

# MA2104 - Multivariable Calculus Suggested Solutions

(Semester 2: AY2018/19)

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**Question 1.** [20 marks]

Let  $P$  be the point  $(3, 3, 3)$  in  $\mathbb{R}^3$ .

(a) Find the distance from  $P$  to the line  $\ell := x = 2y = z$ .

Notice the line passes through the origin.

$$\text{Distance} = \left| \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \times \frac{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}}{3} \right| = \frac{1}{3} \left| \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \right| = \sqrt{2}$$

(b) Let  $S$  be the surface  $z = x^2 - y^2 + 3$ .

(i) Find an equation of the tangent plane  $\pi$  to  $S$  at  $P$ .

Let  $f(x, y) = x^2 - y^2 + 3$ .

$$\begin{aligned} f_x &= 2x & f_y &= -2y \\ f_x(3, 3) &= 6 & f_y(3, 3) &= -6 \\ z - 3 &= 6(x - 3) - 6(y - 3) \\ \pi : z &= 6x - 6y + 3 \end{aligned}$$

(ii) Show that the line  $\ell_1 : x = y, z = 3$  lies in the intersection of  $S$  and  $\pi$ .

To find the intersection, we solve the equation:  $x^2 - y^2 + 3 = 6x - 6y + 3$

$(x + y)(x - y) = 6(x - y)$  is true when  $x = y$ .

When  $x = y$ ,  $z = 3$  for both  $S$  and  $\pi$ ,  $\forall (x, y) \in \mathbb{R}^2$ .

(iii) Find symmetric equations of another line  $\ell_2$  different from  $\ell_1$  that passes through  $P$  and lies in the intersection of  $S$  and  $\pi$ .

If  $x \neq y$ , then the intersection yields  $x + y = 6$ .

Let  $x = 6, y = 0$ , then  $z = 39$ .

$$\ell_2 : \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 6-3 \\ 0-3 \\ 39-3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix} \implies x-3 = -y+3 = \frac{z-3}{12}$$

**Question 2.** [15 marks]

(a) Let  $f(x, y) = x^3 + y^3 + 3xy$ . Find all critical points of  $f$ . At each of these critical points, determine whether  $f$  has a local maximum, a local minimum, or a saddle point.

We first locate the critical points:

$$\begin{aligned} f_x &= 3x^2 + 3y & f_y &= 3y^2 + 3x \\ 3x^2 + 3y &= 0 \implies y = -x^2 & 3x^4 + 3x &= 0 \implies 3x(x^3 + 1) \end{aligned}$$

There are 2 critical points  $(0, 0)$  and  $(-1, -1)$ . Next, we calculate the second partial derivatives  $D(x, y)$ :

$$\begin{aligned} f_{xx} &= 6x & f_{yy} &= 6y & f_{xy} &= 3 \\ D(x, y) &= (6x)(6y) - 9 = 36xy - 9 \\ D(0, 0) &= -9 \implies \text{saddle point} & D(-1, -1) &= 27 \implies \text{local max since } f_{xx}(-1) = -6 \end{aligned}$$

(b) Find the maximum and minimum values of  $f(x, y) = 2x + y$  subject to the constraint  $x^2 + 2xy + 2y^2 = 5$ .

By Lagrange Multipliers,

$$\begin{aligned} 2 &= \lambda(2x + 2y) \\ 1 &= \lambda(2x + 4y) \\ 2\lambda(x + y) &= 4\lambda(x + 2y) \implies x + y = 2x + 4y \implies x = -3y \\ \implies 5 &= 9y^2 - 6y^2 + 2y^2 = 5y^2 \implies y = \pm 1, x = \mp 3 \\ f(-3, 1) &= -5 \implies \text{min} & f(3, -1) &= 5 \implies \text{max} \end{aligned}$$

**Question 3.** [15 marks]

(a) By using transformation  $T(x, y) = (x + y, y - 2x)$ , evaluate the double integral

$$\iint_R \sqrt{x+y} (y-2x)^2 \, dx dy,$$

where  $R$  is the triangle in the  $xy$ -plane with vertices  $A(0,0)$ ,  $B(3,0)$ ,  $C(0,3)$ .

Using the change of variables  $u = x + y$ ,  $v = y - 2x$ , the vertices are transformed to  $A(0,0)$ ,  $B(3,-6)$ ,  $C(3,3)$ .  $AB$  is the line  $v = -2u$  and  $AC$  is the line  $v = u$ .

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 3$$

$$\begin{aligned} \iint_R \sqrt{x+y} (y-2x)^2 dx dy &= \int_0^3 \int_{-2u}^u \sqrt{uv}^2 \frac{1}{3} dv du \\ &= \frac{1}{3} \int_0^3 \left[ \frac{1}{3} \sqrt{uv}^3 \right]_{-2u}^u du \\ &= \int_0^3 u^{\frac{7}{2}} du \\ &= \left[ \frac{2}{9} u^{\frac{9}{2}} \right]_0^3 \\ &= \frac{2}{9} \cdot 3^{\frac{9}{2}} \end{aligned}$$

(b) Let  $\mathbf{F}(x, y, z) = \langle y \sin(z), x \sin(z), xy \cos(z) \rangle$  and let  $C$  be the curve from  $(0,0,0)$  to  $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$  with parametric equations:

$$x = t + t \cos(t), \quad y = t \sin(t), \quad z = t, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Find a potential function of  $\mathbf{F}$ . Hence, or otherwise, evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ .

Find potential function  $f$  such that  $\mathbf{F} = \nabla f$ .

$f_x(x, y, z) = y \sin(z) \implies f(x, y, z) = xy \sin(z) + g(y, z) \implies f_y(x, y, z) = x \sin(z) + g_y(y, z)$   
 But  $f_y(x, y, z) = x \sin(z)$  so  $g_y(y, z) = 0 \implies g(y, z) = h(z) \implies f(x, y, z) = xy \sin(z) + h(z)$   
 $\implies f_z(x, y, z) = xy \cos(z) + h'(z) = xy \cos(z) \implies h'(z) = 0 \implies h(z) = K$ , a constant.  
 Hence,  $f = xy \sin(z)$  (taking  $K = 0$ ).

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) - f(0,0,0) = \frac{\pi^2}{4}$$

**Question 4.** [15 marks]

(a) Using Green's Theorem, evaluate the line integral

$$\oint_C (7y - e^{\sin x}) dx + [9x - \cos(y^3 + 7y)] dy,$$

where  $C$  is the circle of radius 2 centred at point  $(1, 1)$  and is given the counterclockwise orientation.

Let  $P = 7y - e^{\sin x}$ ,  $Q = 9x - \cos(y^3 + 7y)$ .

$$\begin{aligned}\int_C P \, dx + Q \, dy &= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \\ &= \iint_D 9 - 7 \, dA \\ &= 2 \cdot \pi(2)^2 \\ &= 8\pi\end{aligned}$$

(b) Find the volume of the solid bounded below by the cone  $\sqrt{3}z = \sqrt{x^2 + y^2}$  and above by the sphere  $x^2 + y^2 + z^2 = 2z$ .

Using spherical coordinates, the equation of the cone can be written as

$$\sqrt{3}\rho \cos \phi = \sqrt{\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta} = \rho \sin \phi$$

This gives  $\tan \phi = \sqrt{3}$ , or  $\phi = \frac{\pi}{3}$ . The equation of the sphere can be written as

$$\rho^2 = 2\rho \cos \phi \implies \rho = 2 \cos \phi$$

Therefore the description of the solid  $E$  in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \frac{\pi}{3}, 0 \leq \rho \leq 2 \cos \phi\}$$

$$\begin{aligned}\iiint_E dV &= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^{2 \cos \phi} \rho^2 \sin \phi \, d\rho d\phi d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} \sin \phi \left[ \frac{\rho^3}{3} \right]_{\rho=0}^{\rho=2 \cos \phi} d\phi \\ &= \frac{16\pi}{3} \int_0^{\frac{\pi}{3}} \sin \phi \cos^3 \phi \, d\phi \\ &= \frac{16\pi}{3} \left[ -\frac{\cos^4 \phi}{4} \right]_0^{\frac{\pi}{3}} \\ &= \frac{5\pi}{4}\end{aligned}$$

**Question 5.** [15 marks]

(a) Rewrite the following iterated integral in the order  $dydx dz$ :

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) dz dy dx.$$

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) dz dy dx = \iiint_E f(x, y, z) dV$$

where  $E = \{(x, y, z) \mid -1 \leq x \leq 1, x^2 \leq y \leq 1, 0 \leq z \leq 1 - y\}$ . This description of  $E$  enables us to write projections onto the three coordinate planes as follows:

$$\begin{aligned} \text{on the } xy\text{-plane: } & \{(x, y) \mid -1 \leq x \leq 1, x^2 \leq y \leq 1\} \\ & = \{(x, y) \mid 0 \leq y \leq 1, -\sqrt{y} \leq x \leq \sqrt{y}\} \\ \text{on the } yz\text{-plane: } & \{(y, z) \mid x^2 \leq y \leq 1, 0 \leq z \leq 1 - y\} \\ & = \{(y, z) \mid 0 \leq z \leq 1, x^2 \leq y \leq 1 - z\} \\ \text{on the } xz\text{-plane: } & \{(x, z) \mid -1 \leq x \leq 1, 0 \leq z \leq 1 - y\} \\ & = \{(x, z) \mid 0 \leq z \leq 1, -\sqrt{1-z} \leq x \leq \sqrt{1-z}\} \end{aligned}$$

$$E = \{(x, y, z) \mid 0 \leq z \leq 1, -\sqrt{1-z} \leq x \leq \sqrt{1-z}, x^2 \leq y \leq 1 - z\}$$

Thus,

$$\iiint_E f(x, y, z) dV = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} f(x, y, z) dy dx dz$$

(b) Let  $\mathbf{F}(x, y, z) = \langle y^3, x, z^3 \rangle$ . Let  $C$  be the curve of intersection of the surface  $z = xy$  and the cylinder  $x^2 + y^2 = 1$ .  $C$  is oriented in the counterclockwise sense when viewed from above. Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

Using cylindrical coordinates,  $\mathbf{F}(\mathbf{r}(\theta)) = \langle \sin^3 \theta, \cos \theta, \cos^3 \theta \sin^3 \theta \rangle$  and the intersection  $C$  is  $\mathbf{r}(\theta) = \langle \cos \theta, \sin \theta, \cos \theta \sin \theta \rangle$ .  $\mathbf{r}'(\theta) = \langle -\sin \theta, \cos \theta, \cos^2 \theta - \sin^2 \theta \rangle$ .

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta)) \cdot \mathbf{r}'(\theta) d\theta \\ &= \int_0^{2\pi} \begin{pmatrix} \sin^3 \theta \\ \cos \theta \\ \cos^3 \theta \sin^3 \theta \end{pmatrix} \cdot \begin{pmatrix} -\sin \theta \\ \cos \theta \\ \cos^2 \theta - \sin^2 \theta \end{pmatrix} d\theta \\ &= \int_0^{2\pi} -\sin^4 \theta + \cos^2 \theta + \cos^5 \theta \sin^3 \theta - \cos^3 \theta \sin^5 \theta d\theta \\ &= \int_0^{2\pi} -\sin^4 \theta + \cos^2 \theta d\theta \quad \text{since odd functions are symmetric about 0} \\ &= \frac{1}{32} [4\theta + 16 \sin(2\theta) - \sin(4\theta)]_0^{2\pi} \\ &= \frac{\pi}{4} \end{aligned}$$

**Question 6.** [20 marks]

(a) Let  $f(x, y, z)$  and  $g(x, y, z)$  be functions having continuous 2nd order partial derivatives on  $\mathbb{R}^3$ . Let  $\Sigma$  be a smooth oriented surface in  $\mathbb{R}^3$  with boundary  $C$  which is a simple closed curve oriented with the positive orientation. Using Stokes' theorem, or otherwise, prove that

$$\int_C f \nabla g \cdot d\mathbf{r} = \int_{-C} g \nabla f \cdot d\mathbf{r}.$$

$$\int_C g \nabla f \cdot d\mathbf{r} = \iint_{\Sigma} \text{curl}(f \nabla g) \cdot d\mathbf{\Sigma} = \iint_{\Sigma} \nabla \times \begin{pmatrix} fg_x \\ fg_y \\ fg_z \end{pmatrix} \cdot d\mathbf{\Sigma}$$

$$= \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fg_x & fg_y & fg_z \end{vmatrix} \cdot d\mathbf{\Sigma}$$

$$(\text{since } \text{curl}(\nabla f) = \mathbf{0}) = \iint_{\Sigma} [g_y f_z - g_z f_y] \mathbf{i} - (g_x f_z - g_z f_x) \mathbf{j} + (g_x f_y - g_y f_x) \mathbf{k} \cdot d\mathbf{\Sigma}$$

$$\int_C f \nabla g \cdot d\mathbf{r} = \iint_{\Sigma} \nabla \times \begin{pmatrix} fg_x \\ fg_y \\ fg_z \end{pmatrix} \cdot d\mathbf{\Sigma}$$

$$= \iint_{\Sigma} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fg_x & fg_y & fg_z \end{vmatrix} \cdot d\mathbf{\Sigma}$$

$$(\text{since } \text{curl}(\nabla g) = \mathbf{0}) = \iint_{\Sigma} [(f_y g_z - f_z g_y) \mathbf{i} - (f_x g_z - f_z g_x) \mathbf{j} + (f_x g_y - f_y g_x) \mathbf{k}] \cdot d\mathbf{\Sigma}$$

$$= - \int_C g \nabla f \cdot d\mathbf{r} = \int_{-C} g \nabla f \cdot d\mathbf{r}$$

(b) Let  $\mathbf{F}$  be the vector field defined by

$$\mathbf{F}(x, y, z) = \left\langle \frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}, \frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} + z^2 \right\rangle,$$

where  $(x, y, z) \neq (0, 0, 0)$ . Let  $S$  be the ellipsoid  $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$  oriented with the outward pointing normal. Using divergence theorem, or otherwise, evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Let  $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$  where

$$\mathbf{F}_1 = \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \langle x, y, z \rangle \quad \mathbf{F}_2 = \langle 0, 0, z^2 \rangle$$

$S$  is the boundary of the ellipsoid  $E$  given by  $x^2 + \frac{y^2}{4} + \frac{z^2}{9} \leq 1$ .

$$\iiint_S \mathbf{F}_2 \cdot d\mathbf{S} = \iiint_E 2z \, dV = 0 \text{ by symmetry of } E$$

Since  $\mathbf{F}_1$  is undefined at  $(0, 0, 0)$ , we introduce a unit sphere  $T$  centered at  $(0, 0, 0)$  and calculate a modified flux

$$\begin{aligned} \operatorname{div} \mathbf{F}_1 &= \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - x(3x)(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3} + \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - x(3y)(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3} \\ &\quad + \frac{(x^2 + y^2 + z^2)^{\frac{3}{2}} - x(3z)(x^2 + y^2 + z^2)^{\frac{1}{2}}}{(x^2 + y^2 + z^2)^3} \\ &= \frac{3}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \frac{3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0 \end{aligned}$$

$$\iint_S \mathbf{F}_1 \cdot d\mathbf{S} - \iint_T \mathbf{F}_1 \cdot d\mathbf{T} = \iiint_{E'} \operatorname{div} \mathbf{F}_1 \, dV = 0$$

$$\therefore \iint_S \mathbf{F}_1 \cdot d\mathbf{S} = \iint_T \mathbf{F}_1 \cdot d\mathbf{T}$$

$$\mathbf{F}_1 \cdot d\mathbf{T} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot |\langle x, y, z \rangle| = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \cdot \frac{\langle x, y, z \rangle}{\sqrt{x^2 + y^2 + z^2}} = \frac{\rho^2}{\rho^4}$$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_T \mathbf{F}_1 \cdot d\mathbf{T} = \int_0^{2\pi} \int_0^\pi \left( \frac{\rho^2}{\rho^4} \right) \rho^2 \sin \phi \, d\phi d\theta = 4\pi$$

**END OF PAPER**