MATHEMATICS ANALYSIS AND APPROACHES HL
Producing the IB Logo with the Fourier Series

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1 Rationale

I have shown interest in visual arts done through the means of software, with particular experience in 3D modelling and animation in Blender and Cinema 4D.

I never was experienced with drawing, therefore producing digital art on a 2D plane using artistic skill was not of interest to me. However, something that I came across online was the use of the Fourier Series in order to produce vector art, which instantly intrigued me.

While vector art files such as those with the file extension ".svg" relate to mathematics in the sense that it contains multiple graphed mathematical relationships in order to produce an image, the method of using the Fourier Series to produce similar art is more mathematically intriguing, as it proves use just one expression to produce the same result done by the numerous mathematical relationships.

2 Aim

The objective of this investigation is to link Fourier series with complex numbers to create a single series that is capable of reproducing the IB logo on the Argand plane.

3 Plan of Action

This exploration focuses on the following areas of math:

- Integral Calculus
- Series
- Trigonometry
- Complex Analysis
- Vectors

4 Background Information

4.1 Overarching idea of the Fourier Series

A periodic function is one where the output for a particular input equals to the output for the sum of the same input and the value of the function's period. This can be represented mathematically as:

$$\begin{aligned}&f(x) = f(x+P)\\&\text{where } P = \text{ the period of the function}\end{aligned}$$

The sine wave is widely known for being a periodic function for the ease of graphing a sinusoidal wave. However, there are periodic functions that are difficult to graph with an algebraic expression, such as one that alternates between 1 and -1 or one that is shaped as a zig-zag.

This is the motivation behind the Fourier Series, which is to be able to represent period functions that normally can't be represented by an algebraic function.

The idea behind the Fourier Series is to take an infinite sum of varying sinusoidal functions such that a desired periodic function is produced.

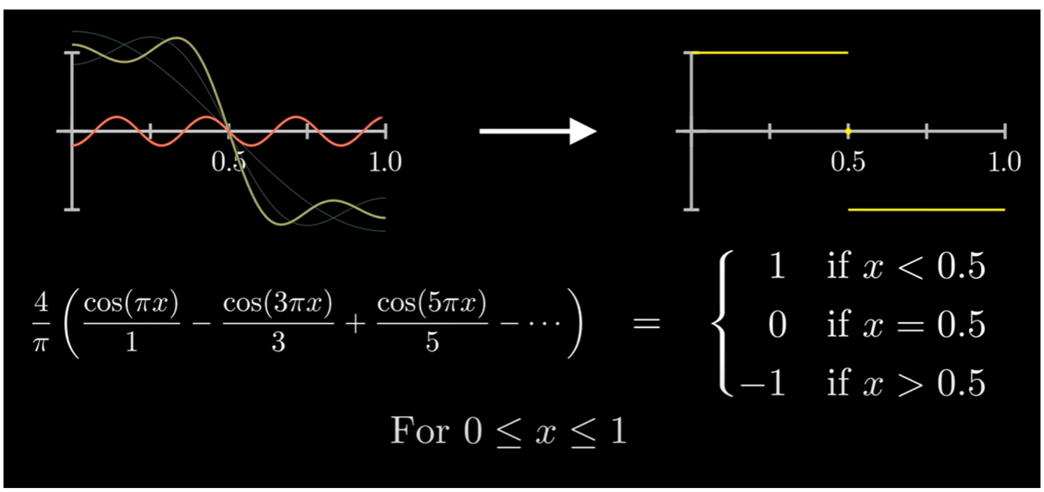

Figure 1: Visualization of the mechanism of the Fourier Series (Sanderson, 2019). The yellow line is the periodic function resulting from the previous iteration, the red line is the sinusoidal function to be added in the next iteration.

4.2 Idea of drawing with the Fourier series explained with the Cartesian Plane

The rule is for eligible drawings to be any that can be drawn by starting at one point on a cartesian plane and, without lifting the hypothetical plane throughout the entire sketch, return to the exact same point.

Defining the variable t as time, $t = 0$ will represent the point in time where the drawing began and $t = 1$ will represent the point in time where the drawing ended.

Each point of the drawing on the plane will be defined by $P(x(t), y(t))$, where $x(t)$ and $y(t)$ are both functions with an input of t that indicates the coordinates after some amount of time passed of the pen's progress through the drawing.

Assuming that any function can be represented as a Fourier Series, then $x(t)$ and $y(t)$ can be any function and therefore, by extracting the coordinates on a graph of any drawing obeying the rule described earlier, the Fourier Series of $x(t)$ and $y(t)$ can be determined and therefore produce the desired drawing for $t \in [0, 1]$

As a simple example that does not require a Fourier Series, we can take a unit circle defined by $x^2 + y^2 = 1$ as the drawing. It quickly becomes evident of what $x(t)$ and $y(t)$ are, as since it is a circle, then $x(t) = \cos(2\pi t)$ and $y(t) = \sin(2\pi t)$.

4.3 Enriched application to draw on the Argand Plane

Euler's formula is defined to be

$$e^{it} = \cos(t) + i\sin(t)$$

Given that both the cosine function and the sine function are included in this formula, a connection between this formula and the Fourier Series becomes evident. The two sinusoidal functions in the formula are the core behind applying the Fourier Series to the Argand Plane.

It is easier to think of Euler's formula to be a vector on the Argand Plane(Sanderson, 2019). With just the formula given above, we have a vector with a length of 1 that rotates counterclockwise for larger values of t and clockwise for smaller values of t.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{n \cdot 2\pi i t}$$

$$c_n = \int_0^1 f(t) e^{-n \cdot 2\pi i t}\, dt$$

References

Bazett, T. (n.d.). *Intro to FOURIER SERIES: The Big Idea - YouTube.* https://www.youtube.com/watch?v=wmCIrpLBFds.

Clastify. (n.d.). *IB Math AA IA examples — Clastify.* https://www.clastify.com/ia/math-aa.

DeCross, M., & Khim, J. (n.d.). *Fourier Series — Brilliant Math & Science Wiki.* https://brilliant.org/wiki/fourier-series/.

Euler's formula. (2023, December). *Wikipedia*.

Sanderson, G. (2019, June). *But what is a Fourier series? From heat flow to drawing with circles.*

Sandlin, D. (2018, December). *What is a Fourier Series? (Explained by drawing circles) - Smarter Every Day 205.*

Tripathi, S. (2023, January). *IB Math IA (Ultimate Guide For 2023) - Nail IB.* https://nailib.com/blog/ib-math-ia.