Supplementary Material for the Paper Titled "Multi-View Clustering Based on Belief Propagation"

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1 MESSAGE SIMPLIFICATION

The within-view messages $\alpha_{ik}^p(\mathbf{c}_i^p)$ and $\beta_{ik}^p(\mathbf{c}_i^p)$, and cross-view messages $\eta_{ii}^{pq}(\mathbf{c}_i^p)$ and $\theta_{ii}^{pq}(\mathbf{c}_i^p)$ are respectively as follows.

1) Within-view messages:

$$\begin{aligned}
&\forall p=1,\dots,m,\quad \forall i=1,\dots,n,\quad \forall k=1,\dots,n\\
\alpha_{ik}^p(\mathbf{c}_i^p) &=\mu_{\delta_k^p\to\mathbf{c}_i^p}(\mathbf{c}_i^p)\\
&= \max_{\substack{j_1,\dots,\\ j_{i-1},j_{i+1}\\ ,\dots,j_n}} \left[\delta_k^p(j_1,\dots,\mathbf{c}_i^p,\dots,j_n)+\sum_{i':i'\neq i}\beta_{i'k}^p(j_{i'})\right]
\end{aligned}\tag{1}$$

$$\begin{aligned}
\beta_{ik}^p(\mathbf{c}_i^p) &=\mu_{\mathbf{c}_i^p\to\delta_k^p}(\mathbf{c}_i^p)\\
&=S^p(i,\mathbf{c}_i^p)+\sum_{k':k'\neq k}\alpha_{ik'}^p(\mathbf{c}_i^p)+\sum_{q=1,q\neq p}^{m}\eta_{ii}^{pq}(\mathbf{c}_i^p)
\end{aligned}\tag{2}$$

2) Cross-view messages:

$$\begin{aligned}
&\forall p,q=1,\dots,m,\ p\neq q,\quad \forall i=1,\dots,n\\
\eta_{ii}^{pq}(\mathbf{c}_i^p) &=\mu_{L^{pq}(\mathbf{c}_i^p,\mathbf{c}_i^q)\to\mathbf{c}_i^p}\\
&=\max_{\mathbf{c}_i^q}\left[L^{pq}(\mathbf{c}_i^p,\mathbf{c}_i^q)+\theta_{ii}^{qp}(\mathbf{c}_i^q)\right]
\end{aligned}\tag{3}$$

$$\begin{aligned}
\theta_{ii}^{pq}(\mathbf{c}_i^p) &=\mu_{\mathbf{c}_i^p\to L^{pq}(\mathbf{c}_i^p,\mathbf{c}_i^q)}\\
&=S^p(i,\mathbf{c}_i^p)+\sum_{k=1}^{n}\alpha_{ik}^p(\mathbf{c}_i^p)+\sum_{q'=1,q'\neq q,q'\neq p}^{m}\eta_{ii}^{pq'}(\mathbf{c}_i^p)
\end{aligned}\tag{4}$$

To derive simple yet functionally equivalent computations, the exemplar consistency constraint embedded in (1) is firstly expanded. However, the expansion contains a great amount of redundant computations, which needs to be eliminated for computational efficiency. To this end, each variable message is taken as the sum of a variable and a constant. By dexterously setting the constants and applying variable substitution, the computations are finally simplified to be simple yet functionally equivalent.

First, according to the definition of the exemplar consistency constraint δ_k^p, the message $\alpha_{ik}^p(\mathbf{c}_i^p)$ in (1) can be expanded as follows,

$$\begin{aligned}
\alpha_{ik}^p(\mathbf{c}_i^p) &= \max_{\substack{j_1,\dots,\\ j_{i-1},j_{i+1}\\ ,\dots,j_n}} \left[\delta_k^p(j_1,\dots,\mathbf{c}_i^p,\dots,j_n)+\sum_{i':i'\neq i}\beta_{i'k}^p(j_{i'})\right]\\
&=\begin{cases}
\sum_{i':i'\neq k}\max_{j'}\beta_{i'k}^p(j'), & \mathbf{c}_i^p=k=i\\[2ex]
\sum_{i':i'\neq k}\max_{j':j'\neq k}\beta_{i'k}^p(j'), & \mathbf{c}_i^p\neq k=i\\[2ex]
\beta_{kk}^p(k)+\sum_{i':i'\notin\{i,k\}}\max_{j'}\beta_{i'k}^p(j'), & \mathbf{c}_i^p=k\neq i\\[2ex]
\max\Big[\beta_{kk}^p(k)+\sum_{i':i'\notin\{i,k\}}\max_{j'}\beta_{i'k}^p(j'), & \\
\quad \max_{j':j'\neq k}\beta_{kk}^p(j')+\sum_{i':i'\notin\{i,k\}}\max_{j':j'\neq k}\beta_{i'k}^p(j')\Big] & \mathbf{c}_i^p\neq k\neq i
\end{cases}
\end{aligned}\tag{5}$$

Although the above expansion addresses the problem associated with the exemplar consistency constraint, it is still too complicated since there exist lots of redundant computations, which needs further simplification.

To this end, we can view each message as the sum of a variable *w.r.t.* $\mathbf{c}_i^p$ and a constant, that is $\alpha_{ik}^p(\mathbf{c}_i^p)=\tilde{\alpha}_{ik}^p(\mathbf{c}_i^p)+\bar{\alpha}_{ik}^p$, $\beta_{ik}^p(\mathbf{c}_i^p)=\tilde{\beta}_{ik}^p(\mathbf{c}_i^p)+\bar{\beta}_{ik}^p$, $\eta_{ii}^{pq}(\mathbf{c}_i^p)=\tilde{\eta}_{ii}^{pq}(\mathbf{c}_i^p)+\bar{\eta}_{ii}^{pq}$ and $\theta_{ii}^{pq}(\mathbf{c}_i^p)=\tilde{\theta}_{ii}^{pq}(\mathbf{c}_i^p)+\bar{\theta}_{ii}^{pq}$. Consequently, the four types of

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messages can be rewritten as follows.

$$\alpha^p_{ik}(\mathbf{c}^p_i) = \begin{cases} \sum_{i':i'\neq k} \max_{j'} \tilde{\beta}^p_{i'k}(j') + \sum_{i':i'\neq k} \bar{\beta}^p_{i'k}, & \mathbf{c}^p_i = k = i \\ \sum_{i':i'\neq k} \max_{j':j'\neq k} \tilde{\beta}^p_{i'k}(j') + \sum_{i':i'\neq k} \bar{\beta}^p_{i'k}, & \mathbf{c}^p_i \neq k = i \\ \tilde{\beta}^p_{kk}(k) + \sum_{i':i'\notin\{i,k\}} \max_{j'} \tilde{\beta}^p_{i'k}(j') + \sum_{i':i'\neq i} \bar{\beta}^p_{i'k}, & \mathbf{c}^p_i = k \neq i \\ \max\Big[\tilde{\beta}^p_{kk}(k) + \sum_{i':i'\notin\{i,k\}} \max_{j'} \tilde{\beta}^p_{i'k}(j'), & \\ \quad \max_{j':j'\neq k} \tilde{\beta}^p_{kk}(j') + \sum_{i':i'\notin\{i,k\}} \max_{j':j'\neq k} \tilde{\beta}^p_{i'k}(j')\Big] & \\ \quad + \sum_{i':i'\neq i} \bar{\beta}^p_{i'k} & \mathbf{c}^p_i \neq k \neq i \end{cases} \tag{6}$$

$$\begin{aligned} \beta^p_{ik}(\mathbf{c}^p_i) =& S^p(i,\mathbf{c}^p_i) + \sum_{k':k'\neq k} \tilde{\alpha}^p_{ik'}(\mathbf{c}^p_i) + \sum_{k':k'\neq k} \bar{\alpha}^p_{ik'} \\ &+ \sum_{q=1,q\neq p}^{m} \tilde{\eta}^{pq}_{ii}(\mathbf{c}^p_i) + \sum_{q=1,q\neq p}^{m} \bar{\eta}^{pq}_{ii} \end{aligned} \tag{7}$$

$$\eta^{pq}_{ii}(\mathbf{c}^p_i) = \max_{\mathbf{c}^q_i} \Big[L^{pq}(\mathbf{c}^p_i, \mathbf{c}^q_i) + \tilde{\theta}^{qp}_{ii}(\mathbf{c}^q_i)\Big] + \bar{\theta}^{qp}_{ii} \tag{8}$$

$$\begin{aligned} \theta^{pq}_{ii}(\mathbf{c}^p_i) =& S^p(i,\mathbf{c}^p_i) + \sum_{k=1}^{n} \tilde{\alpha}^p_{ik}(\mathbf{c}^p_i) + \sum_{k=1}^{n} \bar{\alpha}^p_{ik} \\ &+ \sum_{q'=1,q'\neq q,q'\neq p}^{m} \tilde{\eta}^{pq'}_{ii}(\mathbf{c}^p_i) + \sum_{q'=1,q'\neq q,q'\neq p}^{m} \bar{\eta}^{pq'}_{ii} \end{aligned} \tag{9}$$

By observing (6), if we let constant $\bar{\beta}^p_{ik} = \max_{j:j\neq k} \beta^p_{ik}(j)$, then we have

$$\begin{aligned} \max_{j':j'\neq k} \tilde{\beta}^p_{ik}(j') &= \max_{j':j'\neq k} [\beta^p_{ik}(j') - \bar{\beta}^p_{ik}] \\ &= \max_{j':j'\neq k} \beta^p_{ik}(j') - \bar{\beta}^p_{ik} = 0, \\ \max_{j'} \tilde{\beta}^p_{ik}(j') &= \max(0, \tilde{\beta}^p_{ik}(k)) \end{aligned}$$

Therefore, equation (6) can be simplified as

$$\alpha^p_{ik}(\mathbf{c}^p_i) = \begin{cases} \sum_{i':i'\neq k} \max(0, \tilde{\beta}^p_{i'k}(k)) + \sum_{i':i'\neq k} \bar{\beta}^p_{i'k}, & \mathbf{c}^p_i = k = i \\ \sum_{i':i'\neq k} \bar{\beta}^p_{i'k}, & \mathbf{c}^p_i \neq k = i \\ \tilde{\beta}^p_{kk}(k) + \sum_{i':i'\notin\{i,k\}} \max(0, \tilde{\beta}^p_{i'k}(k)) & \\ \quad + \sum_{i':i'\neq i} \bar{\beta}^p_{i'k}, & \mathbf{c}^p_i = k \neq i \\ \max\Big[0, \tilde{\beta}^p_{kk}(k) + \sum_{i':i'\notin\{i,k\}} \max(0, \tilde{\beta}^p_{i'k}(k))\Big] & \\ \quad + \sum_{i':i'\neq i} \bar{\beta}^p_{i'k} & \mathbf{c}^p_i \neq k \neq i \end{cases} \tag{10}$$

Moreover, we notice that, the value of each expression in the righthand side of (10) does not directly depend on the value of $\mathbf{c}^p_i$, rather only the choice of expression depends on $\mathbf{c}^p_i$. It implies that, there are only two values for message $\alpha^p_{ik}(\mathbf{c}^p_i)$, one for $\mathbf{c}^p_i = k$ and the other for $\mathbf{c}^p_i \neq k$. By taking advantage of this property, we can set the constant $\bar{\alpha}^p_{ik}$ as one of the two values, e.g., $\bar{\alpha}^p_{ik} = \alpha^p_{ik}(\mathbf{c}^p_i : \mathbf{c}^p_i \neq k)$. In this way, we have for all $\mathbf{c}^p_i \neq k$,

$$\tilde{\alpha}^p_{ik}(\mathbf{c}^p_i) = \alpha^p_{ik}(\mathbf{c}^p_i) - \bar{\alpha}^p_{ik} = \alpha^p_{ik}(\mathbf{c}^p_i) - \alpha^p_{ik}(\mathbf{c}^p_i : \mathbf{c}^p_i \neq k) = 0$$

Therefore, we have

$$\begin{aligned} &\sum_{k':k'\neq k} \tilde{\alpha}^p_{ik'}(\mathbf{c}^p_i) \\ &= \begin{cases} \sum_{k':k'\notin\{k,\mathbf{c}^p_i\}} \tilde{\alpha}^p_{ik'}(\mathbf{c}^p_i) + \tilde{\alpha}^p_{i\mathbf{c}^p_i}(\mathbf{c}^p_i) & \\ \qquad = 0 + \tilde{\alpha}^p_{i\mathbf{c}^p_i}(\mathbf{c}^p_i) = \tilde{\alpha}^p_{i\mathbf{c}^p_i}(\mathbf{c}^p_i) & \mathbf{c}^p_i \neq k \\ \sum_{k':k'\neq k} \tilde{\alpha}^p_{ik'}(k) = 0 & \mathbf{c}^p_i = k \end{cases} \end{aligned}$$

Therefore, equation (7) can be simplified as

$$\beta^p_{ik}(\mathbf{c}^p_i) = \begin{cases} S^p(i,\mathbf{c}^p_i) + \sum_{k':k'\neq k} \bar{\alpha}^p_{ik'} & \\ \quad + \sum_{q=1,q\neq p}^{m} \tilde{\eta}^{pq}_{ii}(\mathbf{c}^p_i) + \sum_{q=1,q\neq p}^{m} \bar{\eta}^{pq}_{ii}, & \mathbf{c}^p_i = k \\ S^p(i,\mathbf{c}^p_i) + \tilde{\alpha}^p_{i\mathbf{c}^p_i}(\mathbf{c}^p_i) + \sum_{k':k'\neq k} \bar{\alpha}^p_{ik'} & \\ \quad + \sum_{q=1,q\neq p}^{m} \tilde{\eta}^{pq}_{ii}(\mathbf{c}^p_i) + \sum_{q=1,q\neq p}^{m} \bar{\eta}^{pq}_{ii} & \mathbf{c}^p_i \neq k \end{cases} \tag{11}$$

Similarly, equation (9) can be simplified as

$$\begin{aligned}\theta_{ii}^{pq}(\mathbf{c}_i^p) =& S^p(i,\mathbf{c}_i^p) + \sum_{k=1}^{n}\tilde{\alpha}_{ik}^p(\mathbf{c}_i^p) + \sum_{k=1}^{n}\bar{\alpha}_{ik}^p \\ &+ \sum_{q'=1,q'\neq q,q'\neq p}^{m}\tilde{\eta}_{ii}^{pq'}(\mathbf{c}_i^p) + \sum_{q'=1,q'\neq q,q'\neq p}^{m}\bar{\eta}_{ii}^{pq'} \\ =& S^p(i,\mathbf{c}_i^p) + \sum_{k:k\neq\mathbf{c}_i^p}\tilde{\alpha}_{ik}^p(\mathbf{c}_i^p) + \tilde{\alpha}_{i\mathbf{c}_i^p}^p(\mathbf{c}_i^p) + \sum_{k=1}^{n}\bar{\alpha}_{ik}^p \\ &+ \sum_{q'=1,q'\neq q,q'\neq p}^{m}\tilde{\eta}_{ii}^{pq'}(\mathbf{c}_i^p) + \sum_{q'=1,q'\neq q,q'\neq p}^{m}\bar{\eta}_{ii}^{pq'} \\ =& S^p(i,\mathbf{c}_i^p) + \tilde{\alpha}_{i\mathbf{c}_i^p}^p(\mathbf{c}_i^p) + \sum_{k=1}^{n}\bar{\alpha}_{ik}^p \\ &+ \sum_{q'=1,q'\neq q,q'\neq p}^{m}\tilde{\eta}_{ii}^{pq'}(\mathbf{c}_i^p) + \sum_{q'=1,q'\neq q,q'\neq p}^{m}\bar{\eta}_{ii}^{pq'}\end{aligned} \tag{12}$$

Next, let's solve for $\tilde{\alpha}_{ik}^p(\mathbf{c}_i^p=k) = \alpha_{ik}^p(\mathbf{c}_i^p=k) - \bar{\alpha}_{ik}^p$ and $\tilde{\beta}_{ik}^p(\mathbf{c}_i^p=k) = \beta_{ik}^p(\mathbf{c}_i^p=k) - \bar{\beta}_{ik}^p$, we get

$$\begin{aligned}&\tilde{\alpha}_{ik}^p(\mathbf{c}_i^p=k) = \alpha_{ik}^p(\mathbf{c}_i^p=k) - \bar{\alpha}_{ik}^p \\ &= \begin{cases}\sum_{i':i'\neq k}\max(0,\tilde{\beta}_{i'k}^p(k)) & k=i \\ \left(\tilde{\beta}_{kk}^p(k) + \sum_{i':i'\notin\{i,k\}}\max(0,\tilde{\beta}_{i'k}^p(k))\right) \\ \quad - \max\left[0, \tilde{\beta}_{kk}^p(k) + \sum_{i':i'\notin\{i,k\}}\max(0,\tilde{\beta}_{i'k}^p(k))\right] & k\neq i\end{cases} \\ &= \begin{cases}\sum_{i':i'\neq k}\max(0,\tilde{\beta}_{i'k}^p(k)) & k=i \\ \min\left[0, \tilde{\beta}_{kk}^p(k) + \sum_{i':i'\notin\{i,k\}}\max(0,\tilde{\beta}_{i'k}^p(k))\right] & k\neq i\end{cases}\end{aligned} \tag{13}$$

$$\begin{aligned}&\tilde{\beta}_{ik}^p(\mathbf{c}_i^p=k) = \beta_{ik}^p(\mathbf{c}_i^p=k) - \bar{\beta}_{ik}^p \\ &= \beta_{ik}^p(k) - \max_{j:j\neq k}\beta_{ik}^p(j) \\ &= \Big(S^p(i,\mathbf{c}_i^p) + \sum_{k':k'\neq k}\bar{\alpha}_{ik'}^p + \sum_{q=1,q\neq p}^{m}\tilde{\eta}_{ii}^{pq}(\mathbf{c}_i^p) + \sum_{q=1,q\neq p}^{m}\bar{\eta}_{ii}^{pq}\Big) \\ &\quad - \max_{j:j\neq k}\Big(S^p(i,\mathbf{c}_i^p) + \tilde{\alpha}_{i\mathbf{c}_i^p}^p(\mathbf{c}_i^p) + \sum_{k':k'\neq k}\bar{\alpha}_{ik'}^p \\ &\qquad + \sum_{q=1,q\neq p}^{m}\tilde{\eta}_{ii}^{pq}(\mathbf{c}_i^p) + \sum_{q=1,q\neq p}^{m}\bar{\eta}_{ii}^{pq}\Big) \\ &= S^p(i,k) + \sum_{q=1,q\neq p}^{m}\tilde{\eta}_{ii}^{pq}(k) \\ &\quad - \max_{j:j\neq k}\Big(S^p(i,j) + \tilde{\alpha}_{ij}^p(j) + \sum_{q=1,q\neq p}^{m}\tilde{\eta}_{ii}^{pq}(j)\Big)\end{aligned} \tag{14}$$

Here, we don't need to consider the cases of $\tilde{\alpha}_{ik}^p(\mathbf{c}_i^p\neq k)$ and $\tilde{\beta}_{ik}^p(\mathbf{c}_i^p\neq k)$, since $\tilde{\alpha}_{ik}^p(\mathbf{c}_j^p\neq k)=0$ and the update of messages doesn't depend on $\tilde{\beta}_{ik}^p(\mathbf{c}_i^p\neq k)$.

Moreover, let $\bar{\theta}_{ii}^{pq} = \sum_{k=1}^{n}\bar{\alpha}_{ik}^p + \sum_{q'=1,q'\neq q,q'\neq p}^{m}\bar{\eta}_{ii}^{pq'}$, then according to (12),

$$\tilde{\theta}_{ii}^{pq}(\mathbf{c}_i^p) = S^p(i,\mathbf{c}_i^p) + \tilde{\alpha}_{i\mathbf{c}_i^p}^p(\mathbf{c}_i^p) + \sum_{q'=1,q'\neq q,q'\neq p}^{m}\tilde{\eta}_{ii}^{pq'}(\mathbf{c}_i^p) \tag{15}$$

Therefore, according to (8), if we let $\bar{\eta}_{ii}^{pq} = \bar{\theta}_{ii}^{qp}$, then we have

$$\begin{aligned}\tilde{\eta}_{ii}^{pq}(\mathbf{c}_i^p) &= \max_{\mathbf{c}_i^q}\left[L^{pq}(\mathbf{c}_i^p,\mathbf{c}_i^q) + \tilde{\theta}_{ii}^{qp}(\mathbf{c}_i^q)\right] \\ &= \max_{\mathbf{c}_i^q}\Big[L^{pq}(\mathbf{c}_i^p,\mathbf{c}_i^q) + S^q(i,\mathbf{c}_i^q) + \tilde{\alpha}_{i\mathbf{c}_i^q}^q(\mathbf{c}_i^q) \\ &\quad + \sum_{p'=1,p'\neq p,p'\neq q}^{m}\tilde{\eta}_{ii}^{qp'}(\mathbf{c}_i^q)\Big]\end{aligned} \tag{16}$$

Therefore, message $\tilde{\theta}_{ii}^{pq}(\mathbf{c}_i^p)$ can be eliminated in the final computations.

Consequently, the cluster assignment computation can be simplified accordingly.

$$\begin{aligned}\hat{\mathbf{c}}_i^p &= \arg\max_j\left[S^p(i,j) + \sum_{q=1,q\neq p}^{m}\eta_{ii}^{pq}(j) + \sum_{k=1}^{n}\alpha_{ik}^p(j)\right] \\ &= \arg\max_j\Big[S^p(i,j) + \sum_{q=1,q\neq p}^{m}\tilde{\eta}_{ii}^{pq}(j) + \sum_{q=1,q\neq p}^{m}\bar{\eta}_{ii}^{pq} \\ &\quad + \sum_{k=1}^{n}\tilde{\alpha}_{ik}^p(j) + \sum_{k=1}^{n}\bar{\alpha}_{ik}^p\Big] \\ &= \arg\max_j\left[S^p(i,j) + \sum_{q=1,q\neq p}^{m}\tilde{\eta}_{ii}^{pq}(j) + \sum_{k=1}^{n}\tilde{\alpha}_{ik}^p(j)\right] \\ &= \arg\max_j\left[S^p(i,j) + \sum_{q=1,q\neq p}^{m}\tilde{\eta}_{ii}^{pq}(j) + \tilde{\alpha}_{ij}^p(j)\right]\end{aligned} \tag{17}$$

2 MESSAGE SUMMARY

To summarize, we get the simplified messages as follows:
1. Messages passing within the p-th view

$$\tilde{\alpha}_{ik}^p(k) = \begin{cases}\sum_{i':i'\neq k}\max(0,\tilde{\beta}_{i'k}^p(k)) & k=i \\ \min\left[0, \tilde{\beta}_{kk}^p(k) + \sum_{i':i'\notin\{i,k\}}\max(0,\tilde{\beta}_{i'k}^p(k))\right] & k\neq i\end{cases} \tag{18}$$

$$\begin{aligned}\tilde{\beta}_{ik}^p(k) &= S^p(i,k) + \sum_{q=1,q\neq p}^{m}\tilde{\eta}_{ii}^{pq}(k) \\ &\quad - \max_{j:j\neq k}\Big(S^p(i,j) + \tilde{\alpha}_{ij}^p(j) + \sum_{q=1,q\neq p}^{m}\tilde{\eta}_{ii}^{pq}(j)\Big)\end{aligned} \tag{19}$$

2. Messages passing across any two views: from the q-th view to the p-th view

$$\begin{aligned}\tilde{\eta}_{ii}^{pq}(k) &= \max_{k'}\Big[L^{pq}(k,k') + S^q(i,k') + \tilde{\alpha}_{ik'}^q(k') \\ &\quad + \sum_{p'=1,p'\neq p,p'\neq q}^{m}\tilde{\eta}_{ii}^{qp'}(k')\Big]\end{aligned} \tag{20}$$

The message θ is eliminated.

The simplified cluster assignment computation is

$$\hat{\mathbf{c}}_i^p = \arg\max_j \left[S^p(i,j) + \sum_{q=1, q\neq p}^{m} \tilde{\eta}_{ii}^{pq}(j) + \tilde{\alpha}_{ij}^p(j) \right] \quad (21)$$