MATHEMATICS ANALYSIS AND APPROACHES HL
Producing the IB Logo with the Fourier Series

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1 Rationale

I have shown interest in visual arts done through the means of software, with particular experience in 3D modelling and animation in Blender and Cinema 4D.

I never was experienced with drawing, therefore producing digital art on a 2D plane using artistic skill was not of interest to me. However, something that I came across online was the use of the Fourier Series in order to produce vector art, which instantly intrigued me.

While vector art files such as those with the file extension ".svg" relate to mathematics in the sense that it contains multiple graphed mathematical relationships in order to produce an image, the method of using the Fourier Series to produce similar art is more mathematically intriguing, as it proves use just one expression to produce the same result done by the numerous mathematical relationships.

2 Aim

The objective of this investigation is to link Fourier series with complex numbers to create a single series that is capable of reproducing the IB logo on the Argand plane.

3 Plan of Action

This exploration focuses on the following areas of math:

- Integral Calculus
- Series
- Trigonometry
- Complex Analysis
- Vectors

4 Background Information

4.1 Overarching idea of the Fourier Series

A periodic function is one where the output for a particular input equals to the output for the sum of the same input and the value of the function's period. This can be represented mathematically as:

$$
\begin{aligned}
&f(x) = f(x+P)\\
&\text{where } P = \text{ the period of the function}
\end{aligned}
$$

The sine wave is widely known for being a periodic function for the ease of graphing a sinusoidal wave. However, there are periodic functions that are difficult to graph with an algebraic expression, such as one that alternates between 1 and -1 or one that is shaped as a zig-zag.

This is the motivation behind the Fourier Series, which is to be able to represent period functions that normally can't be represented by an algebraic function.

The idea behind the Fourier Series is to take an infinite sum of varying sinusoidal functions such that a desired periodic function is produced.

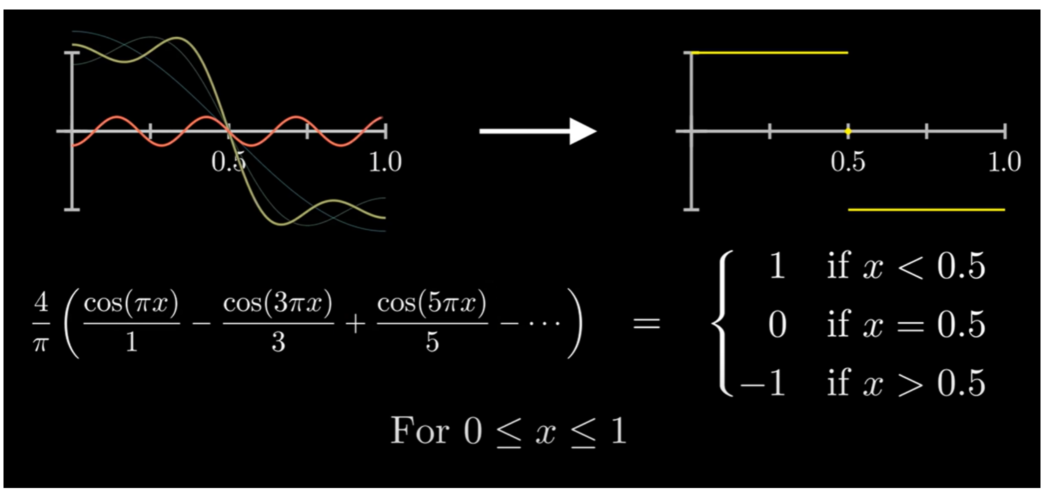

Figure 1: Visualization of the mechanism of the Fourier Series (Sanderson, 2019). The yellow line is the periodic function resulting from the previous iteration, the red line is the sinusoidal function to be added in the next iteration.

4.2 Idea of drawing with the Fourier series explained with the Cartesian Plane

The rule is for eligible drawings to be any that can be drawn by starting at one point on a cartesian plane and, without lifting the hypothetical plane throughout the entire sketch, return to the exact same point.

Defining the variable t as time, $t = 0$ will represent the point in time where the drawing began and $t = 1$ will represent the point in time where the drawing ended.

Each point of the drawing on the plane will be defined by $P(x(t), y(t))$, where $x(t)$ and $y(t)$ are both functions with an input of t that indicates the coordinates after some amount of time passed of the pen's progress through the drawing.

Assuming that any function can be represented as a Fourier Series, then $x(t)$ and $y(t)$ can be any function and therefore, by extracting the coordinates on a graph of any drawing obeying the rule described earlier, the Fourier Series of $x(t)$ and $y(t)$ can be determined and therefore produce the desired drawing for $t \in [0, 1]$

As a simple example that does not require a Fourier Series, we can take a unit circle defined by $x^2 + y^2 = 1$ as the drawing. It quickly becomes evident of what $x(t)$ and $y(t)$ are, as since it is a circle, then $x(t) = \cos(2\pi t)$ and $y(t) = \sin(2\pi t)$.

4.3 Enriched application to draw on the Argand Plane

Euler's formula is defined to be

$$e^{it} = \cos(t) + i\sin(t) \tag{1}$$

Given that both the cosine function and the sine function are included in this formula, a connection between this formula and the Fourier Series becomes evident. The two sinusoidal functions in the formula are the core behind applying the Fourier Series to the Argand Plane.

It is easier to think of Euler's formula to be a vector on the Argand Plane (Sanderson, 2019). With just the formula given above, we have a vector with a length of 1 that rotates counterclockwise when the value that e is raised to is positive and clockwise when the value is negative.

Let's incorporate n and 2π to the power in Equation **(1)** such that we have $e^{n\cdot 2\pi it}$. The purpose of 2π is to simplify what defines a revolution, as now, for every unit of time that t passes, a revolution will be completed. $n, n \in \mathbb{Z}$ will define the frequency and directionality of the rotation of the vector. If $n > 0$, then the vector will rotate counterclockwise and vice versa. If $n = 2$, then the vector would rotate by 2 revolutions for every unit of time that passes.

Lastly, let's multiply the entire power by the variable c_n to get $c_n e^{n\cdot 2\pi it}$. This will not only allow for the vector to be scaled but also for the starting rotation of the vector to be defined. Let's temporarily define c_n to be:

$$c_n = A \cdot z$$

A will indicate the factor to which the vector will be scaled by. Because the original length of the vector was 1, then the factor will directly indicate the length of the vector.

z will indicate the starting rotation of the vector, which would be $e^{i\theta}$, where θ is the starting rotation angle in radians.

It could be imagined that the final Fourier Series that produces a desired drawing is the sum of multiple vectors of different magnitudes rotating a different frequencies indicated by $n, n \in \mathbb{Z}$. Therefore, if $f(t)$ is defined to be the function that represents the drawing, this can be expressed mathematically as:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{n\cdot 2\pi it} \tag{2}$$

The main concern as of right now is how the value of c_n will be determined for a specific drawing.

The easiest way to start off is by first considering $c_0 e^{0\cdot 2\pi it}$. This can be simplified into c_0, but what does this mean? This is the vector that is rotating at a frequency of 0, which means that it is static. It can therefore be defined to be the "centre of mass"(Sanderson, 2019) of the entire function.

If we take discrete intervals of t and obtain the value of $f(t)$, then by taking the average of all those values, we obtain a complex number close to c_0. This is illusrated by Figure 2.

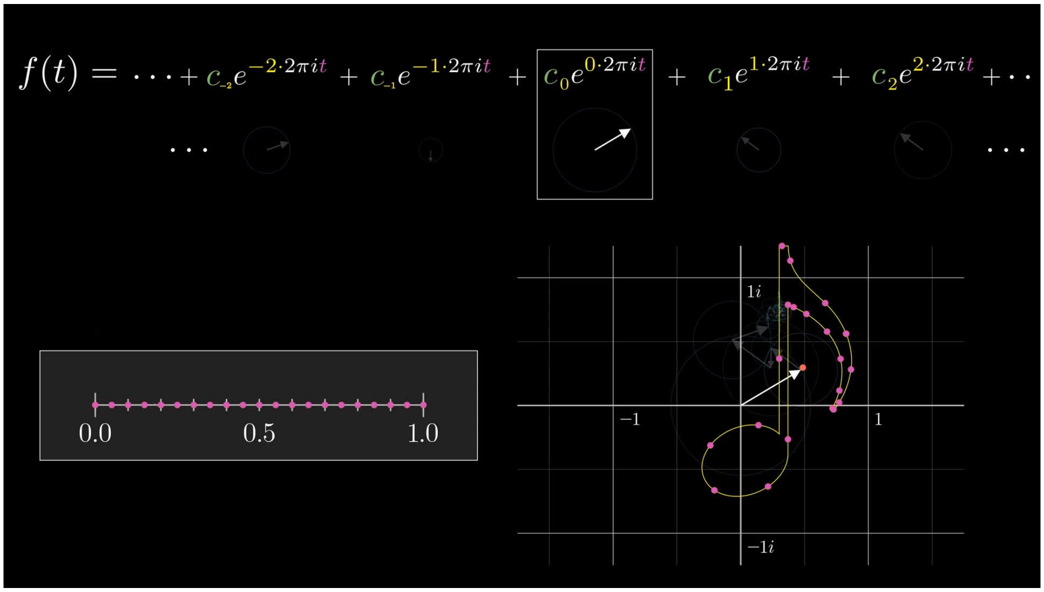

Figure 2: Averaging points throughout $f(t)$ illustrated (Sanderson, 2019). The number line represent t, and the red dots on the musical note represent the resulting values of $f(t)$

With finer and finer intervals of t, the result becomes more and more accurate to the value of c_0, therefore it can be said that:

$$c_0 = \lim_{\Delta t \to 0} \sum_{t=0}^{\frac{1}{\Delta t}} f(t \cdot \Delta t)\Delta t \tag{3}$$

From Equation **(3)**, c_0 can ultimately be expressed as:

$$c_0 = \int_0^1 f(t)\, dt \tag{4}$$

We can use the same technique for all other values of n, but the problem is for all other values of n that are not 0, the vector is rotating, therefore it doesn't make sense to take the average of the rotating vector.

Recalling the thorough definition of $f(t)$ stated earlier in Equation 2, we can substitute Equation 2 into Equation 4.

$$c_0 = \int_0^1 \sum_{n=-\infty}^{\infty} c_n e^{n\cdot 2\pi i t}\, dt$$

$$c_0 = \int_0^1 \left(\cdots + c_{-1}e^{-1\cdot 2\pi i t} + c_0 e^{0\cdot 2\pi i t} + c_1 e^{1\cdot 2\pi i t} + \cdots\right)\, dt$$

$$c_0 = \cdots + \int_0^1 c_{-1}e^{-1\cdot 2\pi i t}\, dt + \int_0^1 c_0 e^{0\cdot 2\pi i t}\, dt + \int_0^1 c_1 e^{1\cdot 2\pi i t}\, dt + \cdots$$

Remember that $\int_0^1 c_0 e^{0\cdot 2\pi i t}\, dt$ was easy to simplify as the power cancels out from being raised to 0. In turn, it can be further evaluated to be just c_0, as shown below:

$$\begin{aligned} &\int_0^1 c_0 e^{0\cdot 2\pi i t}\, dt \\ &= \int_0^1 c_0\, dt \\ &= [c_0 t]_0^1 \\ &= c_0 \end{aligned}$$

Additionally, if all the other integrals were thought of as the the average of all the points produced when its vector rotates by one revolution, then it can be argued that each integral, when evaluated, would be 0.

This idea is illustrated in Figure 3.

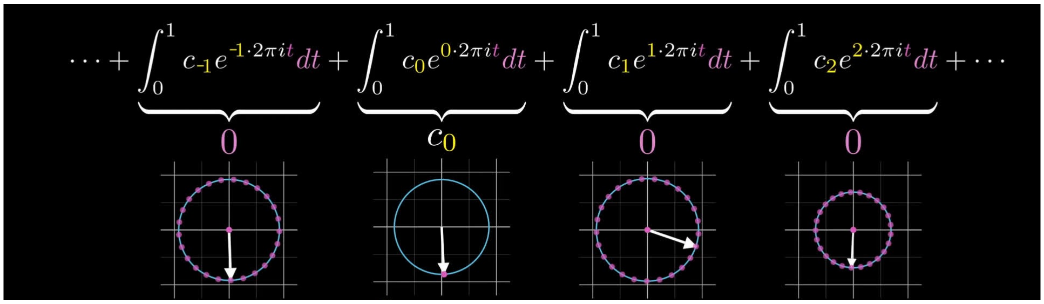

Figure 3: Illustration of averaging all the points on a circle (Sanderson, 2019).

If we were to multiply $f(t)$ by $e^{-n \cdot 2\pi i t}$, then an effect can occur where all of the power's exponents will decrease by n. Ultimately, we will get a similar scenario in the summation of all the integrals where all integrals are evaluated to become 0 except for the integral where $n = 0$, which evaluates to c_0. However, the multiplication of the two powers will cause the integral of some n value to simplify to just c_n.

This mechanism is mathematically shown below.

$$
\begin{aligned}
&n = 1 \\
&c_1 = \int_0^1 f(t) e^{-1 \cdot 2\pi i t}\, dt \\
&c_1 = \cdots + \int_0^1 c_{-1} e^{-1 \cdot 2\pi i t} \cdot e^{-1 \cdot 2\pi i t}\, dt \\
&+ \int_0^1 c_0 e^{0 \cdot 2\pi i t} \cdot e^{-1 \cdot 2\pi i t}\, dt + \int_0^1 c_1 e^{1 \cdot 2\pi i t} \cdot e^{-1 \cdot 2\pi i t}\, dt + \cdots \\
&c_1 = \cdots + \int_0^1 c_{-1} e^{-2 \cdot 2\pi i t}\, dt + \int_0^1 c_0 e^{-1 \cdot 2\pi i t}\, dt + \int_0^1 c_1 e^{0 \cdot 2\pi i t}\, dt + \cdots \\
&c_1 = \cdots + 0 + 0 + c_1 + 0 + 0 + \cdots \\
&c_1 = c_1
\end{aligned}
$$

Therefore, c_n can be defined as:

$$
c_n = \int_0^1 f(t) e^{-n \cdot 2\pi i t}\, dt \tag{5}
$$

Equation 5 is what will be used in order to determine each value of c_n

All concepts of this section come from (Sanderson, 2019).

4.4 Bezier Curves

4.5 Composition of ".svg" files

4.6 Contour integration?

(This is just a placeholder. This concept may enrich this IA, although I do not yet know enough to foresee if understanding this concept will be something that is achievable)

5 Methodology

5.1 Analysis of ".svg" Files

5.2 Computation of parameters

Given that $f(t)$ is able to be determined as a piecewise function from the previous section, the desired summation representing $f(t)$ can be determined.

Referring back to Section 4.3, c_n is defined as:

$$c_n = \int_0^1 f(t)e^{-n \cdot 2\pi it}\, dt$$

This means that every c_n must be determined individually. Therefore, it becomes evident that it is unreasonable to evaluate the infinite sum of $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{n \cdot 2\pi it}$, and that the limits of the summation must be defined.

Let's rewrite $f(t)$ as:

$$f(t) = \sum_{n=-k}^{k} c_n e^{n \cdot 2\pi it}$$

so that k indicates the selected frequency of the two fastest vectors that are spinning in opposite directions to each other.

As $k \to \infty$, $f(t)$ becomes more and more accurate to the original drawing, which will be demonstrated by performing distinct analyses for various values of k.

5.2.1 Approach 1: Numerical Integration

$$c_n = \int_0^1 f(t)e^{-n \cdot 2\pi it}\, dt$$

The integral above can be evaluated for some value of n by taking some small value to represent Δt, in which by summing up the values from $f(t)e^{-n \cdot 2\pi it}$ produced by each increment of t by Δt, a value close to the original integral can be determined (Sanderson, 2019).

$$c_n = \sum_{t=0}^{\frac{1}{\Delta t}} f(t \cdot \Delta t)e^{-n \cdot 2\pi i(t \cdot \Delta t)}\Delta t \tag{6}$$

5.3 Rendering the final image

References

Bazett, T. (n.d.). *Intro to FOURIER SERIES: The Big Idea - YouTube.* https://www.youtube.com/watch?v=wmCIrpLBFds.

Clastify. (n.d.). *IB Math AA IA examples — Clastify.* https://www.clastify.com/ia/math-aa.

DeCross, M., & Khim, J. (n.d.). *Fourier Series — Brilliant Math & Science Wiki.* https://brilliant.org/wiki/fourier-series/.

Euler's formula. (2023, December). *Wikipedia*.

Holmer, F. (2021, August). *The Beauty of Bézier Curves.*

Sanderson, G. (2019, June). *But what is a Fourier series? From heat flow to drawing with circles.*

Sandlin, D. (2018, December). *What is a Fourier Series? (Explained by drawing circles) - Smarter Every Day 205.*

Tripathi, S. (2023, January). *IB Math IA (Ultimate Guide For 2023) - Nail IB.* https://nailib.com/blog/ib-math-ia.