

MA2104 - Multivariable Calculus Suggested Solutions

(Semester 2: AY2018/19)

Written by: Chen YiJia
Audited by: Pan Jing Bin

Question 1. [20 marks]

Let P be the point $(3, 3, 3)$ in \mathbb{R}^3 .

(a) Find the distance from P to the line $\ell := x = 2y = z$.

Notice the line passes through the origin.

$$\text{Distance} = \left| \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \times \frac{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}{3} \right| = \frac{1}{3} \left| \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \right| = \sqrt{2}$$

(b) Let S be the surface $z = x^2 - y^2 + 3$.

(i) Find an equation of the tangent plane π to S at P .

Let $f(x, y) = x^2 - y^2 + 3$.

$$\begin{aligned} f_x &= 2x & f_y &= -2y \\ f_x(3, 3) &= 6 & f_y(3, 3) &= -6 \\ z - 3 &= 6(x - 3) - 6(y - 3) \\ \pi : z &= 6x - 6y + 3 \end{aligned}$$

(ii) Show that the line $\ell_1 : x = y, z = 3$ lies in the intersection of S and π .

To find the intersection, we solve the equation: $x^2 - y^2 + 3 = 6x - 6y + 3$

$(x + y)(x - y) = 6(x - y)$ is true when $x = y$.

When $x = y$, $z = 3$ for both S and π , $\forall (x, y) \in \mathbb{R}^2$.

(iii) Find symmetric equations of another line ℓ_2 different from ℓ_1 that passes through P and lies in the intersection of S and π .

If $x \neq y$, then the intersection yields $x + y = 6$.

Let $x = 6, y = 0$, then $z = 39$.

$$\ell_2 : \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 6-3 \\ 0-3 \\ 39-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix} \implies x - 3 = -y + 3 = \frac{z - 3}{12}$$

Question 2. [15 marks]

(a) Let $f(x, y) = x^3 + y^3 + 3xy$. Find all critical points of f . At each of these critical points, determine whether f has a local maximum, a local minimum, or a saddle point.

We first locate the critical points:

$$\begin{aligned} f_x &= 3x^2 + 3y & f_y &= 3y^2 + 3x \\ 3x^2 + 3y &= 0 \implies y = -x^2 & 3x^4 + 3x &= 0 \implies 3x(x^3 + 1) \end{aligned}$$

There are 2 critical points $(0, 0)$ and $(-1, -1)$. Next, we calculate the second partial derivatives $D(x, y)$:

$$\begin{aligned} f_{xx} &= 6x & f_{yy} &= 6y & f_{xy} &= 3 \\ D(x, y) &= (6x)(6y) - 9 = 36xy - 9 \\ D(0, 0) &= -9 \implies \text{saddle point} & D(-1, -1) &= 27 \implies \text{local min} \end{aligned}$$

(b) Find the maximum and minimum values of $f(x, y) = 2x + y$ subject to the constraint $x^2 + 2xy + 2y^2 = 5$.

By Lagrange Multipliers,

$$\begin{aligned} 2 &= \lambda(2x + 2y) \\ 1 &= \lambda(2x + 4y) \\ 2\lambda(x + y) &= 4\lambda(x + 2y) \implies x + y = 2x + 4y \implies x = -3y \\ \implies 5 &= 9y^2 - 6y^2 + 2y^2 = 5y^2 \implies y = \pm 1, x = \mp 3 \\ f(-3, 1) &= -5 \implies \text{min} & f(3, -1) &= 5 \implies \text{max} \end{aligned}$$

Question 3. [15 marks]

(a) By using transformation $T(x, y) = (x + y, y - 2x)$, evaluate the double integral

$$\iint_R \sqrt{x + y} (y - 2x)^2 \, dx dy,$$

where R is the triangle in the xy -plane with vertices $A(0,0)$, $B(3,0)$, $C(0,3)$.

Using the change of variables $u = x + y$, $v = y - 2x$, the vertices are transformed to $A(0,0)$, $B(3,-6)$, $C(3,3)$. AB is the line $v = -2u$ and AC is the line $v = u$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$\begin{aligned} \iint_R \sqrt{x+y} (y-2x)^2 dx dy &= \int_0^3 \int_{-2u}^u \sqrt{uv}^2 \frac{1}{3} dv du \\ &= \frac{1}{3} \int_0^3 \left[\frac{1}{3} \sqrt{uv}^3 \right]_{-2u}^u du \\ &= \int_0^3 u^{\frac{7}{2}} du \\ &= \left[\frac{2}{9} u^{\frac{9}{2}} \right]_0^3 \\ &= \frac{2}{9} \cdot 3^{\frac{9}{2}} \end{aligned}$$

(b) Let $\mathbf{F}(x, y, z) = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$ and let C be the curve from $(0,0,0)$ to $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ with parametric equations:

$$x = t + t \cos(t), \quad y = t \sin(t), \quad z = t, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Find a potential function of \mathbf{F} . Hence, or otherwise, evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Find potential function f such that $\mathbf{F} = \nabla f$.

$f_x(x, y, z) = y \sin(z) \implies f(x, y, z) = xy \sin(z) + g(y, z) \implies f_y(x, y, z) = x \sin(z) + g_y(y, z)$
 But $f_y(x, y, z) = x \sin(z)$ so $g_y(y, z) = 0 \implies g(y, z) = h(z) \implies f(x, y, z) = xy \sin(z) + h(z)$
 $\implies f_z(x, y, z) = xy \cos(z) + h'(z) = xy \cos(z) \implies h'(z) = 0 \implies h(z) = K$, a constant.
 Hence, $f = xy \cos(z)$ (taking $K = 0$).

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) - f(0, 0, 0) = 0$$

Question 4. [15 marks]

(a) Using Green's Theorem, evaluate the line integral

$$\oint_C (7y - e^{\sin x}) dx + [9x - \cos(y^3 + 7y)] dy,$$

where C is the circle of radius 2 centred at point $(1, 1)$ and is given the counterclockwise orientation.

Let $P = 7y - e^{\sin x}$, $Q = 9x - \cos(y^3 + 7y)$.

$$\begin{aligned} \int_C P \, dx + Q \, dy &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \\ &= \iint_D 9 - 7 \, dA \\ &= 2 \cdot \pi(2)^2 \\ &= 8\pi \end{aligned}$$

(b) Find the volume of the solid bounded below by the cone $\sqrt{3}z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 2z$.

Using spherical coordinates, the equation of the cone can be written as

$$\sqrt{3}\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta} = \rho \sin \phi$$

This gives $\tan \phi = \sqrt{3}$, or $\phi = \frac{\pi}{3}$. The equation of the sphere can be written as

$$\rho^2 = 2\rho \cos \phi \implies \rho = 2 \cos \phi$$

Therefore the description of the solid E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi, \frac{\pi}{3} \leq \phi \leq \frac{\pi}{2}, 0 \leq \rho \leq 2 \cos \phi\}$$

$$\begin{aligned} \iiint_E dV &= \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{2 \cos \phi} \rho^2 \sin \phi \, d\rho d\phi d\theta \\ &= \int_0^{2\pi} d\theta \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin \phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=2 \cos \phi} d\phi \\ &= \frac{16\pi}{3} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin \phi \cos^3 \phi \, d\phi \\ &= \frac{16\pi}{3} \left[-\frac{\cos^4 \phi}{4} \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= \frac{\pi}{12} \end{aligned}$$

Question 5. [15 marks]

(a) Rewrite the following iterated integral in the order $dydx dz$:

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) dz dy dx.$$

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) dz dy dx = \iiint_E f(x, y, z) dV$$

where $E = \{(x, y, z) \mid -1 \leq x \leq 1, x^2 \leq y \leq 1, 0 \leq z \leq 1 - y\}$. This description of E enables us to write projections onto the three coordinate planes as follows:

$$\begin{aligned} \text{on the } xy\text{-plane: } & \{(x, y) \mid -1 \leq x \leq 1, x^2 \leq y \leq 1\} \\ & = \{(x, y) \mid 0 \leq y \leq 1, -\sqrt{y} \leq x \leq \sqrt{y}\} \\ \text{on the } yz\text{-plane: } & \{(y, z) \mid x^2 \leq y \leq 1, 0 \leq z \leq 1 - y\} \\ & = \{(y, z) \mid 0 \leq z \leq 1, x^2 \leq y \leq 1 - z\} \\ \text{on the } xz\text{-plane: } & \{(x, z) \mid -1 \leq x \leq 1, 0 \leq z \leq 1 - y\} \\ & = \{(x, z) \mid 0 \leq z \leq 1, -\sqrt{1-z} \leq x \leq \sqrt{1-z}\} \end{aligned}$$

$$E = \{(x, y, z) \mid 0 \leq z \leq 1, -\sqrt{1-z} \leq x \leq \sqrt{1-z}, x^2 \leq y \leq 1 - z\}$$

Thus,

$$\iiint_E f(x, y, z) dV = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} f(x, y, z) dy dx dz$$

(b) Let $\mathbf{F}(x, y, z) = \langle y^3, x, z^3 \rangle$. Let C be the curve of intersection of the surface $z = xy$ and the cylinder $x^2 + y^2 = 1$. C is oriented in the counterclockwise sense when viewed from above. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Using cylindrical coordinates, $\mathbf{F}(\mathbf{r}(\theta)) = \langle \sin^3 \theta, \cos \theta, \cos^3 \theta \sin^3 \theta \rangle$ and the intersection C is $\mathbf{r}(\theta) = \langle \cos \theta, \sin \theta, \cos \theta \sin \theta \rangle$. $\mathbf{r}'(\theta) = \langle -\sin \theta, \cos \theta, \cos^2 \theta - \sin^2 \theta \rangle$.

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta)) \cdot \mathbf{r}'(\theta) d\theta \\ &= \int_0^{2\pi} \begin{pmatrix} \sin^3 \theta \\ \cos \theta \\ \cos^3 \theta \sin^3 \theta \end{pmatrix} \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \\ \cos^2 \theta - \sin^2 \theta \end{pmatrix} d\theta \\ &= \int_0^{2\pi} -\sin^4 \theta + \cos^2 \theta + \cos^5 \theta \sin^3 \theta - \cos^3 \theta \sin^5 \theta d\theta \\ &= \int_0^{2\pi} -\sin^4 \theta + \cos^2 \theta d\theta \quad \text{since odd functions are symmetric about 0} \\ &= \frac{1}{32} [4\theta + 16 \sin(2\theta) - \sin(4\theta)]_0^{2\pi} \\ &= \frac{\pi}{4} \end{aligned}$$

Question 6. [20 marks]

(a) Let $f(x, y, z)$ and $g(x, y, z)$ be functions having continuous 2nd order partial derivatives on \mathbb{R}^3 . Let Σ be a smooth oriented surface in \mathbb{R}^3 with boundary C which is a simple closed curve oriented with the positive orientation. Using Stokes' theorem, or otherwise, prove that

$$\int_C f \nabla g \cdot d\mathbf{r} = \int_{-C} g \nabla f \cdot d\mathbf{r}.$$

$$\int_C g \nabla f \cdot d\mathbf{r} = \iint_{\Sigma} \text{curl}(f \nabla g) \cdot d\mathbf{\Sigma} = \iint_{\Sigma} \nabla \times \begin{pmatrix} fg_x \\ fg_y \\ fg_z \end{pmatrix} \cdot d\mathbf{\Sigma}$$

$$= \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fg_x & fg_y & fg_z \end{vmatrix} \cdot d\mathbf{\Sigma}$$

$$(\text{since } \text{curl}(\nabla f) = \mathbf{0}) = \iint_{\Sigma} [g_y f_z - g_z f_y] \mathbf{i} - (g_x f_z - g_z f_x) \mathbf{j} + (g_x f_y - g_y f_x) \mathbf{k} \cdot d\mathbf{\Sigma}$$

$$\int_C f \nabla g \cdot d\mathbf{r} = \iint_{\Sigma} \nabla \times \begin{pmatrix} fg_x \\ fg_y \\ fg_z \end{pmatrix} \cdot d\mathbf{\Sigma}$$

$$= \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fg_x & fg_y & fg_z \end{vmatrix} \cdot d\mathbf{\Sigma}$$

$$(\text{since } \text{curl}(\nabla g) = \mathbf{0}) = \iint_{\Sigma} [(f_y g_z - f_z g_y) \mathbf{i} - (f_x g_z - f_z g_x) \mathbf{j} + (f_x g_y - f_y g_x) \mathbf{k}] \cdot d\mathbf{\Sigma}$$

$$= - \int_C g \nabla f \cdot d\mathbf{r} = \int_{-C} g \nabla f \cdot d\mathbf{r}$$

(b) Let \mathbf{F} be the vector field defined by

$$\mathbf{F}(x, y, z) = \left\langle \frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + z^2 \right\rangle,$$

where $(x, y, z) \neq (0, 0, 0)$. Let S be the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ oriented with the outward pointing normal. Using divergence theorem, or otherwise, evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Let $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ where

$$\mathbf{F}_1 = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \langle x, y, z \rangle \quad \mathbf{F}_2 = \langle 0, 0, z^2 \rangle$$

S is the boundary of the ellipsoid E given by $x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1$.

$$\iiint_S \mathbf{F}_2 \cdot d\mathbf{S} = \iiint_E 2z \, dV = 0 \text{ by symmetry of } E$$

Since \mathbf{F}_1 is undefined at $(0, 0, 0)$, we introduce a unit sphere T centered at $(0, 0, 0)$ and calculate a modified flux

$$\begin{aligned} \iint_S \mathbf{F}_1 \cdot d\mathbf{S} - \iint_T \mathbf{F}_1 \cdot d\mathbf{T} &= \iiint_{E'} \nabla \cdot \mathbf{F}_1 \, dV = 0 \\ \therefore \iint_S \mathbf{F}_1 \cdot d\mathbf{S} &= \iint_T \mathbf{F}_1 \cdot d\mathbf{T} \\ \mathbf{F}_1 \cdot d\mathbf{T} &= \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot |\langle x, y, z \rangle| = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} = \frac{\rho^2}{\rho^4} \\ \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_T \mathbf{F}_1 \cdot d\mathbf{T} = \int_0^{2\pi} \int_0^\pi \left(\frac{\rho^2}{\rho^4} \right) \rho^2 \sin \phi \, d\phi d\theta = 4\pi \end{aligned}$$

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