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Correction and improvement on several results in quantitative logic ☆

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ABSTRACT

The aim of this paper is to correct and improve some results obtained in the paper "Quantitative logic" [Information Sciences 179 (2009) 226–247].

1. Introduction

In [1], the authors introduced the concepts of truth degree of a formula, similarity degree and pseudo-metric between formulas, divergence degree and consistency degree of a theory, and hence provided a possible framework for graded approximate reasoning. However, several results in Theorem 8 and Theorem 9 in [1] are incorrect. So, in this note, we will correct them and give the detailed proof processes.

The above mentioned results are related to n-valued Łukasiewicz propositional logic system $Ł_n$, n-valued R_0-type propositional logic system $\mathcal{L}_n^*$ and fuzzy R_0-type propositional logic system $\mathcal{L}^*$. For the convenience of reading, we will use the same notations as in [1,2].

2. Corrections to results in systems $_n$ and $\mathcal{L}_n^*$

Definition 2.1 (*Wang and Zhou [1]*). Let $A = A(p_1, \ldots, p_m)$ be a formula in $F(S)$ containing m atomic formulas $p_1, \ldots, p_m$, and let $\overline{A}(x_1, \ldots, x_m)$ be the truth function induced by A. Define

$$\tau_n(A) = \frac{1}{n^m}\sum_{i=1}^{n-1}\frac{i}{n-1}\left|\overline{A}^{-1}\left(\frac{i}{n-1}\right)\right|,$$

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where $|E|$ denotes the number of elements of the set E. $\tau_n(A)$ is called the degree of the truth of A in n-valued system.

Definition 2.2 (*Wang and Zhou [1]*). Let $A, B \in F(S)$. Define

$$\xi_n(A,B) = \tau_n((A \to B) \wedge (B \to A)).$$

$\xi_n(A,B)$ is called the degree of similarity between A and B. In the sequel we often write ξ with subscripts to explicitly indicate the logic system involved.

Definition 2.3 (*Wang and Zhou [1]*). Let $A, B \in F(S)$. Define

$$\rho_n(A,B) = 1 - \xi_n(A,B).$$

$\rho_n(A,B)$ is called the pseudo-metric between A and B.

Proposition 2.1 (*Theorem 8(iv) in [1]*).

$\xi_n(A,B) = 0$ if and only if one of A and B is a tautology and the other one is a contradiction. ξ_n here is either $\xi_{Ł_n}$ or $\xi_{R_{0n}}$.

The following counterexample shows that Proposition 2.1 is incorrect.

Example 2.1. (1) In system $Ł_n$, take $A = p^n, B = \neg p^n$, where $p \in S$(the set of all atomic formulas), $p^2 = p\&p, p^{k+1} = p^k\&p, k = 2,3,\ldots$, and & is defined by $C\&D = \neg(C \to \neg D), C, D \in F(S)$. Since

$$\forall v \in \Omega,\ v(p) \in Ł_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\},$$

$$v(A) = \begin{cases} 1 & v(p) = 1 \\ 0 & v(p) \neq 1 \end{cases}, \quad v(B) = \begin{cases} 0 & v(p) = 1 \\ 1 & v(p) \neq 1 \end{cases},$$

we get, $\forall v \in \Omega,\ v((A \to B) \wedge (B \to A)) = 0$, thus $\tau_n((A \to B) \wedge (B \to A)) = 0$, i.e., $\xi_{Ł_n}(A,B) = 0$. But there is neither a tautology nor a contradiction in A, B.

(2) In system $\mathcal{L}_n^*$, take $A = (\neg p^2)^2, B = \neg A$.

Since

$$\forall v \in \Omega,\ v(p^2) = \begin{cases} 0 & v(p) \leqslant \frac{1}{2} \\ v(p) & v(p) > \frac{1}{2} \end{cases}, \quad v(\neg p^2) = \begin{cases} 1 & v(p) \leqslant \frac{1}{2} \\ 1 - v(p) & v(p) > \frac{1}{2} \end{cases},$$

we get,

$$v(A) = \begin{cases} 1 & v(p) \leqslant \frac{1}{2} \\ 0 & v(p) > \frac{1}{2} \end{cases}, \quad v(B) = \begin{cases} 0 & v(p) \leqslant \frac{1}{2} \\ 1 & v(p) > \frac{1}{2} \end{cases}.$$

Further we get $\forall v \in \Omega,\ v((A \to B) \wedge (B \to A)) = 0$, thus $\tau_n((A \to B) \wedge (B \to A)) = 0$, i.e., $\xi_{R_{0n}}(A,B) = 0$. But there is neither a tautology nor a contradiction in A, B.

Now we correct Proposition 2.1 as follows:

Theorem 2.1. *$\xi_n(A,B) = 0$ if and only if $A \approx \neg B$, and $\forall v \in \Omega,\ v(A) \in \{0,1\}, v(B) \in \{0,1\}$. ξ_n is either $\xi_{Ł_n}$ or $\xi_{R_{0n}}$.*

Proof.

(1) In system $Ł_n$, $\xi_{Ł_n}(A,B) = 0$,
iff $\tau_n((A \to B) \wedge (B \to A)) = 0$,
iff $\forall v \in \Omega,\ v((A \to B) \wedge (B \to A)) = (1 - v(A) + v(B) \wedge (1 - v(B) + v(A)) \wedge 1 = 0$,
iff $\forall v \in \Omega,\ v(A) - v(B) = 1$, or $v(B) - v(A) = 1$,
iff $\forall v \in \Omega,\ v(A) = 1, v(B) = 0$; or $v(A) = 0, v(B) = 1$,
iff $A \approx \neg B$, and $\forall v \in \Omega,\ v(A) \in \{0,1\}, v(B) \in \{0,1\}$.

(2) In system $\mathcal{L}_n^*$, $\xi_{R_{0n}}(A,B) = 0$, iff $\tau_n((A \to B) \wedge (B \to A)) = 0$,
iff $\forall v \in \Omega,\ v((A \to B) \wedge (B \to A)) = (v(A) \to v(B)) \wedge (v(B) \to v(A)) = 0$,
iff $\forall v \in \Omega,\ v(A) \to v(B) = 0$, or $v(B) \to v(A) = 0$,
iff $\forall v \in \Omega, 1 - v(A) \vee v(B) = 0$, or $1 - v(B) \vee v(A) = 0$,
iff $\forall v \in \Omega,\ v(A) = 1, v(B) = 0$; or $v(A) = 0, v(B) = 1$,
iff $A \approx \neg B$, and $\forall v \in \Omega,\ v(A) \in \{0,1\}, v(B) \in \{0,1\}$. $\square$

Since $\rho_n(A,B) = 1$ if and only if $\xi_n(A,B) = 0$, the following proposition is also incorrect.

Proposition 2.2 (*Theorem 10(i) in [1]*). *$\rho_n(A,B)=1$ if and only if one of A and B is a tautology and the other one is a contradiction.*

As a corollary of Theorem 2.1, Proposition 2.2 can be corrected as follows:

Corollary 2.2. *$\rho_n(A,B)=1$ if and only if $A\approx\neg B$*, and $\forall v\in\Omega$, $v(A)\in\{0,1\}$, $v(B)\in\{0,1\}$.

3. Corrections to results in system $\mathcal{L}^*$

Definition 3.1 (*Wang and Zhou [1]*). Let $A=A(p_1,\dots,p_m)$ be a formula built up from m atomic formulas $p_1,\dots,p_m$, and let $\overline{A}(x_1,\dots,x_m)$ be the truth function induced by the A. Define

$$\tau_\infty(A)=\int_{[0,1]^m}\overline{A}(x_1,\dots,x_m)dx_1\cdots dx_m.$$

$\tau_\infty(A)$ is called the integrated degree of truth of A.

Definition 3.2 (*Wang and Zhou [1]*). Let $A,B\in F(S)$. Define

$$\xi_{R_{0\infty}}(A,B)=\tau_\infty((A\to B)\wedge(B\to A)).$$

$\xi_{R_{0\infty}}(A,B)$ is called the degree of similarity between A and B.

Definition 3.3 (*Wang and Zhou [1]*). Let $A,B\in F(S)$. Define

$$\rho_{R_{0\infty}}(A,B)=1-\xi_{R_{0\infty}}(A,B).$$

$\rho_{R_{0\infty}}(A,B)$ is called the pseudo-metric between A and B.

Proposition 3.1 (*Theorem 9(ii) in [1]*). *$\xi_{R_{0\infty}}(A,B)=0$ if and only if one of $\overline{A}$ and $\overline{B}$ is equal to 1 almost everywhere on $[0,1]^m$ and the other one is equal to 0 almost everywhere on $[0,1]^m$, we assume here that A and B contain the same atomic formulas $p_1,p_2,\dots,p_m$.*

By taking A and B the same as those in Example 2.1(2), we have $\xi_{R_{0\infty}}(A,B)=0$, but "one of $\overline{A}$ and $\overline{B}$ is equal to 1 almost everywhere on [0,1] and the other one is equal to 0 almost everywhere on [0,1]" does not hold. Therefore, the proposition above is incorrect. Now we correct it as follows:

Theorem 3.1. *$\xi_{R_{0\infty}}(A,B)=0$ if and only if $\overline{A}(x)+\overline{B}(x)=1,\overline{A}(x)\in\{0,1\}$*, and *$\overline{B}(x)\in\{0,1\}$ hold almost everywhere on $[0,1]^m$*, where *A* and *B are assumed to contain the same atomic formulas $p_1,p_2,\dots,p_m$.*

Proof. $\xi_{R_{0\infty}}(A,B)=0$, iff $\tau_\infty((A\to B)\wedge(B\to A))=0$,

iff $\overline{(A\to B)\wedge(B\to A)}(x)=0$ holds almost everywhere on $[0,1]^m$,
iff $\overline{A\to B}(x)\wedge\overline{B\to A}(x)=0$ holds almost everywhere on $[0,1]^m$,
iff $\overline{A\to B}(x)=0$, or $\overline{B\to A}(x)=0$ hold almost everywhere on $[0,1]^m$,
iff $\overline{A}(x)\to\overline{B}(x)=0$, or $\overline{B}(x)\to\overline{A}(x)=0$ hold almost everywhere on $[0,1]^m$,
iff $1-\overline{A}(x)\vee\overline{B}(x)=0$, or $1-\overline{B}(x)\vee\overline{A}(x)=0$ hold almost everywhere on $[0,1]^m$,
iff $\overline{A}(x)=1,\overline{B}(x)=0$, or $\overline{A}(x)=0,\overline{B}(x)=1$ hold almost everywhere on $[0,1]^m$,
iff $\overline{A}(x)+\overline{B}(x)=1$, $\overline{A}(x)\in\{0,1\},\overline{B}(x)\in\{0,1\}$ hold almost everywhere on $[0,1]^m$. □

Since $\rho_{R_{0\infty}}(A,B)=1$ if and only if $\xi_{R_{0\infty}}(A,B)=0$, the following proposition is also incorrect.

Proposition 3.2 (*Theorem 10(iii) in [1]*). *$\rho_{R_{0\infty}}(A,B)=1$ if and only if one of $\overline{A}$ and $\overline{B}$ is equal to 1 almost everywhere on $[0,1]^m$ and the other one is equal to 0 almost everywhere on $[0,1]^m$, we assume here that A and B contain the same atomic formulas $p_1,p_2,\dots,p_m$.*

As a corollary of Theorem 3.1, Proposition 3.2 *can be corrected as follows:*

Corollary 2.1. *$\rho_{R_{0\infty}}(A,B)=1$ if and only if $\overline{A}(x)+\overline{B}(x)=1,\overline{A}(x)\in\{0,1\}$*, and *$\overline{B}(x)\in\{0,1\}$ hold almost everywhere on $[0,1]^m$, where A and B are assumed to contain the same atomic formulas $p_1,p_2,\dots,p_m$.*

4. Conclusion

We have shown by several examples that Theorem 8(iv), Theorem 9(ii), Theorem 10(i) and Theorem 10(iii) in [1] are incorrect and corrected them. We have also checked other results based on the propositions above and found them to be all correct. Of course, the original proofs of these results need modification. And it is not difficult to give them new proofs according to the corrected propositions.

References

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