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MATH 445 HW 5
Due Friday 5/11/18
14 C,D
15 A,C

14C Suppose that ℓ is a line, C and D are points on opposite sides of ℓ, and $r = CD$. Prove that $\mathscr{C}(C, R)$ intersects ℓ in exactly two points.

Draw the segment $\overline{CD}$

Since C, D on opposite sides of ℓ, they must intersect at a point, call it P, with $C * P * D$

P on $\overline{CD}$, with $C * P * D$, so $CP < CD$

Then P must be on the interior of $\mathscr{C}(C, r)$

So by theorem 14.6, since ℓ contains point P on the interior of the circle, ℓ is a secant line for $\mathscr{C}$ and thus there are exactly two points where ℓ intersects $\mathscr{C}$

14D Suppose that A, B are distinct points and $r = AB$. Prove that $C = \mathscr{C}(A, R)$ and $D = \mathscr{C}(B, r)$ intersect.

We know that $AB = r$

Then for circle $D = \mathscr{C}(B, r)$, it contains point A

The distance from the point A to itself is 0, which is strictly less than the radius of C, r

Then by definition of interior point, A is an interior point of C on the circle D

Draw the line $\overleftrightarrow{AB}$

Then take the point E on the line $\overleftrightarrow{AB}$ such that $EB = r$, with $A * B * E$

Then $AE = 2r$

Then by definition of exterior point, E is an exterior point of C on the circle D

Then by theorem 14.10, D and C intersect at exactly two points.

15A Prove that a circular region is a convex set.

To show that a circular region is convex, we need to show that given any two points A, B in the circular region, any point P between them are also in the set.

There are three cases:

Case : Points A, B are on the boundary of the region.

Suppose that we are given 2 points A, B

Draw the triangle ΔOAB

Since A, B are on the boundary, the distances $OA = OB = r$

Then ΔOAB is isosceles, and $\angle OAB = \angle OBA$, both acute

We need to show that for any point P between A, B on the segment $\overline{AB}$, $OP < r$

Take a point P on $\overline{AB}$

Without loss of generality, assume that angle $\angle OPB$ is not acute (it is either right or obtuse)

Then since $\angle OBA$ is acute, $\angle OPB > \angle OBA$

Then by scalene inequality, $OP < OB$

Then $OP < r$, so P must be within the circular region.

Case : Points A, B both in interior of circular region.

Draw line $\overleftrightarrow{AB}$, it contains point A in the interior of the circular region, so $\overleftrightarrow{AB}$ intersects the circle at exactly two points, call them A', B'

Then by the previous case, all points in between A', B' on the segment $\overline{A'B'}$ are in the circular region.

Then since $\overline{AB}$ is a subset of $\overline{A'B'}$, then all points between A, B on $\overline{AB}$ are also in the circular region.

Case : Point A on boundary of the circular region, B on interior of circular region.

As in the previous case, draw line $\overleftrightarrow{AB}$, it intersects the cicle at two points, one of them is A, call the other B'

Then $A * B * B'$

Then by the first case, all points between A, B' on the segment $\overline{AB'}$ are in the circular region.

Since $\overline{AB}$ is a subset of $\overline{AB'}$, then all points between A, B on $\overline{AB}$ are also in the circular region.

15C Prove that the area of every sector of a circle of radius r is $\frac{\pi r^2}{360}$ times the measure of the arc that determines it.

a) Given any real number x such that $0 < x < 360$, let $a(x)$ denote the area of any sector of $\mathscr{C}$ whose measure is x. It suffices to show that $a(x) = \pi r^2 \frac{x}{180}$ whenever $0 < x \leq 180$

This can be done, since given fact 1 and fact 2, we can simply find the area of a sector of measure 180, and then add the rest of the are.

b) Show that $a(180) = \frac{\pi r^2}{2}$

Using fact 1, if we take two sectors of the same measure, 180 each, then they must have the same area.

Then each sector $a(180)$ must have the same area, then each must be $\frac{\pi r^2}{2}$

c) Show that if x is a real number and n is a positive integer such that both x and nx are both strictly between 0 and 180, then $a(nx) = n * a(x)$

By induction on n

Base case: $n = 1$, this is the same as the previous step.

Inductive step:

Inductive hypothesis: assume for n that for x, nx strictly between 0 and 180, then $a(nx) = n * a(x)$

Then for $n + 1$, this is $a(nx + x)$

Which is just $(n + 1)a(x)$

d) Show that if n is any positive integer, then $a(180/n) = \frac{\pi r^2}{2n}$

By induction on n

Base case: $n = 1$, this is the same as two steps ago, where $a(180) = \pi r^2/2$

Inductive step:

Inductive hypothesis: assume for positive integer n, $a(180/n) = \frac{\pi r^2}{2n}$

For $n+1$, the sector $a(180)$ is cut into $n+1$ even pieces, then by facts 1 and 2 and the previous step, we have that $(n+1)a(180/(n+1)) = a(180)$, and that each piece $a(180/(n+1))$ is of the same area, $a(180/(n+1)) = \frac{\pi r^2}{2(n+1)}$

e) Show that if m, n are positive integers with $m \leq n$, then $a(180m/n) = m/n\pi r^2/2$

Using the previous step, we know that for $m = 1$, then $a(180/n) = \pi r^2/(2n)$

Then by step c), we have that $a(m180/n) = ma(180/n) = m/n * \pi r^2/2$

f) Show that $a(x) = \pi r^2 x/180$ if x is any real number between 0 and 180.

By contradiction, assume that $a(x) \neq \pi r^2 x/180$

Case: $a(x) < \pi r^2 x/180$

Then $a(x) = \pi r^2 x_0/180$ for some x_0

We know that $\mathbb{Q}$ is dense in $\mathbb{R}$

Then we can choose a rational number $180m/n$ such that $a(180m/n)$ is between $a(x_0), a(x)$

Then we have that the measure $180m/n$ is less than x, but $a(180m/n) > a(x_0) = a(x)$, contradiction

Case: $a(x) > \pi r^2 x/180$

Using similar reasoning, we can come to the contradiction with a measure $180m/n$ being greater than x, but $a(180m/n) < a(x_0) = a(x)$

Then $a(x) = \pi r^2 x/180$ for $0 \leq x \leq 180$

Then $a(x) = \pi r^2 x/360$ for $0 \leq x \leq 360$