MATHEMATICS ANALYSIS AND APPROACHES HL
Producing the IB Logo with the Fourier Series

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1 Introduction and Rationale

I have shown interest in visual arts done through the means of software, with particular experience in 3D modelling and animation in Blender and Cinema 4D. It was the ability for the software to interpret any decision the artist accurately while handling the computations and rendering of the numerous aspects in the artwork through the software's complicated structure. Another related type of digital art is vector art in ".svg" files, which by its own already relate to mathematics in the sense that it contains numerous graphed mathematical relationships in order to produce an image.

I discovered an application of Fourier Series, a summation of sinusoidal functions to represent any periodic function, that were capable of converting these vector art pieces into a single series. I found this to be incredibly intriguing, as it requires just one expression to produce the same result done by the numerous mathematical relationships. It is comparable to what are known as "one-liners" in the programming community, where some take it as a challenge to solve an intricate programming problem using just one line of code. I've always admired those who are able to come up with these concise solutions, and I perceive the idea of representing an ".svg" file as a Fourier Series to be a "one-liner" version in the area of math.

Traditionally, the Fourier Series is in the form of a summation of cosine and sin functions, which in the context of this investigation would necessitate finding two separate Fourier Series for the x and y plane. However, there is also a form of the Fourier Series represented as a summation of e^{it} terms. This investigation will primarily explore the latter form, as not only will it allow for a single Fourier Series to account for both the horizontal and vertical axis but also extend the breadth of this investigation into the realm of complex numbers, ultimately allowing me to render the IB logo through a single summation.

2 Aim and Methodology

The objective of this investigation is to link Fourier Series with complex numbers to create a single series capable of reproducing the IB logo on the Argand plane using the ".svg" vector art file of the IB logo as the original reference and source.

The common sine-cosine form of the Fourier Series involves calculus, series, and trigonometry. To lay the foundations surrounding Fourier Series, an example will be explored in Section 3.1 that will discuss the overarching idea and common applications of Fourier Series exemplified by a simple periodic function as well as how the Fourier Series for that example function would be evaluated.

The topic of complex numbers will then be introduced, where after connecting the relevance of Euler's formula to Fourier Series, an understanding of the mechanism of the Fourier Series using complex numbers will be developed.

Next, Bézier curves will be explored, which involves polynomials to control the "lerp" of the Bézier curve and complex numbers to define a 2 dimensional position on the Argand Plane. It will then become evident that the necessity for Bézier curves comes from its ubiquitous presence in the ".svg" file of the IB logo.

Finally, the render of the IB logo will be done with the aid of computer programming, which is essential in this investigation that involves the computation of the numerous integrals evaluated as Riemann Sums, the computation of the extensive Fourier Series, as well as presenting the resulting graph on the Argand plane through plotted pixels on an image. For this investigation, a JavaScript program will be written with NodeJS as the runtime (*Node.Js*, n.d.). JavaScript NPM packages will offer additional preprogrammed tools, with the following packages used in this investigation:

- "mathjs" (de Jong, n.d.): Allows computation of complex numbers
- "read-excel-file" (Kuchumov, n.d.): Allows reading the Excel spreadsheet containing data of the original IB logo
- "canvas" (Automattic, n.d.): Allows the result to be rendered on an image
- "prompt-sync" (Fragomeni, n.d.): Enables input during the program's runtime through a command line interface, allowing for observations of how the results vary depending on parameters in the Fourier Series

3 Background Information

3.1 Overarching idea of the Fourier Series

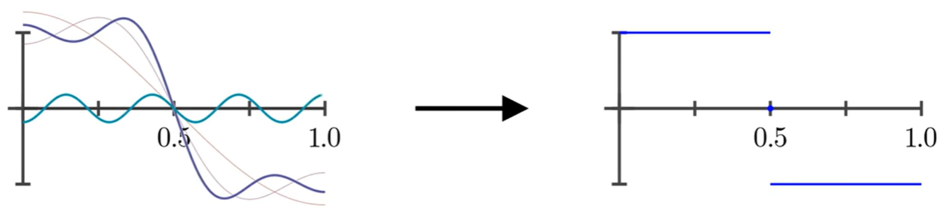
A periodic function is one where the output for a particular input equals to the output for the sum of the same input and the value of the function's period. This can be represented mathematically as:

$$f(x) = f(x + P)$$
$$\text{where } P = \text{ the period of the function}$$

The sine wave is a common example of a periodic function, which can be easily graphed from $y = \sin x$. However, there are periodic functions that are difficult to graph with a simple expression, such as one that alternates between 1 and -1 or one that is shaped as a zigzag.

This is the motivation behind the Fourier Series, which is to be able to represent period functions that normally can't be represented by an algebraic function.

The idea behind the Fourier Series is to take an infinite sum of varying sinusoidal functions such that a desired periodic function is produced.



$$\frac{4}{\pi}\left(\frac{\cos(\pi x)}{1} - \frac{\cos(3\pi x)}{3} + \frac{\cos(5\pi x)}{5} - \cdots\right) = \begin{cases} 1 & \text{if } x < 0.5 \\ 0 & \text{if } x = 0.5 \\ -1 & \text{if } x > 0.5 \end{cases}$$

For $0 \leq x \leq 1$

Figure 1: Visualization of the mechanism of the Fourier Series (Sanderson, 2019). The purple line is the periodic function resulting from the previous iteration, the blue line is the sinusoidal function to be added in the next iteration.

The form of the Fourier Series in Equation 1 is considered to be the sin-cosine form and was the formula used to determine the Fourier Series of the function in Figure 1. This is the most common form of the Fourier Series as it is capable of modelling periodic functions such as sawtooth waves and square waves that can't be expressed in any other way yet are common in real world contexts such as analysis of "periodic signals in experimentation" (*Fourier Series - Definition, Formula, Applications and Examples*, n.d.). It is also crucial in solving partial differential equations such as the heat equation (Sanderson, 2019).

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \tag{1}$$

Where $L =$ Half of the period of the function

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \tag{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \tag{3}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \tag{4}$$

Note: Equations 1 to 4 came from (Tisdell, 2009).

For calculations of a Fourier Series using the sine-cosine form, it is important to consider whether the function of concern is even or odd. This is because integrating an odd function from $-L$ to L would always equal to zero, with the mathematical explanation shown below accompanied by the visual in Figure 2.

$$I_{odd} = \int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx$$

$$\int_{-L}^{0} f(x)\,dx = -\int_{0}^{L} f(x)\,dx$$

$$\therefore I_{odd} = 0$$

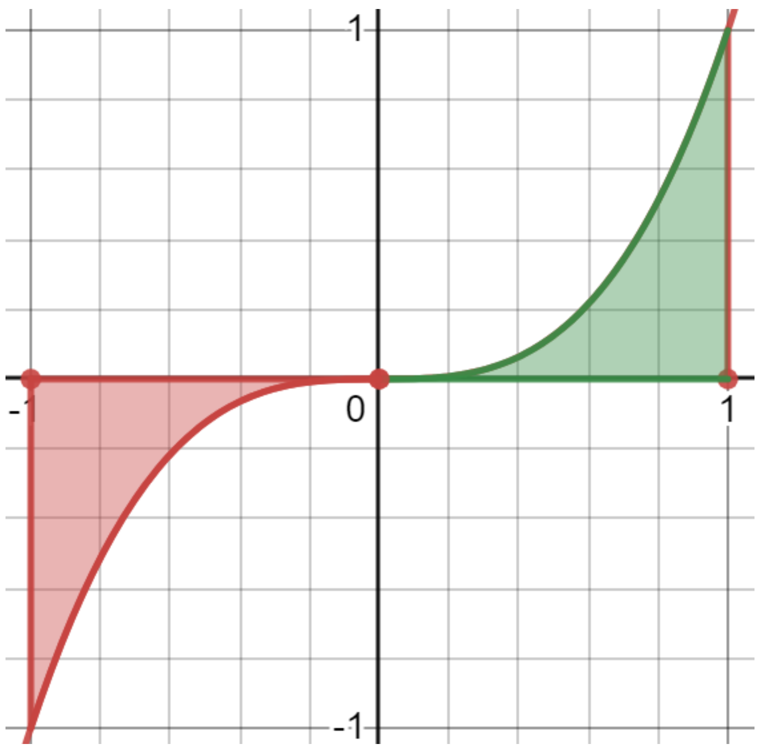


Figure 2: Desmos visual of an integral of an odd function from -1 to 1 equalling to 0 due to the summation of 2 integrals that are equal in magnitude but opposite in signage (*Desmos | Graphing Calculator*, n.d.)

Additionally, any integral of an even function from $-L$ to L can be simplified as shown below, with a visual explanation shown in Figure 3.

$$I_{even} = \int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx$$

$$\int_{-L}^{0} f(x)\,dx = \int_{0}^{L} f(x)\,dx$$

$$\therefore I_{even} = 2\int_{0}^{L} f(x)\,dx$$

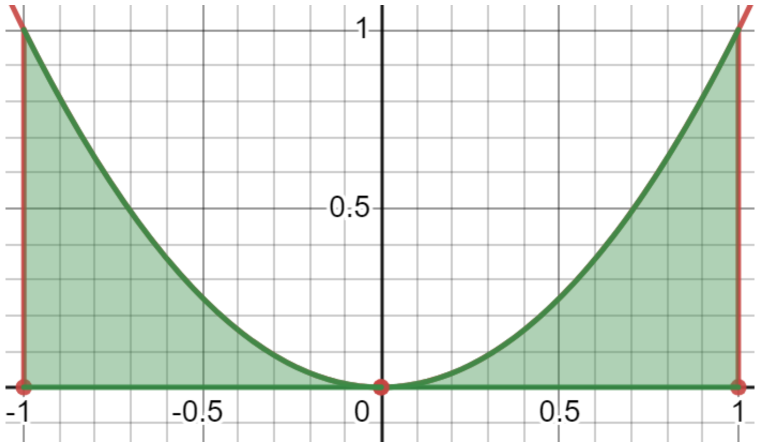


Figure 3: Desmos visual of an integral of an even function from -1 to 1 equalling to double the integral of the same even function from 0 to 1 due to the summation of 2 equal integrals (*Desmos | Graphing Calculator*, n.d.)

Finally, while a_n and b_n both involve multiplying two even/odd functions together, there is a method of predicting the type of function that will result.

- Multiplying even by even produces even because the two same signs of y will emerge for $x < 0$ and $x > 0$
- Multiplying even by odd produces odd because the signage of y for one of $x < 0$ or $x > 0$ must be positive and the other negative
- Multiplying odd by odd produces even because either the signage of y in the two functions will be opposite for both $x < 0$ and $x > 0$ or the signage will be equal, therefore producing a final signage for y of negative and positive respectively.

The determination of the Fourier Series of the function in Figure 1 is shown below.

$$f(x) = \begin{cases} & \vdots \\ -1 & -1.5 < x < -0.5 \\ 1 & -0.5 < x < 0.5 \\ -1 & 0.5 < x < 1.5 \\ & \vdots \end{cases}$$

$$f(x) = f(x+2)$$
$$\therefore 2 = 2L$$
$$L = 1$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \underbrace{f(x)}_{even}\,dx = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \int_{0}^{1} f(x)\,dx = \int_{0}^{0.5} dx - \int_{0.5}^{1} dx = 0$$

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{even} \underbrace{\cos\frac{n\pi x}{L}}_{even}\, dx = \int_{-1}^{1} \underbrace{f(x)\cos(n\pi x)}_{even}\, dx \\
&= 2\int_0^1 f(x)\cos(n\pi x)\, dx \\
&= 2\left(\int_0^{0.5} \cos(n\pi x)\, dx - \int_{0.5}^{1} \cos(n\pi x)\, dx\right) \\
&= 2\left(\frac{\sin(n\pi x)}{n\pi}\Big]_0^{0.5} - \frac{\sin(n\pi x)}{n\pi}\Big]_{0.5}^{1}\right) \\
&= 2\left(\frac{2\sin\left(\frac{n\pi}{2}\right)}{n\pi} - \frac{\sin(n\pi)}{n\pi}\right) \\
&\because n \in \mathbb{N}, \quad a_n = 4\left(\frac{\sin(\frac{n\pi}{2})}{n\pi}\right) \\
a_n &= \begin{cases} 0 & n = 2, 4, 6, \cdots \\ \frac{4}{n\pi} & n = 1, 5, 9, \cdots \\ \frac{-4}{n\pi} & n = 3, 7, 11, \cdots \end{cases} \\
b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{even} \underbrace{\sin\frac{n\pi x}{L}}_{odd}\, dx = \int_{-1}^{1} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{odd}\, dx = 0
\end{aligned}
$$

$$
\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}) \\
&= \frac{4}{\pi}\left(\frac{\cos\pi x}{1} - \frac{3\cos\pi x}{3} + \frac{5\cos\pi x}{5} - \cdots\right)
\end{aligned}
$$

Note: Information regarding even and odd functions and technique to solving a Fourier Series using the sine-cosine formula were of reference to (Tisdell, 2009).

As stated in the aim, this investigation will explore Fourier Series with complex numbers to achieve a 2 dimensional graphic, which will involve a form of the Fourier Series differing from what is in Equation 1.

3.2 Idea of drawing with the Fourier series explained with the Cartesian Plane

The rule is for eligible drawings to be any that can be drawn by starting at one point on a Cartesian plane and, without lifting the hypothetical plane throughout the entire sketch, return to the exact same point.

Defining the variable t as time, $t = 0$ will represent the point in time when the drawing began and $t = 1$ will represent the point in time when the drawing ended.

Each point of the drawing on the plane will be defined by $P(x(t), y(t))$, where $x(t)$ and $y(t)$ are both functions with an input of t that indicates the coordinates after some amount of time passed of the pen's progress through the drawing.

Assuming that any function can be represented as a Fourier Series, then $x(t)$ and $y(t)$ can be any function and therefore, by extracting the coordinates on a graph of any drawing obeying the rule described earlier, the Fourier Series of $x(t)$ and $y(t)$ can be determined and therefore produce the desired drawing for $t \in [0, 1]$

As a simple example that does not require a Fourier Series, we can take a unit circle defined by $x^2 + y^2 = 1$ as the drawing. Without consideration of Fourier Series, it quickly becomes evident of what $x(t)$ and $y(t)$ are. If the attempt is to model a circle on the Cartesian Plane with the x and y coordinates being functions of time (t) where $t \in [0, 1]$ with a period of 1, then by connecting back to trigonometry, this is a unit circle where t has a relationship to the angle in standard position defined by $2\pi t = \theta$. Therefore, $x(t) = \cos(2\pi t)$ and $y(t) = \sin(2\pi t)$ as illustrated in Figure 4.

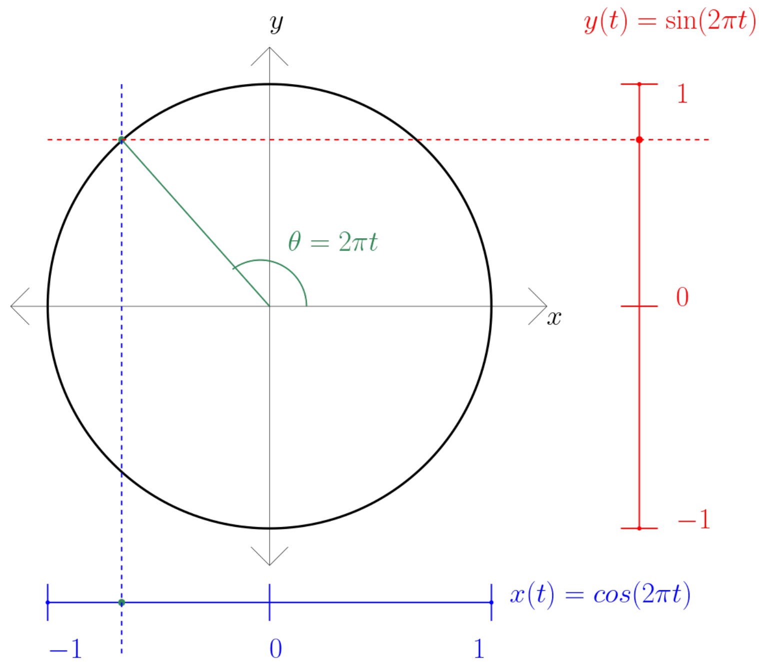


Figure 4: Simple example of the general idea in rendering a graphic using periodic functions on the Cartesian Plane. Figure produced using "Ipe" (*The Ipe Extensible Drawing Editor*, n.d.)

3.3 Enriched application to draw on the Argand Plane

In Section 3.1, it was explained that the overarching idea of Fourier Series is to represent any periodic function as a sum of sinusoidal functions. In Section 3.2, the general approach to rendering a graphic through periodic functions was discussed, using the unit circle as an example. Recall that the final goal is to render a graphic using Fourier Series on the Argand Plane, meaning that a connection must be made between complex numbers and the idea discussed in Section 3.2. Fortunately, Euler's formula as defined in Equation 5 will provide the strong link between complex numbers and the idea surrounding rendering a graphic using the Fourier Series.

$$e^{it} = \cos(t) + i\sin(t) \tag{5}$$

For some value of t, Euler's formula, when graphed on the Argand Plane, will appear as a point where its position along the horizontal (real) axis will equal to the real component of e^{it} which is $\cos(t)$ and its position along the vertical (imaginary) axis will equal to the imaginary component of e^{it} which is $i\sin(t)$. It therefore becomes evident that Euler's formula manifests the same idea shown in Figure 4 of moving a point on a 2 dimensional plane according to a periodic function of time indicating the point's position on the x-axis and another periodic function of time indicating the point's position on the y-axis.

Before proceeding, recall that 2π was multiplied within both sinusoidal functions of $y(t) = \sin(2\pi t)$ and $x(t) = \cos(2\pi t)$ in Section 3.2 such that both periodic functions had a period of 1 to satisfy the restraint that $t \in [0, 1]$. This restraint still applies, therefore 2π must be incorporated to Euler's formula as well as shown below.

$$e^{2\pi it} = \cos(2\pi t) + i\sin(2\pi t) \tag{6}$$

For a complicated graphic, it is easier to think of Euler's formula to be a rotating vector on the Cartesian Plane or the Argand Plane (Sanderson, 2019). For the Cartesian Plane, each vector would be a grouping of one sinusoidal term of the Fourier Series representing $x(t)$ and one sinusoidal term of the Fourier Series representing $y(t)$, with the amplitudes of the two grouped terms equal to each other. This would mean that the initial angle of standard position of the vector will not always be 0 because if they were all 0, then the graphic must begin on the x-axis with a positive x value. The reason behind this varying initial angle will be explored more definitively later on in the section; however, the factor that is important to consider is that this will involve non-zero values of the Fourier Series of $x(t)$ and $y(t)$ when $t = 0$ that is accomplished by the inclusion of amplitudes as well as both sine and cosine terms in Equation 1. This means that for a rotating vector modelled by Euler's formula, there must be variable(s) that allow the definition of both the amplitude of the vector and the initial angle in standard position of the vector.

This will be accomplished by the variable $c_n \in \mathbb{C}$, where n is the frequency of revolution of the rotating vector, which will be incorporated into Equation 6 to get $c_n e^{n \cdot 2\pi it}$. This will not only allow for the vector to be scaled but also for the starting rotation of the vector to be defined due to c_n being an element of the complex numbers. c_n can be expressed as

$$c_n = Ae^{i\theta}$$

A will indicate the factor to which the vector will be scaled by. Because the original length of the vector was 1, then the factor will directly indicate the length of the vector.

$e^{i\theta}$ will indicate the starting rotation of the vector, where θ is the starting rotation angle in radians.

Lastly, it is important to note that when using Euler's formula for Fourier series, $n \in \mathbb{Z}$. In the sine-cosine form of Fourier Series in Equation 1, n was defined by $n \in \mathbb{N}$. Returning to the idea surrounding a grouping of two sinusoidal terms associated with the horizontal and vertical axes separately to create a rotating vector, then a vector that rotates clockwise can be achieved by a particular mixture of parameters between the two sinusoidal functions (e.g. $x(t) = cos(2\pi t)$, $y(t) = sin(-2\pi t) = -sin(2\pi t)$). This gives the reason behind why negative values of n are not necessary in the sine-cosine form of Fourier Series. However, negative n values are necessary to include for Euler's formula, as only the sign on the n value is capable of defining the direction of rotation in Euler's formula.

It could be imagined that the final Fourier Series that produces a desired drawing is the sum of multiple vectors of different magnitudes rotating at different frequencies indicated by $n, n \in \mathbb{Z}$. Therefore, if $f(t)$ is defined to be the function that represents the drawing, this can be expressed mathematically as:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{n \cdot 2\pi i t} \tag{7}$$

The main concern is how the value of c_n will be determined for a specific drawing.

The easiest way to start off is by first considering $c_0 e^{0 \cdot 2\pi i t}$. This can be simplified into c_0, but what does this mean? This is the vector that is rotating at a frequency of 0, which means that it is static. It can therefore be defined to be the "centre of mass"(Sanderson, 2019) of the entire function.

If we take discrete intervals of t and obtain the value of $f(t)$, then by taking the average of all those values, we obtain a complex number close to c_0. This is illustrated by Figure 5.

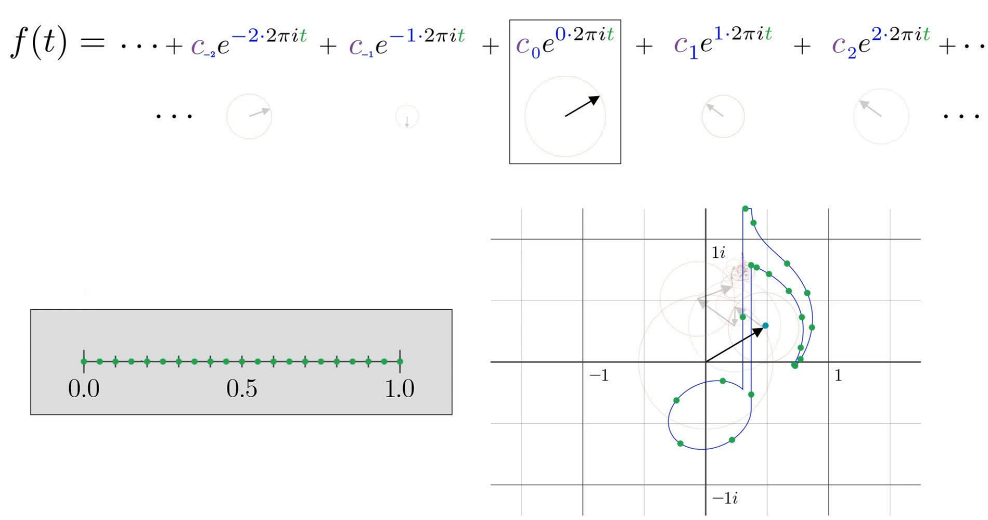


Figure 5: Averaging points throughout $f(t)$ illustrated (Sanderson, 2019). The number line represent t, and the red dots on the musical note represent the resulting values of $f(t)$

With finer and finer intervals of t, the result becomes more and more accurate to the value of c_0. Because multiplying by Δt is the same as dividing by the number of intervals of t, it can be said that the average of c_0 is:

$$c_0 = \lim_{\Delta t \to 0} \sum_{t=0}^{\frac{1}{\Delta t}} f(t \cdot \Delta t)\Delta t \tag{8}$$

From Equation **(8)**, c_0 can ultimately be expressed as:

$$c_0 = \int_0^1 f(t)\, dt \tag{9}$$

We can use the same technique for all other values of n, but the problem is for all other values of n that are not 0, the vector is rotating, therefore it doesn't make sense to take the average of the rotating vector.

Recalling the thorough definition of $f(t)$ stated earlier in Equation 7, we can substitute Equation 7 into Equation 9.

$$\begin{aligned}
c_0 &= \int_0^1 \sum_{n=-\infty}^{\infty} c_n e^{n \cdot 2\pi i t}\, dt \\
c_0 &= \int_0^1 \left(\cdots + c_{-1} e^{-1 \cdot 2\pi i t} + c_0 e^{0 \cdot 2\pi i t} + c_1 e^{1 \cdot 2\pi i t} + \cdots \right) dt \\
c_0 &= \cdots + \int_0^1 c_{-1} e^{-1 \cdot 2\pi i t}\, dt + \int_0^1 c_0 e^{0 \cdot 2\pi i t}\, dt + \int_0^1 c_1 e^{1 \cdot 2\pi i t}\, dt + \cdots
\end{aligned}$$

Remember that $\int_0^1 c_0 e^{0 \cdot 2\pi i t}\, dt$ was easy to simplify as the power cancels out from being raised to 0. In turn, it can be further evaluated to be just c_0, as shown below:

$$\begin{aligned}
&\int_0^1 c_0 e^{0 \cdot 2\pi i t}\, dt \\
&= \int_0^1 c_0\, dt \\
&= c_0 t]_0^1 \\
&= c_0
\end{aligned}$$

Additionally, if all the other integrals were thought of as the average of all the points produced when its vector rotates by one revolution, then it can be argued that each integral, when evaluated, would be 0.

This idea is illustrated in Figure 6.

$$\cdots + \underbrace{\int_0^1 c_{-1}e^{-1\cdot 2\pi it}dt}_{0} + \underbrace{\int_0^1 c_0 e^{0\cdot 2\pi it}dt}_{c_0} + \underbrace{\int_0^1 c_1 e^{1\cdot 2\pi it}dt}_{0} + \underbrace{\int_0^1 c_2 e^{2\cdot 2\pi it}dt}_{0} + \cdots$$

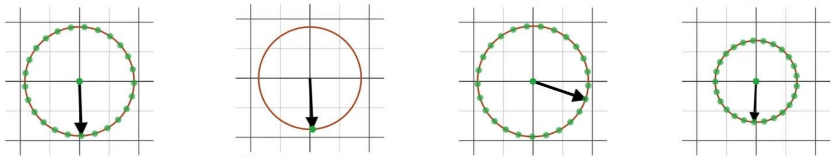


Figure 6: Illustration of averaging all the points on a circle (Sanderson, 2019).

If we were to multiply $f(t)$ by $e^{-n\cdot 2\pi it}$, then an effect can occur where all of the power's exponents will decrease by n. Ultimately, we will get a similar scenario in the summation of all the integrals where all integrals are evaluated to become 0 except for the integral where $n = 0$, which evaluates to c_0. However, the multiplication of the two powers will cause the integral of some n value to simplify to just c_n.

This mechanism is mathematically shown below.

$$
\begin{aligned}
&n = 1 \\
&c_1 = \int_0^1 f(t)e^{-1\cdot 2\pi it}\,dt \\
&c_1 = \cdots + \int_0^1 c_{-1}e^{-1\cdot 2\pi it}\cdot e^{-1\cdot 2\pi it}\,dt \\
&+ \int_0^1 c_0 e^{0\cdot 2\pi it}\cdot e^{-1\cdot 2\pi it}\,dt + \int_0^1 c_1 e^{1\cdot 2\pi it}\cdot e^{-1\cdot 2\pi it}\,dt + \cdots \\
&c_1 = \cdots + \int_0^1 c_{-1}e^{-2\cdot 2\pi it}\,dt + \int_0^1 c_0 e^{-1\cdot 2\pi it}\,dt + \int_0^1 c_1 e^{0\cdot 2\pi it}\,dt + \cdots \\
&c_1 = \cdots + 0 + 0 + c_1 + 0 + 0 + \cdots \\
&c_1 = c_1
\end{aligned}
$$

Therefore, c_n can be defined as:

$$c_n = \int_0^1 f(t)e^{-n\cdot 2\pi it}\,dt \tag{10}$$

Equation 10 is what will be used in order to determine each value of c_n

All concepts of this section come from (Sanderson, 2019).

3.4 Bézier curves

3.4.1 Linear interpolation

Suppose there is a free moving point P on the line drawn between the stationary points P_0 and P_1 as illustrated on Figure 7.

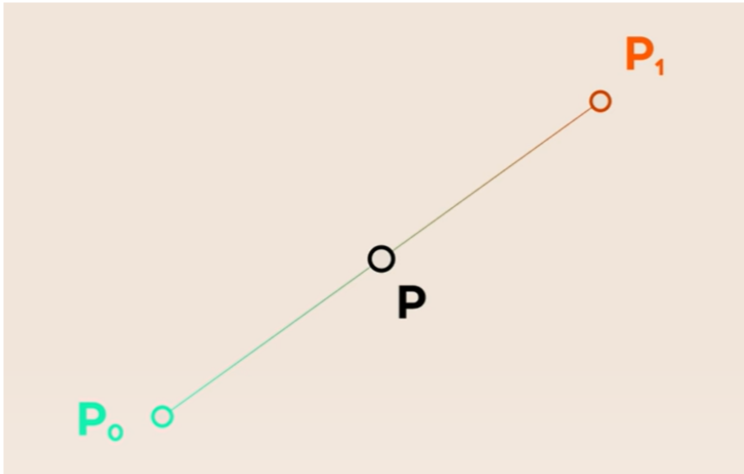


Figure 7: 2 point Bézier (Holmer, 2021).

The position of P along the line will be defined by t, which is thought of to be the percentage that t is along the line, where $t = 100\%$ is located at P_1 and $t = 0\%$ is located at P_0. This process is called linear interpolation (lerp) and can be expressed mathematically as:

$$P = lerp(P_0, P_1, t) = (1 - t)P_0 + tP_1 \tag{11}$$

where $lerp(P_0, P_1, t)$ is the function that represents the process of lerping.

3.4.2 Cubic Bézier curves

While there are many types of Bézier curves, this investigation will only focus on Cubic Bézier curves as they are the only type of Bézier curve used in the IB Logo.

Suppose that instead of just 2 points, there are 4 points on the plane (P_0, P_1, P_2, P_3). The lines are drawn so that P_0 connects to P_1, P_1 connects to P_2, and P_2 connects to P_3 such that P_0 and P_3 are both endpoints.

Each line segment has its own individual moving point, with all moving points "lerping" according to the same universal t value.

Lines can then be drawn between these new moving points, and these points lerp according to the same t value.

This process of adding points onto lines and drawing new lines until just one moving point along a single line is created, which will be the "pen" of the Bézier curve.

This is all illustrated by Figures 8 and 9.

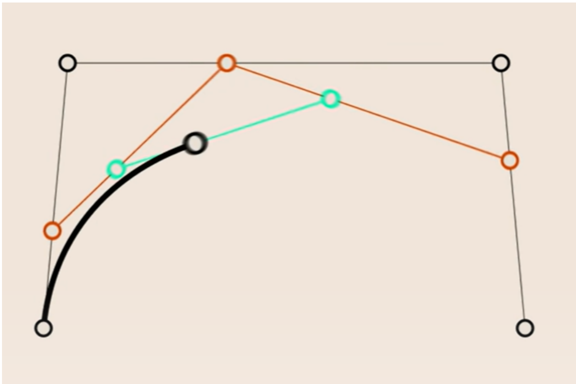


Figure 8: Cubic Bézier around the middle of its lerp (Holmer, 2021).

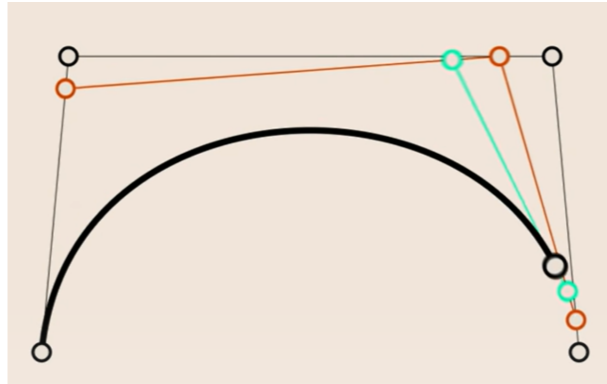


Figure 9: Cubic Bézier nearing the end of its lerp (Holmer, 2021).

3.4.3 Bernstein Polynomial Form

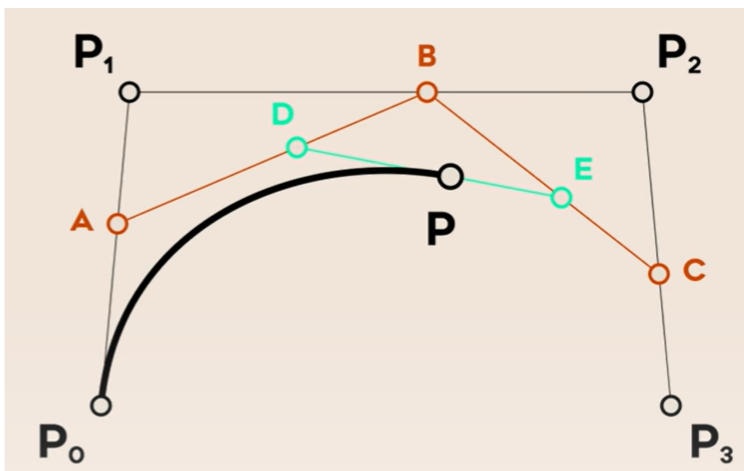


Figure 10: Cubic Bézier curve with labelled points (Holmer, 2021).

We can derive a general formula for P in terms of P_0, P_1, P_2, P_3, t by expanding out all of the lerp functions for each point in a Cubic Bézier curve and substituting when necessary, as shown below. Variables are with reference to Figure 10.

$$
\begin{aligned}
A &= lerp(P_0, P_1, t) \\
B &= lerp(P_1, P_2, t) \\
C &= lerp(P_2, P_3, t) \\
D &= lerp(A, B, t) \\
E &= lerp(B, C, t) \\
P &= lerp(D, E, t)
\end{aligned}
$$

From Equation 11
$lerp(P_0, P_1, t) = (1-t)P_0 + tP_1$

$A = (1-t)P_0 + tP_1$
$B = (1-t)P_1 + tP_2$
$C = (1-t)P_2 + tP_3$
$D = (1-t)A + tB$
$E = (1-t)B + tC$
$P = (1-t)D + tE$

$P = P_0(-t^3 + 3t^2 - 3t + 1) + P_1(3t^3 - 6t^2 + 3t)$
$+ P_2(-3t^3 + 3t^2) + P_3(t^3)$

Therefore, the general formula for P in terms of $P_0, P_1, P_2, P_3, \text{and} t$ is

$$P = P_0(-t^3 + 3t^2 - 3t + 1) + P_1(3t^3 - 6t^2 + 3t) + P_2(-3t^3 + 3t^2) + P_3(t^3) \tag{12}$$

All information regarding Bézier curves come from (Holmer, 2021).

3.5 Composition of ".svg" files

SVG files are "XML-based vector image format for defining two-dimensional graphics" ("SVG", 2023).

An SVG's two-dimensional size is defined by the "viewBox" attribute, which will indicate the width and height of the canvas.

There are many attributes for drawing predefined shapes, but what we are concerned with is the "path" attribute, which allows for custom shapes and is what the IB logo will be composed of.

The "path" attribute will be defined by various curves that are indicated by letters. The letters that are of concern in this investigation are listed below.

- M: Move the pen to somewhere without drawing a line
- C: Cubic Bézier defined by the starting location, 2 control points, and an indicated end location
- L: Draw a vertical line to some coordinate
- V: Draw a vertical line up to some y-coordinate
- Z: Draw a straight line back to the first point of the path

Note that when the letter is lowercase, the coordinates specified will be relative to the current position (ex. $m\ \Delta x\ \Delta y$). If the letter is uppercase, the coordinates specified will be absolute to the canvas (ex. $M\ x\ y$).

All information regarding paths comes from (Mozilla Developer Network, 2023a).

4 Calculation and Computation

4.1 Analysis of the ".svg" File

On a canvas of height 198.426px and width 198.425px, a few lines of the path of the IB Logo is presented below, with the entire path presented in Section A of the appendix (International Baccalaureate Organisation, 2013). **Note: SVG files interpret positive y-values as downwards.**

```
<path fill="url(#SVGID_1_)" d="
M198.425,99.155
c 0,54.833 -45.075,99.271 -100.685,99.271
c -47.27, 0 -86.91 -32.11 -97.74 -75.416
c 19.703 -0.222, 38.391 -4.567, 55.26 -12.149
V 72.226
...
C 68.199, 48.705, 71.582, 47.359, 74.363, 44.669
z"/>
```

Disregarding any "Move" actions, there are 54 pen strokes in total for the IB Logo. This means that for the starting piecewise form of $f(t)$, each piece will have a domain of $\frac{1}{54}(n-1) \leq t < \frac{1}{54}n$, where n represents the index of the current pen stroke.

The contents of this path were moved into an Excel Spreadsheet for the sake of organization, with part of the spreadsheet shown in Figure 11.

	A	B	C	D	E	F	G	H	I	J	K
1		P_0		P_1		P_2		P_3		Time	
2	Operation	X	Y	X	Y	X	Y	X	Y	Initial	Final
3	M	99.2125	0.058								
4	C	99.2125	0.058	99.2125	-54.775	54.1375	-99.213	-1.4725	-99.213	0	0.018519
5	C	-1.4725	-99.213	-48.7425	-99.213	-88.3825	-67.103	-99.2125	-23.797	0.018519	0.037037
6	C	-99.2125	-23.797	-79.5095	-23.575	-60.8215	-19.23	-43.9525	-11.648	0.037037	0.055556
7	V	-43.9525	-11.648	-43.9525	26.987					0.055556	0.074074
8	C	-43.9525	26.987	-43.9525	30.049	-45.0315	32.287	-47.1865	33.703	0.074074	0.092593
9	C	-47.1865	33.703	-49.3375	35.112	-53.1615	35.823	-58.6655	35.823	0.092593	0.111111
10	V	-58.6655	35.823	-58.6655	39.402					0.111111	0.12963
11	C	-58.6655	39.402	-51.3705	39.642	-44.6815	40.067	-38.6025	40.669	0.12963	0.148148
12	C	-38.6025	40.669	-32.5165	41.273	-27.2715	42.182	-22.8625	43.39	0.148148	0.166667
13	C	-22.8625	43.39	-16.7765	43.994	-11.5315	44.903	-7.1225	46.111	0.166667	0.185185
14	L	-7.1225	46.111	-7.0305	-10.002					0.185185	0.203704
15	C	-7.0305	-10.002	-7.0305	-13.184	-7.0305	-16.656	-7.0305	-20.424	0.203704	0.222222
16	C	-7.0305	-20.424	-7.0305	-24.081	-7.0855	-27.674	-7.2085	-31.206	0.222222	0.240741
17	C	-7.2085	-31.206	-7.3305	-34.743	-7.5115	-38.039	-7.7475	-41.101	0.240741	0.259259
18	C	-7.7475	-41.101	-7.9895	-44.166	-8.2875	-46.754	-8.6485	-48.872	0.259259	0.277778
19	C	-8.6485	-48.872	-2.6655	-52.296	3.9725	-55.121	11.2725	-57.362	0.277778	0.296296
20	C	11.2725	-57.362	18.5715	-59.601	25.3875	-60.715	31.7275	-60.715	0.296296	0.314815
21	C	31.7275	-60.715	37.1105	-60.715	42.3165	-59.92	47.3395	-58.328	0.314815	0.333333
22	C	47.3395	-58.328	52.3655	-56.738	56.8235	-54.412	60.7065	-51.338	0.333333	0.351852
23	C	60.7065	-51.338	64.5955	-48.272	67.7395	-44.505	70.1345	-40.016	0.351852	0.37037
24	C	70.1345	-40.016	72.5245	-35.531	73.7195	-30.453	73.7195	-24.79	0.37037	0.388889
25	C	73.7195	-24.79	73.7195	-19.602	72.8215	-14.821	71.0255	-10.455	0.388889	0.407407
26	C	71.0255	-10.455	69.2305	-6.087	66.7125	-2.316	63.4735	0.874	0.407407	0.425926
27	C	63.4735	0.874	60.2395	4.064	56.4135	6.564	51.9795	8.394	0.425926	0.444444
28	C	51.9795	8.394	47.5425	10.228	42.6935	11.143	37.4225	11.143	0.444444	0.462963
29	C	37.4225	11.143	33.4675	11.143	29.4305	10.456	25.2965	9.095	0.462963	0.481481
30	C	25.2965	9.095	21.1575	7.724	17.4175	6.03	14.0655	4.016	0.481481	0.5
31	V	14.0655	4.016	14.0655	17.996					0.5	0.518519
32	C	14.0655	17.996	32.1985	36.68	44.9595	60.467	49.8745	86.966	0.518519	0.537037
33	C	49.8745	86.966	27.4315	96.362	1.4225	97.585	-23.1775	88.378	0.537037	0.555556
34	C	-23.1775	88.378	-66.2455	78.152	-86.144	57.16	-95.8035	32.157	0.555556	0.574074

Figure 11: Path data imported in an Excel Spreadsheet

Using JavaScript under the NodeJS Runtime, this spreadsheet can be read using the NPM Package "read-excel-file" (Kuchumov, n.d.). This package will parse the spreadsheet into an array of rows, with each row being an array of cells. Then, the code presented below will iterate through each row, with access to all the cells in that row indexed by zero-based numbering.

```
readXlsxFile('./SVG_Coordinate_Spreadsheet.xlsx')
.then((rows) => {
    // `rows` is an array of rows
    // each row being an array of cells.
    rows.forEach(row => {
        // iterate through each row, accessing the
        // cells of each row
    });
})
```

For each row, the code checks if the operation for the current row is relevant for drawing. Only "Move" (M) is ignored, and all "Z" operations were written down as "line" (L) operations. Using the data within each row, the parameter of c_n can be computed for each value of n.

4.2 Computation of parameters

Given that $f(t)$ is able to be determined as a piecewise function from the previous section, the desired summation representing $f(t)$ can be determined.

Referring back to Section 3.3, c_n is defined as:

$$c_n = \int_0^1 f(t)e^{-n\cdot 2\pi it}\,dt$$

This means that every c_n must be determined individually. Therefore, it becomes evident that it is unreasonable to evaluate the infinite sum of $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{n\cdot 2\pi it}$, and that the limits of the summation must be defined.

Let's rewrite $f(t)$ as:

$$f(t) = \sum_{n=-k}^{k} c_n e^{n\cdot 2\pi it}$$

so that k indicates the selected frequency of the two fastest vectors that are spinning in opposite directions to each other.

As $k \to \infty$, $f(t)$ becomes more and more accurate to the original drawing, which will be demonstrated by performing distinct analyses for various values of k.

4.2.1 Approach to integration

$$c_n = \int_0^1 f(t)e^{-n\cdot 2\pi it}\,dt$$

The integral above can be evaluated for some value of n by taking some small value to represent Δt, in which by summing up the values from $f(t)e^{-n\cdot 2\pi it}$ produced by each increment of t by Δt, a value close to the original integral can be determined (Sanderson, 2019). This is known as a Riemann Sum ("Riemann Sum", 2023).

$$c_n = \sum_{t=0}^{\frac{1}{\Delta t}} f(t\cdot\Delta t)e^{-n\cdot 2\pi i(t\cdot\Delta t)}\Delta t \tag{13}$$

Because a small value of Δt must be used, c_n must be evaluated through a computer. Additionally, since the original version of $f(t)$ can be thought of as a piecewise function composed of Bézier curves and lines, then Equation 13 must be split up into multiple sums, each accounting for the domain of each piece of the piecewise function ($\frac{1}{54}(n-1) \leq t < \frac{1}{54}n$). From this consideration, Equation 13 can be expressed as the following:

$$\begin{aligned} c_n = &\sum_{t=0}^{(\frac{1}{54})(\frac{1}{\Delta t})} f(t\cdot\Delta t)e^{-n\cdot 2\pi i(t\cdot\Delta t)}\Delta t + \sum_{t=(\frac{1}{54})(\frac{1}{\Delta t})}^{(\frac{2}{54})(\frac{1}{\Delta t})} f(t\cdot\Delta t)e^{-n\cdot 2\pi i(t\cdot\Delta t)}\Delta t \\ &+ \cdots + \sum_{t=(\frac{53}{54})(\frac{1}{\Delta t})}^{(\frac{54}{54})(\frac{1}{\Delta t})} f(t\cdot\Delta t)e^{-n\cdot 2\pi i(t\cdot\Delta t)}\Delta t \end{aligned} \tag{14}$$

Additionally, in order to evaluate the complete range of $f(t)$ for some section, $f(t)$ must take in the complete domain of $[0, 1]$. The reason for this will be elaborated on in Section 4.2.2, but because each section will only account for the same domain defined by $[\frac{1}{54}(n-1), \frac{1}{54}n)$, then $f(t)$ must take an input of a manipulated value of t. To simplify this manipulation, let's make every summation in Equation 14 have limits from 0 to $\frac{1}{54\Delta t}$ and add the initial time of each section to $t \cdot \Delta t$ in the expression $e^{-n \cdot 2\pi i(t \cdot \Delta t)}$ so that the calculation for such expression stays absolute to the universal time passage. The input of $f(t)$ can therefore be multiplied by 54 so that when the upper limit of the summation ($\frac{1}{54\Delta t}$) is multiplied by 54 and Δt, then the resulting value is 1 ($\frac{1}{54\Delta t} \cdot 54\Delta t = 1$). This therefore offers a domain for $f(t)$ from 0 to 1 even if $t \in [\frac{1}{54}(n-1), \frac{1}{54}n)$ for each summation.

Overall, if we let $f_n(t)$ equal to the function $f(t)$ isolated to just the n^{th} section, Equation 13 can be expressed as:

$$\begin{aligned} c_n = &\sum_{t=0}^{\frac{1}{54\Delta t}} f_1(54t \cdot \Delta t)e^{-n \cdot 2\pi i(t \cdot \Delta t + 0)}\Delta t \\ &+ \sum_{t=0}^{\frac{1}{54\Delta t}} f_2(54t \cdot \Delta t)e^{-n \cdot 2\pi i(t \cdot \Delta t + \frac{1}{54})}\Delta t \\ &+ \sum_{t=0}^{\frac{1}{54\Delta t}} f_3(54t \cdot \Delta t)e^{-n \cdot 2\pi i(t \cdot \Delta t + \frac{2}{54})}\Delta t \\ &+ \cdots + \sum_{t=0}^{\frac{1}{54\Delta t}} f_{54}(54t \cdot \Delta t)e^{-n \cdot 2\pi i(t \cdot \Delta t + \frac{53}{54})}\Delta t \end{aligned} \tag{15}$$

Therefore, the approach of the code will be to iterate through each value of n from $-k \leq n \leq k$, and evaluate the Riemann Sum using Equation 15.

The computer program saves a Dictionary composed of a searchable key, which will composed of all of the possible n values within a defined range that comes from a user input specifying what k equals. Then, each key will have an associated value that will start as being defined to be $0 + 0i$, which is achievable through the "complex" object offered by the "mathjs" NPM package (de Jong, n.d.).

Then, for each row of the Excel Spreadsheet (in which the process of iterating through every row was explained in Section 4.1), every possible value for n is analyzed with a for loop, allowing the program to evaluate the value of c_n associated with each row.

The following code is equivalent to Equation 15. **Note: "f" represents f(t) and the calculation for it will differ based on whether a Bézier curve or a Straight Line is being drawn. The calculation for f(t) will be elaborated on in Section 4.2.2.**

```
for(let t = 0; t < 1/sectionCount/dt; t++){
    const f = null;
    const add = math.multiply(dt, math.multiply(f, math.pow(math.e, math↩
        .multiply(math.complex(0, 1), -2 * n * math.pi * (t * dt + row↩
        [9])))));
    CnDict[n] = math.add(CnDict[n], add);
}
```

4.2.2 Calculation for Cubic Bézier curves

Recalling that Equation 12 stated that

$$P = P_0(-t^3 + 3t^2 - 3t + 1) + P_1(3t^3 - 6t^2 + 3t) + P_2(-3t^3 + 3t^2) + P_3(t^3)$$

This was originally meant for when P, P_0, P_1, P_2, P_3 were ordered pairs on a Cartesian Plane. However, because multiplication between a scalar and a Cartesian Vector operates similarly to multiplication between a scalar and a complex number, then P_0, P_1, P_2, P_3 are allowed to be expressed as complex numbers. Given this condition, P would be equivalent to $f(t)$ because P is the Cartesian vector for all points defined by a Bézier curve, and if the result representing P was a complex number, then P is representable as a function of t.

Something crucial to consider is that t in Equation 12 is **not** the time passage in drawing the IB logo but rather the lerp of the Bézier curve from 0 to 1. To distinguish this distinction, the function will instead take in an input j, with the relationship between j and t being established in Section 4.2.1 to be $j = 54t$.

Ultimately, the Bézier curve can be expressed as:

$$f(j) = P_0(-j^3 + 3j^2 - 3j + 1) + P_1(3j^3 - 6j^2 + 3j) + P_2(-3j^3 + 3j^2) + P_3(j^3) \tag{16}$$

The value of $f(j)$ is evaluated through the following code, which is equivalent to Equation 16:

```
const sectionCount = 54;

function subTimeCalc(t){
    return sectionCount * t;
};

const fCubicCalc = (t, r) => {
    const P0 = math.complex(r[1], r[2]);
    const P1 = math.complex(r[3], r[4]);
    const P2 = math.complex(r[5], r[6]);
    const P3 = math.complex(r[7], r[8]);

    let j = subTimeCalc(t);
    const ret =
        math.add(
            math.add(
                math.multiply(P0, (-1*math.pow(j,3) + 3*math.pow(j,2) - ↩
                    3*j + 1)),
                math.multiply(P1, (3*math.pow(j,3) - 6*math.pow(j,2) + ↩
                    3*j))
            ),
            math.add(
                math.multiply(P2, (-3*math.pow(j,3) + 3*math.pow(j,2))),
                math.multiply(P3, (math.pow(j,3)))
            )
        )
    return ret;
};
```

4.2.3 Calculation for Straight Lines

In Section 4.2.2, it was mentioned that t cannot be used as the input of $f(t)$ as the original meaning of t in such context was the lerp progression in the Bézier curve.

While t could be used in the calculation of $f(t)$ for straight lines, it is actually better to continue to use $j = 54t$, as this allows for consistency and will make the derived equations relevant in the section to be way cleaner than if t was used.

Every line will have an initial point at $j = 0$ that can be expressed as $x_0 + iy_0$, as well as a final point at $j = 1$ that can be expressed as $x_1 + iy_1$.

Because the pathway between these two points is a straight line, then two linear functions can be found for x and y that are in terms of j. This means that $f(j)$ can be expressed as follows:

$$f(j) = x(j) + iy(j)$$

The formula for $x(j)$ is derived below.

$$\begin{aligned}
x(j) &= mj + b \\
b &= x_0 \\
m &= \frac{\Delta x}{\Delta j} \\
&= \frac{x_1 - x_0}{1 - 0} \\
&= x_1 - x_0 \\
x(j) &= (x_1 - x_0)j + x_0
\end{aligned}$$

$y(j)$ can be said to have a similar formula, as x_0, x_1 are simply replaced by y_0, y_1, and Δj remains 1.

Ultimately, $f(j)$ can be expressed as follows:

$$f(j) = ((x_1 - x_0)j + x_0) + i((y_1 - y_0)j + y_0) \tag{17}$$

This is evaluated through the following code:

```
const f =
math.add(
    math.add(
        row[1],
        math.multiply(
            subTimeCalc(t * dt),
            row[3] - row[1]
        )
    ),
    math.multiply(
        math.complex(0, 1),
        math.add(
            row[2],
            math.multiply(
```

```
                    subTimeCalc(t * dt),
                    math.add(
                        row[4],
                        -1 * row[2]
                    )
                )
            )
        )
    );
```

4.3 Rendering the final image

Recalling that the end result of $f(t)$ is defined as the equation below, then rendering the final image is just a matter of going through various values of t, summing up $c_n e^{n \cdot 2\pi i t}$ for all values of k, and rendering the result by placing the real parts of $f(t)$ on the horizontal axis on a plane, and placing the imaginary part of $f(t)$ on the vertical axis on a plane.

$$f(t) = \sum_{n=-k}^{k} c_n e^{n \cdot 2\pi i t}$$

Because all possible n values within a given domain and their associated values for c_n were stored in a dictionary, then iterating through the dictionary will allow for accessing all of the c_n values properly linked with their associated n value.

The code that calculates $f(t)$ and renders the result is presented below:

```
let universalTime = 0;

function plotPoint(x, y){
    const adjX = x + 99.2125;
    const adjY = -y + 99.213;
    ctx.strokeStyle = 'rgba(255,0,0,1)';
    ctx.beginPath();
    ctx.lineTo(adjX-1, adjY);
    ctx.lineTo(adjX+1, adjY);
    ctx.stroke();
}

function finalF(){
    while(universalTime <= 1){
        let curr = math.complex(0, 0);
        Object.entries(CnDict).forEach(pair => {
            [currN, cn] = pair;
            //for a specific time
            //calculate f(t)
            //plot
            curr = math.add(
                curr,
                math.multiply(
                    cn,
                    math.pow(
                        math.e,
```

```
                              math.multiply(
                                  math.complex(0, 1),
                                  currN * 2 * math.pi * universalTime
                              )
                          )
                      )
                  )
              });
              plotPoint(curr.re, curr.im);
              universalTime += dt;
          }
      }

      finalF();
      console.log('<img src="' + canvas.toDataURL() + '" />');
```

5 Results

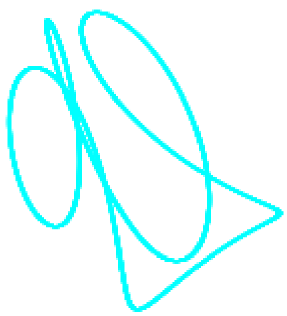


Figure 12: $\Delta t = 0.0001, k = 5$, 10 spinning vectors, 11 total vectors

Figure 13: $\Delta t = 0.0001, k = 25$, 50 spinning vectors, 51 total vectors

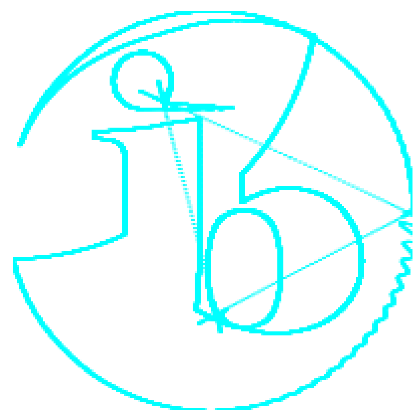


Figure 14: $\Delta t = 0.00001, k = 1000$, 2000 spinning vectors, 2001 total vectors

6 Conclusion

Throughout this investigation, the aim of converting the IB logo into a Fourier Series and in turn rendering the IB logo using its Fourier Series has been accomplished. In addition to that, connecting Fourier Series with complex numbers has enabled the ability to handle a two-dimensional logo without separating it into two components.

Achieving the goal of rendering the IB logo through Fourier Series using the complex plane is a strength of this investigation given that it necessitates the determination of only one Fourier Series that will consider both the horizontal (real) and vertical (imaginary) axes simultaneously.

The primary limitation of my approach to this investigation was the inflexibility in terms of what the computer program could convert into a Fourier Series. Ideally, the program should take in a ".svg" file and automatically convert the path to Bézier curves as representation of what $f(t)$ is prior to being converted to a Fourier Series. This is all due to transferring all the values to an Excel spreadsheet, which makes it easier to see the moving parts but prevents the program from analyzing any kind of vector art without the conversion of an ".svg" file into a spreadsheet and modifications to the code (e.g. number of Bézier curves or lines, size of canvas).

Another limitation is that the method of going through the ".svg" path without considering continuity resulted in the artifacts present in the final result. This also causes the manifestation of "Gibbs Phenomenon" visible in Figure 14 which are extreme peaks and fluctuations resulting from the jumps between discontinuities. This may be prevented in a future investigation by coding a program to begin parsing a separate $f(t)$ function after detection of a significant and sudden change between the previous and current complex values.

Manifestations of sinusoidal waves can also be seen in the results even in the extreme parameters in Figure 14 that took about 10 minutes to compute and render. This indicates the imperfections in Fourier Series, with computing a near perfect render necessitating more terms in the Fourier Series. The primary method of reducing the effects of this limitation is to look for optimizations in the computer program. Additionally, simpler vector artworks will likely manifest these sinusoidal waves to a lower degree.

I expect that I will be able to handle learning Fourier Series in a university course in the future with ease. Not only have I already worked through a calculation for a common Fourier Series in Section 3.1, but the application of Fourier Series using complex numbers to draw the IB logo has given a thorough understanding of the exact mechanisms of the Fourier Series in producing a variety of periodic functions. Therefore, I will be able to stress less about understanding Fourier Series alone and focus on learning how to apply it to partial differential equations in a future university course.

The main area of extension of this investigation is the way in which the integral for c_n was evaluated where it was evaluated as a Riemann Sum. It is possible to evaluate this integral through integration by parts; however, the challenge when incorporating this into the method of this investigation is that because each Bézier curve is evaluated between 0 and 1, then numerous integrals must be evaluated with a unique transformation to each Bézier curve piece part of the original piece-wise $f(t)$ function thereby resulting in more and more complicated integrals for pieces of the function nearing the end boundary.

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A Full SVG Path

From (International Baccalaureate Organisation, 2013).

```
<path fill="url(#SVGID_1_)" d="
M198.425,99.155
c 0,54.833 -45.075,99.271 -100.685,99.271
c -47.27, 0 -86.91 -32.11 -97.74 -75.416
c 19.703 -0.222, 38.391 -4.567, 55.26 -12.149
V 72.226
c 0 -3.062 -1.079 -5.3 -3.234 -6.716
c -2.151 -1.409 -5.975 -2.12 -11.479 -2.12
v -3.579
c 7.295 -0.24, 13.984 -0.665, 20.063 -1.267
c 6.086 -0.604, 11.331 -1.513, 15.74 -2.721
l 0.092, 56.113
c 0, 3.182, 0, 6.654, 0, 10.422
c 0, 3.657 -0.055, 7.25 -0.178, 10.782
c -0.122, 3.537 -0.303, 6.833 -0.539, 9.895
c -0.242, 3.065 -0.54, 5.653 -0.901, 7.771
c 5.983, 3.424, 12.621, 6.249, 19.921, 8.49
c 7.299, 2.239, 14.115, 3.353, 20.455, 3.353
c 5.383, 0, 10.589 -0.795, 15.612 -2.387
c 5.026 -1.59, 9.484 -3.916, 13.367 -6.99
c 3.889 -3.066, 7.033 -6.833, 9.428 -11.322
c 2.39 -4.485, 3.585 -9.563, 3.585 -15.226
c 0 -5.188 -0.898 -9.969 -2.694 -14.335
c -1.795 -4.368 -4.313 -8.139 -7.552 -11.329
c -3.234 -3.19 -7.06 -5.69 -11.494 -7.52
c -4.437 -1.834 -9.286 -2.749 -14.557 -2.749
c -3.955, 0 -7.993, 0.687 -12.126, 2.048
c -4.139, 1.371 -7.879, 3.065 -11.231, 5.079
V 81.217
c 18.133 -18.684, 30.894 -42.471, 35.809 -68.97
c -22.443 -9.396 -48.452 -10.619 -73.052 -1.412
C 32.967, 21.061, 12.981, 42.053, 3.409, 67.056
c 9.221 -26.919, 30.12 -49.704, 59.155 -60.579
c 24.245 -9.068, 49.573 -8.27, 71.696, 0.248
c 0.002 -0.021, 0.011 -0.045, 0.014 -0.063
C 171.81, 21.082, 198.425, 57.046, 198.425, 99.155
z
M 100.598, 149.993
c 2.746, 4.661, 7.408, 6.986, 13.987, 6.986
c 6.564, 0, 11.233 -2.325, 13.985 -6.986
c 2.749 -4.652, 4.123 -11.988, 4.123 -22.013
```

```
c 0 -10.014 -1.374 -17.353 -4.123 -22.008
c -2.752 -4.657 -7.421 -6.979 -13.985 -6.979
c -6.58, 0 -11.242, 2.322 -13.987, 6.979
c -2.75, 4.655 -4.124, 11.994 -4.124, 22.008
C 96.474, 138.005, 97.848, 145.341, 100.598, 149.993
z
M 74.363, 44.669
c 2.793 -2.686, 4.185 -6.045, 4.185 -10.084
c 0 -4.037 -1.392 -7.397 -4.185 -10.086
c -2.781 -2.692 -6.165 -4.037 -10.129 -4.037
c -3.968, 0 -7.351, 1.345 -10.137, 4.037
c -2.79, 2.688 -4.181, 6.049 -4.181, 10.086
c 0, 4.039, 1.391, 7.398, 4.181, 10.084
c 2.786, 2.69, 6.168, 4.036, 10.137, 4.036
C 68.199, 48.705, 71.582, 47.359, 74.363, 44.669
z"/>
```

B Full Calculation and Graphing Program Code

```
const math = require('mathjs')
const prompt = require("prompt-sync")({ sigint: true });
const readXlsxFile = require('read-excel-file/node')

const { createCanvas, loadImage } = require('canvas')
const canvas = createCanvas(198.425, 198.426)
const ctx = canvas.getContext('2d')


const dt = parseFloat(prompt("Enter \Delta t: "));
const k = parseFloat(prompt("Enter abs(k): "));

const sectionCount = 54;

function subTimeCalc(t){
    return sectionCount * t;
};

const fCubicCalc = (t, r) => {
    const P0 = math.complex(r[1], r[2]);
    const P1 = math.complex(r[3], r[4]);
    const P2 = math.complex(r[5], r[6]);
    const P3 = math.complex(r[7], r[8]);

    let j = subTimeCalc(t);
    const ret =
        math.add(
            math.add(
                math.multiply(P0, (-1*math.pow(j,3) + 3*math.pow(j,2) - 3*j ↩
                    + 1)),
```

```
                math.multiply(P1, (3*math.pow(j,3) - 6*math.pow(j,2) + 3*j))
            ),
            math.add(
                math.multiply(P2, (-3*math.pow(j,3) + 3*math.pow(j,2))),
                math.multiply(P3, (math.pow(j,3)))
            )
        )
    return ret;
};

const CnDict = {};
for(let n = -k; n <= k; n++){
    CnDict[n] = math.complex(0, 0);
}

function plotPoint(x, y){
    const adjX = x + 99.2125;
    const adjY = -y + 99.213;
    ctx.strokeStyle = 'rgba(255,0,0,1)';
    ctx.beginPath();
    ctx.lineTo(adjX-1, adjY);
    ctx.lineTo(adjX+1, adjY);
    ctx.stroke();
}

readXlsxFile('./SVG_Coordinate_Spreadsheet.xlsx').then((rows) => {
    // `rows` is an array of rows
    // each row being an array of cells.
    rows.forEach(row => {
        if(row[0] == 'C'){
            for(let n = -k; n <= k; n++){
                //summation
                for(let t = 0; t < 1/sectionCount/dt; t++){
                    const f = fCubicCalc(t * dt, row);
                    const add = math.multiply(dt, math.multiply(f, math.pow(↩
                        math.e, math.multiply(math.complex(0, 1), -2 * n * ↩
                        math.pi * (t * dt + row[9])))));
                    CnDict[n] = math.add(CnDict[n], add);
                }
            }
        }
        else if(row[0] == 'V'){
            for(let n = -k; n <= k; n++){
                for(let t = 0; t < 1/sectionCount/dt; t++){
                    const f =
                    math.add(
                        row[1],
                        math.multiply(
                            math.complex(0, 1),
                            math.add(
                                row[2],
                                math.multiply(
                                    subTimeCalc(t * dt),
                                    math.add(
```

```
                                row[4],
                                -1 * row[2]
                            )
                        )
                    )
                )
            );
            const add =
            math.multiply(dt,
                math.multiply(
                    f,
                    math.pow(math.e, math.multiply(math.complex(0, ↩
                        1), -2 * n * math.pi * (t * dt + row[9])))
                )
            );
            CnDict[n] = math.add(CnDict[n], add);
        }
    }
}
else if(row[0] == 'L'){
    for(let n = -k; n <= k; n++){
        for(let t = 0; t < 1/sectionCount/dt; t++){
            const f = math.add(
                math.add(
                    row[1],
                    math.multiply(
                        subTimeCalc(t * dt),
                        row[3] - row[1]
                    )
                ),
                math.multiply(
                    math.complex(0, 1),
                    math.add(
                        row[2],
                        math.multiply(
                            subTimeCalc(t * dt),
                            math.add(
                                row[4],
                                -1 * row[2]
                            )
                        )
                    )
                )
            );
            const add =
            math.multiply(
                dt,
                math.multiply(
                    math.pow(math.e, math.multiply(math.complex(0, ↩
                        1), -2 * n * math.pi * (t * dt + row[9]))),
                    f
                )
            )
            CnDict[n] = math.add(CnDict[n], add);
```

```
                }
            }
        }
    });

    let universalTime = 0;
    function finalF(){
        while(universalTime <= 1){
            let curr = math.complex(0, 0);
            Object.entries(CnDict).forEach(pair => {
                [currN, cn] = pair;
                //for a specific time
                //calculate f(t)
                //plot
                curr = math.add(
                    curr,
                    math.multiply(
                        cn,
                        math.pow(
                            math.e,
                            math.multiply(
                                math.complex(0, 1),
                                currN * 2 * math.pi * universalTime
                            )
                        )
                    )
                )
            });
            plotPoint(curr.re, curr.im);
            universalTime += dt;
        }
    }

    finalF();
    console.log('<img src="' + canvas.toDataURL() + '" />');
})
```