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UPPER BOUND OF DISCREPANCIES OF DIVISORS COMPUTING MINIMAL LOG DISCREPANCIES ON SURFACES

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ABSTRACT. Fix a subset $I \subseteq \mathbb{R}_{>0}$ such that

$$\gamma = \inf\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in I\} > 0.$$

We give a explicit upper bound $\ell(\gamma) \in O(1/\gamma^2)$ as $\gamma \to 0$, such that for any smooth surface A of arbitrary characteristic with a closed point 0 and an $\mathbb{R}$-ideal $\mathfrak{a}$ with exponents in I, there always exists a prime divisor E over A computing the minimal log discrepancy of $(A, \mathfrak{a})$ at 0 and with its log discrepancy $k_E + 1 \leq \ell(\gamma)$.

1. Introduction

Let A be a smooth variety over an algebraically closed field k and $0 \in A$ be a closed point. Let $\mathfrak{a}$ be an $\mathbb{R}$-ideal on A, that is, formal product $\mathfrak{a} = \prod_{j=1}^{r} \mathfrak{a}_j^{\lambda_j}$, where each λ_j is a non-negative real number and each $\mathfrak{a}_j$ is a non-zero coherent ideal sheaf on A. Denote by $\mathrm{mld}(0; \mathrm{A}, \mathfrak{a})$ the minimal log discrepancy (mld, for short) of the pair $(A, \mathfrak{a})$ at 0 and denote by $a(E; A, \mathfrak{a})$ the log discrepancy of E with repect to $(A, \mathfrak{a})$. We say a prime divisor E with the center 0 computes $\mathrm{mld}(0; \mathrm{A}, \mathfrak{a})$ if $a(E; A, \mathfrak{a})$ equals to $\mathrm{mld}(0; \mathrm{A}, \mathfrak{a})$ or is negative. Mustaţă and Nakamura [MN] posed a conjecture, says Mustaţă-Nakamura conjecture (MN conjecture, for short) on the boundness of the discrepancy of divisors computing mld. Although the original statement is more general, we state the conjecture only for smooth varieties since we will focus on smooth surfaces in this paper.

Conjecture 1.1 (MN conjecture for smooth varieties)**.** *Let A be a smooth variety of dimension N over an algebraically closed field with a closed point* 0*. Given a finite subset I of the positive real numbers, there exists a positive integer $\ell_{N,I}$ depending only on N and I such that for any $\mathbb{R}$-ideal $\mathfrak{a}$ with exponents in I, there exists a prime divisor E over A that computes* $\mathrm{mld}(0; A, \mathfrak{a})$ *and such that its log discrepancy $k_E + 1 \leq \ell_{N,I}$.*

MN conjecture is important in birational geometry. It is proved in [MN] that this conjecture implies the ACC conjecture for mld in characteristic 0. Then Kawakita [Ka, Theorem 4.6] proves the converse also holds for threefolds. Besides, MN conjecture also plays an important role on basic properties of singularities, for example, it guarantees lower semi-continuity of Mather-Jacobian mld and also stability of Mather-Jacobian log canonicity under small deformations which are not known for positive characteristic (see Theorem 1.3 and Proposition 1.7 in [Is1]).

For surfaces over the base field of characteristic 0, it is proved by Mustaţă and Nakamura [MN] that MN conjecture holds. In this case, Alexeev [Al, Lemma 3.7] proves that it still holds even when I is just a DCC set but not a finite set, under the assumption that $\mathfrak{a}$ is locally principle (i.e. an $\mathbb{R}$-divisor) and $\mathrm{mld}(0; A, \mathfrak{a}) \geq 0$. One can see [CH, Theorem B.1] for a proof of Alexeev's result. Han and Luo [HL, Theorem 1.3] prove it in a more general setting: I is a subset of $\mathbb{R}_{>0}$ such that

$$\gamma = \inf\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in I\} > 0.$$

Note that this condition is satisfied for any DCC sets (see [HL, Lemma 3.2]). They also give a explicit upper bound which only depends on γ.

Theorem 1.2. [HL, Theorem 1.3] *Given a subset I of the positive real numbers such that*

$$\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in I\} \subseteq [\gamma, +\infty).$$

for some $\gamma \in (0, 1]$. Let X be a smooth surface over $\mathbb{C}$ with a closed point 0 and B an effective $\mathbb{R}$-divisors on X with coefficients in I such that $\mathrm{mld}(0; X, B) \geq 0$. *Then there exists a prime divisor E over X that computes* $\mathrm{mld}(0; X, B)$ *and with its log discrepancy $k_E + 1 \leq 2^{N_0}$, where*

$$N_0 = \left\lfloor 1 + \frac{32}{\gamma^2} + \frac{1}{\gamma} \right\rfloor.$$

The upper bound they give grows roughly like $2^{1/\gamma^2}$ when γ tends to 0. In this paper, we will use a completely different approach to give a smaller bound which belongs to $O(\frac{1}{\gamma^2})$ as $\gamma \to 0$.

Our idea comes from Ishii [Is2]. In the paper, Ishii proves that MN conjecture holds for any smooth surface A of arbitrary characteristic and she points out that in surfaces case the upper bound in the conjecture can be calculated by using toric geometry. Indeed, she proves that for every $\mathbb{R}$-ideal $\mathfrak{a}$ on a smooth surface A there is a monomial $\mathbb{R}$-ideal $\mathfrak{a}_*$ on $\mathbb{A}_k^2$ with same exponents as $\mathfrak{a}$, such that $\mathrm{mld}(0; \mathrm{A}, \mathfrak{a}) = \mathrm{mld}(0; \mathbb{A}_{\mathrm{k}}^2, \mathfrak{a})$ and $a(E; A, \mathfrak{a}) \leq a(E; \mathbb{A}_{\mathrm{k}}^2, \mathfrak{a}_*)$ for any prime divisor E with center 0 (as there is a natural bijection between the set of prime divisors over A with the center 0 and that over $\mathbb{A}_k^2$ with the center 0, we can identify them). Thus every prime divisor computing $\mathrm{mld}(0; \mathbb{A}_{\mathrm{k}}^2, \mathfrak{a}_*)$ also computes $\mathrm{mld}(0; \mathrm{A}, \mathfrak{a})$. Then the problem is reduced to the one on the pairs of monomial ideals on $\mathbb{A}_k^2$ and can be solved by combinatorics.

Our main theorem is

Theorem 1.3. *Let A be a smooth surface over an algebraically closed field of arbitrary characteristic and let 0 be a closed point on A. Given a subset I of the positive real numbers, denote $e = \inf I$ and*

$$\gamma = \inf\left\{\sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in I\right\}.$$

Suppose $\gamma > 0$, then

(1) *for any $\mathbb{R}$-ideal $\mathfrak{a}$ with exponents in I such that* $\mathrm{mld}(0; A, \mathfrak{a}) \geq 0$, *there exists a prime divisor E over A that computes* $\mathrm{mld}(0; A, \mathfrak{a})$ *and such that its log discrepancy*

$$k_E + 1 \leq \max\left\{\left\lfloor \frac{\gamma+1}{e\gamma} \right\rfloor + \left\lceil \frac{\gamma+1}{e} \right\rceil, 2\right\},$$

(2) *for any $\mathbb{R}$-ideal $\mathfrak{a}$ with exponents in I such that* $\operatorname{mld}(0; A, \mathfrak{a}) = -\infty$*, there exists a prime divisor E over A that computes* $\operatorname{mld}(0; A, \mathfrak{a})$ *and such that its log discrepancy*
$$k_E + 1 \leq \left\lfloor \frac{\gamma+1}{e\gamma} \right\rfloor + \left\lceil \frac{\gamma+1}{e} \right\rceil + 1.$$

Remark 1.4. Note that we always have $e \geq \gamma$, so we can replace e by γ in the bound.

Over a smooth surface, every exceptional divisor E can be obtained by a finite sequence of blowing-ups of points and its discrepancy k_E is equal to the number of necessary blowing-ups of points to obtain E. Therefore we obtain a upper bound of the number of necessary blowing-ups of points to get a divisor computing the mld on surfaces.

The following two examples indicate that our bound is optimal. The proofs of the examples can be found in Section 5.

Example 1.5. Fix a positive integer $n \geq 2$. Denote
$$e = \frac{1}{n-1} + \frac{1}{n^2}.$$
Let $I = \{e\}$. Then $e = \inf I$ and
$$\gamma := \inf \left\{ \sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in I \right\} = \frac{n-1}{n^2}.$$
By calculation, the bound in Theorem 1.3 (1) is
$$n^2 + n - 1.$$
Let $\mathfrak{a} = (x^{n^2}, y^{n-1})^e$ on $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$. Then $\operatorname{mld}(0; \mathbb{A}^2_{\mathrm{k}}, \mathfrak{a}) = 0$ and the toric divisor corresponding to the vector $(n-1, n^2)$ computes the mld with its log discrepancy equal to $n^2 + n - 1$. Moreover, any prime divisor that computes the mld satisfies that its log discrepancy $\geq n^2 + n - 1$. Therefore the bound is optimal.

Example 1.6. Fix a positive integer n. Let $I = \{1/n\}$. Then $e := \inf I = 1/n$ and
$$\gamma := \inf \left\{ \sum_i n_i b_i - 1 > 0 \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in I \right\} = \frac{1}{n}.$$
By calculation, the bound in Theorem 1.3 (2) is
$$(n+1)^2 + 1.$$
Let $\mathfrak{a} = (x^{n^2+n+1}, y^{n+1})^{1/n}$ on $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$. Then $\operatorname{mld}(0; \mathbb{A}^2_{\mathrm{k}}, \mathfrak{a}) = -\infty$ and the toric divisor corresponding to the vector $(n+1, n^2+n+1)$ computes the mld with its log discrepancy equal to $(n+1)^2 + 1$. Moreover, any prime divisor that computes the mld satisfies that its log discrepancy $\geq (n+1)^2 + 1$. Therefore the bound is optimal.

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2. Minimal log discrepancy

Let k ba an algebraically closed field of arbitrary characteristic.

Definition 2.1. Let A be a smooth variety over k and $\mathfrak{a} = \prod_{j=1}^{r} \mathfrak{a}_j^{\lambda_j}$ an $\mathbb{R}$-ideal on A. For a prime divisor E over A, the log discrepancy of $(A, \mathfrak{a})$ is defined to be

$$a(E; A, \mathfrak{a}) = k_E + 1 - \sum_{j=1}^{r} \lambda_j \mathrm{val}_E(\mathfrak{a}_j).$$

The minimal log discrepancy of the pair $(A, \mathfrak{a})$ at a closed point 0 is given by

$$\mathrm{mld}(0; A, \mathfrak{a}) = \inf\{a(E; A, \mathfrak{a}) \mid E \text{ is a prime divisor with center } 0\}.$$

We say a prime divisor E over A with center 0 computes $\mathrm{mld}(0; A, \mathfrak{a})$ if

$$a(E; A, \mathfrak{a}) = \begin{cases} \mathrm{mld}(0; A, \mathfrak{a}), & \text{if} \quad \mathrm{mld}(0; \mathrm{A}, \mathfrak{a}) \geq 0, \\ < 0, & \text{if} \quad \mathrm{mld}(0; \mathrm{A}, \mathfrak{a}) = -\infty. \end{cases}$$

Definition 2.2. An $\mathbb{R}$-ideal $\mathfrak{a} = \prod_{j=1}^{r} \mathfrak{a}_j^{\lambda_j}$ on $\mathbb{A}_k^N$ is called a monomial $\mathbb{R}$-ideal if each $\mathfrak{a}_j$ is generated by monomials.

Let A be a smooth surface over k with a closed point 0 and an $\mathbb{R}$-ideal $\mathfrak{a}$ on A. In the proof of Theorem 1.4 in [Is2], Ishii proves that there is a regular system of parameters x, y of $\mathcal{O}_{A,0}$ and a monomial $\mathbb{R}$-ideal $\mathfrak{a}_*$ on $\mathbb{A}_k^2$ with same exponents as $\mathfrak{a}$ such that

(1) $\mathrm{mld}(0; \mathrm{A}, \mathfrak{a}) = \mathrm{mld}(0; \mathbb{A}_{\mathrm{k}}^2, \mathfrak{a}_*)$,
(2) if we identify prime divisors over A with center 0 and that over $\mathbb{A}_k^2$ with center 0 in terms of the étale morphism from A to $\mathbb{A}_k^2$ induced by parameters x, y, then
$$a(E; A, \mathfrak{a}) \leq a(E; \mathbb{A}_k^2, \mathfrak{a}_*)$$
for any prime divisor E over A with center 0 (or over $\mathbb{A}_k^2$ with center 0).

Hence every prime divisor computing $\mathrm{mld}(0; \mathbb{A}_{\mathrm{k}}^2, \mathfrak{a}_*)$ also computes $\mathrm{mld}(0; \mathrm{A}, \mathfrak{a})$. Hence the problem is reduced into the one on the pairs of monomial $\mathbb{R}$-ideals on $\mathbb{A}_k^2$.

At the end of this section let's introduce some notations that will be used in the following sections:

(1) For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$ $(\boldsymbol{a} \neq \boldsymbol{b})$, we denote the unique line passing through $\boldsymbol{a}$ and $\boldsymbol{b}$ by $\overline{\boldsymbol{ab}}$.
(2) Given a line L not paralleling to the y-axis in $\mathbb{R}^2$, we decompose $\mathbb{R}^2$ into three parts
$$\mathbb{R}^2 = L^+ \cup L \cup L^-,$$
where
$$\begin{aligned} L^+ = \{(x_0, y_0) \in \mathbb{R}^2 \,|y_0 > & \text{ the second coordinate of} \\ & \text{the intersection point of } x = x_0 \text{ and } L\}; \\ L^- = \{(x_0, y_0) \in \mathbb{R}^2 \,|y_0 < & \text{ the second coordinate of} \\ & \text{the intersection point of } x = x_0 \text{ and } L\}. \end{aligned}$$
(3) We write $\mathbf{1}$ for the vector $(1, 1)$ and $\mathbf{0}$ for the vector $(0, 0)$.
(4) For $\boldsymbol{a} \in \mathbb{R}_{\geq 0}^2$, we denote its first coordinate by $\boldsymbol{a}_x$ and its second coordinate by $\boldsymbol{a}_y$.

(5) Let λ be a positive real number. For any real number a, we denote
$$\lceil a \rceil_\lambda = \min\{n\lambda \mid n \in \mathbb{Z} \text{ and } n\lambda \geq a\},$$
$$\lfloor a \rfloor_\lambda = \max\{n\lambda \mid n \in \mathbb{Z} \text{ and } n\lambda \leq a\}.$$
The absence of subscripts means $\lambda = 1$. It's not hard to check that
$$\frac{\lceil a \rceil_\lambda}{\lambda} = \left\lceil \frac{a}{\lambda} \right\rceil \quad \text{and} \quad \frac{\lfloor a \rfloor_\lambda}{\lambda} = \left\lfloor \frac{a}{\lambda} \right\rfloor.$$

(6) Let $B = \sum b_i B_i$ be a divisor on a variety where the B_i are prime divisors. Let ϵ be a real number, then we denote
$$B_{\leq \epsilon} = \sum_{b_i \leq \epsilon} b_i B_i \quad \text{and} \quad B_{< \epsilon} = \sum_{b_i < \epsilon} b_i B_i.$$

3. Newton polygon

Let $\mathfrak{a} = \prod_{j=1}^r \mathfrak{a}_j^{\lambda_j}$ be a monomial $\mathbb{R}$-ideal on $\mathbb{A}_k^2$ and write Supp $\mathfrak{a}$ for the set
$$\left\{ \sum_j \lambda_j (a_j, b_j) \in \mathbb{R}^2_{\geq 0} \mid (a_j, b_j) \text{ is the exponent of a monomial in } \mathfrak{a}_j \right\}.$$
We denote by $\Gamma(\mathfrak{a})$ the convex hull of (Supp $\mathfrak{a} + \mathbb{R}^2_{\geq 0}$) in $\mathbb{R}^2_{\geq 0}$, which is called the Newton polygon of $\mathfrak{a}$. Then $\Gamma(\mathfrak{a})$ has finite vertex and every compact 1-dimensional facets has slope less than 0.

Lemma 3.1. *Let $\mathfrak{a} = \prod_{j=1}^r \mathfrak{a}_j^{\lambda_j}$ be a monomial $\mathbb{R}$-ideal on $\mathbb{A}_k^2$. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2_{\geq 0}$ are two vertices of a 1-dimensional compact facets of $\Gamma(\mathfrak{a})$ such that $\boldsymbol{a}_y > \boldsymbol{b}_y$, then there exists $0 < \alpha \leq 1$ and $j = 1, \cdots, r$ such that $(\boldsymbol{a} - \boldsymbol{c}) \in (\lambda_j \mathbb{Z})^2$ where $\boldsymbol{c} = (1-\alpha)\boldsymbol{a} + \alpha \boldsymbol{b}$.*

Proof. To avoid complicated notations, here we only give a proof of the case that $r = 2$, i.e. $\mathfrak{a} = \mathfrak{a}_1^{\lambda_1}\mathfrak{a}_2^{\lambda_2}$. A similar argument works in the general case. Let $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ (resp. $\boldsymbol{b}_1, \cdots, \boldsymbol{b}_m$) be vertices of $\Gamma(\mathfrak{a}_1)$ (resp. $\Gamma(\mathfrak{a}_2)$) such that $(\boldsymbol{a}_1)_y > \cdots > (\boldsymbol{a}_n)_y$ (resp. $(\boldsymbol{b}_1)_y > \cdots > (\boldsymbol{b}_m)_y$). Let $\boldsymbol{a}, \boldsymbol{b}$ be two vertices of a 1-dimensional compact facets of $\Gamma(\mathfrak{a})$ such that $\boldsymbol{a}_y > \boldsymbol{b}_y$. Then we can write $\boldsymbol{a} = \lambda_1 \boldsymbol{a}_{i_1} + \lambda_2 \boldsymbol{b}_{j_1}$ and $\boldsymbol{b} = \lambda_1 \boldsymbol{a}_{i_2} + \lambda_2 \boldsymbol{b}_{j_2}$ for some $i_1, i_2 = 1, \cdots, n$ and some $j_1, j_2 = 1, \cdots, m$. Since $\boldsymbol{a}_y > \boldsymbol{b}_y$, either $i_1 < i_2$ or $j_1 < j_2$.

Let $\boldsymbol{f} = \lambda_1 \boldsymbol{a}_{i_1} + \lambda_2 \boldsymbol{b}_{j_2}$, $\boldsymbol{g} = \lambda_1 \boldsymbol{a}_{i_2} + \lambda_2 \boldsymbol{b}_{j_1}$. Since $\boldsymbol{a}, \boldsymbol{b}$ are vertices of $\Gamma(\mathfrak{a})$, both $\boldsymbol{f}$ and $\boldsymbol{g} \in \overline{\boldsymbol{ab}} \cup \overline{\boldsymbol{ab}}^+$. On the other hands, $\boldsymbol{f} + \boldsymbol{g} = \boldsymbol{a} + \boldsymbol{b}$, this implies that both $\boldsymbol{f}, \boldsymbol{g} \in \overline{\boldsymbol{ab}}$. It follows from the fact that $\boldsymbol{a}, \boldsymbol{b}$ are vertices that both
$$\boldsymbol{f}, \boldsymbol{g} \in \{(1-\alpha)\boldsymbol{a} + \alpha \boldsymbol{b} \mid 0 \leq \alpha \leq 1\}.$$

If $i_1 < i_2$, then $\boldsymbol{a} - \boldsymbol{g} = \lambda_1(\boldsymbol{a}_{i_1} - \boldsymbol{a}_{i_2}) \neq 0$, thus $\boldsymbol{g} \in \{(1-\alpha)\boldsymbol{a} + \alpha\boldsymbol{b} \mid 0 < \alpha \leq 1\}$ and $\boldsymbol{a} - \boldsymbol{g} \in (\lambda_1 \mathbb{Z})^2$.

If $j_1 < j_2$, then $\boldsymbol{a} - \boldsymbol{f} = \lambda_2(\boldsymbol{b}_{j_1} - \boldsymbol{b}_{j_2}) \neq 0$, thus $\boldsymbol{f} \in \{(1-\alpha)\boldsymbol{a} + \alpha\boldsymbol{b} \mid 0 < \alpha \leq 1\}$ and $\boldsymbol{a} - \boldsymbol{f} \in (\lambda_2 \mathbb{Z})^2$. $\square$

Lemma 3.2. *Let $\mathfrak{a} = \prod_{j=1}^r \mathfrak{a}_j^{\lambda_j}$ be a monomial $\mathbb{R}$-ideal in $\mathbb{A}_k^2$. If $\boldsymbol{a}$ is a vertex of $\Gamma(\mathfrak{a})$, then*
$$\boldsymbol{a}_x, \boldsymbol{a}_y \in \left\{ \sum_i n_i b_i \mid n_i \in \mathbb{Z}_{\geq 0}, b_i \in I \right\}.$$

Proof. This is an immediate consequence of the definition of the Newton polygon of a monomial $\mathbb{R}$-ideals. $\square$

Let $\mathfrak{a}$ be a monomial $\mathbb{R}$-ideal on $\mathbb{A}_k^2$ with Newton polygon Γ. For any $\boldsymbol{p} \in \mathbb{N}^2$, we denote by $E_{\boldsymbol{p}}$ the prime toric divisor over $\mathbb{A}_k^2$ which corresponds to the 1-dimensional cone $\boldsymbol{p}\mathbb{R}_{\geq 0}$, then we have $k_{E_{\boldsymbol{p}}} + 1 = \langle \boldsymbol{p}, \boldsymbol{1} \rangle$ and $\mathrm{val}_{E_{\boldsymbol{p}}}(\mathfrak{a}) = \langle \boldsymbol{p}, \Gamma \rangle$, where $\langle \boldsymbol{p}, \Gamma \rangle$ is defined as

$$\langle \boldsymbol{p}, \Gamma \rangle = \inf\{\langle \boldsymbol{p}, \boldsymbol{q} \rangle \mid \boldsymbol{q} \in \Gamma\}.$$

Therefore, $a(E_{\boldsymbol{p}}; \mathbb{A}_k^2, \mathfrak{a}) = \langle \boldsymbol{p}, \boldsymbol{1} \rangle - \langle \boldsymbol{p}, \Gamma \rangle$.

Lemma 3.3. *There exists $\boldsymbol{p} \in \mathbb{N}^2$ such that $E_{\boldsymbol{p}}$ computes $\mathrm{mld}(0; \mathbb{A}_k^2, \mathfrak{a})$. That is to say, if $\mathrm{mld}(0; \mathbb{A}_k^2, \mathfrak{a}) \geq 0$, there exists $\boldsymbol{p} \in \mathbb{N}^2$ such that*

$$\langle \boldsymbol{p}, \boldsymbol{1} \rangle - \langle \boldsymbol{p}, \Gamma \rangle = \inf\{\langle \boldsymbol{q}, \boldsymbol{1} \rangle - \langle \boldsymbol{q}, \Gamma \rangle \mid \boldsymbol{q} \in \mathbb{N}^2\} = \mathrm{mld}(0; \mathbb{A}_k^2, \mathfrak{a})$$

and if $\mathrm{mld}(0; \mathbb{A}_k^2, \mathfrak{a}) = -\infty$, there exists $\boldsymbol{p} \in \mathbb{N}^2$ such that

$$\langle \boldsymbol{p}, \boldsymbol{1} \rangle - \langle \boldsymbol{p}, \Gamma \rangle < 0.$$

Proof. Take a toric log resolution of the pair $(\mathbb{A}_k^N, \mathfrak{a} \cdot \mathfrak{m}_0)$, where $\mathfrak{m}_0$ is the maximal ideal of the origin. Then there exists a toric divisor on the resolution computing $\mathrm{mld}(0; \mathbb{A}_k^2, \mathfrak{a})$. $\square$

Lemma 3.4. *The followings are equivalent:*
(1) $\mathrm{mld}(0; \mathbb{A}_k^2, \mathfrak{a}) \geq 0$,
(2) $\boldsymbol{1} \in \Gamma$.

Proof. If $\boldsymbol{1} \in \Gamma$, choose $\boldsymbol{p} \in \mathbb{N}^2$ such that $E_{\boldsymbol{p}}$ computes $\mathrm{mld}(0; \mathbb{A}_k^2, \mathfrak{a})$. Since $\boldsymbol{1} \in \Gamma$, we have $\langle \boldsymbol{p}, \boldsymbol{1} \rangle - \langle \boldsymbol{p}, \Gamma \rangle \geq 0$, which implies that $\mathrm{mld}(0; \mathbb{A}_k^2, \mathfrak{a}) \geq 0$.

If $\boldsymbol{1} \notin \Gamma$, there exists a compact 1-dimensional facet of Γ with vertices $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{1} \in \overline{\boldsymbol{ab}}^-$. Write L for $\overline{\boldsymbol{ab}}$, then L has negative slope. After some perturbations, we may suppose that $\boldsymbol{1} \in L^-$, $\Gamma \subseteq L^+$, L still has negative slope and L passes through two integral points $\boldsymbol{c}$ and $\boldsymbol{d}$. We may suppose that $\boldsymbol{c}_y > \boldsymbol{d}_y$ and $\boldsymbol{c}_x < \boldsymbol{d}_x$. Let $\boldsymbol{p} = (\boldsymbol{c}_y - \boldsymbol{d}_y, \boldsymbol{d}_x - \boldsymbol{c}_x) \in \mathbb{N}^2$, then $\langle \boldsymbol{p}, \boldsymbol{1} \rangle < \langle \boldsymbol{p}, \Gamma \rangle$. Thus $a(E_{\boldsymbol{p}}; \mathbb{A}_k^2, \mathfrak{a}) < 0$, which implies $\mathrm{mld}(0; \mathbb{A}_k^2, \mathfrak{a}) = -\infty$. $\square$

4. Proof of the main theorem

Lemma 4.1. *Let λ be a positive real number. If $a, b \in \lambda\mathbb{Z}$ satisfy $1 < a \leq b \leq 2$, then*

$$\left\lfloor \frac{a}{a-1} \right\rfloor_\lambda + a \geq \left\lfloor \frac{b}{b-1} \right\rfloor_\lambda + b.$$

Proof. Let n be an integer such that $1 < n\lambda \leq (n+1)\lambda \leq 2$, then

$$\frac{n\lambda}{n\lambda - 1} - \frac{(n+1)\lambda}{(n+1)\lambda - 1} = \frac{\lambda}{(n\lambda - 1)((n+1)\lambda - 1)} \geq \lambda.$$

Hence

$$\left\lfloor \frac{n\lambda}{n\lambda - 1} \right\rfloor_\lambda \geq \left\lfloor \frac{(n+1)\lambda}{(n+1)\lambda - 1} \right\rfloor_\lambda + \lambda,$$

which implies that

$$\left\lfloor \frac{n\lambda}{n\lambda - 1} \right\rfloor_\lambda + n\lambda \geq \left\lfloor \frac{(n+1)\lambda}{(n+1)\lambda - 1} \right\rfloor_\lambda + (n+1)\lambda.$$

$\square$

Lemma 4.2. *Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2_{\geq 0}$ such that*

(1) *there is $\gamma \in \mathbb{R}_{>0}$ such that $1 + \gamma \leq \boldsymbol{a}_y \leq 2$,*
(2) *$\boldsymbol{a}_x < \boldsymbol{b}_x$ and $\boldsymbol{a}_y > \boldsymbol{b}_y$,*
(3) *$\mathbf{1} \in \overline{\boldsymbol{ab}} \cup \overline{\boldsymbol{ab}}^+$,*
(4) *there is $\lambda \in \mathbb{R}_{>0}$ such that $\boldsymbol{a} - \boldsymbol{b} \in (\lambda\mathbb{Z})^2$.*

Then

$$\frac{\boldsymbol{a}_y - \boldsymbol{b}_y + \boldsymbol{a}_x - \boldsymbol{b}_x}{\lambda} \leq \left\lfloor \frac{\gamma+1}{\lambda\gamma} \right\rfloor + \left\lceil \frac{\gamma+1}{\lambda} \right\rceil.$$

Proof. Since $\mathbf{1} \in \overline{\boldsymbol{ab}} \cup \overline{\boldsymbol{ab}}^+$, we have

$$\boldsymbol{b}_x - \boldsymbol{a}_x \leq \frac{(1-\boldsymbol{a}_x)(\boldsymbol{a}_y - \boldsymbol{b}_y)}{\boldsymbol{a}_y - 1}.$$

Note that $\boldsymbol{b}_x - \boldsymbol{a}_x \in \lambda\mathbb{Z}$ and $\boldsymbol{a}_y \geq \gamma + 1$, we have

$$\begin{aligned} \boldsymbol{b}_x - \boldsymbol{a}_x &\leq \left\lfloor \frac{(1-\boldsymbol{a}_x)(\boldsymbol{a}_y - \boldsymbol{b}_y)}{\boldsymbol{a}_y - 1} \right\rfloor_\lambda \\ &\leq \left\lfloor \frac{\boldsymbol{a}_y}{\boldsymbol{a}_y - 1} \right\rfloor_\lambda \\ &\leq \left\lfloor \frac{\gamma+1}{\gamma} \right\rfloor_\lambda. \end{aligned} \tag{4.1}$$

If $\boldsymbol{a}_y - \boldsymbol{b}_y \leq \gamma + 1$, then

$$\begin{aligned} \frac{\boldsymbol{a}_y - \boldsymbol{b}_y + \boldsymbol{a}_x - \boldsymbol{b}_x}{\lambda} &\leq \frac{1}{\lambda}(\gamma + 1 + \left\lfloor \frac{\gamma+1}{\gamma} \right\rfloor_\lambda) \\ &= \left\lceil \frac{\gamma+1}{\lambda} \right\rceil + \left\lfloor \frac{\gamma+1}{\lambda\gamma} \right\rfloor, \end{aligned}$$

If $\boldsymbol{a}_y - \boldsymbol{b}_y > \gamma + 1$, let $l = \lceil \gamma + 1 \rceil_\lambda$, then $l \leq \boldsymbol{a}_y - \boldsymbol{b}_y$ since $\boldsymbol{a}_y - \boldsymbol{b}_y \in \lambda\mathbb{Z}$. Apply Lemma 4.1 to conclude that

$$\left\lfloor \frac{\boldsymbol{a}_y - \boldsymbol{b}_y}{\boldsymbol{a}_y - \boldsymbol{b}_y - 1} \right\rfloor_\lambda + \boldsymbol{a}_y - \boldsymbol{b}_y \leq \left\lfloor \frac{l}{l-1} \right\rfloor_\lambda + l. \tag{4.2}$$

It follows from (4.1) that

$$\boldsymbol{b}_x - \boldsymbol{a}_x \leq \left\lfloor \frac{\boldsymbol{a}_y - \boldsymbol{b}_y}{\boldsymbol{a}_y - \boldsymbol{b}_y - 1} \right\rfloor_\lambda. \tag{4.3}$$

Finally, (4.2), (4.3) and the fact that $l = \lceil \gamma + 1 \rceil_\lambda \geq \gamma + 1$ imply

$$\begin{aligned} \frac{1}{\lambda}(\boldsymbol{a}_y - \boldsymbol{b}_y + \boldsymbol{b}_x - \boldsymbol{a}_x) &\leq \frac{1}{\lambda}(\left\lfloor \frac{\boldsymbol{a}_y - \boldsymbol{b}_y}{\boldsymbol{a}_y - \boldsymbol{b}_y - 1} \right\rfloor_\lambda + \boldsymbol{a}_y - \boldsymbol{b}_y) \\ &\leq \frac{1}{\lambda}(\left\lfloor \frac{l}{l-1} \right\rfloor_\lambda + l) \\ &\leq \frac{1}{\lambda}(\left\lfloor \frac{\gamma+1}{\gamma} \right\rfloor_\lambda + \lceil \gamma + 1 \rceil_\lambda) \\ &\leq \left\lfloor \frac{\gamma+1}{\lambda\gamma} \right\rfloor + \left\lceil \frac{\gamma+1}{\lambda} \right\rceil. \end{aligned}$$

□

Lemma 4.3. *Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2_{\geq 0}$. Suppose $\boldsymbol{a}_y > 1$ and $\boldsymbol{a} \in \triangle$, where*

$$\triangle := \{(x, y) \in \mathbb{R}^2_{\geq 0} \mid y \leq 2 - x\}.$$

(1) *If $\boldsymbol{b} \notin \triangle$, $\boldsymbol{b}_x > \boldsymbol{a}_x$ and $1 \leq \boldsymbol{b}_y < \boldsymbol{a}_y$, then $\boldsymbol{1} \in \overline{\boldsymbol{ab}}^-$.*
(2) *If $\boldsymbol{c} \notin \triangle$, $\boldsymbol{c}_x < \boldsymbol{a}_x$ and $\boldsymbol{c}_y > \boldsymbol{a}_y$, then $\boldsymbol{1} \in \overline{\boldsymbol{ca}}^+$.*

Proof. It's not hard to check by plotting the graph. □

Proof of Theorem 1.3 (1). By the argument in Section 2, we may suppose that $A = \mathbb{A}^2_k$ and $\mathfrak{a} = \prod_{j=1}^r \mathfrak{a}_j^{\lambda_j}$ is a monomial $\mathbb{R}$-ideal on A. Denote the Newton polygon $\Gamma(\mathfrak{a})$ by Γ. Since $\mathrm{mld}(0; A, \mathfrak{a}) \geq 0$, by Lemma 3.4 we have $\boldsymbol{1} \in \Gamma$, which implies that no vertices of Γ locate in $(1, +\infty) \times (1, +\infty)$. Let $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_{n+m+t}$ be vertices of Γ such that $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n \in [0, 1] \times (1, +\infty)$, $\boldsymbol{a}_{n+1}, \cdots, \boldsymbol{a}_{n+m} \in [0, 1] \times [0, 1]$, $\boldsymbol{a}_{n+m+1}, \cdots, \boldsymbol{a}_{n+m+t} \in (1, +\infty) \times [0, 1]$ and $(\boldsymbol{a}_1)_y > \cdots > (\boldsymbol{a}_{n+m+t})_y$. Then

$$\langle \boldsymbol{q}, \boldsymbol{1} \rangle - \langle \boldsymbol{q}, \Gamma \rangle = \max_{1 \leq i \leq n+m+t} \{\langle \boldsymbol{q}, \boldsymbol{1} - \boldsymbol{a}_i \rangle\} \quad \text{for any } \boldsymbol{q} \in \mathbb{N}^2.$$

For convenience, we denote

$$\boldsymbol{a}_0 = ((\boldsymbol{a}_1)_x, +\infty) \quad \text{and} \quad \boldsymbol{a}_{n+m+t+1} = (+\infty, (\boldsymbol{a}_{n+m+t})_y).$$

Define

$$\boldsymbol{b}_i = \Big((\boldsymbol{a}_i)_y - (\boldsymbol{a}_{i+1})_y, (\boldsymbol{a}_{i+1})_x - (\boldsymbol{a}_i)_x\Big)$$

for $i = 0, \cdots, n + m + t$, then

$$(\boldsymbol{b}_{n+m+t})_y/(\boldsymbol{b}_{n+m+t})_x > \cdots > (\boldsymbol{b}_0)_y/(\boldsymbol{b}_0)_x.$$

Note that $(\boldsymbol{b}_{n+m+t})_y/(\boldsymbol{b}_{n+m+t})_x = +\infty$ and $(\boldsymbol{b}_0)_y/(\boldsymbol{b}_0)_x = 0$. It's not hard to see that if $\boldsymbol{q} \in \mathbb{N}^2$ satisfies

$$(\boldsymbol{b}_{i-1})_y/(\boldsymbol{b}_{i-1})_x \leq \boldsymbol{q}_y/\boldsymbol{q}_x \leq (\boldsymbol{b}_i)_y/(\boldsymbol{b}_i)_x,$$

then

$$\langle \boldsymbol{q}, \boldsymbol{1} \rangle - \langle \boldsymbol{q}, \Gamma \rangle = \langle \boldsymbol{q}, \boldsymbol{1} - \boldsymbol{a}_i \rangle$$

for $i = 1, \cdots, n + m + t$. There are following two cases:

(1) $(\boldsymbol{b}_n)_y/(\boldsymbol{b}_n)_x < 1 < (\boldsymbol{b}_{n+m})_y/(\boldsymbol{b}_{n+m})_x$. Then $m > 0$ and there exists $i_0 \in \{n + 1, \cdots, n + m\}$ such that $\langle \boldsymbol{1}, \boldsymbol{1} \rangle - \langle \boldsymbol{1}, \Gamma \rangle = \langle \boldsymbol{1}, \boldsymbol{1} - \boldsymbol{a}_{i_0} \rangle$. For any $\boldsymbol{q} \in \mathbb{N}^2$, since $\boldsymbol{1} - \boldsymbol{a}_{i_0} \in [0, 1] \times [0, 1]$, we have

$$\langle \boldsymbol{q}, \boldsymbol{1} \rangle - \langle \boldsymbol{q}, \Gamma \rangle \geq \langle \boldsymbol{q}, \boldsymbol{1} - \boldsymbol{a}_{i_0} \rangle \geq \langle \boldsymbol{1}, \boldsymbol{1} - \boldsymbol{a}_{i_0} \rangle = \langle \boldsymbol{1}, \boldsymbol{1} \rangle - \langle \boldsymbol{1}, \Gamma \rangle.$$

Therefore $E_{\boldsymbol{1}}$ with log discrepancy 2 computes $\mathrm{mld}(0, A, \mathfrak{a})$.

(2) $(\boldsymbol{b}_n)_y/(\boldsymbol{b}_n)_x \geq 1$ or $(\boldsymbol{b}_{n+m})_y/(\boldsymbol{b}_{n+m})_x \leq 1$. We may suppose the former holds (if not we can replace Γ by its reflection along the diagram), then $n > 0$ (since $(\boldsymbol{b}_0)_y/(\boldsymbol{b}_0)_x$=0) and

$$(\boldsymbol{a}_{n+1})_x - (\boldsymbol{a}_n)_x \geq (\boldsymbol{a}_n)_y - (\boldsymbol{a}_{n+1})_y. \tag{4.4}$$

It follows from $\boldsymbol{1} \in \Gamma$ that

$$\boldsymbol{1} \in \overline{\boldsymbol{a}_n \boldsymbol{a}_{n+1}}^+ \cup \overline{\boldsymbol{a}_n \boldsymbol{a}_{n+1}}. \tag{4.5}$$

Hence $\boldsymbol{a}_{n+1} \neq (+\infty, (\boldsymbol{a}_n)_y)$ since $(\boldsymbol{a}_n)_y > 1$. That is to say, $m + t > 0$. Therefore, we do not need to worry about that $\boldsymbol{a}_n$ or $\boldsymbol{a}_{n+1}$ is an infinite point.

Apply Lemma 3.2 to obtain that $(\boldsymbol{a}_n)_y > 1+\gamma$. It follows from (4.4) and (4.5) that $(\boldsymbol{a}_n)_y \le 2$. By Lemma 3.1, there exists $j = 1, \cdots, r$ and $0 < \alpha \le 1$ such that $\alpha \boldsymbol{b}_n/\lambda_j \in \mathbb{N}^2$. Denote $\alpha \boldsymbol{b}_n/\lambda_j$ by $\boldsymbol{b}'$. We apply Lemma 4.2 to conclude that

$$\begin{aligned} \boldsymbol{b}'_x + \boldsymbol{b}'_y &\le \left\lfloor \frac{\gamma+1}{\lambda_j \gamma} \right\rfloor + \left\lceil \frac{\gamma+1}{\lambda_j} \right\rceil \\ &\le \left\lfloor \frac{\gamma+1}{e\gamma} \right\rfloor + \left\lceil \frac{\gamma+1}{e} \right\rceil . \end{aligned}$$

The second inequality follows from that $\lambda_j \ge e$. Since $\boldsymbol{b}'_y/\boldsymbol{b}'_x = (\boldsymbol{b}'_n)_y/(\boldsymbol{b}'_n)_x$,

$$\langle \boldsymbol{b}', \boldsymbol{1}\rangle - \langle \boldsymbol{b}', \Gamma\rangle = \langle \boldsymbol{b}', \boldsymbol{1} - \boldsymbol{a}_n\rangle = \langle \boldsymbol{b}', \boldsymbol{1} - \boldsymbol{a}_{n+1}\rangle. \tag{4.6}$$

Let $\boldsymbol{p} \in \mathbb{N}^2$ such that $E_{\boldsymbol{p}}$ computes $\mathrm{mld}(0, \mathbb{A}, \mathfrak{a})$, i.e.,

$$\langle \boldsymbol{p}, \boldsymbol{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle = \inf\{\langle \boldsymbol{q}, \boldsymbol{1}\rangle - \langle \boldsymbol{q}, \Gamma\rangle \mid \boldsymbol{q} \in \mathbb{N}^2\}. \tag{4.7}$$

We may suppose that

$$\langle \boldsymbol{p}, \boldsymbol{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle < \langle \boldsymbol{b}', \boldsymbol{1}\rangle - \langle \boldsymbol{b}', \Gamma\rangle. \tag{4.8}$$

Indeed, if not, then $E_{\boldsymbol{b}'}$ computes the mld with its discrepancy satisfying the inequality and the proof is completed.

As $k_{E_{\boldsymbol{p}}} + 1 = \boldsymbol{p}_x + \boldsymbol{p}_y$, it is enough to show that $\boldsymbol{p}_x \le \boldsymbol{b}'_x$ and $\boldsymbol{p}_y \le \boldsymbol{b}'_y$. There are four following subcases:

(2a) $\boldsymbol{p}_y/\boldsymbol{p}_x \le \boldsymbol{b}'_y/\boldsymbol{b}'_x$. Since

$$\langle \boldsymbol{p}, \boldsymbol{1} - \boldsymbol{a}_n\rangle \le \langle \boldsymbol{p}, \boldsymbol{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle < \langle \boldsymbol{b}', \boldsymbol{1}\rangle - \langle \boldsymbol{b}', \Gamma\rangle = \langle \boldsymbol{b}', \boldsymbol{1} - \boldsymbol{a}_n\rangle,$$

we have

$$(\boldsymbol{p}_x - \boldsymbol{b}'_x)(1 - (\boldsymbol{a}_n)_x) < (\boldsymbol{p}_y - \boldsymbol{b}'_y)((\boldsymbol{a}_n)_y - 1). \tag{4.9}$$

Note that $1 - (\boldsymbol{a}_n)_x \ge 0$ and $(\boldsymbol{a}_n)_y - 1 > 0$. We claim that $\boldsymbol{p}_x \le \boldsymbol{b}'_x$. Indeed, if this is not the case, then (4.9) implies that

$$\frac{1 - (\boldsymbol{a}_n)_x}{(\boldsymbol{a}_n)_y - 1} < \frac{\boldsymbol{p}_y - \boldsymbol{b}'_y}{\boldsymbol{p}_x - \boldsymbol{b}'_x} \le \frac{\boldsymbol{b}'_y}{\boldsymbol{b}'_x}.$$

The last equality comes from $\boldsymbol{p}_y/\boldsymbol{p}_x \le \boldsymbol{b}'_y/\boldsymbol{b}'_x$. However, (4.5) implies that

$$\frac{1 - (\boldsymbol{a}_n)_x}{(\boldsymbol{a}_n)_y - 1} \ge \frac{(\boldsymbol{a}_{n+1})_x - (\boldsymbol{a}_n)_x}{(\boldsymbol{a}_n)_y - (\boldsymbol{a}_{n+1})_y} = \frac{\boldsymbol{b}'_y}{\boldsymbol{b}'_x}$$

which leads to a contradiction. Therefore, $\boldsymbol{p}_x \le \boldsymbol{b}'_x$. Then $\boldsymbol{p}_y \le \boldsymbol{b}'_y$ since $\boldsymbol{p}_y/\boldsymbol{p}_x \le \boldsymbol{b}'_y/\boldsymbol{b}'_x$.

(2b) $\boldsymbol{p}_y/\boldsymbol{p}_x > \boldsymbol{b}'_y/\boldsymbol{b}'_x$ and $\boldsymbol{p}_y \le \boldsymbol{b}'_y$. Then $\boldsymbol{p}_x \le \boldsymbol{b}'_x$.

(2c) $\boldsymbol{p}_y/\boldsymbol{p}_x > \boldsymbol{b}'_y/\boldsymbol{b}'_x$, $\boldsymbol{p}_y > \boldsymbol{b}'_y$ and $\boldsymbol{p}_x \le \boldsymbol{b}'_x$. Let $\boldsymbol{p}' = (\boldsymbol{p}_x, \boldsymbol{b}'_y)$, then $\boldsymbol{p}'_y/\boldsymbol{p}'_x \ge \boldsymbol{b}'_y/\boldsymbol{b}'_x$. Therefore there exists $j_0 \in \{n+1, \cdots, n+m+t\}$ such that

$$\langle \boldsymbol{p}', \boldsymbol{1}\rangle - \langle \boldsymbol{p}', \Gamma\rangle = \langle \boldsymbol{p}', \boldsymbol{1} - \boldsymbol{a}_{j_0}\rangle.$$

Since $1 - (\boldsymbol{a}_{j_0})_y \ge 0$, we have

$$\langle \boldsymbol{p}', \boldsymbol{1} - \boldsymbol{a}_{j_0}\rangle \le \langle \boldsymbol{p}, \boldsymbol{1} - \boldsymbol{a}_{j_0}\rangle \le \langle \boldsymbol{p}, \boldsymbol{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle \le \langle \boldsymbol{p}', \boldsymbol{1}\rangle - \langle \boldsymbol{p}', \Gamma\rangle.$$

The last inequality comes from (4.7). This implies that $\langle \boldsymbol{p}, \boldsymbol{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle = \langle \boldsymbol{p}', \boldsymbol{1}\rangle - \langle \boldsymbol{p}', \Gamma\rangle$. We therefore obtain a prime divisor $E_{\boldsymbol{p}'}$ computing the minimal log discrepancy and its log discrepancy $\boldsymbol{p}'_x + \boldsymbol{p}'_y \le \boldsymbol{b}'_x + \boldsymbol{b}'_y$.

(2d) $\boldsymbol{p}_y/\boldsymbol{p}_x > \boldsymbol{b}'_y/\boldsymbol{b}'_x$, $\boldsymbol{p}_y > \boldsymbol{b}'_y$ and $\boldsymbol{p}_x > \boldsymbol{b}'_x$. It follows from

$$\langle \boldsymbol{p}, \mathbf{1} - \boldsymbol{a}_{n+1}\rangle \le \langle \boldsymbol{p}, \mathbf{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle < \langle \boldsymbol{b}', \mathbf{1}\rangle - \langle \boldsymbol{b}', \Gamma\rangle = \langle \boldsymbol{b}', \mathbf{1} - \boldsymbol{a}_{n+1}\rangle,$$

that

$$(\boldsymbol{p}_x - \boldsymbol{b}'_x)((\boldsymbol{a}_{n+1})_x - 1) > (\boldsymbol{p}_y - \boldsymbol{b}'_y)(1 - (\boldsymbol{a}_{n+1})_y). \tag{4.10}$$

Note that $1 - (\boldsymbol{a}_{n+1})_y \ge 0$. If $1 - (\boldsymbol{a}_{n+1})_y = 0$, then (4.5) implies that $(\boldsymbol{a}_{n+1})_x \le 1$. On the other hand, $\boldsymbol{p}_x - \boldsymbol{b}'_x > 0$. This contradicts (4.10).

If $1 - (\boldsymbol{a}_{n+1})_y > 0$, then (4.10) implies that

$$\frac{(\boldsymbol{a}_{n+1})_x - 1}{1 - (\boldsymbol{a}_{n+1})_y} > \frac{\boldsymbol{p}_y - \boldsymbol{b}'_y}{\boldsymbol{p}_x - \boldsymbol{b}'_x} > \frac{\boldsymbol{b}'_y}{\boldsymbol{b}'_x}.$$

The last equality comes from $\boldsymbol{p}_y/\boldsymbol{p}_x > \boldsymbol{b}'_y/\boldsymbol{b}'_x$. However, (4.5) implies that

$$\frac{(\boldsymbol{a}_{n+1})_x - 1}{1 - (\boldsymbol{a}_{n+1})_y} \le \frac{(\boldsymbol{a}_{n+1})_x - (\boldsymbol{a}_n)_x}{(\boldsymbol{a}_n)_y - (\boldsymbol{a}_{n+1})_y} = \frac{\boldsymbol{b}'_y}{\boldsymbol{b}'_x}$$

which leads to a contradiction. □

Proof of Theorem 1.3 (2). By the argument in Section 2, we may suppose that $A = \mathbb{A}^2_k$ and $\mathfrak{a} = \prod_{j=1}^r \mathfrak{a}_j^{\lambda_j}$ is a monomial $\mathbb{R}$-ideal on A. Denote the Newton polygon $\Gamma(\mathfrak{a})$ by Γ. Since $\mathrm{mld}(0; A, \mathfrak{a}) = -\infty$, by Lemma 3.4 we have $\mathbf{1} \notin \Gamma$. We may suppose that Γ is convenient, i.e Γ meets both x-axis and y-axis. Indeed, if this is not the case, we can replace each $\mathfrak{a}_i$ by the ideal generated by $\mathfrak{a}_i, x^m$ and y^m for a large enough integer m. Then we obtain a monomial $\mathbb{R}$-ideal with its Newton polygon convenient and containing the original one. Every divisor computing the mld of the new $\mathbb{R}$-ideal also computes that of the original one, therefore we may replace the original one by the new one. Let $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_k$ be vertices of Γ such that $(\boldsymbol{a}_1)_y > \cdots > (\boldsymbol{a}_k)_y$. Denote $\boldsymbol{a}_0 = ((\boldsymbol{a}_1)_x, +\infty)$ and $\boldsymbol{a}_{k+1} = (+\infty, (\boldsymbol{a}_k)_y)$ for convenience.

If $\langle \mathbf{1}, \Gamma\rangle > 2$, then $\langle \mathbf{1}, \mathbf{1}\rangle - \langle \mathbf{1}, \Gamma\rangle < 0$, thus $E_{\mathbf{1}}$ with log discrepancy 2 computes $\mathrm{mld}(0; A, \mathfrak{a})$ and the inequality holds.

If $\langle \mathbf{1}, \Gamma\rangle \le 2$, there exists $i_0 = 1, \cdots k$ such that $\boldsymbol{a}_{i_0} \in \triangle$, where

$$\triangle := \{(x, y) \in \mathbb{R}^2_{\ge 0} \mid y \le 2 - x\}.$$

Since $\mathbf{1} \notin \Gamma$, we have $\boldsymbol{a}_{i_0} \notin [0,1] \times [0,1]$. So we may assume that $(\boldsymbol{a}_{i_0})_y > 1$ (if not we can replace Γ by its reflection along the diagonal). Let

$$j_0 = \max\{i = 1, \cdots k \mid \boldsymbol{a}_i \in \triangle\},$$

then $j_0 \ge i_0$. We claim that $(\boldsymbol{a}_{j_0})_y > 1$. Indeed, if this is not the case, then

$$\mathbf{1} \in \{(1-\alpha)\boldsymbol{a}_{i_0} + \alpha \boldsymbol{a}_{j_0} + \beta \boldsymbol{q} \mid 0 \le \alpha \le 1, \beta \ge 0, \boldsymbol{q} = (1, 0)\}$$

which follows from $(\boldsymbol{a}_{i_0})_y > 1, (\boldsymbol{a}_{j_0})_y \le 1$ and $\boldsymbol{a}_{i_0}, \boldsymbol{a}_{j_0} \in \triangle$. This contradicts the fact $\mathbf{1} \notin \Gamma$. Therefore we have $j_0 < k$, since $(\boldsymbol{a}_k)_y = 0$ which follows from that Γ is convenient.

By the definition of j_0, we have $\boldsymbol{a}_{j_0+1} \notin \triangle$. We claim that $\mathbf{1} \in \overline{\boldsymbol{a}_{j_0}\boldsymbol{a}_{j_0+1}}^-$. In fact, if not, by Lemma 4.3 (1) we have $(\boldsymbol{a}_{j_0+1})_y < 1$. Then

$$\mathbf{1} \in \{(1-\alpha)\boldsymbol{a}_{j_0} + \alpha \boldsymbol{a}_{j_0+1} + \beta \boldsymbol{q} \mid 0 \le \alpha \le 1, \beta \ge 0, \boldsymbol{q} = (1, 0)\}$$

which follows from $(\boldsymbol{a}_{j_0})_y > 1, (\boldsymbol{a}_{j_0+1})_y < 1$ and $\mathbf{1} \in \overline{\boldsymbol{a}_{j_0}\boldsymbol{a}_{j_0+1}}^+ \cup \overline{\boldsymbol{a}_{j_0}\boldsymbol{a}_{j_0+1}}$. This contradicts the fact $\mathbf{1} \notin \Gamma$.

Let

$$l_0 = \min\{i = 1, \cdots, k-1 \mid \mathbf{1} \in \overline{\boldsymbol{a}_i \boldsymbol{a}_{i+1}}^-\}.$$

Then $l_0 \leq j_0$. We claim that $\boldsymbol{a}_{l_0} \in \triangle$. In fact, if not, then $l_0 < j_0$ and by Lemma 4.3 (2) we have $\mathbf{1} \in \overline{\boldsymbol{a}_{l_0}\boldsymbol{a}_{j_0}}^+$, which yields $\mathbf{1} \in \overline{\boldsymbol{a}_{l_0}\boldsymbol{a}_{l_0+1}}^+$ and leads to a contradiction.

Denote $\boldsymbol{a}_{l_0}$ (resp. $\boldsymbol{a}_{l_0+1}$) by $\boldsymbol{a}$ (resp. $\boldsymbol{b}$). Then $\boldsymbol{a} \in \triangle$, $\mathbf{1} \in \overline{\boldsymbol{ab}}^-$ and $\boldsymbol{a}_y = (\boldsymbol{a}_{l_0})_y \geq (\boldsymbol{a}_{j_0})_y > 1$. Apply Lemma 3.2 to obtain that $\boldsymbol{a}_y \geq 1 + \gamma$. By Lemma 3.1, there exists $0 < \alpha \leq 1$ and $j = 1, \cdots, r$ such that $(\boldsymbol{a} - \boldsymbol{c}) \in (\lambda_j \mathbb{Z})^2$ where $\boldsymbol{c} = (1-\alpha)\boldsymbol{a} + \alpha \boldsymbol{b}$. Denote λ_j by λ, then $\lambda \geq e$. Let

$$n = \max\{n' \in \mathbb{Z} \mid \mathbf{1} \in \overline{\boldsymbol{ac'}}^-, \text{ where } \boldsymbol{c'} = (\boldsymbol{c}_x - n'\lambda, \boldsymbol{c}_y)\}.$$

and $\boldsymbol{d} = (\boldsymbol{c}_x - n\lambda, \boldsymbol{c}_y)$. Then $\mathbf{1} \in \overline{\boldsymbol{ad}}^-$ and $\boldsymbol{b} \in \overline{\boldsymbol{ad}} \cup \overline{\boldsymbol{ad}}^+$.

Let $\boldsymbol{f} = \boldsymbol{a}_{l_0-1}$ (recall that $\boldsymbol{a} = \boldsymbol{a}_{l_0}$). If $l_0 = 1$, then $\boldsymbol{f} = (\boldsymbol{a}_x, +\infty)$, hence $\boldsymbol{f} \in \overline{\boldsymbol{ad}}^+$. If $l_0 > 1$, by the definition of l_0, we have $\mathbf{1} \notin \overline{\boldsymbol{fa}}^-$ while $\mathbf{1} \in \overline{\boldsymbol{ad}}^-$. It follows that $\boldsymbol{f} \in \overline{\boldsymbol{ad}}^+$.

Since $\boldsymbol{f} = \boldsymbol{a}_{l_0-1}, \boldsymbol{b} = \boldsymbol{a}_{l_0+1} \in \overline{\boldsymbol{ad}} \cup \overline{\boldsymbol{ad}}^+$ and $\boldsymbol{a} = \boldsymbol{a}_{l_0}$, we have $\Gamma \subseteq \overline{\boldsymbol{ad}} \cup \overline{\boldsymbol{ad}}^+$. On the other hand, $\mathbf{1} \in \overline{\boldsymbol{ad}}^-$. This implies that $\langle \boldsymbol{p}, \mathbf{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle < 0$ where

$$\boldsymbol{p} = \frac{(\boldsymbol{a}_y - \boldsymbol{d}_y, \boldsymbol{d}_x - \boldsymbol{a}_x)}{\lambda} \in \mathbb{N}^2.$$

Hence $E_{\boldsymbol{p}}$ computes $\mathrm{mld}(0; A, \mathfrak{a})$ with log discrepancy $\boldsymbol{p}_x + \boldsymbol{p}_y$.

Let $\boldsymbol{d'} = (\boldsymbol{d}_x - \lambda, \boldsymbol{d}_y)$. If $\boldsymbol{d}'_x \leq \boldsymbol{a}_x$, then $\boldsymbol{d}_x - \boldsymbol{a}_x \leq \lambda$. It follows from $\boldsymbol{a} \in \triangle$ and $\mathbf{1} \in \overline{\boldsymbol{ad}}^-$ that the slope of $\overline{\boldsymbol{ad}} > -1$. Thus $\boldsymbol{a}_y - \boldsymbol{d}_y < \boldsymbol{d}_x - \boldsymbol{a}_x$. Therefore,

$$\boldsymbol{p}_x + \boldsymbol{p}_y = \frac{\boldsymbol{d}_x - \boldsymbol{a}_x + \boldsymbol{a}_y - \boldsymbol{d}_y}{\lambda} \leq 2,$$

which implies the inequality we want.

If $\boldsymbol{d}'_x > \boldsymbol{a}_x$, by the definition of $\boldsymbol{d}$, we have $\mathbf{1} \in \overline{\boldsymbol{ad'}} \cup \overline{\boldsymbol{ad'}}^+$. Applying Lemma 4.2 we obtain

$$\frac{\boldsymbol{a}_y - \boldsymbol{d}'_y + \boldsymbol{d}'_x - \boldsymbol{a}_x}{\lambda} \leq \left\lfloor \frac{\gamma+1}{\lambda\gamma} \right\rfloor + \left\lceil \frac{1+\gamma}{\lambda} \right\rceil,$$

which implies that

$$\begin{aligned} \boldsymbol{p}_x + \boldsymbol{p}_y &= \frac{\boldsymbol{a}_y - \boldsymbol{d}_y + \boldsymbol{d}_x - \boldsymbol{a}_x}{\lambda} \\ &\leq \left\lfloor \frac{\gamma+1}{\lambda\gamma} \right\rfloor + \left\lceil \frac{\gamma+1}{\lambda} \right\rceil + 1 \\ &\leq \left\lfloor \frac{\gamma+1}{e\gamma} \right\rfloor + \left\lceil \frac{\gamma+1}{e} \right\rceil + 1. \end{aligned}$$

□

5. Proofs of examples

Lemma 5.1. *Let $\mathfrak{a}$ be a monomial $\mathbb{R}$-ideal on $\mathbb{A}^2_k$ which supports on the origin. Suppose that there exists a positive number ℓ such that any toric prime divisor that computes* $\mathrm{mld}(0; \mathbb{A}^2_{\mathrm{k}}, \mathfrak{a})$ *satisfies that its log discrepancy $\geq \ell$. Then the same conclusion holds for all prime divisors that compute the mld.*

Proof. Take a toric log resolution $\pi : M \to \mathbb{A}^2_k$ of the pair $(\mathbb{A}^2_k, \mathfrak{a})$. Then for any prime divisor E over $\mathbb{A}^2_k$, we have

$$a(E; \mathbb{A}^2_k, \mathfrak{a}) = a(E; M, B)$$

for

$$B = \sum_F (1 - a(F; \mathbb{A}^2_k, \mathfrak{a}))F,$$

where F runs all prime divisors on M whose center on $\mathbb{A}^2_k$ is 0. Note that B is log smooth.

Let E be a prime divisor over $\mathbb{A}^2_k$ with the center 0 that computes the mld. If E lies on M, then E is a toric divisor and the conclusion holds. If E is exceptional over M, we claim that the center of E on M is contained in some divisor on M that computes the mld. Indeed, if this is not the case, there are the following two cases:

(1) $\mathrm{mld}(0; \mathbb{A}^2_{\mathrm{k}}, \mathfrak{a}) = -\infty$. Then $a(E; M, B) = a(E; M, B_{\le 1}) \ge 0$ since B is log smooth. This contradicts that E computes the mld.

(2) $\mathrm{mld}(0; \mathbb{A}^2_{\mathrm{k}}, \mathfrak{a}) = \epsilon \ge 0$. Then $a(E; M, B) = a(E; M, B_{<1-\epsilon}) > 2\epsilon$ since B is log smooth. This contradicts that E computes the mld.

Therefore the center of E on M is contained in some divisor on M that computes the mld. Since every divisor on M computing the mld satisfies that its log dicrepancy $\ge \ell$, the log dicrepancy of $E \ge \ell$. $\square$

Lemma 5.2. *Fix a positive integer $n \ge 2$. Let Γ be the Newton polygon of the monomial $\mathbb{R}$-ideal $\mathfrak{a} = (x^{n^2}, y^{n-1})^e$ on $\mathbb{A}^2_k$ where $e = 1/(n-1) + 1/n^2$ and $\boldsymbol{p}$ be the vector $(n-1, n^2)$. Then $\langle \boldsymbol{p}, \mathbf{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle = 0$ and for any $\boldsymbol{q} \in \mathbb{N}^2$ we have*

$$\langle \boldsymbol{q}, \mathbf{1}\rangle - \langle \boldsymbol{q}, \Gamma\rangle \ge 0$$

Moreover, if the equality holds, then $\langle \boldsymbol{q}, \mathbf{1}\rangle \ge \langle \boldsymbol{p}, \mathbf{1}\rangle = n^2 + n - 1$.

Proof. By direct calculation, we obtain $\langle \boldsymbol{p}, \mathbf{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle = 0$. By definition,

$$\langle \boldsymbol{q}, \Gamma\rangle = \min\left\{en^2\boldsymbol{q}_x, e(n-1)\boldsymbol{q}_y\right\}.$$

Since

$$\langle \boldsymbol{q}, \mathbf{1}\rangle = \boldsymbol{q}_x + \boldsymbol{q}_y = \frac{n-1}{n^2+n-1}(en^2\boldsymbol{q}_x) + \frac{n^2}{n^2+n-1}(e(n-1)\boldsymbol{q}_y),$$

we have either

$$en^2\boldsymbol{q}_x \le \langle \boldsymbol{q}, \mathbf{1}\rangle \le e(n-1)\boldsymbol{q}_y$$

or

$$e(n-1)\boldsymbol{q}_y \le \langle \boldsymbol{q}, \mathbf{1}\rangle \le en^2\boldsymbol{q}_x.$$

Therefore, $\langle \boldsymbol{q}, \mathbf{1}\rangle \ge \langle \boldsymbol{q}, \Gamma\rangle$ and the equality holds if and only if

$$en^2\boldsymbol{q}_x = \langle \boldsymbol{q}, \mathbf{1}\rangle = e(n-1)\boldsymbol{q}_y.$$

Note that n^2 is coprime with $n-1$. Hence in this case $\boldsymbol{q}$ is a multiple of $\boldsymbol{p} = (n-1, n^2)$, which implies that $\langle \boldsymbol{q}, \mathbf{1}\rangle \ge \langle \boldsymbol{p}, \mathbf{1}\rangle$. $\square$

Proof of Example 1.5. It follows from Lemma 5.1 and Lemma 5.2. $\square$

Lemma 5.3. *Fix a positive integer n. Let Γ be the Newton polygon of the monomial $\mathbb{R}$-ideal $(x^{n^2+n+1}, y^{n+1})^{1/n}$ on $\mathbb{A}^2_k$ and $\boldsymbol{p}$ be the vector $(n+1, n^2+n+1)$. Then $\langle \boldsymbol{p}, \mathbf{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle < 0$ and for any $\boldsymbol{q} \in \mathbb{N}^2$ such that $\langle \boldsymbol{q}, \mathbf{1}\rangle - \langle \boldsymbol{q}, \Gamma\rangle < 0$, we have $\langle \boldsymbol{q}, \mathbf{1}\rangle \ge \langle \boldsymbol{p}, \mathbf{1}\rangle = (n+1)^2 + 1$.*

Proof. By direct calculation, we obtain $\langle \boldsymbol{p}, \mathbf{1}\rangle - \langle \boldsymbol{p}, \Gamma\rangle = -1/n$. Let $\boldsymbol{q} \in \mathbb{N}^2$ such that $\langle \boldsymbol{q}, \mathbf{1}\rangle - \langle \boldsymbol{q}, \Gamma\rangle < 0$. By definition,

$$\langle \boldsymbol{q}, \Gamma\rangle = \min\left\{\frac{n^2+n+1}{n}\boldsymbol{q}_x, \frac{n+1}{n}\boldsymbol{q}_y\right\}.$$

If $(n+1)\boldsymbol{q}_y \leq (n^2+n+1)\boldsymbol{q}_x$, then

$$\boldsymbol{q}_y \leq \left(n + \frac{1}{n+1}\right)\boldsymbol{q}_x$$

and

$$\langle \boldsymbol{q}, \mathbf{1}\rangle - \langle \boldsymbol{q}, \Gamma\rangle = \boldsymbol{q}_x - \frac{1}{n}\boldsymbol{q}_y < 0. \tag{5.1}$$

Hence

$$n\boldsymbol{q}_x < \boldsymbol{q}_y \leq \left(n + \frac{1}{n+1}\right)\boldsymbol{q}_x.$$

Note that $\boldsymbol{q}_x$ and $\boldsymbol{q}_y$ are integers, we have $\boldsymbol{q}_x \geq n+1$. Hence (5.1) implies that $\boldsymbol{q}_y \geq n^2+n+1$.

If $(n+1)\boldsymbol{q}_y \geq (n^2+n+1)\boldsymbol{q}_x$, then

$$\boldsymbol{q}_y \geq \left(n + \frac{1}{n+1}\right)\boldsymbol{q}_x > n\boldsymbol{q}_x \tag{5.2}$$

and

$$\langle \boldsymbol{q}, \mathbf{1}\rangle - \langle \boldsymbol{q}, \Gamma\rangle = \boldsymbol{q}_y - \left(n + \frac{1}{n}\right)\boldsymbol{q}_x < 0.$$

Hence

$$n\boldsymbol{q}_x < \boldsymbol{q}_y < \left(n + \frac{1}{n}\right)\boldsymbol{q}_x.$$

Note that $\boldsymbol{q}_x$ and $\boldsymbol{q}_y$ are integers, we have $\boldsymbol{q}_x \geq n+1$. Hence (5.2) implies that $\boldsymbol{q}_y \geq n^2+n+1$. □

Proof of Example 1.6. It follows from Lemma 5.1 and Lemma 5.3.

□

References

[Al] V. Alexeev, *Two dimensional terminations*, DukeMath. J., **69**(3), (1993), 527-545. 1

[CH] G. Chen and J. Han, *Boundedness of (ϵ, n)-complements for surfaces*, preprint 2020, arXiv: 2002.02246v1. 1

[HL] J. Han, Y. Luo ,*On boundedness of divisors computing minimal log discrepancies for surfaces*, preprint 2020, arXiv:2005.09626v2. 1, 1.2

[Is1] S. Ishii, *Finite determination conjecture for Mather-Jacobian minimal log discrepancies and its applications*, Eur. J. Math. **4**, (2018), 1433-1475. 1

[Is2] S. Ishii, *The minimal log discrepancies on a smooth surface in positive characteristic*, Math. Zeitschrift, (2020). 1, 2

[Ka] M. Kawakita, *On equivalent conjectures for minimal log discrepancies on smooth threefolds*, preprint 2018, arXiv:1803.02539.

[MN] M. Mustaţă and Y. Nakamura, *A boundedness conjecture for minimal log discrepancies on a fixed germ*, AMS Contemporary Mathematics, **712**, (2018), 287-306. 1
1, 1