

Shaping chaos in bilayer graphene cavities

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Bilayer graphene (BLG) cavities, where electrons are confined in finite graphene flakes, provide a suitable platform to study quantum chaos in condensed matter systems. Trigonally warped Fermi surfaces in BLG cavities lead to a rich variety of quantum phenomena. In this work, we investigate the effect of the rotation angle on the spectral statistics of BLG cavities. We find that rotation leads to level repulsion following Wigner–Dyson statistics, while corresponding eigenstate analysis indicates a transition from near-integrability to spatially uncorrelated random waves. Analysis of the semiclassical ray-dynamics with the Floquet theory reveals that the rotation angle drives the system towards chaos. These findings establish an avenue to quantum chaotic phenomena in BLG cavities with potential applications in quantum device engineering.

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I. INTRODUCTION

Most physical systems are nonintegrable and exhibit chaotic dynamics in their classical limit [1]. Billiard models have long served as a paradigm to study such behavior, where the dynamical complexity arises solely from the geometry of the confining boundary [2]. In the quantum regime, a billiard is typically understood as an idealized cavity, and is described by the Schrödinger equation with appropriate boundary conditions. Its spectral properties reveal how classical chaos manifests in quantum mechanics [3–5]. The semiclassical correspondence, established through periodic orbit theory and the Bohigas–Giannoni–Schmit conjecture, connects irregular classical trajectories with universal spectral fluctuations governed by random matrix theory [6, 7]. Over the past decades, billiards have thus served as a testbed for exploring a wide range of phenomena, from spectral statistics and level correlations to wavefunction localization and the emergence of scars in chaotic eigenstates [8, 9].

These results, however, are all established for Schrödinger dynamics. In contrast, Berry and Mondragon (1987) introduced a relativistic “neutrino” billiard [10], and this framework was subsequently extended to graphene, whose low-energy quasiparticles obey the Dirac equation [11–13]. Graphene-based cavities represent a fundamentally different class of quantum billiards. In monolayer graphene, the linear Dirac dispersion and the pseudospin degree of freedom [14, 15] introduce relativistic boundary effects [16] absent in conventional quantum billiards. Building on this, theoretical studies have

investigated spectral statistics in monolayer graphene cavities[17–20], and, more recently, experiments have directly observed quantum scars associated with quantum chaos in graphene devices [21].

Despite the extensive work on monolayer systems, the quantum dynamics of BLG cavities remain largely unexplored. BLG represents an equally intriguing material platform: unlike monolayer graphene, its low-energy quasiparticles follow a quadratic dispersion resembling that of the Schrödinger equation, yet the electronic structure is further enriched by pronounced trigonal warping [22]. This combination produces a highly anisotropic Fermi surface, so that the resulting behavior becomes exceptionally sensitive to lattice orientation, gating [23, 24], and strain features [25] that have motivated broad interest in BLG for transport and device studies [26, 27].

These same properties render BLG particularly promising for quantum chaos research. The trigonal warping-induced anisotropy allows the underlying semiclassical dynamics to be tuned far more flexibly than in monolayer graphene or conventional two-dimensional electron gases. Recent studies have shown that the trigonal warping-induced anisotropy of the Fermi surface strongly modifies the stability of periodic orbits in semiclassical ray-dynamics and Poincaré analyses, revealing semiclassical phase space patterns entirely distinct from those of conventional billiards [28, 29]. Beyond its dynamical implications, the same anisotropy has been exploited in electron-optical applications such as Veselago lensing, highlighting the unique transport behavior of bilayer graphene cavities [30].

However, these investigations have remained almost entirely semiclassical. The quantum mechanical properties of BLG cavities, such as their eigenstate morphology, spectral statistics, and the extent to which the semi-

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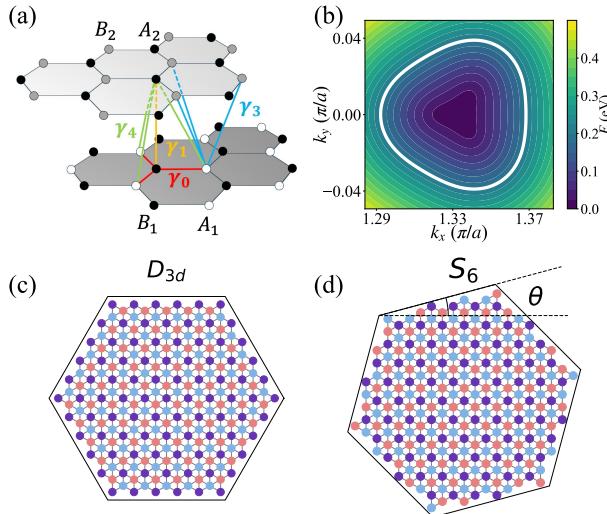


FIG. 1. (a) Schematic diagram of the bilayer graphene tight-binding model showing atoms and hopping. (b) K valley of BLG's band structure, where the white curve represents the Fermi surface at $E = 0.2$ eV, showing trigonal warping due to interlayer hopping γ_3 . (c) and (d) show diagrams of BLG cut into a hexagonal shape, where red and blue indicate the non-dimer atoms A_1 and B_2 , and purple indicates the dimer atoms B_1 and A_2 . In (c), the boundary is aligned with the inner lattice, and the point group of the atoms is D_{3d} . In (d), the boundary is rotated by 15° relative to the inner lattice, and the point group of the atoms is S_6 . The cavities we study contain millions of atoms.

classical features persist or break down at the quantum level, are essentially unknown. Addressing these issues is crucial for understanding BLG cavities not only as electron-optical devices [30] but also as a platform for systematically investigating quantum chaotic phenomena in strongly anisotropic materials [31].

Here, we explore ways to control quantum chaos within BLG cavities. In Sec. II, we introduce a tight-binding model of the bilayer graphene cavity and describe how the subspaces are separated according to symmetry. In Sec. III, we present the statistical analysis of the eigenenergy spectrum. In Sec. IV, we display the eigenstates statistics using correlation function. In Sec. V, we highlight the semiclassical dynamics, and in Sec. VI we discuss and conclude.

II. MODEL

A. Tight-binding model of finite-size BLG cavity

To explore finite bilayer graphene cavities, we use a tight-binding model, namely AB-stacked bilayer graphene with a standard Slonczewski–Weiss–McClure description shown in Fig. 1(a). This model includes the dominant in-plane hopping γ_0 , the interlayer dimer hopping γ_1 , the skew interlayer hopping γ_3 responsible for

trigonal warping, and the additional interlayer skew hopping γ_4 [22].

$$H_{\text{TB}} = \sum_{\langle i,j \rangle_{\text{intra}}} \gamma_0 c_i^\dagger c_j + \sum_{\langle B_1, A_2 \rangle} \gamma_1 c_{B_1}^\dagger c_{A_2} + \sum_{\langle A_1, B_2 \rangle} \gamma_3 c_{A_1}^\dagger c_{B_2} + \sum_{\langle A_1, A_2 \rangle} \gamma_4 c_{A_1}^\dagger c_{A_2} + \sum_{\langle B_1, B_2 \rangle} \gamma_4 c_{B_1}^\dagger c_{B_2} + \text{H.c.} \quad (1)$$

Here c_i^\dagger and c_i are the electron creation and annihilation operators on site i , respectively, and $\langle \cdot, \cdot \rangle$ denotes the corresponding in-plane or interlayer nearest-neighbor pairs as indicated in Fig. 1(a). The low-energy band structure of AB stacked bilayer graphene exhibits an anisotropic convex Fermi surface in the K valley, illustrated in Fig. 1(b), where the trigonal warping effect induced by γ_3 is clearly visible even at energies of a few hundred meV. We focus on the energy range from 0.18 eV to 0.25 eV above the quadratic band touching (QBT) point, where the Fermi surface is clearly warped and the system resides in the semiclassical regime, with an average wavelength of approximately $\lambda = 20$ nm.

The cavity is constructed by cutting BLG into a regular hexagon shape, as shown in Fig. 1(c). The circumradius is set to $r = 400$ nm, corresponding to a characteristic size of $L \simeq 800$ nm. Since the lattice constant of BLG is $a = 0.246$ nm, the system satisfies the condition $L \gg \lambda \gg a$. To investigate the influence of lattice–boundary misalignment and the interplay between the cavity geometry and the trigonal warping effect, we introduce a rotated BLG cavity with rotation angle θ , as illustrated in Fig. 1(d). We compare the quantum chaotic behavior of unrotated and rotated cavities. Owing to rotational symmetry, it is sufficient to consider θ in the range 0° to 30° , where intermediate angles correspond to misaligned configurations. In our numerical calculations, the hexagonal cavities contain between 1.5×10^6 and 2.5×10^6 lattice sites, which allows us to resolve fully quantum spectral statistics in a regime where semiclassical or continuum approximations would normally be invoked.

Based on the tight-binding Hamiltonian in Eq. (1), the eigenvalue problem of our finite-size BLG cavity can be formulated as follows:

$$H_{\text{TB}} \psi_n(\mathbf{r}) = E_n \psi_n(\mathbf{r}), \quad (2)$$

where $E_n(\mathbf{r})$ denote the eigenenergies and $\psi_n(\mathbf{r})$ the corresponding eigenstates.

B. Symmetry analysis

The symmetry of the tight-binding Hamiltonian differs between the two configurations we study. When the boundary is aligned with the crystalline axes, the tight-binding Hamiltonian respects the full D_{3d} symmetry of the BLG lattice, as shown in Fig. 1(c). This group contains a threefold rotation C_3 , inversion I , and three vertical mirror operations σ_v .

Since $[\hat{H}, \mathcal{R}_{2\pi/3}] = 0$, the eigenstates of \hat{H} can be chosen as simultaneous eigenstates of the 120° rotation operator $\mathcal{R}_{2\pi/3}$, namely

$$\mathcal{R}_{2\pi/3}\psi_n^{(m)}(\mathbf{r}) = e^{i2\pi m/3}\psi_n^{(m)}(\mathbf{r}), \quad m = 0, \pm 1. \quad (3)$$

The rotation-invariant sector $m = 0$ further splits under inversion \mathcal{I} and mirror reflection σ_v into four one-dimensional irreducible representations. To make these parity properties explicit, we label the corresponding eigenstates as $\psi_n^{(0,p_i,p_\sigma)}$, where $p_i = \pm 1$ and $p_\sigma = \pm 1$ denote the eigenvalues under inversion and mirror operations, respectively:

$$\mathcal{I}\psi_n^{(0,p_i,p_\sigma)} = p_i \psi_n^{(0,p_i,p_\sigma)}, \quad \sigma_v \psi_n^{(0,p_i,p_\sigma)} = p_\sigma \psi_n^{(0,p_i,p_\sigma)} \quad (4)$$

In this notation, $(p_i, p_\sigma) = (+, +)$, $(+, -)$, $(-, +)$, and $(-, -)$ correspond to the irreducible representations A_{1g} , A_{2g} , A_{1u} , and A_{2u} , respectively. The correspondence between the eigenvalues of the symmetry operations (m , p_i , and p_σ) and the irreducible representations is summarized in Table I (left).

TABLE I. Symmetry sectors used for D_{3d} (left) and S_6 (right). Listed are the irreducible subspace's dimension, rotation sector m , and inversion/mirror parities p_i/p_σ .

Group	Irrep	Dim.	m	p_i	p_σ	Group	Irrep	Dim.	m	p_i
D_{3d}	A_{1g}	1	0	+1	+1	S_6	A_g	1	0	+1
	A_{2g}	1	0	+1	-1		A_u	1	0	-1
	A_{1u}	1	0	-1	+1		E_g	2	± 1	+1
	A_{2u}	1	0	-1	-1		E_u	2	± 1	-1
	E_g	2	± 1	+1	0					
	E_u	2	± 1	-1	0					

When the boundary is rotated by an angle θ as shown in Fig. 1(d), mirror symmetry is broken, and the symmetry group of the tight-binding Hamiltonian reduces to S_6 . The corresponding symmetry properties and irreducible representations are listed in Table I (right). Based on the symmetry analysis, we can study the eigenstates independently in each subspace in the following sections.

C. Dynamical properties

Before analyzing the quantum spectrum, it is useful to recall the dynamical properties of the geometric cavity itself. A regular hexagon is a prototypical pseudointegrable polygonal billiard. Its interior angles are rational multiples of π , and unfolding the trajectories produces a higher-genus translation surface rather than the invariant tori of integrable systems [32, 33]. Such systems have zero Lyapunov exponents but display nontrivial ergodic and mixing behavior originating from the corner singularities [34–37]. Their quantum spectra therefore exhibit intermediate, often semi-Poisson statistics [33, 38–42].

This geometric baseline plays an important role in interpreting the BLG cavities studied here. As we will show, misalignment between the BLG lattice and the

hexagonal boundary breaks the residual pseudointegrable structure, and, together with the strong trigonal warping of the BLG dispersion, drives the system into a fully chaotic regime.

III. LEVEL STATISTICS

To characterize the spectral properties for each orientation angle θ , we compute the ordered eigenvalues $\{E_i\}$ of the tight-binding Hamiltonian and analyze their fluctuations after unfolding. Nearest-neighbor spacings are defined as $s_i = E_{i+1} - E_i$, and the resulting spacing distribution $P(s)$ is compared with the standard Poisson and Wigner–Dyson forms of random-matrix theory [6]. We further evaluate the ratio of consecutive spacings, $r_i = \frac{s_{i+1}}{s_i}$, $\tilde{r}_i = \min(r_i, 1/r_i)$, which takes the characteristic values $\langle \tilde{r} \rangle_{\text{P}} \simeq 0.386$, $\langle \tilde{r} \rangle_{\text{GOE}} \simeq 0.531$, and $\langle \tilde{r} \rangle_{\text{GUE}} \simeq 0.600$ [43].

Long-range correlations of spectrum are quantified using the Dyson–Mehta spectral rigidity [44], which is defined as

$$\Delta_3(L) = \frac{1}{L} \min_{\{A,B\}} \int_E^{E+L} [N(E') - AE' - B]^2 dE' \quad (5)$$

where $N(E)$ is the unfolded staircase function. While integrable spectra show a linear increase of $\Delta_3(L)$ and chaotic spectra exhibit the logarithmic behavior predicted by random-matrix theory, the orientation-dependent evolution of Δ_3 provides a complementary measure of long-range correlations in bilayer graphene cavities. To quantify the degree of chaos in the system, we compute the index $\langle \Delta_3(L) \rangle$, defined as the average of $\Delta_3(L)$ over the range $L \in [10, 40]$. For reference, the typical values of $\langle \Delta_3 \rangle$ in the three limiting cases are approximately $\langle \Delta_3 \rangle_{\text{Poisson}} \simeq 1.67$, $\langle \Delta_3 \rangle_{\text{GOE}} \simeq 0.319$, and $\langle \Delta_3 \rangle_{\text{GUE}} \simeq 0.184$. These benchmarks provide a quantitative basis for tracking how the spectral statistics evolve with respect to rotation angle θ .

We compute a total of 2,978 eigenstates within the energy range of 0.18–0.25 eV for the system and perform the spectral statistics within each irreducible representation separately as introduced in Sec. II. Fig. 2 shows the symmetry-resolved spectral statistics for the unrotated cavity. The three sectors exhibit markedly different levels of spectral correlation. The (A_{1u}, A_{2g}) subspectrum has $\langle \tilde{r} \rangle = 0.379$, in agreement with the Poisson value and consistent with essentially integrable behavior.

By contrast, the (A_{1g}, A_{2u}) and (E_u, E_g) subspectra show stronger level repulsion at small spacings together with an approximately exponential tail with $\langle \tilde{r} \rangle = 0.486$ and 0.463 respectively, indicating more chaotic behavior than the (A_{1u}, A_{2g}) subspaces [6]. Their spacing distributions lie between Poisson and Wigner–Dyson limits and are well described by a semi-Poisson form, characteristic of pseudointegrable polygonal cavities [45]. We can fit

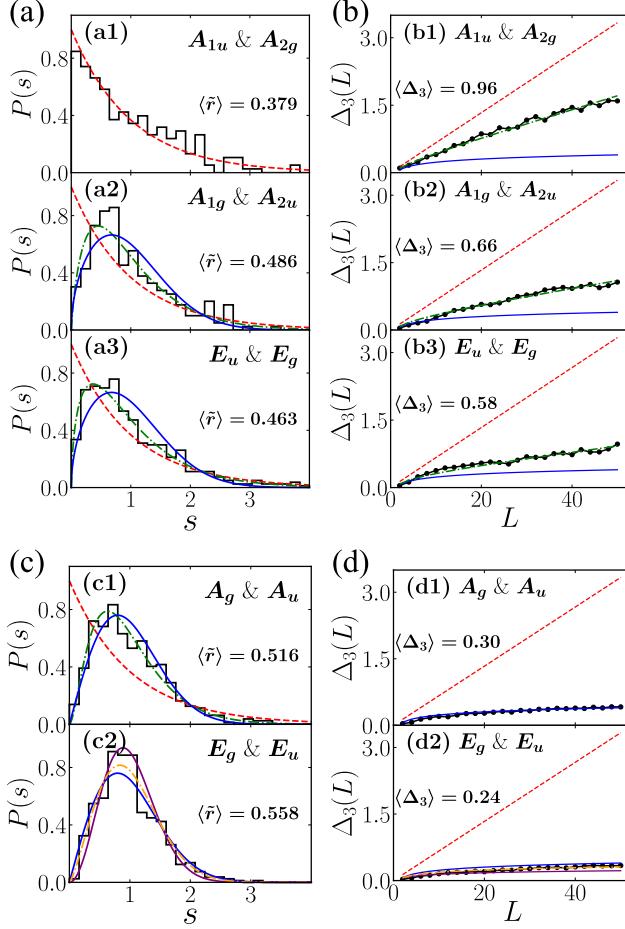


FIG. 2. (a) and (b) show the level-spacing statistics and spectral rigidity for the unrotated cavity, with the corresponding $\langle \tilde{r} \rangle$ values and averaged rigidity indicated in each panel. The red dashed line denotes the Poisson distribution, the blue solid line represents the Wigner–Dyson distribution, and the green dash-dotted line shows the fitted semi-Poisson distribution. (c) and (d) present the same analyses for the cavity rotated by 15° . The purple solid line represents the GUE distribution, and the orange dash-dotted line denotes the fitted GOE–GUE distribution.

the distribution by taking the semi-Poisson form [46–48]

$$P_{SP}(\beta, s) = A_\beta s^\beta e^{-(\beta+1)s}, \quad A_\beta = \frac{(\beta+1)^{\beta+1}}{\Gamma(\beta+1)} \quad (6)$$

with $\beta = 0.83$ for the (A_{1g}, A_{2u}) subspectrum and $\beta = 0.57$ for the (E_u, E_g) subspectrum, indicating intermediate statistics between Poisson and Wigner–Dyson limits.

The rigidity curves $\Delta_3(L)$ in Fig. 2(b) show the same ordering. The (A_{1u}, A_{2g}) sector exhibits an almost linear increase of $\Delta_3(L)$ versus L , as expected for weakly correlated spectra, whereas the other two sectors develop noticeably reduced slopes that lie between the Poisson and GOE predictions. Although the long-range rigidity is more pronounced than suggested by the nearest-

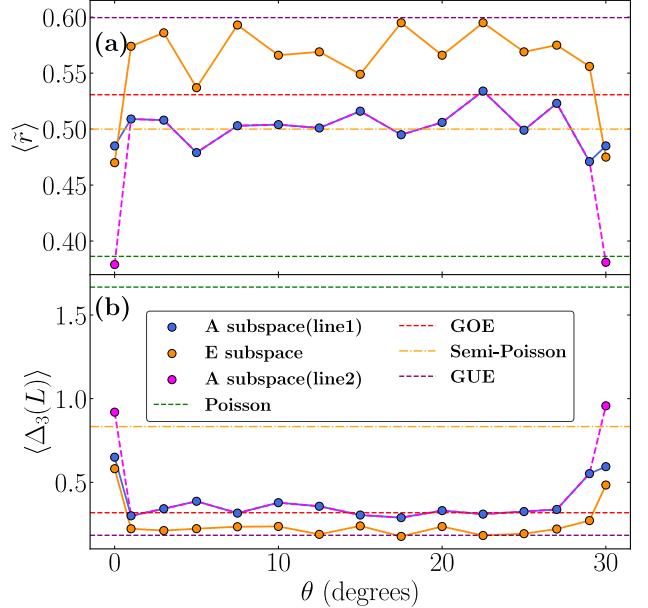


FIG. 3. (a) and (b) present the dependence of the r -value and the average spectral rigidity on the rotation angle θ for the A and E subspaces, respectively. In the A subspaces, two lines are shown: for the unrotated cases $\theta = 0^\circ$ and 30° , the blue solid line corresponds to the pseudointegrable sector, while the purple dashed line represents the integrable sector.

neighbor statistics, all three sectors preserve a consistent correlation hierarchy. The integrable (A_{1u}, A_{2g}) sector has an average rigidity of $\langle \Delta_3(L) \rangle = 0.94$, larger than $\langle \Delta_3(L) \rangle = 0.66$ and 0.59 for the other two pseudointegrable sectors. Thus, the unrotated BLG cavity contains both effectively integrable subspaces and pseudointegrable ones. These will serve as the baseline for assessing how lattice–boundary misalignment modifies the spectral statistics.

Next, we focus on the spectral statistics of a cavity rotated by 15° , as shown in Fig. 2(c) and (d). The level spacing distributions in Fig. 2(c) of the (A_g, A_u) sector remain in an intermediate form, yielding $\langle \tilde{r} \rangle = 0.516$. In the (E_g, E_u) sector, originating from the two-dimensional rotational pair $m = 1, 2$, the spacing statistics lie between GOE and GUE: the exponential tail is clearly suppressed, and the average ratio $\langle \tilde{r} \rangle = 0.558$ indicates fully chaotic behavior. The mixed GOE–GUE fitted curves using the formula in Ref. [49] provide an excellent description of the data, demonstrating that the rotation eliminates the nearly integrable subspaces present at $\theta = 0^\circ$ and drives the system into a chaotic regime. The spectral rigidity shown in Fig. 2(d) also indicates a more chaotic behavior compared to the unrotated case. The rigidity of the (A_g, A_u) sector, with $\langle \Delta_3(L) \rangle = 0.31$, is well described by the GOE prediction, whereas that of the (E_g, E_u) sector, with $\langle \Delta_3(L) \rangle = 0.24$, lies between the GOE and GUE limits. Both the level spacing distribution and the rigidity

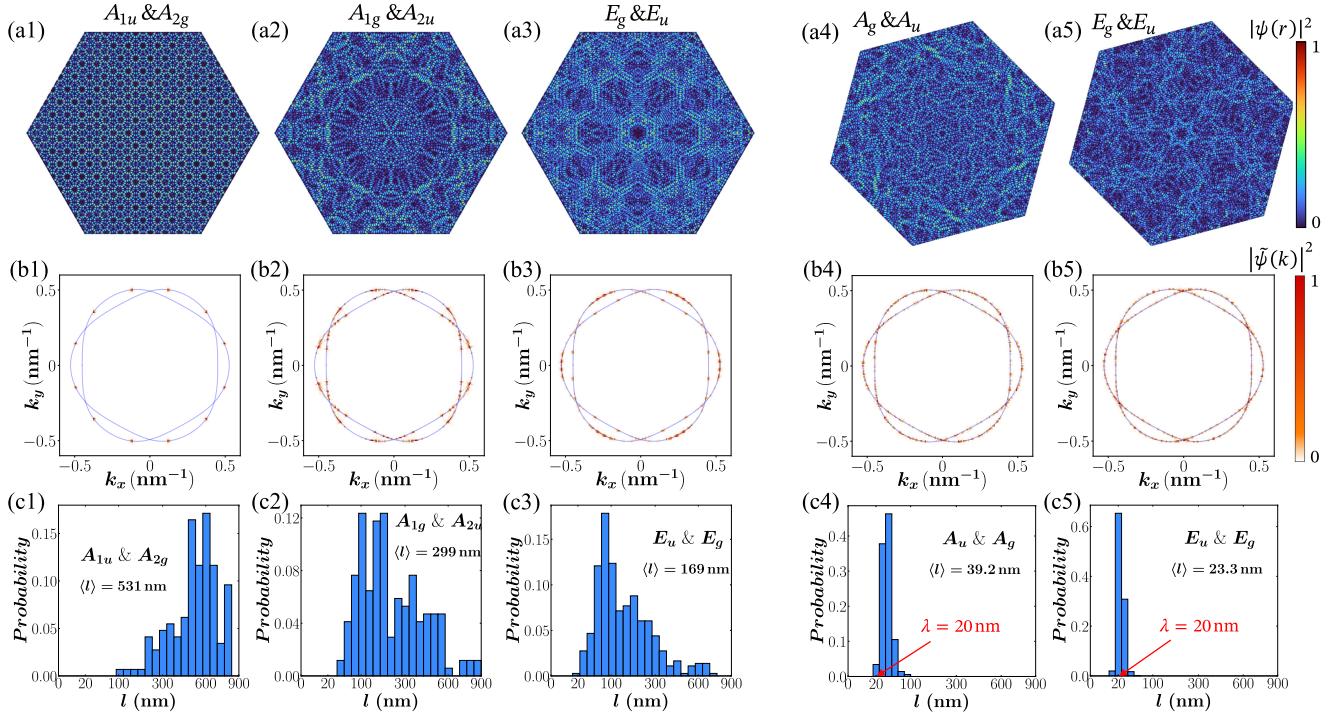


FIG. 4. The figures illustrate the wavefunctions and their associated statistical properties. Panels (a1)–(a3) show representative probability density distributions $|\psi(\mathbf{r})|^2$ of eigenstates in unrotated cavities for different symmetry subspaces, while panels (a4) and (a5) present the corresponding results for the rotated case. Panels (b1)–(b5) display the associated distributions $|\tilde{\psi}(\mathbf{k})|^2$ in momentum space with trigonal warped Fermi surface at K and K' . Panels (c1)–(c5) summarize the statistical behavior for the unrotated and rotated cavities across the symmetry subspaces.

ity indicate that rotation enhances level repulsion while reducing long-range spectral correlations, consistent with the onset of quantum chaos.

Fig. 3 illustrates how the spectral statistics evolve as a function of the rotation angle θ . The variation of $\langle \tilde{r} \rangle$ with θ is shown in Fig. 3(a). We analyze the spectral behavior in both the A and E subspaces. For the A subspaces, line 1 begins and ends in the intermediate regime at $\theta = 0^\circ$ and $\theta = 30^\circ$, indicating that $\langle \tilde{r} \rangle$ remains at an intermediate value across all angles, fluctuating around the semi-Poisson prediction. More distinctly, line 2 starts and ends in the integrable regime, with $\langle \tilde{r} \rangle$ close to the Poisson limit; as θ increases, $\langle \tilde{r} \rangle$ rises rapidly, reflecting the loss of integrability due to rotation. Meanwhile, the curve corresponding to the E subspaces evolves from an intermediate $\langle \tilde{r} \rangle$ value to a fully chaotic one, approaching values between the GOE and GUE predictions.

The long-range correlations, quantified by the averaged rigidity $\langle \Delta_3 \rangle$, show a similarly stable behavior as shown in Fig. 3(b). Once the integrable channel present at $\theta = 0^\circ$ and $\theta = 30^\circ$ is removed, the rigidity remains close to the GOE/GUE values across the entire angular interval. Together, these results demonstrate that the enhancement of chaotic correlations induced by lattice-boundary misalignment is robust with respect to the rotation angle, and no fine-tuned orientation is required to maintain the chaotic character of the spectrum.

IV. EIGENSTATES

After the eigenenergy spectrum statistics, we turn our attention to examine how the misalignment between the lattice and the boundary influences the eigenstates. The structure of eigenstates is strongly linked to the underlying classical dynamics. In integrable systems, eigenfunctions typically exhibit regular standing wave patterns [49]. In contrast, chaotic systems follow Berry's random wave model [50] and may display quantum scars, i.e. localized enhancements along unstable periodic classical orbits [8]. Pseudointegrable systems lie in between: their eigenstates are neither fully regular nor completely random and often depend on symmetry sectors [42].

To quantify how the rotation influences the overall behavior of the system's eigenstates, we perform a correlation function analysis to characterize the degree of randomness for the eigenstates, defined as $C(\mathbf{r}, \mathbf{r}') = \langle \psi^*(\mathbf{r})\psi(\mathbf{r}') \rangle$. According to Berry's random wave model for chaotic cavities, the wavefunction $\psi(\mathbf{r})$ can be modeled as a superposition of plane waves with random phases but fixed wavenumber k : $\psi(\mathbf{r}) = \sum_{n=1}^N a_n e^{ik_n \cdot \mathbf{r}}$, the correlation function takes the form $C(r) = J_0(kr)$ [50].

In our model, the correlation function is generally anisotropic and depends on both the magnitude and

direction of \mathbf{r} . To obtain a scalar measure from the anisotropic correlation function, we define $C(r) \equiv \langle \max_{\mathbf{r}'} C(\mathbf{r}, \mathbf{r}') \rangle_{\mathbf{r}}$ for a fixed $r = |\mathbf{r}'|$. We analyze the correlation function by fitting it to the form $C(r) = J_0(2\pi \frac{r}{l})$, where l represents the correlation length, characterizing the spatial correlation of the eigenfunction. We use correlation length as a relative, heuristic measure of spatial correlation for the eigenstates. For a fully chaotic system, the correlation length l is the wavelength λ of the wavefunction. In our case, the wavelength is approximately $\lambda \approx 20 for energies around 0.2 eV. We select 1,056 eigenstates near 0.2 eV and perform statistical analysis for each symmetry subspace based on their correlation lengths l . The average correlation length $\langle l \rangle$ then provides a measure of how chaotic each subspace is.$

To illustrate the characteristic patterns of the eigenstates, we present the probability density distributions of representative eigenstates from each symmetry subspace in Fig. 4(a1)–(a5). Their momentum space structure is further analyzed via the Fourier transform $\tilde{\psi}(\mathbf{k}) = \int d^2 \mathbf{r} e^{-i\mathbf{k} \cdot \mathbf{r}} \psi(\mathbf{r})$ with the results shown in Fig. 4(b1)–(b5), where the density $|\tilde{\psi}(\mathbf{k})|^2$ and the trigonal warped Fermi surfaces in the K and K' valleys are displayed.

For the unrotated cavity, the probability density distributions of representative wavefunctions are shown in Fig. 4(a1)–(a3). The eigenstate in the (A_{1u}, A_{2g}) sector displayed in Fig. 4(a1) exhibits regular standing wave patterns, characteristic of regular eigenstates [5]. Its momentum space density shown in Fig. 4(b1) is localized at discrete points on the Fermi surface, reflecting and explaining the regular spatial structure of the eigenstate. The other sectors (A_{1g}, A_{2u}) and (E_u, E_g) exhibit more irregular patterns compared to the (A_{1u}, A_{2g}) sector, as shown in Fig. 4(a2) and (a3). This behavior can also be understood from the momentum space distributions in Fig. 4(b2) and (b3), where the eigenstates are composed of a larger number of plane wave components.

Fig. 4(a4) and (a5) show the probability density distributions of representative wavefunctions from the (A_u, A_g) and (E_u, E_g) sectors in the rotated cavities. The density profiles exhibit highly irregular structures with no indication of periodic orbits. The corresponding momentum space analysis in Fig. 4(b4) and (b5) reveals an almost uniform and continuous distribution of $|\tilde{\psi}(\mathbf{k})|^2$ along the warped Fermi surface. In contrast to Berry's random wave model, where the wavefunction is composed of plane waves with a fixed wavenumber but random propagation directions, the rotated cavities host random waves formed by plane waves whose wave vectors lie on the trigonal warped Fermi surface.

The statistics for different subspaces are shown in Fig. 4(c1)–(c3). Overall, all subspaces exhibit average correlation lengths much larger than the wavelength $\lambda = 20, indicating that the unrotated system is non-chaotic. For the (A_{1u}, A_{2g}) sector in Fig. 4(c1), which is shown to be integrable from the spectral statistics, the average correlation length is $\langle l \rangle_{(A_{1u}, A_{2g})} = 531,$$

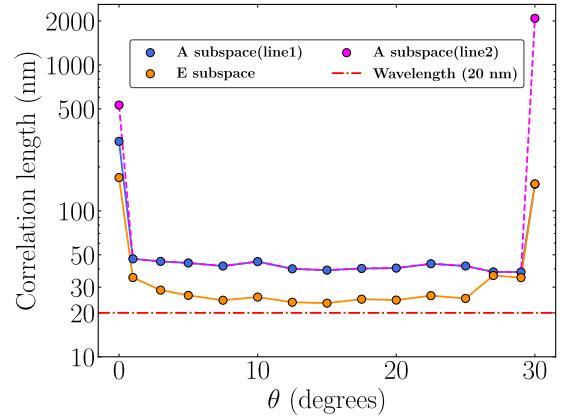


FIG. 5. The figure shows how the correlation length varies with the rotation angle θ for the A and E subspaces. In the A subspaces, two curves are plotted. The blue solid curve corresponds to the pseudointegrable sector at $\theta = 0^\circ$ and 30° for the unrotated cavities, while the purple dashed curve denotes the integrable one.

much larger than the wavelength λ , signaling the presence of strong spatial correlations. For the (A_{1g}, A_{2u}) and (E_u, E_g) sectors in Fig. 4(c2) and (c3), which have been shown to be pseudointegrable from the spectral analysis, the average correlation lengths are smaller than that of the (A_{1u}, A_{2g}) sector. Nevertheless, the average correlation lengths, $\langle l \rangle_{(A_{1g}, A_{2u})} = 299 and $\langle l \rangle_{(E_u, E_g)} = 169, remain larger than the wavelength, indicating that these sectors do not exhibit fully chaotic behavior.$$

The change in correlation length statistics from the unrotated to the rotated cavity is remarkable. We consider a cavity rotated by $\theta = 15^\circ$, as shown in Fig. 4(c4) and (c5). The correlation lengths for both the (A_u, A_g) and (E_u, E_g) sectors decrease significantly compared to the unrotated case, with $\langle l \rangle_{(A_u, A_g)} = 39.2 and $\langle l \rangle_{(E_u, E_g)} = 23.3. Notably, in the (E_u, E_g) subspace, the average correlation length is close to the wavelength, and most eigenstates have correlation lengths around $\lambda = 20. This shows that the rotation of the BLG cavity induces quantum chaos, as it causes the eigenstates to become spatially uncorrelated.$$$

Fig. 5 summarizes the dependence of the average correlation length on the rotation angle θ for the different symmetry subspaces. It is evident that the correlation length decreases rapidly to the scale of the wavelength upon rotation. This observation further supports our conclusion that the misalignment between the lattice and the boundary drives the system from a pseudointegrable regime to one exhibiting fully developed quantum chaos signatures.

V. SEMICLASSICAL DYNAMICS

To highlight the geometric consequences of trigonal warping, we employ a simplified semiclassical ray-dynamics model based on a continuum Hamiltonian. This bulk description captures the dominant warping-induced anisotropy while omitting atomistic edge effects, and therefore serves as a heuristic tool for illustrating how lattice-boundary misalignment shapes the effective cavity dynamics.

At the energies considered here ($|E| \ll \gamma_1$), the electron motion in AB-stacked BLG is well captured by the two-band McCann–Fal’ko Hamiltonian [22], whose warped dispersion produces a strongly anisotropic Fermi contour. The group velocity $\mathbf{v}(\mathbf{k}) = \nabla_{\mathbf{k}} E(\mathbf{k})$ therefore inherits the characteristic threefold anisotropy, while the Berry curvature vanishes and no anomalous velocity arises [51].

Between collisions the band energy is conserved. When a trajectory hits a hard wall with normal $\hat{\mathbf{n}}$, the outgoing momentum \mathbf{k}' satisfies energy conservation and the reversal of the normal component of the group velocity:

$$\mathbf{v}(\mathbf{k}') = \mathbf{v}(\mathbf{k}) - 2[\mathbf{v}(\mathbf{k}) \cdot \hat{\mathbf{n}}]\hat{\mathbf{n}}, \quad E(\mathbf{k}') = E(\mathbf{k}). \quad (7)$$

Unlike geometric mirror reflection, this rule is momentum dependent and highly anisotropic. Nevertheless, because the warped Fermi contour remains strictly convex in our energy window, the resulting reflection map is smooth and area preserving. Our numerical evaluation yields a maximal Lyapunov exponent $\lambda_{\max} = 0$, confirming the absence of true classical chaos. Trigonal warping therefore does not generate exponential sensitivity but instead controls the ergodicity of the motion.

The Poincaré sections in Fig. 6 are constructed by recording the arclength coordinate s and incidence angle θ whenever a ray intersects a fixed reference edge of the hexagon, over 5×10^5 collisions. When the boundary is aligned with a crystalline C_3 axis, the warped reflection rule is commensurate with the boundary symmetry. As a result, the warped reflection rule effectively reduces to ordinary specular reflection [28]. Successive reflections recur along a small set of preferred directions. Consecutive reflections therefore repeat only a small set of incidence angles, and the resulting Poincaré section collapses into a few discrete lines rather than filling the available phase space. The motion is confined to these invariant lines, consistent with pseudo-integrable dynamics.

For comparison, panel (c) in Fig. 6 shows a mirror-reflecting irrational hexagon. The Poincaré points fill the allowed domain uniformly, demonstrating genuinely ergodic behavior. In contrast, the Poincaré section in panel(b) exhibits stratified, layer-like structures, reflecting the quasi-ergodic character of the warped BLG dynamics.

After rotating the cavity by 15° , this commensurability is lost. The warped reflection rule becomes incommensurate with the boundary, and incidence angles no longer

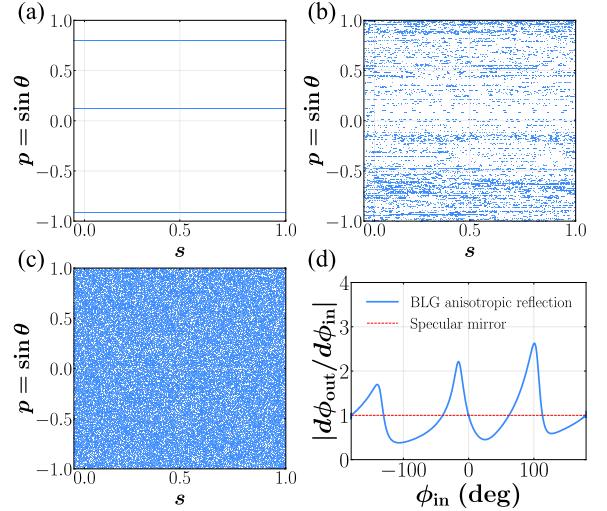


FIG. 6. Poincaré sections constructed by recording, for every trajectory crossing a chosen reference edge, the boundary coordinate s (position along the edge) and the incidence angle θ (relative to the boundary normal). (a) With the boundary aligned to crystalline C_3 axis, the map collapses onto a few invariant curves characteristic of pseudo-integrable motion. (b) After rotating the cavity by 15° , the map spreads quasi-ergodically over the accessible phase space. (c) For comparison, a mirror reflecting irrational hexagon shows uniformly filled Poincaré points, demonstrating true ergodicity without crystalline commensurability. (d) The angular magnification $|d\phi_{\text{out}}/d\phi_{\text{in}}|$ for BLG’s anisotropic reflection (blue) compared with the constant unit magnification of specular mirror reflection (red dashed). The variations in BLG magnification quantify the underlying warping responsible for the structures in panels (a) and (b).

recur coherently. Successive reflections gradually populate the entire accessible phase space, producing quasi-ergodic spreading [52].

VI. DISCUSSIONS AND CONCLUSION

The emergence of quantum chaos upon misaligning the boundary and the lattice can be understood from a semiclassical perspective. We emphasize that the *chaos* here refers to quantum chaos, indicated by Wigner–Dyson spectral statistics and random wave eigenstates. Although we use a simplified semiclassical ray-dynamics construction, this model is intended to highlight the geometric consequences of the warped dispersion. No simple semiclassical description fully captures the atomistic boundary physics of the lattice model.

The simplified semiclassical picture nonetheless provides a meaningful interpretation of the observed transition. The rotated cavity exhibits quasi-ergodic phase space spreading, and such semiclassical ergodicity is known to be sufficient to generate Wigner–Dyson statis-

tics even in the absence of strong classical chaos or positive Lyapunov exponents. Indeed, studies of weakly mixing or non-hyperbolic systems, such as the quantum triangle map [52], triangular cavities [53], and parabolic triangle maps [54], have shown that classical quasi-ergodicity alone is sufficient to produce GOE or GUE level statistics in the semiclassical limit, highlighting the role of classical mixing properties in shaping quantum spectral behavior.

In addition, for a rotated cavity defined on a lattice, one might worry that rough edges are responsible for the onset of chaos. When the cavity is rotated relative to the underlying lattice, the boundary becomes a mixture of zigzag and armchair segments. Although in monolayer graphene, such mixed edges modify the intervalley scattering processes and consequently affect the spectral correlations and eigenstate properties in certain cavity geometries [36, 55], our control tests show that boundary roughness alone does not generate quantum chaos in the bilayer hexagon cavity. We find that rotating a monolayer cavity, which reduces the symmetry only from D_6 to C_6 , does not lead to GOE or GUE statistics, and the spectrum remains semi-Poisson. Even in bilayer graphene, an unrotated cavity with artificially introduced boundary perturbations affecting roughly ten percent of each edge continues to exhibit semi-Poisson statistics characteristic of pseudointegrable dynamics. By contrast, in bilayer graphene a rotation as small as 0.1° , corresponding to shifting the lattice by only one atomic spacing and producing negligible geometric deformation, is already sufficient to drive the spectrum to Wigner–Dyson statistics. These comparisons indicate that boundary roughness alone is insufficient to generate quantum chaos in this system, and that the onset of Wigner–Dyson statistics in bilayer graphene is instead closely associated with the mismatch between the lattice orientation and the confining boundary.

The bilayer graphene cavity model in our study can be experimentally realized. Such cavities can be obtained either by patterning BLG into discrete shapes, creating hard-wall confinement [11, 55, 56], or by using electrostatic gates to define a soft confinement potential, allowing tunable control over the dot occupancy and tunnel barriers [57–59]. Low temperature transport measurements, such as Coulomb blockade and excited state spec-

troscopy, can probe energy level statistics and wavefunction distributions, enabling experimental access to signatures of quantum chaotic behavior [60]. Both approaches offer complementary ways to explore and control quantum chaos in BLG-based systems.

For future studies of bilayer graphene cavities, we can focus on controlling quantum behavior by tuning various parameters of the system. One promising direction is to introduce an external electric field, which can modify electron dynamics and selectively generate or tune specific wavefunction patterns. The interplay among the electric field, cavity geometry, and trigonal warping may give rise to a rich variety of dynamical regimes and quantum phenomena. In particular, it may enable the controlled manipulation of superscars in classically pseudointegrable cavities. Other factors, such as an applied magnetic field or an interlayer potential difference, can also be used to tune the system by altering its symmetry and internal electronic dynamics.

In conclusion, we have shown that misalignment between the bilayer graphene lattice and the hexagonal boundary fundamentally reshapes both the spectral and eigenstate properties of the cavity. Rotating the boundary removes the high symmetry subspaces in the unrotated cavity, produces Wigner–Dyson distribution of level spacing, and drives the eigenstates’ correlation lengths toward the wavelength scale, signaling the emergence of fully chaotic eigenstates. The semiclassical analysis confirms that this transition originates from the loss of commensurability between the warped Fermi surface and the polygonal boundary, which generates ergodicity in phase space and correspondingly causes quantum chaos. Overall, our results demonstrate that the boundary-lattice interplay provides an effective and experimentally accessible route for engineering and controlling quantum chaos in bilayer graphene cavities.

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