delta ABC = \Delta PQR$

Then $\angle PQR \cong \angle ABC = 90$

But this contradicts that ΔABC is not a right triangle.

Then ΔABC must be a right triangle, with hypotenuse $AC = c$