4D $\mathcal{N}=2$ SCFT AND SINGULARITY THEORY PART IV: ISOLATED RATIONAL GORENSTEIN NON-COMPLETE INTERSECTION SINGULARITIES WITH AT LEAST ONE-DIMENSIONAL DEFORMATION AND NONTRIVIAL T^2

BINGYI CHEN, STEPHEN S.-T. YAU, SHING-TUNG YAU, AND HUAIQING ZUO

ABSTRACT. We study the miniversal deformations of minimally elliptic two-dimensional singularities of multiplicities of 5, 6 and 7. By restricting the miniversal deformations on the line transverse to the discriminant locus, we construct many new three-dimensional isolated rational Gorenstein singularities with one-dimensional equisingular deformation and nontrivial T^2. In fact the three-dimensional isolated rational Gorenstein singularities constructed from minimally elliptic singularities of multiplicity 5 has four-dimensional family of deformation, of which one-dimensional family is equisingular in the sense of Hilbert polynomial. On the other hand, the three-dimensional isolated rational Gorenstein singularities constructed from minimally elliptic singularities of multiplicity 6 and 7 respectively has nontrivial T^2 and has one-dimensional equisingular family of deformation. These singularities define many new interesting four dimensional $N=2$ superconformal field theories.

Keywords. rational singularity, Gorenstein singularity, deformation.
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1. INTRODUCTION

This is the fourth of a series of papers in which we try to classify four-dimensional $\mathcal{N}=2$ superconformal field theories (SCFTs) using classification of singularity.

In [XY], we conjecture that any three-dimensional isolated rational Gorenstein graded singularity should define a $\mathcal{N}=2$ SCFT. A complete list of hypersurface singularities was obtained in [YY], and this immediately gives us a large number of new four-dimensional $\mathcal{N}=2$ SCFTs.

Four-dimensional (4d) $\mathcal{N}=2$ superconformal field theory (SCFT) can be defined using type IIB string theory on following background

$$R^{1,3}\times X \tag{1}$$

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Here X is conjectured to be an isolated rational Gorenstein singularity [XY] with a good $\mathbb{C}^*$ action, and we take string coupling $g_s \to 0$ and go to infrared limit [SV, GKP]. These singularities naturally appear in the degeneration of compact Calabi-Yau three manifolds, and in fact general definition of Calabi-Yau variety allows such singularity [Gr].

4d $\mathcal{N}=2$ SCFT has a $SU(2)_R \times U(1)_R$ R symmetries, and there are two kinds of half-BPS operators $E_{r,(0,0)}$ and $\hat{B}_1$ [DO]. The Coulomb branch deformations are described as follows [ALLM]:

- Deformation using half-BPS operator $E_{r,(0,0)}$:

$$\delta S = \lambda \int d^4x dQ^4 E_{r,(0,0)} + c.c. \tag{2}$$

- Deformation using half-BPS operator $\hat{B}_1$:

$$\delta S = m \int d^4x Q^2 \hat{B}_1 + c.c. \tag{3}$$

- We can also turn on expectation value of operator $E_{r,(0,0)}$: $u_r = \langle E_{r,(0,0)} \rangle$.

A central question of understanding 4d $\mathcal{N}=2$ SCFT is to understand the low energy physics for general deformations parameterized by (λ, m, u_r). The low energy physics is best captured by the Seiberg-Witten geometry [SW]. Usually Seiberg-Witten geometry is described by a family of Riemann surface, and it is conjectured in [XY] that more general Coulomb branch geometry can be captured by the miniversal deformation [GLS] of a rational Gorenstein threefold singularity X with $\mathbb{C}^*$ action. Gorenstein means that there is a canonical nowhere zero holomorphic 3-form Ω defined outside the singular locus of X, and rational means that the weight of Ω under the $\mathbb{C}^*$ action is positive. Roughly speaking, a deformation is a flat morphism $\pi: Y \to S$, with $\pi^{-1}(0)$ isomorphic to the singularity X, and a miniversal deformation essentially captures all the deformations. Let $X \hookrightarrow \mathscr{X} \to S$ be a miniversal deformation of X, where $S \subset \mathbb{C}^\mu$ and μ is the dimension of T^1_X. If T^2_X is trivial then $S = \mathbb{C}^\mu$ (see the definitions of T^1_X and T^2_X in section 4). Let $\lambda_\alpha, \alpha = 1, \cdots, \mu$ be the coordinate of $\mathbb{C}^\mu$. The $\mathbb{C}^*$ action of X induce a $\mathbb{C}^*$ action on $\mathscr{X}$ and S, so is $\mathbb{C}^\mu$. The scaling dimension of λ_α is defined to be the ratio of the weight of λ_α to the weight of the canonical 3-form Ω under the $\mathbb{C}^*$ action. If the scaling dimension of λ_α larger than 1, then it is a Coulomb branch operator.

Therefore the study of 4d $\mathcal{N}=2$ SCFT is reduced to the study of singularity X and its miniversal deformation. We have classified such X which can be described by complete intersection [XY, YY, CX1] and by quotient singularity [CX2]. The physical aspects of these 4d $\mathcal{N}=2$ SCFTs are studied in [XY1, XY2, XY3, XYY, WX].

The purpose of this note is to study isolated rational Gorenstein non-complete intersection singularities and their deformations. One of the main results of this paper is to construct many new three-dimensional isolated rational Gorenstein non-complete intersection singularities with nontrivial T^1 and T^2. Therefore the corresponding $4d$ theory has Coulomb branch. On the other hand, all the complete intersection examples studied in [XY, YY, CX1] have a non-trivial miniversal deformation and therefore a non-trivial Coulomb branch, however they have trivial T^2.

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2. Preliminaries

Let $\pi\colon M \to V$ be a resolution of the normal two-dimensional Stein space V. We assume that p is the only singularity of V. Let $\pi^{-1}(p) = A = \cup A_i$, $1 \le i \le n$, be the decomposition of the exceptional set A into irreducible components.

A cycle $D = \Sigma d_i A_i$, $1 \le i \le n$ is an integral combination of the A_i, with d_i an integer. There is a natural partial ordering denoted by $<$, between cycles defined by comparing the coefficients. We let supp $D = \cup A_i$, $d_i \ne 0$, denote the support of D.

Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on M. Let $\mathcal{O}(-D)$ be the sheaf of germs of holomorphic functions on M which vanish to order d_i on A_i. Let $\mathcal{O}_D$ denote $\mathcal{O}/\mathcal{O}(-D)$. Define

$$\chi(D) := \dim H^0(M, \mathcal{O}_D) - \dim H^1(M, \mathcal{O}_D). \tag{2.1}$$

The Riemann-Roch theorem [Se, Proposition IV.4, p. 75] says

$$\chi(D) = -\frac{1}{2}(D^2 + D \cdot K), \tag{2.2}$$

where K is the canonical divisor on M. $D \cdot K$ may be defined as follows. Let ω be a meromorphic 2-form on M. Let (ω) be the divisor of ω. Then $D \cdot K = D \cdot (\omega)$ and this number is independent of the choice of ω. In fact, let g_i be the geometric genus of A_i, i.e., the genus of the desingularization of A_i. Then the adjunction formula [Se, Proposition IV, 5, p. 75] says

$$A_i \cdot K = -A_i^2 + 2g_i - 2 + 2\delta_i \tag{2.3}$$

where δ_i is the "number" of nodes and cusps on A_i. Each singular point on A_i other than a node or cusp counts as at least two nodes. It follows immediately from (2.2) that if B and C are cycles, then

$$\chi(B + C) = \chi(B) + \chi(C) - B \cdot C. \tag{2.4}$$

Definition 2.1. Associated to π a unique fundamental cycle Z [Ar, pp. 131-132] such that $Z > 0$, $A_i \cdot Z \leq 0$ all A_i and such that Z is minimal with respect to those two properties.

The fundamental cycle Z can be computed from the intersection as follows via a computation sequence for Z in the sense of Laufer [La1, Proposition 4.1, p. 607].

$$\begin{aligned} Z_0 = 0, Z_1 = A_{i_1}, Z_2 = Z_1 + A_{i_2}, \dots, Z_j = Z_{j-1} + A_{i_j}, \dots, \\ Z_\ell = Z_{\ell-1} + A_{i_\ell} = Z \end{aligned}$$

where A_{i_1} is arbitrary and $A_{i_j} \cdot Z_{j-1} > 0$, $1 < j \leq \ell$.

Lemma 2.1. [La2] *Let Z_k be part of a computation sequence for Z and such that $\chi(Z_k) = 0$. Then* $\dim H^1(M, \mathcal{O}_D) \leq 1$ *for all cycles D such that $0 \leq D \leq Z_k$. Also $\chi(D) \geq 0$.*

3. Minimally elliptic singularities

In this section we shall recall some of the properties of minimally elliptic singularities which we need for our construction.

Definition 3.1. A cycle $E > 0$ is *minimally elliptic* if $\chi(E) = 0$ and $\chi(D) > 0$ for all cycles D such that $0 < D < E$.

Wagreich [Wa] defined the singularity p to be elliptic if $\chi(D) \geq 0$ for all cycles $D \geq 0$ and $\chi(F) = 0$ for some cycles $F > 0$. He proved that this definition is independent of the resolution. It is easy to see that under this hypothesis, $\chi(Z) = 0$. The converse is also true [La2]. Henceforth, we shall adopt the following definition:

Definition 3.2. p is said to be weakly elliptic if $\chi(Z) = 0$.

The following proposition and lemma hold for weakly elliptic singularities.

Proposition 3.1. [La2] *Suppose that $\chi(D) \geq 0$ for all cycles $D > 0$. Let $B = \Sigma b_i A_i$ and $C = \Sigma c_i A_i$, $1 \leq i \leq n$, be any cycles such that $0 < B, C$ and $\chi(B) = \chi(C) = 0$. Let $F = \Sigma \min(b_i, c_i) A_i$, $1 \leq i \leq n$. Then $F > 0$ and $\chi(F) = 0$. In particular, there exists a unique minimally elliptic cycle E.*

Lemma 3.1. [La2] *Let E be a minimally elliptic cycle. Then for $A_i \subset \operatorname{supp} E$, $A_i \cdot E = -A_i \cdot K$. Suppose additionally that π is the minimal resolution. Then E is the fundamental cycle for the singularity having* supp E *as its exceptional set. Also, if E_k is part of a computation sequence for E as a fundamental cycle and $A_j \subset \operatorname{supp}\,(E - E_k)$, then the computation sequence may be continued past E_k so as to terminate at $E = E_\ell$ with $A_{i_\ell} = A_j$.*

Theorem 3.1. [La2] *Let $\pi\colon M \to V$ be the minimal resolution of the normal two-dimensional variety V with one singular point p. Let Z be the fundamental cycle on the exceptional set $A = \pi^{-1}(p)$. Then the following are equivalent:*

(1) *Z is a minimally elliptic cycle,*
(2) *$A_i \cdot Z = -A_i \cdot K$ for all irreducible components A_i in A,*
(3) *$\chi(Z) = 0$ and any connected proper subvariety of A is the exceptional set for a rational singularity.*

In [La2], Laufer introduced the notion of minimally elliptic singularity.

Definition 3.3. Let p be a normal two-dimensional singularity. p is minimally elliptic if the minimal resolution $\pi\colon M \to V$ of a neighborhood of p satisfies one of the conditions of Theorem 3.1.

Theorem 3.2. [La2] *Let V be a Stein normal two-dimensional space with p as its only singularity. Let $\pi\colon M \to V$ be a resolution of V. Then p is minimally elliptic singularity if and only if $H^1(M, \mathcal{O}) = \mathbb{C}$ and $\mathcal{O}_{V,p}$ is a Gorenstein ring.*

Theorem 3.3. [La2] *Let p be a minimally elliptic singularity. Let $\pi\colon M \to V$ be a resolution of a Stein neighborhood V of p with p as its only singular point. Let m be the maximal ideal in $\mathcal{O}_{V,p}$. Let Z be the fundamental cycle on $A = \pi^{-1}(p)$.*

(1) *If $Z^2 \le -2$, then $\mathcal{O}(-Z) = m\mathcal{O}$ on A.*
(2) *If $Z^2 = -1$, and π is the minimal resolution or the minimal resolution with non-singular A_i and normal crossings, $\mathcal{O}(-Z)/m\mathcal{O}$ is the structure sheaf for an embedded point.*
(3) *If $Z^2 = -1$ or -2, then p is a double point.*
(4) *If $Z^2 = -3$, then for all integers $n \ge 1$, $m^n \approx H^0\big(A, \mathcal{O}(-nZ)\big)$ and $\dim m^n/m^{n+1} = -nZ^2$.*
(5) *If $-3 \le Z^2 \le -1$, then p is a hypersurface singularity.*
(6) *If $Z^2 = -4$, then p is a complete intersection and in fact a tangential complete intersection.*

(7) *If $Z^2 \leq -5$, then p is a non-complete intersection.*

4. T^1 and T^2

For an affine scheme $Y = \mathrm{Spec}A$, there are two important A-modules, T^1_Y and T^2_Y. These modules play an important role in the deformation theory in analytic and algebraic geometry. In case Y admits a versal deformation, T^1_Y may be identified as the Zariski tangent space of the versal deformation; i.e. it is the space of infinitesimal deformations. T^2_Y contains the obstructions for extending deformations of Y to larger base spaces.

Let $Y \subseteq \mathbb{C}^{w+1}$ be given by equations $f_1, \cdots, f_m$, i.e. its ring of regular functions equals $A = P/I$ with $P = \mathbb{C}[z_0, \cdots, z_w]$ and $I = (f_1, \cdots, f_m)$. Then, using $d : I/I^2 \to A^{w+1}$ ($d(f_i) := (\frac{\partial f_i}{\partial z_0}, \cdots, \frac{\partial f_i}{\partial z_w})$), the vector space T^1_Y equals

$$T^1_Y = \mathrm{Hom}_A(I/I^2, A)/\mathrm{Hom}_A(A^{w+1}, A).$$

Let $\mathcal{R} \subseteq P^m$ denotes the P-module of relations between the equations $f_1, \cdots, f_m$. It contains the so-called Koszul relations $\mathcal{R}_0 :=< f_i e^j - f_j e^i >$ as a submodule.

Now, T^2_Y can be obtained as

$$T^2_Y = \mathrm{Hom}_P(\mathcal{R}/\mathcal{R}_0, A)/\mathrm{Hom}_P(P^m, A).$$

Remark 4.1. For isolated complete intersection singularities $\mathcal{R} = \mathcal{R}_0$, so $T^2 = 0$.

5. Self-intersection number $Z^2 = -5$

In this section, we will construct some three-dimensional isolated rational Gorenstein non-complete intersection singularities based on minimally elliptic singularities with $Z^2 = -5$.

Definition 5.1. Let (V, q) be a normal surface singularity. If the exceptional curve A of the minimal resolution of (V, q) is a nonsingular elliptic curve, then (V, q) is called a simple elliptic singularity.

A simple elliptic singularity is a minimally elliptic singularity, and hence a Gorenstein singularity.

Now consider a simple elliptic singularity with $A \cdot A = -5$. More explicitly, let A be a nonsingular elliptic curve, p a point on A and M the total space of the line bundle corresponding to the divisor $-5p$. A can be identified as the 0-section of M and its self-intersection number in M is -5. It's well known that A is exceptional in M, that is to say, there exists an analytic variety V and a proper map $\pi : M \to V$ such that $\pi(A)$ is a point q in V and the restriction of π on $M - A$ is biholomorphic. Then (V, q) is a two-dimensional simple elliptic singularity. Obviously the fundamental cycle of $\pi : M \to V$ is A. Since the self-intersection number of the fundamental cycle of (V, q) is -5, so it follows from Theorem 3.3 that (V, q) is a non-complete intersection singularity.

Next we will calculate the defining equations for (V, q). First we need a finite set of generators of $\Gamma(A, \mathcal{O})$, the ring of holomorphic functions which are defined on a neighbourhood of A in M. The sections of M can be identified with meromorphic functions on A which have a zero of order at least 5 at $p \in A$. The complex plane $\mathbb{C}$ is the universal covering of A and A can be identified as $\mathbb{C}/\Lambda$ where Λ is a lattice in $\mathbb{C}$. Let z be a coordinate of $\mathbb{C}$ such that $z = 0$ project onto p. Since the restriction of M to $A - q$ is a trivial bundle, we may let (z, t) be coordinates for M over $A - q$ and (z, t') coordinates for M near q, with z the coordinate for $\mathbb{C}$ and t, t' fibre coordinates. The transition functions are given by

$$\begin{aligned} t &= t' z^5, \\ z &= z. \end{aligned} \tag{5.5}$$

Given $f \in \Gamma(A, \mathcal{O})$, on the first chart it can be written as

$$f(t, z) = f_0(z) + t f_1(z) + t^2 f_2(z) + \cdots + t^i f_i(z) + \cdots .$$

On the second chart, by (5.5) it can be written as

$$f(t, z) = f_0(z) + t' z^5 f_1(z) + (t')^2 z^{10} f_2(z) + \cdots + (t')^i z^{5i} f_i(z) + \cdots .$$

Hence f_i are meromorphic functions on A which have poles of order at most $5i$ at $p \in A$ for each $i \geq 0$. In particular, f_0 is a constant function. Denote by S_i the set of doubly periodic meromorphic function f on $\mathbb{C}$ (i.e. $f(z) = f(z + \omega)$ for any $\omega \in \Lambda$) which is holomorphic outside the lattice points and has poles of order i at the lattice points (a zero of order k is viewed as a pole of order $-k$). Then f_i may be identified as an element in $\cup_{k \leq 5i} S_k$.

Let us recall some facts about the Weierstrass elliptic function. Suppose the lattice Λ is generated by ω_1 and ω_2, i.e, $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$. The Weierstrass's elliptic function $\mathfrak{p}$ is a meromorphic function on $\mathbb{C}$ with periods ω_1 and ω_2 defined as

$$\mathfrak{p}(z) = \frac{1}{z^2} + \sum_{n^2 + m^2 \neq 0} \left\{ \frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right\}.$$

Then $\mathfrak{p}$ is in S_2 and $\mathfrak{p}'$ (derived function of $\mathfrak{p}$) is in S_3. $\cup_{k<0} S_k$ and S_1 are empty. All elements in $\cup_{k \in \mathbb{Z}} S_k$ can be written as linear combinations of products of $\mathfrak{p}$ and $\mathfrak{p}'$. In a punctured neighborhood of the origin, the Laurent series expansion of $\mathfrak{p}(z)$ is

$$\mathfrak{p}(z) = z^{-2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6),$$

where

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} (m\omega_1 + n\omega_2)^{-4},$$

$$g_3 = 140 \sum_{(m,n)\neq(0,0)} (m\omega_1 + n\omega_2)^{-6}.$$

It's well known that $\mathfrak{p}(z)$ satisfies the following differential equation

$$(\mathfrak{p}')^2 - 4(\mathfrak{p})^3 + g_2\mathfrak{p} + g_3 = 0. \tag{5.6}$$

Proposition 5.1. *For any positive integer i and any element f in $\cup_{k\leq 5i} S_k$, $t^i f(z)$ can be written as a linear combination of products of the following five functions*

$$t, \quad t\mathfrak{p}(z), \quad t\mathfrak{p}'(z), \quad t\mathfrak{p}^2(z), \quad t\mathfrak{p}(z)\mathfrak{p}'(z). \tag{5.7}$$

Proof. We prove the claim by induction on the order d of poles of $f(z)$ at the lattices points (d can not be 1 and $d \leq 5i$). If $d = 0$, then f is a constant, then the claim holds. Assume the claim holds for $0, 2, 3, \cdots, d-1$. Write $d = 5k+r$ where k, r are non-negative integers and $0 < r \leq 5$. Since $d \leq 5i$, we have $k \leq i - 1$. There are two cases.

a) $r \neq 1$, then there is an element e in $\{\mathfrak{p}(z), \mathfrak{p}'(z), \mathfrak{p}^2(z), \mathfrak{p}(z)\mathfrak{p}'(z)\}$ such that $e(\mathfrak{p}(z)\mathfrak{p}'(z))^k$ has poles of order d at the lattice points. Hence $f(z) - ce\big(\mathfrak{p}(z)\mathfrak{p}'(z)\big)^k$ has poles of order less than d at the lattice points for some coefficient c. By inductive assumption,

$$t^i f(z) - c\big(te\big)\big(t\mathfrak{p}(z)\mathfrak{p}'(z)\big)^k t^{i-k-1}$$

can be generated by (5.7), hence the claim holds.

b) $r = 1$. Then $k \geq 1$ since $d \neq 1$. $f(z) - c\big(\mathfrak{p}(z)\big)^3\big(\mathfrak{p}(z)\mathfrak{p}'(z)\big)^{k-1}$ has poles of order less than d at the lattice points for some coefficient c. By inductive assumption,

$$t^i f(z) - c\big(t\mathfrak{p}(z)\big)\big(t\mathfrak{p}^2(z)\big)\big(t\mathfrak{p}(z)\mathfrak{p}'(z)\big)^{k-1} t^{i-k-1}$$

can be generated by (5.7), hence the claim holds. $\square$

By the above claim, we have $\Gamma(A, \mathcal{O})$ is generated by (5.7). Next we need to explore relations of the generators. Consider a homomorphism

$$\rho : \mathbb{C}[e_1, e_2, e_3, e_4, e_5] \to \Gamma(A, \mathcal{O})$$

$$e_1 \mapsto t, \qquad e_2 \mapsto t\mathfrak{p}(z), \qquad e_3 \mapsto t\mathfrak{p}'(z),$$

$$e_4 \mapsto t\mathfrak{p}^2(z), \qquad e_5 \mapsto t\mathfrak{p}(z)\mathfrak{p}'(z).$$

Proposition 5.2. *The kernel of ρ is generated by*

$$\begin{aligned}
&e_1e_5 - e_2e_3,\\
&e_1e_4 - e_2^2,\\
&e_2e_5 - e_3e_4, \qquad\qquad (5.8)\\
&e_3^2 - 4e_2e_4 + g_2e_1e_2 + g_3e_1^2,\\
&e_3e_5 - 4e_4^2 + g_2e_2^2 + g_3e_1e_2.
\end{aligned}$$

Proof. By equation (5.6), it is easy to check that elements in (5.8) are in the kernel of ρ. Denote by I the ideal generated by (5.8). Given $f \in \mathrm{Ker}(\rho)$, we need to prove $f \in I$. Write $f = \sum_{i=0}^{n} f_i$ where f_i is a homogeneous polynomial of degree i. Since $\rho(f) = \sum_{i=0}^{n} \rho(f_i) = 0$ and $t^i \mid \rho(f_i)$ and $t^{i+1} \nmid \rho(f_i)$, we have $\rho(f_i) = 0$ for each i. Hence we may assume that f is a homogeneous polynomial of degree n without loss of generality. Associated to each variable e_i a weight $wt(e_i)$, which is equal to the order of poles at the lattice points of the periodic meromorphic function $\rho(e_i)/t$. Then

$$wt(e_1) = 0 \quad wt(e_2) = 2 \quad wt(e_3) = 3 \quad wt(e_4) = 4 \quad wt(e_5) = 5. \qquad (5.9)$$

We may write $f = \sum_{i=0}^{m} f_i$ where f_i is weighted homogeneous of weight i with respect to the weight system (5.9). Since $\rho(f_i)/t^n$ has poles of degree i at the lattice point, $\{\rho(f_i)\}_{i=1}^m$ is linear independent. Hence $\rho(f) = \sum_{i=0}^{n} \rho(f_i) = 0$ implies that $\rho(f_i) = 0$ for each i. Hence we may assume that f is homogeneous of degree n and weighted homogeneous of weight m without loss of generality.

We will prove the claim by induction on the weight m. If $m = 0$, then $f = ce_1^n$ for some coefficient c. Since $\rho(f) = ct^n = 0$ in $\Gamma(A, \mathcal{O})$, we have $c = 0$ and $f = 0$, hence the claim holds. Assume the claim holds for $0, 1, 2, \ldots, m-1$. There are two cases.

a) Each term h of f satisfies the following condition

$$(*) \quad \text{the support of } h \text{ is } \{e_i, e_{i+1}\} \text{ or } \{e_i\} \text{ for some } i$$

here the support of a monomial means the set of variables which appear in the monomial. Since f is homogeneous and $wt(e_i) < wt(e_{i+1})$ for any i, the weights of terms in f are pairwise distinct. And since f is weighted homogeneous, f has only one term, hence $\rho(f) = 0$ implies $f = 0$.

b) There exists a term h in f which does not satisfy Condition $(*)$, for example, $h = e_3e_5\widetilde{h}$, then $h - 4e_4^2\widetilde{h} + (g_2e_2^2 + g_3e_1e_2)\widetilde{h} \in I$ (see the fifth equation in (5.8)). The weights of terms in $(g_2e_2^2 + g_3e_1e_2)\widetilde{h}$ are less than $wt(f) = m$. From this example, it's not hard to see that there exist polymonomials f_1, f_2 such that

1) $f - f_1 - f_2 \in I$;

2) each term in f_1 satisfies condition $(*)$;

3) the weights of terms in f_2 are less than $wt(f) = m$.

By the inductive assumption, $f_2 \in I$ and by case (a) $f_1 = 0$, hence $f \in I$. $\square$

By Proposition 5.1 and Proposition 5.2, we have that $(V, q) \hookrightarrow \mathbb{C}^5$ is defined by

$$\begin{aligned} &xv - yz = 0, \\ &xu - y^2 = 0, \\ &yv - zu = 0, \\ &z^2 - 4yu + g_2xy + g_3x^2 = 0, \\ &zv - 4u^2 + g_2y^2 + g_3xy = 0. \end{aligned} \tag{5.10}$$

where x, y, z, u, v are coordinates of $\mathbb{C}^5$ and g_2, g_3 depend on the elliptic curve A.

The next step is to calculate the miniversal deformation of (V, q) for general g_2, g_3 using Singular ([DGPS] a computer algebra system for polynomial computations). By calculation, the miniversal deformation (V, q) is defined as follows. Let $\mathscr{X} \subseteq \mathbb{C}^5 \times \mathbb{C}^6$ be the subvariety defined by

$$\begin{aligned} &-yz + xv - Az - g_2Cx + 4Cu + Eyv + Fy + AEv + AF = 0, \\ &-y^2 + xu - Ay + Cz + Dx = 0, \\ &-zu + yv + g_3Cx - Dz + Euv + Fu + BC + DEv + DF = 0, \\ &g_3x^2 + g_2xy + z^2 - 4yu + Bx - Ezv - Fz = 0, \\ &g_3xy + g_2y^2 - 4u^2 + zv + g_3Ax + g_2Ay + By - g_2Cz - 4Du + AB = 0. \end{aligned} \tag{5.11}$$

where x, y, z, u, v are coordinates of $\mathbb{C}^5$ and A, B, C, D, E, F are coordinated of the base space $\mathbb{C}^6$. Then $\phi : (\mathscr{X}, 0) \to (\mathbb{C}^6, 0)$ is the miniversal deformation of (V, q) for general g_2, g_3 ("general" means that there exists a Zariski open set $U \subset \mathbb{C}^2$ such that if $(g_2, g_3) \in U$ then the above statement holds), where ϕ is induced by the projection $\mathbb{C}^5 \times \mathbb{C}^6 \to \mathbb{C}^6$.

Consider a line $i : l \hookrightarrow \mathbb{C}^6$ in the base space, which is given by $C = s$ and $A = B = D = E = F = 0$. Restrict the deformation to the line l, then we get a three-dimensional germ $(X, 0) \subseteq (\mathbb{C}^6, 0)$ which is defined by

$$\begin{aligned} &f_1 = -yz + xv - g_2sx + 4su = 0, \\ &f_2 = -y^2 + xu + sz = 0, \\ &f_3 = -zu + yv + g_3sx = 0, \\ &f_4 = g_3x^2 + g_2xy + z^2 - 4yu = 0, \\ &f_5 = g_3xy + g_2y^2 - 4u^2 + zv - g_2sz = 0. \end{aligned} \tag{5.12}$$

where x, y, z, u, v, s are coordinates of $\mathbb{C}^6$ and g_2, g_3 depend on the ellptic curve A.

Using SINGULAR, we can check that for general g_2, g_3, $(X, 0)$ is an isolated singularity.

$\dim(V, q) = \dim(X, 0) - 1$ and $(V, q) = (X, 0) \cap \{s = 0\}$, hence (V, q) is non-complete intersection implies that $(X, 0)$ is non-complete intersection.

The following theorem tells us that $(X, 0)$ is Gorenstein.

Theorem 5.1. [Is] *Let $\pi : (\mathscr{X}, x) \to (C, 0)$ be a one-parameter deformation of a Gorenstein singularity (X, x), then $(\mathscr{X}, x)$ is a Gorenstein singularity.*

Finally we need to check that $(X, 0)$ is a rational singularity for general g_2, g_3. Since $(X, 0)$ is a homogeneous Gorenstein singularity, there is a homogeneous nowhere zero holomorphic 3-form on $X - \{0\}$ near 0. In order to prove that $(X, 0)$ is rational, we only need to check that the degree of the 3-form is positive.

We will construct a nowhere zero holomorphic 3-form on $X - \{0\}$. We firstly recall some notations. Let $S \hookrightarrow \mathbf{A}_k^N = Spec(k[x_1, \cdots, x_N])$ be a reduced subscheme of pure dimension n, where k is a field. Let $f_1, \cdots, f_d$ be generators of the ideal I_S of S. Consider $F_1, \cdots, F_d$ with $F_i = \sum_{j=1}^{d} a_{i,j} f_j$ for general $a_{i,j} \in k$ such that the following three conditions are satisfied. Denote by T the scheme defined by the ideal $I_T = (F_1, \cdots, F_r)$ where $r = N - n$.

1) Each irreducible component of T has dimension n, so T is complete intersection.

2) S is a closed subscheme of T and $S = T$ at the generic point of every irreducible component of S.

3) Some minor Δ of the Jacobian matrix of $F_1, \cdots, F_r$ does not vanish at the generic point of any irreducible component of S. Without loss of generality, we may assume that $\Delta = \det(\partial F_i / x_j)_{i,j \le r}$.

Let ω_S (resp. ω_T) be the canonical sheaf of S (resp. T). Note that the stalk of ω_S at the generic point of an irreducible component Z of S is $\Omega^n_{K_Z/k}$, where K_Z is the residue field at the generic point of Z. Denote by K the product of the residue fields of the generic points of the irreducible components of S. Then we have the localization map $\omega_X \to \Omega^n_{K/k}$.

Proposition 5.3. [EM] *With the above notation, there are canonical morphisms $u : \omega_S \to \omega_T \mid_S$ and $w : \omega_T \mid_S \to \Omega^n_{K/k}$ with the following properties:*

a) w is injective and identifies $\omega_T \mid_S$ with $\mathcal{O}_S \cdot \Delta^{-1} dx_{r+1} \wedge \cdots \wedge dx_N$.

b) u is injective and the image of $w \circ u$ is $((I_T : I_S) + I_S)/I_S \cdot \Delta^{-1} dx_{r+1} \wedge \cdots \wedge dx_N$.

c) $w \circ u$ is the localization map.

Now we look at the three dimension singularity $(X, 0)$ we have constructed, which is defined by the ideal $I_X = (f_1, f_2, \cdots, f_5)$ in (5.12). Let U be the close subscheme of $\mathbb{C}^6$ defined by ideal $I_U = (f_3, f_4, f_5)$ then X is a close subvariety of U. By using SINGULAR, we compute the dimension of $(U, 0)$ is 3 for general g_2, g_3, hence $(U, 0)$ is complete intersection for general g_2, g_3. Using MAPLE, we compute $(I_U : I_X)$ and find that $((I_U : I_X) + I_X)/I_X$ is generated by one element $g_2y + g_3x$ as an ideal of $\mathbb{C}[x, y, z, u, v, s]/I_X$. Let

$$\Delta = \det \begin{pmatrix} \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z \\ \partial f_4/\partial x & \partial f_4/\partial y & \partial f_4/\partial z \\ \partial f_5/\partial x & \partial f_5/\partial y & \partial f_5/\partial z \end{pmatrix}.$$

It can be checked that $\Delta - (g_2y + g_3x)(g_2sv - 2g_2uy - 3g_3y^2 - v^2) \in I_X$. By Proposition 5.3,

$$\frac{du \wedge dv \wedge ds}{g_2sv - 2g_2uy - 3g_3y^2 - v^2}$$

is a nowhere zero 3-form on $X - 0$ near 0. Since the degree of the 3-form is positive, $(X, 0)$ is a rational singularity. Hence $(X, 0)$ is a three-dimensional isolated homogeneous rational Gorenstein singularity which is non-complete intersection for general g_2, g_3.

Using SINGULAR, we calculate the multiplicity of $(X, 0)$ which is 5 for general parameters g_2, g_3. We compute the miniversal deformation of $(X, 0)$ for general g_2, g_3. By calculation, for general g_2, g_3, $\dim T^1 = 4$, $\dim T^2 = 0$ and the miniversal deformation of (V, q) is

$$\pi : (\mathscr{X}, 0) \to (\mathbb{C}^4, 0).$$

The total space $\mathscr{X}$ is a subvariety of $\mathbb{C}^6 \times \mathbb{C}^4$ which is defined by

$$\begin{aligned} &- yz + xv - g_2xs + 4us - Az + Dy + AD = 0, \\ &- y^2 + xu + zs - Ay + Cx = 0, \\ &- zu + yv + g_3xs + Bs - Cz + Du + CD = 0, \\ &g_3x^2 + g_2xy + z^2 - 4yu + Bx - Dz = 0, \\ &g_3xy + g_2y^2 - 4u^2 + zv - g_2zs + g_3Ax + g_2Ay + By - 4Cu \\ &+ AB = 0, \end{aligned} \tag{5.13}$$

where x, y, z, u, v, s are coordinates of $\mathbb{C}^6$ and A, B, C, D are coordinates of $\mathbb{C}^4$. π is induced by the projection $\mathbb{C}^6 \times \mathbb{C}^4 \to \mathbb{C}^4$.

6. SELF-INTERSECTION NUMBER $Z^2 = -6$

This section we consider a simple elliptic singularity with $Z^2 = A \cdot A = -6$. Let A be a smooth elliptic curve, p a point on A and M the total space of the line bundle corresponding to the divisor $-6p$. Collapse A to a point, we get a simple elliptic singularity

(V,q). By Theorem 3.3, we know that it is non-complete intersection. Similarly as Section 5, we can prove that $\Gamma(A,\mathcal{O})$ is generated by 6 elements

$$\begin{aligned}
&x = t, \quad y = t\mathfrak{p}(z), \quad z = t\mathfrak{p}'(z), \quad u = t\mathfrak{p}^2(z), \\
&v = t\mathfrak{p}(z)\mathfrak{p}'(z), \quad w = t(\mathfrak{p}'(z))^2.
\end{aligned}$$

And their relations are generated by

$$\begin{aligned}
&xw - z^2, \\
&xv - yz, \\
&xu - y^2, \\
&yw - zv, \\
&yv - zu, \\
&z^2 - 4yu + g_2xy + g_3x^2, \\
&zw - 4uv + g_2yz + g_3xz, \\
&zv - 4u^2 + g_2y^2 + g_3xy, \\
&uw - v^2.
\end{aligned} \tag{6.14}$$

Hence $(V,q) \subseteq \mathbb{C}^6$ is defined by equations in (6.14). By using Singular, we compute the miniversal deformation of (V,q) for general g_2, g_3. The miniversal deformation of (V,q) is

$$\pi : (\mathscr{X}, 0) \to (\mathscr{B}, 0).$$

The base space $\mathscr{B} \subseteq \mathbb{C}^7$ is defined by

$$\begin{aligned}
&4B^2 - g_2CF - g_2F^2 + 4BG + 16g_2ABC + 8g_2ABF + \\
&\qquad 8g_2ACG + 4g_2AFG + 16g_2^2A^2C^2 + 16g_2^2A^2CF + 4g_2^2A^2F^2 = 0, \\
&BD + -g_2CE + -g_2EF +^2 g_2ACD + g_2ADF = 0, \\
&-4BE + DF - 4EG + -8g_2ACE + -4g_2AEF = 0,
\end{aligned} \tag{6.15}$$

where A, B, C, D, E, F, G are coordinates of $\mathbb{C}^7$. The total space $\mathscr{X}$ is a subvariety of $\mathbb{C}^6 \times \mathbb{C}^7$ (its defining equations are too complicate to list here) and π is induced by the projection $\mathbb{C}^6 \times \mathbb{C}^7 \to \mathbb{C}^7$.

Consider a line $i : l \hookrightarrow \mathscr{B}$ in the base space, which is given by $C = -F = s$ and $A = B = D = E = G = 0$. Restrict the deformation to the line l, then we get a three-dimensional germ $(X,0) \subseteq \mathbb{C}^7$ which is defined by the ideal $I_X = (f_1, f_2, \cdots, f_9)$, where

$$
\begin{aligned}
f_1 &= -4su + wx - z^2, \\
f_2 &= vx - sz - yz, \\
f_3 &= ux - sy - y^2, \\
f_4 &= -g_3sx - g_2sy + wy - vz, \\
f_5 &= vy - uz, \\
f_6 &= g_3x^2 - 4uy + g_2xy + z^2, \\
f_7 &= -4uv + wz + g_3xz + g_2yz, \\
f_8 &= -4u2 + g_3sx + g_2sy + g_3xy + g_2y^2 + vz, \\
f_9 &= -g_3s^2 - g_2su - v^2 + uw - g_3sy.
\end{aligned}
\tag{6.16}
$$

where x, y, z, u, v, w, s are coordinates of $\mathbb{C}^7$.

Using Singular, we can check that for general g_2, g_3, $(X, 0)$ is an isolated singularity. Theorem 5.1 tells us that $(X, 0)$ is Gorenstein. (V, q) is non-complete intersection implies that $(X, 0)$ is non-complete intersection.

Let U be the close subscheme of $\mathbb{C}^7$ defined by ideal $I_U = (f_1, f_2, f_3, f_4)$ then X is a close subvariety of U. By using Singular, we compute the dimension of U is 3 for general g_2, g_3, hence U is complete intersection for general g_2, g_3. By using Maple, we find that $((I_U : I_X) + I_X)/I_X$ is generated by one element sx as an ideal of $\mathbb{C}[x, y, z, u, v, w, s]/I_X$. Let

$$
\Delta = \det \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z & \partial f_1/\partial u \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z & \partial f_2/\partial u \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z & \partial f_3/\partial u \\ \partial f_4/\partial x & \partial f_4/\partial y & \partial f_4/\partial z & \partial f_4/\partial u. \end{pmatrix}.
$$

It can be checked that $\Delta - (sx)(g_2sw - 2g_2vz - 3g_3z^2 - w^2) \in I_X$. By Proposition 5.3,

$$
\frac{dv \wedge dw \wedge ds}{g_2sw - 2g_2vz - 3g_3z^2 - w^2}
$$

is a nowhere zero 3-form on $X - \{0\}$ near 0. Since the degree of the 3-form is positive, so $(X, 0)$ is a rational singularity. Hence $(X, 0)$ is a three-dimensional isolated homogeneous rational Gorenstein singularity which is non-complete intersection for general g_2, g_3.

We use Singular to compute T^1 and T^2 of $(X, 0)$ and find that dim $T^1 = 3$ and dim $T^2 = 2$ for general g_2, g_3.

Using Singular, we calculate the multiplicity of $(X, 0)$ which is 6 for general parameters g_2, g_3. We compute the miniversal deformation of $(X, 0)$ for general g_2, g_3. By calculation, for general g_2, g_3, dim $T^1 = 3$, dim $T^2 = 2$ and the miniversal deformation of

(V, q) is

$$\pi : (\mathscr{X}, 0) \to (\mathscr{B}, 0).$$

The total space $\mathscr{X}$ is a subvariety of $\mathbb{C}^7 \times \mathbb{C}^3$ which is defined by

$$\begin{aligned}
&- z^2 + xw - 4us + 4Ay + 4As + 4Bv = 0, \\
&- yz + xv - zs - g_2 Bx + 4Bu = 0, \\
&- y^2 + xu - ys + Bz + Cx = 0, \\
&- zv + yw - g_3 xs - g_2 ys + 4Au + g_2 Bz - 4g_3 B^2 + 4AC = 0, \\
&- zu + yv + g_3 Bx - Cz = 0, \\
&g_3 x^2 + g_2 xy + z^2 - 4yu = 0, \\
&g_3 xz + g_2 yz - 4uv + zw + 4Av - 4g_3 By - 4g_3 Bs = 0, \\
&g_3 xy + g_2 y^2 - 4u^2 + zv + g_3 xs + g_2 ys - g_2 Bz - 4Cu = 0, \\
&- v^2 + uw - g_3 ys - g_2 us - g_3 s^2 + g_3 Ax + g_2 Ay + Aw \\
&- g_3 Bz + g_2 Bv + Cw - g_2 Cs = 0,
\end{aligned} \tag{6.17}$$

where x, y, z, u, v, w, s are coordinates of $\mathbb{C}^7$ and A, B, C are coordinates of $\mathbb{C}^3$. The base space $\mathscr{X} \subseteq \mathbb{C}^3$ is defined by

$$\begin{aligned}
&A^2 + AC = 0, \\
&AB + BC = 0,
\end{aligned} \tag{6.18}$$

and π is induced by the projection $\mathbb{C}^7 \times \mathbb{C}^3 \to \mathbb{C}^3$.

7. SELF-INTERSECTION NUMBER $Z^2 = -7$

This section we consider a simple elliptic singularity with $Z^2 = A \cdot A = -7$. Let A be a smooth elliptic curve, p a point on A and M the total space of the line bundle corresponding to the divisor $-7p$. Collapse A to a point, then we get a simple elliptic singularity (V, q). By Theorem 3.3, we know that it is non-complete intersection. Similarly as in Section 5, we can prove that $\Gamma(A, \mathcal{O})$ is generated by 7 elements

$$\begin{aligned}
&x = t, \quad y = t\mathfrak{p}(z), \quad z = t\mathfrak{p}'(z), \quad u = t\mathfrak{p}^2(z), \\
&v = t\mathfrak{p}(z)\mathfrak{p}'(z), \quad w = t(\mathfrak{p}'(z))^2, \quad r = t\mathfrak{p}^2(z)\mathfrak{p}'(z).
\end{aligned}$$

And their relations are generated by

$$\begin{aligned}
& xw - z^2. \\
& xv - yz. \\
& xu - y^2, \\
& yw - zv, \\
& yv - zu, \\
& z^2 - 4yu + g_2 xy + g_3 x^2, \\
& zw - 4uv + g_2 yz + g_3 xz, \\
& zv - 4u^2 + g_2 y^2 + g_3 xy, \\
& uw - v^2, \\
& xr - yv, \\
& yr - uv, \\
& zr - uw, \\
& vw - 4ur + g_2 yv + g_3 xv, \\
& w^2 - 4vr + g_2 yw + g_3 xw.
\end{aligned} \tag{7.19}$$

Hence $(V, q) \subseteq \mathbb{C}^7$ is defined by equations in (7.19). Unfortunately, the program to calculate its universal deformation for general g_2, g_3 takes too long time because it has too many defining equations, so we only calculate for the special case that $g_2 = 1$ and $g_3 = 0$. By calculation, the miniversal deformation of (V, q) (for $g_2 = 1$ and $g_3 = 0$) is

$$\pi : (\mathscr{X}, 0) \to (\mathscr{B}, 0).$$

The base space $\mathscr{B} \subseteq \mathbb{C}^7$ is defined by

$$\begin{aligned}
& 4A^2 + 8AF + 4F^2 - 4CG - 8EG - DH + BE^2 = 0, \\
& -4AG - 4FG - CH - EH + BDG = 0, \\
& AE - DG + GH = 0, \\
& AD + CE + 2E^2 + DF - AH - FH - BEG = 0, \\
& 2AC + 3AE + CF + EF + GH - ABG - BFG = 0, \\
& -AD + CE + E^2 + 4G^2 - BEG = 0, \\
& A^2 + AF + CG + 2EG - BG^2 = 0.
\end{aligned} \tag{7.20}$$

where A, B, C, D, E, F, G, H are coordinates of $\mathbb{C}^8$. The total space $\mathscr{X}$ is a subvariety of $\mathbb{C}^7 \times \mathbb{C}^8$ (its defining equations are too complicate to list here) and ϕ is induced by the projection $\mathbb{C}^7 \times \mathbb{C}^8 \to \mathbb{C}^8$.

Consider a line $i : l \hookrightarrow \mathscr{B}$ in the base space, which is given by $C = s$ and $A = B = D = E = F = G = H = 0$. Restrict the deformation to the line l, then we get a three-dimensional germ $(X, 0) \in \mathbb{C}^8$ which is defined by the ideal $I_X = (f_1, f_2, \cdots, f_{14})$, where

$$\begin{aligned}
f_1 &= -4sv + wx - z^2, \\
f_2 &= -4su + sx + vx - yz, \\
f_3 &= ux - y^2 - sz, \\
f_4 &= wy - sz - vz, \\
f_5 &= vy - uz, \\
f_6 &= -4uy + xy + z^2, \\
f_7 &= -4uv + wz + yz, \\
f_8 &= -4u^2 + y^2 + sz + vz, \\
f_9 &= -sv - v^2 + uw, \\
f_{10} &= -sw + rx - sy - vy, \\
f_{11} &= -su - uv + ry, \\
f_{12} &= -uw + rz, \\
f_{13} &= -4ru + sw + vw + sy + vy, \\
f_{14} &= -4rv + w^2 + wy.
\end{aligned} \tag{7.21}$$

where x, y, z, u, v, w, r, s are coordinates of $\mathbb{C}^8$. Using SINGULAR, we can check that $(X, 0)$ is an isolated singularity. Theorem 5.1 tells us that $(X, 0)$ is Gorenstein. (V, q) is non-complete intersection implies that $(X, 0)$ is non-complete intersection.

Let U be the close subscheme of $\mathbb{C}^8$ defined by ideal $I_U = (f_1, f_2, f_3, f_4, f_{14})$ then X is a close subvariety of U. Using SINGULAR, we compute the dimension of U is 3, hence U is complete intersection. Using MAPLE, we find that $((I_U : I_X) + I_X)/I_X$ is generated by one element $swx - sz^2$ as an ideal of $\mathbb{C}[x, y, z, u, v, w, r, s]/I_X$. Let

$$\Delta = \det \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z & \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z & \partial f_2/\partial u & \partial f_2/\partial v \\ \partial f_3/\partial x & \partial f_3/\partial y & \partial f_3/\partial z & \partial f_3/\partial u & \partial f_3/\partial v \\ \partial f_4/\partial x & \partial f_4/\partial y & \partial f_4/\partial z & \partial f_4/\partial u & \partial f_4/\partial v \\ \partial f_{14}/\partial x & \partial f_{14}/\partial y & \partial f_{14}/\partial z & \partial f_{14}/\partial u & \partial f_{14}/\partial v \end{pmatrix}.$$

It can be checked that $\Delta - (16r^2 - 4s^2 + 4v^2 - 16rz)(swx - sz^2) \in I_X$. By Proposition 5.3,

$$\frac{dw \wedge dr \wedge ds}{16r^2 - 4s^2 + 4v^2 - 16rz}$$

is a nowhere zero 3-form on $X - \{0\}$ near 0. Since the degree of the 3-form is positive, so $(X, 0)$ is a rational singularity. Hence $(X, 0)$, for $g_2 = 1$ and $g_3 = 0$, is a three-dimensional isolated homogeneous rational Gorenstein singularity which is non-complete intersection.

With the help of Singular, we compute the semi-universal deformation of $(X, 0)$ for $g_2 = 1$ and $g_3 = 0$. By calculation, dim $T^1 = 2$, dim $T^2 = 4$ and the miniversal deformation of (V, q) is

$$\pi : (\mathscr{X}, 0) \to (\mathscr{B}, 0).$$

The total space $\mathscr{X}$ is a subvariety of $\mathbb{C}^8 \times \mathbb{C}^2$ which is defined by

$$\begin{aligned}
&-z^2 + xw - 4vs - 4Ay + Bx - 4Bu = 0,\\
&-yz + xv + xs - 4us - Bz = 0,\\
&-y^2 + xu - zs + Ax - By = 0,\\
&-zv + yw - zs - 4Au = 0,\\
&-zu + yv - Az = 0,\\
&xy + z^2 - 4yu = 0,\\
&yz - 4uv + zw - 4Av + Bz = 0,\\
&y^2 - 4u^2 + zv + zs - 4Au + By = 0,\\
&-v^2 + uw - vs - Ay = 0,\\
&-yv + xr - ys - ws - Az - Bv - Bs = 0,\\
&-uv + yr - us - 2Av - 2As = 0,\\
&-uw + zr + Ay - Aw = 0,\\
&yv + vw - 4ur + ys + ws + Bv + Bs = 0,\\
&yw + w^2 - 4vr - 4Au + Bw = 0,
\end{aligned} \tag{7.22}$$

where x, y, z, u, v, w, r, s are coordinates of $\mathbb{C}^8$ and A, B are coordinates of $\mathbb{C}^2$. The base space $\mathscr{X} \subseteq \mathbb{C}^2$ is defined by

$$\begin{aligned}
AB &= 0,\\
A^2 &= 0,
\end{aligned} \tag{7.23}$$

and π is induced by the projection $\mathbb{C}^8 \times \mathbb{C}^2 \to \mathbb{C}^2$.

References

[Ar] M. Artin, *On isolated rational singularities of surfaces*, Amer. J. Math. **88** (1996), 129–136.

[ALLM] P. Argyres, M. Lotito, Y. L and M. Martone, *Geometric constraints on the space of N=2 SCFTs I: physical constraints on relevant deformations,* arXiv:1505.04814 [hep-th].

[CX1] B. Chen, D. Xie, S. T. Yau, S. S.-T. Yau and H. Zuo, *4d $\mathcal{N}=2$ SCFT and singularity theory Part II: complete intersection,* Adv. Theor. Math. Phys. **21** (2017), no. 1, 121-145.

[CX2] B. Chen, D. Xie, S. S.-T. Yau, S. T. Yau and H. Zuo, *4d $\mathcal{N}=2$ SCFT and singularity theory Part III: rigid singularity,* [arXiv:1712.00464v1 [hep-th]].

[DGPS] W. Decker, G.-M. Greuel, and G. Pfister, H. Schönemann, Singular 4-1-0 – A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2016).

[DO] F. A. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry,* Annals Phys. **307**, 41 (2003) [hep-th/0209056].

[EM] L. Ein and M. Mustata, *Jet schemes and singularities,* Algebraic Geometry-Seattle 2005. Part 2, 505-546, Proc. Sympos. Pure Math., **80**, Part 2, Amer. Math. Soc., Providence, RI, 2009.

[Gr] M. Gross, *Primitive Calabi-Yau threefolds,* Journal of Differential Geometry **45** (1997), no.2, 288-318.

[GKP] A. Giveon, D. Kutasov and O. Pelc, *Holography for noncritical superstrings,* JHEP **9910**, 035 (1999) [hep-th/9907178].

[GLS] G.M. Greuel, , C. Lossen, and E.I. Shustin, *Introduction to singularities and deformations,* Springer Science & Business Media, 2007.

[Is] S. Ishii, *Introduction to singularities,* Springer, Tokyo, 2014. viii+223 pp. ISBN: 978-4-431-55080-8; 978-4-431-55081-5.

[La1] H. Laufer, *On rational singularities,* Amer. J. Math. **94** (1972), 597–608.

[La2] H. Laufer, *On minimally elliptic singularities,* Amer. J. Math. **99** (1977), 1257–1295.

[Se] J. P. Serre, *Groupes algebriques et corps de classes,* Actualités Scientifiques et Industrielles, no. 1264, Hermann, Paris, 1959.

[SV] A. D. Shapere and C. Vafa, *BPS structure of Argyres-Douglas superconformal theories,* hep-th/9910182.

[SW] N. Seiberg and E. Witten, *Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,* Nucl. Phys. B **426**, 19 (1994) Erratum: [Nucl. Phys. B **430**, 485 (1994)] [hep-th/9407087].

[Wa] P. Wagreich, *Elliptic singularities of surfaces*, Amer. J. Math. **92** (1970), 419–454.

[WX] Y. Wang, D. Xie, S. S.-T. Yau and S. T. Yau, *4d N=2 SCFT from Complete Intersection Singularity,* Adv. Theor. Math. Phys. **21**, 801 (2017) [arXiv:1606.06306 [hep-th]].

[XY] D. Xie and S. T. Yau, *4d $\mathcal{N}=2$ SCFT and singularity theory Part I: classification,* preprint, arXiv:math/1510.01324, (2015).

[XY1] D. Xie and S. T. Yau, *Semicontinuity of 4d $\mathcal{N}=2$ spectrum under renormalization group flow,* JHEP **1603**, 094 (2016) [arXiv:1510.06036 [hep-th]].

[XY2] D. Xie and S. T. Yau, *New $\mathcal{N}=2$ dualities,* arXiv:1602.03529 [hep-th].

[XY3] D. Xie and S. T. Yau, *Argyres-Douglas matter and $\mathcal{N}=2$ dualities,* arXiv:1701.01123 [hep-th].

[XYY] D. Xie, W. Yan and S. T. Yau, *Chiral algebra of Argyres-Douglas theory from M5 brane,* arXiv:1604.02155 [hep-th].

[YY] S. S.-T. Yau and Y. Yu, *Algebraic classification of rational CR structures on topological 5-sphere with transversal holomorphic S^1 action in $\mathbb{C}^4$,* Math. Nachr. **246** (2002), no. 1, 207-233.

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P. R. China.
Email address: chenby16@mails.tsinghua.edu.cn

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P. R. China.
Email address: yau@uic.edu

Department of Mathematics, Center of Mathematical Sciences and Applications, Jefferson Physical Laboratory, Harvard University, Cambridge, 02138, USA
Email address: yau@math.harvard.edu

Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P. R. China.
Email address: hqzuo@mail.tsinghua.edu.cn