PHYSICS SL
Barton's Pendulum

Candidate Code:
Session: May 2024
Page Count:

Contents

1 Research Question

What is the relationship between the amplitude of the forced pendulum /rad and the angular frequency of the driver pendulum /rad s^{-1} in a Barton's pendulum.

2 Introduction

Pendulums undergo a repeating cycle of energy transfer from only potential energy to only kinetic energy back to only potential energy. This process causes a pendulum system to undergo simple harmonic motion, therefore pendulums have properties such as period and frequency of oscillation defined that are dependent on the length of the pendulum.

What this also means is that any periodic external force will or will not resonate with the pendulum. This applies to various real-life scenarios and problems, from something as little as the frequency that a parent should push their child on a swing to whether gusts of wind are capable of driving an idle wrecking ball to dangerous.

Intuitively, if an external periodic force is resonant with the pendulum, then the extent that the pendulum's amplitude will reach will be at its maximum. However, one should question exactly how this maximum amplitude grows as the frequency of the external force approaches the resonant frequency of the pendulum.

This could easily be investigated by suspending a driving pendulum and a forced pendulum on the same string. This allows for an easy way to provide an external periodic force through the driver pendulum, allowing for the frequency of the external force to be manipulated through changing the length of the driver pendulum and removing the necessity of a motorized instrument.

3 Background information

3.1 Friction in a pendulum

Because a pendulum is not sliding along a surface, then friction cannot be defined by the formula $F_f = \mu F_N = \mu_d R$. However, a pendulum behaves alike to a particle experiencing frictional force that is proportional to its velocity (*Frictional Force Is Proportional to the Velocity*, 2021). It can therefore be defined as

$$F_f = -\gamma v \tag{1}$$

Where γ = the coefficient of friction of an object experiencing friction through drag /kg s^{-1}

Note that the negative sign is necessary, as the direction of translational velocity of the pendulum opposes the direction of frictional force.

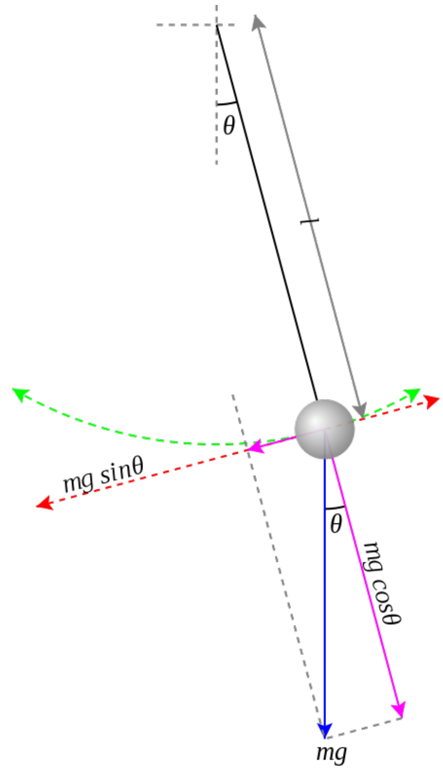

Figure 1: Pendulum labelled with components of gravitational force (Krishnavedala, 25 January 2013, 12:56:47).

With reference to Figure 1, the displacement along the green arc (s) can be defined below using the definition of a radian.

$$s = l\theta \tag{2}$$

Where l = the length of the forced pendulum /m

Differentiating both sides of Equation 2, the following relationship is obtained.

$$\dot{s} = v = l\dot{\theta} \tag{3}$$

This means that Equation 1 can be rewritten as

$$F_f = -\gamma l\dot{\theta} \tag{4}$$

3.2 External periodic force on a pendulum

The external periodic force on a pendulum can be modelled as follows

$$F_e = A_e \cos(\Omega t) \tag{5}$$

Where F_e = the external periodic force acting on the pendulum /N
A_e = the amplitude of the external periodic force /N
Ω = the angular frequency of the external periodic force /rad s^{-1}
t = duration of time passed after a reference moment in time /s

3.3 The Damped, Driven Pendulum

In Figure 1, the component of gravitational force parallel to the instantaneous velocity of the pendulum is $mg\sin\theta$. However, when letting F_g equal to such component of gravitational force, it must equal to the negative of $mg\sin\theta$ such that the directionality of F_g opposes the directionality of the horizontal component of s.

$$F_g = -mg\sin\theta \tag{6}$$

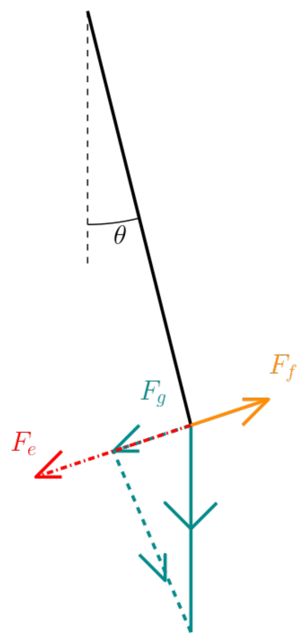

Figure 2: A pendulum with relevant forces labelled

With reference to Figure 2, Equation 7 can be derived as follows.

$$\begin{aligned} F_{net} &= ma \\ F_g + F_f + F_e &= ma \\ -mg\sin\theta - \gamma l\dot{\theta} + A_e\cos(\Omega t) &= m\ddot{s} \qquad \text{(A)} \end{aligned}$$

$$\begin{aligned} m\ddot{s} &= -mg\sin\theta \\ l\ddot{\theta} &= -g\sin\theta \qquad \text{(B)} \\ \ddot{\theta} &= -\frac{g}{l}\sin\theta \\ \because \omega &= \sqrt{\frac{g}{l}} \\ \ddot{\theta} &= -\omega^2\sin\theta \qquad \text{(C)} \end{aligned}$$

$$\begin{aligned} &\text{Substituting B into A} \\ ml\ddot{\theta} - \gamma l\dot{\theta} + A_e\cos(\Omega t) &= m\ddot{s} \qquad \text{(D)} \\ &\text{Substituting C into D} \\ -ml\omega^2\sin\theta - \gamma l\dot{\theta} + A_e\cos(\Omega t) &= m\ddot{s} \end{aligned}$$

$$
\begin{aligned}
\because \ddot{s} &= l\ddot{\theta} \\
A_e \cos(\Omega t) &= ml\ddot{\theta} + \gamma l\dot{\theta} + ml\omega^2 \sin\theta \\
\text{Letting } C &= \frac{A_e}{ml} \\
\lambda &= \frac{\gamma}{m}
\end{aligned}
$$

$$
\ddot{\theta} + \lambda\dot{\theta} + \omega^2 \sin\theta = C\cos(\Omega t) \tag{7}
$$

A formula to determine the amplitude of a damped, driven pendulum can be then determined.

$$
\begin{aligned}
&\text{Using the small angle approximation of } \sin\theta \approx \theta \\
&\ddot{\theta} + \lambda\dot{\theta} + \omega^2\theta = C\cos(\Omega t) \\
&\ddot{z} + \lambda\dot{z} + \omega^2 z = Ce^{i\Omega t} \\
&\text{The ansatz for } z \text{ is } Ae^{i\Omega t} \text{ (Chasnov, 2022b)} \\
&\dot{z} = i\Omega Ae^{i\Omega t}, \quad \ddot{z} = -\Omega^2 Ae^{i\Omega t} \\
&-\Omega^2 A + i\lambda\Omega A + \omega^2 A = C \\
A &= \frac{C}{(\omega^2 - \Omega^2) + i\lambda\Omega} \times \frac{(\omega^2 - \Omega^2) - i\lambda\Omega}{(\omega^2 - \Omega^2) - i\lambda\Omega} \\
A &= \frac{C\left[(\omega^2 - \Omega^2) - i\lambda\Omega\right]}{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2} \\
&\text{Using the exponential form of complex numbers: } a + bi = re^{i\phi} \\
&(\omega^2 - \Omega^2) - i\lambda\Omega = e^{i\phi}\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2} \\
A &= \frac{Ce^{i\phi}}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}} \\
\theta(t) &= \Re(Ae^{i\Omega t}) \\
\theta(t) &= \Re(e^{i\Omega t}\frac{Ce^{i\phi}}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}}) \\
\theta(t) &= \Re(e^{i(\Omega t + \phi)})\frac{C}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}} \\
\theta(t) &= \left(\frac{C}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}}\right)\cos(\Omega t + \phi)
\end{aligned}
$$

$$
A = \frac{C}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}} \tag{8}
$$

Equation 8 assumes that:

- Due to the small angle approximation made, θ does not exceed 13.99°
- The string connected to the weight is massless

All derivations and ideas within Section 3 were with reference to (Chasnov, 2022b).

4 Hypothesis

When the length of the driver pendulum equals to the length of the forced pendulum, then the amplitude of the forced pendulum will be at its maximum as the two pendulums are in resonance.

As the length of the driver pendulum approaches the length of the forced pendulum, then the amplitude of the forced pendulum will approach that maximum from lower values.

A more detailed hypothesis can be made by referring to and modifying Equation 8.

$$
\begin{aligned}
A &= \frac{C}{\sqrt{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2}} \\
A^2 &= \frac{C^2}{(\omega^2 - \Omega^2)^2 + \lambda^2\Omega^2} \\
A^2 &= \frac{C^2}{\Omega^4 + (\lambda^2 - 2\omega^2)\Omega^2 + \omega^4} \\
\frac{1}{A^2} &= \frac{\Omega^4 + (\lambda^2 - 2\omega^2)\Omega^2 + \omega^4}{C^2}
\end{aligned}
$$

$$\frac{1}{A^2} = \frac{1}{C^2}\Omega^4 + \frac{\lambda^2 - 2\omega^2}{C^2}\Omega^2 + \frac{\omega^4}{C^2} \tag{9}$$

Unfortunately, Equation 9 cannot be manipulated further to achieve a linearized form. However, that doesn't remove the possibility of analysis despite its form.

If $\frac{1}{A^2}$ is graphed as a function of Ω^2, a parabola opening upwards should be expected as none of the variables in $C = \frac{A_e}{ml}$ can be negative. Additionally, the question of in which quadrant does the vertex lay can be answered by taking the derivative of Equation 9 with respect to $o = \Omega^2$ and solving for o when the derivative equals to 0.

$$
\begin{aligned}
\frac{1}{A^2} &= \frac{1}{C^2}o^2 + \frac{\lambda^2 - 2\omega^2}{C^2}o + \frac{\omega^4}{C^2} \\
\frac{\partial}{\partial o}\left(\frac{1}{A^2}\right) &= \frac{2}{C^2}o + \frac{\lambda^2 - 2\omega^2}{C^2} = 0 \\
o = \Omega^2 &= \omega^2 - \frac{\lambda}{2}
\end{aligned}
$$

With reference to Equation 1 that states $F_f = -\gamma v$, it may be assumed that $F_f \ll v$, therefore causing $\lambda = \frac{\gamma}{m}$ to be insignificant. Because both ω and A can only be positive given their contexts, the vertex of $\frac{1}{A^2}$ as a function of Ω^2 must be in the first quadrant.

The specific relationship of A as a function of Ω can be viewed by graphing such a function through Equation 9. This is shown in Figure 3, which the shape of the curve makes sense given that there is a maximum at the resonant angular frequency.

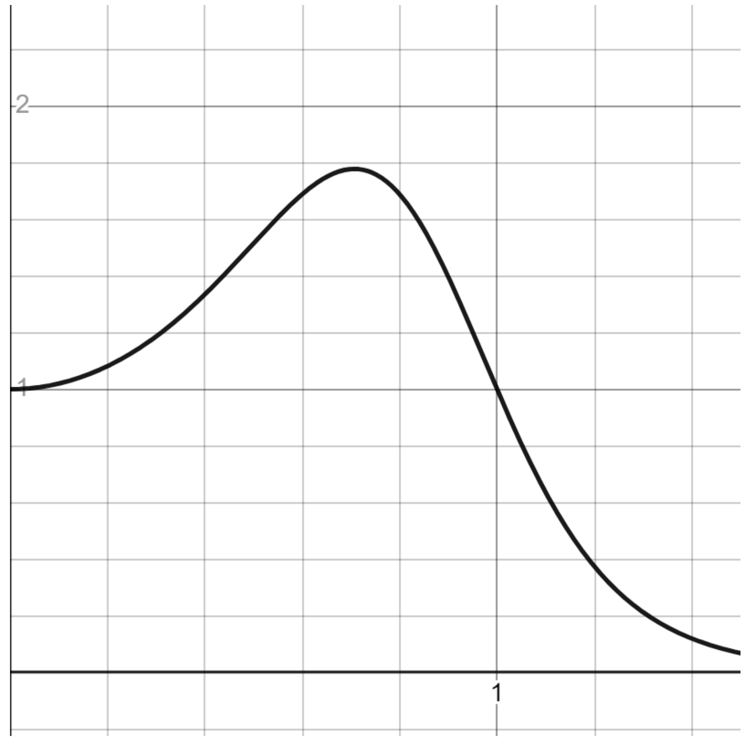

Figure 3: Amplitude of forced pendulum as a function of angular frequency of the driver pendulum graph using Desmos to present a hypothesis of the function's curve shape

5 Materials

- (3) Retort Stands
- (2) C-clamps
- (4) Clamps
- (2) Rulers
- (2) Meter Sticks
- (1) 200 g weight
- (1) 50 g weight
- Tape
- String
- Paper
- Scissors
- Highlighter
- Phone Camera

6 Variables

6.1 Independent Variable

The independent variable is the angular frequency of the driver pendulum /rad s^{-1}. In this lab, the independent variable will be changed by shortening the length of the string. This change will be measured by PASCO Capstone's distance measurement tool.

6.2 Dependent Variable

The dependent variable is the amplitude of the forced pendulum /rad. The dependent variable will be measured using PASCO Capstone's object tracking tool.

6.3 Controlled Variables

The first controlled variable will be the starting amplitude of the driver pendulum. This variable will be controlled by setting up a meter stick with its foot pivoted on a retort stand at a consistent position while leaning on the meter stick sitting on top of the clamps between the two primary retort stands, where each trial involved pushing the driver pendulum up to the angle of that leaning meter stick. This variable must be controlled in order to ensure that any changes in the amplitude of the forced pendulum between trials isn't a result of varying initial amplitude of the driver pendulum and therefore inconsistent total energy in the system. This controlled variable is also seen in Equation 8 from $C = \frac{A_e}{ml}$, where A_e is the amplitude of the external periodic force.

The second controlled variable will be the tension in the master string that both pendulums are suspended from. This variable will be controlled by using the same string of a consistent length and maintaining the positions of the primary retort stands with C-clamps. This variable must be controlled to ensure that the degree of energy exchange between the two pendulums is consistent between trials. Additionally, overly loose strings will cause the degree of oscillation in the master string to drastically effect the data.

The third controlled variable is the length of the forced pendulum. This variable will be controlled by not exchanging the string of the forced pendulum between trials. This variable must be controlled to ensure that the angular frequency of the forced pendulum seen in Equation 8 as ω remains consistent. It also ensures that C remains consistent as $C = \frac{A_e}{ml}$, where l is the length of the forced pendulum.

The fourth controlled variable is the mass of the forced pendulum. This variable will be controlled by avoiding the exchange of weights of different masses for the forced pendulum between trials. This variable must be controlled to ensure that the parameters of $C = \frac{A_e}{ml}$ and $\lambda = \frac{\gamma}{m}$ remain consistent.

7 Experimental Protocol

A string was cut using scissors to a length that is slightly longer than a meter stick. A good portions of the ends were folded and twisted to create a loop at the ends. Two clamps were positioned near the top of two retort stands where the loops were wrapped around the clamps' screws.

The retort stands were separated as much as possible such that the master string would have as much tension as possible. The retort stands were then secured using the C-clamps. With the bases of the two stands oriented with the rods as far away as possible from each other, the edges of the bases were observed to be 53 cm apart.

The 200 g weight was dedicated to be the driver pendulum and the 50 g weight was dedicated to be the forced pendulum. The motivation of this configuration was that if the driver pendulum had significantly more mass to it, then it would be resistant to complete energy transfers to the forced pendulum. A slip of paper was coloured with pink highlighter and was wrapped around the 50 g weight. The purpose of doing this was so PASCO Capstone can easily track the forced pendulum.

A string of some arbitrary length was cut and was attached to the forced pendulum and suspended from the master using loops. A triangular ruler was then taped onto the third retort stand, in which the height of the triangular ruler was 11 cm. Using 2 clamps to secure the phone onto one of the primary retort stands, the forced pendulum and the third retort stand were aligned in the same plane, and a picture was taken using the phone. The motivation of this was to analyse that image using PASCO to determine a more accurate length that considers the centre of mass of the weight.

A longer string length was then attached to the 200 g weight for the driving pendulum and suspended on the master string. The forced pendulum was shifted along the master string towards the side of the retort stand with the phone, and the driver pendulum was suspended on the side of the other retort stand. The third retort stand was aligned along the plane of the driver pendulum. A meter stick was placed on top of the two clamps securing the master string and another meter stick was set to lean against the first meter stick with its foot sitting against the base of the third retort stand. The motivation of this set up was to maintain a consistent initial amplitude of the driver pendulum, therefore a 15 cm ruler sat between the two retort stands to ensure that the distance between the two bases are consistent.

Each trial involved setting the phone to record the motion of the forced pendulum after releasing the driver pendulum. Changing the driver pendulum's length involved increasing the size of the loop such that the overall length is reduced. If the length reaches the point where there is too much excess string, then some string was cut off to allow this process to continue.

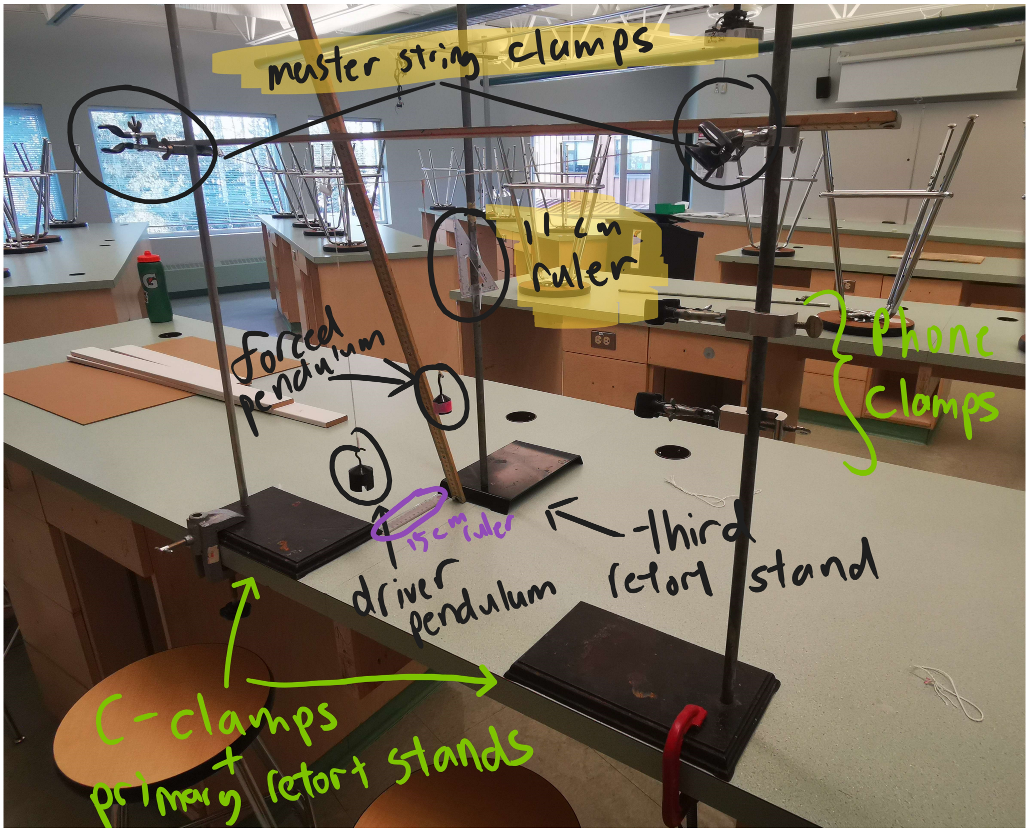

Figure 4: Lab experiment apparatus setup

8 Raw Data

8.1 Qualitative Observations

- As the angular frequency of the pendulum approached the resonant angular frequency of the forced pendulum, the driver pendulum experiences progressively more complete energy transfers. This was observed as temporary absence of potential and kinetic energy in the driver pendulum when the forced pendulum experiences oscillations of maximum amplitude.
- Despite the method of controlling the tension of the master string by avoiding exchange between different master strings, the string was observed to lose tension overtime.
- When the length of the driver pendulum was 0.156 m, the master string failed and collapsed.

8.2 Quantitative Data

Table 1 presents the raw data collected from the experiment that are necessary to determine the angular frequency of the driver pendulum and the maximum amplitude of the forced pendulum for each trial.

Table 1: Average, maximum, and minimum horizontal oscillation position and maximum and minimum vertical oscillation position as functions of the length of driver pendulum

Length of driver pendulum /m	Average of horizontal oscillation position /mm	Maximum of horizontal oscillation position /mm	Minimum of horizontal oscillation position /mm	Maximum of vertical oscillation position /mm	Minimum of vertical oscillation position /mm
0.374 ± 0.007	-2.24 ± 0.04	39.5 ± 0.7	-45.0 ± 0.8	2.50 ± 0.05	-7.9 ± 0.1
0.353 ± 0.006	-15.1 ± 0.3	39.7 ± 0.7	-67 ± 1	4.62 ± 0.08	-4.32 ± 0.08
0.320 ± 0.006	-10.2 ± 0.2	62 ± 1	-86 ± 2	2.50 ± 0.05	-7.9 ± 0.1
0.297 ± 0.005	-13.2 ± 0.2	70 ± 1	-95 ± 2	7.2 ± 0.1	-7.7 ± 0.1
0.269 ± 0.005	-14.1 ± 0.3	104 ± 2	-129 ± 2	17.6 ± 0.3	-6.6 ± 0.1
0.250 ± 0.005	-12.9 ± 0.2	134 ± 2	-157 ± 3	26.1 ± 0.5	-9.1 ± 0.2
0.240 ± 0.004	-13.7 ± 0.2	177 ± 3	-204 ± 4	51.6 ± 0.9	-6.6 ± 0.1
0.229 ± 0.004	-13.8 ± 0.3	205 ± 4	-229 ± 4	75 ± 1	-9.3 ± 0.2
0.203 ± 0.004	-13.1 ± 0.2	193 ± 4	-211 ± 4	63 ± 1	-9.1 ± 0.2
0.168 ± 0.003	-15.0 ± 0.3	114 ± 2	-138 ± 3	25.1 ± 0.5	-9.6 ± 0.2
0.156 ± 0.003	-18.0 ± 0.3	91 ± 2	-129 ± 2	21.9 ± 0.4	-7.3 ± 0.1
0.153 ± 0.003	-18.6 ± 0.3	88 ± 2	-122 ± 2	20.6 ± 0.4	-7.7 ± 0.1
0.130 ± 0.002	-10.8 ± 0.2	69 ± 1	-90 ± 2	13.3 ± 0.2	-6.6 ± 0.1
0.212 ± 0.004	8.0 ± 0.1	197 ± 4	-233 ± 4	74 ± 1	-7.7 ± 0.1
0.115 ± 0.002	-17.9 ± 0.3	57 ± 1	-88 ± 2	10.7 ± 0.2	-7.6 ± 0.1

The uncertainty for each value was dependent on the uncertainty of the ruler used for PASCO Capstone to use as a reference length. Because the calibration of the ruler relied on orienting points based off of pixels, the confidence and uncertainty of one position along the ruler is the increment of length for each tick on the ruler (± 1 mm). However, because the overall calibration of the ruler's length involves two points, then the final uncertainty for the 11.0 cm ruler is ± 0.2 cm.

Because this uncertainty is relative to the 11.0 cm ruler, then the uncertainties for the distance measurements according to PASCO Capstone will be the percent uncertainty of the ruler. A sample calculation is shown below for the length of the driver pendulum of the first trial.

$$
\begin{aligned}
\text{Let } l_e &= \text{the length of the external driver pendulum /m} \\
l_r &= \text{the length of the reference ruler /cm} \\
\Delta l_e &= l_e \left(\frac{\Delta l_r}{l_r} \right) \\
&= (0.374\,\text{m}) \left(\frac{0.2\,\text{cm}}{11.0\,\text{cm}} \right) \\
&= 0.007\,\text{m}
\end{aligned}
$$

9 Processed Data

The lengths of the driver pendulums will be used to determine the angular frequency of the driver pendulum for each trial. A sample calculation is shown below for the length of the driver pendulum of the first trial.

From (Gonzales, 2022)

$$\Omega = \sqrt{\frac{g}{l_e}}$$
$$= \sqrt{\frac{9.81\,\mathrm{m\,s^{-2}}}{0.374\,\mathrm{m} \pm 0.007\,\mathrm{m}}}$$
$$= 5.12\,\mathrm{rad\,s^{-1}} \pm 0.05\,\mathrm{rad\,s^{-1}}$$

$$\Delta\Omega = \Omega\left(\frac{1}{2}\cdot\frac{\Delta l_e}{l_e}\right)$$
$$= (5.12\,\mathrm{rad\,s^{-1}})\left(\frac{1}{2}\cdot\frac{0.007\,\mathrm{m}}{0.374\,\mathrm{m}}\right)$$
$$= 0.05\,\mathrm{rad\,s^{-1}}$$

Next, based on the maximum, minimum, and average positions of the pendulum along the horizontal and vertical axis can be used to determine the pendulum's maximum horizontal and vertical displacement from equilibrium. For the horizontal axis, its maximum distance from equilibrium is the greater value between $|X_{max} - X_{average}$ and $|X_{min} - X_{average}|$. For the vertical axis, its maximum distance from equilibrium is $|Y_{max} - Y_{min}|$.

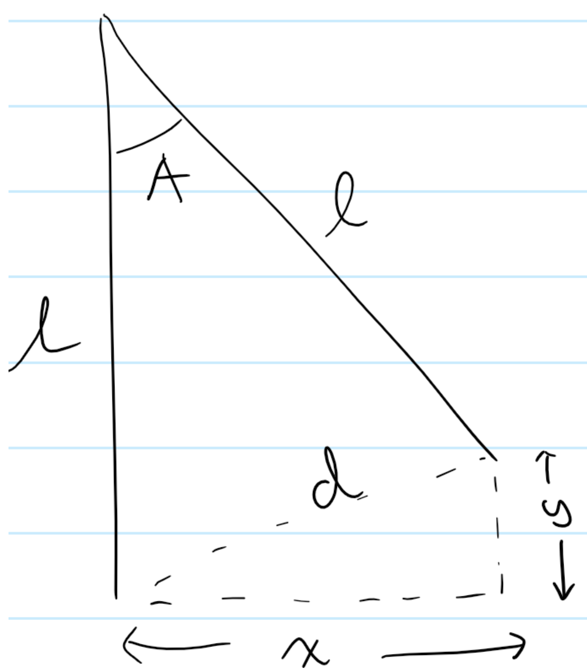

Figure 5: Forced pendulum at its maximum amplitude labelled with relevant variables to determine the value of the maximum amplitude

The amplitude of the forced pendulum /rad can be determined by taking the maximum horizontal oscillation amplitude and the maximum vertical oscillation amplitude and using cosine law to determine the angle that the forced pendulum has rotated about its pivot. The calculation for this for the first trial is shown below with reference to Figure 5.

$$\text{Let } l = \text{the length of the forced pendulum}$$
$$d^2 = 2l^2 - 2l^2 \cos A, \quad d^2 = x^2 + y^2$$
$$A = \arccos\left(1 - \frac{x^2+y^2}{2l^2}\right)$$
$$= \arccos\left(1 - \frac{(4.27\,\text{cm} \pm 0.09\,\text{cm})^2 + (1.04\,\text{cm} \pm 0.02\,\text{cm})^2}{2(26.1\,\text{cm} \pm 0.5\,\text{cm})^2}\right)$$
$$= 0.168\,\text{rad}$$
Using Equation 10 to propagate the uncertainty
$$A = 0.168\,\text{rad} \pm 0.006\,\text{rad}$$

From (*Uncertainties and Error Propagation*, n.d.)

$$\Delta f(a_1, a_2, a_3, \cdots) = \sum \left|\frac{\partial f}{\partial a_n}\right| \Delta a_n \tag{10}$$

Where f = the multivariable function whose uncertainty is to be found
a_n = the nth variable in f

Table 2 presents the processed data that will provide the necessary values to graph the relationship between the forced pendulum's maximum amplitude and the angular frequency of the driver pendulum.

Table 2: Maximum horizontal and vertical displacement from equilibrium of forced pendulum and maximum amplitude of forced pendulum as a function of the angular frequency of the driver pendulum

Angular frequency of driver pendulum /rad s^{-1}	Forced pendulum's maximum horizontal displacement from equilibrium /cm	Forced pendulum's maximum vertical displacement from equilibrium /cm	Forced pendulum's maximum amplitude /rad
5.12 ± 0.05	4.27 ± 0.09	1.04 ± 0.02	0.168 ± 0.006
5.27 ± 0.05	5.5 ± 0.1	0.89 ± 0.02	0.213 ± 0.008
5.54 ± 0.05	7.6 ± 0.2	1.04 ± 0.02	0.30 ± 0.01
5.75 ± 0.05	8.3 ± 0.2	1.49 ± 0.03	0.33 ± 0.01
6.04 ± 0.05	11.9 ± 0.2	2.42 ± 0.04	0.47 ± 0.02
6.26 ± 0.06	14.6 ± 0.3	3.51 ± 0.06	0.58 ± 0.02
6.39 ± 0.06	19.1 ± 0.3	5.8 ± 0.1	0.78 ± 0.03
6.55 ± 0.06	21.9 ± 0.4	8.5 ± 0.2	0.93 ± 0.04
6.96 ± 0.06	20.6 ± 0.4	7.2 ± 0.1	0.86 ± 0.03
7.64 ± 0.07	12.9 ± 0.2	3.48 ± 0.06	0.52 ± 0.02
7.93 ± 0.07	11.1 ± 0.3	2.92 ± 0.05	0.44 ± 0.02
8.00 ± 0.07	10.7 ± 0.2	2.83 ± 0.05	0.43 ± 0.02
8.69 ± 0.08	8.0 ± 0.1	1.99 ± 0.04	0.32 ± 0.01
6.80 ± 0.06	24.1 ± 0.4	8.1 ± 0.1	1.02 ± 0.04
9.24 ± 0.08	7.5 ± 0.1	1.83 ± 0.03	0.29 ± 0.01

Table 3 presents the reciprocal of the forced pendulum's maximum amplitude squared as a function of the square of the angular frequency of the driver pendulum. Note that Equation 10 was used to propagate the uncertainties of the values in the table.

Table 3: Reciprocal of the forced pendulum's maximum amplitude squared as a function of the square of the angular frequency of the driver pendulum

Square of angular frequency of driver pendulum /rad^2 s^{-2}	Reciprocal of the forced pendulum's maximum amplitude squared /rad^{-2}
26.24 ± 0.09	35 ± 3
27.8 ± 0.1	22 ± 2
30.7 ± 0.1	11 ± 1
33.0 ± 0.1	9.5 ± 0.7
36.4 ± 0.1	4.6 ± 0.3
39.2 ± 0.1	2.9 ± 0.2
40.8 ± 0.1	1.6 ± 0.1
42.9 ± 0.1	1.16 ± 0.09
48.4 ± 0.1	1.3 ± 0.1
58.4 ± 0.1	3.8 ± 0.3
62.9 ± 0.1	5.1 ± 0.4
64.0 ± 0.1	5.5 ± 0.4
75.6 ± 0.2	10.1 ± 0.7
46.2 ± 0.1	0.97 ± 0.08
85.3 ± 0.2	11.5 ± 0.8

10 Analysis

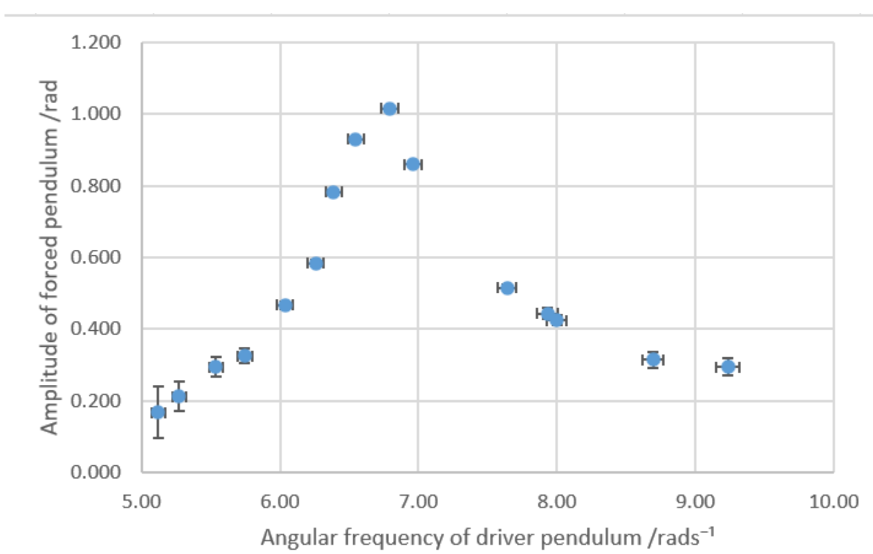

Figure 6: Amplitude of forced pendulum /rad as a function of angular frequency of driver pendulum /rad s^{-1} to confirm the hypothesis to the research question

As seen in Figure 6, a portion of the hypothesis has been proven, where there is a maximum in the graph at when the driver pendulum's angular frequency resonates with the forced pendulum's angular frequency. Further analysis can be done given the graph in Figure 7.

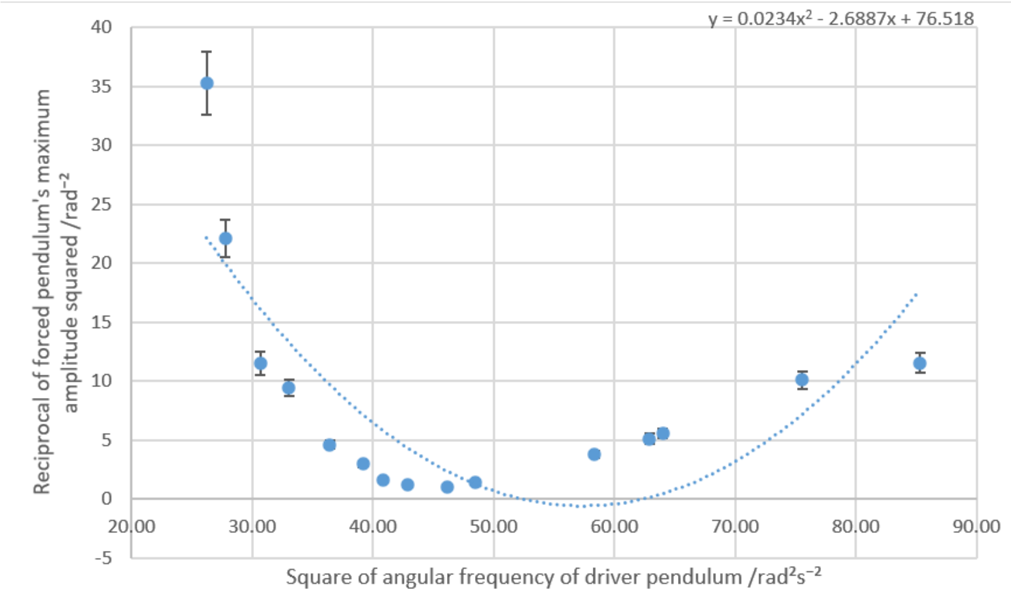

Figure 7: Reciprocal of forced pendulum's maximum amplitude squared $/\mathrm{rad}^{-2}$ as a function of square of angular frequency of driver pendulum $/\mathrm{rad}^2\,\mathrm{s}^{-2}$ to indicate error in data

Evidently, this trend does not fit the parabolic model discussed earlier. However, further inspection may suggest that the first portion follows a trend resembling one that is quadratic, with the final 5 data points breaking the trend. By restricting the data points to not include the final 5 values, a graph is produced that better fits the parabolic model, as seen in Figure 8.

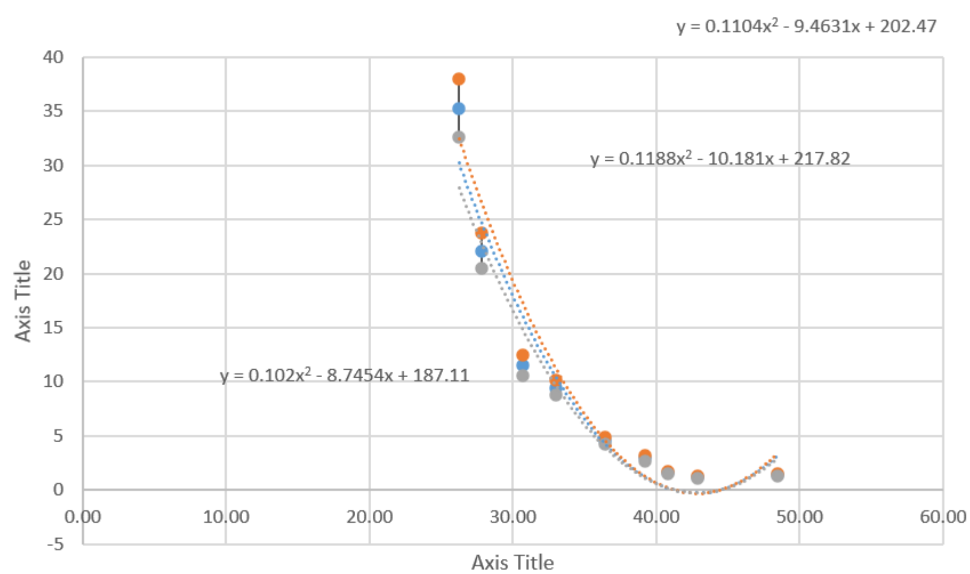

Figure 8: Reciprocal of forced pendulum's maximum amplitude squared $/\mathrm{rad}^{-2}$ as a function of square of angular frequency of driver pendulum $/\mathrm{rad}^2\,\mathrm{s}^{-2}$ to focus on the portion of the data that best follows the parabolic model

The parameters of the best-fit parabola can be used in conjunction with Equation 8 to quantify the data's accuracy. Specifically, the parameters can be used to experimentally determine the forced pendulum's angular frequency, which is comparable to a theoretical value obtained by $\omega = \sqrt{\frac{g}{l}}$.

The following parabolic relationship is formed from the topmost trend and lowest trend maximized and minimized respectively according to y-intercept which is the most significant parameter to determining the experimental angular frequency of the forced pendulum.

$$\frac{1}{A^2} = (0.110\,\mathrm{s}^4 \pm 0.008\,\mathrm{s}^4)\Omega^4 - (9.5\,\mathrm{rad}^2\,\mathrm{s}^{-2} \pm 0.7\,\mathrm{rad}^2\,\mathrm{s}^{-2})\Omega^2 + (2.0\times10^2\,\mathrm{rad}^4 \pm 0.2\times10^2\,\mathrm{rad}^4) \tag{11}$$

The experimental and theoretical angular frequencies of the forced pendulum can be determined as shown below.

$$\begin{aligned}
\frac{1}{C^2} &= 0.110\,\mathrm{s}^4 \pm 0.008\,\mathrm{s}^4, \quad \frac{\omega^4}{C^2} = 2.0\times10^2\,\mathrm{rad}^4 \pm 0.2\times10^2\,\mathrm{rad}^4 \\
\omega^4\left(\frac{1}{C^2}\right) &= 2.0\times10^2\,\mathrm{rad}^4 \pm 0.2\times10^2\,\mathrm{rad}^4 \\
\omega^4(0.110\,\mathrm{s}^4 \pm 0.008\,\mathrm{s}^4) &= 2.0\times10^2\,\mathrm{rad}^4 \pm 0.2\times10^2\,\mathrm{rad}^4 \\
\omega &= 6.5\,\mathrm{rad}\,\mathrm{s}^{-1} \pm 0.3\,\mathrm{rad}\,\mathrm{s}^{-1}
\end{aligned}$$

$$\begin{aligned}
\omega_{theoretical} &= \sqrt{\frac{g}{l}} \\
&= \sqrt{\frac{9.81\,\mathrm{m}\,\mathrm{s}^{-2}}{0.261\,\mathrm{m} \pm 0.005\,\mathrm{m}}} \\
&= 6.13\,\mathrm{rad}\,\mathrm{s}^{-1} \pm 0.06\,\mathrm{rad}\,\mathrm{s}^{-1}
\end{aligned}$$

11 Evaluation

References

15.6 Forced Oscillations | University Physics Volume 1. (n.d.). Retrieved 2023-12-14, from `https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/15-6-forced-oscillations/`

Chasnov, J. (2022a, January). *10: The Simple Pendulum.* Retrieved 2024-01-15, from `https://math.libretexts.org/Bookshelves/Scientific_Computing_Simulations_and_Modeling/Scientific_Computing_(Chasnov)/II%3A_Dynamical_Systems_and_Chaos/10%3A_The_Simple_Pendulum`

Chasnov, J. (2022b, January). *11: The Damped, Driven Pendulum.* Retrieved 2024-01-15, from `https://math.libretexts.org/Bookshelves/Scientific_Computing_Simulations_and_Modeling/Scientific_Computing_(Chasnov)/II%3A_Dynamical_Systems_and_Chaos/11%3A_The_Damped%2C_Driven_Pendulum`

Frictional force is proportional to the velocity [Forum Post]. (2021, September). Retrieved 2024-03-03, from `https://physics.stackexchange.com/q/663846`

Gibbs, K. (2013). *Resonance.* Retrieved 2023-12-13, from `https://www.schoolphysics.co.uk/age16-19/Mechanics/Simple%20harmonic%20motion/text/Resonance_/index.html`

Gonzales, J. (2022, August). *Measurement Uncertainty - Squares, Cubes, and Square Roots.* Retrieved 2024-03-08, from `https://www.youtube.com/watch?v=QR1qJX_bEG8`

Krishnavedala. (25 January 2013, 12:56:47). *English: Diagram depicting the forces acting on a simple pendulum suspended in gravity.* Retrieved 2024-03-03, from `https://commons.wikimedia.org/wiki/File:Pendulum_gravity.svg`

NeuroEng. (2021, September). *Answer to "Frictional force is proportional to the velocity".* Retrieved 2024-03-03, from https://physics.stackexchange.com/a/663854

Pendulum (mechanics) - Wikipedia. (n.d.). Retrieved 2024-03-03, from https://en.wikipedia.org/wiki/Pendulum_(mechanics)

Small-angle approximation. (2023, December). *Wikipedia*. Retrieved 2024-03-03, from https://en.wikipedia.org/w/index.php?title=Small-angle_approximation&oldid=1191246922

Tolani, L. (2021, September). *Answer to "Frictional force is proportional to the velocity".* Retrieved 2024-03-03, from https://physics.stackexchange.com/a/663849

Uncertainties and Error Propagation. (n.d.). Retrieved 2024-03-08, from https://www.geol.lsu.edu/jlorenzo/geophysics/uncertainties/Uncertaintiespart2.html