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G52ACE 2017-18
Shortest Paths:
Floyd-Warshall (FW)

All-Pairs Shortest Paths

- Suppose that wanted to find the SP between **all** pairs of start and end nodes
  - We could run Dijkstra from every node
  - Would then be a factor of O(|V|) worse than Dijkstra, e.g. O(|V| * (|V| * log(|V|) +|E|))

- But may be better to run a specific algorithm
  - We will do the "Floyd-Warshall" (FW) algorithm

FW All-Pairs SPs

- The basic method has similarities to the methods for "change giving" in an earlier lecture
  - "Build best answers for a set of coins, and then add the effects of coins one at a time"
- "Build the optimal answers using a subset of the nodes. Then add the effects of other nodes one at a time"

FW All-Pairs SPs: data structure

Main data structure definition:

d(i,j,k) =

shortest distance between nodes i and j,

but using only the nodes {1,…,k} as potential allowed intermediary points

- E.g. d(2,5,3) =
  "shortest distance from n2 to n5 using only {n1,n2,n3} as potential intermediate points'."
  - Usage of all or any of these intermediate points is not forced, but other points, such as n4 …, cannot be used

FW: Initialisation of data structure

Initialisation:

d(i,j,0) = best distance between nodes i and j, but not using any intermediate nodes,

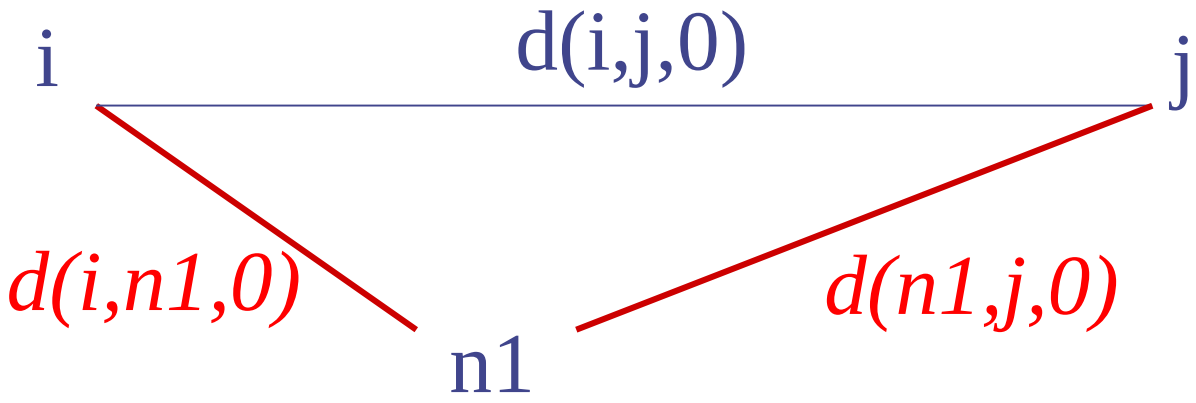
so only using a single edge, Hence

d(i,j,0) = w(i,j) if there is an edge i to j
= Inf otherwise

where Inf is 'infinity', and could be implemented as a special value, or a number large enough to act as infinity (bigger than all other numbers encountered), etc.

FW All-Pairs SPs

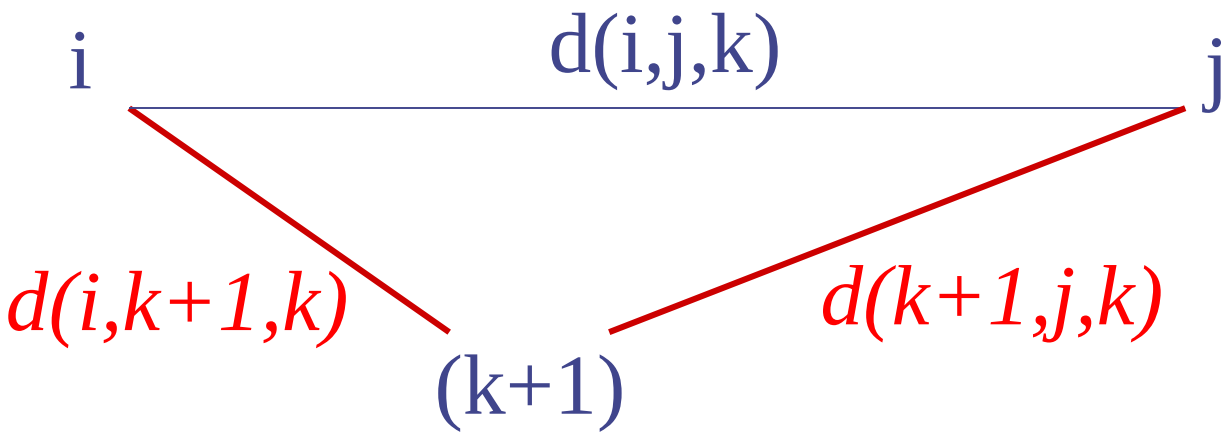
- Now suppose that we add the node 'n1' to the set of nodes that can be intermediates, i.e. consider k = 1
- Best path is now the best of "either direct, or via n1."
- d(i,j,1) = min (d(i,j,0), w(i,n1) + w(n1,j))
  = min (d(i,j,0), d(i,n1,0) + d(n1,j,0))



FW All-Pairs SPs

- Now suppose that we add the new node "(k+1)" to the set of "via nodes" that can be intermediates, but have already considered k of them
- Best path is now either direct using only the k 'via nodes' already accounted for, or else also via node 'k+1' (and using the previous k via's)

$$d(i,j,k+1) = \min (d(i,j,k),\ \ d(i,k+1,k) + d(k+1,j,k))$$



FW equations

d(i,j,0) = w(i,j) if there is an edge i to j

= Inf otherwise

d(i,j,k+1) = min (d(i,j,k),

d(i,k+1,k) + d(k+1,j,k))

Note:

- The rhs depends only on d(-,-,k)
- The lhs gives d(-,-,k+1)
- So solve by "start at k=0 and iteratively increase k"

(These are an example of "Belmann Equations" and can be solved using DP.)

FW equations – ADDITION

k is a dummy variable, and so we could equally well write

$$d(i,j,k) = \min (\; d(i,j,k-1), \; d(i,k,k-1) + d(k,j,k-1) \;)$$

(This version is more compatible with the FW code that was provided.)

Of course, in practice, when implementing, then have to be more careful about the range of k. So that it correctly captures the idea of

- Start with no "via nodes" allowed
- Add "via nodes" one at a time

Self-edges?

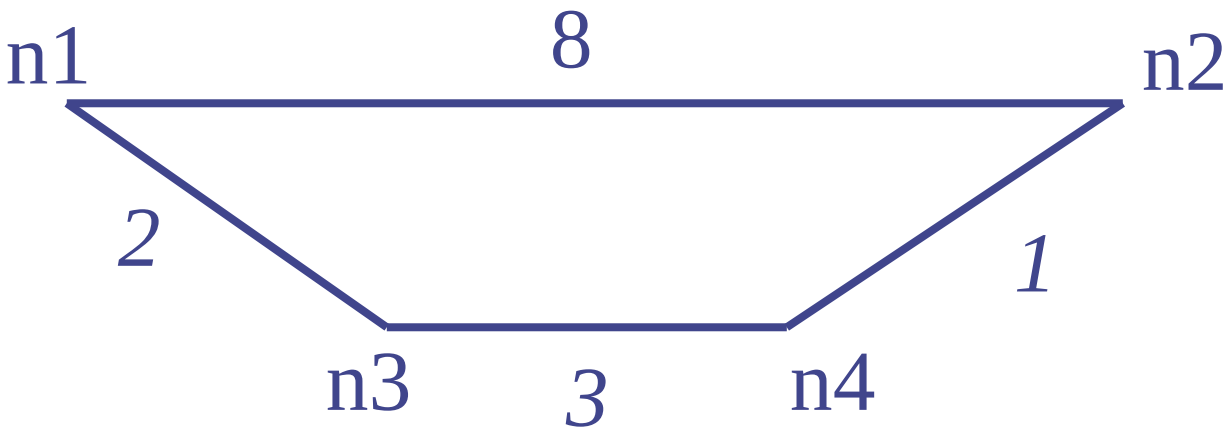
- The FW format does not in itself make any assumptions about d(i,i); whether self-edges are always assumed to exist.

- However, we will assume that

$$d(i, i) = 0 \quad \text{forall } i$$

  - that is, travelling from a node A too itself is possible at cost 0.
  - This is for simplicity, and also makes it compatible with Dijkstra which (typically) has the same assumption

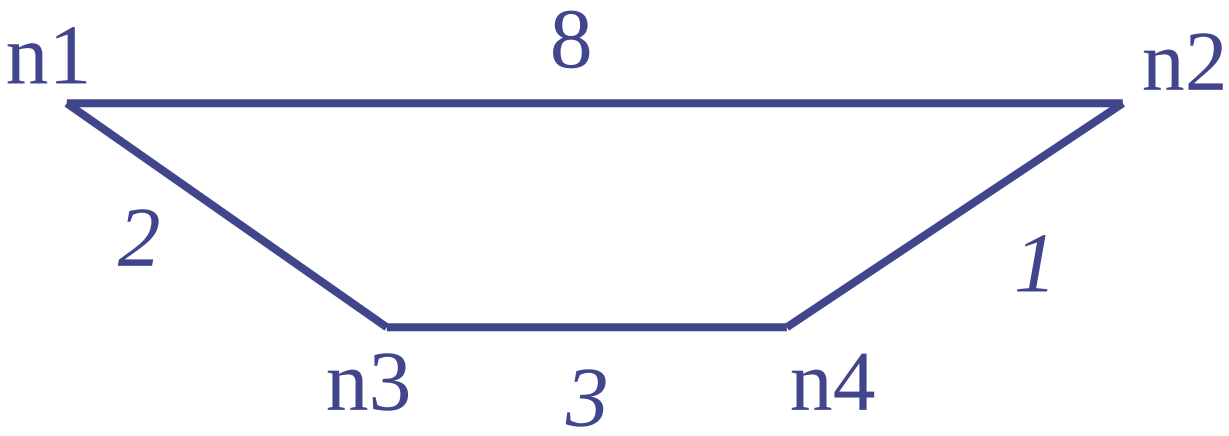
FW example : d(i,j,0)



- Initial d(i,j,0)
  - direct edges only
- Edges are undirected, hence matrix is symmetric

d(i,j,0)	n1	n2	n3	n4
n1	0	8	2	Inf
n2	8	0	Inf	1
n3	2	Inf	0	3
n4	Inf	1	3	0

FW example : d(i,j,1), partial



Set of intermediates = { n1 }

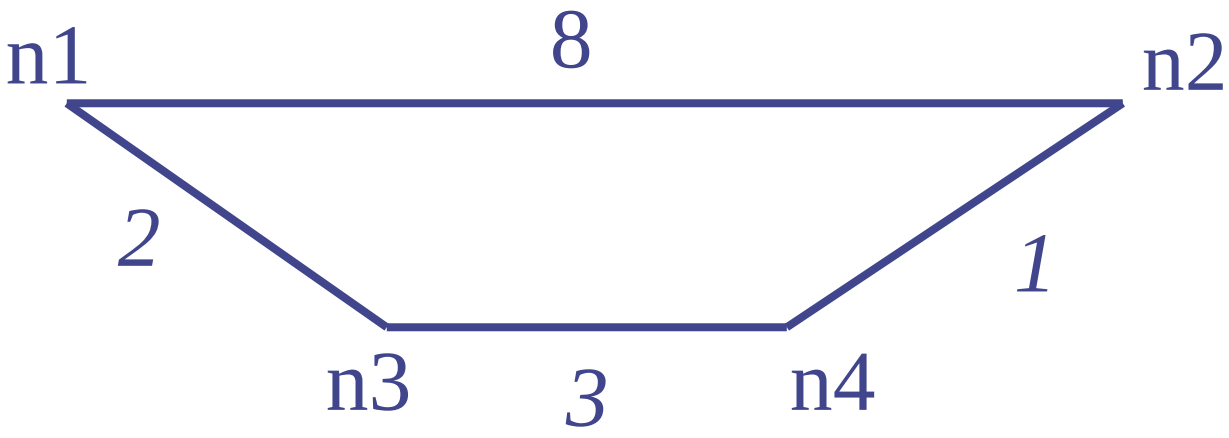
Example:

d(n3,n2,1)
= min(Inf, d(n3,n1,0) + d(n1,n2,0))
= min(Inf, 2+8) = 10

(We can "see" the optimal will be 3+1 via n4, but we have not considered n4 yet)

d(i,j,1)	n1	n2	n3	n4
n1	0	8	2	Inf
n2	8	0	~~**Inf**~~ **10**	1
n3	2	~~**Inf**~~ **10**	0	3
n4	Inf	1	3	0

FW example : d(i,j,1)



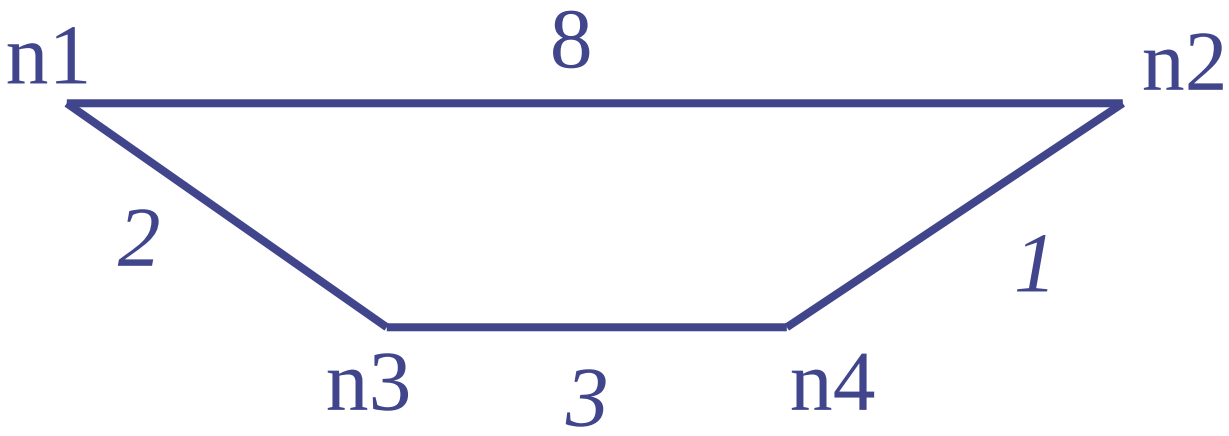
Set of intermediates = { n1 }

Filling in the other entries – check that no other entries change

The matrix is symmetric so will now just give the lower-triangle.

d(i,j,1)	n1	n2	n3	n4
n1	0			
n2	8	0		
n3	2	10	0	
n4	Inf	1	3	0

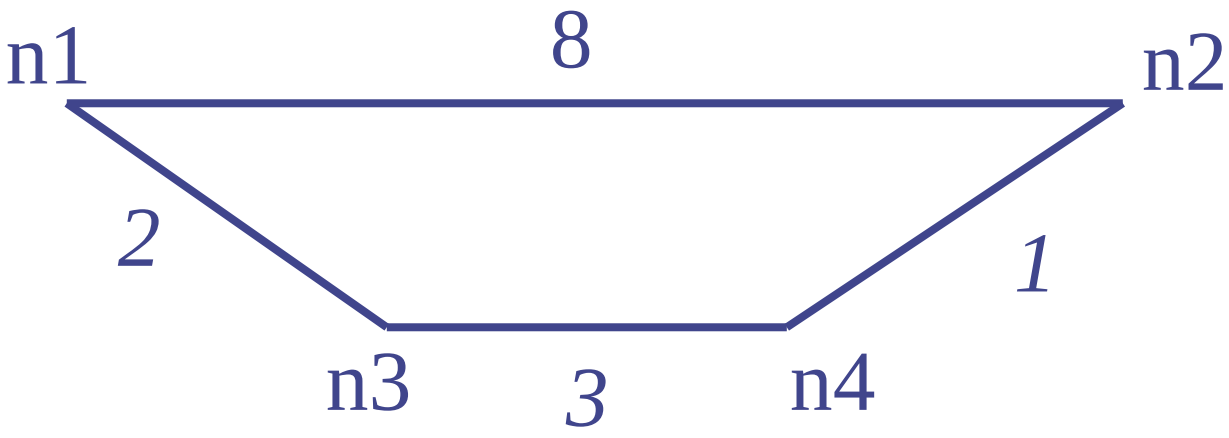
FW example : d(i,j,2)



Set of intermediates = { n1, n2 }
Node n2 is new, so e.g.
d(n4,n1,2)
= min(Inf, d(n4,n2,1) + d(n2,n1,1))
= min(Inf, 1+8) = 9

d(i,j,2)	n1	n2	n3	n4
n1	0			
n2	8	0		
n3	2	10	0	
n4	~~Inf~~ 9	1	3	0

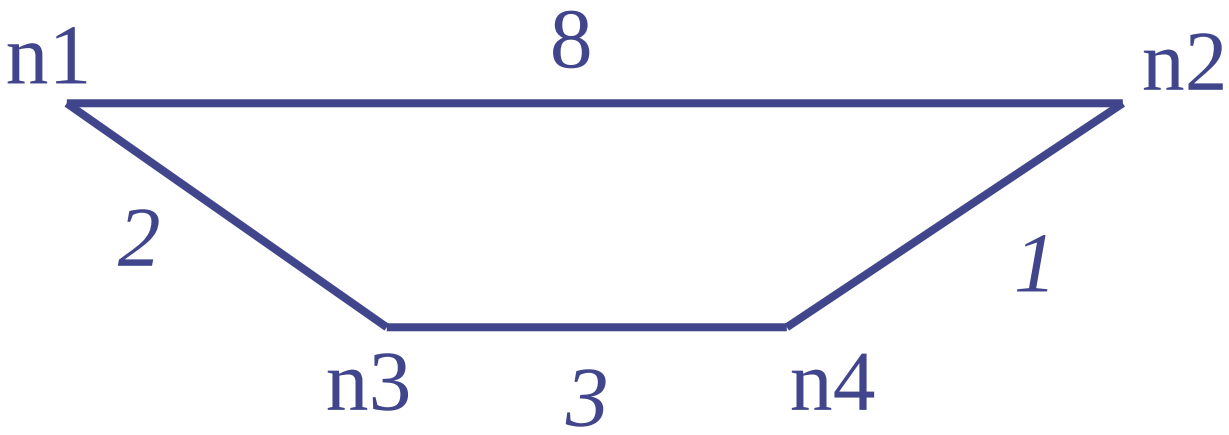
FW example : d(i,j,3)



Set of intermediates = { n1, n2, n3 }
Node n3 is new, so e.g.
d(n4,n1,3)
= min(9, d(n4,n3,2) + d(n3,n1,2))
= min(9, 3+2) = 5

d(i,j,3)	n1	n2	n3	n4
n1	0			
n2	8	0		
n3	2	10	0	
n4	~~9~~ 5	1	3	0

FW example : d(i,j,4)



Set of intermediates = {n1,n2,n3,n4}
Node n4 is new, so e.g.
d(n1,n2,4)
= min(8, d(n1,n4,3) + d(n4,n2,3))
= min(8, 5+1) = 6

d(i,j,4)	n1	n2	n3	n4
n1	0			
n2	~~8~~ 6	0		
n3	2	~~10~~ 4	0	
n4	5	1	3	0

Finished as all intermediates are now accounted for

FW code & complexity

The main loop after initialisation is:

foreach k = 1, … (so "foreach $n_k \in V$")
 foreach i ∈ V
  foreach j ∈ V
   d(i,j,k+1) = min(d(i,j,k), d(i,k+1,k) + d(k+1,j,k))

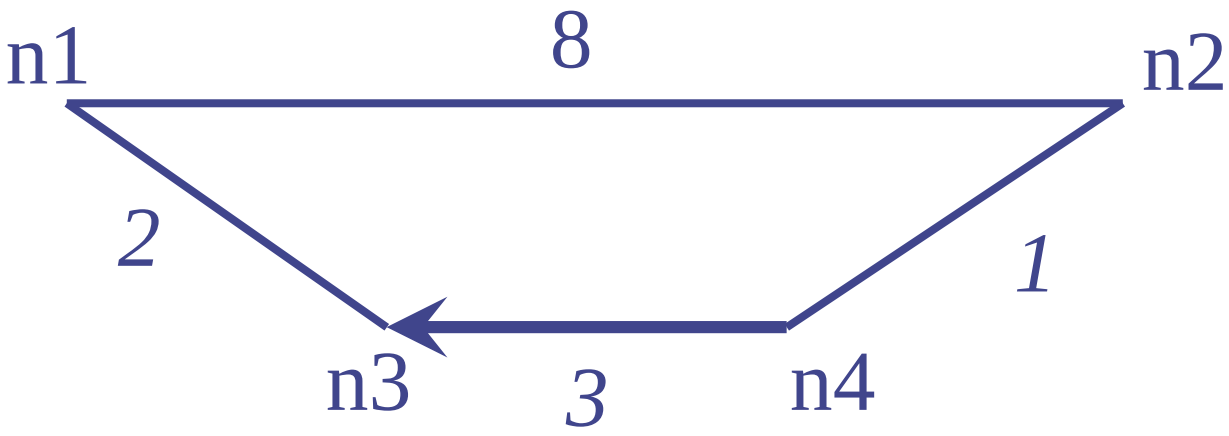
Note that is it is vital that 'k' is the outer loop.
Have 3 nested loops, of ranges |V|. **Hence is O($|V|^3$)**

(If the graph is sparse, then |E| << $|V|^2$, so then "all-starts Dijkstra" may be better.)

FW on digraphs

- FW also works the same on directed graphs
  - The initial matrix d(i,j,0) need not be symmetric, but then the remaining calculations use exactly the same formulas

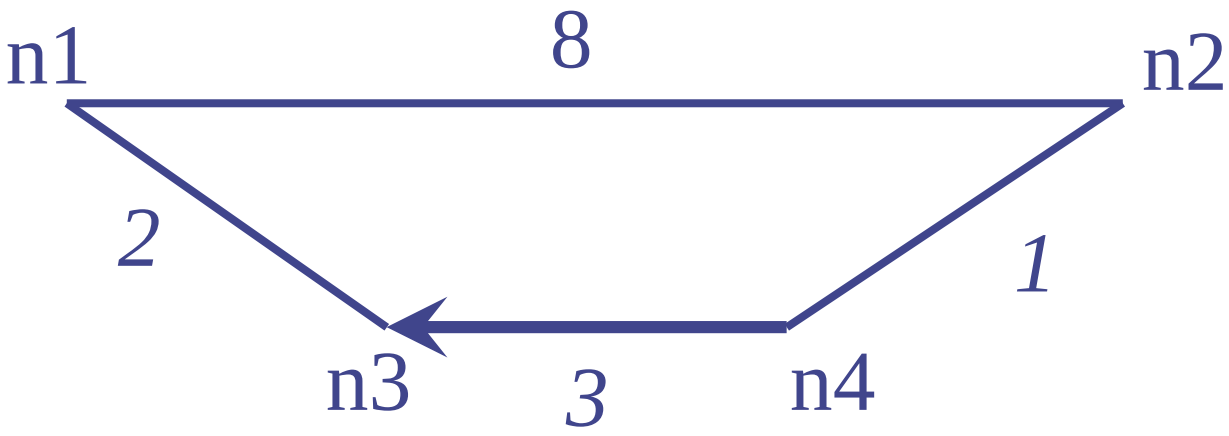
FW digraph example : d(i,j,0)



- Initial d(i,j,0)
  - direct edges only
  - edges with no arrows are bidirectional
- Matrix is no longer symmetric

d(i,j,0)	n1	n2	n3	n4
n1	0	8	2	Inf
n2	8	0	Inf	1
n3	2	Inf	0	**Inf**
n4	Inf	1	3	0

FW example : d(i,j,1), partial



Set of intermediates = { n1 }

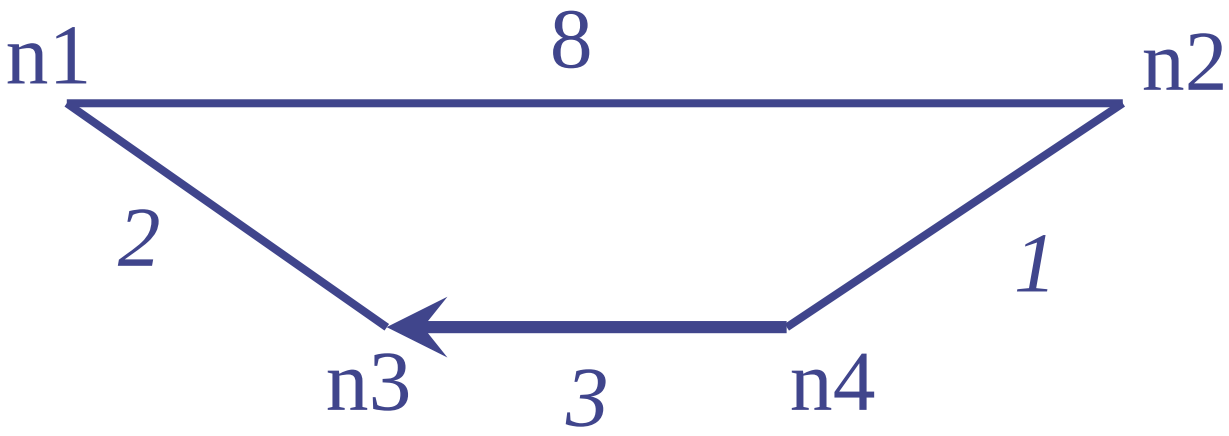
Example:

d(n3,n2,1)
= min(Inf, d(n3,n1,0) + d(n1,n2,0))
= min(Inf, 2+8) = 10

(We can "see" the optimal will be 3+1 via n4, but we have not considered n4 yet)

d(i,j,1)	n1	n2	n3	n4
n1	0	8	2	Inf
n2	8	0	~~**Inf**~~ **10**	1
n3	2	~~**Inf**~~ **10**	0	Inf
n4	Inf	1	3	0

FW example : d(i,j,1)

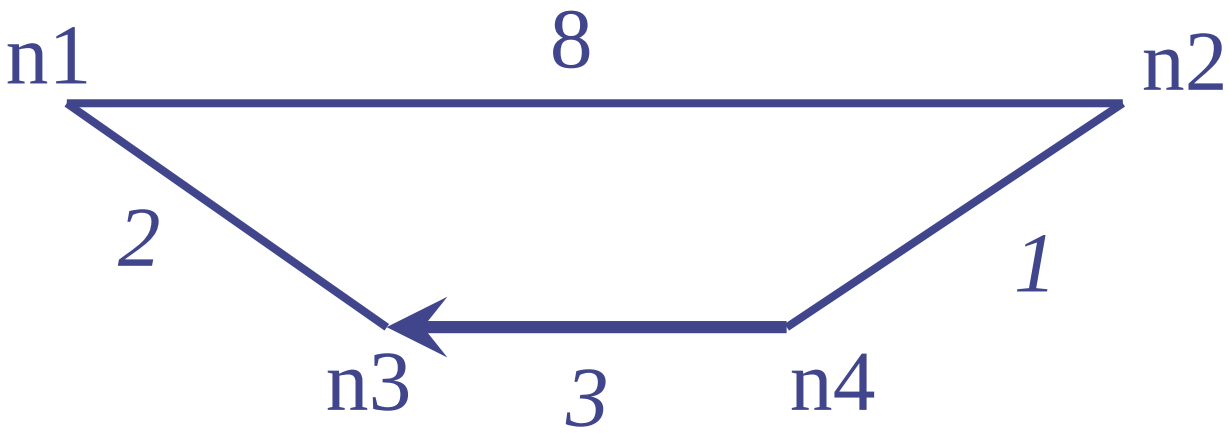


Set of intermediates = { n1 }

Filling in the other entries – check that no other entries change

d(i,j,1)	n1	n2	n3	n4
n1	0	8	2	Inf
n2	8	0	10	1
n3	2	10	0	Inf
n4	Inf	1	3	0

FW example : d(i,j,2)



Set of intermediates = { n1, n2 }

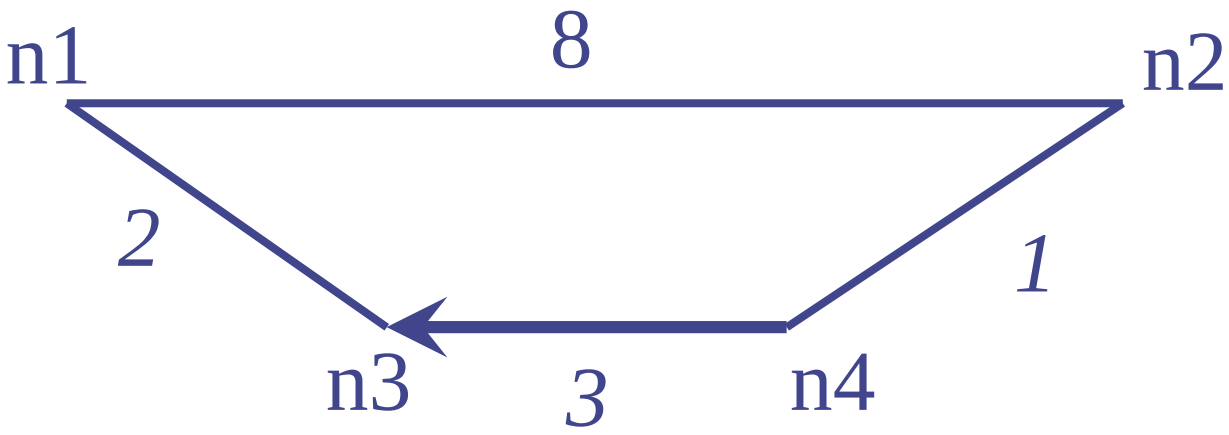
Node n2 is new, so e.g.

d(n4,n1,2)
= min(Inf, d(n4,n2,1) + d(n2,n1,1))
= min(Inf, 1+8) = 9 etc

d(n3,n4,2)
= min(Inf, d(n3,n2,1) + d(n2,n4,1))
= min(Inf, 10+1) = 11

d(i,j,2)	n1	n2	n3	n4
n1	0	8	2	~~Inf~~ 9
n2	8	0	10	1
n3	2	10	0	~~Inf~~ 11
n4	~~Inf~~ 9	1	3	0

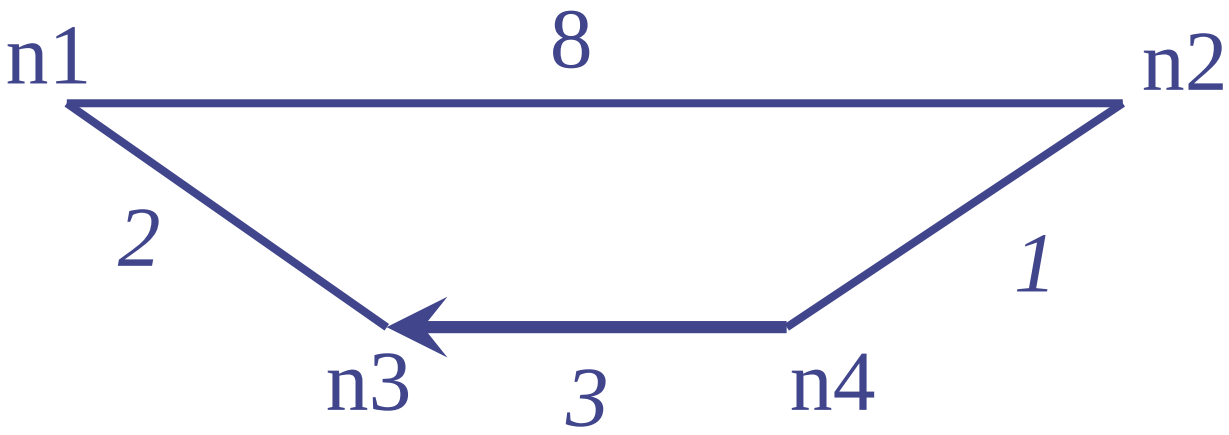
FW example : d(i,j,3)



Set of intermediates = { n1, n2, n3 }
Node n3 is new, so e.g.
d(n4,n1,3)
= min(9, d(n4,n3,2) + d(n3,n1,2))
= min(9, 3+2) = 5
but
d(n1,n4,3)
= min(9, d(n1,n3,2) + d(n3,n4,2))
= min(9, 2+Inf) = 9

d(i,j,3)	n1	n2	n3	n4
n1	0	8	2	**9**
n2	8	0	10	1
n3	2	10	0	11
n4	~~9~~ **5**	1	3	0

FW example : d(i,j,4)



Set of intermediates = {n1,n2,n3,n4}
Node n4 is new, so e.g.
d(n2,n1,4)
= min(8, d(n2,n4,3) + d(n4,n1,3))
= min(8, 1 + 5) = 6

d(i,j,4)	n1	n2	n3	n4
n1	0	8	2	9
n2	~~8~~ **6**	0	~~10~~ **4**	1
n3	2	10	0	11
n4	5	1	3	0

Finished as all intermediates are now accounted for

FW with negative edges

- FW even works if some (directed) edge weights are negative
  - BUT it is essential that there are no cycles of total negative weight
  - Otherwise simply repeatedly following around the negative cycle may reduce lengths to be as negative as desired, so there is no shortest path

- Offline Exercise: repeat the previous example with d(n4,n3) = -1 (instead of +3)
  - The cycle n4-n3-n1-n2-n4 has weight -1+2+8+1=10 which is allowed

FW: Recall - key idea

Uses that SP does satisfy a nice decomposition of

- If P(A,B) is a shortest path, and goes via M then P(A,M) is optimal for A to M and P(M B) is optimal for M to B

- Hence uses a version of "dynamic programming"

Exercise

- You are **highly** recommended to
  - create some small to medium graphs (directed and undirected) and work through both (Dijkstra and FW) algorithms
  - repeat working examples until you **understand** them fully and can do it 'by hand' quickly and easily
    - Both algorithms are classics, and similar ideas appear in many other algorithms

Minimum Expectations

- Know and understand definition of shortest path, Dijkstra's algorithm, and Floyd-Warshall algorithm
- Be able to apply them, by hand, to small graphs