BERGMAN FUNCTIONS AND THE EQUIVALENCE PROBLEM OF SINGULAR DOMAINS

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Dedicate to Professor Joseph J. Kohn on the occasion of his 88th birthday

ABSTRACT. In this article, we use the Bergman function, which is introduced by the second author in [Ya], to study the equivalence problem of bounded complete Reinhardt domains in the singular variety $\widetilde{V} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1u_4 = u_2u_3\}$.

1. INTRODUCTION

In several complex variables, one of the most fundamental question asks that whether there is a biholomorphic map between two given domains in $\mathbb{C}^n$. When $n = 1$, the Riemann mapping theorem tells us that any simply connected domain in $\mathbb{C}$ is holomorphically equivalent to either $\mathbb{C}$ or the unit disk. However, in higher dimension case, there are lots of domains which is topologically equivalent to the ball but not holomorphic equivalent to the ball, for example, $\Delta^n = \{(z_1, \cdots, z_n) \mid |z_i| < 1\}$. About 100 years ago, Poincaré observed that the interior complex structure of a domain D in $\mathbb{C}^n$ is close related to the partial complex structure of its boundary ∂D, which is called the CR structure. This structure was studied systematically by Cartan [Ca] and later by Chern and Moser [CM], Tanaka [Ta], Webster [We], etc. A fundamental result by the work of Fefferman [Fe] asserts that a biholomorphic mapping between two bounded smooth strictly pseudoconvex domains induces a CR-equivalence between their boundaries. Therefore, two bounded smooth strongly pseudoconvex domains in $\mathbb{C}^n$ are holomorphic equivalent if and only if their boundaries are CR equivalent.

However, the fundermental question that whether two given strictly pseudoconvex CR manifolds are CR equivalent is still unsolved. In 1974, Boutel de Monvel [Bo] (see also Kohn [Ko]) proved that any compact strictly pseudoconvex CR manifold of dimension ≥ 5 can be embedded in $\mathbb{C}^N$ for N large enough. A beautiful result by Harvey-Lawson [HL] asserts that any embeddable strongly pseudoconvex CR manifold is the boundary of a complex variety with only isolated normal singularity, which is called the Stein filling of the CR manifold. One can use the structures of singularities in the Stein fillings to distinguish the structures of the CR manifolds (see Theorem 3.1 of [Ya]). Hence if two CR manifolds bound nonisomorphic singularities, then they are not CR equivalent. However, when two CR manifolds bound isomorphic singularities, i.e. they lie on the same variety V, it is

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difficult to distinguish them, or equivalently, distinguish the singular domains they bound. If V is smooth, this difficult problem is just the classical problem mentioned above and has been considered by many leading mathematicians. On the other hand, if V is singular, the CR equivalence problem is wide open. In [Ya], the second author introduced a novel method, the so called Bergman function, to attack this problem. This is a new biholomorphic invariant for singular varieties, which puts lots of restrictions on bioholomorphic maps. We can use Bergman functions to construct many numerical invariants and determine the automorphism groups of the singular varieties. Using this new technique, Du and the second author [DY] solved the biholomorphic equivalence problem for bounded complete Reinhardt domains in the two dimensional A_n-variety. This technique can also be used to attack the equivalence problem for smooth complete Reinhardt domains in $\mathbb{C}^n$ (see [DGY]).

In this paper, we use Bergman functions to solve the equivalence problem for bounded complete Reinhardt domains in the 3-dimensional singular variety $\widetilde{V} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1u_4 = u_2u_3\}$. Biholomorphic maps between two such domains not only have to preserve Bergman functions, but also have to leave the variety $\widetilde{V}$ invariant, thus the set of biholomorphic maps is dramatically small (see Theorem 3.5 and Theorem 3.7). Indeed, we prove that any biholomorphic map between two such domains in $\widetilde{V}$ must be the restriction of a linear automorphism of $\mathbb{C}^4$. This is an analogue of Cartan's well known result that any automorphism of a bounded circular domain in $\mathbb{C}^n$ which fixes 0 must be linear.

We construct a numerical invariant ν_V for any bounded complete Reinhardt domain V in the singular variety $\widetilde{V}$. If V_1 is biholomorphic to V_2, we show in Theorem 4.3 that $\nu_{V_1} = \nu_{V_2}$ or $\nu_{V_1}\nu_{V_2} = 1$. This invariant reflects the symmetry of the domain. We say V is asymmetric if $\nu_V \neq 1$. In this case the set of biholomorphic maps is much smaller. Indeed, we prove that any biholomorphic map between two asymmetric domains must be of the special form: permutation of coordinate modulo scalar multiplication (see Theorem 4.2). Using this theorem we construct lots of numerical invariants for asymmetric domains and give a sufficient and necessary condition for the equivalence of two such domains (see Theorem 4.4).

As an application, in Theorem 5.2 and 5.4 we solve the equivalence problem of two four parameter families $V^k_{a,b,c,d} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1u_4 = u_2u_3, a|u_1|^{2k} + b|u_1|^{2k} + c|u_1|^{2k} + d|u_1|^{2k} < \varepsilon\}$, where $a, b, c, d > 0$, ε is a fixed positive constant and $k = 1, 2$. We show that

$$V_1 = V^k_{a_1,b_1,c_1,d_1} \simeq V_1 = V^k_{a_1,b_1,c_1,d_1} \Longleftrightarrow \nu_{V_1} = \nu_{V_2} \text{ or } \nu_{V_1}\nu_{V_2} = 1$$
$$\Longleftrightarrow \frac{a_1d_1}{b_1c_1} = \frac{a_2d_2}{b_2c_2} \text{ or } \frac{a_1d_1}{b_1c_1} = \frac{b_2c_2}{a_2d_2}$$

for $k = 1, 2$.

Let $\widetilde{M}$ be the total slace of the vector bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ on $\mathbf{P}^1$, then $\widetilde{M}$ is a open Calabi-Yau manifold of three dimension (i.e. the canonical bundle is trivial). It is well known that if we blow down the 0-section in $\widetilde{M}$ to a point then we obtain the singular variety $\widetilde{V} = \{u_1u_4 = u_2u_3\}$. That is to say, there is a resolution map $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ such that the 0-section in $\widetilde{M}$ is the exceptional set. A biholomorphism between $\widetilde{\pi}^{-1}(V_1)$ and $\widetilde{\pi}^{-1}(V_2)$ will induce a biholomorphism between V_1 and V_2 (see the end of Section 4). Therefore we obtain a necessary condition for the equivalence problem of a class of open Calabi-Yau manifolds of three dimension. However, the equivalence of V_1 and V_2 may not imply that of $\widetilde{\pi}^{-1}(V_1)$ and $\widetilde{\pi}^{-1}(V_2)$ (see Counter-Example 4.9). Thus this is not a sufficient condition.

Remark 1.1. Our Counter-Example 4.9 says that in general for 1-convex manifold M, there may exist two strictly pseudoconvex open subsets U_1 and U_2 containing the maximal compact analytic subset of M such that U_1 and U_2 are not biholomorphic equivalent although the boundaries of U_1 and U_2 are CR biholomorphic equivalent.

Our paper is organized as follows. In Section 2, we recall some basic definitions and results about Bergman functions in [Ya] and [DY]. In Section 3, we write down explicitly the k-th order Bergman functions for bounded complete Reinhardt domains in $\widetilde{V} = \{u_1u_4 = u_2u_3\}$. In Section 4, we introduce the definition of asymmetric domains and determine all possible biholomorphisms between two asymmetric domains. Then we give a sufficient and necessary condition for the equivalence of two such domains. In Section 5, as an application, we solve the equivalence problem of two concrete four parameter families.

2. Bergman function

In this section, we will recall some basic definitions and results about Bergman functions in [Ya] and [DY].

We first recall the definition of the Bergman kernel. Let M be a complex manifold of dimension n. Let F_M be the set of all L^2 integrable holomorphic n-forms on M. Then F_M is a separable complex Hilbert space under the inner product

$$\langle \phi_1, \phi_2 \rangle_M = (\sqrt{-1})^{n^2} \int_M \phi_1 \wedge \overline{\phi_2}.$$

The corresponding norm $\sqrt{\langle \phi, \phi \rangle_M}$ will be denote by $||\phi||_M$. Let $\{\omega_i\}$ be a complete orthonormal basis for F_M. Then

$$K_M = \sum \omega_j \wedge \overline{\omega_j}$$

converges uniformly on compact subsets to a $2n$-form on M, which is independent of choice of the orthonormal basis of F_M (see [Kr]). K_M is called the Bergman kernel of M.

Definition 2.1. ([Ya],[DY]) Let M be a complex manifold with a reduced divisor E. Denote by $F^{(k)}_{M,E}$ the set of all L^2 integrable holomorphic n-forms on M vanishing as least the k-th order on each irreducible component of E. Let $\{\omega^{(k)}_j\}$ be a complete orthonormal basis of $F^{(k)}_{M,E}$. The Bergman kernel vanishing on E of k-th order is defined to be

$$K^{(k)}_{M,E} = \sum \omega^{(k)}_j \wedge \overline{\omega^{(k)}_j}.$$

And the k-th order Bergman function $B^{(k)}_{M,E}$ on M is defined to be

$$B^{(k)}_{M,E} = \frac{K^{(k)}_{M,E}}{K_M}.$$

The proof of the following two lemmas can be found in [Ya]. For the convenience of readers, we will give a proof here.

Lemma 2.2. *$K_{M,E}^{(k)}$ and $B_{M,E}^{(k)}$ are independent of the choice of the complete orthonormal basis of $F_{M,E}^{(k)}$.*

Proof. Let $\{\omega_i\}$ and $\{\widetilde{\omega_i}\}$ be two complete orthonormal bases of $F_{M,E}^{(k)}$. Let $\{\alpha_i\}$ be a complete orthonormal base of $(F_{M,E}^{(k)})^{\perp}$ in F_M, then both $\{\alpha_i\} \cup \{\omega_i\}$ and $\{\alpha_i\} \cup \{\widetilde{\omega_i}\}$ are complete orthonormal bases of F_M. Since the Bergman kernel K_M is independent of the choice of the complete orthonormal basis, we have

$$\sum \alpha_i \wedge \overline{\alpha_i} + \sum \omega_i \wedge \overline{\omega_i} = \sum \alpha_i \wedge \overline{\alpha_i} + \sum \widetilde{\omega_i} \wedge \overline{\widetilde{\omega_i}}.$$

Thus $\sum \omega_i \wedge \overline{\omega_i} = \sum \widetilde{\omega_i} \wedge \overline{\widetilde{\omega_i}}$, which implies that $K_{M,E}^{(k)}$ and $B_{M,E}^{(k)}$ are independent of the choice of the complete orthonormal basis. □

Lemma 2.3. *If $f : M_1 \to M_2$ is boholomorphic and $f(E_1) = E_2$, where E_i is a reduced divisor on M_i for $i = 1, 2$, then*

$$f^*(B_{M_2,E_2}^{(k)}) = B_{M_1,E_1}^{(k)}.$$

Proof. Let $\{\omega_j^{(k)}\}$ be a complete orthonormal basis of $F_{M_2,E_2}^{(k)}$. Since f is biholomorphic, $\{f^*(\omega_j^{(k)})\}$ is a complete orthonormal basis of $F_{M_1,E_1}^{(k)}$, which implies that $f^*(K_{M_2,E_2}^{(k)}) = K_{M_1,E_1}^{(k)}$. As a consequence, $f^*(B_{M_2,E_2}^{(k)}) = B_{M_1,E_1}^{(k)}$. □

Let V be a Stein variety with only one normal isolated singularity p. Let $\pi : M \to V$ be a divsorial resolution of p (i.e. the exceptional set $\pi^{-1}(p)$ is a divisor on M). Denote the exceptional divisor by E, then E is connected since (V, p) is normal. For convenience, we denote $K_{M,E}^{(k)}$ (resp. $B_{M,E}^{(k)}$) by $K_M^{(k)}$ (resp. $B_M^{(k)}$) in the following. Since E is compact and connected, using the maximum principle for harmonic functions we conclude that $B_M^{(k)}$ is a constant function on E.

Definition 2.4. ([Ya],[DY]) In the above notation, the k-th order Bergman function $B_V^{(k)}$ on V is defined to be the push forward of $B_M^{(k)}$ by the map π.

The argument in the proof of [LYY, Theorem 1] can be used to prove the following lemma.

Lemma 2.5. *Let M be a complex manifold with a reduced divisor E. Let A be a submanifold of codimension ≥ 2 which is contained in E. Let $\pi : \widetilde{M} \to M$ be the blow up of M along A. Then we have $K_{\widetilde{M}}^{(k)} = \pi^*(K_M^{(k)})$. Consequently, $B_{\widetilde{M}}^{(k)} = \pi^*(B_M^{(k)})$.*

Proof. Since π is birational, the map $\pi^* : F_M \to F_{\widetilde{M}}$ is injective and preserves inner product. Take $\widetilde{\omega} \in F_{\widetilde{M}}$, then $\widetilde{\omega}$ defines a n-form ω on $M \setminus A$. Since A is a submanifold of codimension ≥ 2, ω extends to a holomorphic n-form on M. Clearly $\pi^*(\omega) = \widetilde{\omega}$, hence π^* is surjective.

Next we prove that $\pi^*(F_M^{(k)}) = F_{\widetilde{M}}^{(k)}$. It is clear that $\pi^*(F_M^{(k)}) \subset F_{\widetilde{M}}^{(k)}$. If $\omega \in F_M \setminus F_M^{(k)}$, then there exists a irreducible component E_i of E such that the vanishing order of ω on E_i is less than k. Let $\widetilde{E_i}$ be the strict transformation of E_i, then the vanishing order of $\pi^*(\omega)$ on $\widetilde{E_i}$ is less than k, which implies that $\pi^*(\omega) \notin F_{\widetilde{M}}^{(k)}$. Therefore, $\pi^*(F_M^{(k)}) = F_{\widetilde{M}}^{(k)}$.

Let $\{\omega_i\}$ be a complete orthonormal basis of $F^{(k)}_M$, then $\{\pi^*(\omega_i)\}$ is a complete orthonormal basis of $F^{(k)}_{\widetilde{M}}$. Hence

$$K^{(k)}_{\widetilde{M}} = \sum \pi^*\omega_j \wedge \overline{\pi^*\omega_j} = \pi^*(\sum \omega_j \wedge \overline{\omega_j}) = \pi^*(K^{(k)}_M).$$

□

Let $M_i \to V$ be two divisorial resolutions of the singularity of V. By Hironaka's theorem [Hi], there exists a resolution $\widetilde{\pi} : \widetilde{M} \to V$ such that $\widetilde{M}$ can be obtained from $M_i, i = 1, 2$, by successive blowing up along submanifolds in the exceptional set. In view of Lemma 2.5, $B^{(k)}_V$ is independent of the choice of resolutions.

Theorem 2.6. ([Ya],[DY]) *Let V be a Stein variety with only one normal isolated singularity p. The k-th order Bergman function $B^{(k)}_V$ on V is invariant under biholomorphic maps.*

Proof. Let $f : V_1 \to V_2$ be a biholomorphic morphism. Let $\pi_2 : M_2 \to V_2$ be a divisorial resolution of V_2 with the exceptional divisor E_2. Consider the base change of $\pi_2 : M_2 \to V_2$ by f

$$\begin{tikzcd} M_1 \arrow[r, "g"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\square"] & M_2 \arrow[d, "\pi_2"] \\ V_1 \arrow[r, "f"] & V_2 \end{tikzcd} .$$

Denote $g^{-1}(E_2)$ by E_1. Then $\pi_1 : M_1 \to V_1$ is also a divisorial resolution of V_1 with the exceptional divisor E_1. Since f is biholomorphic, $g : M_1 \to M_2$ is also biholomorphic. By Lemma 2.3, we have $g^*(B^{(k)}_{M_2}) = B^{(k)}_{M_1}$. Hence $f^*(B^{(k)}_{V_2}) = f^*(\pi_2)_*(B^{(k)}_{M_2}) = (\pi_1)_* g^*(B^{(k)}_{M_2}) = B^{(k)}_{V_1}$. □

3. Bounded complete Reinhardt domains

In this section, we will use Bergman functions to study the equivalent problem for bounded complete Reinhardt domains in the singular variety

$$\widetilde{V} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1u_2 = u_3u_4\}.$$

An explicit resolution $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ can be given in terms of coordinate charts and transition functions as follows:

Coordinates charts: $\widetilde{U}_1 = \mathbb{C}^3 = \{(x, y_1, y_2)\}$ and $\widetilde{U}_2 = \mathbb{C}^3 = \{(w, z_1, z_2)\}$.

Transition functions: $\begin{cases} z_1 = xy_1 \\ z_2 = xy_2 \\ w = 1/x \end{cases}$ or $\begin{cases} y_1 = wz_1 \\ y_2 = wz_2 \\ x = 1/w \end{cases}$.

Resolution maps: $\widetilde{\pi}(x, y_1, y_2) = (y_1, y_2, xy_1, xy_2)$ and $\widetilde{\pi}(w, z_1, z_2) = (wz_1, wz_2, z_1, z_2)$.

Exceptional set $E = \widetilde{\pi}^{-1}(0)$: $E \cap \widetilde{U}_1 = \{y_1 = y_2 = 0\}$ and $E \cap \widetilde{U}_2 = \{z_1 = z_2 = 0\}$.

It is easy to see that $\widetilde{M}$ is the total place of the vector bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$ on $\mathbf{P}^1$ and the exceptional set E is the 0-section. However, since E has codimension 2, $\widetilde{\pi} : \widetilde{M} \to \widetilde{V}$ is not a divisorial resolution. So we need to consider the blow up of $\widetilde{M}$ along E

$$\widetilde{p} : \widetilde{W} \to \widetilde{M}.$$

Then $\widetilde{\pi} \circ \widetilde{p} : \widetilde{W} \to \widetilde{V}$ is a divisorial resolution of $\widetilde{V}$.

An open subset $D \subseteq \mathbb{C}^n$ is called a complete Reinhardt domain if, whenever $(z_1, \cdots, z_n) \in D$, then $(\tau_1 z_1, \cdots, \tau_n z_n) \in D$ for any complex numbers τ_j with $|\tau_j| \leq 1$.

Definition 3.1. An open subset V in the singular variety $\widetilde{V} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1 u_4 = u_2 u_3\}$ is called a bounded complete Reinhardt domain if there exists a bounded complete Reinhardt domain D in $\mathbb{C}^4$ such that $V = \widetilde{V} \cap D$.

From now on, we always suppose V to be a bounded complete Reinhardt domain in $\widetilde{V}$. Let $M = \widetilde{\pi}^{-1}(V)$, $W = \widetilde{p}^{-1}(M)$, $U_i = M \cap \widetilde{U}_i$ for $i = 1, 2$, $\pi = \widetilde{\pi}|_M : M \to V$ and $p = \widetilde{p}|_W : W \to M$.

Next we will calculate the k-th Bergman function for V. In term of the transition functions, we have

$$x^a y_1^b y_2^c dx \wedge dy_1 \wedge dy_2 = -w^{b+c-a} z_1^b z_2^c dw \wedge dz_1 \wedge dz_2.$$

Hence for any non-negative integers a, b and c, the 3-form $x^a y_1^b y_2^c dx \wedge dy_1 \wedge dy_2$ is holomorphic on M if and only if $b + c \geq a$. Thus

$$\{x^a y_1^b y_2^c dx \wedge dy_1 \wedge dy_2 \mid a, b, c \in \mathbb{Z}_{\geq 0}, b + c \geq a\} \tag{3.1}$$

is a complete basis of F_M.

Proposition 3.2. *In the above notation, let $\phi_{abc} = x^a y_1^b y_2^c dx \wedge dy_1 \wedge dy_2$, $a, b, c \in \mathbb{Z}_{\geq 0}$. Then*

$$\left\{ \frac{\phi_{abc}}{||\phi_{abc}||_M} \mid b + c \geq a \right\}$$

is a complete orthonormal basis of F_M. As a result,

$$\left\{ \frac{p^*(\phi_{abc})}{||\phi_{abc}||_M} \mid b + c \geq a \right\}$$

is a complete orthonormal basis of F_W.

Proof. We only need to prove that $\langle \phi_{abc}, \phi_{def} \rangle_M = 0$ for any $(a, b, c) \neq (d, e, f)$. Suppose $V = D \cap \widetilde{V}$, where D is a bounded Reinhardt domain in $\mathbb{C}^4$. Then $(u_1, u_2, u_3, u_4) \in D$ if and only if $(|u_1|, |u_2|, |u_3|, |u_4|) \in D$. Recall that the resolution map is given by $\widetilde{\pi}(x, y_1, y_2) = (y_1, y_2, xy_1, xy_2)$, so the chart $U_1 = M \cap \widetilde{U_1}$ is given by

$$\{(x, y_1, y_2) \in \mathbb{C}^3 \mid (|y_1|, |y_2|, |x||y_1|, |x||y_2|) \in D\}.$$

Since $M \backslash U_1$ is of measure zero, we may compute integrals on M using the (x, y_1, y_2) coordinate on the chart U_1 alone. Hence

$$\int_M \phi_{abc} \overline{\phi_{def}} = \int_{(|y_1|, |y_2|, |x||y_3|, |x||y_4|) \in D} x^a y_1^b y_2^c \overline{x^d y_1^e y_2^f} dx dy_1 dy_2 \overline{dx dy_1 dy_2}. \tag{3.2}$$

Write $x = se^{i\delta}$, $y_1 = r_1 e^{i\theta_1}$ and $y_2 = r_2 e^{i\theta_2}$. Then $dx\overline{dx} = (-2i) s ds d\delta$, $dy_1 \overline{dy_1} = (-2i) r_1 dr_1 d\theta_1$ and $dy_2 \overline{dy_2} = (-2i) r_2 dr_2 d\theta_2$ (here $i = \sqrt{-1}$). Hence

$$\begin{aligned} \int_M \phi_{abc} \overline{\phi_{def}} = (-2i)^3 \int_{(r_1, r_2, sr_1, sr_2) \in D} & s^{a+d+1} r_1^{b+e+1} r_2^{c+f+1} ds dr_1 dr_2 \cdot \\ & \int_0^{2\pi} e^{i(a-d)\delta} d\delta \int_0^{2\pi} e^{i(b-e)\theta_1} d\theta_1 \int_0^{2\pi} e^{i(c-f)\theta_2} d\theta_2. \end{aligned} \tag{3.3}$$

Since $\int_0^{2\pi} e^{inx}dx = 0$ for any integer number $n \neq 0$, the above integral is equal to 0 for any $(a,b,c) \neq (d,e,f)$. $\square$

Since $p : W \to M$ is the blow up of M along the submanifold $\{y_1 = y_2 = 0\} \cup \{z_1 = z_2 = 0\}$ of codimension 2, $p^*(dx \wedge dy_1 \wedge dy_2)$ vanishes on the exceptional divisor in W of order 1. Thus $p^*(\phi_{abc})$ vanishes on the exceptional divisor in W of order $b+c+1$. By Proposition 3.2, the Bergman kernel vanishing on the exceptional divisor of k-th order $K_W^{(k)}$ and the Bergman kernel K_W are given by

$$K_W^{(k)} = p^*(\Theta_M^{(k)} dx \wedge dy_1 \wedge dy_2 \wedge d\overline{x} \wedge d\overline{y_1} \wedge d\overline{y_2}) \tag{3.4}$$

and

$$K_W = p^*(\Theta_M^{(0)} dx \wedge dy_1 \wedge dy_2 \wedge d\overline{x} \wedge d\overline{y_1} \wedge d\overline{y_2}) \tag{3.5}$$

where

$$\Theta_M^{(k)} = \sum_{b+c \geq \max\{k-1,a\}} \frac{|x|^{2a}|y_1|^{2b}|y_2|^{2c}}{||\phi_{abc}||_M^2}. \tag{3.6}$$

Therefore, the k-th order Bergman function on W is given by

$$B_W^{(k)} = p^* \left(\frac{\Theta_M^{(k)}}{\Theta_M^{(0)}} \right). \tag{3.7}$$

Recall that the π is given by $(u_1,u_2,u_3,u_4) = (y_1, y_2, xy_1, xy_2)$. Hence

$$\pi^*(u_1^{i_1} u_2^{i_2} u_3^{i_3} u_4^{i_4}) = x^{i_3+i_4} y_1^{i_1+i_3} y_2^{i_2+i_4}. \tag{3.8}$$

If $i_3 + i_4 = j_3 + j_4$, $i_1 + i_3 = j_1 + j_3$ and $i_2 + i_4 = j_3 + j_4$, then $\pi^*(u_1^{i_1} u_2^{i_2} u_3^{i_3} u_4^{i_4}) = \pi^*(u_1^{j_1} u_2^{j_2} u_3^{j_3} u_4^{j_4})$ on M, which implies that $u_1^{i_1} u_2^{i_2} u_3^{i_3} u_4^{i_4} = u_1^{j_1} u_2^{j_2} u_3^{j_3} u_4^{j_4}$ on V. For example, $u_1u_4 = u_2u_3$ on V.

Define a equivalence relation on $\mathbb{Z}^4_{\geq 0}$:

$$\begin{aligned}(i_1,i_2,i_3,i_4) \sim (j_1,j_2,j_3,j_4) &\iff \\ &i_3 + i_4 = j_3 + j_4, i_1 + i_3 = j_1 + j_3 \text{ and } i_2 + i_4 = j_3 + j_4.\end{aligned}$$

Denote by $[i_1,i_2,i_3,i_4]$ the equivalence class of (i_1,i_2,i_3,i_4).

Theorem 3.3. *In the above notations, the k-th order Bergman function for V is given by*

$$B_V^{(k)} = \frac{\Theta_V^{(k)}}{\Theta_V^{(0)}}, \tag{3.9}$$

where

$$\Theta_V^{(k)} = \sum_{[i_1,i_2,i_3,i_4] \in S_k} \frac{|u_1|^{2i_1}|u_2|^{2i_2}|u_3|^{2i_3}|u_4|^{2i_4}}{\psi^V_{[i_1,i_2,i_3,i_4]}}, \tag{3.10}$$

$$S_k = \{[i_1,i_2,i_3,i_4] \in \mathbb{Z}^4_{\geq 0}/\sim \mid i_1 + i_2 + i_3 + i_4 \geq k-1\}, \tag{3.11}$$

$$\psi^V_{[i_1,i_2,i_3,i_4]} = ||\phi_{i_3+i_4, i_1+i_3, i_2+i_4}||_M^2. \tag{3.12}$$

Proof. Recall that π is given by $(u_1,u_2,u_3,u_4) = (y_1,y_2,xy_1,xy_2)$. The conclusion follows from (3.6) and (3.7). $\square$

In particular, for $k=2$, using the above theorem, we have

$$B_V^{(2)} = \frac{\Theta_V^{(2)}}{\frac{1}{\psi^V_{[0,0,0,0]}} + \Theta_V^{(2)}} = \frac{\psi^V_{[0,0,0,0]}\Theta_V^{(2)}}{1+\psi^V_{[0,0,0,0]}\Theta_V^{(2)}}. \tag{3.13}$$

Proposition 3.4. $\psi^V_{[0,0,0,0]}\Theta_V^{(2)}$ *is invariant under biholomorphic maps.*

Proof. The proposition follows from Theorem 2.6 and the equation (3.13). □

Denote

$$\varphi^V_{[i_1,i_2,i_3,i_4]} = \frac{\psi^V_{[0,0,0,0]}}{\psi^V_{[i_1,i_2,i_3,i_4]}}. \tag{3.14}$$

Then

$$\psi^V_{[0,0,0,0]}\Theta_V^{(2)} = \sum_{[i_1,i_2,i_3,i_4]\in S_2} \varphi^V_{[i_1,i_2,i_3,i_4]}|u_1|^{2i_1}|u_2|^{2i_2}|u_3|^{2i_3}|u_4|^{2i_4} \tag{3.15}$$

where $S_2 = \{[i_1,i_2,i_3,i_4] \in \mathbb{Z}^4_{\geq 0}/\sim \mid i_1+i_2+i_3+i_4 \geq 1\}$.

For short, we denote

$$\begin{aligned} \varphi^1_V &= \varphi^V_{[1,0,0,0]}, \quad \varphi^2_V = \varphi^V_{[0,1,0,0]}, \\ \varphi^3_V &= \varphi^V_{[0,0,1,0]}, \quad \varphi^4_V = \varphi^V_{[0,0,0,1]}. \end{aligned} \tag{3.16}$$

Theorem 3.5. *Let $V_i, i=1,2$ be two bounded complete Reinhardt domains in $\widetilde{V} = \{(u_1,u_2,u_3,u_4) \in \mathbb{C}^4 \mid u_1u_4 = u_2u_3\}$. Suppose that $\Psi : V_1 \to V_2$ is a biholomorphic map, then Ψ is the restriction of a linear map $\ell : \mathbb{C}^4 \to \mathbb{C}^4$, that is to say, Ψ is given by*

$$\Psi(u_1,u_2,u_3,u_4) = (\sum_{i=1}^4 a_{1i}u_i, \sum_{i=1}^4 a_{2i}u_i, \sum_{i=1}^4 a_{3i}u_i, \sum_{i=1}^4 a_{4i}u_i).$$

Moreover, the linear map ℓ satisfies the following conditions

$$\sum_{k=1}^4 \varphi^k_{V_2}|a_{ki}|^2 = \varphi^i_{V_1} \quad \text{for } i=1,2,3,4, \tag{3.17}$$

$$\sum_{k=1}^4 \varphi^j_{V_2} a_{ki}\overline{a_{kj}} = 0 \quad \text{for } i \neq j \in \{1,2,3,4\}, \tag{3.18}$$

$$\det(a_{ij}) \neq 0. \tag{3.19}$$

Proof. Write $\Psi = (\Psi_1,\Psi_2,\Psi_3,\Psi_4)$ and

$$\Psi_i(u_1,u_2,u_3,u_4) = \ell_i(u_1,u_2,u_3,u_4) + f_i(u_1,u_2,u_3,u_4),$$

where ℓ_i is a linear function and f_i is the nonlinear part for $i = 1, 2, 3, 4$. By Proposition 3.4, we have $\psi^{V_1}_{[0,0,0,0]}\Theta^{(2)}_{V_1} = \Psi^*(\psi^{V_2}_{[0,0,0,0]}\Theta^{(2)}_{V_2})$ on V_1, i.e.,

$$
\begin{aligned}
&\sum_{i=1}^4 \varphi^i_{V_1}|u_i|^2 + \text{ higher order terms}\\
(3.20)\qquad &=\sum_{i=1}^4 \varphi^i_{V_2}|\ell_i + f_i|^2 + \text{ higher order terms}\\
&=\sum_{i=1}^4 \varphi^i_{V_2}(|\ell_i|^2 + |f_i|^2 + \ell_i\overline{f_i} + \overline{\ell_i}f_i) + \text{ higher order terms}
\end{aligned}
$$

modulo $u_1u_4 - u_2u_3$. Comparing $(1, k)$-terms for $k \geq 2$ (here (p, q)-term means $a_p\overline{b_q}$ where a_p is a monomial of degree p and b_q is a monomial of degree q), we have

$$
(3.21)\qquad \sum_{i=1}^4 \varphi^i_{V_2}\ell_i\overline{f_i} = 0.
$$

Write $\ell_i(u_1, u_2, u_3, u_4) = a_{i1}u_1 + a_{i2}u_2 + a_{i3}u_3 + a_{i4}u_4$ for $i = 1, 2, 3, 4$. Since Ψ is isomorphic, we have $\ell = (\ell_1, \ell_2, \ell_3, \ell_4)$ induces a isomorphism between the Zariski tangent spaces of V_1 and V_2 at 0, which implies $A = (a_{ij})$ is invertible. (3.21) implies that

$$
\sum_{j=1}^4(\sum_{i=1}^4 a_{ij}\varphi^i_{V_2}\overline{f_i})u_j = 0.
$$

Hence $\sum_{i=1}^4 a_{ij}\varphi^i_{V_2}\overline{f_i} = 0$ for $j = 1, 2, 3, 4$. Since $A = (a_{ij})$ is invertible, we have $\varphi^i_{V_2}\overline{f_i}$=0 for all i. By the definition of $\varphi^i_{V_2}$, we know that $\varphi^i_{V_2} \neq 0$, which implies that $f_i = 0$ for $i = 1, 2, 3, 4$. Hence Ψ is induced by a linear map.

Since $f_i = 0$ for all i, (3.20) implies that

$$
(3.22)\qquad \sum_{i=1}^4 \varphi^i_{V_1}|u_i|^2 + \text{ higher order terms } = \sum_{i=1}^4 \varphi^i_{V_2}|\ell_i|^2 + \text{ higher order terms.}
$$

Substitute ℓ_i by $a_{i1}u_1 + a_{i2}u_2 + a_{i3}u_3 + a_{i4}u_4$, then by comparing the coefficients of terms $|u_i|^2$ for $i = 1, 2, 3, 4$ and terms $u_i\overline{u_j}$ for $i \neq j \in \{1, 2, 3, 4\}$ in (3.22), we obtain (3.17) and (3.18). □

Lemma 3.6. *Let $V_i, i = 1, 2$ be two bounded complete Reinhardt domains in $\widetilde{V} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1u_4 = u_2u_3\}$. Suppose that $\Psi : V_1 \to V_2$ is a biholomorphic map, then Ψ is the restriction of a linear map $\ell = (\ell_1, \ell_2, \ell_3, \ell_4) : \mathbb{C}^4 \to \mathbb{C}^4$ such that*

$$
\ell_1\ell_4 - \ell_2\ell_3 = c(u_1u_4 - u_2u_3),
$$

where c is a nonzero constant. That is to say, Ψ is given by

$$
\Psi(u_1, u_2, u_3, u_4) = (\sum_{i=1}^4 a_{1i}u_i, \sum_{i=1}^4 a_{2i}u_i, \sum_{i=1}^4 a_{3i}u_i, \sum_{i=1}^4 a_{4i}u_i)
$$

such that

$$a_{1i}a_{4i} - a_{3i}a_{2i} = 0 \quad \textit{for } i = 1, 2, 3, 4, \tag{3.23}$$

$$a_{1i}a_{4j} - a_{2i}a_{3j} - a_{3i}a_{2j} + a_{4i}a_{1j} = 0 \tag{3.24}$$
$$\textit{for } i, j \in \{1, 2, 3, 4\} \textit{ such that } \{i, j\} \neq \{1, 4\}, \{2, 3\},$$

$$a_{11}a_{44} - a_{21}a_{34} - a_{31}a_{24} + a_{41}a_{14}+ \tag{3.25}$$
$$a_{12}a_{43} - a_{22}a_{33} - a_{32}a_{23} + a_{42}a_{13} = 0.$$

Proof. By Theorem 3.5, Ψ is induce by a linear map $\ell = (\ell_1, \ell_2, \ell_3, \ell_4) : \mathbb{C}^4 \to \mathbb{C}^4$. Since ℓ induce a isomorphic from V_1 to V_2, we have

$$\ell^*(u_1u_4 - u_2u_3) = (u_1u_4 - u_2u_3)g \tag{3.26}$$

where g is an invertible element in the ring of convergent power series $\mathbb{C}\{u_1, u_2, u_3, u_4\}$. Denote the constant term of g by c, then $c \neq 0$. By looking at the quadratic part of (3.26), we obtain

$$\ell^*(u_1u_4 - u_2u_3) = c(u_1u_4 - u_2u_3).$$

Thus

$$\ell_1\ell_4 - \ell_2\ell_3 = c(u_1u_4 - u_2u_3).$$

Substitute ℓ_i by $a_{i1}u_1 + a_{i2}u_2 + a_{i3}u_3 + a_{i4}u_4$ and compare the coefficients, we obtain (3.23), (3.24) and (3.25). □

Theorem 3.7. *Let V_1 and V_2 be two bounded Reinhardt domains in $\widetilde{V} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1u_4 = u_2u_3\}$. The biholomrphic map $\Psi : V_1 \to V_2$ must be of the following form*

$$\Psi(u_1, u_2, u_3, u_4) = (\sum_{i=1}^{4} a_{1i}u_i, \sum_{i=1}^{4} a_{2i}u_i, \sum_{i=1}^{4} a_{3i}u_i, \sum_{i=1}^{4} a_{4i}u_i)$$

and there exist complex numbers $a_1, \cdots, a_4$ and $b_1, \cdots, b_4$ such that

$$A = (a_{ij}) = \begin{pmatrix} a_1b_1 & a_1b_2 & a_2b_1 & a_2b_2 \\ a_1b_3 & a_1b_4 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_4b_1 & a_4b_2 \\ a_3b_3 & a_3b_4 & a_4b_3 & a_4b_4 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \tag{3.27}$$

or

$$A = (a_{ij}) = \begin{pmatrix} a_1b_1 & a_2b_1 & a_1b_2 & a_2b_2 \\ a_1b_3 & a_2b_3 & a_1b_4 & a_2b_4 \\ a_3b_1 & a_4b_1 & a_3b_2 & a_4b_2 \\ a_3b_3 & a_4b_3 & a_3b_4 & a_4b_4 \end{pmatrix} = \tag{3.28}$$
$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\otimes$ means the Kronecker product of two matrices and $*$ means the matrix multiplication.

Proof. By (3.23), we have $a_{1i}a_{4i} - a_{3i}a_{2i} = 0$ for $i = 1, 2, 3, 4$. Hence there exist (b_{1i}, b_{2i}) and s_i, t_i such that

$$(a_{1i}, a_{2i}) = s_i(b_{1i}, b_{2i}), \quad (a_{3i}, a_{4i}) = t_i(b_{1i}, b_{2i}) \tag{3.29}$$

for $i = 1, 2, 3, 4$. Note that $(s_i, t_i) \neq (0, 0)$ since $A = (a_{ij})$ is invertible. For $i, j \in \{1, 2, 3, 4\}$ such that $\{i, j\} \neq \{1, 4\}, \{2, 3\}$, by (3.24) we have

$$a_{1i}a_{4j} - a_{2i}a_{3j} - a_{3i}a_{2j} + a_{4i}a_{1j} = 0. \tag{3.30}$$

Equations (3.29) and (3.30) imply that

$$(s_i t_j - s_j t_i)(b_{1i}b_{2j} - b_{2i}b_{1j}) = 0 \tag{3.31}$$

for $i, j \in \{1, 2, 3, 4\}$ such that $\{i, j\} \neq \{1, 4\}, \{2, 3\}$.

Claim 3.8. (a) *There do not exist mutually distinct $i, j, k \in \{1, 2, 3, 4\}$ such that $[s_i : t_i] = [s_j : t_j] = [s_k : t_k]$.*

(b) *There do not exist mutually distinct $i, j, k \in \{1, 2, 3, 4\}$ such that $[s_i : t_i]$, $[s_j : t_j]$ and $[s_k : t_k]$ are mutually distinct.*

Here $[\alpha_1 : \beta_1] = [\alpha_2 : \beta_2]$ means $\alpha_1\beta_2 = \alpha_2\beta_1$.

Proof of claim 3.8. (a) Assume that there exist mutually distinct i, j, k such that $[s_i : t_i] = [s_j : t_j] = [s_k : t_k]$. Then there exist c_i, c_j, c_k such that $(c_i, c_j, c_k) \neq (0, 0, 0)$ and

$$\begin{aligned} c_i s_i(b_{i1}, b_{i2}) + c_j s_j(b_{j1}, b_{j2}) + c_k s_k(b_{k1}, b_{k2}) &= 0, \\ c_i t_i(b_{i1}, b_{i2}) + c_j t_j(b_{j1}, b_{j2}) + c_k t_k(b_{k1}, b_{k2}) &= 0. \end{aligned} \tag{3.32}$$

Equations (3.29) and (3.32) imply that

$$\begin{aligned} c_i(a_{i1}, a_{i2}, a_{i3}, a_{i4}) &+ c_j(a_{j1}, a_{j2}, a_{j3}, a_{j4}) \\ &+ c_k(a_{k1}, a_{k2}, a_{k3}, a_{k4}) = 0, \end{aligned}$$

which contradicts the fact that $A = (a_{ij})$ is invertible.

(b) Assume that there exist mutually distinct i, j, k such that $[s_i : t_i]$, $[s_j : t_j]$ and $[s_k : t_k]$ are mutually distinct. Since i, j, k are mutually distinct, there must be two distinct pairs $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\} \subset \{i, j, k\}$ ($\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$) which are not equal to $\{1, 4\}$ or $\{2, 3\}$. By (3.31), we have

$$[b_{1\alpha_1} : b_{2\alpha_1}] = [b_{1\beta_1} : b_{2\beta_1}] \quad \text{and} \quad [b_{1\alpha_2} : b_{2\alpha_2}] = [b_{1\beta_2} : b_{2\beta_2}],$$

which implies that

$$[b_{1i} : b_{2i}] = [b_{1j} : b_{2j}] = [b_{1k} : b_{2k}].$$

Thus there exists c_i, c_j, c_k such that $(c_i, c_j, c_k) \neq (0, 0, 0)$ such that

$$\begin{aligned} c_i b_{1i}(s_i, t_i) + c_j b_{1j}(s_j, t_j) + c_k b_{1k}(s_k, t_k) &= 0, \\ c_i b_{2i}(s_i, t_i) + c_j b_{2j}(s_j, t_j) + c_k b_{2k}(s_k, t_k) &= 0. \end{aligned} \tag{3.33}$$

Equations (3.29) and (3.33) imply that

$$\begin{aligned} c_i(a_{i1}, a_{i2}, a_{i3}, a_{i4}) &+ c_j(a_{j1}, a_{j2}, a_{j3}, a_{j4}) \\ &+ c_k(a_{k1}, a_{k2}, a_{k3}, a_{k4}) = 0, \end{aligned}$$

which contradicts the fact that $A = (a_{ij})$ is invertible. $\square$

Returning to the proof of Theorem 3.7. By the above claim, there are three following cases:

(1) $[s_1 : t_1] = [s_2 : t_2] \neq [s_3 : t_3] = [s_4 : t_4]$. Then there exist (s, t) and c_1, c_2 such that such that $(s_1, t_1) = c_1(s, t)$, $(s_2, t_2) = c_2(s, t)$ and there exists $(\widetilde{s}, \widetilde{t})$ and $\widetilde{c_1}, \widetilde{c_2}$ such that $(s_3, t_3) = \widetilde{c_1}(\widetilde{s}, \widetilde{t})$, $(s_4, t_4) = \widetilde{c_2}(\widetilde{s}, \widetilde{t})$. Since $[s_2 : t_2] \neq [s_3 : t_3]$, we have

$$s\widetilde{t} - \widetilde{s}t \neq 0. \tag{3.34}$$

By (3.31), we have $[b_{11} : b_{21}] = [b_{13} : b_{23}]$ and $[b_{12} : b_{22}] = [b_{14} : b_{24}]$. Then there exists (p, q) and d_1, d_2 such that such that $(b_{11}, b_{21}) = d_1(p, q)$, $(b_{13}, b_{23}) = d_2(p, q)$ and there exists $(\widetilde{p}, \widetilde{q}))$ and $\widetilde{d}_1, \widetilde{d}_2$ such that $(b_{12}, b_{22}) = \widetilde{d}_1(\widetilde{p}, \widetilde{q})$, $(b_{14}, b_{24}) = \widetilde{d}_2(\widetilde{p}, \widetilde{q})$. Since $[b_{11} : b_{21}] \neq [b_{12} : b_{22}]$ (if not then $A = (a_{ij})$ is not invertible), we have

$$p\widetilde{q} - \widetilde{p}q \neq 0. \tag{3.35}$$

Then we have

$$A = (a_{ij}) = \begin{pmatrix} c_1 d_1 sp & c_2 \widetilde{d}_1 s\widetilde{p} & \widetilde{c}_1 d_2 \widetilde{s}p & \widetilde{c}_2 \widetilde{d}_2 \widetilde{s}\widetilde{p} \\ c_1 d_1 sq & c_2 \widetilde{d}_1 s\widetilde{q} & \widetilde{c}_1 d_2 \widetilde{s}q & \widetilde{c}_2 \widetilde{d}_2 \widetilde{s}\widetilde{q} \\ c_1 d_1 tp & c_2 \widetilde{d}_1 t\widetilde{p} & \widetilde{c}_1 d_2 \widetilde{t}p & \widetilde{c}_2 \widetilde{d}_2 \widetilde{t}\widetilde{p} \\ c_1 d_1 tq & c_2 \widetilde{d}_1 t\widetilde{q} & \widetilde{c}_1 d_2 \widetilde{t}q & \widetilde{c}_2 \widetilde{d}_2 \widetilde{t}\widetilde{q} \end{pmatrix}. \tag{3.36}$$

Equations (3.25) and (3.36) imply that

$$\begin{aligned} & c_1 d_1 sp \widetilde{c}_2 \widetilde{d}_2 \widetilde{t}\widetilde{q} - c_1 d_1 sq \widetilde{c}_2 \widetilde{d}_2 \widetilde{t}\widetilde{p} - c_1 d_1 tp \widetilde{c}_2 \widetilde{d}_2 \widetilde{s}\widetilde{q} + c_1 d_1 tq \widetilde{c}_2 \widetilde{d}_2 \widetilde{s}\widetilde{p} + \\ & \qquad c_2 \widetilde{d}_1 s\widetilde{p} \widetilde{c}_1 d_2 \widetilde{t} q - c_2 \widetilde{d}_1 s\widetilde{q} \widetilde{c}_1 d_2 \widetilde{t} p - c_2 \widetilde{d}_1 t\widetilde{p} \widetilde{c}_1 d_2 \widetilde{s} q + \widetilde{c}_1 d_2 \widetilde{t} q \widetilde{c}_1 d_2 \widetilde{s} p \\ = & (s\widetilde{t} - \widetilde{s}t)(p\widetilde{q} - \widetilde{p}q)(c_1 d_1 \widetilde{c}_2 \widetilde{d}_2 - c_2 \widetilde{d}_1 \widetilde{c}_1 d_2) = 0. \end{aligned}$$

By (3.34) and (3.35), we have

$$c_1 d_1 \widetilde{c}_2 \widetilde{d}_2 - c_2 \widetilde{d}_1 \widetilde{c}_1 d_2 = 0.$$

Hence there exists (α, β) and γ_1, γ_2 such that $(c_1 d_1, c_2 \widetilde{d}_1) = \gamma_1(\alpha, \beta)$ and $(\widetilde{c}_1 d_2, \widetilde{c}_2 \widetilde{d}_2) = \gamma_2(\alpha, \beta)$. Hence

$$\begin{aligned} A = (a_{ij}) &= \begin{pmatrix} \gamma_1 \alpha sp & \gamma_1 \beta s\widetilde{p} & \gamma_2 \alpha \widetilde{s}p & \gamma_2 \beta \widetilde{s}\widetilde{p} \\ \gamma_1 \alpha sq & \gamma_1 \beta s\widetilde{q} & \gamma_2 \alpha \widetilde{s}q & \gamma_2 \beta \widetilde{s}\widetilde{q} \\ \gamma_1 \alpha tp & \gamma_1 \beta t\widetilde{p} & \gamma_2 \alpha \widetilde{t}p & \gamma_2 \beta \widetilde{t}\widetilde{p} \\ \gamma_1 \alpha tq & \gamma_1 \beta t\widetilde{q} & \gamma_2 \alpha \widetilde{t}q & \gamma_2 \beta \widetilde{t}\widetilde{q} \end{pmatrix} \\ &= \begin{pmatrix} \gamma_1 s & \gamma_2 \widetilde{s} \\ \gamma_1 t & \gamma_2 \widetilde{t} \end{pmatrix} \otimes \begin{pmatrix} \alpha p & \beta \widetilde{p} \\ \alpha q & \beta \widetilde{q} \end{pmatrix}. \end{aligned}$$

Therefore, (3.27) holds.

(2) $[s_1 : t_1] = [s_3 : t_3] \neq [s_2 : t_2] = [s_4 : t_4]$. Using a similar argument in the case (1), we can prove that (3.28) holds.

(3) $[s_1 : t_1] = [s_4 : t_4] \neq [s_2 : t_2] = [s_3 : t_3]$. Then $[s_1 : t_1] \neq [s_2 : t_2]$ and $[s_1 : t_1] \neq [s_3 : t_3]$. By (3.31) we obtain $[b_{11} : b_{21}] = [b_{12} : b_{22}] = [b_{13} : b_{23}]$, which contradicts the fact that $A = (a_{ij})$ is invertible (see the argument in the proof of Claim 3.8(b)). Hence this case will not occur. □

4. Asymmetric domains

We introduce a numerical invariant for a bounded complete Reinhardt domain in $\widetilde{V}$ which reflects the symmetry of the domain.

Definition 4.1. Let V be a bounded complete Reinhardt domain in $\widetilde{V} = \{u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1 u_4 = u_2 u_3\}$. We define

$$\nu_V = \frac{\varphi_V^1 \varphi_V^4}{\varphi_V^2 \varphi_V^3}.$$

We say V is asymmetric if $\nu_V \neq 1$.

By the definition of φ_V^i (see (3.12),(3.14) and (3.16)), we have

$$\nu_V = \frac{\psi^V_{[0,1,0,0]}\psi^V_{[0,0,1,0]}}{\psi^V_{[1,0,0,0]}\psi^V_{[0,0,0,1]}} = \frac{||\phi_{001}||^2_M||\phi_{110}||^2_M}{||\phi_{010}||^2_M||\phi_{101}||^2_M}. \tag{4.1}$$

The set of biholomorphic morphisms between two asymmetric domains is dramatically small and we can determine all possible biholomorphisms. Let G be the subgroup of the symmetric group S_4 generated by three element $(1,4)$, $(2,3)$ and $(1,2)(3,4)$. Then G has eight elements:

$$1234, 4231, 1324, 4321, 2143, 3142, 2413, 3412,$$

here $abcd$ means the element σ in G such that $\sigma(1) = a, \sigma(2) = b, \sigma(3) = c, \sigma(4) = d$.

Theorem 4.2. *Let V_1 and V_2 be two bounded Reinhardt domains in $\widetilde{V_n} = \{u_1u_4 = u_2u_3\}$. If $\nu_{V_1} \neq 1$, then the biholomrphic map $\Psi : V_1 \to V_2$ must be of the following form*

$$\Psi(u_1, u_2, u_3, u_4) = (c_1u_{\sigma(1)}, c_2u_{\sigma(2)}, c_3u_{\sigma(3)}, c_4u_{\sigma(4)}) \tag{4.2}$$

where $\sigma \in G$ and such that

$$c_1c_4 = c_2c_3 \tag{4.3}$$

and

$$\varphi^{\sigma(i)}_{V_1} = \varphi^i_{V_2}|c_i|^2 \tag{4.4}$$

for $i = 1, 2, 3, 4$.

Proof. By Theorem 3.7, Ψ is given by

$$\Psi(u_1, u_2, u_3, u_4) = (\sum_{i=1}^4 a_{1i}u_i, \sum_{i=1}^4 a_{2i}u_i, \sum_{i=1}^4 a_{3i}u_i, \sum_{i=1}^4 a_{4i}u_i)$$

such that one of the following cases occurs:

(1) $A = (a_{ij}) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$. By (3.17), we have

$$\begin{pmatrix} |a_1|^2 & |a_3|^2 \\ |a_2|^2 & |a_4|^2 \end{pmatrix} \begin{pmatrix} \varphi^1_{V_2} & \varphi^2_{V_2} \\ \varphi^3_{V_2} & \varphi^4_{V_2} \end{pmatrix} \begin{pmatrix} |b_1|^2 & |b_2|^2 \\ |b_3|^2 & |b_4|^2 \end{pmatrix} = \begin{pmatrix} \varphi^1_{V_1} & \varphi^2_{V_1} \\ \varphi^3_{V_1} & \varphi^4_{V_1} \end{pmatrix}. \tag{4.5}$$

As $\nu_{V_1} \neq 1$, we have $\varphi^1_{V_1}\varphi^4_{V_1} - \varphi^2_{V_1}\varphi^3_{V_1} \neq 0$. Hence $\varphi^1_{V_2}\varphi^4_{V_2} - \varphi^2_{V_2}\varphi^3_{V_2} \neq 0$, $|a_1|^2|a_4|^2 - |a_2|^2|a_3|^2 \neq 0$ and $|b_1|^2|b_4|^2 - |b_2|^2|b_3|^2 \neq 0$.

By (3.18) we have

$$\begin{pmatrix} |a_1|^2 & |a_3|^2 \\ |a_2|^2 & |a_4|^2 \end{pmatrix} \begin{pmatrix} \varphi^1_{V_2} & \varphi^2_{V_2} \\ \varphi^3_{V_2} & \varphi^4_{V_2} \end{pmatrix} \begin{pmatrix} b_1\overline{b_2} \\ b_3\overline{b_4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.6}$$

and

$$\begin{pmatrix} a_1\overline{a_2} & a_3\overline{a_4} \end{pmatrix} \begin{pmatrix} \varphi^1_{V_2} & \varphi^2_{V_2} \\ \varphi^3_{V_2} & \varphi^4_{V_2} \end{pmatrix} \begin{pmatrix} |b_1|^2 & |b_2|^2 \\ |b_3|^2 & |b_4|^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}. \tag{4.7}$$

Since $\varphi^1_{V_2}\varphi^4_{V_2} - \varphi^2_{V_2}\varphi^3_{V_2} \neq 0$, $|a_1|^2|a_4|^2 - |a_2|^2|a_3|^2 \neq 0$ and $|b_1|^2|b_4|^2 - |b_2|^2|b_3|^2 \neq 0$, we have

$$a_1\overline{a_2} = a_3\overline{a_4} = b_1\overline{b_2} = b_3\overline{b_4} = 0.$$

Hence

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}, \tag{4.8}$$

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} b_1 & 0 \\ 0 & b_4 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & b_2 \\ b_3 & 0 \end{pmatrix}. \tag{4.9}$$

Therefore, we have

$$A = \begin{pmatrix} a_1b_1 & 0 & 0 & 0 \\ 0 & a_1b_4 & 0 & 0 \\ 0 & 0 & a_4b_1 & 0 \\ 0 & 0 & 0 & a_4b_4 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & a_2b_1 & 0 \\ 0 & 0 & 0 & a_2b_4 \\ a_3b_1 & 0 & 0 & 0 \\ 0 & a_3b_4 & 0 & 0 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} 0 & a_1b_2 & 0 & 0 \\ a_1b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4b_2 \\ 0 & 0 & a_4b_3 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 & a_2b_2 \\ 0 & 0 & a_2b_3 & 0 \\ 0 & a_3b_2 & 0 & 0 \\ a_3b_3 & 0 & 0 & 0 \end{pmatrix}.$$

(2) $A = (a_{ij}) = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \otimes \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} * \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1. \end{pmatrix}$. By using a similar argument in case (1), we can prove that

$$A = \begin{pmatrix} a_1b_1 & 0 & 0 & 0 \\ 0 & 0 & a_1b_4 & 0 \\ 0 & a_4b_1 & 0 & 0 \\ 0 & 0 & 0 & a_4b_4 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & a_2b_1 & 0 & 0 \\ 0 & 0 & 0 & a_2b_4 \\ a_3b_1 & 0 & 0 & 0 \\ 0 & 0 & a_3b_4 & 0 \end{pmatrix}$$

$$\text{or } \begin{pmatrix} 0 & 0 & a_1b_2 & 0 \\ a_1b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4b_2 \\ 0 & a_4b_3 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 & a_2b_2 \\ 0 & a_2b_3 & 0 & 0 \\ 0 & 0 & a_3b_2 & 0 \\ a_3b_3 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, there exists a $\sigma \in G$ such that

$$\Psi(u_1, u_2, u_3, u_4) = (a_{1\sigma(1)}u_{\sigma(1)}, a_{2\sigma(2)}u_{\sigma(2)}, a_{3\sigma(3)}u_{\sigma(3)}, a_{4\sigma(4)}u_{\sigma(4)}).$$

and $a_{1\sigma(1)}a_{4\sigma(4)} = a_{2\sigma(2)}a_{3\sigma(3)}$. By (3.17), it is easy to see that $\varphi_{V_1}^{\sigma(i)} = \varphi_{V_2}^{i}|a_{i\sigma(i)}|^2$ for $i = 1, 2, 3, 4$. □

Theorem 4.3. *Let V_1 and V_2 be two bounded complete Reinhardt domains in $\widetilde{V} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1u_4 = u_2u_3\}$ such that V_1 is biholomorphic to V_2. Then $\nu_{V_1} = \nu_{V_2}$ or $\nu_{V_1}\nu_{V_2} = 1$.*

Proof. There are three following cases:

(1) $\nu_{V_1} = \nu_{V_2} = 1$, then the conclusion holds.

(2) $\nu_{V_1} \neq 1$. Take a biholomorphic map $\Psi : V_1 \to V_2$, then Ψ has the form in Theorem 4.2. By (4.3) and (4.4), we have

$$\begin{aligned}\frac{\varphi^1_{V_2}\varphi^4_{V_2}}{\varphi^2_{V_2}\varphi^3_{V_2}} &= \frac{|c_1|^2|c_4|^2}{|c_2|^2|c_3|^2}\frac{\varphi^1_{V_2}\varphi^4_{V_2}}{\varphi^2_{V_2}\varphi^3_{V_2}} \\ &= \frac{(|c_1|^2\varphi^1_{V_2})(|c_4|^2\varphi^4_{V_2})}{(|c_2|^2\varphi^2_{V_2})(|c_3|^2\varphi^3_{V_2})} \\ &= \frac{\varphi^{\sigma(1)}_{V_1}\varphi^{\sigma(4)}_{V_1}}{\varphi^{\sigma(2)}_{V_1}\varphi^{\sigma(3)}_{V_1}},\end{aligned} \tag{4.10}$$

If $\sigma = (1,4)$ or $(2,3)$, then

$$\frac{\varphi^{\sigma(1)}_{V_1}\varphi^{\sigma(4)}_{V_1}}{\varphi^{\sigma(2)}_{V_1}\varphi^{\sigma(3)}_{V_1}} = \frac{\varphi^1_{V_1}\varphi^4_{V_1}}{\varphi^2_{V_1}\varphi^3_{V_1}}.$$

If $\sigma = (1,2)(3,4)$, then

$$\frac{\varphi^{\sigma(1)}_{V_1}\varphi^{\sigma(4)}_{V_1}}{\varphi^{\sigma(2)}_{V_1}\varphi^{\sigma(3)}_{V_1}}\frac{\varphi^1_{V_1}\varphi^4_{V_1}}{\varphi^2_{V_1}\varphi^3_{V_1}} = 1.$$

Since G is generated by these three elements, by the definition of ν_{V_1} and ν_{V_2} we have

$$\nu_{V_1} = \nu_{V_2} \quad \text{or} \quad \nu_{V_1}\nu_{V_2} = 1.$$

(3) If $\nu_{V_2} \neq 1$, then take a biholomorphic map $\Psi' : V_2 \to V_1$. Using a similar argument in case (2), we obtain the conclusion. □

Next we give a criteria to see whether two asymmetric domains in $\widetilde{V}$ are biholomorphic.

Theorem 4.4. *Let V_1 and V_2 be two bounded complete Reinhardt domains in $\widetilde{V} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1u_4 = u_2u_3\}$ such that ν_{V_1} or $\nu_{V_2} \neq 1$.*

(1) *If $V_1 \simeq V_2$, then there exists $\sigma \in G$ such that*

$$\frac{\varphi^1_{V_2}\varphi^4_{V_2}}{\varphi^2_{V_2}\varphi^3_{V_2}} = \frac{\varphi^{\sigma(1)}_{V_1}\varphi^{\sigma(4)}_{V_1}}{\varphi^{\sigma(2)}_{V_1}\varphi^{\sigma(3)}_{V_1}} \tag{4.11}$$

and

$$\frac{\varphi^{V_2}_{[i_1,i_2,i_3,i_4]}}{(\varphi^1_{V_2})^{i_1}(\varphi^2_{V_2})^{i_2}(\varphi^3_{V_2})^{i_3}(\varphi^4_{V_2})^{i_4}} = \frac{\varphi^{V_1}_{[i_{\sigma^{-1}(1)},i_{\sigma^{-1}(2)},i_{\sigma^{-1}(3)},i_{\sigma^{-1}(4)}]}}{(\varphi^1_{V_1})^{i_{\sigma^{-1}(1)}}(\varphi^2_{V_1})^{i_{\sigma^{-1}(2)}}(\varphi^3_{V_1})^{i_{\sigma^{-1}(3)}}(\varphi^4_{V_1})^{i_{\sigma^{-1}(4)}}} \tag{4.12}$$

for any $i_1, i_2, i_3, i_4 \in \mathbb{Z}^4_{\geq 0}$.

(2) *Suppose V_i is strictly pseudoconvex with smooth boundary or pseudoconvex with analytic boundary for $i = 1, 2$, then the converse of the statement in (1) holds.*

Proof. (1) In the proof of Theorem 4.3 we have already proved (4.11) (see (4.10)). Let $\Psi : V_1 \to V_2$ be a boholomorphic map. Then Ψ must be of the form in Theorem 4.2. By (3.15), (4.2) and (4.4), we have

(4.13)
$$\begin{aligned}
\Psi^*(\psi^{V_2}_{[0,0,0,0]}\Theta^{(2)}_{V_2}) &= \Psi^*\left(\sum_{[i_1,i_2,i_3,i_4]\in S_2}\varphi^{V_2}_{[i_1,i_2,i_3,i_4]}\prod_{k=1}^4|u_k|^{2i_k}\right)\\
&=\sum_{[i_1,i_2,i_3,i_4]\in S_2}\varphi^{V_2}_{[i_1,i_2,i_3,i_4]}\prod_{k=1}^4\left(|c_k|^{2i_k}||u_{\sigma(k)}|^{2i_k}\right)\\
&=\sum_{[i_1,i_2,i_3,i_4]\in S_2}\varphi^{V_2}_{[i_1,i_2,i_3,i_4]}\prod_{k=1}^4\frac{(\varphi^{\sigma(k)}_{V_1})^{i_k}}{(\varphi^k_{V_2})^{i_k}}\prod_{k=1}^4|u_{\sigma(k)}|^{2i_k}\\
&=\sum_{[i_1,i_2,i_3,i_4]\in S_2}\varphi^{V_2}_{[i_1,i_2,i_3,i_4]}\prod_{k=1}^4\frac{(\varphi^{k}_{V_1})^{i_{\sigma^{-1}(k)}}}{(\varphi^k_{V_2})^{i_k}}\prod_{k=1}^4|u_k|^{2i_{\sigma^{-1}(k)}},
\end{aligned}$$

(4.14)
$$\begin{aligned}
\psi^{V_1}_{[0,0,0,0]}\Theta^{(2)}_{V_1} &= \sum_{[i_1,i_2,i_3,i_4]\in S_2}\varphi^{V_1}_{[i_1,i_2,i_3,i_4]}\prod_{k=1}^4\left(|u_k|^{2i_k}\right)\\
&=\sum_{[i_1,i_2,i_3,i_4]\in S_2}\varphi^{V_1}_{[i_{\sigma^{-1}(1)},i_{\sigma^{-1}(2)},i_{\sigma^{-1}(3)},i_{\sigma^{-1}(4)}]}\prod_{k=1}^4|u_k|^{2i_{\sigma^{-1}(k)}}.
\end{aligned}$$

By using Proposition 3.4 and comparing the coefficients, we obtain (4.12).

(2) If there is a σ such that (4.11) and (4.12) hold. Define a automorphism of $\mathbb{C}^4$ as follows:

$$\Psi(u_1,u_2,u_3,u_4)=\left(\sqrt{\frac{\varphi^{\sigma(1)}_{V_1}}{\varphi^1_{V_2}}}u_{\sigma(1)},\sqrt{\frac{\varphi^{\sigma(2)}_{V_1}}{\varphi^2_{V_2}}}u_{\sigma(2)},\sqrt{\frac{\varphi^{\sigma(3)}_{V_1}}{\varphi^3_{V_2}}}u_{\sigma(3)},\sqrt{\frac{\varphi^{\sigma(4)}_{V_1}}{\varphi^4_{V_2}}}u_{\sigma(4)}\right).$$

By (4.11), we have

$$\sqrt{\frac{\varphi^{\sigma(1)}_{V_1}}{\varphi^1_{V_2}}}\sqrt{\frac{\varphi^{\sigma(4)}_{V_1}}{\varphi^4_{V_2}}}=\sqrt{\frac{\varphi^{\sigma(2)}_{V_1}}{\varphi^2_{V_2}}}\sqrt{\frac{\varphi^{\sigma(3)}_{V_1}}{\varphi^3_{V_2}}}.$$

Denote the value of the above equality by c. Then

$$\begin{aligned}
\Psi^*(u_1u_4-u_2u_3) &= (\sqrt{\frac{\varphi^{\sigma(1)}_{V_1}}{\varphi^1_{V_2}}}\sqrt{\frac{\varphi^{\sigma(4)}_{V_1}}{\varphi^4_{V_2}}}u_{\sigma(1)}u_{\sigma(4)}-\sqrt{\frac{\varphi^{\sigma(2)}_{V_1}}{\varphi^2_{V_2}}}\sqrt{\frac{\varphi^{\sigma(3)}_{V_1}}{\varphi^3_{V_2}}}u_{\sigma(2)}u_{\sigma(3)})\\
&= c(u_1u_4-u_2u_3).
\end{aligned}$$

Hence Ψ induce a automorphism of $\widetilde{V}$. It follows from (4.12), (4.13) and (4.14) that $\psi^{V_1}_{[0,0,0,0]}\Theta^{(2)}_{V_1}=\Psi^*(\psi^{V_2}_{[0,0,0,0]}\Theta^{(2)}_{V_2})$. Then by (3.13), we have

$$B^{(2)}_{V_1}(u_1,u_2,u_3,u_4)=B^{(2)}_{V_2}(\Psi(u_1,u_2,u_3,u_4)). \tag{4.15}$$

Next we will prove that V_1 is biholomorpohic to V_2 by using a similar argument in the proof of Theorem B in [DY]. By Fornaess's Lemma (See Lemma 4.7 below), there exists a dense set in the boudary of M_i such that the Bergman kernel blows up at the points in this dense set. Hence the Bergman function $B^{(2)}_{V_i}$ is equal to 1 in a dense subset of ∂V_i. Since $B^{(2)}_{V_i} < 1$ in interior of V_i and Ψ preserve the level set of Bergman functions, Ψ sends a dense sunset of ∂V_1 to a dense subset of ∂V_2. By continuity, Ψ sends ∂V_1 to ∂V_2, thus it induces a biholomorphic map from V_1 to V_2. $\square$

One can find the proof of the following two lemmas for dimension 2 case in [DY]. After small modifications we can generalize it to higher dimension case.

Lemma 4.5. (Henkin [He], Ramirez [Ra]) *Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$. Let p be a point in the boundary of D. Then there exists an L^2 holomorphic function on D which blows up only at p.*

Proof. There exists a holomorphic function f defined on a neighborhood of $\overline{D}$ such that $f(p) = 0$ and $|f(q) - f(p)| \geq |q-p|^2$ for any $q \in \overline{D}$ ([He],[Ra]). Let $F = \frac{1}{f^{\frac{2n-1}{4}}}$, then

$$\int_D |F|^2 = \int_D \frac{1}{|f|^{\frac{2n-1}{2}}} \leq \int_D \frac{1}{|z-p|^{2n-1}} < \infty$$

and F blow ups only at p. $\square$

Remark 4.6. To find such function f as in the proof of the previous lemma, all we need are that p is a strictly pseudoconvex boundary point and that $\overline{D}$ has a Stein neighborhood basis.

Lemma 4.7. (Fornaess) *Let D be a pseudoconvex domain with real analytic boundary in $\mathbb{C}^n$. Let $E = \{p \in \partial D \mid$ there exists $g \in L^2(D)$ that blows up only at $p\}$. Then E is dense in the boundary of D, and the Bergman kernel of D blows up at points in E.*

Proof. Since D has a real analytic boundary, strictly pseudoconvex boundary points are dense in ∂D. Moreover $\overline{D}$ has a Stein neighbourhood basis (see [DF]). Therefore the lemma follows from the previous lemma and remark. $\square$

By Theorem 4.4, we can construct many invariants for V such that $\nu_V \neq 1$. Define an action of G on $\mathbb{Z}^4_{\geq 0}$ as follows:

$$\sigma(i_1, i_2, i_3, i_4) = (i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, i_{\sigma^{-1}(3)}, i_{\sigma^{-1}(4)})$$

for any $\sigma \in G$. The action of G on $\mathbb{Z}^4$ naturally induces an action of G on the ring $\mathbb{C}[x_s \mid s \in \mathbb{Z}^4_{\geq 0}]$:

$$\sigma(x_s) = x_{\sigma(s)}$$

for any $\sigma \in G$ and any $s \in \mathbb{Z}^4_{\geq 0}$.

Theorem 4.8. *Let V a bounded complete Reinhardt domain in $\widetilde{V} = \{(u_1, u_2, u_3, u_4) \in \mathbb{C}^4 \mid u_1u_4 = u_2u_3\}$ such that $\nu_V \neq 1$. Let f be an invariant polynomial in $\mathbb{C}[x_s \mid s \in \mathbb{Z}^4_{\geq 0}]$ under the action of G. If we replace each x_s ($s = (i_1, i_2, i_3, i_4) \in \mathbb{Z}^4_{\geq 0}$) in the expression of f by*

$$\frac{\varphi^V_{[i_1,i_2,i_3,i_4]}}{(\varphi^1_V)^{i_1}(\varphi^2_V)^{i_2}(\varphi^3_V)^{i_3}(\varphi^4_V)^{i_4}},$$

then we obtain an invariant for asymmetric domains.

Proof. It follows from Theorem 4.4 directly. $\square$

For example,

$$(1)\frac{\varphi^V_{[1,1,1,1]}}{\varphi^1_V\varphi^2_V\varphi^3_V\varphi^4_V},\qquad (2)\frac{\varphi^V_{[1,0,0,1]}}{\varphi^1_V\varphi^4_V}+\frac{\varphi^V_{[0,1,1,0]}}{\varphi^2_V\varphi^3_V},$$
$$(3)\frac{\varphi^V_{[1,1,0,0]}}{\varphi^1_V\varphi^2_V}+\frac{\varphi^V_{[1,0,1,0]}}{\varphi^1_V\varphi^3_V}+\frac{\varphi^V_{[0,1,0,1]}}{\varphi^2_V\varphi^4_V}+\frac{\varphi^V_{[0,0,1,1]}}{\varphi^3_V\varphi^4_V}$$

are invariants for asymmetric domains.

Recall that $\widetilde{\pi}:\widetilde{M}\to\widetilde{V}$ is a resolution of $\widetilde{V}$, where $\widetilde{M}$ is the total space of the vector bundle $\mathcal{O}_{\mathbf{P}^1}(-1)\oplus\mathcal{O}_{\mathbf{P}^1}(-1)$ on $\mathbf{P}^1$ and $\widetilde{V}=\{(u_1,u_2,u_3,u_4)\in\mathbb{C}^4\mid u_1u_4=u_2u_3\}$. Denote the exceptional set in $\widetilde{M}$ by E. Let $M_i, i=1,2$ be the pull back by $\widetilde{\pi}$ of a bounded complete Reinhardt domain $V_i, i=1,2$ in $\widetilde{V}$ such that ν_{V_1} or $\nu_{V_2}\neq 1$. If $f:M_1\to M_2$ is biholomorphic, then f induces a biholomorphism between $M_1\backslash E$ and $M_2\backslash E$, therefore induces a biholomorphism between $V_1\backslash 0$ and $V_2\backslash 0$. Since V_1 and V_2 are normal and 0 has codimension 3, this biholomorphism can be extended to an isomorphism between V_1 and V_2. Therefore, the condition in Theorem 4.4(1) is a necessary condition of the equivalence between M_1 and M_2. However, this is not a sufficient condition. There exists a counter example in which $V_1\cong V_2$ but $M_1\ncong M_2$.

Counter-Example 4.9. Let

$$V_1=\widetilde{V}\cap\{|u_1|^2+2|u_2|^2+|u_3|^2+|u_4|^2+\epsilon|u_1^2u_2|^2+\epsilon|u_3^3u_4|^2<1\}$$

and

$$V_2=\widetilde{V}\cap\{|u_1|^2+2|u_2|^2+|u_3|^2+|u_4|^2+\epsilon|u_4^2u_2|^2+\epsilon|u_3^3u_1|^2<1\}.$$

where ϵ is sufficiently small positive number. Let $V_0=\widetilde{V}\cap\{|u_1|^2+2|u_2|^2+|u_3|^2+|u_4|^2<1\}$. In next section we will see that $\nu_{V_0}\neq 1$. By continuity it follows that $\nu_{V_1}\neq 1$ and $\nu_{V_2}\neq 1$. Theorem 4.2 tells us that any biholomorphism between V_1 and V_2 is of the special form: permutation of coordinate modulo scalar multiplication. Then by the definitions of V_1 and V_2 we can see that the unique biholomorphism from V_1 to V_2 is

$$f(u_1,u_2,u_3,u_4)=(u_4,u_2,u_3,u_1).$$

Suppose there is a biholomorphism g from M_1 to M_2, then $\widetilde{\pi}\circ g=f\circ\widetilde{\pi}$. Recall that $\widetilde{M}$ has two charts (x,y_1,y_2) and (w,z_1,z_2) with transition functions $y_1=wz_1$, $y_2=wz_2$ and $x=1/w$. $\widetilde{\pi}$ is given by $\widetilde{\pi}(x,y_1,y_2)=(y_1,y_2,xy_1,xy_2)$ and $\widetilde{\pi}(w,z_1,z_2)=(wz_1,wz_2,z_1,z_2)$. Therefore we have

$$g^*(x)=g^*\widetilde{\pi}^*(u_3/u_1)=\widetilde{\pi}^*f^*(u_3/u_1)=\widetilde{\pi}^*(u_3/u_4)=(xy_1)/(xy_2)=y_1/y_2.$$

Note that the set of indeterminacy of $x=1/w$ is empty while that of y_1/y_2 is not empty. Thus g is not a biholomorphism, which leads to a contradiction.

5. Application

As an application, in this section we will study the equivalence problem for two four parameter families

$$V^k_{a,b,c,d}=\{(u_1,u_2,u_3,u_4)\in\mathbb{C}^4\mid u_1u_4=u_2u_3, a|u_1|^{2k}+b|u_2|^{2k}+c|u_3|^{2k}+d|u_4|^{2k}<\varepsilon\},$$

where $a, b, c, d > 0$, ε is a fixed positive constant and $k = 1, 2$.

Case $k = 1$**:**
Let $V = V^1_{a,b,c,d} \subseteq \widetilde{V}$, using formula (3.3) we have

$$\begin{aligned}
||\phi_{\alpha\beta\gamma}||^2_M =&(\sqrt{-1})^9 \int_M \phi_{\alpha\beta\gamma}\overline{\phi_{\alpha\beta\gamma}} \\
=&2^3(2\pi)^3 \int_{ar_1^2+br_2^2+c(sr_1)^2+d(sr_2)^2<\varepsilon} s^{2\alpha+1}r_1^{2\beta+1}r_2^{2\gamma+1}dsdr_1dr_2 \\
=&64\pi^3 \int_{(a+s^2c)r_1^2+(b+s^2d)r_2^2<\varepsilon} s^{2\alpha+1}r_1^{2\beta+1}r_2^{2\gamma+1}dsdr_1dr_2.
\end{aligned}$$

Wirte $\widetilde{r_1} = (\sqrt{a+s^2c})r_1$ and $\widetilde{r_2} = (\sqrt{b+s^2d})r_2$, then

$$\begin{aligned}
||\phi_{\alpha\beta\gamma}||^2_M =&64\pi^3 \int_{s\geq 0}\int_{\widetilde{r_1}^2+\widetilde{r_2}^2<\varepsilon} s^{2\alpha+1}\left(\frac{\widetilde{r_1}}{\sqrt{a+s^2c}}\right)^{2\beta+1}\left(\frac{\widetilde{r_2}}{\sqrt{b+s^2d}}\right)^{2\gamma+1}\cdot \\
&ds\frac{dr_1}{\sqrt{a+s^2c}}\frac{dr_2}{\sqrt{b+s^2d}} \\
=&64\pi^3 \int_0^\infty \frac{s^{2\alpha+1}}{(a+s^2c)^{\beta+1}(b+s^2d)^{\gamma+1}}ds \int_{\widetilde{r_1}^2+\widetilde{r_2}^2<\varepsilon} \widetilde{r_1}^{2\beta+1}\widetilde{r_2}^{2\gamma+1}d\widetilde{r_1}d\widetilde{r_2}.
\end{aligned}$$

Write $\widetilde{s} = s^2$ then

$$||\phi_{\alpha\beta\gamma}||^2_M =32\pi^3 \int_0^\infty \frac{\widetilde{s}^\alpha}{(a+\widetilde{s}c)^{\beta+1}(b+\widetilde{s}d)^{\gamma+1}}d\widetilde{s} \int_{\widetilde{r_1}^2+\widetilde{r_2}^2<\varepsilon} \widetilde{r_1}^{2\beta+1}\widetilde{r_2}^{2\gamma+1}d\widetilde{r_1}d\widetilde{r_2}.$$

Denote

$$A_{\alpha\beta\gamma} = \int_0^\infty \frac{\widetilde{s}^\alpha}{(a+\widetilde{s}c)^{\beta+1}(b+\widetilde{s}d)^{\gamma+1}}d\widetilde{s}$$

and

$$B_{\beta\gamma} = \int_{\widetilde{r_1}^2+\widetilde{r_2}^2<\varepsilon} \widetilde{r_1}^{2\beta+1}\widetilde{r_2}^{2\gamma+1}d\widetilde{r_1}d\widetilde{r_2},$$

then $||\phi_{\alpha\beta\gamma}||^2 = 32\pi^3 A_{\alpha\beta\gamma}B_{\beta\gamma}$. By calculation, we have

$$\begin{aligned}
A_{000} &= \frac{1}{bc-ad}\ln(\frac{bc}{ad}), \\
A_{010} &= \left(\frac{1}{a(bc-ad)} - \frac{d}{(bc-ad)^2)}\ln(\frac{bc}{ad})\right), \\
A_{001} &= \left(\frac{1}{b(ad-bc)} - \frac{c}{(ad-bc)^2)}\ln(\frac{ad}{bc})\right), \\
A_{110} &= \frac{1}{c}A_{000} - \frac{a}{c}A_{010}, \\
A_{101} &= \frac{1}{d}A_{000} - \frac{b}{d}A_{001}.
\end{aligned}$$

Hence by (4.1)

$$\begin{aligned}
\nu_V &= \frac{||\phi_{001}||^2_M||\phi_{110}||^2_M}{||\phi_{010}||^2_M||\phi_{101}||^2_M} = \frac{A_{001}A_{110}}{A_{010}A_{101}} \\
&= \frac{bc(\frac{ad}{bc} - 1 - \ln(\frac{ad}{bc}))^2}{ad(\frac{bc}{ad} - 1 - \ln(\frac{bc}{ad})^2}.
\end{aligned}$$

Denote

$$f(x) = \begin{cases} \frac{x(\frac{1}{x}-1-\ln\frac{1}{x})^2}{(x-1-\ln x)^2}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases}$$

then $\nu_V = f(\frac{bc}{ad})$. It is easy to see that $f(x)f(1/x) = 1$ for any $x > 0$ and f is continuous.

Lemma 5.1. *$f(x)$ is a strictly decreasing function when $x > 0$.*

Proof. As $f(x)f(1/x) = 1$, it is enough to check that $f(x)$ is strictly decreasing when $x > 1$. By calculation,

$$f'(x) = \frac{(\frac{1}{x} - 1 + \ln x)(-6 + 3x + \frac{3}{x} - x\ln x + \frac{\ln x}{x} - (\ln x)^2)}{(x - 1 - \ln x)^3}.$$

Let

$$g(x) = -6 + 3x + \frac{3}{x} - x\ln x + \frac{\ln x}{x} - (\ln x)^2.$$

We claim that $g(x) < 0$ if $x > 1$. Indeed, $g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x\right)\frac{(x+1)^2}{x^2} < 0$ if $x > 1$ and $g(1) = 0$. Hence $g(x) < 0$ for $x > 1$, which implies that $f'(x) < 0$ if $x > 1$. □

Theorem 5.2. *Let $V_1 = V^1_{a_1,b_1,c_1,d_1}$ and $V_2 = V^1_{a_2,b_2,c_2,d_2}$. The followings are equivalent.*

(1) $V_1 \simeq V_2$;

(2) $\nu_{V_1} = \nu_{V_2}$ or $\nu_{V_1}\nu_{V_2} = 1$;

(3) $\frac{a_1d_1}{b_1c_1} = \frac{a_2d_2}{b_2c_2}$ or $\frac{a_1d_1}{b_1c_1} = \frac{b_2c_2}{a_2d_2}$.

Proof. (1) $\Rightarrow$ (2): it follows from Theorem 4.3.

(2) $\Rightarrow$ (3): If $\nu_{V_1} = \nu_{V_2}$ then $f(\frac{b_1c_1}{a_1d_1}) = f(\frac{b_2c_2}{a_2d_2})$, which implies that $\frac{b_1c_1}{a_1d_1} = \frac{b_2c_2}{a_2d_2}$ since $f(x)$ is strictly decreasing. If $\nu_{V_1} = 1/\nu_{V_2}$, then $f(\frac{b_1c_1}{a_1d_1}) = 1/f(\frac{b_2c_2}{a_2d_2}) = f(\frac{a_2d_2}{b_2c_2})$, which implies that $\frac{b_1c_1}{a_1d_1} = \frac{a_2d_2}{b_2c_2}$.

(3) $\Rightarrow$ (1): If $\frac{a_1d_1}{b_1c_1} = \frac{a_2d_2}{b_2c_2}$, then $\frac{a_1d_1}{a_2d_2} = \frac{b_1c_1}{b_2c_2}$. Denote $p = \frac{a_1d_1}{a_2d_2} = \frac{b_1c_1}{b_2c_2}$. Let

$$\Psi = (\sqrt{\frac{a_1}{a_2}}u_1, \sqrt{\frac{b_1}{b_2}}u_2, \sqrt{\frac{c_1}{c_2}}u_3, \sqrt{\frac{d_1}{d_2}}u_4).$$

Then

$$\Psi^*(u_1u_4 - u_2u_3) = \sqrt{\frac{a_1}{a_2}}\sqrt{\frac{d_1}{d_2}}u_1u_4 - \sqrt{\frac{c_1}{c_2}}\sqrt{\frac{b_1}{b_2}}u_3u_4 = \sqrt{p}(u_1u_4 - u_2u_3)$$

and

$$\Psi^*(a_2|u_1|^2 + b_2|u_2|^2 + c_2|u_3|^2 + d_2|u_4|^2) = a_1|u_1|^2 + b_1|u_2|^2 + c_1|u_3|^2 + d_1|u_4|^2.$$

Hence Ψ is a biholomorphic map from V_1 to V_2.

If $\frac{a_1d_1}{b_1c_1} = \frac{b_2c_2}{a_2d_2}$, then let

$$\Psi = (\sqrt{\frac{b_1}{a_2}}u_2, \sqrt{\frac{a_1}{b_2}}u_1, \sqrt{\frac{d_1}{c_2}}u_4, \sqrt{\frac{c_1}{d_2}}u_3).$$

Using a similar argument in the previous case, we can prove that Ψ is a biholomorphic map from V_1 to V_2. □

Case $k=2$**:**

Let $V=V^2_{a,b,c,d}\subseteq\widetilde{V}$, using formula (3.3) we have

$$\begin{aligned}
||\phi_{\alpha\beta\gamma}||^2_M=&(\sqrt{-1})^9\int_M\phi_{\alpha\beta\gamma}\overline{\phi_{\alpha\beta\gamma}}\\
=&2^3(2\pi)^3\int_{ar_1^4+br_2^4+c(sr_1)^4+d(sr_2)^4<\varepsilon}s^{2\alpha+1}r_1^{2\beta+1}r_2^{2\gamma+1}dsdr_1dr_2\\
=&64\pi^3\int_{(a+s^4c)r_1^4+(b+s^4d)r_2^4<\varepsilon}s^{2\alpha+1}r_1^{2\beta+1}r_2^{2\gamma+1}dsdr_1dr_2.
\end{aligned}$$

Write $\widetilde{r_1}=(\sqrt[4]{a+s^4c})r_1$ and $\widetilde{r_2}=(\sqrt[4]{b+s^4d})r_2$, then

$$\begin{aligned}
||\phi_{\alpha\beta\gamma}||^2_M=&64\pi^3\int_{s\geq 0}\int_{\widetilde{r_1}^4+\widetilde{r_2}^4<\varepsilon}s^{2\alpha+1}\left(\frac{\widetilde{r_1}}{\sqrt[4]{a+s^4c}}\right)^{2\beta+1}\left(\frac{\widetilde{r_2}}{\sqrt[4]{b+s^4d}}\right)^{2\gamma+1}\cdot\\
&ds\frac{dr_1}{\sqrt[4]{a+s^4c}}\frac{dr_2}{\sqrt[4]{b+s^4d}}\\
=&64\pi^3\int_0^\infty\frac{s^{2\alpha+1}}{\sqrt{a+s^4c}^{\beta+1}\sqrt{b+s^4d}^{\gamma+1}}ds\int_{\widetilde{r_1}^4+\widetilde{r_2}^4<\varepsilon}\widetilde{r_1}^{2\beta+1}\widetilde{r_2}^{2\gamma+1}d\widetilde{r_1}d\widetilde{r_2}.
\end{aligned}$$

Write $\widetilde{s}=s^2$ then

$$||\phi_{\alpha\beta\gamma}||^2_M=32\pi^3\int_0^\infty\frac{\widetilde{s}^{\alpha}}{\sqrt{a+\widetilde{s}^2c}^{\beta+1}\sqrt{b+\widetilde{s}^2d}^{\gamma+1}}d\widetilde{s}\int_{\widetilde{r_1}^4+\widetilde{r_2}^4<\varepsilon}\widetilde{r_1}^{2\beta+1}\widetilde{r_2}^{2\gamma+1}d\widetilde{r_1}d\widetilde{r_2}.$$

Denote

$$A_{\alpha\beta\gamma}=\int_0^\infty\frac{\widetilde{s}^{\alpha}}{\sqrt{a+\widetilde{s}^2c}^{\beta+1}\sqrt{b+\widetilde{s}^2d}^{\gamma+1}}d\widetilde{s}$$

and

$$B_{\beta\gamma}=\int_{\widetilde{r_1}^4+\widetilde{r_2}^4<\varepsilon}\widetilde{r_1}^{2\beta+1}\widetilde{r_2}^{2\gamma+1}d\widetilde{r_1}d\widetilde{r_2},$$

then $||\phi_{\alpha\beta\gamma}||^2=32\pi^3A_{\alpha\beta\gamma}B_{\beta\gamma}$. By calculation, there are the following three cases:

(1) $bc>ad$, then

$$\begin{aligned}
A_{010}=&\frac{1}{\sqrt{a}\sqrt{bc-ad}}\left(\arctan\frac{\sqrt{bc-ad}}{\sqrt{ad}}\right),\\
A_{001}=&\frac{1}{\sqrt{b}\sqrt{bc-ad}}\left(\operatorname{arctanh}\frac{\sqrt{bc-ad}}{\sqrt{bc}}\right),\\
A_{110}=&\frac{1}{\sqrt{c}\sqrt{bc-ad}}\left(\operatorname{arctanh}\frac{\sqrt{bc-ad}}{\sqrt{bc}}\right),\\
A_{101}=&\frac{1}{\sqrt{d}\sqrt{bc-ad}}\left(\arctan\frac{\sqrt{bc-ad}}{\sqrt{ad}}\right),\\
\nu_V=&\frac{||\phi_{001}||^2_M||\phi_{110}||^2_M}{||\phi_{010}||^2_M||\phi_{101}||^2_M}=\frac{A_{001}A_{110}}{A_{010}A_{101}}\\
=&\frac{\sqrt{ad}\left(\operatorname{arctanh}\sqrt{1-\frac{ad}{bc}}\right)^2}{\sqrt{bc}\left(\arctan\sqrt{\frac{bc}{ad}-1}\right)^2}.
\end{aligned}$$

(2) $bc = ad$, then

$$
\begin{aligned}
A_{010} &= \frac{1}{a\sqrt{d}}, \quad A_{001} = \frac{1}{b\sqrt{c}},\\
A_{110} &= \frac{1}{c\sqrt{b}}, \quad A_{101} = \frac{1}{d\sqrt{a}},\\
\nu_V &= \frac{||\phi_{001}||_M^2||\phi_{110}||_M^2}{||\phi_{010}||_M^2||\phi_{101}||_M^2} = \frac{A_{001}A_{110}}{A_{010}A_{101}} = \left(\frac{ad}{bc}\right)^{\frac{3}{2}} = 1.
\end{aligned}
$$

(3) $bc < ad$, then

$$
\begin{aligned}
A_{010} &= \frac{1}{\sqrt{a}\sqrt{ad-bc}}\left(\text{arctanh}\frac{\sqrt{ad-bc}}{\sqrt{ad}}\right),\\
A_{001} &= \frac{1}{\sqrt{b}\sqrt{ad-bc}}\left(\arctan\frac{\sqrt{ad-bc}}{\sqrt{bc}}\right),\\
A_{001} &= \frac{1}{\sqrt{c}\sqrt{ad-bc}}\left(\arctan\frac{\sqrt{ad-bc}}{\sqrt{bc}}\right),\\
A_{010} &= \frac{1}{\sqrt{d}\sqrt{ad-bc}}\left(\text{arctanh}\frac{\sqrt{ad-bc}}{\sqrt{ad}}\right),\\
\nu_V &= \frac{||\phi_{001}||_M^2||\phi_{110}||_M^2}{||\phi_{010}||_M^2||\phi_{101}||_M^2} = \frac{A_{001}A_{110}}{A_{010}A_{101}}\\
&= \frac{\sqrt{ad}\left(\arctan\sqrt{\frac{ad}{bc}-1}\right)^2}{\sqrt{bc}\left(\text{arctanh}\sqrt{1-\frac{bc}{ad}}\right)^2}.
\end{aligned}
$$

Define function

$$
f(x) = \begin{cases} \frac{\left(\text{arctanh}\sqrt{1-\frac{1}{x}}\right)^2}{\sqrt{x}\left(\arctan\sqrt{x-1}\right)^2}, & \text{if } x > 1\\ 1, & \text{if } x = 1\\ \frac{\left(\arctan\sqrt{\frac{1}{x}-1}\right)^2}{\sqrt{x}\left(\text{arctanh}\sqrt{1-x}\right)^2}, & \text{if } x < 1 \end{cases}
$$

then $\nu_V = f(\frac{bc}{ad})$. It is easy to see that $f(x)f(1/x) = 1$ for any $x > 0$ and f is continuous.

Lemma 5.3. *$f(x)$ is a strictly decreasing function when $x > 0$.*

Proof. As $f(x)f(1/x) = 1$, it is enough to check that $f(x)$ is strict decreasing when $x > 1$. By calculation, when $x > 0$, we have

$$
f'(x) = \left(\frac{\arctan\sqrt{x-1}}{\sqrt{1-\frac{1}{x}}} - \frac{1}{2}\text{arctanh}\sqrt{1-\frac{1}{x}}\arctan\sqrt{x-1} - \frac{\text{arctanh}\sqrt{1-\frac{1}{x}}}{\sqrt{x-1}}\right) \cdot \frac{\text{arctanh}\sqrt{1-\frac{1}{x}}}{x\sqrt{x}(\arctan\sqrt{x-1})^3}.
$$

Let

$$g(x) = \left(\frac{\arctan\sqrt{x-1}}{\sqrt{1-\frac{1}{x}}} - \frac{1}{2}\operatorname{arctanh}\sqrt{1-\frac{1}{x}}\arctan\sqrt{x-1} - \frac{\operatorname{arctanh}\sqrt{1-\frac{1}{x}}}{\sqrt{x-1}} \right).$$

We claim that $g(x) < 0$ if $x > 1$. Indeed,

$$g'(x) = \frac{(x+1)\sqrt{x-1}}{4(x-1)^2\sqrt{x}} \left(-\arctan\sqrt{x-1} + \frac{1}{\sqrt{x}}\operatorname{arctanh}\sqrt{1-\frac{1}{x}} \right) < 0$$

for $x > 1$ and $g(1) = 0$. Hence $g(x) < 0$ if $x > 1$, which implies that $f'(x) < 0$ if $x > 1$. $\square$

Theorem 5.4. *Let $V_1 = V^2_{a_1,b_1,c_1,d_1}$ and $V_2 = V^2_{a_2,b_2,c_2,d_2}$. The followings are equivalent.*

(1) $V_1 \simeq V_2$;
(2) $\nu_{V_1} = \nu_{V_2}$ or $\nu_{V_1}\nu_{V_2} = 1$;
(3) $\frac{a_1d_1}{b_1c_1} = \frac{a_2d_2}{b_2c_2}$ or $\frac{a_1d_1}{b_1c_1} = \frac{b_2c_2}{a_2d_2}$.

Proof. (1) $\Rightarrow$ (2): it follows from Theorem 4.3.

(2) $\Rightarrow$ (3): If $\nu_{V_1} = \nu_{V_2}$ then $f(\frac{b_1c_1}{a_1d_1}) = f(\frac{b_2c_2}{a_2d_2})$, which implies that $\frac{b_1c_1}{a_1d_1} = \frac{b_2c_2}{a_2d_2}$ since $f(x)$ is strictly decreasing. If $\nu_{V_1} = 1/\nu_{V_2}$, then $f(\frac{b_1c_1}{a_1d_1}) = 1/f(\frac{b_2c_2}{a_2d_2}) = f(\frac{a_2d_2}{b_2c_2})$, which implies that $\frac{b_1c_1}{a_1d_1} = \frac{a_2d_2}{b_2c_2}$.

(3) $\Rightarrow$ (1): If $\frac{a_1d_1}{b_1c_1} = \frac{a_2d_2}{b_2c_2}$, then $\frac{a_1d_1}{a_2d_2} = \frac{b_1c_1}{b_2c_2}$. Denote $p = \frac{a_1d_1}{a_2d_2} = \frac{b_1c_1}{b_2c_2}$. Let

$$\Psi = (\sqrt[4]{\frac{a_1}{a_2}}u_1, \sqrt[4]{\frac{b_1}{b_2}}u_2, \sqrt[4]{\frac{c_1}{c_2}}u_3, \sqrt[4]{\frac{d_1}{d_2}}u_4).$$

Then

$$\Psi^*(u_1u_4 - u_2u_3) = \sqrt[4]{\frac{a_1}{a_2}}\sqrt[4]{\frac{d_1}{d_2}}u_1u_4 - \sqrt[4]{\frac{c_1}{c_2}}\sqrt[4]{\frac{b_1}{b_2}}u_3u_4 = \sqrt[4]{p}(u_1u_4 - u_2u_3)$$

and

$$\Psi^*(a_2|u_1|^4 + b_2|u_2|^4 + c_2|u_3|^4 + d_2|u_4|^4) = a_1|u_1|^4 + b_1|u_2|^4 + c_1|u_3|^4 + d_1|u_4|^4.$$

Hence Ψ is a biholomorphic map from V_1 to V_2.

If $\frac{a_1d_1}{b_1c_1} = \frac{b_2c_2}{a_2d_2}$, then let

$$\Psi = (\sqrt[4]{\frac{b_1}{a_2}}u_2, \sqrt[4]{\frac{a_1}{b_2}}u_1, \sqrt[4]{\frac{d_1}{c_2}}u_4, \sqrt[4]{\frac{c_1}{d_2}}u_3).$$

Using a similar argument in the previous case, we can prove that Ψ is a biholomorphic map from V_1 to V_2. $\square$

Corollary 5.5. *Let $M^k_{a,b,c,d} = \{(x, y_1, y_2) \in \mathbb{C}^3 \mid a|y_1|^{2k} + b|y_2|^{2k} + c|xy_1|^{2k} + d|xy_2|^{2k} < \varepsilon\} \cup \{(w, z_1, z_2) \in \mathbb{C}^3 \mid a|wz_1|^{2k} + b|wz_2|^{2k} + c|z_1|^{2k} + d|z_2|^{2k} < \varepsilon\}$ where $z_1 = xy_1$, $z_2 = xy_2$ and $w = 1/x$. Then $M^k_{a,b,c,d}$ is a four parameter family of open Calabi-Yau manifolds. We have*

$$M^k_{a_1,b_1,c_1,d_1} \simeq M^k_{a_2,b_2,c_2,d_2} \Longleftrightarrow \frac{a_1d_1}{b_1c_1} = \frac{a_2d_2}{b_2c_2} \text{ or } \frac{a_1d_1}{b_1c_1} = \frac{b_2c_2}{a_2d_2}$$

for $k = 1, 2$.

Proof. $\Longrightarrow$: $M^k_{a,b,c,d}$ is the pull back of $V^k_{a,b,c,d}$ by $\widetilde{\pi}:\widetilde{M}\to\widetilde{V}$. Since a biholomorphism between $M^k_{a_1,b_1,c_1,d_1}$ and $M^k_{a_2,b_2,c_2,d_2}$ induces a biholomorphism between $V^k_{a_1,b_1,c_1,d_1}$ and $V^k_{a_2,b_2,c_2,d_2}$, it follows from Theorem 5.2 and Theorem 5.4.

$\Longleftarrow$: If $\frac{a_1d_1}{b_1c_1}=\frac{a_2d_2}{b_2c_2}$ or $\frac{a_1d_1}{b_1c_1}=\frac{b_2c_2}{a_2d_2}$, from the proof of Theorem 5.2 and Theorem 5.4 we see that there is a biholomorphism Ψ between $V^k_{a_1,b_1,c_1,d_1}$ and $V^k_{a_2,b_2,c_2,d_2}$ of the form $\Psi(u_1,u_2,u_3,u_4)=(u_1,u_2,u_3,u_4)$ or $\Psi(u_1,u_2,u_3,u_4)=(u_2,u_1,u_4,u_3)$ modulo scalar multiplication. So we only need to check that $\Psi(u_1,u_2,u_3,u_4)=(u_2,u_1,u_4,u_3)$ induces a automorphism of $\widetilde{M}$. Indeed, it induces the following automorphism:

$$\begin{cases} x=x\\ y_1=y_2\\ y_2=y_1\end{cases} \quad \textit{and} \quad \begin{cases} w=w\\ z_1=z_2\\ z_2=z_1\end{cases}.$$

where (x,y_1,y_2) and (z,w_1,w_2) are two charts of $\widetilde{M}$ with transition functions $z_1=xy_1$, $z_2=xy_2$ and $w=1/x$. $\square$

References

[Bo] Boutet De Monvel, *Integration des equation de Cauchy-Riemann*, Seminaire Goulaouic-Lions-Schwartz, Ex ré IX, 1974-1975.

[Ca] E. Cartan, *Sur la geometrie pseudo-conforme des hypersurfaces de deux variables complexes, I-II*, Ann. Math. Pura Appl. (4) **11** (1932), 17-90; and Ann. Scuola Norm. Sup. Pisa (2) **1** (1932), 333-354.

[CM] S.S. Chern and J. Moser, *Real hypersurfaces in complex manifolds*, ActaMath. **133** (1974), 219-271.

[DF] K. Diederich and J.E. Fornaess, *Pseudoconvex domains with real analytic boundary*, Ann. of Math. **107** (1978), 371-384.

[DGY] R. Du, Y. Gao and S. S.-T. Yau, *Explicit construction of moduli space of bounded complete Reinhardt domains in $\mathbb{C}^n$*, Comm. Anal. Geom., Vol. **18**, No. 3(2010), 601-626.

[DY] R. Du and S.S.-T. Yau, *Higher order Bergman functions and explicit construction of moduli space for complete Reinhardt domains*, J. Diff. Geom. **82**(3) (2009), 567-610.

[Fe] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1-65.

[He] G. M. Henkin, *Integral representations of functions holomorphic in strictly pseudoconvex domains and some applications*, Math. USSR Sb. **11** (1969), 597-616.

[Hi] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II*, Ann. of Math. **79** (1964), 109-326.

[HL] R. Harvey and B. Lawson, *On boundaries of complex analytic varieties I*, Ann. of Math. **102** (1975), 233-290.

[Ko] J. Kohn, *The range of the tangential Cauchy-Riemann operator*, Duke Math. J. **53** (1986), 525-545.

[Kr] S. Krantz, *Function Theory of Several Complex Variables*, John Wiley & Sons, New York, 1982.

[LYY] H.-S. Luk, S.S.-T. Yau and L.-Y. Leh, *Bergman kernels on resolutions of isolated singularities*, Math. Res. Lett. **8** (2001), 303-319.

[Ra] E. Ramirez, *Ein Divisionsproblem und Randintegraldarstellungen in der kom- plexen Analysis*, Math. Ann. **184** (1970), 172-187.

[Ta] N. Tanaka, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan J. Math. **2** (1976), 131-190.

[We] S. M. Webster, Pseudohermitian structures on a real hypersurface, J. Diff. Geometry **13** (1978), 25-41.

[Ya] S. S.-T. Yau, *Global invariants for strongly pseudoconvex varieties with isolated singularities: Bergman functions*, Math. Res. Lett. **11** (2004), 809-832.

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