Working in Progress

Distributionally Robust Scheduling Problem with Uncertain Release Time and Completion Time

Xun Zhang
Antai College of Economics and Management, Shanghai Jiao Tong University, China, xunzhang@sjtu.edu.cn

Du Chen
Antai College of Economics and Management, Shanghai Jiao Tong University, China, chendu2017@sjtu.edu.cn

This paper gives

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1. Introduction

Scheduling critical resourses to jobs (or activities) is one of the most important problems in the field of production, service, manufacturing and transportation systems. To obtain the best performance in the scheduling problem, the decision-maker has to decide the optimal sequence of the jobs to be processed. In the past several decades, numerous literature study scheduling problem from all kinds of perspective. For a comprehensive and in-depth survey of this field, we refer to the books by Leung (2004) and Pinedo (2008).

However, most of exsiting literature investigate deterministic scheduling problems. The more practical problem is scheduling problem with uncertain parameters, such as release times, processing times, and due dates. In presence of uncertainty in the optimization problem, stochastic programming, robust optimization and distributionally robust optimization are three main approaches to obtain a reliable solution. In stochastic programming, the uncertain parameters are represented as random variables with a known distribution, which is hard to estimate in reality. Robust optimization do not need to know any information about distribution, except the support. But the solution from Robust optimization might provide an over-conservative solution. Distributionlly robust optimization (DRO) places in a intermediate position, DRO usually assume that limited information are known a prior, such as mean and variance, which can be easily estimated. In this paper, we adopt a distributionally robust optimization for our problem.

This paper deal with the scheduling problem with uncertain job parameters (SPwU), i.e., uncertain processing time and/or uncertain release time. The objective of the probem are the makespan and total completion time, which are two of the most important measures to optimise. Minimising the total completion time tends to minimise the average waiting time of the customers, whereas minimising the makespan tends to minimise the maximum waiting time of the customers. In contrast to the deterministic scheduling problem, studies on SPwU are on the rise in recent years (e.g. see Kasperski (2005),Chang et al. (2017), Daniels and Kouvelis (1995), Drwal and Rischke (2016), Kasperski (2005)). However, most of them only assume that the processing time is stochastic and ignore the stochastic release time. solving the scheduling problem with release time is a more challenging task. To the best of our knowledge, only a few papers present their solutions of SPeW. Bachtler et al. (2020) investigate a single machine makespan scheduling problem with uncertain release date, which take values within known intervals. Yue et al. (2018) study a similar problem, but the objective is maximum waiting time (MWT). Liu et al. (2020) apply sample average approximate method to solve a parallel machine scheduling problem with uncertain release time and processing time.

The rest of this paper is organized as follows. After introduction, we formulate our scheduling problem as a distributionally robust model with a specified ambiguity set. In section 3, we provide a tractable reformulation to solve the total completion time problem. In section 4, we show that the makespan problem can also be solved in the same way. We extend our problem to the parallel machine scheduling in section 5. The conculsion are summarize in section 6.

2. Problem Formulation

2.1. Deterministic Single Machine Scheduling Problem

Consider a single machine scheduling problem that a set $\mathscr{J} = \{1, 2, \ldots, n\}$ of jobs are processed on one machine. The processing time of job j is denoted as $p_j, j \in \{1, \ldots, n\}$, which is ready to process at its release time r_j. For the simplicity, we denote $\boldsymbol{p} = \{p_1, \ldots, p_n\}$ and $\boldsymbol{r} = \{r_1, \ldots, r_n\}$. In this paper, we mainly concerned with non-preemptive scheduling discipline, which means that a machine can process at most one job at a time. Therefore, the set of the feasible sequence $\mathscr{X}$ can be expressed as follows,

$$\mathscr{X} = \left\{ \boldsymbol{x} | \begin{array}{c} \sum_{i=1}^{n} x_{ij} = 1, \forall j \in \{1, \ldots, n\} \\ \sum_{j=1}^{n} x_{ij} = 1, \forall i \in \{1, \ldots, n\} \\ x_{ij} \in \{0, 1\}, \forall i, j \in \{1, \ldots, n\} \end{array} \right\}$$

In this expression, $x_{ij} = 1$ stands for job j is processed at the i-th position of the sequence, and $x_{ij} = 0$ otherwise. The first row in above set forces that one job can only be processed at one position, the second row requires that one position should be occupied by only one job, the third row means that the job cannot be divided.

Given a feasible sequence $\boldsymbol{x}$, we denote the start time of the job at the i-th position by t_i, then the completion time of the job at the i-th position is $c_i = t_i + \sum_{j=1}^{n} x_{ij} p_j$. Therefore, the makespan (schedule length) can be expressed as $C_{max}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{r}) = t_n + \sum_{j=1}^{n} x_{nj} p_j$, and the total completion time can be expressed as $T(\boldsymbol{x},\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{n}\left[t_i + \sum_{j=1}^{n} x_{ij}p_j\right]$. In this paper, our goal is to find a sequence such that the makespan or total completion time minimum. In general, we denote $f(\boldsymbol{x},\boldsymbol{p},\boldsymbol{r})$ as our objective function, which can be the makespan $C_{max}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{r})$ or total completion time $T(\boldsymbol{x},\boldsymbol{p},\boldsymbol{r})$. Then, our problem can be formulated as a 0-1 integer programming model:

$$Q(\boldsymbol{x},\boldsymbol{p},\boldsymbol{r}) = \min_{\boldsymbol{x},\boldsymbol{t}} \; f(\boldsymbol{x},\boldsymbol{p},\boldsymbol{r}) \tag{1}$$

$$\text{s.t } t_i \geq \sum_{j=1}^{n} x_{ij}r_j, \forall i \in \{1,\ldots,n\} \tag{2}$$

$$t_i \geq t_{i-1} + \sum_{j=1}^{n} x_{i-1,j}p_j, \forall i \in \{2,\ldots,n\} \tag{3}$$

In this formulation, the decision variable is $\boldsymbol{x}$ and $\boldsymbol{t}$ and the objective (1) is the makespan or total completion time. The constraint (2) means that the start time at the i-th the position should greater or equal to the release time of the job at the i-th position , the constraint (3) means that the start time at the i-th position should greater or equal to the completion time of the job at the previous position.

2.2. Distributionally Robust Single Machine Scheduling Problem

In practice, the processing time and release time are usually stochastic, the exact value and distribution are merely known a prior. The only available information we can obtain is some statistical information. In facing this stochastic setting, model $Q(\boldsymbol{x},\boldsymbol{p},\boldsymbol{r})$ is incapable to provide a efficient schedule. To differentiate from the deterministic case, the random processing time and release time are denoted by $\tilde{\boldsymbol{p}} = \{\tilde{p}_1,\ldots,\tilde{p}_n\} \in \mathbb{R}_+^n$ and $\tilde{\boldsymbol{r}} = \{\tilde{r}_1,\ldots,\tilde{r}_n\} \in \mathbb{R}_+^n$. We assume that random variable $(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{r}})$ follows an distribution $\mathbb{P}$, which is unknown in advance but the the first moment and the second moment are available. Let $\tilde{\boldsymbol{z}} = [\tilde{\boldsymbol{p}},\tilde{\boldsymbol{r}}]^T$, then we denote mean vector and variance vector by$\boldsymbol{\mu} = [\boldsymbol{\mu}^p,\boldsymbol{\mu}^r]^T$ and $\boldsymbol{\sigma} = [\boldsymbol{\sigma}^p,\boldsymbol{\sigma}^r]^T$, respectively. Therefore, an ambiguity set that includes all feasible distributions can be expressed as follows,

$$\mathscr{F}_{\mathrm{MM}} = \left\{ \mathbb{P} \in \mathscr{P}_0\left(\mathbb{R}^{2n}\right) \;\middle|\; \begin{array}{l} \tilde{\mathbf{z}} \sim \mathbb{P} \\ \mathbb{E}_{\mathbb{P}}[\tilde{\mathbf{z}}] = \boldsymbol{\mu} \\ \mathbb{E}_{\mathbb{P}}\left[(\tilde{z}_i - \mu_i)^2\right] \leq \sigma_i^2, \forall i \in \{1,\ldots,2n\} \\ \mathbb{P}\left[\tilde{\mathbf{z}} \in \mathbb{R}_+^{2n}\right] = 1 \end{array} \right\}$$

Now, we are able to provide our distributionally robust model. Given the first stage decision variable $\boldsymbol{x}$, when the random variable $\tilde{\boldsymbol{z}}$ is realized, the second stage problem is to determine the start time such the objective minimum, which can formulated as follows:

$$Q(\boldsymbol{x},\tilde{\boldsymbol{z}}) = \min_{\boldsymbol{t}} \; f(\boldsymbol{x},\tilde{\boldsymbol{z}}) \tag{4}$$

$$\text{s.t } t_i \geq \sum_{j=1}^{n} x_{ij}\tilde{z}_{n+j}, \forall i \in \{1,\ldots,n\} \tag{5}$$

$$t_i \geq t_{i-1} + \sum_{j=1}^{n} x_{i-1,j}\tilde{z}_j, \forall i \in \{2,\ldots,n\} \tag{6}$$

In the first stage, we aim to find a sequence, which immune all distribution in ambiguity set $\mathscr{F}_{\mathrm{MM}}$, to minimize the objective. Hence, we formulate our scheduling problem as the following mini-max model,

$$\min_{\boldsymbol{x}\in\mathscr{X}} \sup_{\mathbb{P}\in\mathscr{F}_{\mathrm{MM}}} \mathbb{E}_{\mathbb{P}}\left[Q(\mathbf{x},\tilde{\mathbf{z}})\right] \tag{7}$$

In this model, we try to find a schedule that minimize the maximal expected makespan or total completion time over the distribution $\mathbb{P}\in\mathscr{F}_{\mathrm{MM}}$.

3. The Total Completion Time Problem

In this section, we mainly focus on solving the total completion time problem. Since the distributionally robust model (7) is hard to solve directly, our methods heavily based on the linear decison rule. The linear decision rule is one of most powerful tools to solve the two stage distributionally robust problem. We will illustrate how the linear decision rule be used in this problem.

3.1. Linear Decision Rule

In this section, we first formulate a new ambiguity set by introducing an auxiliary random variable $\tilde{\boldsymbol{u}}$. Then, we propose a linear decision rule based on the new ambiguity set to approximate original problem (7).

Consider the original ambiguity set, we introduce an epigraphical random varibale $\tilde{\boldsymbol{u}}$ for the term $(\tilde{z}_i-\mu_i)^2, i\in\{1,\ldots,2n\}$, and then we present a new ambiguity set $\mathscr{G}$,

$$\mathscr{G} = \left\{ \mathbb{P}\in\mathscr{P}_0\left(\mathbb{R}^{2n}\times\mathbb{R}^{2n}\right) \;\middle|\; \begin{array}{l} (\tilde{\mathbf{z}},\tilde{\mathbf{u}})\sim\mathbb{P} \\ \mathbb{E}_{\mathbb{P}}[\tilde{\mathbf{z}}]=\boldsymbol{\mu} \\ \mathbb{E}_{\mathbb{P}}\left[\tilde{u}_i\right]\leq\sigma_i^2, \forall i\in\{1,\ldots,2n\} \\ \mathbb{P}\left[(\tilde{\mathbf{z}},\tilde{\mathbf{u}})\in\mathscr{W}_{\mathrm{MM}}\right]=1 \end{array} \right\}$$

and

$$\mathscr{W}_{\mathrm{MM}} = \left\{ (\mathbf{z},\mathbf{u})\in\mathbb{R}^{2n}\times\mathbb{R}^{2n} \;\middle|\; \begin{array}{l} \mathbf{z}\geq\mathbf{0} \\ \mathrm{Case}_{[2n]} \end{array} \right\}$$

with $\mathrm{Case}_{[2n]}: \quad \sqrt{(z_i-\mu_i)^2+\left(\frac{u_i-1}{2}\right)^2}\leq\frac{u_i+1}{2}, \quad \forall i\in\{1,\ldots,2n\}$

Proposition 1. *The ambiguity set, $\mathscr{F}_{\mathrm{MM}}$, is equivalent to the set of marginal distribution of $\tilde{\mathbf{z}}$ under $\mathbb{P}$, for all $\mathbb{P}\in\mathscr{G}$.*

Proof Proposition 1 is a direct result of Proposition 1 in Bertsimas et al. (2019). □

As a consequence of the Proposition 1, we equally reformulate model (7) as below,

$$\min_{\boldsymbol{x}\in\mathscr{X}} \sup_{\mathbb{P}\in\mathscr{G}} \mathbb{E}_{\mathbb{P}}\left[Q(\mathbf{x},(\tilde{\mathbf{z}},\tilde{\mathbf{u}}))\right] \tag{8}$$

However, the model (8) is still hard to solve. However, we note that the second stage decision variable $\boldsymbol{t}$ can be seen as a mapping of the random variable $(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{u}})$, hence, model (8) can be reformulated as follows:

$$\begin{aligned}
&\min_{x\in\mathscr{X}} \sup_{\mathbb{P}\in\mathscr{G}} \mathbb{E}_{\mathbb{P}}\left(\sum_{i=1}^{n}\left[t_i(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{u}})+\sum_{j=1}^{n} x_{ij}\tilde{z}_j\right]\right)\\
&\qquad \text{s.t. } t_i(\boldsymbol{z},\boldsymbol{u}) \geq \sum_{j=1}^{n} x_{ij} z_{n+j}, \forall (\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{\mathrm{MM}}, \forall i\in\{1,\ldots,n\}\\
&\qquad\quad t_i(\boldsymbol{z},\boldsymbol{u}) \geq t_{i-1}(\boldsymbol{z},\boldsymbol{u})+\sum_{j=1}^{n} x_{i-1,j} z_j, \forall (\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{\mathrm{MM}}, \forall i\in\{2,\ldots,n\}\\
&\qquad\quad \boldsymbol{t}\in\mathscr{R}^{4n,n}
\end{aligned}$$

Where $\boldsymbol{t}(\boldsymbol{z},\boldsymbol{u}):\mathbb{R}^{4n}\to\mathbb{R}^n$ is a decision measure function. Note that we do not require $\boldsymbol{t}(\boldsymbol{z},\boldsymbol{u})$ to satisfies any explicity form. In general, above optimization model is an intractable model.

Now, we are ready to present the linear decision rule, which means that the decision measure function has to be a linear function of random variables. By adding this constraint, we could obtain an upper bound of original optimization model. Specifically, the decision variable $\boldsymbol{t}$ is defined as follows,

$$\boldsymbol{t}\in\mathscr{L}^{4n,n}=\left\{\boldsymbol{t}\in\mathscr{R}^{4n,n} \mid \begin{array}{l}\exists \boldsymbol{t}^0,\boldsymbol{t}^{1i},\boldsymbol{t}^{2i}\in\mathbb{R}^n,\forall i\in\{1,\ldots,2n\}\\ \boldsymbol{t}(\boldsymbol{z},\boldsymbol{u})=\boldsymbol{t}^0+\sum_{i=1}^{2n}\left(\boldsymbol{t}^{1i}z_i+\boldsymbol{t}^{2i}u_i\right)\end{array}\right\}$$

Incoporate lifted linear decision rule into model (8), we have the following programming,

$$\begin{aligned}
[LS]\ &\min_{x\in\mathscr{X}} \sup_{\mathbb{P}\in\mathscr{G}} \mathbb{E}_{\mathbb{P}}\left(\sum_{i=1}^{n}\left[t_i(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{u}})+\sum_{j=1}^{n} x_{ij}\tilde{z}_j\right]\right)\\
&\qquad \text{s.t. } t_i(\boldsymbol{z},\boldsymbol{u}) \geq \sum_{j=1}^{n} x_{ij} z_{n+j}, \forall (\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{MM}, \forall i\in\{1,\ldots,n\}\\
&\qquad\quad t_i(\boldsymbol{z},\boldsymbol{u}) \geq t_{i-1}(\boldsymbol{z},\boldsymbol{u})+\sum_{j=1}^{n} x_{i-1,j} z_j, \forall (\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{MM}, \forall i\in\{2,\ldots,n\}\\
&\qquad\quad \boldsymbol{t}\in\mathscr{L}^{4n\times n}
\end{aligned} \tag{9}$$

Next, we aim to provide a tractable reformulation of the problem (9).

3.2. Tractable Reformulation

In this section, we reformulate problem (9) into a mixed integer second order cone programming, which can be solved quickly by the state-of-art commerical solver, such as Gurobi or Mosek. We start our reformulation with considering the inner problem of model (9). Specifically, we first try to reformulate the problem $\sup_{\mathbb{P}\in\mathscr{G}} \mathbb{E}_{\mathbb{P}}\left(\sum_{i=1}^{n}\left[t_i(\tilde{\boldsymbol{z}},\tilde{\boldsymbol{u}})+\sum_{j=1}^{n} x_{ij}\tilde{z}_j\right]\right)$, which can be rewritten as follows optimization problem,

$$
\begin{aligned}
\sup \ & \int_{(\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{MM}} \sum_{i=1}^{n}\left(t_i((\boldsymbol{z},\boldsymbol{u}))+\sum_{j=1}^{n} x_{ij}z_j\right) dF((\boldsymbol{z},\boldsymbol{u})) \\
\text{s.t.} \ & \int_{(\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{MM}} \boldsymbol{z}dF((\boldsymbol{z},\boldsymbol{u}))=\boldsymbol{\mu} \\
& \int_{(\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{MM}} u_i dF((\boldsymbol{z},\boldsymbol{u}))\le \sigma_i^2, \forall i\in\{1,\dots,2n\} \\
& \int_{(\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{MM}} dF((\boldsymbol{z},\boldsymbol{u}))=1 \\
& dF((\boldsymbol{z},\boldsymbol{u}))\ge 0
\end{aligned}
\tag{10}
$$

Here the decision variable is the function $F((\boldsymbol{z},\boldsymbol{u}))$. To solve this problem, we provide the following proposition.

PROPOSITION 2. *The problem* (9) *is equivalent to the following model,*

$$
\begin{aligned}
\inf \ & \boldsymbol{s}^T\boldsymbol{\mu}+\sum_{i=1}^{2n} o_i\sigma_i^2+v \\
s.t. \ & v-\sum_{i=1}^{n} t_i^0 \ge \sum_{j=1}^{2n}\left(\frac{1}{2}(\alpha_j-\beta_j)-\mu_j\tau_j\right) \\
& \tau_j \le s_j-\sum_{i=1}^{n} t_i^{1j}-\sum_{i=1}^{n} x_{ij}, j\in\{1,\dots,n\} \\
& \tau_j \le s_j-\sum_{i=1}^{n} t_i^{1j}, j\in\{n+1,\dots,2n\} \\
& \frac{1}{2}(\alpha_j+\beta_j)=o_j-\sum_{i=1}^{n} t_i^{2j}, j\in\{1,\dots,2n\} \\
& \begin{bmatrix}\alpha_j\\ \beta_j\\ \tau_j\end{bmatrix} \succeq_Q 0, \forall j\in\{1,\dots,2n\} \\
& \mathbf{o}\ge\mathbf{0}
\end{aligned}
\tag{11}
$$

where $\boldsymbol{s},\boldsymbol{o},v,\boldsymbol{t},\alpha_j,\beta_j$ *and* $\tau_j, j\in\{1,\dots,2n\}$ *are decision variables.*

Proof The dual problem of problem (10) can be expressed as follows,

$$
\inf \ \boldsymbol{s}^T\boldsymbol{\mu}+\sum_{i=1}^{2n} o_i\sigma_i^2+v \tag{12}
$$

$$
\text{s.t.} \ \boldsymbol{s}^T\mathbf{z}+\sum_{i=1}^{2n} o_iu_i+v \ge \sum_{i=1}^{n}\left(t_i(\mathbf{z},\mathbf{u})+\sum_{j=1}^{n} x_{ij}z_j\right), \forall(\mathbf{z},\mathbf{u})\in\mathscr{W}_{\mathrm{MM}} \tag{13}
$$

$$\mathbf{o} \geq \mathbf{0}$$
$$\boldsymbol{t} \in \mathscr{L}^{n\times 4n}$$

where $\boldsymbol{s}, o_i$ and v are the dual variable corresponding to the first constraint, the second constraint and the third constraint, respectively. Note that above problem is still an intractable optimization problem as there is an infinite number of inequality constraints in (13). We now rewrite constraint (13) by considering the linear formulation of the decision variable $\boldsymbol{t}$

$$\begin{aligned}
&\boldsymbol{s}^T\mathbf{z} + \sum_{i=1}^{2n} o_i u_i + v \geq \sum_{i=1}^{n}\left(t_i(\boldsymbol{z},\mathbf{u}) + \sum_{j=1}^{n} x_{ij}z_j\right), \forall(\mathbf{z},\mathbf{u}) \in \mathscr{W}_{\mathrm{MM}} \\
\Leftrightarrow &\boldsymbol{s}^T\mathbf{z} + \sum_{i=1}^{2n} o_i u_i + v \geq \sum_{i=1}^{n}\left(t_i^0 + \sum_{j=1}^{2n}(t_i^{1j}z_j + t_i^{2j}u_j) + \sum_{j=1}^{n} x_{ij}z_j\right), \forall(\mathbf{z},\mathbf{u}) \in \mathscr{W}_{\mathrm{MM}} \\
\Leftrightarrow &v - \sum_{i=1}^{n} t_i^0 \geq \sum_{i=1}^{n}\left(\sum_{j=1}^{2n}(t_i^{1j}z_j + t_i^{2j}u_j) + \sum_{j=1}^{n} x_{ij}z_j\right) - \boldsymbol{s}^T\mathbf{z} - \sum_{i=1}^{2n} o_i u_i, \forall(\mathbf{z},\mathbf{u}) \in \mathscr{W}_{\mathrm{MM}} \\
\Leftrightarrow &v - \sum_{i=1}^{n} t_i^0 \geq \max_{(\mathbf{z},\mathbf{u})\in\mathscr{W}_{\mathrm{MM}}} \left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{2n}(t_i^{1j}z_j + t_i^{2j}u_j) + \sum_{j=1}^{n} x_{ij}z_j\right) - \boldsymbol{s}^T\mathbf{z} - \sum_{i=1}^{2n} o_i u_i\right\}
\end{aligned} \tag{14}$$

Then, we write the maximum problem in the expression (14),

$$\begin{aligned}
\max_{(\mathbf{z},\mathbf{u})} \ &\left\{\sum_{i=1}^{n}\left(\sum_{j=1}^{2n}(t_i^{1j}z_j + t_i^{2j}u_j) + \sum_{j=1}^{n} x_{ij}z_j\right) - \boldsymbol{s}^T\mathbf{z} - \sum_{i=1}^{2n} o_i u_i\right\} \\
\text{s.t } &\boldsymbol{z} \geq 0 \\
&\sqrt{(z_i - \mu_i)^2 + \left(\frac{u_i - 1}{2}\right)^2} \leq \frac{u_i + 1}{2}, \quad \forall i \in \{1, \ldots, 2n\}
\end{aligned} \tag{15}$$

Take dual of this problem, we could obtain the following problem,

$$\begin{aligned}
\inf \ &\sum_{j=1}^{2n}\left(\frac{1}{2}(\alpha_j - \beta_j) - \mu_j\tau_j\right) \\
\text{s.t. } &\tau_j \leq s_j - \sum_{i=1}^{n} t_i^{1j} - \sum_{i=1}^{n} x_{ij}, j \in \{1, \ldots, n\} \\
&\tau_j \leq s_j - \sum_{i=1}^{n} t_i^{1j}, j \in \{n+1, \ldots, 2n\} \\
&\frac{1}{2}(\alpha_j + \beta_j) = o_j - \sum_{i=1}^{n} t_i^{2j}, j \in \{1, \ldots, 2n\} \\
&\begin{bmatrix}\alpha_j \\ \beta_j \\ \tau_j\end{bmatrix} \succeq_Q 0, \forall j \in \{1, \ldots, 2n\}
\end{aligned} \tag{16}$$

where α_j, β_j and $\tau_j, \forall j \in \{1, \ldots, 2n\}$, are the decision variables. Replace (15) with (15) in (14) and incoporate these results in (12), the desire result are obtained immediately. $\square$

As a result of the Proposition 2, the model (11) can be expressed as follows,

$$\inf\ \boldsymbol{s}^T\boldsymbol{\mu}+\sum_{i=1}^{2n} o_i\sigma_i^2+v \tag{17}$$

$$\text{s.t. } t_i(\boldsymbol{z},\boldsymbol{u})\geq\sum_{j=1}^{n} x_{ij}z_{n+j}, \forall(\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{\text{MM}}, \forall i\in\{1,\ldots,n\} \tag{18}$$

$$t_i(\boldsymbol{z},\boldsymbol{u})\geq t_{i-1}(\boldsymbol{z},\boldsymbol{u})+\sum_{j=1}^{n} x_{i-1,j}z_j, \forall(\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{\text{MM}}, \forall i\in\{2,\ldots,n\} \tag{19}$$

$$v-\sum_{i=1}^{n} t_i^0\geq\sum_{j=1}^{2n}\left(\frac{1}{2}(\alpha_j-\beta_j)-\mu_j\tau_j\right)$$

$$\tau_j\leq s_j-\sum_{i=1}^{n} t_i^{1j}-\sum_{i=1}^{n} x_{ij}, j\in\{1,\ldots,n\}$$

$$\tau_j\leq s_j-\sum_{i=1}^{n} t_i^{1j}, j\in\{n+1,\ldots,2n\}$$

$$\frac{1}{2}(\alpha_j+\beta_j)=o_j-\sum_{i=1}^{n} t_i^{2j}, j\in\{1,\ldots,2n\}$$

$$\begin{bmatrix}\alpha_j\\ \beta_j\\ \tau_j\end{bmatrix}\succeq_Q 0, \forall j\in\{1,\ldots,2n\}$$

$$\mathbf{o}\geq\mathbf{0}, \mathbf{x}\in\mathscr{X}$$

$$\boldsymbol{t}\in\mathscr{R}^{4n,n}$$

Note that we still have infinity constraints in (18) and (19). We now prepare to provide their robust counterparts. We first rewrite constraint (18) as follows,

$$t_i(\boldsymbol{z},\boldsymbol{u})\geq\sum_{j=1}^{n} x_{ij}z_{n+j}, \forall(\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{\text{MM}}, \forall i\in\{1,\ldots,n\}$$
$$\Leftrightarrow t_i^0\geq\max_{(\boldsymbol{z},\boldsymbol{u})\in\mathscr{W}_{\text{MM}}}\left\{\sum_{j=1}^{n} x_{ij}z_{n+j}-\sum_{j=1}^{2n}(t_i^{1j}z_i+t_i^{2j}u_j)\right\}, \forall i\in\{1,\ldots,n\}$$

Consider the problem

$$\begin{aligned}\max_{(\boldsymbol{z},\boldsymbol{u})}\ &\left\{\sum_{j=1}^{n} x_{ij}z_{n+j}-\sum_{j=1}^{2n}(t_i^{1j}z_i+t_i^{2j}u_j)\right\}\\ \text{s.t }&\boldsymbol{z}\geq 0\\ &\sqrt{(z_i-\mu_i)^2+\left(\frac{u_i-1}{2}\right)^2}\leq\frac{u_i+1}{2}, \quad \forall i\in\{1,\ldots,2n\}\end{aligned}$$

Take dual, we obtain the following optimization problem,

$$\min\ \sum_{j=1}^{2n}\left(\frac{1}{2}(a_j^i-b_j^i)-\mu_jc_j^i\right) \tag{20}$$

$$
\begin{aligned}
& c_j^i \le t_i^{1j}, \forall j \in \{1,\dots,n\} \\
& c_j^i \le t_i^{1j} - x_{ij}, \forall j \in \{n+1,\dots,2n\} \\
& \frac{1}{2}(a_j^i + b_j^i) = t_i^{2j}, j \in \{1,\dots,2n\} \\
& \begin{bmatrix} a_j^i \\ b_j^i \\ c_j^i \end{bmatrix} \succeq_Q 0, \forall j \in \{1,\dots,2n\}
\end{aligned}
$$

Next, we rewrite the (19) as follows,

$$
\begin{aligned}
& t_i(\boldsymbol{z},\boldsymbol{u}) \ge t_{i-1}(\boldsymbol{z},\boldsymbol{u}) + \sum_{j=1}^{n} x_{i-1,j} z_j, \forall (\boldsymbol{z},\boldsymbol{u}) \in \mathscr{W}_{\mathrm{MM}}, \forall i \in \{2,\dots,n\} \\
\Leftrightarrow & t_i^0 - t_{i-1}^0 \ge \max_{(\boldsymbol{z},\boldsymbol{u}) \in \mathscr{W}_{\mathrm{MM}}} \left\{ \sum_{j=1}^{2n} \left[(t_{i-1}^{1j} - t_i^{1j}) z_i + (t_{i-1}^{2j} - t_i^{2j}) u_j) \right] + \sum_{j=1}^{n} x_{i-1,j} z_j, \forall i \in \{2,\dots,n\} \right\}
\end{aligned}
$$

Consider the problem

$$
\begin{aligned}
\max_{(\boldsymbol{z},\boldsymbol{u})} \quad & \left\{ \sum_{j=1}^{2n} \left[(t_{i-1}^{1j} - t_i^{1j}) z_i + (t_{i-1}^{2j} - t_i^{2j}) u_j) \right] + \sum_{j=1}^{n} x_{i-1,j} z_j \right\} \\
& \boldsymbol{z} \ge 0 \\
& \sqrt{(z_i - \mu_i)^2 + \left(\frac{u_i - 1}{2} \right)^2} \le \frac{u_i + 1}{2}, \quad \forall i \in \{1,\dots,2n\}
\end{aligned}
$$

Take dual, we obtain

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{2n} \left(\frac{1}{2}(d_j^{i-1} - e_j^{i-1}) - \mu_j f_j^{i-1} \right) \\
& f_j^{i-1} \le t_i^{1j} - t_{i-1}^{1j} - x_{i-1,j}, j \in \{1,\dots,n\} \\
& f_j^{i-1} \le t_i^{1j} - t_{i-1}^{1j}, j \in \{n+1,\dots,2n\} \\
& \frac{1}{2}(d_j^{i-1} + e_j^{i-1}) = t_i^{2j} - t_{i-1}^{2j}, \forall j \in \{1,\dots,2n\} \\
& \begin{bmatrix} d_j^{i-1} \\ e_j^{i-1} \\ f_j^{i-1} \end{bmatrix} \succeq_Q 0, \forall j \in \{1,\dots,2n\}
\end{aligned}
\tag{21}
$$

Incoporate these results into problem (17), we obtain the main result of this paper.

Theorem 1. $[LS]$ *is equivalent to the model* $[LS - SOCP]$

$$
\begin{aligned}
[LS - SOCP] \ \inf \ & \boldsymbol{s}^T \boldsymbol{\mu} + \sum_{i=1}^{2n} o_i \sigma_i^2 + v \\
\text{s.t. } & v - \sum_{i=1}^{n} t_i^0 \ge \sum_{j=1}^{2n} \left(\frac{1}{2}(\alpha_j - \beta_j) - \mu_j \tau_j \right) \\
& \tau_j \le s_j - \sum_{i=1}^{n} t_i^{1j} - \sum_{i=1}^{n} x_{ij}, j \in \{1,\dots,n\}
\end{aligned}
$$

$$
\begin{aligned}
&\tau_j \leq s_j - \sum_{i=1}^{n} t_i^{1j}, j \in \{n+1,\ldots,2n\} \\
&\frac{1}{2}(\alpha_j + \beta_j) = o_j - \sum_{i=1}^{n} y_i^{2j}, j \in \{1,\ldots,2n\} \\
&t_i^0 \geq \sum_{j=1}^{2n} \left(\frac{1}{2}(a_j^i - b_j^i) - \mu_j c_j^i \right), \forall i \in \{1,\ldots,n\} \\
&c_j^i \leq t_i^{1j}, \forall i \in \{1,\ldots,n\}, j \in \{1,\ldots,n\} \\
&c_j^i \leq t_i^{1j} - x_{ij}, \forall i \in \{1,\ldots,n\}, j \in \{n+1,\ldots,2n\} \\
&\frac{1}{2}(a_j^i + b_j^i) = t_i^{2j}, \forall i \in [n], j \in \{1,\ldots,2n\} \\
&\begin{bmatrix} a_j^i \\ b_j^i \\ c_j^i \end{bmatrix} \succeq_Q 0, \forall i \in \{1,\ldots,n\}, \forall j \in \{1,\ldots,2n\}, \begin{bmatrix} \alpha_j \\ \beta_j \\ \tau_j \end{bmatrix} \succeq_Q 0, \forall j \in \{1,\ldots,2n\} \\
&t_i^0 - t_{i-1}^0 \geq \sum_{j=1}^{2n} \left(\frac{1}{2}(d_j^{i-1} - e_j^{i-1}) - \mu_j f_j^{i-1} \right), \forall i \in \{2,\ldots,n\} \\
&f_j^{i-1} \leq t_i^{1j} - t_{i-1}^{1j} - x_{i-1,j}, \forall i \in \{2,\ldots,n\}, j \in \{1,\ldots,n\} \\
&f_j^{i-1} \leq t_i^{1j} - t_{i-1}^{1j}, \forall i \in \{2,\ldots,n\}, j \in \{n+1,\ldots,2n\} \\
&\frac{1}{2}(d_j^{i-1} + e_j^{i-1}) = t_i^{2j} - t_{i-1}^{2j}, \forall i \in \{2,\ldots,n\}, \forall j \in \{1,\ldots,2n\} \\
&\begin{bmatrix} d_j^{i-1} \\ e_j^{i-1} \\ f_j^{i-1} \end{bmatrix} \succeq_Q 0, \forall i \in \{2,\ldots,n\}, \forall j \in \{1,\ldots,2n\} \\
&\mathbf{o} \geq \mathbf{0}, \mathbf{x} \in \mathscr{X}
\end{aligned}
$$

where the $\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{o}, v, \boldsymbol{t}^0, \boldsymbol{t}^1, \boldsymbol{t}^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\tau}$, $\boldsymbol{a}^i, \boldsymbol{b}^i, \boldsymbol{c}^i, \forall i \in \{1,\ldots,n\}$, and $\boldsymbol{d}^{i-1}, \boldsymbol{e}^{i-1}, \boldsymbol{f}^{i-1}, \forall i \in \{2,\ldots,n\}$ are the decision variables.

3.3. Two Special Problems

In above section, we provide a tractable reformulation for the general problem, that is both processing time and release time are stochastic. In fact, our methods can also tackle the problem which either processing time or release time is stochastic. In this section, we show how to reformulate them into a tractable optimization problem.

3.3.1. Stochastic Release Time In this subsection, suppose release time is stochastic and the processing time is fixed, the second stage problem can be expressed as follows,

$$
\begin{aligned}
Q(\boldsymbol{x}, \tilde{\boldsymbol{r}}) = \min_{\boldsymbol{t}} \quad & \sum_{i=1}^{n} \left[t_i + \sum_{j=1}^{n} x_{ij} p_j \right] \\
\text{s.t } & t_i \geq \sum_{j=1}^{n} x_{ij} \tilde{r}_j, \forall i \in \{1,\ldots,n\}
\end{aligned}
$$

$$t_i \geq t_{i-1} + \sum_{j=1}^{n} x_{i-1,j} p_j, \forall i \in \{2,\ldots,n\}$$

Then, the distributionally robust model can be formulated below,

$$\min_{\boldsymbol{x}\in\mathscr{X}} \sup_{\mathbb{P}\in\mathscr{G}} \mathbb{E}_{\mathbb{P}}\left[Q(\mathbf{x},\tilde{\mathbf{r}})\right] \tag{22}$$

Using the linear decision rule, model (22) can be approximated by the following optimization model. Note that we omit the details of the reformulation, since it is similar with the reformulation of the model $[LS-SOCP]$

$$\begin{aligned}
[SR] \ \inf \ & \boldsymbol{s}^T\boldsymbol{\mu}^r + \sum_{i=1}^{n} o_i(\sigma_i^r)^2 + v \\
\text{s.t.} \ & v - \sum_{i=1}^{n} t_i^0 - \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}p_j \geq \sum_{j=1}^{n}\left(\frac{1}{2}(\alpha_j-\beta_j) - \mu_j^r\tau_j\right) \\
& \tau_j \leq s_j - \sum_{i=1}^{n} t_i^{1j}, \forall j \in \{1,\ldots,n\} \\
& \frac{1}{2}(\alpha_j+\beta_j) = o_j - \sum_{i=1}^{n} t_i^{2j}, \forall j \in \{1,\ldots,n\} \\
& t_i^0 \geq \sum_{j=1}^{n}\left(\frac{1}{2}(a_j^i - b_j^i) - \mu_j^r c_j^i\right), \forall i \in \{1,\ldots,n\} \\
& c_j^i \leq t_i^{1j} - x_{ij}, \forall i \in \{1,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \frac{1}{2}(a_j^i + b_j^i) = t_i^{2j}, \forall i \in \{1,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \begin{bmatrix} a_j^i \\ b_j^i \\ c_j^i \end{bmatrix} \succeq_Q 0, \forall i \in \{1,\ldots,n\}, \forall j \in \{1,\ldots,n\}, \begin{bmatrix} \alpha_j \\ \beta_j \\ \tau_j \end{bmatrix} \succeq_Q 0, \forall j \in \{1,\ldots,n\} \\
& t_i^0 - t_{i-1}^0 - \sum_{j=1}^{n} x_{i-1,j}p_j \geq \sum_{j\in[n]}\left(\frac{1}{2}(d_j^{i-1} - e_j^{i-1}) - \mu_j^r f_j^{i-1}\right), \forall i \in \{2,\ldots,n\} \\
& f_j^{i-1} \leq t_i^{1j} - t_{i-1}^{1j}, \forall i \in \{2,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \frac{1}{2}(d_j^{i-1} + e_j^{i-1}) = t_i^{2j} - t_{i-1}^{2j}, \forall i \in \{2,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \begin{bmatrix} d_j^{i-1} \\ e_j^{i-1} \\ f_j^{i-1} \end{bmatrix} \succeq_Q 0, \forall i \in \{2,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \mathbf{o} \geq \mathbf{0}, \mathbf{x} \in \mathscr{X}
\end{aligned}$$

3.3.2. Stochastic Processing Time In this subsection, we assume that only processing time is stochastic, the second stage problem is written as follows,

$$Q(\boldsymbol{x},\tilde{\boldsymbol{p}}) = \min_{\boldsymbol{t}} \ \sum_{i=1}^{n}\left[t_i + \sum_{j=1}^{n} x_{ij}\tilde{p}_j\right]$$

$$
\begin{aligned}
\text{s.t } t_i &\geq \sum_{j=1}^{n} x_{ij} r_j, \forall i \in \{1,\ldots,n\} \\
t_i &\geq t_{i-1} + \sum_{j=1}^{n} x_{i-1,j} \tilde{p}_j, \forall i \in \{2,\ldots,n\}
\end{aligned}
$$

Then, the distributionally robust model can be formulated below,

$$
\min_{\boldsymbol{x}\in\mathscr{X}} \sup_{\mathbb{P}\in\mathscr{G}} \mathbb{E}_{\mathbb{P}}\left[Q(\mathbf{x},\tilde{\mathbf{p}})\right] \tag{23}
$$

This problem also can be approximated by a mixed integer second order cone programming, which is expressed below,

$$
\begin{aligned}
[SP]\ \inf\ & \boldsymbol{s}^T \boldsymbol{\mu}^p + \sum_{i=1}^{n} o_i (\sigma_i^p)^2 + v \\
\text{s.t. } & v - \sum_{i=1}^{n} t_i^0 \geq \sum_{j=1}^{n} \left(\frac{1}{2}(\alpha_j - \beta_j) - \mu_j^p \tau_j \right) \\
& \tau_j \leq s_j - \sum_{i=1}^{n} (t_i^{1j} - x_{ij}), \forall j \in \{1,\ldots,n\} \\
& \frac{1}{2}(\alpha_j + \beta_j) = o_j - \sum_{i=1}^{n} t_i^{2j}, \forall j \in \{1,\ldots,n\} \\
& t_i^0 - \sum_{j=1}^{n} x_{ij} r_j \geq \sum_{j=1}^{n} \left(\frac{1}{2}(a_j^i - b_j^i) - \mu_j^p c_j^i \right), \forall i \in \{1,\ldots,n\} \\
& c_j^i \leq t_i^{1j}, \forall i \in \{1,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \frac{1}{2}(a_j^i + b_j^i) = t_i^{2j}, \forall i \in \{1,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \begin{bmatrix} a_j^i \\ b_j^i \\ c_j^i \end{bmatrix} \succeq_Q 0, \forall i \in \{1,\ldots,n\}, \forall j \in \{1,\ldots,n\}, \begin{bmatrix} \alpha_j \\ \beta_j \\ \tau_j \end{bmatrix} \succeq_Q 0, \forall j \in \{1,\ldots,n\} \\
& t_i^0 - t_{i-1}^0 \geq \sum_{j\in[n]} \left(\frac{1}{2}(d_j^{i-1} - e_j^{i-1}) - \mu_j^p f_j^{i-1} \right), \forall i \in \{2,\ldots,n\} \\
& f_j^{i-1} \leq t_i^{1j} - t_{i-1}^{1j} - x_{i-1,j}, \forall i \in \{2,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \frac{1}{2}(d_j^{i-1} + e_j^{i-1}) = t_i^{2j} - t_{i-1}^{2j}, \forall i \in \{2,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \begin{bmatrix} d_j^{i-1} \\ e_j^{i-1} \\ f_j^{i-1} \end{bmatrix} \succeq_Q 0, \forall i \in \{2,\ldots,n\}, \forall j \in \{1,\ldots,n\} \\
& \mathbf{o} \geq \mathbf{0}, \mathbf{x} \in \mathscr{X}
\end{aligned}
$$

3.4. Numerical experiments

In this section, we conduct numerical experiments to validate the effectness of the distributionally robust scheduling models. Due the limited space, we only consider two types of stochastic cases in our numerical experiments. In section 3.4.2, we assume that processing time has no randomness and only release time is stochastic. In section 3.4.3, we assume that both processing time and release time are stochastic. Our experiments are conducted on a ThinkPad X1 PC, with an Intel (R) Core (TM) i7-10710 U CPU running at 1.10 GigaHertz and a 16.00 Gigabyte of memory. The overall framework was implemented with the programming language Python 3.7, and the optimization models are solved by Gurobi.

3.4.1. Experiment Description For our experiment, we choose number of jobs n are choose from the $\{8,9,10\}$. In the case of 3.4.2, the true processing time $\boldsymbol{p}$ are drawn from $U[10,50]$, while in the case of 3.4.3, we draw the true mean processing time μ_p from $U[10,50]$. The true mean release time μ_r are generated from the $[0, n*30]$ for both cases. We define $cv = \frac{\sigma}{\mu}$ by the coefficient of variation, which is chosen from the set $\{0.3, 0.7, 1.0\}$. Therefore, the variance of stochastic processing time and release time are $\boldsymbol{\sigma}_p = cv * \boldsymbol{\mu}_p$, $\boldsymbol{\sigma}_r = cv * \boldsymbol{\mu}_r$, respectively. Note that, for the simplicity, we do not set different cv for release time and processing time, and the the cv is same among all jobs. To ensure fairness, we generate K historical data from a given distribution with true mean and variance. We consider three types of commonly used distribution: truncated normal distribution, log-normal distribution and gamma distribution. Specifically, for a given distribution, in section 3.4.2, we generate K observations of release time $\{\mathbf{r}^1, \ldots, \mathbf{r}^K\}$ and processing time $\mathbf{p}$ are known in advance. In section 3.4.3, we observe K records of release time and processing time, i.e., $\{(\mathbf{p}^1, \mathbf{r}^1), \ldots, (\mathbf{p}^K, \mathbf{r}^K)\}$. Then, we are able to calculate sample mean $\hat{\boldsymbol{\mu}}_r, \hat{\boldsymbol{\mu}}_p$ and sample variance $\hat{\boldsymbol{\sigma}}_r, \hat{\boldsymbol{\sigma}}_p$ for release time and processing time, which are used to solve the distributionally robust model.

We compare the performance of our distributionally robust model and the deterministic model, as well as sample average approximation (SAA). In deterministic model, we solve the model (4) by replacing the random variable with their sample mean value. SAA has been widely used in stochastic optimization problems. In our problem, SAA can be formulated as a determinsitic model, which is expressed as follows,

$$
\begin{aligned}
[SAA]\ \min_{\boldsymbol{t}^k, \boldsymbol{x}}\ & \frac{1}{K} \sum_{k=1}^{K} \sum_{i=1}^{n} \left[t_i^k + \sum_{j=1}^{n} x_{ij} \hat{p}_j^k \right] \\
\text{s.t}\ & t_i^k \geq \sum_{j=1}^{n} x_{ij} r_j^k, \forall i \in \{1, \ldots, n\}, \forall k \in \{1, \ldots, K\} \\
& t_i^k \geq t_{i-1}^k + \sum_{j=1}^{n} x_{i-1,j} \hat{p}_j^k, \forall i \in \{1, \ldots, n\}, \forall k \in \{1, \ldots, K\} \\
& \boldsymbol{x} \in \mathscr{X}
\end{aligned}
$$

Note that $\hat{p}_j^k$ is same for all $k \in \{1, \ldots, K\}$, i.e., $\hat{p}_j^k = p_j$ for all $k \in \{1, \ldots, K\}$ in section 3.4.2, but in section 3.4.3, the $\hat{p}_j^k$ is different for different $k \in \{1, \ldots, K\}$, i.e., $\hat{p}_j^k = p_j^k, k \in \{1, \ldots, K\}$. In the distributionally robust model, i.e., $[LS-SOCP]$, based on the sample mean and sample variance. Via

Table 1 **Total Completion Time Results under Stochastic Release Time When cv = 0.3**

Dist	n	SAA				DRO			
		Gap^m_{saa}	Gap^{75}_{saa}	Gap^{95}_{saa}	Gap^{99}_{saa}	Gap^m_{dro}	Gap^{75}_{dro}	Gap^{95}_{dro}	Gap^{99}_{dro}
Log-Normal	8	0.41%	0.76%	1.51%	1.90%	0.39%	0.94%	2.19%	3.14%
	9	0.85%	1.52%	3.06%	4.11%	0.87%	1.71%	3.38%	4.33%
	10	1.66%	2.67%	4.71%	5.80%	1.61%	2.79%	5.03%	6.34%
Normal	8	1.05%	1.82%	3.50%	4.54%	0.91%	2.16%	4.35%	5.66%
	9	1.25%	1.90%	3.77%	4.69%	1.19%	2.03%	4.31%	5.58%
	10	1.35%	1.97%	3.08%	3.86%	1.37%	2.36%	3.96%	5.11%
Gamma	8	1.13%	1.65%	3.48%	4.61%	1.11%	1.71%	3.59%	4.89%
	9	1.08%	1.63%	2.68%	3.14%	0.94%	1.82%	3.12%	3.75%
	10	1.50%	2.15%	3.40%	4.19%	1.38%	2.39%	3.90%	4.75%

solving deterministic model, SAA and distributionally robust model, we obtain their corresponding sequence, i.e., Seq^d, Seq^s, and Seq^{dro}, respectively.

To evaluate performance of Seq^d, Seq^s, and Seq^{dro}, we compute their corresponding total completion time as follows. We randomly draw 10,000 samples from a given distribution. For each sample, we compute their total completion time corresponding to Seq^d, Seq^s, and Seq^{dro}, respectively. Then, we calculate the following four measures over the 10,000 samples:

- means, denoted by M_d, M_s and M_{dro}
- t-th percentiles, denoted by PT^t_d, PT^t_s and PT^t_{dro} for $t = 75, 95$ and $t = 99$, respectively

Then, we use the values of four measure for determinsitc as our benchmark and compute the gap between SAA and deterministic model and the gap between distributionally robust model and deterministic model. Specifically, they can be compute by following expressions:

$$Gap^m_{saa} = \frac{M_d - M_s}{M_d}, Gap^m_{dro} = \frac{M_d - M_{dro}}{M_d}$$
$$Gap^t_{saa} = \frac{PT^t_d - PT^t_s}{PT^t_d}, Gap^t_{dro} = \frac{PT^t_d - PT^t_{dro}}{PT^t_d}, t = 75, 95, 99$$

For each fixed number of jobs and types of distribution, we generate 20 instances. The average results over 20 instances in each cases are calculated.

3.4.2. Numerical Results on Stochastic Release Time All numerical results are reported in Table 1, 2, and Table 3. Note that performance of deterministic model are the benchmark, we do not provide their results. As shown in these three tables, both SAA and DRO dominate the deterministic model in all measurements. The reason is the deterministic do not account for the variance in the model.

3.4.3. Stochastic release time and processing time In this section, the processing time is stochastic too. Hence, we generate the mean processing time $\mu^j_p, j \in \{1, \ldots, n\}$ from the uniform

Table 2 Total Completion Time Results under Stochastic Release Time when cv = 0.7

Dist	n	SAA				DRO			
		Gap^m_{saa}	Gap^{75}_{saa}	Gap^{95}_{saa}	Gap^{99}_{saa}	Gap^m_{dro}	Gap^{75}_{dro}	Gap^{95}_{dro}	Gap^{99}_{dro}
Log-Normal	8	1.81%	2.96%	6.42%	8.61%	1.50%	2.97%	7.35%	9.58%
	9	1.63%	3.27%	5.35%	6.25%	1.69%	3.89%	7.32%	9.12%
	10	2.27%	3.25%	6.98%	10.15%	2.44%	4.11%	9.65%	13.67%
Normal	8	2.66%	4.49%	7.68%	9.29%	2.61%	4.77%	8.38%	10.18%
	9	1.91%	3.02%	4.87%	5.78%	1.89%	3.38%	5.70%	6.87%
	10	1.74%	2.93%	5.35%	6.59%	1.77%	3.52%	6.46%	8.10%
Gamma	8	2.22%	3.56%	5.89%	7.23%	2.40%	4.49%	9.40%	12.34%
	9	3.37%	6.04%	8.31%	8.97%	3.71%	6.58%	10.06%	11.62%
	10	1.73%	2.60%	3.99%	4.97%	1.31%	3.18%	5.78%	7.24%

Table 3 Total Completion Time Results under Stochastic Release Time when cv = 1

Dist	n	SAA				DRO			
		Gap^m_{saa}	Gap^{75}_{saa}	Gap^{95}_{saa}	Gap^{99}_{saa}	Gap^m_{dro}	Gap^{75}_{dro}	Gap^{95}_{dro}	Gap^{99}_{dro}
Log-Normal	8	2.60%	4.27%	8.48%	10.98%	3.16%	5.32%	11.90%	15.98%
	9	2.37%	3.43%	8.59%	13.07%	2.53%	3.80%	10.30%	16.58%
	10	2.98%	4.16%	8.60%	11.91%	3.02%	4.78%	9.53%	12.63%
Normal	8	5.08%	8.05%	12.81%	15.00%	5.05%	8.33%	13.76%	16.30%
	9	3.77%	5.67%	9.78%	12.10%	3.55%	5.92%	11.34%	14.22%
	10	2.57%	4.03%	7.55%	9.23%	2.49%	4.32%	8.48%	10.78%
Gamma	8	3.63%	5.33%	9.52%	11.95%	3.57%	6.42%	12.20%	16.04%
	9	3.72%	5.66%	11.56%	14.53%	3.80%	6.17%	13.15%	17.06%
	10	5.86%	8.29%	14.73%	18.68%	5.49%	8.43%	15.96%	20.51%

Table 4 Total Completion Time Results under Stochastic Release and Processing Time

Dist	cv	SAA				DRO			
		Gap^m_{saa}	Gap^{75}_{saa}	Gap^{95}_{saa}	Gap^{99}_{saa}	Gap^m_{dro}	Gap^{75}_{dro}	Gap^{95}_{dro}	Gap^{99}_{dro}
Log-Normal	0.3	0.85%	1.15%	2.97%	4.71%	0.61%	1.09%	3.64%	5.94%
	0.7	1.52%	2.65%	6.00%	7.84%	0.77%	2.32%	6.26%	8.42%
	1	0.84%	1.72%	3.82%	5.53%	0.52%	1.73%	5.25%	7.53%
Normal	0.3	0.56%	0.85%	1.21%	1.55%	0.42%	0.96%	1.47%	1.88%
	0.7	2.52%	3.73%	5.76%	6.96%	2.27%	3.79%	6.22%	7.77%
	1	3.36%	4.04%	5.69%	6.83%	3.10%	4.02%	5.79%	6.80%
Gamma	0.3	0.42%	0.83%	1.81%	2.60%	0.38%	0.86%	1.92%	2.75%
	0.7	1.93%	3.16%	6.72%	9.03%	1.90%	3.56%	7.24%	9.56%
	1	2.24%	3.78%	6.38%	7.59%	2.04%	4.05%	8.84%	11.47%

distribution over $[10, 50]$ and the standard deviation of processing time $\delta_p^j = \Delta * \mu_p^j, j \in \{1, \dots, n\}$. The mean release time of jobs are drawn from uniform distribution over $[0, 30n]$ and the standard deviation of release time $\delta_r^j = \Delta * \mu_r^j, j \in \{1, \dots, n\}$.

4. The Makespan Problem

In this section, we show that linear decision rule can be used to solve makespan problem. In fact, the only difference between total completion time and makespan problem is the objective function.

4.1. Tractable Reformulation

In this section, we provide a tractable reformulation for the makespan problem where both processing time and release time are stochastic. Under the lifted ambiguity set, the distributionally robust makespan problem can be approximately expressed,

$$
\begin{aligned}
[DROML] \ & \min_{x \in \mathscr{X}} \sup_{\mathbb{P} \in \mathscr{G}} \mathbb{E}_{\mathbb{P}} \left(t_n(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{u}}) + \sum_{j=1}^{n} x_{nj} \tilde{z}_j \right) \\
& \text{s.t. } t_i(\boldsymbol{z}, \boldsymbol{u}) \geq \sum_{j=1}^{n} x_{ij} z_{n+j}, \forall (\boldsymbol{z}, \boldsymbol{u}) \in \mathscr{W}_{MM}, \forall i \in \{1, \ldots, n\} \\
& \quad t_i(\boldsymbol{z}, \boldsymbol{u}) \geq t_{i-1}(\boldsymbol{z}, \boldsymbol{u}) + \sum_{j=1}^{n} x_{i-1,j} z_j, \forall (\boldsymbol{z}, \boldsymbol{u}) \in \mathscr{W}_{MM}, \forall i \in \{1, \ldots, n\} \\
& \quad \boldsymbol{t}(\cdot) \in \mathscr{L}^{4n \times n}
\end{aligned}
\tag{24}
$$

Using the same technical that we have used in the total completion time, we give the equivalent model

$$
\begin{aligned}
[DROML-S] \ & \inf \ \boldsymbol{s}^T \boldsymbol{\mu} + \sum_{i=1}^{2n} o_i \sigma_i^2 + v \\
& \text{s.t. } v - t_n^0 \geq \sum_{j \in [2n]} \left(\frac{1}{2}(-\beta_j + \alpha_j) - \mu_j \tau_j \right) \\
& \quad -\tau_j \geq t_n^{1j} + x_{nj} - s_j, j \in \{1, \ldots, n\} \\
& \quad -\tau_j \geq t_n^{1j} - s_j, j \in \{n+1, \ldots, 2n\} \\
& \quad -\frac{1}{2}(\alpha_j + \beta_j) = t_i^{2j} - o_j, j \in [2n] \\
& \quad t_i^0 \geq \sum_{j \in [2n]} \left(\frac{1}{2}(-b_j^i + a_j^i) - \mu_j c_j^i \right), \forall i \in [n] \\
& \quad -c_j^i \geq -t_i^{1j}, \forall i \in [n], j \in \{1, \ldots, n\} \\
& \quad -c_j \geq x_{ij} - t_i^{1j}, \forall i \in [n], j \in \{n+1, \ldots, 2n\} \\
& \quad -\frac{1}{2}(a_j^i + b_j^i) = -t_i^{2j}, \forall i \in [n], j \in \{n+1, \ldots, 2n\} \\
& \quad \begin{bmatrix} a_j^i \\ b_j^i \\ c_j^i \end{bmatrix} \succeq_Q 0, \forall i \in [n], \forall j \in [2n], \begin{bmatrix} \alpha_j \\ \beta_j \\ \tau_j \end{bmatrix} \succeq_Q 0, \forall j \in [2n] \\
& \quad t_i^0 - t_{i-1}^0 \geq \sum_{j \in [2n]} \left(\frac{1}{2}(-e_j^{i-1} + d_j^{i-1}) - \mu_j f_j^{i-1} \right), \forall i \in \{2, \ldots, n\} \\
& \quad -f_j^{i-1} \geq t_{i-1}^{1j} - t_i^{1j} + x_{i-1,j}, \forall i \in \{2, \ldots, n\}, j \in \{1, \ldots, n\}
\end{aligned}
$$

$$
\begin{aligned}
& -f_j^{i-1} \geq t_{i-1}^{1j} - t_i^{1j}, \forall i \in \{2,\ldots,n\}, j \in \{n+1,\ldots,2n\} \\
& -\frac{1}{2}(d_j^{i-1} + e_j^{i-1}) = t_{i-1}^{2j} - t_i^{2j}, \forall i \in \{2,\ldots,n\}, \forall j \in [2n] \\
& \mathbf{o} \geq \mathbf{0}, \mathbf{x} \in \mathscr{X}
\end{aligned}
$$

THEOREM 2. *$[DROML]$ is equivalent to the model $[DROML-S]$.*

Proof We omit the proof since it is similar with the proof of Theorem 1. □

4.2. Two Special Cases

Jus like the special cases in total completion time problem, linear decision rule can be applied in the setting where either processing time or release time is stochastic. We do not provide the specifical demonstration here.

5. Parallel Machine Scheduling Problems

In fact, our approach also can be used to solve the parallel machine scheduling problem.

6. Conclusion

tbd

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