MA2104 - Multivariable Calculus Suggested Solutions

(Semester 2: AY2018/19)

Written by: Chen YiJia
Audited by: Pan Jing Bin

Question 1. [20 marks]
Let P be the point $(3,3,3)$ in $\mathbb{R}^3$.

(a) Find the distance from P to the line $\ell := x = 2y = z$.

Notice the line passes through the origin.

$$\text{Distance} = \left| \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \times \frac{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}{3} \right| = \frac{1}{3} \left| \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \right| = \sqrt{2}$$

(b) Let S be the surface $z = x^2 - y^2 + 3$.

(i) Find an equation of the tangent plane π to S at P.

Let $f(x,y) = x^2 - y^2 + 3$.

$$f_x = 2x \qquad f_y = -2y$$
$$f_x(3,3) = 6 \qquad f_y(3,3) = -6$$
$$z - 3 = 6(x-3) - 6(y-3)$$
$$\pi : z = 6x - 6y + 3$$

(ii) Show that the line $\ell_1 : x = y, z = 3$ lies in the intersection of S and π.

To find the intersection, we solve the equation: $x^2 - y^2 + 3 = 6x - 6y + 3$
$(x+y)(x-y) = 6(x-y)$ is true when $x = y$.
When $x = y,\ z = 3$ for both S and π, $\forall (x,y) \in \mathbb{R}^2$.

(iii) Find symmetric equations of another line ℓ_2 different from ℓ_1 that passes through P and lies in the intersection of S and π.

If $x \neq y$, then the intersection yields $x + y = 6$.
Let $x = 6, y = 0$, then $z = 39$.

$$\ell_2 : \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 6-3 \\ 0-3 \\ 39-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix} \implies x - 3 = -y + 3 = \frac{z-3}{12}$$

Question 2. [15 marks]

(a) Let $f(x, y) = x^3 + y^3 + 3xy$. Find all critical points of f. At each of these critical points, determine whether f has a local maximum, a local minimum, or a saddle point.

We first locate the critical points:

$$f_x = 3x^2 + 3y \qquad f_y = 3y^2 + 3x$$
$$3x^2 + 3y = 0 \implies y = -x^2 \qquad 3x^4 + 3x = 0 \implies 3x(x^3 + 1)$$

There are 2 critical points $(0, 0)$ and $(-1, -1)$. Next, we calculate the second partial derivatives $D(x, y)$:

$$f_{xx} = 6x \qquad f_{yy} = 6y \qquad f_{xy} = 3$$
$$D(x, y) = (6x)(6y) - 9 = 36xy - 9$$
$$D(0, 0) = -9 \implies \text{saddle point} \qquad D(-1, -1) = 27 \implies \text{local min}$$

(b) Find the maximum and minimum values of $f(x, y) = 2x + y$ subject to the constraint $x^2 + 2xy + 2y^2 = 5$.

By Lagrange Multipliers,

$$2 = \lambda(2x + 2y)$$
$$1 = \lambda(2x + 4y)$$
$$2\lambda(x + y) = 4\lambda(x + 2y) \implies x + y = 2x + 4y \implies x = -3y$$
$$\implies 5 = 9y^2 - 6y^2 + 2y^2 = 5y^2 \implies y = \pm 1, x = \mp 3$$
$$f(-3, 1) = -5 \implies min \qquad f(3, -1) = 5 \implies max$$

Question 3. [15 marks]

(a) By using transformation $T(x, y) = (x + y, y - 2x)$, evaluate the double integral

$$\iint_R \sqrt{x + y}\ (y - 2x)^2\ dxdy,$$

where R is the triangle in the xy-plane with vertices $A(0,0)$, $B(3,0)$, $C(0,3)$.

Using the change of variables $u = x + y$, $v = y - 2x$, the vertices are transformed to $A(0,0)$, $B(3,-6)$, $C(3,3)$. AB is the line $v = -2u$ and AC is the line $v = u$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$\begin{aligned} \iint_R \sqrt{x+y}\,(y-2x)^2\,dxdy &= \int_0^3 \int_{-2u}^{u} \sqrt{u}v^2 \frac{1}{3}\,dvdu \\ &= \frac{1}{3}\int_0^3 \left[\frac{1}{3}\sqrt{u}v^3\right]_{-2u}^{u} du \\ &= \int_0^3 u^{\frac{7}{2}}\,du \\ &= \left[\frac{2}{9}u^{\frac{9}{2}}\right]_0^3 \\ &= \frac{2}{9}\cdot 3^{\frac{9}{2}} \end{aligned}$$

(b) Let $\mathbf{F}(x,y,z) = \langle y\sin(z), x\sin(z), xy\cos(z)\rangle$ and let C be the curve from $(0,0,0)$ to $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ with parametric equations:

$$x = t + t\cos(t),\ y = t\sin(t),\ z = t, \qquad 0 \le t \le \frac{\pi}{2}.$$

Find a potential function of $\mathbf{F}$. Hence, or otherwise, evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.

Find potential function f such that $\mathbf{F} = \nabla f$.

$f_x(x,y,z) = y\sin(z) \implies f(x,y,z) = xy\sin(z) + g(y,z) \implies f_y(x,y,z) = x\sin(z) + g_y(y,z)$
But $f_y(x,y,z) = x\sin(z)$ so $g_y(y,z) = 0 \implies g(y,z) = h(z) \implies f(x,y,z) = xy\sin(z) + h(z)$
$\implies f_z(x,y,z) = xy\cos(z) + h'(z) = xy\cos(z) \implies h'(z) = 0 \implies h(z) = K$, a constant.
Hence, $f = xy\cos(z)$ (taking $K = 0$).

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2}) - f(0,0,0) = 0$$

Question 4. [15 marks]

(a) Using Green's Theorem, evaluate the line integral

$$\oint_C (7y - e^{\sin x})\,dx \ + [9x - \cos(y^3 + 7y)]dy,$$

where C is the circle of radius 2 centred at point $(1,1)$ and is given the counterclockwise orientation.

Let $P = 7y - e^{\sin x}$, $Q = 9x - \cos(y^3 + 7y)$.

$$\begin{aligned}\int_C P\ dx\ + Q\ dy &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA \\ &= \iint_D 9 - 7\ dA \\ &= 2 \cdot \pi(2)^2 \\ &= 8\pi\end{aligned}$$

(b) Find the volume of the solid bounded below by the cone $\sqrt{3}z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 2z$.

Using spherical coordinates, the equation of the cone can be written as

$$\sqrt{3}\rho\cos\phi = \sqrt{\rho^2 sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} = \rho\sin\phi$$

This gives $\tan\phi = \sqrt{3}$, or $\phi = \frac{\pi}{3}$. The equation of the sphere can be written as

$$\rho^2 = 2\rho\cos\phi \implies \rho = 2\cos\phi$$

Therefore the description of the solid E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{3},\ 0 \le \rho \le 2\cos\phi\}$$

$$\begin{aligned}\iiint_E dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{2\cos\phi} \rho^2\sin\phi\ d\rho d\phi d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} \sin\phi \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=2\cos\phi} d\phi \\ &= \frac{16\pi}{3}\int_0^{\frac{\pi}{3}} \sin\phi\cos^3\phi\ d\phi \\ &= \frac{16\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{3}} \\ &= \frac{5\pi}{4}\end{aligned}$$

Question 5. [15 marks]

(a) Rewrite the following iterated integral in the order $dydxdz$:

$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)dzdydx.$$

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} f(x,y,z)dzdydx = \iiint_E f(x,y,z)\ dV$$

where $E = \{(x,y,z)\,|-1 \le x \le 1, x^2 \le y \le 1, 0 \le z \le 1-y\}$. This description of E enables us to write projections onto the three coordinate planes as follows:

$$\begin{aligned}
\text{on the } xy\text{-plane:} \quad & \{(x,y)\,|-1 \le x \le 1,\ x^2 \le y \le 1\} \\
& =\{(x,y)\mid 0 \le y \le 1,\ -\sqrt{y} \le x \le \sqrt{y}\} \\
\text{on the } yz\text{-plane:} \quad & \{(y,z)\mid x^2 \le y \le 1,\ 0 \le z \le 1-y\} \\
& =\{(y,z)\mid 0 \le z \le 1,\ x^2 \le y \le 1-z\} \\
\text{on the } xz\text{-plane:} \quad & \{(x,z)\,|-1 \le x \le 1,\ 0 \le z \le 1-y\} \\
& =\{(x,z)\mid 0 \le z \le 1,\ -\sqrt{1-z} \le x \le \sqrt{1-z}\}
\end{aligned}$$

$$E = \{(x,y,z)\mid 0 \le z \le 1,\ -\sqrt{1-z} \le x \le \sqrt{1-z},\ x^2 \le y \le 1-z\}$$

Thus,

$$\iiint_E f(x,y,z)\ dV = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} f(x,y,z)dydxdz$$

(b) Let $\mathbf{F}(x,y,z) = \langle y^3, x, z^3\rangle$. Let C be the curve of intersection of the surface $z = xy$ and the cylinder $x^2 + y^2 = 1$. C is oriented in the counterclockwise sense when viewed from above. Evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r}.$$

Using cylindrical coordinates, $\mathbf{F}(\mathbf{r}(\theta)) = \langle \sin^3\theta, \cos\theta, \cos^3\theta\sin^3\theta\rangle$ and the intersection C is $\mathbf{r}(\theta) = \langle\cos\theta, \sin\theta, \cos\theta\sin\theta\rangle$. $\mathbf{r}'(\theta) = \langle -\sin\theta, \cos\theta, \cos^2\theta - \sin^2\theta\rangle$.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta))\cdot\mathbf{r}'(\theta)\ d\theta \\
&= \int_0^{2\pi} \begin{pmatrix} \sin^3\theta \\ \cos\theta \\ \cos^3\theta\sin^3\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ \cos^2 - \sin^2\theta \end{pmatrix} d\theta \\
&= \int_0^{2\pi} -\sin^4\theta + \cos^2\theta + \cos^5\theta\sin^3\theta - \cos^3\theta\sin^5\theta\ d\theta \\
&= \int_0^{2\pi} -\sin^4\theta + \cos^2\theta\ d\theta \ \ \text{since odd functions are symmetric about 0} \\
&= \frac{1}{32}\left[4\theta + 16\sin(2\theta) - \sin(4\theta)\right]_0^{2\pi} \\
&= \frac{\pi}{4}
\end{aligned}$$

Question 6. [20 marks]

(a) Let $f(x, y, z)$ and $g(x, y, z)$ be functions having continuous 2nd order partial derivatives on $\mathbb{R}^3$. Let Σ be a smooth oriented surface in $\mathbb{R}^3$ with boundary C which is a simple closed curve oriented with the positive orientation. Using Stokes' theorem, or otherwise, prove that

$$\int_C f\nabla g \cdot d\mathbf{r} = \int_{-C} g\nabla f \cdot d\mathbf{r}.$$

$$\begin{aligned}
\int_C g\nabla f \cdot d\mathbf{r} &= \iint_\Sigma \text{curl } (f\nabla g) \cdot d\mathbf{\Sigma} = \iint_\Sigma \nabla \times \begin{pmatrix} gf_x \\ gf_y \\ gf_z \end{pmatrix} \cdot d\mathbf{\Sigma} \\
&= \iint_\Sigma \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ gf_x & gf_y & gf_z \end{vmatrix} \cdot d\mathbf{\Sigma} \\
(\text{since curl}(\nabla f) = \mathbf{0}) &= \iint_\Sigma [g_y f_z - g_z f_y]\mathbf{i} - (g_x f_z - g_z f_x)\mathbf{j} + (g_x f_y - g_y f_x)\mathbf{k} \cdot d\mathbf{\Sigma} \\
\int_C f\nabla g \cdot d\mathbf{r} &= \iint_\Sigma \nabla \times \begin{pmatrix} fg_x \\ fg_y \\ fg_z \end{pmatrix} \cdot d\mathbf{\Sigma} \\
&= \iint_\Sigma \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fg_x & fg_y & fg_z \end{vmatrix} \cdot d\mathbf{\Sigma} \\
(\text{since curl}(\nabla g) = \mathbf{0}) &= \iint_\Sigma [(f_y g_z - f_z g_y)\mathbf{i} - (f_x g_z - f_z g_x)\mathbf{j} + (f_x g_y - f_y g_x)\mathbf{k}] \cdot d\Sigma \\
&= -\int_C g\nabla f \cdot d\mathbf{r} = \int_{-C} g\nabla f \cdot d\mathbf{r}
\end{aligned}$$

(b) Let $\mathbf{F}$ be the vector field defined by

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} + z^2 \right\rangle,$$

where $(x,y,z) \neq (0,0,0)$. Let S be the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ oriented with the outward pointing normal. Using divergence theorem, or otherwise, evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Let $\mathbf{F} = \mathbf{F_1} + \mathbf{F_2}$ where

$$\mathbf{F_1} = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} \langle x, y, z \rangle \qquad \mathbf{F_2} = \langle 0, 0, z^2 \rangle$$

S is the boundary of the ellipsoid E given by $x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1$.

$$\iint_S \mathbf{F_2} \cdot d\mathbf{S} = \iiint_E 2z \; dV = 0 \text{ by symmetry of } E$$

Since $\mathbf{F_1}$ is undefined at $(0,0,0)$, we introduce a unit sphere T centered at $(0,0,0)$ and calculate a modified flux

$$\iint_S \mathbf{F_1} \cdot d\mathbf{S} - \iint_T \mathbf{F_1} \cdot d\mathbf{T} = \iiint_{E'} \nabla \cdot \mathbf{F_1} \; dV = 0$$

$$\therefore \iint_S \mathbf{F_1} \cdot d\mathbf{S} = \iint_T \mathbf{F_1} \cdot d\mathbf{T}$$

$$\mathbf{F_1} \cdot d\mathbf{T} = \frac{\langle x,y,z \rangle}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot |\langle x,y,z \rangle| = \frac{\langle x,y,z \rangle}{(x^2+y^2+z^2)^{\frac{3}{2}}} \cdot \frac{\langle x,y,z \rangle}{\sqrt{x^2+y^2+z^2}} = \frac{\rho^2}{\rho^4}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_T \mathbf{F_1} \cdot d\mathbf{T} = \int_0^{2\pi} \int_0^{\pi} \int_0^1 \left(\frac{\rho^2}{\rho^4} \right) \rho^2 \sin\phi \; d\rho d\phi d\theta = 4\pi$$

END OF PAPER