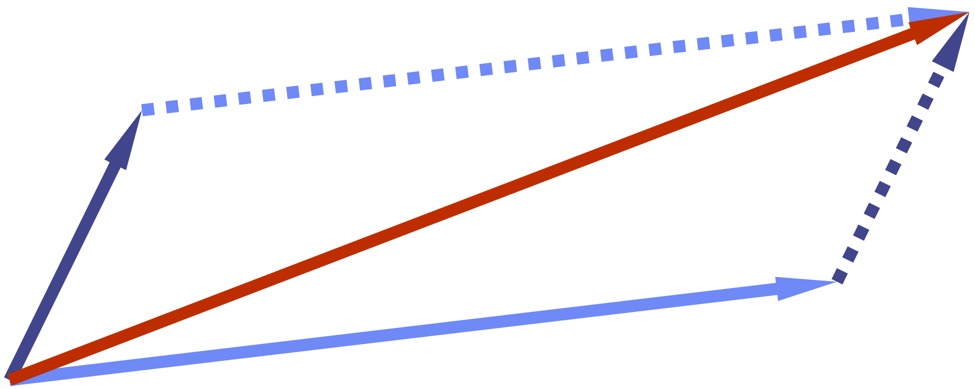
G52ACE 2017-18: The Vector ADT and CDT & Amortised Analysis



Andrew Parkes

http://www.cs.nott.ac.uk/~pszajp/

Vector ADT

- The “Vector” is an Abstract Data Type corresponding to generalising the notion of the “Array” (concrete data type)
- Key idea:
  - The “index” of an entry in an array can be thought of as the “number of elements preceding it
  - E.g. in A[2], two elements, A[0], A[1] precede it
  - In these lectures it is then called “rank”

The Vector ADT

- The Vector ADT is based on the array CDT, and stores a sequence of arbitrary objects
- An element can be accessed, inserted or removed by specifying its **rank, i.e. the number of elements preceding it**
- An exception is thrown if an incorrect rank is specified (e.g., a negative rank)

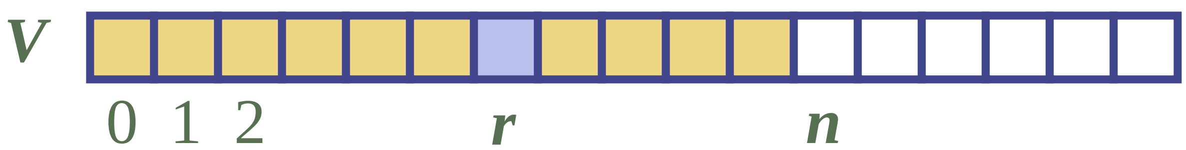
- Main vector operations:
  - object elemAtRank(integer r): returns the element at rank r without removing it
  - object replaceAtRank(integer r, object o): replace the element at rank with o and return the old element
  - insertAtRank(integer r, object o): insert a new element o to have rank r
  - object removeAtRank(integer r): removes and returns the element at rank r
- Additional operations size() and isEmpty()

Applications of Vectors

- Direct applications
  - Sorted collection of objects (elementary database)
- Indirect applications
  - Auxiliary data structure for algorithms
  - Component of other data structures

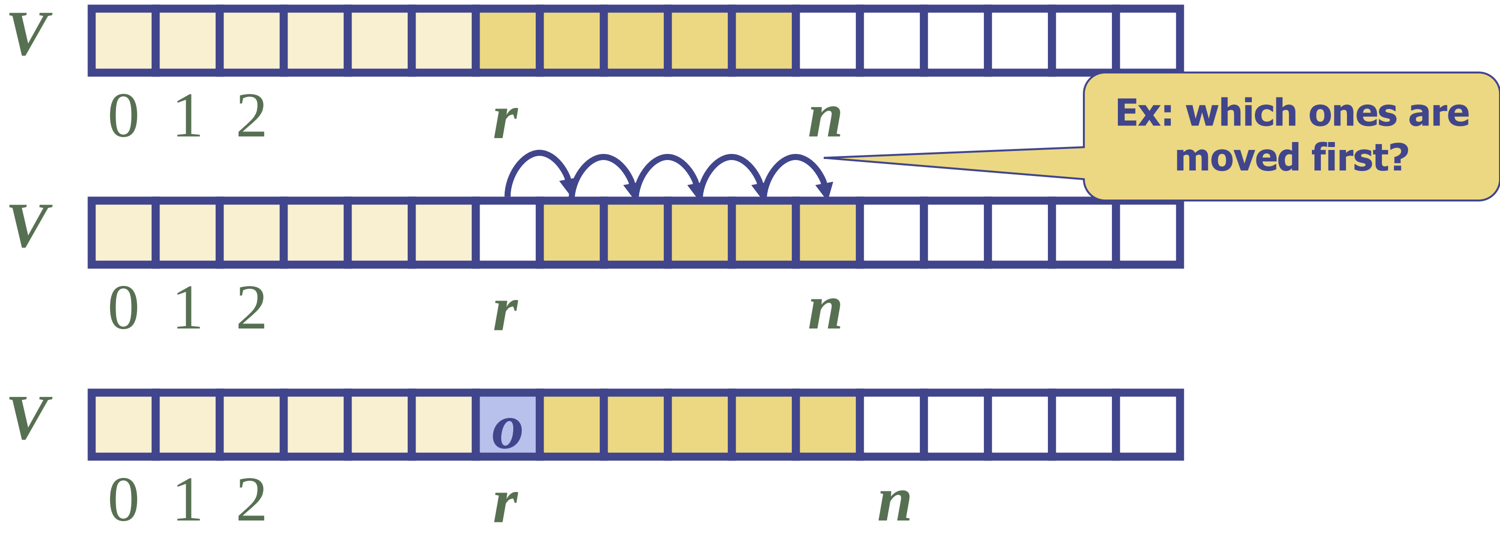
Array-based Vector

- Use an array V of size N as the CDT
- A variable n keeps track of the size of the vector (number of elements currently stored)
- Operation ***elemAtRank***(r) is implemented in $O(1)$ time by simply \returning $V[r]$



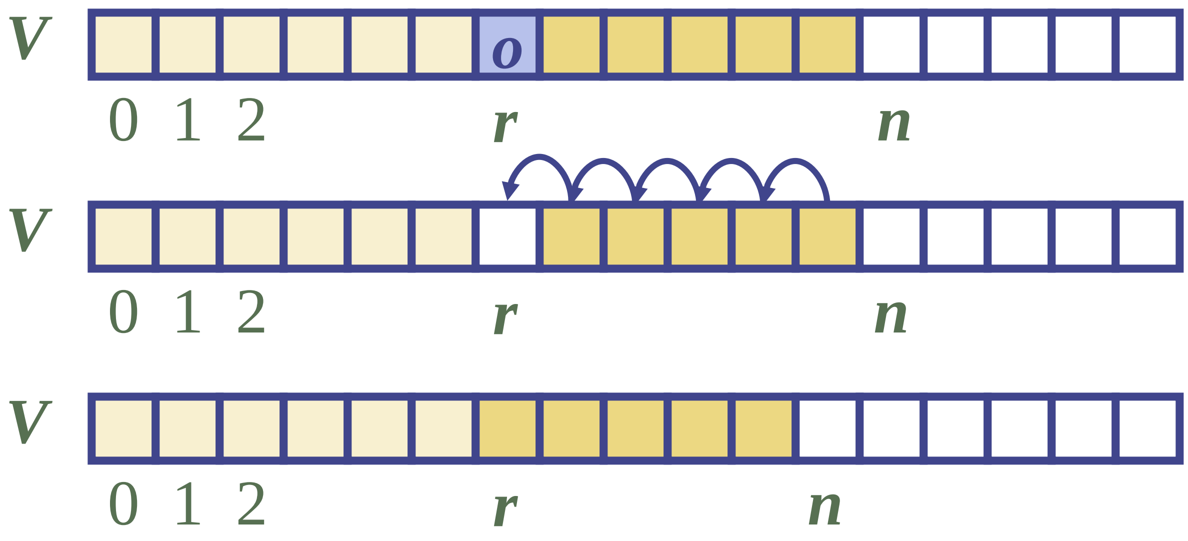
Insertion

- In operation *insertAtRank*(r, o), we need to make room for the new element by shifting forward the $n - r$ elements $V[r], \ldots, V[n-1]$
- In the worst case ($r = 0$), this takes $O(n)$ time



Deletion

- In operation ***removeAtRank***(r), we need to fill the hole left by the removed element by shifting backward the $n - r - 1$ elements $V[r+1]$, …, $V[n-1]$
- In the worst case ($r = 0$), this takes $O(n)$ time



Performance

- In the array based implementation of a Vector
  - The space used by the data structure is $O(n)$
  - ***size***, ***isEmpty***, ***elemAtRank*** and ***replaceAtRank*** run in $O(1)$ time
  - ***insertAtRank*** and ***removeAtRank*** run in $O(n)$ time
- If we use the array in a circular fashion (see lectures on queues), ***insertAtRank***(0) and ***removeAtRank***(0) run in $O(1)$ time
- In an ***insertAtRank*** operation, when the array is full, instead of throwing an exception, we can replace the array with a larger one

Growable Array-based Vector

- In a push (insertAtRank(t)) operation, when the array is full, instead of throwing an exception, we can replace the array with a larger one
- How large should the new array be?
  - **incremental strategy**: increase size by a constant c
  - **doubling strategy**: double the size

Algorithm ***push***(***o***)
  if $t = S.length - 1$ **then**
    $A \leftarrow$ new array of size …
    for $i \leftarrow 0$ **to** t **do**
      $A[i] \leftarrow S[i]$
    $S \leftarrow A$
    $t \leftarrow t + 1$
    $S[t] \leftarrow o$

Comparison of the Strategies

- We compare the incremental strategy and the doubling strategy by analyzing the total time $T(n)$ needed to perform a series of n push operations
- We assume that we start with an empty stack represented by an array of size 1
- We call **amortized** time of a push operation the average time taken by a push over the series of operations, i.e., $T(n)/n$

Meaning of "Amortize"

- See http://www.thefreedictionary.com/amortize or similar if you are not familiar with the word.
- It refers to writing off, or paying off, debts over a period of time.
- Similar to the way a mortgage for a house is paid back over many years, as opposed to needing to pay all in one go.

General Remarks on Amortised Analysis

- Suppose some individual operation (such as ‘push’) takes time T in the worst-case
- Suppose do a **sequence** of operations, and there are s such operations taking time T_s
- Then sT is an upper-bound for the total time
  - however, such an upper-bound might not ever occur.
- The time T_s might well be o(sT) even in the worst-case
  - the average time per operation, T_s /s is sometimes the most relevant quantity in practice

Exercise Question:

- Why is amortised analysis different from the average case analysis?

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- Why is amortised analysis different from the average case analysis?
- Answer:
  - (long) **real sequence** of dependent operations
  - vs.
  - **Set** of (possibly independent) operations
- Note the usual big-Oh family is still used to describe amortised analysis
  - it is just describing functions
  - the functions are different measures of the runtime cost
    - "Worst case of a sequence" not "worst case of a single operation"

Incremental: Example

- Take c=3, with start capacity of 3, then a sequence of pushes might have costs for each push as follows:

1,1,1,3+1,1,1,6+1,1,1,9+1,1,1,12+1,1,1,
15+1,1,1,18+1,...

So a constant fraction of the pushes have cost O(n). Average is O(n).

Example: In detail

- [- , - , -]   n=0   Starting point
  - empty, but capacity=3
  - "-" means "not yet used"
- Suppose do a sequence of "push(0)", we get the sequence of costs (primitive operations):
  1. [0 , - , -]   n=1, **cost=1**
  2. [0 , 0 , -]   n=2, **cost=1**
  3. [0 , 0 , 0]   n=3, **cost=1**
  4. [0 , 0 , 0, 0 , - , -]   n=4, **cost=3+1**
     - cost= "3 for the copy" + "1 for the push(0)"
  5. [0 , 0 , 0, 0 , 0 , -]   n=5, **cost=1**
  6. [0 , 0 , 0, 0 , 0 , 0]   n=6, **cost=1**
  7. [0 , 0 , 0, 0 , 0 , 0 , 0 , - , -]   n=7, **cost=6+1**
     - cost= "6 for the copy" + "1 for the push(0)"
  8. etc, etc...

Incremental Strategy Analysis

- We replace the array $\boldsymbol{k} = \boldsymbol{n}/\boldsymbol{c}$ times
- Each "replace" costs the current size
- The total time $\boldsymbol{T}(\boldsymbol{n})$ of a series of $\boldsymbol{n}$ push operations is proportional to

$$\boldsymbol{n} + \boldsymbol{c} + 2\boldsymbol{c} + 3\boldsymbol{c} + 4\boldsymbol{c} + \ldots + \boldsymbol{kc} =$$

$$\boldsymbol{n} + \boldsymbol{c}(1 + 2 + 3 + \ldots + \boldsymbol{k}) =$$

$$\boldsymbol{n} + \boldsymbol{ck}(\boldsymbol{k} + 1)/2$$

- Since $\boldsymbol{c}$ is a constant, $\boldsymbol{T}(\boldsymbol{n})$ is $\boldsymbol{O}(\boldsymbol{n} + \boldsymbol{k}^2)$, i.e., $\boldsymbol{O}(\boldsymbol{n}^2)$
- The amortized time of a push operation is $\boldsymbol{O}(\boldsymbol{n})$

Doubling: Example

- With start capacity of 3, then a sequence of pushes starting from an empty vector, might have costs for each push in turn of

1,1,1,3+1,1,1,6+1,1,1,1,1,1,12+1,1,1,1,1,
1,1,...

So the fraction of pushes having cost O(n) reduces with n. Average is ??.

Doubling: Example

- With start capacity of 2, then a sequence of pushes might have costs

1,1,2+1,1,4+1,1,1,1,8+1,1,1,1,1,1,1,1,1,1,...,16+1,...

For every push of cost O(n) we will be able to do another O(n) pushes of cost O(1) before having to resize again.

So the O(n) cost on resizing can be 'amortised' over n other O(1) operations and give an average of O(1) per operation.

Doubling: Example

- With start capacity of 2, then a sequence of pushes might have costs

  1,1,2+1,1,4+1,1,1,1, ...

- The cost of the doubling can be "spread" over the later operations and so might be counted as

  1,1,1+1,1+1,1+1,1+1,1+1,1+1,1,...

where the red is a cost that has been moved.

Analogy: save £1/day "for a rainy day"

- This 'view' makes it clearer that the net effect will just be a (rough) doubling of the original costs

Doubling Strategy Analysis

- Exercise (online): for ***n*** ‘pushes’ how many times is the array grown?
- For simplicity assume n is a power of 2
  - We replace the array $\boldsymbol{k} = \log_2 \boldsymbol{n}$ times

- The total time $\boldsymbol{T}(\boldsymbol{n})$ of a series of ***n*** push operations is proportional to

$$\boldsymbol{n} + 1 + 2 + 4 + 8 + \ldots + 2^{\boldsymbol{k-1}}$$

Examples

- n=1 #doublings=0
  - lg n = 0
- n=2 #doublings=1
  - lg n = 1
- n=4 #doublings=2
  - lg n = 2

Example: n=4

‘|’ is ‘End of stack’ marker

- start [| _]
  - capacity=1, size=0
- push(A) [A |]
- push(B) [A B |]
  - needed: 1 double – 1 copies
- push(C) [A B C | _]
  - needed: 1 double – 2 copies
- push(D) [A B C D |]

Exercise (offline): Do this is full detail; as done earlier for the incremental strategy. Code it using counting as in autumn lab!!

Recall: Geometric sums

Want to find S

$$S = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{\boldsymbol{k}}$$

Standard trick:

$$2S = \quad 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1}$$

So

$$2S - S = 2^{k+1} - 1$$

Hence $\qquad S = 2^{k+1} - 1$

I.e. "the next term minus one"

Doubling Strategy Analysis

- We replace the array $k = \log_2 n$ times
- The total time $T(n)$ of a series of n push operations is proportional to

$$n + 1 + 2 + 4 + 8 + \ldots + 2^{k-1} =$$

$$n + 2^k - 1 = 2n - 1$$

- $T(n)$ is $O(n)$
- The amortized time of a push operation is $O(1)$
- That is, no worse than if all the needed memory was pre-assigned!

Exercises/discussion

- When might you still want to use incremental increase rather than doubling?
- Why is it doubling rather than tripling? Or increasing by some other constant factor?
  - Try redoing the analysis with a arbitrary growth factor *b*
  - Try implementing and doing experimental comparisons!
    - Should be quick and easy to code
    - The best way to really learn it properly.

C/C++ comments

- Doubling vector size might be made even more efficient if use realloc rather than malloc or new
  - Internally it can (often not always) just extend the space allocated to the array, and so avoid the need for a copy
  - It can use “memcpy” which is direct copy rather than via individuals

Expectations

- Stacks, Queues, Arrays, Vectors, Lists are all very common and basic data-structures.
- You should
  - be familiar with their usage
  - be able to implement them
  - be able to readily analyse the complexity of the various operations.

Expectations

- Be able to use the big-Oh analyses of CDTs in order to decide which one is appropriate for a particular application

EXERCISE: OFFLINE

- Implement a Vector using both strategies for size increase of the array:
    - Constant addition of size
    - Doubling the size
- Test it out e.g.
    - Do timing tests
    - Do counting of operations
- Do your results match the analysis of the lectures!?

Comments

- This concludes “Part 1” of the module in sense that have covered many of the basic concepts and ideas:
  - big-Oh family (big-Oh, Omega, Theta, and little-oh)
  - ADT vs. CDT
  - ADTs: stacks, queues, vectors
  - CDTs: arrays, lists
  - Simple sorting, merge-sort and quicksort
  - Recurrence and master theorem
  - Using big-Oh analyses to help with design decisions
- Much of the rest of the module is
  - exploiting and extending these ideas
  - data structures and algorithms with better big-Oh behaviours (in some needed circumstances) than can obtain with arrays and lists