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MATH 395 HW 3
6.1(a,b), 6.6, 6.10, 6.20, 8(a,b), 42

6.1 a, b Two fair dice are rolled. Find the joint probability mass function of X, Y when

a) X is the largest value obtained on any die, and Y is the sum of the values

b) X is the value on the first die, and Y is the larger of the two values

6.6 A bin of 5 transistors is known to contain 2 that are defective. The transistors are to be tested, one at a time, until the defective ones are identified. Denote by N_1 the number of tests made until the first is identified, and N_2 the number of additional tests until the second defective is identified. Find the joint probability mass function of N_1 and N_2

We know that $1 < N_1 < 4, N_1 < N_2 \leq 5$, since N_2 must come after N_1, and N_1 must be at least 1, but cannot be 5, or else there aren't 2 defective

5 transistors total, 2 defective, $\binom{5}{2} = 10$ possible outcomes, each equally likely

So joint probability is uniform distribution with 10 possibilities, all equally likely

So for specific x, y, we have

$P(N_1 = x, N_2 = y) = \frac{1}{10}, 1 \leq x < y \leq 5$

6.10 The joint probability density function of X, Y is given by $f(x, y) = e^{-(x+y)} 0 \leq x < \infty, 0 \leq y < \infty$

$f_X(x) = e^{-x}, f_Y(y) = e^{-y}, f_X(x) * f_Y(y) = f_{X,Y}(x, y)$, independent

a) $P(X < Y)$?

$f_{X,Y}(x, y) = e^{-(x+y)}$

$P(X < Y) = \int_0^\infty \int_0^y e^{-(x+y)} dxdy$

$= \int_0^\infty -e^{-(x+y)}|_0^y dy$

$= \int_0^\infty -e^{-2y} + e^{-y} dy$

$= \frac{e^{-2y}}{2} - e^{-y}|_0^\infty = [0 - 0] - [\frac{1}{2} - 1] = \frac{1}{2}$

b) $P(X < a)$?

$P(X < a) = \int_0^a f_X(x)$

$= \int_0^a e^{-x} dx$

$= -e^{-x}|_0^a$

$= -e^{-a} + 1$

6.20 The joint density of X, Y is given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X, Y independent?

$f_Y(y) = \int_0^\infty xe^{-(x+y)} dx$

$= e^{-y} \int_0^\infty xe^{-x} dx$

Using $\Gamma(2) = 1$, $f_Y(y) = e^{-y}$

$f_X(x) = \int_0^\infty xe^{-(x+y)} dy$

$f_X(x) = xe^{-x} \int_0^\infty e^{-y} dy$

$f_X(x) = xe^{-x}[0 + 1] = xe^{-x}$

And we know X, Y independent if $f_Y(y) * f_X(x) = f_{X,Y}(x, y)$

This is $e^{-y} * xe^{-x} = xe^{-(x+y)}$, so X, Y are independent.

If instead, $f_{X,Y}(x, y)$ were given by

$f_{X,Y}(x,y) = \begin{cases} 2 & 0 < x < y, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

Would X, Y be independent?

$f_X(x) = \int_x^1 2d$

$= 2y|_x^1 = 2 - 2x, 0 < x < 1$

$f_Y(y) = \int_0^y 2dx = 2x|_0^y = 2y, 0 < y < 1$

$f_X(x) * f_Y(y) \neq f_{X,Y}(x,y)$

So X, Y not independent

6.8 a, b The joint probability density function of X, Y is given by

$f_{X,Y}(x,y) = c(y^2 - x^2)e^{-y}, -y \leq x \leq y, 0 < y < \infty$

a) Find c

We know that the total probability has to be equal to 1

$1 = \int_0^\infty \int_{-y}^y c(y^2 - x^2)e^{-y}dxdy$

$\frac{1}{c} = \int_0^\infty e^{-y}[\int_{-y}^y (y^2 - x^2)dx]dy$

$\frac{1}{c} = \int_0^\infty e^{-y}[(y * y^2 - \frac{1}{3}y^3) - (y * y^2 + \frac{1}{3}y^3)]dy$

$\frac{1}{c} = \int_0^\infty e^{-y}[\frac{4}{3}y^3]dy$

$\frac{3}{4c} = \int_0^\infty e^{-y}y^3dy$

Using $\Gamma(4) = 3! = 6$, this is $\frac{3}{4c} = 6$

$c = \frac{1}{8}$

b) Find the marginal densities of X, Y

Find $f_X(x)$

$f_X(x) = \frac{1}{8}\int_{|x|}^\infty (y^2 - x^2)e^{-y}dy$

$= [\frac{1}{8}\int_{|x|}^\infty y^2e^{-y}dy] - [\frac{1}{8}x^2\int_{|x|}^\infty e^{-y}]$

$= [\frac{1}{8}\int_{|x|}^\infty y^2e^{-y}dy] + [\frac{1}{8}x^2e^{-y}]_{|x|}^\infty$

$= [\frac{1}{8}\int_{|x|}^\infty y^2e^{-y}dy] + [0 - \frac{1}{8}x^2e^{-|X|}]$

$= [\frac{1}{8}\int_{|x|}^\infty y^2e^{-y}dy] - [\frac{1}{8}x^2e^{-|X|}]$

$= \frac{1}{8}[-y^2e^{-y} - 2ye^{-y} - 2e^{-y}]_{|x|}^\infty - [\frac{1}{8}x^2e^{-|X|}]$

$= \frac{1}{8} * 0 - \frac{1}{8}[-|x|^2e^{-|x|} - 2|x|e^{-|x|} - 2e^{-|x|}] - [\frac{1}{8}x^2e^{-|x|}]$

And $x^2 = |x|^2$, so terms $x^2e^{-|x|}, |x|^2e^{-|x|}$ cancel out

$= \frac{1}{8}[2|x|e^{-|x|} - 2e^{-|x|}]$

$= \frac{1}{4}[1 + |x|]e^{-|x|}, -\infty < x < \infty$

Find $f_Y(y)$

$f_Y(y) = \frac{1}{8}\int_{-y}^y (y^2 - x^2)e^{-y}dx$

$= \frac{1}{8}\int_{-y}^y y^2e^{-y} - x^2e^{-y}dx$

$= \frac{1}{8}[xy^2e^{-y} - \frac{1}{3}x^3e^{-y}]_{-y}^{y}$

$= \frac{1}{8}[(y^3e^{-y} - \frac{1}{3}y^3e^{-y}) - (-y^3e^{-y} + \frac{1}{3}y^3e^{-y})]$

$= \frac{1}{6}y^3e^{-y}, 0 < y < \infty$

42 The joint density of X, Y is given by

$f_{X,Y}(x,y) = c(x^2 - y^2)e^{-x}, 0 \leq x < \infty, -x \leq y \leq x$

We know that $f_{Y|X}(x|y) = \frac{f_{X,Y}(x,y)}{f_x(x)}$

We found in exercise 6.8a that $c = \frac{1}{8}$

So we want the conditional distribution of Y given $X = x$

We found in exercise 6.8b that $f_X(x) = c\frac{4}{3}x^3$

So $f_{Y|X}(y|x) = \frac{c(x^2-y^2)e^{-x}}{c\frac{4}{3}x^3e^{-x}}$

So $f_{Y|X}(y|x) = \frac{3}{4}\frac{x^2-y^2}{x^3}, -x < y < x$

To find conditional cdf, we simply take the integral of the conditional pdf

That is $F_{Y|X}(y|x) = \int_{-x}^{y} \frac{3}{4}\frac{x^2-y^2}{x^3}dy$

$= \frac{3}{4}\int_{-x}^{y} x^{-1} - \frac{y^2}{x^3}dy$

$= \frac{3}{4}[x^{-1}y - \frac{y^3}{3x^3}]_{-x}^{y}$

$= \frac{3}{4}[(yx^{-1} - \frac{y^3}{3x^3}) - (-1 + \frac{1}{3})] = \frac{3}{4}[\frac{y}{x} - \frac{y^3}{3x^3}) + \frac{2}{3}]$

$= \frac{3y}{4x} - \frac{y^3}{4x^3} + \frac{1}{2}$