Computers & Industrial Engineering 140 (2020) 106214

Contents lists available at ScienceDirect

Computers & Industrial Engineering

journal homepage: www.elsevier.com/locate/caie

Prepositioning network design for disaster reliefs: Stochastic models and Ψ-expander models comparison

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ARTICLE INFO

Keywords:
Distribution network design
Ψ-expander
Process flexibility

ABSTRACT

Disaster relief network, if properly designed, can save thousands of lives. Disaster relief network design is hard due to the unavailability of demand information.

In this paper, we study a network design problem for disaster relief purpose under several different demand information settings: a stochastic model with known demand distributions and a Ψ-expander model with very limited information. Ψ-expander, a sparse but highly connected graph, is a concept derived from Expander Graph, which is widely used in Graph Theory and Computer Science areas. We prove that one special case of the Ψ-expander is non-convex. An approximation model is proposed to handle the non-convexity. We also design a cutting plane procedure to make the approximation model computationally tractable. Numerical study shows that the Ψ-expander approximation model, compared with stochastic models, can significantly reduce the penalty cost. With a slightly lower inventory level, it can also reach an almost same fill rate as that of an exact Ψ-expander does and achieve the lowest total cost when the penalty cost is dominating.

1. Introduction

Natural disasters and catastrophes have always been a part of the world. Even with advanced response mechanism to disasters, losses can still be huge. To better understand the developing periods of a disaster, Gupta, Starr, Farahani, and Matinrad (2016) generally divided time phases of disasters into three types: before the disaster, during the disaster, and after the disaster. The study on disaster relief network design is a popular topic for disaster management before the disaster, in which decision-makers try to minimize economic loss and save lives as much as possible by designing a proper relief network. To design a network appropriately, three aspects of decisions should be concerned during the preparedness phase:

- Choose several critical locations from potential areas to establish relief warehouses.
- Deploy relief materials at chosen locations.
- Form connections between supply nodes and points of demand.

However, since the real demand of relief materials can only be known in the response phase which is after an occurrence of a disaster, properly handling the above three issues during the preparedness phase was hindered by this kind of uncertainty. Moreover, as pointed out by Tufekci and Wallace (1998), regarding the preparedness phase before a disaster and the response phase after a disaster as two separable parts would cause suboptimality. Hence, designing an integrated relief network with the consideration of the post-disaster phase is necessary. But the difficulty is that little information on post-disaster demand is available due to the lack of historical data. It is nearly impossible to estimate the impact of a disaster accurately. To sum up, designing a proper relief network with limited information during the preparedness phase is necessary but challenging.

The study on disaster relief network design started in 1958. It was firstly proposed by Baumol and Wolfe (1958), which describes a heuristic for a warehouse location model under a single-sourcing setting. However, they only considered how to choose locations with known demand distributions. After that, a large number of variants were introduced to deal with the above three problems jointly or partially jointly. Most works concentrate on how to decide inventory levels and appropriately choose locations. For example, Daskin, Coullard, and Shen (2002) formulates a non-linear integer programming problems to decide inventory levels and locations jointly. Rawls and Turnquist (2011) describes a stochastic mixed integer programming formulation with service quality constraints to determine the inventory levels. Drezner (2004) provides a plan for the location of casualty collection points, minimizing the accumulated distances between collection points and demand points. Hu, Han, and Meng (2017) consider a two-stage stochastic problem for integrating decisions on pre-disaster inventory

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https://doi.org/10.1016/j.cie.2019.106214
Received 28 March 2019; Received in revised form 3 November 2019; Accepted 3 December 2019
Available online 15 December 2019

level and post-disaster procurement quantity in humanitarian relief. Some works focus on a specific kind of disasters, exploiting the characteristic of that disaster. Chang, Tseng, and Chen (2007) establishes a network for flood disasters with uncertainty. With the help of short-term hurricane forecasts information, Davis, Samanlioglu, Qu, and Root (2013) studies a network of cooperative warehouses. Other recent works manage to consider more realistic settings. For example, Zhang, Liu, Yu, Ruan, and Chan (2019) considers secondary disasters with the conditional probability based scenario tree. Zhang, Zhang, Zhang, Wei, and Deng (2013) explores the route selection problem in emergency logistics management by taking into consideration travel time and the path length. Pradhananga, Mutlu, Pokharel, Holguín-Veras, and Seth (2016) proposes an integrated resource allocation and distribution model, and they assume that disasters lead to the loss of part of the prepositioned items. Vitoriano, Ortuño, Tirado, and Montero (2011) incorporates the occurrence of disruptions of roads. And Horner and Downs (2010) studies a problem with different facility types. A very recent paper by Klibi, Ichoua, and Martel (2018) describes a multi-phase modeling framework for the strategic problem of designing emergency supply networks to support disaster relief. Dynamics is considered during deployment, sustainment, recovery, and redeployment stages, and a compound propagation process is adopted to formulate the impact of natural disasters. The scenario-based solution approach is applied to evaluate the performance of the proposed model. Some practical implementations also have been reported. Duran, Gutierrez, and Keskinocak (2011), McLay and Moore (2012), and Dibene et al. (2017) show three cases developed with location-inventory models in practice. Hoyos, Morales, and Akhavan-Tabatabaei (2015) conducted a literature survey on OR models with stochastic components. We refer interested readers to their work for a comprehensive understanding of location-inventory problems in disaster management with OR models.

All the models in the above works can be generalized as risk-neutral two-stage stochastic programming problems. Although this kind of stochastic programming problems is widely accused of assuming wrong distributions or scenarios because of the lack of historical data, especially in the field of disaster management, they are still accepted as an efficient approach to describing complex situations without concerning the issue of tractability. Contrast to risk-neutral stochastic models, another widely used tools is the risk-averse robust optimization, in which only partial information is known. For example, only the lower boundary, the mean, and the upper boundary of demand are known. Robust models can fit the reality that the severity of a disaster is hard to predict due to the scarcity of historical data. However, because of the lack of information, designing a network with robust models is not as easy as that with stochastic models. For the disaster management purpose, several pioneer works have been conducted during recent years. Ben-Tal, Do Chung, Mandala, and Yao (2011) adopts a min–max criterion to dynamically assign emergency response when the demand uncertainty is time-dependent. Najafi, Eshghi, and Dullaert (2013) develops a robust approach for the earthquake response by assuming that information about the transportation network is easily accessible. Liu, Lei, Wu, and Zhang (2019) jointly considers both relief commodities and injured people to minimize unmet demand with a multi-commodity and multi-period problem. Rolling horizon-based framework is adopted to allow adjustments on plans according to the real-time updated information. However, these works focus on operations in the after-disaster phase, which is not suitable for the prepositioning purpose. Chen, Zhao, Wang, and Dessouky (2016) proposed a warehouse location problem when warehouses can cooperate with warehouses in other regions, where they attempt to minimize the maximum expected cost of any region. Alinaghian, Aghaie, and Sabbagh (2019) proposed a mathematical model for the location of temporary relief centers, where all affected areas are required to be covered by at least one temporary relief center. By dynamically designing the allocation route for helicopters, they attempt to minimize the last designated temporary relief center. Hasani and Mokhtari (2018) leveraged a mixed-integer linear programming model to address the problem of designing a relief network. A comprehensive scenario-based robust approach is proposed. Liu, Li, and Zhang (2019) considered the location problem of emergency medical service station with joint chance constraints requiring the possibility that owned ambulances can simultaneous fulfill medical service must achieve α at least. Moment information is utilized to describe the ambiguity set over the distribution of demand. Rezaei-Malek, Tavakkoli-Moghaddam, Zahiri, and Bozorgi-Amiri (2016) proposed an approach for designing a robust network with perishable commodities. By allowing decision makers to renew perishable commodities, their bi-objective model attempted to minimize the average response time as well as the total cost.

Besides the location-inventory decisions, service radius is also a key factor in disaster management. For instance, the first 72 h after an earthquake is crucial to saving lives. Many recent works had taken into this factor into consideration. Salman and Yücel (2015) considered a location problem when road disruptions randomly happen with dependency. They attempted to maximize the expected demand coverage within a specified distance. An et al. (2015) proposed a mixed-integer nonlinear program model to conduct a plan of emergency service facility location. Facility coverage ability is taken into consideration. Mohamadi and Yaghoubi (2017) and Charles, Lauras, Van Wassenhove, and Dupont (2016) regarded time limits as a restriction in their works for location problems. Khalilpourazari and Khamseh (2017) introduced a multi-objective model to decide facility locations and allocation decisions for an emergency blood supply network. Coverage radius for blood collection facilities is involved when facing the devastating impact of an earthquake.

A recent work by Ni, Shu, and Song (2018) proposes a min-max robust model to jointly choose locations and inventory levels. Because of the uncertainty in demands and the possibility of disruptions of roads and warehouses, solving the robust model by a commercial solver is impossible. To deal with this issue, they develop a computationally tractable approach. Li, Shu, Song, Zhang, and Zheng (2017) studies a network design problem with joint chance constraint requiring a certain service level, and they simultaneously determine locations, inventory levels, and links by reformulating their original model into the Ψ-expander. Meanwhile, they obtain a structure-free inventory level through two kinds of relaxations, which leads to conservative results. Inspired by their work, we want to exploit the concept of Ψ-expander from its definition to simultaneously decide locations, inventory levels, and links for the disaster relief purpose during the preparedness period, since a Ψ-expander network structure guarantees that demand can be fully fulfilled or at least $\Psi \times 100\%$ resource can be utilized (Chou, Teo, & Zheng, 2008).

For further disaster management literature reviews, readers are recommended to refer to Anaya-Arenas, Renaud, and Ruiz (2014) and Gupta et al. (2016). Table 1 summarizes the characteristic of mentioned works. From Table 1, it is easy to discover that the only work that jointly deals with location selections, inventory level decisions, and roads formation is Li et al. (2017). Instead of modeling a problem with joint chance constraints and then reformulating it into Ψ-expander, we explore the definition of Ψ-expander and divide it into two types of constraints. Since our work focus on handling the uncertainty over demand, we reasonably ignore some factors. In Section 4, we show some extensions to demonstrate the compatibility of our model. Moreover, we also follow the assumptions below:

1. Quick response is reified as a service radius. For example, the first 72 h after an earthquake are crucial to saving lives, beyond that time limit the utilization of arrived relief materials is under satisfaction. Therefore a warehouse can only provide relief packages to all demand nodes within its service radius after a disaster happens.
2. Newly established warehouses do not suffer from disruptions caused by disasters, as well as relief materials. Since warehouses are built for relief purpose, the resistibility to disasters should be high enough. As a result, relief supply does not get influence, too.
3. The break of roads does not be taken into consideration. Because

Table 1
Characteristics of reviewed literature on network design for disaster relief purpose.

	Objective*	Periods	Commodity	Sourcing	Location◇	Inventory◇	Link◇
Stochastic							
Daskin et al. (2002)	1.2.3	Single	Single	Multi	✓	✓	
Drezner (2004)	3	Single	Single	Multi	✓		
Chang et al. (2007)	1.2.3.5	Single	Multi	Multi	✓	✓	
Horner and Downs (2010)	3.4	Single	Multi	Multi			✓
Vitoriano et al. (2011)	3	Single	Single	Multi			✓
Rawls and Turnquist (2011)	1.2.3.5	Two	Multi	Multi		✓	
Davis et al. (2013)	3.5	Two	Single	Multi	✓	✓	
Zhang et al. (2013)	3	Single	Single	Single			✓
An et al. (2015)	1.3.	Two	Single	Multi	✓		✓
Salman and Yücel (2015)	Demand Coverage	Single	Single	Multi	✓		✓
Pradhananga et al. (2016)	1.2.3.5	Two	Multi	Multi	✓	✓	
Hu et al. (2017)	2.3.5	Two	Single	Multi	✓	✓	
Khalilpourazari and Khamseh (2017)	1.2.3.	Multi	Single	Multi	✓	✓	
Klibi et al. (2018)	2.3	Multi	Multi	Multi	✓	✓	
Zhang et al. (2019)	3.5	Three	Multi	Multi	✓		
Robust							
Ben-Tal et al. (2011)	2.5	Multi	Single	Multi		✓	
Najafi et al. (2013)	2.5	Multi	Multi	Multi		✓	
Rezaei-Malek et al. (2016)	1.3.5	Two	Multi	Multi	✓	✓	
Chen et al. (2016)	1.3	Single	Single	Multi	✓		
Li et al. (2017)	1.2.4	Two	Single	Multi	✓	✓	✓
Hasani and Mokhtari (2018)	1.2.3.5	Multi	Multi	Multi	✓	✓	
Ni et al. (2018)	1.2.4.5	Two	Single	Multi	✓	✓	
Liu, Li et al. (2019)	1.3	Two	Single	Multi	✓	✓	
Liu, Lei et al. (2019)	5	Multi	Multi	Multi	✓		
Alinaghian et al. (2019)	Time	Multi	Single	Single	✓		✓
Our work	1.2.3.5	Two	Multi	Multi	✓	✓	✓

◇ Last three columns represent whether the corresponding work handles the problem.
* *1* for the facility setup cost, *2* for the holding cost, *3* for the transportation cost, *4* for the cost of constructing links, *5* for the unmet demand penalty.

service radius has considered the estimated time of cleaning obstacles on roads with deployed equipment in the warehouse, the disruption of roads does not need to be incorporated into our models.

4. All the road links have unlimited transportation capacity. Capacity limitations also have been considered when the decision maker determines the service level for each node.

In this paper, we study a relief network problem through two streams, i.e., the risk-neutral two-stage stochastic problem and the risk-averse Ψ-expander model. Our main contribution is that, for disaster relief purpose, we are the first one to choose locations, decide inventory level for each selected warehouse, and construct links between nodes simultaneously by decomposing Ψ-expander. We prove that one special case of Ψ-expander constraint is non-convex. Also, we propose a solvable approximation model with two types of constraints derived from the definition of Ψ-expander. A constraint generation algorithm is introduced to handle the problem that there are exponentially many constraints in the approximation model. Through the numerical experiment, we found that stochastic models with presumed demand distribution can perform better when the penalty cost for unmet demand is relatively inconsequential. Moreover, without losing much conservativeness possessed by Ψ-expander models, the Ψ-expander approximation model can significantly reduce the penalty cost by storing a little bit more relief materials than the stochastic models. Therefore, it can reach an almost same fill rate as that of an exact Ψ-expander does and achieve the lowest total cost when the penalty cost is dominating.

2. Network design problem

Consider a network design problem with a set of potential supply nodes denoted by $\mathcal{W} = \{1, 2, \ldots, m\}$ and a set of demand nodes denoted by $\mathcal{R} = \{1, 2, \ldots, n\}$. Each supply node $i \in \mathcal{W}$ is a potential location to set up a disaster relief warehouse, and each demand node $j \in \mathcal{R}$ stands for a region that might be attacked by a disaster. Warehouse i is able to deliver relief materials to all adjacent demand nodes within its service radius r_i immediately after a disaster happens. Correspondingly, a demand node can receive relief materials from several warehouses whose service radius is larger than the distance between the warehouse and the demand node. We denote $\Gamma_{\mathbf{r}}(j)$ as the set of adjacent warehouses for demand node j when warehouses' service radius is $\mathbf{r} = (r_1, r_2, \ldots, r_m)$. Moreover, $\Gamma_{\mathbf{r}}(\mathcal{S})$ is the set of adjacent warehouses for any subsets of demand nodes $\mathcal{S} \in \mathcal{R}$ when service radius is $\mathbf{r}$. One direct result is $|\Gamma_{\mathbf{r}}(\mathcal{S}_1 \cup \mathcal{S}_2)| \leqslant |\Gamma_{\mathbf{r}}(\mathcal{S}_1)| + |\Gamma_{\mathbf{r}}(\mathcal{S}_2)|, \forall\, \mathcal{S}_1, \mathcal{S}_2 \in \mathcal{R}$ because of common warehouses. We use $|\mathcal{S}|$ to represent the cardinality of set $\mathcal{S}$. As to potential roads, if service radius $\mathbf{r}$ is given, we denote $\mathcal{E}(\mathbf{r})$ as the set of all potential roads in the network. After warehouses are built in some locations, the actual set of roads would be a subset to $\mathcal{E}(\mathbf{r})$.

There are different categories of relief material, such as clothes, water, food, included in a set K. Because of the scarcity of historical data, we only know that the demand ranges from D_j^{kL} to D_j^{kU} ($D_j^{kL} \leqslant D_j^{kU}$) for each demand node $j (j = 1, 2, \ldots, m)$ for each category $k (k = 1, 2, \ldots, |K|)$, where D_j^{kL} is the lower bound, and D_j^{kU} is the upper bound, which means demand is in a box uncertainty set. Please note that we do not assume any distributions upon these demands, and for this reason, our proposed Ψ-expander models possess the ethos of robust optimization. For other benchmark models, we further assume $D_j^{k\mu}$, the mean demand of material k in demand node j, is known.

In the pre-disaster period, we want to jointly make decision on three following issues:

1. choose locations from $\mathcal{W}$ to set up warehouses.
2. decide the inventory level I_i^k for each selected location i for each type of material k.
3. construct links between warehouses and points of demand according to service radius $\mathbf{r}$.

In general, there are some costs we need to consider. In the pre-disaster period, building a stable warehouse at location i needs a setup cost f_i. Reserving one unit material k at location i also needs to pay the holding cost h_i^k. After a disaster, transporting one unit material k from warehouse i to demand node j requires a transportation cost t_{ij}^k, where $k \in K$, $(i, j) \in \mathcal{E}(\mathbf{r})$. And there is a penalty cost p_j^k if one unit demand at demand node j for material k is unmet. Different from the unmet demand in a commercial situation where the unmet demand is only regarded as lost sales, the unmet demand in disaster relief environment can cause a more severe consequence, like irreversible sickness or even death. Reasonably, the penalty cost would be higher than that in a commercial problem.

There are some other parameters for a certain model. For example, to adopt Sample Average Approximation (SAA) method for the stochastic model, we draw some samples from the demand variable matrix $\widetilde{\mathbf{D}}$ (with shape $|K| \times |R|$), where each element $\widetilde{D_j^k}$ is a random variable representing the demand for material k at demand node j. Detailed explanations will be given in Section 2.1. And for Ψ-expander model, three extra parameters (Ψ, s_1, s_2) are incorporated. We will introduce these three parameters in Section 2.2 formally.

To explicitly describe our models, we summarize all parameters and decision variables as followings. Due to the predetermined service radius for each potential location, once the locations to build warehouses are chosen, the links radiating out from these locations are constructed. That is the reason why there are only two kinds of decision variables with respect to the pre-disaster period. In addition, ***BoldType***are used for vectors or matrix.

General Parameters

$\mathcal{R}$	A set of demand locations. $\mathcal{R} = \{1, 2, ..., n\}$
$\mathcal{W}$	A set of potential locations to set up warehouses. $\mathcal{W} = \{1, 2, ..., m\}$
K	A set of relief materials. $K = \{1, 2, ..., \lvert K \rvert\}$
f_i	Fixed cost to set up a warehouse at location i, $\forall\, i \in \mathcal{W}$
h_i^k	Holding cost for per unit of k^{th} relief material in location i, $\forall\, i \in \mathcal{W}$, $\forall\, k \in K$
p_j^k	Penalty cost for per unit of k^{th} unmet demand in location j, $\forall\, j \in \mathcal{R}$, $\forall\, k \in K$
t_{ij}^k	Transportation cost for per unit delivered material from warehouse i to demand location j for k^{th} material, $\forall\, (i, j) \in \mathcal{E}(\mathbf{r})$, $\forall\, k \in K$
r_i	The service radius of warehouse i, $\forall\, i \in \mathcal{W}$
$D_j^{k\mu}$	The mean value of location j's demand for k^{th} material, $\forall\, j \in \mathcal{R}$, $\forall\, k \in K$
D_j^{kU}	The upper bound of location j's demand for k^{th} material, $\forall\, j \in \mathcal{R}$, $\forall\, k \in K$
D_j^{kL}	The lower bound of location j's demand for k^{th} material, $\forall\, j \in \mathcal{R}$, $\forall\, k \in K$
$\mathcal{E}(\mathbf{r})$	The set of potential service routes in the network when the service radius is $\mathbf{r} = (r_1, r_2 ..., r_n)$
$\Gamma_{\mathbf{r}}(\mathcal{S})$	The set of warehouses that are adjacent to at least one demand node in $\mathcal{S}$ when service radius is $\mathbf{r}$, $\forall\, \mathcal{S} \subseteq \mathcal{R}$.

Parameters for stochastic models

$\widetilde{\mathbf{D}}$	Random variable matrix for the demand
$\widetilde{D_j^k}$	Random variable for the demand of k^{th} type materials at region j, $\widetilde{D_j^k} \in \left[D_j^{kL}, D_j^{kU}\right]$

Parameters for Ψ-expander models

s_1	The cardinality restriction for set $\mathcal{S}$ in the Case 1 (see Section 2.2)
s_2	The cardinality restriction for set $\mathcal{S}$ in the Case 2 (see Section 2.2)
Ψ	One parameter for Ψ-expander. (see Section 2.2)

Decision Variables

I_i^k	k^{th} material's inventory level for potential warehouse i, $\forall\, i \in \mathcal{W}$
Z_i	Indicator whether location i is selected to build up a warehouse, $\forall\, i \in \mathcal{W}$
x_{ij}^k	The number of k^{th} material delivered from warehouse i to location j after a disaster happens.

2.1. Two-stage stochastic model

When the demand is uncertain, and the decision maker is risk-neutral, the network design problem can be formulated as a two-stage stochastic model as much literature did. The point that distinguishes our work from others is that our models basically focus on dealing with the uncertainty in demand, and we can simultaneously choose locations, decide inventory levels for different materials, and construct links between nodes with the help of service radius. Our risk-neutral two-stage stochastic programming model is:

$$(\mathbf{StoM})\ \min_{I_i^k, Z_i} \sum_{k \in K} \sum_{i \in \mathcal{W}} h_i^k I_i^k + \sum_{i \in \mathcal{W}} f_i Z_i + \mathbb{E}\left[G\left(\mathbf{I}, \widetilde{\mathbf{D}}\right)\right]$$

$$\text{s. t.}\quad I_i^k \leqslant M Z_i, \quad \forall\, i \in \mathcal{W},\ \forall\, k \in K$$

$$I_i^k \geqslant 0, \quad \forall\, i \in \mathcal{W},\ \forall\, k \in K$$

$$Z_i \in \{0, 1\}, \quad \forall\, i \in \mathcal{W}$$

where

$$G\left(\mathbf{I}, \widetilde{\mathbf{D}}\right) = \min_{x_{ij}^k} \sum_{k \in K} \sum_{j \in \mathcal{R}} p_j^k \left(\widetilde{D_j^k} - \sum_{i \in \mathcal{W}} x_{ij}^k\right) + \sum_{k \in K} \sum_{i \in \mathcal{W}} \sum_{j \in \mathcal{R}} t_{ij}^k x_{ij}^k$$

$$\text{s. t.}\quad \sum_{j \in \mathcal{R}} x_{ij}^k \leqslant I_i^k, \quad \forall\, i \in \mathcal{W},\ \forall\, k \in K$$

$$\sum_{i \in \mathcal{W}} x_{ij}^k \leqslant \widetilde{D_j^k}, \quad \forall\, j \in \mathcal{R},\ \forall\, k \in K$$

$$x_{ij}^k = 0, \quad \mathit{if}\ (i, j) \notin \mathcal{E}(\mathbf{r}), \quad \forall\ i \in \mathcal{W},\ \forall\, j \in \mathcal{R}, \forall\, k \in K$$

$$x_{ij}^k \geqslant 0, \quad \mathit{if}\ (i, j) \in \mathcal{E}(\mathbf{r}), \quad \forall\ i \in \mathcal{W},\ \forall\, j \in \mathcal{R}, \forall\, k \in K$$

In (**StoM**) model, $(\mathbf{I}, \mathbf{Z})$ is the first-stage decision variable. $\mathbf{I}$ is the matrix of inventory level, and $\mathbf{Z}$ is a vector of binary variables indicating that whether a warehouse should be set up at node i. The objective function for the first stage includes the fixed setup cost and the holding cost, plus the expected cost of the second stage. The first constraint, in which M is a large number, ensures that relief materials can be only reserved at the location where a warehouse is established. Other constraints are standard conditions. Based on the first-stage decision $\mathbf{I}$ and the realization of demand $\widetilde{\mathbf{D}}$, we can plan the post-disaster allocation strategy x_{ij}^k, which is the second-stage decision. In the second stage, we need to consider the penalty cost incurred by the unmet demand and the transportation cost. The first constraint for the second stage requires that the k^{th} materials distributed from warehouse i should be less than materials it owns, and the second constraint ensures that the k^{th} materials delivered to affected region j are not more than it needs. x_{ij}^k is the number of the k^{th} relief materials delivered from warehouse i to demand node j. $\mathcal{E}(\mathbf{r})$ is the road accessibility graph based on given radius vector $\mathbf{r}$. Therefore, the third constraint for the second stage guarantees that no material is distributed through path (i, j) when it is not accessible, and the fourth constraint is self-evident. By solving (**StoM**) via sample average approximation method (SAA), we are able to choose locations to launch warehouses, decide the inventory level for each warehouse, and construct links between nodes at the same time.

2.2. Ψ-expander model

Ψ-expander is one ideal and powerful tool to handle the problem caused by the scarcity of demand information. It can design a relatively sparse but highly connected graph with very limited information, usually with the lower boundary, the mean and the upper boundary. The advantage of Ψ-expander can exactly solve the problem we meet in designing a network for the disaster relief purpose, especially when the uncertainty of demand is a nonnegligible factor, or when the decision maker is risk-averse. Hence, we design the relief network with Ψ-expander. Before going to our model formulation, we want to introduce the definition of Ψ-expander firstly under the single-item setting.

Definition 1 (*Chou, Chua, Teo, and Zheng (2011)*). Given Ψ, where $0 < \Psi \le 1$, a Ψ-expander in the process flexibility problem is a bipartite graph in $\mathcal{W} \times \mathcal{R}$ with

$$\sum_{i\in\Gamma(\mathcal{S})} I_i \geqslant min\left\{\sum_{j\in\mathcal{S}} D_j^U, \Psi \sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L\right\}, \quad \forall\ \mathcal{S} \subseteq \mathcal{R}$$

The left-hand side of the inequality is the summation of inventories of a set of warehouses that can cover at least one affected area in the set $\mathcal{S} \subseteq \mathcal{R}$. And the right-hand side is a minimum term, which is complicated. To better understand the definition of Ψ-expander, we should categorize the definition into two cases.

- Case 1: $\sum_{j\in\mathcal{S}} D_j^U \leqslant \Psi \sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L$
  For subset $\mathcal{S} \in \mathcal{R}$ satisfies the above inequality, the definition reduces to: $\sum_{j\in\mathcal{S}} D_j^U \leqslant \sum_{i\in\Gamma(\mathcal{S})} I_i$. This inequality means all demand in the subset $\mathcal{S}$ can be fully fulfilled by adjacent warehouses even when every demand nodes require the maximal number of products.
- Case2: $\sum_{j\in\mathcal{S}} D_j^U \geqslant \Psi \sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L$
  For subset $\mathcal{S} \in \mathcal{R}$ satisfies the above inequality, the definition reduces to: $\sum_{i\in\Gamma(\mathcal{S})} I_i \geqslant \Psi \sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L$ which implies that the capacity connected to such a subset $\mathcal{S}$ is also large enough so that at least Ψ proportion of the total capacity is utilized in the worst case.

One nice property of Ψ-expander is that a Ψ-expander network structure guarantees that demand can be fully fulfilled or at least $\Psi \times 100\%$ resource can be utilized (Chou et al. (2011)). Therefore, with a larger Ψ, the structure is more flexible. When $\Psi = 1$, the Ψ-expander can perform as well as a full flexible system does.

Taking the advantage of Ψ-expander, we can model relief network design problem with multi-commodity as the following, if we set $\Psi = 1$:

$$(\mathbf{ExpM}) \min_{I_i^k, Z_i} \sum_{k\in K} \sum_{i\in\mathcal{W}} h_i I_i^k + \sum_{i\in\mathcal{W}} f_i Z_i$$

$$\text{s. t.} \quad \min\left\{\sum_{j\in\mathcal{S}} D_j^{kU}, \sum_{i\in\mathcal{W}} I_i^k - \sum_{j\notin\mathcal{S}} D_j^{kL}\right\} \leqslant \sum_{i\in\Gamma_{\mathbf{r}}(\mathcal{S})} I_i^k, \quad \forall\ \mathcal{S} \subseteq \mathcal{R},\ \forall\ k \in K \tag{1}$$

$$I_i^k \leqslant MZ_i, \quad \forall\ i \in \mathcal{W},\ \forall\ k \in K$$
$$I_i^k \geqslant 0, \quad \forall\ i \in \mathcal{W},\ \forall\ k \in K$$
$$Z_i \in \{0, 1\}, \quad \forall\ i \in \mathcal{W}$$

In (**ExpM**), $(\mathbf{I}, \mathbf{Z})$ are decision variables and have the same meaning as that of (**StoM**). Constraint (1) ensures that our network is a special case of Ψ-expander for each kind of materials. The remaining constraints are the same as (**StoM**). We ignore the post-disaster stage because the property of Ψ-expander inherently guarantees a high fill rate, which is the target of the post-disaster stage. Please note that constraint (1) is separable in terms of category k, which implies that it designs a Ψ-expander for each category. In the following analysis, for simplicity, we only consider ONE kind of material if notation k does not exist. However, this formulation is trivial because $(\mathbf{I}, \mathbf{Z}) = (\mathbf{0}, \mathbf{0})$ is obviously the optimal solution, and the Ψ-expander constraint is non-convex when $\Psi = 1$.

Proposition 1. $\min\left\{\sum_{j\in\mathcal{S}} D_j^U, \sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L\right\} \leqslant \sum_{i\in\Gamma(\mathcal{S})} I_i, \forall\ \mathcal{S} \subseteq \mathcal{R}$ *is a non-convex constraint.*

Proof. We prove Proposition 1 by contradiction. Firstly, suppose that 1-expander constraint is convex, so the convex combination, $\lambda(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{I}^1) + (1-\lambda)(\boldsymbol{x}^2, \boldsymbol{y}^2, \boldsymbol{I}^2)$, $0 \leqslant \lambda \le 1$, of two points that are in set A is in A for any $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{I}^1), (\boldsymbol{x}^2, \boldsymbol{y}^2, \boldsymbol{I}^2) \in A$, where

$$A := \left\{ \left(\mathbf{x}, \mathbf{y}, \mathbf{I} \right) \middle| \begin{array}{l} \min\left\{\sum_{j\in\mathcal{R}} D_j^U y_j, \sum_{i\in\mathcal{W}} I_i - \sum_{j\in\mathcal{R}} D_j^L \left(1 - y_j\right)\right\} \\ \quad - \sum_{i\in\mathcal{W}} I_i x_i \le 0; \\ x_i \geqslant y_j,\ if\ (i, j) \in \mathcal{E}(\mathbf{r}); \\ 0 \leqslant y_j \leqslant 1, \quad \forall\ j = 1, 2, \ldots m; \\ 0 \leqslant x_i \leqslant 1,\ I_i \ge 0, \quad \forall\ i = 1, 2, \ldots n \end{array} \right\}$$

Obviously, $(\boldsymbol{x}^1, \boldsymbol{y}^1, \boldsymbol{I}^1) = (\mathbf{1}, \mathbf{1}, \mathbf{0})$ is in set A.

Consider a set, $\mathcal{S}$, of demand nodes with $|S| \neq n$ and $|\Gamma(S)| \neq m$. Let

$$x_i^2 = \begin{cases} 1, & \text{if } i \in \mathcal{S} \\ 0, & \text{if } i \notin \mathcal{S} \end{cases}, \quad y_j^2 = \begin{cases} 1, & \text{if } j \in \Gamma(\mathcal{S}) \\ 0, & \text{if } j \notin \Gamma(\mathcal{S}) \end{cases},$$

$$I_i^2 = \begin{cases} \dfrac{\sum_{j\in\mathcal{S}} D_j^U}{|\Gamma(\mathcal{S})|}, & \text{if } i \in \Gamma(\mathcal{S}) \\ \dfrac{2\sum_{j\notin\mathcal{S}} D_j^L + \epsilon}{m - |\Gamma(\mathcal{S})|}, & \text{if } i \notin \Gamma(\mathcal{S}) \end{cases}$$

where ϵ is a small-enough positive number. Now, the term

$$\min\left\{\sum_{j\in\mathcal{R}} D_j^U y_j^2, \sum_{i\in\mathcal{W}} I_i - \sum_{j\in\mathcal{R}} D_j^L \left(1 - y_j^2\right)\right\} - \sum_{i\in\mathcal{W}} I_i x_i^2$$
$$= \min\left\{\sum_{j\in\mathcal{S}} D_j^U, \sum_{i\in\Gamma(\mathcal{S})} \frac{\sum_{j\in\mathcal{S}} D_j^U}{|\Gamma(\mathcal{S})|} + \sum_{i\notin\Gamma(\mathcal{S})} \frac{2\sum_{j\notin\mathcal{S}} D_j^L + \epsilon}{m - |\Gamma(\mathcal{S})|} - \sum_{j\notin\mathcal{S}} D_j^L\right\} - \sum_{i\in\Gamma(\mathcal{S})} \frac{\sum_{j\in\mathcal{S}} D_j^U}{|\Gamma(\mathcal{S})|} = \min\left\{0, \sum_{j\in\mathcal{S}} D_j^L + \epsilon\right\} = 0 \le 0$$

Hence, $(\boldsymbol{x}^2, \boldsymbol{y}^2, \boldsymbol{I}^2)$ is also in set A.

Now, consider the convex combination of $(\mathbf{x}^1, \mathbf{y}^1, \mathbf{I}^1)$ and $(\mathbf{x}^2, \mathbf{y}^2, \mathbf{I}^2)$. If we set $\lambda = \frac{1}{2}$, the convex combination is

$$x_i^3 = \begin{cases} 1, & \text{if } i \in \mathcal{S} \\ \frac{1}{2}, & \text{if } i \notin \mathcal{S} \end{cases}, \quad y_j^3 = \begin{cases} 1, & \text{if } j \in \Gamma(\mathcal{S}) \\ \frac{1}{2}, & \text{if } j \notin \Gamma(\mathcal{S}) \end{cases},$$

$$I_i^3 = \begin{cases} \dfrac{\frac{1}{2}\sum_{j\in\mathcal{S}} D_j^U}{|\Gamma(\mathcal{S})|}, & \text{if } i \in \Gamma(\mathcal{S}) \\ \dfrac{\sum_{j\notin\mathcal{S}} D_j^L + \frac{1}{2}\epsilon}{m - |\Gamma(\mathcal{S})|}, & \text{if } i \notin \Gamma(\mathcal{S}) \end{cases}$$

The following calculation shows that the convex combination is not in set A:

$$\min\left\{\sum_{j\in\mathcal{R}} D_j^U y_j^3, \sum_{i\in\mathcal{W}} I_i - \sum_{j\in\mathcal{R}} D_j^L \left(1 - y_j^3\right)\right\} - \sum_{i\in\mathcal{W}} I_i x_i^3$$
$$= \min\left\{\sum_{j\in\mathcal{S}} D_j^U + \frac{1}{2}\sum_{j\notin\mathcal{S}} D_j^U, \frac{1}{2}\sum_{j\in\mathcal{S}} D_j^U + \sum_{j\notin\mathcal{S}} D_j^L + \frac{1}{2}\epsilon - \frac{1}{2}\sum_{j\notin\mathcal{S}} D_j^L\right\} - \frac{1}{2}\sum_{j\in\mathcal{S}} D_j^U - \frac{1}{2}\sum_{j\notin\mathcal{S}} D_j^L - \frac{1}{4}\epsilon$$
$$= \min\left\{\frac{1}{2}\sum_{j\in\mathcal{R}} D_j^U - \frac{1}{2}\sum_{j\notin\mathcal{S}} D_j^L - \frac{1}{4}\epsilon, \frac{1}{4}\epsilon\right\} > 0.$$

The convex combination $(\boldsymbol{x}^3, \boldsymbol{y}^3, \boldsymbol{I}^3)$ is not in set A, we get a contradiction. Therefore the assumption is incorrect, i.e, constraint (1) is non-convex. $\square$

Instead of solving (**ExpM**), we propose an approximation model on the basis of the definition of Ψ-expander and solve it. According to the definition, for one set $\mathcal{S}$, there are two cases:

- Case 1: $\sum_{j\in\mathcal{S}} D_j^U \leqslant \sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L$
- Case 2: $\sum_{j\in\mathcal{S}} D_j^U > \sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L$

In Case 1, we assume that $\sum_{j\in\mathcal{S}} D_j^U$ is smaller than $\sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L$, and under this assumption, we can derive that $\sum_{j\in\mathcal{S}} D_j^U \leqslant \sum_{i\in\Gamma(\mathcal{S})} I_i$ by replacing the minimum term in constraint (1) with $\sum_{j\in\mathcal{S}} D_j^U$. The inequality, $\sum_{j\in\mathcal{S}} D_j^U \leqslant \sum_{i\in\Gamma(\mathcal{S})} I_i$, is a variant of well-known Hall's Condition in graph theory. Case 1 ensures that these warehouses that are adjacent to demand nodes in set $\mathcal{S}$ should reserve enough relief materials, so that even the worst situation happens these warehouses still have the capability to provide all needed supplies to demand nodes in set $\mathcal{S}$. Hence, the constraint under case 1 ensure that there are enough relief materials stored in our warehouses.

In Case 2, we assume that $\sum_{j\in\mathcal{S}} D_j^U$ is larger than $\sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L$. By replacing the minimum term with $\sum_{i\in\mathcal{W}} I_i - \sum_{j\notin\mathcal{S}} D_j^L$, constraint (1) is equivalently transformed to $\sum_{i\notin\Gamma(\mathcal{S})} I_i \leqslant \sum_{j\notin\mathcal{S}} D_j^L$. The inequality generates an upper limitation on the amount of relief packages among warehouses that are adjacent to demand nodes which are not in the set $\mathcal{S}$. Hence, redundant packages will not be stored or will be redistributed to other warehouses. It is clearly that constraints under case 2 lessen unnecessary inventory levels and balance the distribution of materials.

The two cases require the network to have certain properties from different aspects. More specifically, case 1 ensures that there are enough materials in our built warehouses, and case 2 help lessen unnecessary inventory levels. That implies that we can separate constraint (1) into two kinds of constraints under different cases. By limiting the cardinality of set $\mathcal{S}$, we get an approximation model as following, if we take multi-commodity into consideration:

$$\left(\textbf{ExpM} - \textbf{Approx}\right) \min_{I_i^k, Z_i} \sum_{k\in K}\sum_{i\in\mathcal{W}} h_i^k I_i^k + \sum_{i\in\mathcal{W}} f_i Z_i$$

$$\text{s. t.} \quad \sum_{j\in\mathcal{S}} D_j^{kU} \leqslant \sum_{i\in\Gamma_{\mathbf{r}}(\mathcal{S})} I_i^k, \quad \forall\ \mathcal{S}\in\mathcal{R}, \left|\mathcal{S}\right| \leqslant s_1^k,\ \forall\ k\in K \tag{2}$$

$$\sum_{i\notin\Gamma_{\mathbf{r}}(\mathcal{S})} I_i^k \leqslant \sum_{j\notin\mathcal{S}} D_j^{kL}, \quad \forall\ \mathcal{S}\in\mathcal{R}, \left|\mathcal{S}\right| \geqslant s_2^k,\ \forall\ k\in K \tag{3}$$

$$I_i^k \leqslant MZ_i, \quad \forall\ i\in\mathcal{W},\ \forall\ k\in K$$
$$I_i^k \geq 0, \quad \forall\ i\in\mathcal{W},\ \forall\ k\in K$$
$$Z_i\in\{0,1\}, \quad \forall\ i\in\mathcal{W}$$

Constraint (2) and (3) are corresponding results derived from Case 1 and Case 2, respectively. Constraint (2) requires that enough relief packages should be stored at warehouses, while constraint (3) reduces excessive inventory levels and prevents packages from being put together in several warehouses. Although the designed network is not exactly a Ψ-expander, it still possesses the property of the high connectivity. For example, constraint (2) guarantees that for set $\mathcal{S}$, even in the worst situation, demand can be entirely fulfilled by warehouses in $\Gamma_{\mathbf{r}}(\mathcal{S})$. Cardinality constraints are added for set $\mathcal{S}$ in both constraint (2) and (3) for each type of material. For constraint (2), s_1^k represents the importance of corresponding material. Larger s_1^k means the material is more crucial and less shortage is allowed. For example, $s_1^k = n$ implies that for all possible subsets of points of demand, their adjacent warehouses should hold enough material even in the worst case, which connotes that no shortage is allowed. And when $s_1^k = 1$, adjacent warehouses of each point of demand should hold inventory that can cover the maximal demand. However, since one warehouse is able to deliver goods to several nodes, the shortage is more likely to occur if demands are high in all nodes at the same time. For constraint (3), s_2^k expresses the extent of trying to decline redundant materials. Smaller s_2^k leads to that more redundant materials are removed. A similar analysis can be done, and we omit it here. Please note that, if $s_1^k < s_2^k$ holds, there would be no overlap among constraint (2) and (3) in terms of the cardinality of set $\mathcal{S}$. However, the contradictory aims of the two parameters may cause infeasibility if inappropriate values are set. For the sake of feasibility and better performance, we suggest $|s_1^k - s_2^k|$ should not be too large, and $s_1^k \geqslant s_2^k$ should hold.

3. Numerical experiment

3.1. Benchmark

3.1.1. Deterministic model

For a better comparison between the performances of two proposed models, we regard a deterministic model, which does not consider the uncertainty at all, as a benchmark. The deterministic model can be formulated as:

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$$(\textbf{DetM}) \min_{I_i^k, Z_i, x_{ij}^k} \sum_{k\in K}\sum_{i\in\mathcal{W}} h_i^k I_i^k + \sum_{i\in\mathcal{W}} f_i Z_i + \sum_{k\in K}\sum_{j\in\mathcal{R}} p_j^k \left(D_j^{k\mu} - \sum_{i\in\mathcal{W}} x_{ij}^k\right) + \sum_{k\in K}\sum_{i\in\mathcal{W}}\sum_{j\in\mathcal{R}} t_{ij}^k x_{ij}^k$$

$$\text{s. t.} \quad I_i^k \leqslant MZ_i, \quad \forall\ i\in\mathcal{W}, \quad \forall\ k\in K$$
$$\sum_{j\in\mathcal{R}} x_{ij}^k \leqslant I_i^k, \quad \forall\ i\in\mathcal{W},\ \forall\ k\in K$$
$$\sum_{i\in\mathcal{W}} x_{ij}^k = D_j^{k\mu}, \quad \forall\ j\in\mathcal{R},\ \forall\ k\in K$$
$$I_i^k \geqslant 0, \quad \forall\ i\in\mathcal{W},\ \forall\ k\in K$$
$$Z_i\in\{0,1\}, \quad \forall\ i\in\mathcal{W}$$
$$x_{ij}^k = 0, \quad \text{if } (i,j)\notin\mathcal{E}(\mathbf{r}), \quad \forall\ i\in\mathcal{W},\ \forall\ j\in\mathcal{R},\ \forall\ k\in K$$
$$x_{ij}^k \geqslant 0, \quad \text{if } (i,j)\in\mathcal{E}(\mathbf{r}), \quad \forall\ i\in\mathcal{W},\ \forall\ j\in\mathcal{R},\ \forall\ k\in K$$

The objective function includes both pre- and post-period costs, where the first summation is the cost of holding relief package, the second summation is the expenditure of building warehouses. The third one incurred by unmet demand, and the forth is the transportation cost. Constraints are almost the same as those in the two-stage stochastic model, that is relief material can only be stored at warehouses and can only be delivered through linked roads. The mean demand of each node is treated as the realized demand. Therefore the problem is reduced to a mixed integer programming problem, and it can be solved efficiently.

3.1.2. Model with predetermined total inventory level

Besides, the model in Li et al. (2017) is compared with our models since their work is the only one that has almost same assumptions as ours. They formulated their problems with a joint chance constraint requires that at least $1-\epsilon$ percentage of points of demand are provided with sufficient relief supply. Reformulation is conducted to transform the original computationally intractable problem into a problem of constructing Ψ-expander with a predetermined structure-free inventory level. More specifically, the predetermined inventory level is $\sum_{i\in\mathcal{W}} I_i = \sum_{j\in\mathcal{R}} \mu_j + \sqrt{\frac{\left(-\ln\epsilon\right)\sum_{j\in\mathcal{R}}\left(D_j^U - D_j^L\right)^2}{2}}$. Since this is a structure-free number, their results would be conservative. Formally, under a multi-commodity setting, we introduce the model as (LiM).

$$
\begin{aligned}
(LiM)\ \min_{I_i^k, Z_i}\ & \sum_{k\in K}\sum_{i\in\mathcal{W}} h_i^k I_i^k + \sum_{i\in\mathcal{W}} f_i Z_i\\
\text{s. t. }\ & \min\left\{\sum_{j\in\mathcal{S}} D_j^{kU},\ TI^k\left(\epsilon,\mathcal{R}\right) - \sum_{j\notin\mathcal{S}} D_j^{kL}\right\} \leqslant \sum_{i\in\Gamma_{\mathbf{r}}(\mathcal{S})} I_i^k,\ \forall\ \mathcal{S}\subseteq\mathcal{R},\ \forall\ k\in K\\
& TI^k\left(\epsilon,\mathcal{R}\right) = \sum_{j\in\mathcal{R}} \mu_j^k + \sqrt{\frac{\left(-\ln\epsilon\right)\sum_{j\in\mathcal{R}}\left(D_j^{kU} - D_j^{kL}\right)^2}{2}},\quad \forall\ k\in K\\
& I_i^k \leqslant MZ_i,\quad \forall\ i\in\mathcal{W},\ \forall\ k\in K\\
& I_i^k \geqslant 0,\quad \forall\ i\in\mathcal{W},\ \forall\ k\in K\\
& Z_i\in\{0,1\},\quad \forall\ i\in\mathcal{W}
\end{aligned}
$$

The objective function includes the cost of establishing warehouses and the cost of holding relief materials. The first constraint is the special case of Ψ-expander when $\Psi = 1$. The second constraint is the predetermined inventory level proposed by Li et al. (2017), where ϵ is a relatively small number representing that at least 1-ϵ percentage of points of demand can receive enough materials even in the worst case. Others are standard constraints as previous models.

3.2. Constraints Generation

Note that (**ExpM-Approx**) have exponentially many constraints, which makes the model computationally intractable. To deal with this issue, we propose a constraint generation method by iteratively solving well-defined subproblems and adding valid constraint into the model. In each iteration, we examine whether the current solution of the primary problem is optimal. If no, a valid inequality will be generated and added into the model. If yes, the procedure terminates, and we solve the current primary problem to gain the optimal solution. More specifically, taking constraint (2) as an example and ignoring the index k, we can define a set function $g(*)$ on the demand nodes set $\mathcal{R}$:

$$g(\mathcal{S}) = \sum_{i\in\Gamma_{\mathbf{r}}(\mathcal{S})} I_i - \sum_{j\in\mathcal{S}} D_j^U$$

where I_i is the solution to the primary problem.

The subproblem then becomes $\min_{\mathcal{S}:\mathcal{S}\subseteq\mathcal{R},|\mathcal{S}|\leqslant s_1} g(\mathcal{S})$, and it can be equivalently reformulated with binary variables as follows:

$$
\begin{aligned}
\left(\mathbf{SP1}\right)\ \min_{x_i,y_j}\ & \sum_{i\in\mathcal{W}} I_i x_i - \sum_{j\in\mathcal{R}} D_j^U y_j\\
\text{s. t. }\ & \sum_{j\in\mathcal{R}} y_j \leqslant s_1\\
& x_i \geqslant y_j,\quad \forall\ (i,j)\in\mathcal{E}(\mathbf{r})\\
& x_i,\ y_j\in\{0,1\},\quad \forall\ i\in\mathcal{W},\ j\in\mathcal{R}
\end{aligned}
$$

In each iteration, we solve the subproblem, and if the current subproblem's objective function $g(\mathcal{S}^*) < 0$, we add the constraint $\sum_{j\in\mathcal{S}^*} D_j^U \leqslant \sum_{i\in\Gamma_{\mathbf{r}}(\mathcal{S}^*)} I_i$ into the primary model. Constraint generation procedure terminates when the subproblem's objective function $g(\mathcal{S}^*) \geqslant 0$.

The constraint generation method for constraint (3) is almost the same, in which we define the subproblem as:

$$
\begin{aligned}
\left(\mathbf{SP2}\right)\ \min_{x_i,y_j}\ & \sum_{j\in\mathcal{R}} D_j^L\left(1-y_j\right) - \sum_{i\in\mathcal{W}} I_i\left(1-x_i\right)\\
\text{s. t. }\ & \sum_{j\in\mathcal{R}} y_j \geqslant s_2\\
& x_i \geqslant y_j,\quad \forall\ (i,j)\in\mathcal{E}(\mathbf{r})\\
& x_i,\ y_j\in\{0,1\},\quad \forall\ i\in\mathcal{W},\ j\in\mathcal{R}
\end{aligned}
$$

To sum up, the constraint generation algorithm for (**ExpM-Approx**) can be described in Algorithm 1. For simplicity of presentation, we define a constraints set $C_1^1 := \left\{\sum_{j\in\mathcal{S}} D_j^{kU} \leqslant \sum_{i\in\Gamma_{\mathbf{r}}(\mathcal{S})} I_i^k \middle| \mathcal{S}\in\mathcal{R},\ \left|\mathcal{S}\right| = 1,\ \forall\ k\in K\right\}$, which contains all constraints in Case 1 when the cardinality of $\mathcal{S}$ is equal to 1. Similarly, we can define $C_2^n := \left\{\sum_{i\notin\Gamma_{\mathbf{r}}(\mathcal{S})} I_i^k \leqslant \sum_{j\notin\mathcal{S}} D_j^{kL} \middle| \mathcal{S}\in\mathcal{R},\ \left|\mathcal{S}\right| = n,\ \forall\ k\in K\right\}$ as the constraint set for Case 2. $g_1^{kt}(*)$ and $g_2^{kt}(*)$ are used to express objective function values of two subproblems in t-th iteration for k-th material. Please note that, in Algorithm 1, $y_{1,j}^{kt}$ and $y_{2,j}^{kt}$ depend on material type k while $x_{1,i}^t$ and $x_{2,i}^t$ does not, since $x_{1,i}^t$ and $x_{2,i}^t$ are derived from decision variable Z_i.

Algorithm 1. Constraints Generation for **ExpM-Approx**

```
set C_1^1 and C_2^n as initial constraints
t ← 1
repeat
  Solve ExpM-Approx for (I^t, Z^t)
  Solve SP1 with (I^t, Z^t) for (x_1^t, y_1^t) and g_1^kt(*)
  Solve SP2 with (I^t, Z^t) for (x_2^t, y_2^t) and g_2^kt(*)
  if g_1^kt(*) < 0 then
    Add Σ_{i∈W} I_i^k x_{1,i}^t ≤ Σ_{j∈R} D_j^kU y_{1,j}^kt into ExpM-Approx
  end if
  if g_2^kt(*) < 0 do
    Add Σ_{j∈R} D_j^kL (1 − y_{2,j}^kt) ≤ Σ_{i∈W} I_i (1 − x_{2,i}^t) into ExpM-Approx
  end if
  t ← t + 1
until g_1^kt(*) ≥ 0 & g_2^kt(*) ≥ 0
```

3.3. Numerical study: Yushu Earthquake case study

We use the Yushu Earthquake as a case to evaluate the performance of stochastic models and Ψ-expander models. Yushu Earthquake is a M_s7.1 earthquake that happened at Yushu County in Qinghai Province, PR China, in 2010. Its affected area can be abstractly depicted as Fig. 1. The number above each link is the normalized distance between two affected areas. By classical shortest path algorithm, it is easy to obtain the distances between every node pairs. Based on that, if the distance between two nodes is less than the service radius, then relief materials can be delivered through the corresponding shortest path after an earthquake happened. In this numerical experiment, we partially refer to the parameters in Ni et al. (2018) as an input setting, including the fixed setup cost, the holding cost, the penalty cost, and the transportation cost. The former three kinds of cost are shown in Table 2, while the transportation costs, which are half of Ni et al. (2018), are shown in Table 3. Besides, the service radius $\mathbf{r} = \mathbf{1}$. Please note that larger $\mathbf{r}$ implies that warehouses can serve more points of demand, which

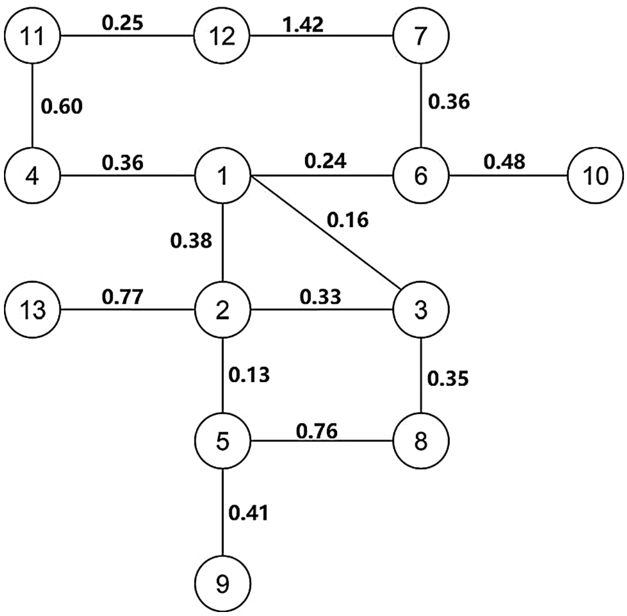

Fig. 1. Yushu affected area and normalized distances between nodes.

improves the pooling effect but weakens the ability to deal with the worst situation.

3.3.1. Single item

Firstly, single item setting is considered, i.e. $k = 1$. In this situation, a small number of materials, including food, water, cloth, medicine, can be packed up as one relief package. After the earthquake happened, the only materials needed to allocate is the relief package. With this assumption, we can easily obtain the crucial advantage of Ψ-expander without overlooking the uncertainty over demand. The demand of each affected area is subjected to a box uncertainty set with lower boundary $D^L = 50$, mean value $D^\mu = 100$, and upper boundary $D^U = 150$. One extra multiplier over the penalty cost c_p is added to conduct sensitivity analysis. For example, when $c_p = 2$, the penalty cost for per unmet demand is doubled.

Procedures for evaluating performance are listed as follows. Firstly, we solve models through different methods to obtain a corresponding relief network, as well as the first-stage cost. (**DetM**) is solved by setting the mean of demand as the truly realized demand. We use Monte Carlo sampling techniques to solve (**StoM**) with 200 scenarios (see supplementary materials) randomly drawn from normal distribution $N(100, 10)$ predicted by imaginary experts. And constraints generation method is applied to solve (**ExpM-Approx**) and (**LiM**) because of exponentially many of constraints. We set $s_1 = 10$, $s_2 = 5$ for (**ExpM-Approx**), and $\epsilon = 0.1$ for (**LiM**). Secondly, 3000 demand scenarios (see supplementary materials) is randomly generated from four different distribution families ranging $[D_j^L, D_j^U]$, which means the sample size equals 750 for each distribution family. These distributions include normal distributions $N(D_j^\mu, 10)$, $N(D_j^\mu, 15)$, $N(D_j^\mu, 20)$, $N(D_j^\mu, 25)$, $N(D_j^\mu, 30)$, uniform distribution $\mathrm{U}[D_j^L, D_j^U]$, triangular distribution $\mathrm{T}(D_j^L, D_j^U, D_j^\mu)$, and two-point distribution $P(D = D_j^L) = P(D = D_j^U) = 0.5$. With four kinds of distributions, we can better simulate the unpredictable consequences caused by the disaster in reality. Thirdly, for each demand scenario, the after-disaster allocation decision is obtained by solving the allocation problem $G(\mathbf{I}, \mathbf{d})$. At the same time, the transportation cost and penalty cost can be calculated. The final comparison is conducted on the average cost of 3000 scenarios.

Table 4 shows four networks designed by different models when $c_p = 2$. It suggests that node 2 and 3 should be chosen to build up warehouses by all four models. (**Exmp-Approx**) and (**LiM**) select node 4 to build the third warehouse, and (**DetM**) and (**StoM**) select node 12. Besides the difference in warehouse locations, the total amount of relief packages is gradually increasing from 1300 to 1687. This change is reasonable since (**DetM**) is insensitive to the uncertainty while (**LiM**) is the most conservative one.

After obtaining designed networks by models, simulations are conducted based on 3000 demand scenarios for each network, respectively. Table 5 illustrates the comparison of various costs. The first column to the fourth column respectively represents the fixed cost of establishing warehouses, the holding cost, the penalty cost, and the transportation cost. The last column represents the improvement in total cost compared to (**DetM**). It is clear that the holding cost is gradually increasing from top to bottom. Since the stock quantity of relief materials is increasing, as shown in Table 4, the tendency of the holding cost is consistent with that. Consequently, after one disaster happens, more relief packages are delivered to affected areas when we adopt (**ExpM-Approx**) or (**LiM**), leading to a higher transportation cost and a dramatic decline in the penalty cost. In terms of total costs, even though improvements are all positive, (**ExpM-Approx**) gains the greatest improvement. This is because the network designed by (**ExpM-Approx**) is a derivative of Ψ-expander, and the high connectivity inherited from Ψ-expander ensures a low proportion of unmet demand. On the other hand, the approximation model gets rid of predetermined structure-free inventory level, and that empowers it to have a chance to pursue a lower inventory level, leading to a network that is not as conservative as an exact Ψ-expander.

Besides the cost, there are some other indicators evaluating designed networks from diverse perspectives. For instance, the type 2 service level is also an essential indicator. It is a quantity-oriented indicator, measuring the proportion of met demand. Another widely used indicator is the type 1 service level. It is an event-oriented indicator, representing the ratio of how many affected areas have been fully satisfied. Table 6 displays the results in terms of service levels. The first two columns contain type 1 service level on average and in the worst case. The following two columns show type 2 service level. It demonstrates that (**ExpM-Approx**) and (**LiM**) can achieve higher service level than (**DetM**) and (**StoM**) on average. More specifically, the average type 2 service levels of two networks derived from Ψ-expander almost reach 100%, which means, on average, almost all demand can be fulfilled. In contrast, (**DetM**) and (**StoM**) can only fulfill 96.5% and 97.9% demand respectively. As to type 1 service level, it is clear that the last two models have remarkably improved the performance on average. This is because when one subset of demand nodes satisfies the definition of Case 1, warehouses that are adjacent to demand nodes in the subset should store enough relief materials even in the worst case. As a result of that, type 1 service level, which is event-oriented, would be boosted. Same results can be found if the worst case is considered. Please note that, regarding service levels, (**LiM**) is slightly better than

Table 2
Setup cost, holding cost and penalty cost.

Node	1	2	3	4	5	6	7	8	9	10	11	12	13
f	203	193	130	117	292	174	130	157	134	161	234	220	170
h	3.40	2.33	2.00	2.69	2.63	3.44	3.43	3.53	2.33	2.50	3.37	2.84	3.76
p	11.48	14.32	12.14	10.19	12.01	14.90	9.42	11.91	10.68	11.24	13.1	11.09	10.18

Table 3
Transportation cost.

Node	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0	0.45	0.18	0.4	0.7	0.26	0.66	0.51	1.1	0.79	1.05	2.13	1.3
2	0.45	0	0.39	0.85	0.25	0.7	1.10	0.77	0.7	1.23	1.13	2.57	0.85
3	0.18	0.39	0	0.57	0.64	0.43	0.83	0.38	1.09	0.96	1.23	2.30	1.24
4	0.4	0.85	0.57	0	1.1	0.66	1.06	0.95	1.55	1.19	0.65	2.53	0.93
5	0.7	0.25	0.64	1.1	0	0.96	1.36	0.62	0.45	1.49	1.39	2.83	1.11
6	0.26	0.7	0.43	0.66	0.96	0	0.4	0.81	1.41	0.53	1.31	1.87	1.56
7	0.66	1.10	0.83	1.06	1.36	0.4	0	1.21	1.81	0.93	1.71	1.47	1.97
8	0.51	0.77	0.38	0.95	0.62	0.81	1.21	0	1.07	1.35	1.61	2.68	1.62
9	1.1	0.7	1.09	1.55	0.45	1.41	1.81	1.07	0	1.94	1.84	3.28	1.56
10	0.79	1.23	0.96	1.19	1.49	0.53	0.93	1.35	1.94	0	1.85	2.4	2.09
11	1.05	1.13	1.23	0.65	1.39	1.31	1.71	1.61	1.84	1.85	0	3.18	0.28
12	2.13	2.57	2.30	2.53	2.83	1.87	1.47	2.68	3.28	2.4	3.18	0	3.43
13	1.3	0.85	1.24	0.93	1.11	1.56	1.97	1.62	1.56	2.09	0.28	3.43	0

Table 4
Designed network by different models (single item).

Node	1	2	3	4	5	6	7	8	9	10	11	12	13
DetM		500	700									100	
StoM		473	751									114	
ExpM-Approx		300	1050	150									
LiM		300	1237	150									

Table 5
Cost comparison of four models.

Model	f	h	p	t	Total Cost	Improvement (vs. DetM)
DetM	543	2849	979	625	4996	–
StoM	543	2928	617	643	4731	5.3%
ExpM-Approx	440	3203	65	913	4621	7.5%
LiM	440	3577	6	917	4940	1.1%

Table 6
Service level comparison.

Model	Type1 service level		Type2 service level	
	Average	Worst	Average	Worst
DetM	91.92%	61.54%	96.55%	70.27%
StoM	94.25%	61.54%	97.86%	72.32%
ExpM-Approx	99.63%	76.92%	99.79%	81.08%
LiM	99.95%	84.62%	99.98%	91.19%

Table 7
Cost comparison under different c_p.

c_p	Model	f	h	p	t	Total Cost
1	DetM	543	2849	489	625	4506
	StoM	543	2901	357	639	4440
	ExpM-Approx	440	3203	36	908	4587
	LiM	440	3577	3	917	4937
2	DetM	543	2849	979	625	4996
	StoM	543	2928	617	643	4731
	ExpM-Approx	440	3203	66	913	4621
	LiM	440	3577	6	917	4940
3	DetM	543	2849	1468	625	5485
	StoM	543	2948	819	646	4956
	ExpM-Approx	440	3203	97	914	4654
	LiM	440	3577	9	917	4943
4	DetM	543	2849	1958	625	5975
	StoM	543	2959	1026	647	5175
	ExpM-Approx	440	3203	129	914	4686
	LiM	440	3577	12	917	4946
5	DetM	543	2849	2447	625	6464
	StoM	543	2965	1245	647	5400
	ExpM-Approx	440	3203	162	914	4719
	LiM	440	3577	15	917	4948

(**ExpM-Approx**), which is the direct consequence of reserving much more relief packages. However, this is not economical according to Table 5, since the extra holding cost generated by reserving more materials does not be equally countervailed by the reduce over penalty cost, leading to a higher total cost. All in all, without holding redundant relief packages, (**ExpM-Approx**) constructs an as highly connected network as exact Ψ-expander does, significantly reducing the penalty cost and achieving the greatest improvement.

Next, we change c_p to explore the sensitivity of our results to the penalty cost parameters. Following the same procedure, we conducted simulations with the same parameters, except the multiplier c_p. Table 7 shows how the aggregated expenditure changes when c_p varies. Same as before, the columns f, h, p, and t refer to the fixed cost, the holding cost, the penalty cost, and the transportation cost respectively.

It is easy to detect that the penalty cost is the only cost that enormously increases when c_p becomes larger. More specifically, the penalty costs of (**DetM**) and (**StoM**) boosts precipitously, while the penalty costs of (**ExpM-Approx**) and (**LiM**) increase a little. Moreover, when c_p becomes larger, the penalty cost of (**DetM**), (**ExpM-Approx**), and (**LiM**) proportionally increases while that of (**StoM**) increases at a slower rate compared to c_p. The main reason lies in the number of unmet demand. (**StoM**) tries to remain more inventory when c_p goes larger while the other three models remain the same inventory level no matter how c_p changes. However, without considering the possible worst case, (**StoM**) still causes a high penalty cost. In contrast, (**ExpM-Approx**) and (**LiM**) are risk-averse and conservative, and they reserve much more relief materials. Hence, the amount of unmet demand would be paltry, causing a negligible penalty cost.

Fig. 2 illustrates the total cost under different models. The vertical axis is the ratio of the total cost of the corresponding model to the total cost of (**DetM**) under different c_p. In other words, the cost of (**DetM**) is regarded as 1. The horizontal axis is the value of c_p, and we gradually change it from 1 to 5. Fig. 2 reveals that when c_p becomes larger, the total cost of (**ExpM-Approx**) and (**LiM**) decreases. Although (**StoM**) has the same trend, the rate of it is lower than the two others. This is because c_p is the measurement of the severity of unmet demand. Large c_p implies a severe consequence over unmet demand. When c_p goes larger, (**ExpM-Approx**) and (**LiM**) always almost ensure that there are enough relief packages. On the contrary, (**StoM**) does not store enough relief packages, and the penalty cost soars when c_p becomes larger. The result is consistent with the well-known fact that a conservative model would perform much better when shortages are intolerable. Therefore, when the penalty cost is a significant and unignorable factor, with inherent

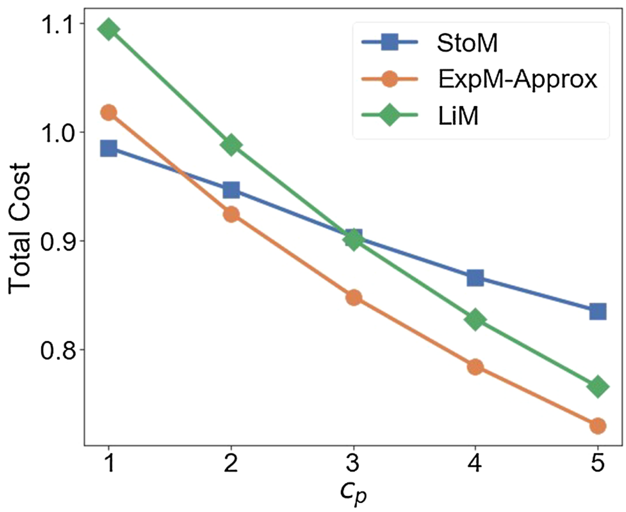

Fig. 2. Sensitivity analysis on c_p.

conservative characteristics, (**ExpM-Approx**) and (**LiM**) would perform better than stochastic models.

3.3.2. Multiple items

In this part, numerical experiments are conducted with a multi-items setting. It is quite common that various relief materials should be delivered to affected regions, including water, food, clothes, etc. And due to the differences over geological structures and climates among affected regions, the proportion of needed relief materials would vary. For example, people in frigid areas need more clothes while people in sweltering areas would need more water. In this situation, relief packages cannot satisfyingly meet heterogeneous demands. Therefore, considerations on the multi-items setting are necessary.

We explore the performances of our model when $k = 3$, that means there are three kinds of relief materials. Procedures of simulations are the same as that in the single-item situation. Besides, as shown in the first three columns in Table 8, extra multiplier for each kind of materials are incorporated to scale input parameters. c_h^k, c_p^k, and c_t^k are the multipliers for the holding cost, the penalty cost, and the transportation cost, respectively. For instance, when $k = 1$, the multiplier $c_p^k = 3$, that means the penalty cost for each unmet demand of the first category of material would be tripled. And since $c_p^3 = 1$, the penalty cost is same as Table 2. The last three columns in Table 8 are the parameters of the uncertainty set for each relief material. Same as the single-item setting, we assume that all affected regions have a consistent uncertainty set. For example, the demand for the first kind of material ranges from 40 to 160 for all affected regions. In addition, one extra multiplier over the penalty cost for all kinds of materials c_p is still added to conduct sensitivity analysis. Other unmentioned parameters remain the same.

Table 9 shows the networks designed by different models when there are 3 varied items and $c_p = 3$. Numbers in the table indicate the necessary amounts of corresponding materials. It is clear that (**DetM**) and (**StoM**) select six locations to build warehouses while the other two models, which are based on Ψ-expander, only choose three locations. The nuance is a direct consequence caused by the different ways in balancing the holding cost, the fixed setup cost, and the penalty cost.

Table 8
Parameter changes for different kinds of materials.

k	c_h^k	c_p^k	c_t^k	D_j^{kL}	$D_j^{k\mu}$	D_j^{kU}
1	1	3	1	40	100	160
2	1	2	1	50	100	150
3	1	1	1	60	100	140

Because, for (**DetM**) and (**StoM**), building the extra three warehouses at the cost of paying setup cost would offset the extra penalty cost when warehouses are no built there. However, for the other two expander-based models, the penalty cost is not taken into consideration, and the objective is minimizing the setup cost and the holding cost; thus, fewer warehouses are built. Besides, in terms of total inventory level, there is an increase from top to bottom since the conservativeness is increasing. Moreover, within one model, the material with a larger range generally has a higher inventory level to cope with the fluctuation.

Then, following the same simulation procedure in single item setting, we conduct the numerical study with 3000 scenarios. Table 10 reveals the various kinds of cost for different categories of materials. The columns 'f,h,t' and 'p' are referred to the fixed setup cost, the holding cost, the transportation cost, and the penalty cost respectively. The column 'k' represents different kinds of materials. The almost same trend can be found that the holding cost is increasing while the penalty cost is decreasing from top to bottom. This is because the conservativeness of models is becoming heavier, directly leading to higher inventory levels and lower unmet penalty costs. Please note that, although (**ExpM-Approx**) store a little bit more relief materials compared with (**StoM**), the decreases in the penalty cost caused by unmet demand are large enough to offset the increased holding cost caused by storing more materials, resulting in a lower total cost. But for (**LiM**), when compared with (**StoM**), much more materials are stored so that the decline in penalty cost does not exceed the increased holding cost, achieving a relatively lower improvement. In one word, under the multi-item setting, with only a little bit more inventory than (**StoM**), (**ExpM-Approx**) can significantly reduce the penalty cost without sacrificing much conservativeness on the basis of (**LiM**).

Next, we change c_p to conduct sensitivity analysis as we have done under the single-item setting. Fig. 3 shows the total costs for various models under different c_p. The horizontal axis is the adopted values c_p, and the vertical axis is normalized total cost according to the total cost of (**DetM**), which means the total cost of (**DetM**) is regarded as 1 for each c_p. The almost same results can be discovered as that of Fig. 2. More specifically, (**ExpM-Approx**) can always perform better than (**LiM**) because of the decreased inventory level. However, with c_p becoming larger, the total costs of (**ExpM-Approx**) and (**LiM**) almost converge. This is because the penalty cost gradually dominates all other factors, becoming an essential contributor to the total cost. Thus, the saving cost by decreasing inventory levels in (**ExpM-Approx**) will gradually exceed by the extra unmet penalty cost. As a result of that, the improvement of (**Exmp-Approx**) is diminished to the same level of (**LiM**). Another same result is that when c_p is relatively small, (**StoM**) has the best performance, which means when the penalty cost is not a predominant factor, a stochastic model, even with an inaccurate prediction on demand, would have the chance to achieve the lowest cost. However, in a disaster relief situation where the direct consequence of delivering insufficient relief materials is losing lives, the penalty cost definitely is the predominant factor. Thus, (**ExpM-Approx**), a highly connected network without redundant relief supplies, would be recommended.

4. Extensions

Based on the proposed Ψ-expander approximation model, several extended models can be obtained by incorporating more constraints and more objective function terms, so that the model would better match the reality. In general, there are two directions to extend our models. First, warehouse types can be incorporated to reflect different capacities and different purposes of warehouses in reality. For example, a warehouse in a metropolitan would be more versatile than one in a rural area considering the importance of that metropolitan and the population of residents there. Another direction to extend our basic model is taking into consideration the potential loss of reserved relief materials because of severe natural disasters.

Table 9
Designed network by different models (multiple item).

Model	Node	1	2	3	4	5	6	7	8	9	10	11	12	13
DetM	k = 1		300	500	200					100	100		100	
	k = 2		300	500	200					100	100		100	
	k = 3		300	500	200					100	100		100	
StoM	k = 1		299	555	196					111	105		109	
	k = 2		300	542	196					110	106		107	
	k = 3		298	533	195					111	107		107	
ExpM-Approx	k = 1		320	1120	160									
	k = 2		300	1050	150									
	k = 3		280	980	140									
LiM	k = 1		320	1285	160									
	k = 2		300	1237	150									
	k = 3		280	1190	140									

Table 10
Cost comparison of four models (multiple items).

Model	f	k	h	t	p	Total Cost	Improvement (vs. DetM)
DetM	955	1	3004	850	4568	20453	–
		2	3004	832	2392		
		3	3004	807	1037		
StoM	955	1	3165	857	2146	16584	18.9%
		2	3135	836	1093		
		3	3115	810	472		
ExpM-Approx	440	1	3416	1806	227	16159	21.0%
		2	3203	1834	170		
		3	2989	1832	242		
LiM	440	1	3746	1811	52	16759	18.1%
		2	3577	1840	7		
		3	3409	1873	4		

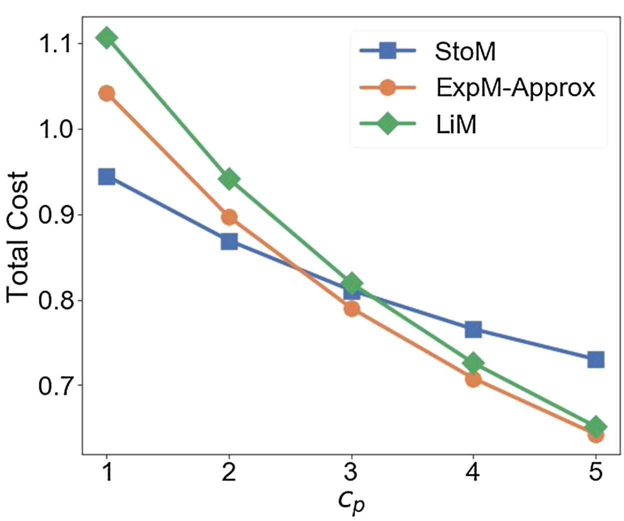

Fig. 3. Sensitivity analysis on c_p (multiple items).

4.1. Warehouse with different capacities

Suppose that there are a set of different warehouse types $C\,(C = \{1, 2, ..., |C|\})$. Each type of warehouse can hold at most $c_l(\ \forall\ l \in C)$ units relief material aggregately no matter which type of material it is. Correspondingly, for each type of warehouse, the setup cost $f_i^l(\ \forall\ i \in \mathcal{W}, \forall\ l \in C)$ is various. With these new requirements, we can slightly modify our proposed model with extra binary decision variables H_i^l, which represents whether the l^{th} type of warehouse is chosen at location i. Only when the building indicator Z_i is 1, then H_i^l is possible to be 1. To follow this relationship, we adopt the Big-M method. We formally introduce the Ψ-expander approximation model with diverse warehouses as following:

$$\left(\textbf{ExpM}-\textbf{Approx}-\textbf{Ex1}\right)\ \min_{I_i^k, H_i^l, Z_i} \sum_{k\in K}\sum_{i\in\mathcal{W}} h_i^k I_i^k + \sum_{i\in\mathcal{W}}\sum_{l\in C} f_i^l H_i^l$$

$$\text{s. t.}\quad \sum_{j\in\mathcal{S}} D_j^{kU} \leqslant \sum_{i\in\Gamma_{\mathbf{r}}(\mathcal{S})} I_i^k,\quad \forall\ \mathcal{S}\in\mathcal{R},\ \left|\mathcal{S}\right| \leqslant s_1^k,\ \forall\ k\in K$$

$$\sum_{i\notin\Gamma_{\mathbf{r}}(\mathcal{S})} I_i^k \leqslant \sum_{j\notin\mathcal{S}} D_j^{kL},\quad \forall\ \mathcal{S}\in\mathcal{R},\ \left|\mathcal{S}\right| \geqslant s_2^k,\ \forall\ k\in K$$

$$I_i^k \leqslant MZ_i,\quad \forall\ i\in\mathcal{W},\ \forall\ k\in K,\ \forall\ l\in C$$

$$\sum_{k\in K} I_i^k \leqslant c_l + M\left(1 - H_i^l\right),\quad \forall\ i\in\mathcal{W},\ \forall\ l\in C \tag{4}$$

$$\sum_{l\in C} H_i^l = Z_i,\quad \forall\ i\in\mathcal{W} \tag{5}$$

$$I_i^k \geq 0,\quad \forall\ i\in\mathcal{W},\ \forall\ k\in K$$
$$Z_i \in \{0, 1\},\quad \forall\ i\in\mathcal{W}$$
$$H_i^l \in \{0, 1\},\quad \forall\ i\in\mathcal{W},\ \forall\ l\in C$$

The second term in the objective function is an expansion to the total setup cost in (**ExpM-Approx**), which considers all possible warehouse selections and sum the setup cost up. The first, second, and third constraints are the same as (**ExpM-Approx**), which requires the designed network is an approximated Ψ-expander. The fourth constraint, the inequality (4), expresses the relationship between the chosen warehouse and the capacity. For instance, if $H_i^l = 1$, then the relief materials stored as location i with warehouse l cannot exceed the capacity c_l of warehouse type l. On the other side, if $H_i^l = 0$, then the constraint is loose enough that almost cannot be bounding. The fifth constraint, the equality (5), shows that only when we select location i as one place to establish a warehouse (i.e., $Z_i = 1$), then we can ponder on the type of warehouses, and obviously only one warehouse can be built at the same location. The remaining constraints are standard non-

Table 11
Scaling factors for setup cost.

Node	1	2	3	4	5	6	7	8	9	10	11	12	13	Capacity
Type 1	1	1	1	1	1	1	1	1	1	1	1	1	1	1000
Type 2	2	2	2	2	2	2	2	2	2	2	2	2	2	2000
Type 3	3	3	3	3	3	3	3	3	3	3	3	3	3	3000
Type 4	4	4	4	4	4	4	4	4	4	4	4	4	4	4000

Table 12
Designed network by ExpM-Approx-Ex2.

Node	1	2	3	4	5	6	7	8	9	10	11	12	13
k = 1		356	1244	178									
k = 2		333	1167	167									
k = 3		311	1089	156									

negative or binary constraints.

With the same setting in Section 3.3.2, we conduct an additional numerical study for this extension. Assume that there are four types of warehouses, and the scaling factors of warehouse setup cost are given in Table 11, along with their capacities.

By solving 4.1 with the proposed Algorithm 1, we can obtain the decisions on places, types of warehouses, and inventory levels. Table 12 shows the designed network when there are different types of warehouses. The column, named 2, 3, 4, respectively, represents the locations chosen by (**ExpM-Approx-Ex1**), other nodes are omitted. And due to the importance of different locations, our extended model can automatically select the warehouse with enough storage space to build.

4.2. Loss of goods

It is common in reality that relief materials stored in even stable warehouses can get lost due to horrible natural disasters. Without considering the loss, available relief materials on hand after a disaster would be in shortage. To better deal with this problem, we can incorporate one additional parameter θ ($\theta \leqslant 1$) into (**ExpM-Approx**) with respect to inventory levels I_i^k to slightly boost the number of reserved relief materials. θ can be explained as the remaining proportion of prepositioning materials after a disaster happens. The smaller θ means fewer relief materials are expected to be left. For example, when $\theta = 0.5$, we expect only half of the prepositioning inventory would remain. Formally, we introduce this problem as our second extension:

$$\left(\mathbf{ExpM-Approx-Ex2}\right) \min_{I_i^k, Z_i} \sum_{k\in K}\sum_{i\in\mathcal{W}} h_i^k \theta I_i^k + \sum_{i\in\mathcal{W}} f_i Z_i$$

$$\text{s. t.} \sum_{j\in\mathcal{S}} D_j^{kU} \leqslant \sum_{i\in\Gamma_{\mathbf{r}}(\mathcal{S})} \theta I_i^k, \quad \forall\, \mathcal{S}\in\mathcal{R},\ \left|\mathcal{S}\right| \leqslant s_1^k,\ \forall\, k\in K \tag{6}$$

$$\sum_{i\notin\Gamma_{\mathbf{r}}(\mathcal{S})} \theta I_i^k \leqslant \sum_{j\notin\mathcal{S}} D_j^{kL}, \quad \forall\, \mathcal{S}\in\mathcal{R},\ \left|\mathcal{S}\right| \geqslant s_2^k,\ \forall\, k\in K \tag{7}$$

$$I_i^k \leqslant MZ_i, \quad \forall\, i\in\mathcal{W},\ \forall\, k\in K$$
$$I_i^k \ge 0, \quad \forall\, i\in\mathcal{W},\ \forall\, k\in K$$
$$Z_i \in \{0, 1\}, \quad \forall\, i\in\mathcal{W}$$

The only difference between the extension and the model (**ExpM-Approx**) in Section 2.2 is the existence of parameter θ in the first two constraints (i.e., (6) and (7)). Solving this model by Algorithm 1 is easy to conduct. With the same setting in Section 3.3.2 and $\theta = 0.9$, we get the designed network work as follows:

Table 12 shows the designed network by model (**ExpM-Approx-Ex2**). The chosen locations are still node 2, 3, and 4, which are the same as that of (**ExpM-Approx**) in Section 3.3.2. However, the total inventory level now is 5001, whose 90% is the total inventory level of (**ExpM-Approx**). That is the direct consequence by introducing the parameter θ into our proposed model. If experts can accurately estimate the parameter θ, then our proposed Ψ-expander approximation model would match the reality better.

5. Conclusion

Disaster relief network design is a popular topic in the field of disaster management nowadays. A properly designed network can not only prevent people, whose lives are threatened after a disaster happened, from being in a shortage of relief materials but also can save as much money as possible for strategy makers. However, because of the scarcity of demand information, it is hard to design a proper network. In this paper, we study two prevalent methods of designing networks with different demand information, i.e., stochastic models with potentially inaccurately predicted demand information and a special case of Ψ-expander models with a box uncertainty set. To make the Ψ-expander tractable and easy to solve, we propose an approximation model accompanying with a constraint generation algorithm. In the numerical study, single-item setting and multi-item setting are taken into consideration to emulate the reality better. Moreover, a deterministic model and the model by Li et al. (2017) is compared with our proposed models. The numerical study demonstrates that stochastic models, with predicted demand information, can achieve lower total costs when the penalty cost c_p is relatively small, while the Ψ-expander approximation model, only with boundaries and mean values information, can perform much better when c_p is large enough. Besides, Ψ-expander approximation model can design an as highly connected network as an exact Ψ-expander, but without a predetermined total inventory level. Two extension models demonstrate that the Ψ-expander is compatible with many realistic requirements. For the disaster relief purpose, we highly recommend the Ψ-expander approximation model to design the relief network.

In future work, we will explore the property of Ψ-expander in a theoretical way and find out the relationship between Ψ-expander and its approximation model. Besides that, we do not take into consideration the break of roads caused by disasters, which almost exists in the real world. So, we will also consider a more realistic setting in future works. At last, we hope our work can attract more attention to disaster management, and we also hope the concept of Ψ-expander can be widely used for manufacturing problems.

Acknowledgement

This research is sponsored by Shanghai “Excellent Academic Leader” Plan (Grant No. 16XD1401700), and by Shanghai “Shu Guang” Project (Grant No. 17SG16).

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