10.1 1. For a real number a, verify that $(x^2 + a) = x^2 + 2a + a^2$

Done.

2. For a real number b, conclude that the polynomial $x^2 + bx + \frac{b}{4}$ is the square of a degree one polynomial

$x^2 + bx + \frac{b}{4} = (x + \frac{b}{2})^2$

3. For real numbers b, c, rewrite $x^2 + bx + c$ by adding and subtracting $\frac{b^2}{4}$ and find that solving the equation $x^2 + bx + c = 0$ is equivalent to solving an equation of the form $(x + \frac{b}{2})^2 = \frac{d}{4}$

$x^2 + bx + c = 0 \leftrightarrow (x + \frac{b}{2})^2 - \frac{b^2}{4} + c = 0$

$(x + \frac{b}{2})^2 = \frac{b^2}{4} - c$

So solving for $x^2 + bx + c = 0$ is the same as solving for $(x + \frac{b}{2})^2 = \frac{d}{4}$ for $d = b^2 - 4c$

4. Deduce that if $d = 0$, then $x^2 + bx + c$ factors as $(x + \frac{b}{2})$, and one solution to $x^2 + bx + c = 0$ is $x = -\frac{b}{2}$

We know that solving $x^2 + bx + c = 0$ is equivalent to solving for $(x + \frac{b}{2})^2 = \frac{d}{4}$. So when $d = 0$, $(x + \frac{b}{2})^2 = 0$, so it factors as $(x + \frac{b}{2})(x + \frac{b}{2})$, then one of the roots is $x = -\frac{b}{2}$

5. Deduce that if d is negative, then there is no real solution in $\mathbb{R}$ to the equation $x^2 + bx + c = 0$, and is irreducible in $\mathbb{R}[x]$

$(x + \frac{b}{2})^2 = \frac{d}{4}$, for d negative

So $(x + \frac{b}{2}) = \pm\sqrt{\frac{-d}{4}}i$.

So there are no real solutions, so it is irreducible in $\mathbb{R}[x]$

6. Deduce that if d is positive, then there are 2 real solutions to $x^2 + bx + c = 0$. Write them out explicitly in terms of b, c

$(x + \frac{b}{2})^2 = \frac{d}{4}$

$(x + \frac{b}{2}) = \pm\sqrt{\frac{d}{4}}$

$x = -\frac{b}{2} \pm \frac{\sqrt{d}}{2}$

So $x = -\frac{b}{2} \pm \frac{\sqrt{b^2-4c}}{2}$