PHYSICS SL
Barton's Pendulum

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Contents

1 Research Question

What is the relationship between the amplitude of the forced pendulum /rad and the angular frequency of the driver pendulum /rad s^{-1} in a Barton's pendulum.

2 Introduction

Pendulums undergo a repeating cycle of energy transfer from only potential energy to only kinetic energy back to only potential energy. This process causes a pendulum system to undergo simple harmonic motion, therefore pendulums have properties such as period and frequency of oscillation defined that are dependent on the length of the pendulum.

What this also means is that any periodic external force will or will not resonate with the pendulum. This applies to various real-life scenarios and problems, from something as little as the frequency that a parent should push their child on a swing to whether gusts of wind are capable of driving an idle wrecking ball to dangerous.

Intuitively, if an external periodic force is resonant with the pendulum, then the extent that the pendulum's amplitude will reach will be at its maximum. However, one should question exactly how this maximum amplitude grows as the frequency of the external force approaches the resonant frequency of the pendulum.

This could easily be investigated by suspending a driving pendulum and a forced pendulum on the same string. This allows for an easy way to provide an external periodic force through the driver pendulum, allowing for the frequency of the external force to be manipulated through changing the length of the driver pendulum and removing the necessity of a motorized instrument.

3 Background information

3.1 Friction in a pendulum

Because a pendulum is not sliding along a surface, then friction cannot be defined by the formula $F_f = \mu F_N = \mu_d R$. However, a pendulum behaves alike to a particle experiencing frictional force that is proportional to its velocity (*Frictional Force Is Proportional to the Velocity*, 2021). It can therefore be defined as

$$F_f = -\gamma v \tag{1}$$

Where $\gamma =$ the coefficient of friction of an object experiencing friction through drag

Note that the negative sign is necessary, as the direction of translational velocity of the pendulum opposes the direction of frictional force.

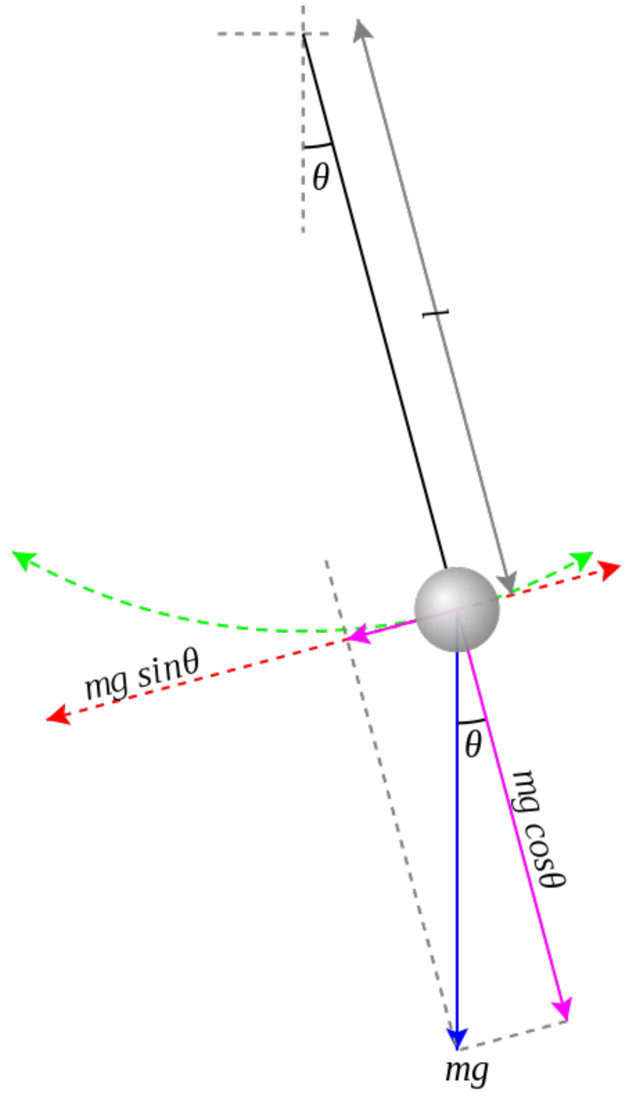

Figure 1: Pendulum labelled with components of gravitational force (Krishnavedala, 25 January 2013, 12:56:47).

With reference to Figure 1, the displacement along the green arc (s) can be defined below using the definition of a radian.

$$s = l\theta \tag{2}$$

Where $l =$ the length of the forced pendulum /m

Differentiating both sides of Equation 2, the following relationship is obtained.

$$\dot{s} = v = l\dot{\theta} \tag{3}$$

This means that Equation 1 can be rewritten as

$$F_f = -\gamma l\dot{\theta} \tag{4}$$

3.2 External periodic force on a pendulum

The external periodic force on a pendulum can be modelled as follows

$$F_e = A_e \cos(\Omega t) \tag{5}$$

$$\begin{aligned} \text{Where } F_e &= \text{the external periodic force acting on the pendulum /N} \\ A_e &= \text{the amplitude of the external periodic force /N} \\ \Omega &= \text{the angular frequency of the external periodic force /rad s}^{-1} \\ t &= \text{duration of time passed after a reference moment in time /s} \end{aligned}$$

3.3 The Damped, Driven Pendulum

In Figure 1, the component of gravitational force parallel to the instantaneous velocity of the pendulum is $mg \sin\theta$. However, when letting F_g equal to such component of gravitational force, it must equal to the negative of $mg \sin\theta$ such that the directionality of F_g opposes the directionality of the horizontal component of s.

$$F_g = -mg \sin\theta \tag{6}$$

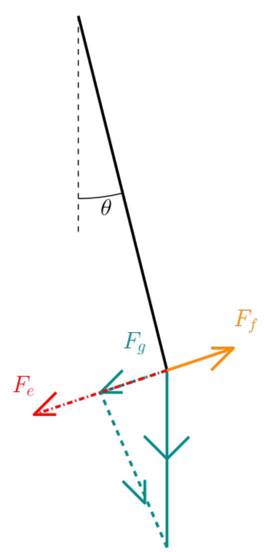

Figure 2: A pendulum with relevant forces labelled

With reference to Figure 2, Equation 7 can be derived as follows.

$$\begin{aligned}
F_{net} &= ma \\
F_g + F_f + F_e &= ma \\
-mg\sin\theta - \gamma l\dot{\theta} + A_e\cos(\Omega t) &= m\ddot{s} \qquad \text{(A)}
\end{aligned}$$

$$\begin{aligned}
m\ddot{s} &= -mg\sin\theta \\
l\ddot{\theta} &= -g\sin\theta \qquad \text{(B)} \\
\ddot{\theta} &= -\frac{g}{l}\sin\theta \\
\because \omega &= \sqrt{\frac{g}{l}} \\
\ddot{\theta} &= -\omega^2\sin\theta \qquad \text{(C)}
\end{aligned}$$

$$\begin{aligned}
&\text{Substituting B into A} \\
ml\ddot{\theta} - \gamma l\dot{\theta} + A_e\cos(\Omega t) &= m\ddot{s} \qquad \text{(D)} \\
&\text{Substituting C into D} \\
-ml\omega^2\sin\theta - \gamma l\dot{\theta} + A_e\cos(\Omega t) &= m\ddot{s} \\
\because \ddot{s} &= l\ddot{\theta} \\
A_e\cos(\Omega t) &= ml\ddot{\theta} + \gamma l\dot{\theta} + ml\omega^2\sin\theta \\
\text{Letting } C &= \frac{A_e}{ml} \\
\lambda &= \frac{\gamma}{m}
\end{aligned}$$

$$\ddot{\theta} + \lambda\dot{\theta} + \omega^2\sin\theta = C\cos(\Omega t) \qquad (7)$$

A formula to determine the amplitude of a damped, driven pendulum can be then determined.

$$\begin{aligned}
&\text{Using the small angle approximation of } \sin\theta \approx \theta \\
&\ddot{\theta} + \lambda\dot{\theta} + \omega^2\theta = C\cos(\Omega t) \\
&\ddot{z} + \lambda\dot{z} + \omega^2 z = Ce^{i\Omega t} \\
&\text{The ansatz for } z \text{ is } Ae^{i\Omega t} \text{ (Chasnov, 2022b)} \\
&\dot{z} = i\Omega Ae^{i\Omega t}, \quad \ddot{z} = -\Omega^2 Ae^{i\Omega t} \\
-\Omega^2 A + i\lambda\Omega A + \omega^2 A &= C \\
A &= \frac{C}{(\omega^2-\Omega^2)+i\lambda\Omega} \times \frac{(\omega^2-\Omega^2)-i\lambda\Omega}{(\omega^2-\Omega^2)-i\lambda\Omega} \\
A &= \frac{C\left[(\omega^2-\Omega^2)-i\lambda\Omega\right]}{(\omega^2-\Omega^2)^2+\lambda^2\Omega^2}
\end{aligned}$$

Using the exponential form of complex numbers: $a + bi = re^{i\phi}$

$$(\omega^2 - \Omega^2) - i\lambda\Omega = e^{i\phi}\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}$$

$$A = \frac{Ce^{i\phi}}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}}$$

$$\theta(t) = \Re(Ae^{i\Omega t})$$

$$\theta(t) = \Re(e^{i\Omega t}\frac{Ce^{i\phi}}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}})$$

$$\theta(t) = \Re(e^{i(\Omega t + \phi)})\frac{C}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}}$$

$$\theta(t) = \left(\frac{C}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}}\right)\cos(\Omega t + \phi)$$

$$A = \frac{C}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}} \tag{8}$$

Equation 8 assumes that:

- Due to the small angle approximation made, θ does not exceed 13.99°

All derivations and ideas within Section 3 were with reference to (Chasnov, 2022b).

4 Hypothesis

When the length of the driver pendulum equals to the length of the forced pendulum, then the amplitude of the forced pendulum will be at its maximum as the two pendulums are in resonance.

As the length of the driver pendulum approaches the length of the forced pendulum, then the amplitude of the forced pendulum will approach that maximum from lower values.

A more detailed hypothesis can be made by referring to and modifying Equation 8.

$$A = \frac{C}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}}$$

$$A^2 = \frac{C^2}{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}$$

$$A^2 = \frac{C^2}{\Omega^4 + (\lambda^2 - 2\omega^2)\Omega^2 + \omega^4}$$

$$\frac{1}{A^2} = \frac{\Omega^4 + (\lambda^2 - 2\omega^2)\Omega^2 + \omega^4}{C^2}$$

$$\frac{1}{A^2} = \frac{1}{C^2}\Omega^4 + \frac{\lambda - 2\omega^2}{C^2}\Omega^2 + \frac{\omega^4}{C^2} \tag{9}$$

Unfortunately, Equation 9 cannot be manipulated further to achieve a linearized form. However, that doesn't remove the possibility of analysis despite its form.

If $\frac{1}{A^2}$ is graphed as a function of Ω^2, a parabola opening upwards should be expected as none of the variables in $C = \frac{A_e}{ml}$ can be negative. Additionally, the question of in which quadrant does the vertex lay can be answered by taking the derivative of Equation 9 with respect to $o = \Omega^2$ and solving for o when the derivative equals to 0.

$$\begin{aligned}\frac{1}{A^2} &= \frac{1}{C^2}o^2 + \frac{\lambda - 2\omega^2}{C^2}o + \frac{\omega^4}{C^2} \\ \frac{\partial}{\partial o}\left(\frac{1}{A^2}\right) &= \frac{2}{C^2}o + \frac{\lambda - 2\omega^2}{C^2} = 0 \\ o = \Omega^2 &= \omega^2 - \frac{\lambda}{2}\end{aligned}$$

With reference to Equation 1 that states $F_f = -\gamma v$, it may be assumed that $F_f \ll v$, therefore causing $\lambda = \frac{\gamma}{m}$ to be insignificant. Because both ω and A can only be positive given their contexts, the vertex of $\frac{1}{A^2}$ as a function of Ω^2 must be in the first quadrant.

The specific relationship of A as a function of Ω can be viewed by graphing such a function through Equation 9. This is shown in Figure 3, which the shape of the curve makes sense given that there is a maximum at the resonant angular frequency.

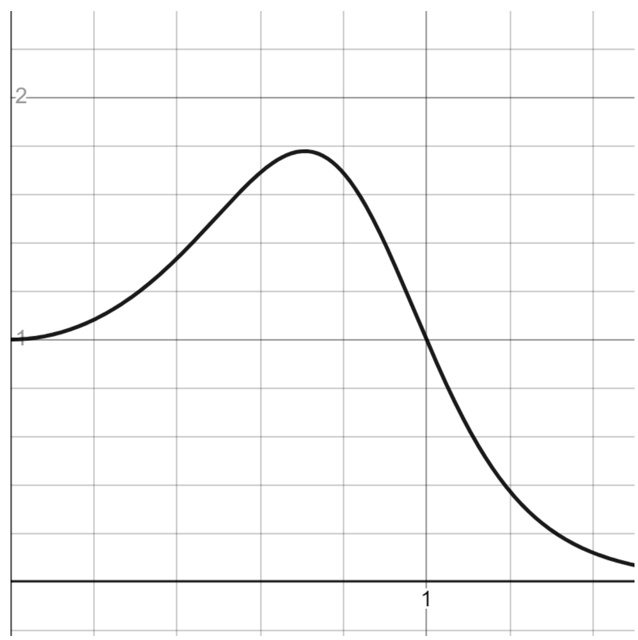

Figure 3: Amplitude of forced pendulum as a function of angular frequency of the driver pendulum graph using Desmos to present a hypothesis of the function's curve shape

5 Materials

- (3) Retort Stands
- (2) C-clamps
- (4) Clamps
- (2) Rulers
- (2) Meter Sticks
- (1) 200 g weight
- (1) 50 g weight
- Tape
- String
- Paper
- Scissors
- Highlighter
- Phone Camera

6 Variables

6.1 Manipulated Variable

The manipulated variable is the angular frequency of the driver pendulum /rad s^{-1}. In this lab, the manipulated variable will be changed by shortening the length of the string. This change will be measured by PASCO Capstone's distance measurement tool.

6.2 Responding Variable

The responding variable is the amplitude of the forced pendulum /rad. The responding variable will be measured using PASCO Capstone's object tracking tool.

6.3 Controlled Variables

The first controlled variable will be the starting amplitude of the driver pendulum. This variable will be controlled by setting up a meter stick with its foot pivoted on a retort stand at a consistent position while leaning on the meter stick sitting on top of the clamps between the two primary retort stands, where each trial involved pushing the driver pendulum up to the angle of that leaning meter stick. This variable must be controlled in order to ensure that any changes in the amplitude of the forced pendulum between trials isn't a result of varying initial amplitude of the driver pendulum and therefore inconsistent total energy in the system. This controlled variable is also seen in Equation 8 from $C = \frac{A_e}{ml}$, where A_e is the amplitude of the external periodic force.

The second controlled variable will be the tension in the master string that both pendulums are suspended from. This variable will be controlled by using the same string of a consistent length and maintaining the positions of the primary retort stands with C-clamps. This variable must be controlled to ensure that the degree of energy exchange between the two pendulums is consistent between trials. Additionally, overly loose strings will cause the degree of oscillation in the master string to drastically effect the data.

The third controlled variable is the length of the forced pendulum. This variable will be controlled by not exchanging the string of the forced pendulum between trials. This variable must be controlled to ensure that the angular frequency of the forced pendulum seen in

Equation 8 as ω remains consistent. It also ensures that C remains consistent as $C = \frac{A_e}{ml}$, where l is the length of the forced pendulum.

The fourth controlled variable is the mass of the forced pendulum. This variable will be controlled by avoiding the exchange of weights of different masses for the forced pendulum between trials. This variable must be controlled to ensure that the parameters of $C = \frac{A_e}{ml}$ and $\lambda = \frac{\gamma}{m}$ remain consistent.

7 Experimental Protocol

8 Raw Data

9 Processed Data

10 Analysis

11 Evaluation

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