Shot noise from which-path detection in a chiral Majorana interferometer

C. W. J. Beenakker

Instituut-Lorentz, Universiteit Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands

(Dated: May 2025)

We calculate the full counting statistics of charge transfer in a chiral Majorana interferometer — a setup where a Dirac mode (an electron-hole mode) is split into two Majorana modes that encircle a number of $h/2e$ vortices in a topological superconductor. Without any coupling to the environment it is known that the low-energy charge transfer is deterministic: An electron is transferred either as an electron or as a hole, dependent on the parity of the vortex number. We show that a stochastic contribution appears if which-path information leaks into the environment, producing the shot noise of random $2e$ charge transfers with binomial statistics. The Fano factor (dimensionless ratio of shot noise power and conductance) increases without bound as the which-path detection probability tends to unity.

arXiv:2505.23670v1 [cond-mat.mes-hall] 29 May 2025

I. INTRODUCTION

Because a Majorana fermion has a real wave function, only phase shifts 0 or π are allowed in a scattering process [1, 2]. This restriction is at the basis of the $\mathbb{Z}_2$ interferometer in a topological superconductor [3–10]: A chiral Dirac mode splits into a pair of chiral Majorana modes, which recombine after having encircled a number N_v of $h/2e$ vortices (see Fig. 1). Depending on the $\mathbb{Z}_2$-valued parity of N_v, a low-energy electron (charge $+e$) is transferred through the interferometer either as a $+e$ electron or as a $-e$ hole. (In the latter case the missing $2e$ charge is absorbed by the superconducting condensate.) This charge transfer is noiseless, fully deterministic.

In an electronic Aharonov-Bohm interferometer the oscillatory magnetic field dependence of transport properties is damped by decoherence due to coupling to the environment. The mechanism is which-path detection [11, 12]: If the pathway taken by the electron through the interferometer leaves a trace in the environment, no quantum interference of different pathways can occur. Here we wish to examine this effect in the $\mathbb{Z}_2$ interferometer. We find that which-path detection adds a stochastic contribution to the charge transfer statistics, producing shot noise as a signature of decoherence in Majorana interferometry.

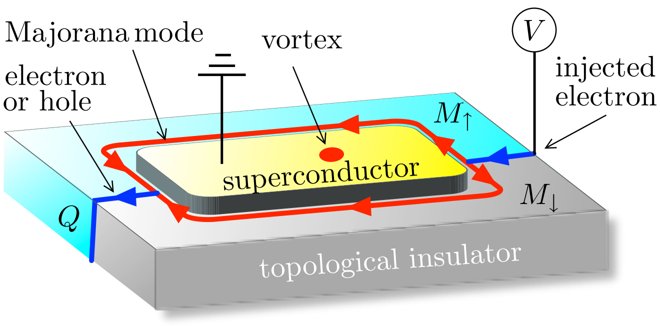

FIG. 1. Layout of the $\mathbb{Z}_2$ interferometer [3, 4]. A three-dimensional topological insulator is covered by magnetic insulators with opposite polarization ($M_\downarrow$ and $M_\uparrow$) and by a superconductor. A pair of chiral Majorana modes flow along the edge between superconductor and magnet. A bias voltage V injects an electron into a superposition of the two Majorana modes, which then recombine either as an electron or as a hole after having encircled a certain number of $h/2e$ vortices. The transferred charge Q depends on the $\mathbb{Z}_2$-valued parity of the number of vortices.

The present study is an application to a topological superconductor of a general method which I recently developed with Jin-Fu Chen [13]. A chiral interferometer is modeled by a monitored quantum channel [14], in which unitary propagation alternates with weak measurements of the occupation number. It was shown in Ref. 13 for a quantum Hall interferometer that the generalized Levitov-Lesovik formula [15, 16] following from this quantum-information based approach preserves the binomial form of the charge transfer probability expected for Fermi statistics [17], thereby removing a shortcoming of the dephasing-probe model of decoherence [18–22].

Here we find as well that the full counting statistics is binomial, with random charge transfers of size $2e$ rather than e, associated with the presence of a superconducting condensate. The Fano factor F, the dimensionless ratio of shot noise power and conductance [17], of the Majorana interferometer is anomalous, very different from the quantum Hall effect: We find that F increases monotonically from zero to arbitrarily large values as the which-path detection becomes more and more effective.

II. FULL COUNTING STATISTICS OF THE $\mathbb{Z}_2$ INTERFEROMETER

The central object for the full counting statistics is the cumulant generating function $C(\xi)$ of the number of electron charges transferred in a time t_{counting}, in the limit $t_{\text{counting}} \to \infty$. We assume elastic scattering, so we can consider separately the contribution of each energy E to the cumulants,

$$C(\xi) = \sum_E \ln F(\xi, E), \quad F(\xi, E) = \langle e^{\xi Q(E)} \rangle, \tag{2.1}$$

with $F(\xi, E)$ the moment generating function of the charge $Q(E)$ transferred at energy E. Charge is mea-

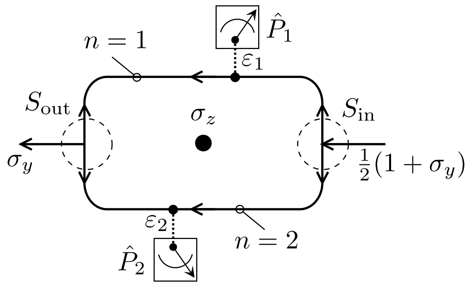

FIG. 2. Scattering geometry corresponding to Fig. 1. The operators $\hat{P}_1$ and $\hat{P}_2$ are weak measurements of the occupation number of chiral Majorana modes $n=1,2$. The vortex (black dot) inserts a π phase shift between the two modes (Pauli matrix σ_z). At the entrance the matrix $(1+\sigma_y)/2$ projects onto an incoming electron, at the exit the matrix σ_y measures the transferred charge. All of this is in the Majorana basis, in the electron-hole basis the role of σ_y and σ_z is interchanged.

sured in units of the electron charge e and energy is discretized in units of $\delta E = h/t_{\rm counting}$. At zero temperature and a bias voltage $V>0$, energies in the range $0<E<eV$ contribute. We will now focus on a single $E>0$, and then at the end sum over all energies.

Following the general method of Ref. 13, we consider a weak measurement of the occupancy of the Majorana mode in each of the arms of the interferometer: mode number $n=1$ in the upper arm, mode number $n=2$ in the lower arm, see Fig. 2. The measurement of mode n interpolates with a weight factor ε_n between the identity $\hat{I}$ and a projection onto a filled ($\hat{P}_{+,n}$) or empty ($\hat{P}_{-,n}$) mode:

$$\begin{aligned}&\hat{P}_{+,n}=\delta_n\hat{I}+\varepsilon_n a_n^\dagger a_n,\;\;\hat{P}_{-,n}=\delta_n\hat{I}+\varepsilon_n a_n a_n^\dagger,\\&\delta_n=\tfrac{1}{2}\big(\sqrt{2-\varepsilon_n^2}-\varepsilon_n\big),\;\;0\leq\varepsilon_n\leq 1.\end{aligned}\tag{2.2}$$

The operators $a_n, a_n^\dagger$ are Majorana fermion operators at $E>0$, with anticommutation relations [23]

$$\{a_n^\dagger,a_m\}=\delta_{nm},\;\;\{a_n,a_m\}=0.\tag{2.3}$$

The coefficients ε_n,δ_n are chosen such that

$$\hat{P}_{+,n}^2+\hat{P}_{-,n}^2=\hat{I}.\tag{2.4}$$

The which-path measurement (2.2) is preceded and followed by unitary propagation, with scattering operators $\hat{S}_{\rm in}$ and $\hat{S}_{\rm out}$.

The incoming modes are in thermal equilibrium at inverse temperature β,

$$\hat{\rho}_{\rm in}=Z^{-1}e^{-\beta a^\dagger H a},\;\;Z=\operatorname{Tr}e^{-\beta a^\dagger H a},\tag{2.5}$$

with single-particle Hamiltonian H. (We collect the operators $a_n, a_n^\dagger$ in vectors $a, a^\dagger$, so that $a^\dagger H a=\sum_{n,m}a_n^\dagger H_{nm}a_m$.) The outgoing modes have the density matrix of a monitored quantum channel [13],

$$\begin{aligned}&\hat{\rho}_{\rm out}=\sum_{s_1,s_2=\pm}\hat{\mathcal{K}}_{s_1,s_2}\hat{\rho}_{\rm in}\hat{\mathcal{K}}_{s_1,s_2}^\dagger,\\&\hat{\mathcal{K}}_{s_1,s_2}=\hat{S}_{\rm out}\hat{P}_{s_1,1}\hat{P}_{s_2,2}\hat{S}_{\rm in}.\end{aligned}\tag{2.6}$$

The Kraus operators $\hat{\mathcal{K}}_\pm$ satisfy the sum rule

$$\sum_{s_1,s_2=\pm}\hat{\mathcal{K}}_{s_1,s_2}^\dagger\hat{\mathcal{K}}_{s_1,s_2}=\hat{I},\tag{2.7}$$

in view of Eq. (2.4) and the unitarity of $\hat{S}$. This ensures that $\operatorname{Tr}\hat{\rho}_{\rm out}=\operatorname{Tr}\hat{\rho}_{\rm in}=1$; the quantum channel (2.6) is a completely-positive trace-preserving map [14].

The elastic scattering operator $\hat{S}$ at energy E is the exponent of a quadratic form in the fermionic operators,

$$\hat{S}=e^{ia^\dagger L(E)a},\tag{2.8}$$

with $L(E)$ a Hermitian 2×2 matrix. The unitary matrix $S(E)=e^{iL(E)}$ is the single-particle scattering matrix. The two scattering operators $\hat{S}_{\rm in}$ and $\hat{S}_{\rm out}$ are thus represented by a pair of unitary 2×2 matrices $S_{\rm in}$ and $S_{\rm out}$.

The charge operator $\hat{Q}$ in the Majorana basis is

$$\hat{Q}=i\big(a_2^\dagger a_1-a_1^\dagger a_2\big)=a^\dagger\sigma_y a.\tag{2.9}$$

(The Pauli matrix σ_y would be σ_z in the electron-hole basis.) The moment generating function $F(\xi,E)$ of the transferred charge at energy E is given by

$$\begin{aligned}F(\xi,E)&=\operatorname{Tr}\hat{\rho}_{\rm out}e^{\xi\hat{Q}}\\&=Z^{-1}\sum_{s_1,s_2=\pm}\hat{\mathcal{K}}_{s_1,s_2}e^{-\beta a^\dagger H a}\hat{\mathcal{K}}_{s_1,s_2}^\dagger e^{\xi a^\dagger\sigma_y a}.\end{aligned}\tag{2.10}$$

At this stage it is helpful to assume $\varepsilon_n\neq 1$. (We can later reach $\varepsilon_n=1$ by taking the limit.) For $0\leq\varepsilon_n<1$ the projectors $\hat{P}_{\pm,n}$ have a Gaussian representation [13],

$$\begin{aligned}&\hat{P}_{\pm,n}=c_{\pm,n}e^{\pm\gamma_n a_n^\dagger a_n},\;\;\gamma_n=\ln(1+\varepsilon_n/\delta_n),\\&c_{+,n}=\delta_n,\;\;c_{-,n}=\varepsilon_n+\delta_n.\end{aligned}\tag{2.11}$$

Traces of products of Gaussian operators can be evaluated by means of Klich's trace-determinant relation [24],

$$\operatorname{Tr}e^{a^\dagger A_1 a}e^{a^\dagger A_2 a}\cdots e^{a^\dagger A_p a}=\operatorname{Det}\big(1+e^{A_1}e^{A_2}\cdots e^{A_p}\big),\tag{2.12}$$

$$\begin{aligned}\Rightarrow F(\xi,E)&=\sum_{s_1,s_2=\pm}c_{s_1}^2c_{s_2}^2\operatorname{Det}(1+e^{-\beta H})^{-1}\\&\quad\times\operatorname{Det}\big(1+\mathcal{S}_{s_1,s_2}e^{-\beta H}\mathcal{S}_{s_1,s_2}^\dagger e^{\xi\sigma_y}\big)\\&=\sum_{s_1,s_2=\pm}c_{s_1}^2c_{s_2}^2\operatorname{Det}\big(1+\mathcal{N}_{\rm in}[\mathcal{S}_{s_1,s_2}^\dagger e^{\xi\sigma_y}\mathcal{S}_{s_1,s_2}-1]\big).\end{aligned}\tag{2.13}$$

We have defined

$$\begin{aligned}&\mathcal{S}_{s_1,s_2}=S_{\rm out}e^{s_1\gamma_1|1\rangle\langle 1|}e^{s_2\gamma_2|2\rangle\langle 2|}S_{\rm in},\\&\mathcal{N}_{\rm in}=(1+e^{\beta H})^{-1}.\end{aligned}\tag{2.14}$$

In the zero-temperature limit, and for positive bias voltage, only electrons are injected, so that $\mathcal{N}_{\rm in}$ is a projector,

$$\mathcal{N}_{\rm in}=\tfrac{1}{2}(1+\sigma_y),\tag{2.15}$$

within the energy range $0 < E < eV$.

Eq. (2.13) generalizes the Levitov-Lesovik formula for full counting statistics [15, 16], to include the effects of weak measurements [13].

III. BINOMIAL CHARGE TRANSFER STATISTICS

The key difference between the quantum Hall interferometer studied in Ref. 13 and the Majorana interferometer considered here is the constraint of particle-hole symmetry, which requires that

$$S(-E) = S^*(E). \tag{3.1}$$

Both $S_{\rm in}$ and $S_{\rm out}$ contain contributions from the junction where the two Majorana modes combine into a Dirac mode (an electron-hole mode). We represent each junction by a point scatterer, of dimension small compared to $\hbar v_{\rm F}/eV$, with $v_{\rm F}$ the Fermi velocity. The scattering matrix of the junction may then be evaluated at the Fermi level, $E = 0$, when it is an element $e^{i\alpha\sigma_y}$ of SO(2).

Propagation along the interferometer does not couple the Majorana modes, the relative phase shift at energy E is $e^{ik\delta L\sigma_z}$, with $k = \frac{1}{2}E/\hbar v_{\rm F}$ and $\delta L = L_1 - L_2$ the path length difference along the two arms. Each $h/2e$ vortex in addition contributes a π relative phase shift, so a factor $\sigma_z^{N_v}$ for N_v vortices. All together we have

$$\begin{aligned} S_{\rm in} &= \sigma_z^{N_v} e^{ik\delta L\sigma_z} e^{i\alpha_{\rm in}\sigma_y}, \\ S_{\rm out} &= e^{i\alpha_{\rm out}\sigma_y}. \end{aligned} \tag{3.2}$$

Since the phase shifts $\propto \sigma_z$ commute with the projective measurements, we may associate them either with $S_{\rm in}$ or $S_{\rm out}$ — their order relative to the measurements does not matter.

Evaluation of the determinant (2.13) gives the result

$$F(\xi, E) = \cosh\xi + s_v(1-p_1)(1-p_2)\cos(2k\delta L)\sinh\xi, \tag{3.3}$$

where $s_v = (-1)^{N_v}$ is the vortex parity and $p_n = \varepsilon_n^2$ is the probability of which-path detection in mode n. Notice that the dependence on p_2 drops out once $p_1 = 1$, which is as it should be: Since the injected electron passes through the interferometer either in Majorana mode 1 or in Majorana mode 2, once the occupation of mode 1 is measured, it no longer matters whether the occupation of mode 2 is measured or not.

A final simplification applies at low voltages $V \ll \hbar v_{\rm F}/e\delta L$. We can then replace $\cos(2k\delta L) \approx 1$ in the energy range $0 < E < eV$, and the cumulant generating function takes the form

$$\begin{aligned} C(\xi) &= N_{\rm in}\ln\bigl[\cosh\xi + s_v(1-p_1)(1-p_2)\sinh\xi\bigr], \\ &= N_{\rm in}\bigl(s_v\xi + \ln[1-\mathcal{T}+\mathcal{T}e^{-2s_v\xi}]\bigr), \end{aligned} \tag{3.4a}$$

$$\mathcal{T} = \tfrac{1}{2}(p_1 + p_2 - p_1p_2). \tag{3.4b}$$

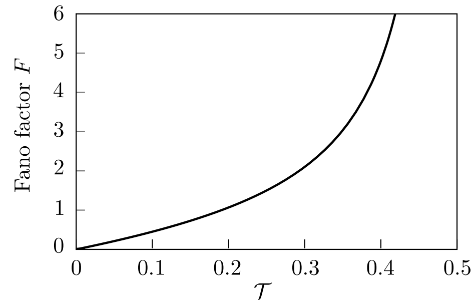

FIG. 3. Fano factor as a function of $\mathcal{T} = \frac{1}{2}(p_1 + p_2 - p_1p_2)$, computed from Eq. (3.10). This dimensionless ratio of shot noise power and conductance increases without bounds as the which-path detection probability tends towards unity (as $\mathcal{T} \to 1/2$).

The number

$$N_{\rm in} = eV/\delta E = eVt_{\rm counting}/h \tag{3.5}$$

is the number of electrons injected in the interferometer during the counting time $t_{\rm counting}$, in the large-time limit when the discreteness of $N_{\rm in}$ can be ignored [25].

Eq. (3.4) combines a deterministic transfer of s_v electron charges and a stochastic transfer of $-2s_v$ charges with probability $\mathcal{T} \in (0, 1/2)$. The corresponding probability distribution function of the transferred charge Q is binomial,

$$P(Q) = \binom{N_{\rm in}}{\frac{1}{2}(N_{\rm in} - s_vQ)}\mathcal{T}^{\frac{1}{2}(N_{\rm in}-s_vQ)}(1-\mathcal{T})^{\frac{1}{2}(N_{\rm in}+s_vQ)}, \tag{3.6a}$$

$$Q \in \{-N_{\rm in}, -N_{\rm in}+2, \ldots N_{\rm in}-2, N_{\rm in}\}. \tag{3.6b}$$

In the full detection limit $p_1 \to 1$ or $p_2 \to 1$, when $\mathcal{T} \to 1/2$, this becomes independent of the vortex number parity s_v,

$$P(Q) = 2^{-N_{\rm in}}\binom{N_{\rm in}}{\frac{1}{2}(N_{\rm in} - Q)}, \text{ if } \mathcal{T} = \tfrac{1}{2}. \tag{3.7}$$

The mean and variance of the transferred charge are

$$\overline{Q} = s_vN_{\rm in}(1-2\mathcal{T}) = s_vN_{\rm in}(1-p_1)(1-p_2), \tag{3.8}$$

$$\operatorname{Var}Q = 4N_{\rm in}\mathcal{T}(1-\mathcal{T}) = N_{\rm in}\bigl(1-(1-p_1)^2(1-p_2)^2\bigr). \tag{3.9}$$

In a transport measurement one can access these as the electrical conductance and shot noise power [17]. Their dimensionless ratio, the Fano factor

$$F = \frac{\operatorname{Var}Q}{|\overline{Q}|} = \frac{\mathcal{T}(1-\mathcal{T})}{1-2\mathcal{T}}, \tag{3.10}$$

diverges as $\mathcal{T} \to 1/2$ (see Fig. 3).

IV. CONCLUSION

In summary, we have investigated the effect on the $\mathbb{Z}_2$ interferometer [3, 4] of a weak measurement, capable of partially or fully distinguishing whether a Majorana fermion propagates through the upper or lower arm of the interferometer. Such a measurement fundamentally alters the interference by introducing which-path information, thereby degrading the coherence between the two paths.

We find that a which-path measurement adds a stochastic contribution with binomial statistics to the deterministic charge transfer. This contribution is measurable as a shot noise of the electrical current passed through the interferometer in response to a bias voltage. We note that shot noise of an unpaired Majorana mode is known to have a *trinomial* statistics [9, 26], distinct from the binomial form found here.

We obtained simple expressions for the full counting statistics, dependent only on the probability of the which-path detection, by working with a high-level description of the measurement as a Gaussian quantum channel [13]. In the present context one source of which-path information is provided by the charge fluctuations of a chiral Majorana mode, coupled to the electromagnetic environment. One would need a microscopic model of this coupling, to find out how effective the which-path measurement is in a realistic geometry. We hope that the striking effect of the coupling presented here, a divergent Fano factor, will motivate such a modeling.

ACKNOWLEDGMENTS

This work was supported by the Netherlands Organisation for Scientific Research (NWO/OCW), as part of Quantum Limits (project number SUMMIT.1.1016). I have benefited from discussions with A. R. Akhmerov and F. Hassler.

[1] A. Altland and M. R. Zirnbauer, *Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures*, Phys. Rev. B **55**, 1142 (1997).

[2] C. W. J. Beenakker and L. P. Kouwenhoven, *A road to reality with topological superconductors*, Nature Phys. **12**, 618 (2016).

[3] L. Fu and C. L. Kane, *Probing neutral Majorana fermion edge modes with charge transport*, Phys. Rev. Lett. **102**, 216403 (2009).

[4] A. R. Akhmerov, J. Nilsson, and C. W. J. Beenakker, *Electrically detected interferometry of Majorana fermions in a topological insulator*, Phys. Rev. Lett. **102**, 216404 (2009).

[5] K. T. Law, P. A. Lee, and T. K. Ng, *Majorana fermion induced resonant Andreev reflection* Phys. Rev. Lett. **103**, 237001 (2009).

[6] G. Strübi, W. Belzig, M. Choi, and C. Bruder, *Interferometric and noise signatures of Majorana fermion edge states in transport experiments*, Phys. Rev. Lett. **107**, 136403 (2011).

[7] J. Li, G. Fleury, and M. Büttiker, *Scattering theory of chiral Majorana fermion interferometry*, Phys. Rev. B **85**, 125440 (2012).

[8] S. Park, J. E. Moore, and H.-S. Sim, *Absence of Aharonov-Bohm effect of chiral Majorana fermion edge states*, Phys. Rev. B **89**, 161408(R) (2014).

[9] G. Strübi, W. Belzig, T. L. Schmidt, and C. Bruder, *Full counting statistics of Majorana interferometers*, Physica E **74**, 489 (2015).

[10] D. S. Shapiro, A. D. Mirlin, and A. Shnirman, *Microwave response of a chiral Majorana interferometer*, Phys. Rev. B **104**, 035434 (2021).

[11] E. Buks, R. Schuster, M. Heiblum, D. Mahalu, and V. Umansky, *Dephasing in electron interference by a 'which-path' detector*, Nature **391**, 871 (1998).

[12] D. Sprinzak, E. Buks, M. Heiblum, and H. Shtrikman, *Controlled dephasing of electrons via a phase sensitive detector*, Phys. Rev. Lett. **84**, 5820 (2000).

[13] C. W. J. Beenakker and Jin-Fu Chen, *Monitored quantum transport: full counting statistics of a quantum Hall interferometer*, arXiv:2504.07773.

[14] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2010).

[15] L. S. Levitov and G. B. Lesovik, *Charge distribution in quantum shot noise*, JETP Lett. **58**, 230 (1993).

[16] L. S. Levitov, H.-W. Lee and G. B. Lesovik, *Electron counting statistics and coherent states of electric current*, J. Math. Phys. **37**, 10 (1996).

[17] Ya. M. Blanter and M. Büttiker, *Shot noise in mesoscopic conductors*, Phys. Reports **336**, 1 (2000).

[18] F. Marquardt and C. Bruder, *Influence of dephasing on shot noise in an electronic Mach-Zehnder interferometer*, Phys. Rev. Lett. **92**, 056805 (2004); Phys. Rev. B **70**, 125305 (2004).

[19] V. S.-W. Chung, P. Samuelsson, and M. Büttiker, *Visibility of current and shot noise in electrical Mach-Zehnder and Hanbury Brown Twiss interferometers*, Phys. Rev. B **72**, 125320 (2005).

[20] S. Pilgram, P. Samuelsson, H. Förster, and M. Büttiker, *Full-counting statistics for voltage and dephasing probes*, Phys. Rev. Lett. **97**, 066801 (2006).

[21] H. Förster, P Samuelsson, S. Pilgram, and M. Büttiker, *Voltage and dephasing probes in mesoscopic conductors: A study of full-counting statistics*, Phys. Rev. B **75**,035340 (2007).

[22] A. Helzel, L. V. Litvin, I. P. Levkivskyi, E. V. Sukhorukov, W. Wegscheider, and C. Strunk, *Counting statistics and dephasing transition in an electronic Mach-Zehnder interferometer*, Phys. Rev. B **91**, 245419 (2015).

[23] Without the restriction to positive energies the anticommutator (2.3) of two annihilation operators does not vanish, because of the particle-hole symmetry relation $a(E) = a^\dagger(-E)$. This plays a role when inelastic scat-

tering couples positive and negative energies, see C. W. J. Beenakker, *Annihilation of colliding Bogoliubov quasiparticles reveals their Majorana nature*, Phys. Rev. Lett. **112**, 070604 (2014).

[24] I. Klich, *An elementary derivation of Levitov's formula*, in: *Quantum Noise in Mesoscopic Physics*, NATO Science Series II, **97**, 397 (2003).

[25] F. Hassler, M. V. Suslov, G. M. Graf, M. V. Lebedev, G. B. Lesovik, and G. Blatter, *Wave-packet formalism of full counting statistics*, Phys. Rev. B **78**, 165330 (2008).

[26] N. V. Gnezdilov, B. van Heck, M. Diez, J. A. Hutasoit, and C. W. J. Beenakker, *Topologically protected charge transfer along the edge of a chiral p-wave superconductor*, Phys. Rev. B **92**, 121406(R) (2015).