J. Math. Anal. Appl. 397 (2013) 671–677

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On nonlinear functional spaces based on triangular conorms

Dong Qiu [a,*], Weiquan Zhang [b], Cheng Li [c]

[a] College of Mathematics and Physics, Chongqing University of Posts and Telecommunications, Nanan, Chongqing, 400065, PR China
[b] School of Information Engineering, Guangdong Medical College, Dongguan, Guangdong, 523808, PR China
[c] School of Mathematics, Shandong University, Shandong 250100, PR China

ARTICLE INFO

Article history:
Received 17 January 2012
Available online 17 August 2012
Submitted by B. Bongiorno

Keywords:
Triangular conorms
Nonlinear functional spaces
Upper-closure
Regular subspaces
Normal subspaces

ABSTRACT

In this paper, we discuss the nonlinear functional spaces based on triangular conorms. Particularly, we discuss the properties of the upper-closures of the regular subspaces of the nonlinear functional space based on a continuous triangular conorm. Furthermore, we prove that with respect to a strict triangular conorm, a subset of the nonlinear functional space is an upper-complete normal subspace if and only if the family of all sets whose characteristic functionals are contained in the given subset of the nonlinear functional space is a sigma-algebra.

1. Introduction

Originally functional analysis could be understood as a unifying abstract treatment of important aspects of linear mathematical models for problems in science, but the latter receded more and more into the background during the intensive theoretical investigations. Numerous questions in physics, chemistry, biology, and economics lead to nonlinear problems. Thus nonlinear functional analysis is an important branch of modern mathematics.

Triangular norms and conorms were first introduced in the context of probabilistic metric spaces [1–4], based on some ideas presented in [5]. They also play an important role in decision making [6–9], in statistics [10,11] as well as in the theories of non-additive measures [12] and cooperative games [13,14]. Some parameterized families of t-norms turn out to be solutions of well-known nonlinear functional equations [15]. In many problems with uncertainty as in the theory of probabilistic metric spaces [1,2,4], multi-valued logics [16,17], general measures [12,18] often we work with many operations different from the usual addition and multiplication of reals. Some of them are triangular norms, triangular conorms, pseudo-additions, pseudo-multiplications, etc. [19–21].

In this paper, we will discuss the nonlinear functional spaces based on triangular conorms. Particularly, we will discuss the properties of the upper-closures of the regular subspaces of the nonlinear functional space based on a continuous triangular conorm. Furthermore, we will prove that with respect to a strict t-conorm, a subset of the nonlinear functional space is an upper-complete normal subspace if and only if the family of all sets whose characteristic functionals are contained in the given subset of the nonlinear functional space is a σ-algebra.

* Corresponding author.
E-mail addresses: dongqiu@yahoo.cn, qiudong@cqupt.edu.cn (D. Qiu).
URL: http://slxy.cqupt.edu.cn/teacher.php?id=qiudong (D. Qiu).

doi:10.1016/j.jmaa.2012.08.034

2. Preliminaries

Throughout this paper, X will denote a nonempty set, $P(X)$ the corresponding power set and $\mathbb{R}$ the real numbers with their usual topology.

A pair $(X, \mathscr{S})$ consisting of a non-empty set X and a σ-algebra $\mathscr{S}$ of subsets of X is called a measurable space. The most basic measurable space in the whole of what follows is $([0, 1], \mathscr{B}_{[0,1]})$, where $[0, 1]$ is the unit interval equipped with a natural topology induced from that of $\mathbb{R}$ and $\mathscr{B}_{[0,1]}$ denotes the Borel σ-algebra of $[0, 1]$. A functional $f : X \to [0, 1]$ is said to be a measurable functional if $f^{-1}(\mathscr{B}_{[0,1]}) \subseteq \mathscr{S}$ [22].

The set of all functionals from X to $[0, 1]$ will be denoted by $\mathcal{F}(X)$. For each $a \in [0, 1]$ the constant functional in $\mathcal{F}(X)$ with value a will also be denoted by a. It will be clear from the context which usage is intended. A functional $f \in \mathcal{F}(X)$ is said to be elementary if the set of values $f(X)$ of f is a finite subset of $[0, 1]$ and the set of such elementary functionals will be denoted by $\mathcal{E}(X)$. For each $A \subseteq X$ define the characteristic functional $I_A \in \mathcal{E}(X)$ as

$$I_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

In general, for each binary operation $\star$ on $[0, 1]$, one can define a corresponding operation on $\mathcal{F}(X)$, also denoted by $\star$, pointwise by letting $(f \star g)(x) = f(x) \star g(x)$ for all $x \in X$. A subset $\mathcal{A}$ of $\mathcal{F}(X)$ is $\star$-closed if $f \star g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$. The total order on $[0, 1]$ induces a partial order $\le$ on $\mathcal{F}(X)$ defined pointwise by stipulating that $f \le g$ if and only if $f(x) \le g(x)$ for all $x \in X$. Thus $(\mathcal{F}(X), \le)$ is a poset, and whenever we consider $\mathcal{F}(X)$ as a poset then it will always be with respect to this partial order. Let $\mathcal{A}$ be a subset of $\mathcal{F}(X)$; the poset $\mathcal{A}$ is said to be upper-complete if $\lim_n f_n \in \mathcal{A}$ for each increasing sequence $\{f_n\}_{n\ge1}$ from $\mathcal{A}$; the poset $\mathcal{A}$ is said to be lower-complete if $\lim_n f_n \in \mathcal{A}$ for each decreasing sequence $\{f_n\}_{n\ge1}$ from $\mathcal{A}$, where the limit of the monotonic functional sequence $\{f_n\}_{n\ge1}$ is given by $(\lim_n f_n)(x) = \lim_n f_n(x)$ for all $x \in X$.

Definition 2.1 ([23]). A triangular norm (t-norm for short) is a binary operation $\top$ on the unit interval $[0, 1]$, i.e., a function $\top : [0, 1]^2 \to [0, 1]$, such that for all $a, b, c, d \in [0, 1]$ the following four axioms are satisfied:

(T-1) $a\top 1 = a$ (boundary condition).
(T-2) $a\top b \le c\top d$ whenever $a \le c$ and $b \le d$ (monotonicity).
(T-3) $a\top b = b\top a$ (commutativity).
(T-4) $a\top(b\top c) = (a\top b)\top c$ (associativity).

A t-norm $\top$ is said to be continuous if it is a continuous function in $[0, 1]^2$. The following are examples of t-norms: $a \wedge b = \min(a, b)$; $a\top_P b = a \cdot b$; $a\top_L b = \max(a + b - 1, 0)$.

Definition 2.2 ([23]). A triangular conorm (t-conorm for short) is a binary operation $\perp$ on the unit interval $[0, 1]$, i.e., a function $\perp : [0, 1]^2 \to [0, 1]$, such that for all $a, b, c, d \in [0, 1]$ the following four axioms are satisfied:

(S-1) $a\perp 0 = a$ (boundary condition).
(S-2) $a\perp b \le c\perp d$ whenever $a \le c$ and $b \le d$ (monotonicity).
(S-3) $a\perp b = b\perp a$ (commutativity).
(S-4) $a\perp(b\perp c) = (a\perp b)\perp c$ (associativity).

A t-conorm $\perp$ is said to be continuous if it is a continuous function in $[0, 1]^2$; a t-conorm $\perp$ is called strict if $\perp$ is continuous and strictly monotone [23]. The following are examples of t-conorms: $a \vee b = \max(a, b)$; $a \perp_P b = a + b - ab$; $a \perp_L b = \min(a+b, 1)$. Because of the associative property, the t-conorm $\perp$ can be extended by induction to n-ary operation by setting

$$\mathop{\perp}_{i=1}^{n} x_i = \left(\mathop{\perp}_{i=1}^{n-1} x_i\right) \perp x_n.$$

For any continuous t-conorm $\perp$ and $a, b \in [0, 1]$ with $b \ge a$, since $a\perp 0 = a$ and $a\perp 1 = 1$, there exists at least one point c such that $b = a\perp c$. If t-conorm $\perp$ is strict, then there exists only one point c such that $b = a\perp c$ for all $a, b \in [0, 1]$ with $a < 1$. Thus we have the following concepts.

Definition 2.3. For any continuous t-conorm $\perp$ and $a, b \in [0, 1]$ with $b \ge a$, the para-complement set $b -_\perp a$ is a nonempty set of all points c such that $b = a\perp c$.

Definition 2.4. For any continuous t-conorm $\perp$, if $f, g \in \mathcal{F}(X)$ with $g \le f$ then define the para-complement set $f -_\perp g$ as the set of all those functionals h such that $f(x) = g(x)\perp h(x)$ for all $x \in X$.

Definition 2.5 ([24]). For any strict t-conorm $\perp$ and $a, b \in [0, 1]$ with $b \ge a$, the complement $b -'_\perp a$ is defined as

$$b -'_\perp a = \begin{cases} c \in [0, 1], \quad \text{such that } b = a\perp c, & \text{if } a < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.6. For any strict t-conorm $\perp$, if $f, g \in \mathcal{F}(X)$ with $g \leq f$ then define the complement functional $f -'_{\perp} g$ pointwise as $(f -'_{\perp} g)(x) = f(x) -'_{\perp} g(x)$ for all $x \in X$.

Definition 2.7. For any $a \in [0, +\infty)$ and $f \in \mathcal{F}(X)$, the scalar multiplication $\otimes$ is defined as

$$(a \otimes f)(x) = \begin{cases} af(x), & \text{if } af(x) < 1, \\ 1, & \text{otherwise}, \end{cases}$$

i.e., $(a \otimes f)(x) = af(x) \wedge 1$.

Definition 2.8. For any t-conorm $\perp$, a non-empty subset $\mathcal{S}$ of $\mathcal{F}(X)$ is said to be a functional space with respect to $\perp$, denoted by $(\mathcal{S}, \perp)$, if $(a \otimes f)\perp(b \otimes g) \in \mathcal{S}$ for all $f, g \in \mathcal{S}$ and $a, b \in [0, +\infty)$.

It is clear that $(\mathcal{F}(X), \perp)$ is the greatest functional space with respect to any t-conorm $\perp$. Thus the functional space $(\mathcal{S}, \perp)$ with $\mathcal{S} \subset \mathcal{F}(X)$ is also called a subspace of $(\mathcal{F}(X), \perp)$. If $(\mathcal{S}, \perp)$ is a functional space with respect to $\perp$ then we just write $\mathcal{S}$ instead of $(\mathcal{S}, \perp)$ whenever $\perp$ can be determined from the context.

Definition 2.9. For each subset $\mathcal{A}$ of $\mathcal{F}(X)$ the upper-closure of $\mathcal{A}$, denoted by $\hat{\mathcal{A}}$, is the set of all elements of $\mathcal{F}(X)$ having the form $\lim_n f_n$ for some increasing sequence $\{f_n\}_{n\geq 1}$ from $\mathcal{A}$.

It follows from Definition 2.9 that $\mathcal{A} \subseteq \hat{\mathcal{A}}$ and $\mathcal{A} = \hat{\mathcal{A}}$ if and only if $\mathcal{A}$ is upper-complete.

Definition 2.10. For any continuous t-conorm $\perp$, a subspace $(\mathcal{S}, \perp)$ will be called para-complemented if for all $f, g \in \mathcal{S}$ with $g \leq f, f -_{\perp} g \subseteq \mathcal{S}$; for any strict t-conorm $\perp$, a subspace $(\mathcal{S}, \perp)$ will be called complemented if for all $f, g \in \mathcal{S}$ with $g \leq f, f -'_{\perp} g \in \mathcal{S}$.

Definition 2.11. For any continuous t-conorm $\perp$, a para-complemented subspace $(\mathcal{S}, \perp)$ is regular if it contains 1 and is closed under both $\vee$ and $\wedge$; for any strict t-conorm $\perp$, a complemented subspace $(\mathcal{S}, \perp)$ is normal if it contains 1 and is closed under both $\vee$ and $\wedge$.

3. Nonlinear functional spaces based on continuous triangular conorms

In this section we will discuss the properties of the upper-closures of regular subspaces of the nonlinear functional space $\mathcal{F}(X)$ based on a continuous t-conorm $\perp$.

Theorem 3.1. *Let $\mathcal{A}$ be a $\vee$-closed subset of $\mathcal{F}(X)$ and let $\{f_n\}_{n\geq 1}$ be an increasing sequence from $\hat{\mathcal{A}}$ with $f = \lim_n f_n$. Then there exists an increasing sequence $\{g_n\}_{n\geq 1}$ from $\mathcal{A}$ with $g_n \leq f_n$ for all $n \geq 1$ and $\lim_n g_n = f$. In particular, this implies $\hat{\mathcal{A}}$ is upper-complete.*

Proof. For each f_n there exists an increasing sequence $\left\{f_{(n,m)}\right\}_{m\geq 1}$ from $\mathcal{A}$ with $f_n = \lim_m f_{(n,m)}$. For each $m \geq 1$ let $g_m = f_{(1,m)} \vee \cdots \vee f_{(m,m)}$; then $g_m \in \mathcal{A}$, since $\mathcal{A}$ is $\vee$-closed, and

$$g_m = f_{(1,m)} \vee \cdots \vee f_{(m,m)} \leq f_{(1,m)} \vee \cdots \vee f_{(m,m)} \vee f_{(m,m+1)} \leq g_{m+1},$$

since $\vee$ is monotone. Therefore $\{g_m\}_{m\geq 1}$ is an increasing sequence from $\mathcal{A}$; put $g = \lim_m g_m$. Now $f_{(n,m)} \leq f_{(1,m)} \vee \cdots \vee f_{(m,m)} = g_m \leq g$ for all $m \geq n \geq 1$; thus $f_n = \lim_m f_{(n,m)} \leq g$ for all $n \geq 1$ and hence $f \leq g$. But on the other hand,

$$g_n = f_{(1,n)} \vee \cdots \vee f_{(n,n)} \leq f_1 \vee \cdots \vee f_n = f_n,$$

and so in particular $g = \lim_n g_n \leq \lim_n f_n = f$, i.e., $g = f$. $\square$

Theorem 3.2. *Let $\star$ be a continuous operation on $[0, 1]$ and $\mathcal{A}$ a $\star$-closed subset of $\mathcal{F}(X)$. Then $\hat{\mathcal{A}}$ is $\star$-closed. In particular, $\hat{\mathcal{A}}$ is $\vee$-closed.*

Proof. Let $f, g \in \hat{\mathcal{A}}$ and $\{f_n\}_{n\geq 1}$, $\{g_n\}_{n\geq 1}$ be increasing sequences from $\mathcal{A}$ with $f = \lim_n f_n$ and $g = \lim_n g_n$. Then $f_n \star g_n \in \mathcal{A}$, since $\mathcal{A}$ is $\star$-closed. In addition for each point $x \in X$,

$$\lim_n (f_n \star g_n)(x) = \lim_n (f_n(x) \star g_n(x)) = (\lim_n f_n(x)) \star (\lim_n g_n(x)) = f(x) \star g(x) = (f \star g)(x),$$

since $\star$ be a continuous operation on $[0, 1]$. Thus $f \star g \in \hat{A}$, since $\hat{\mathcal{A}}$ is complete. This shows that $\hat{\mathcal{A}}$ is $\star$-closed, and in particular $\hat{\mathcal{A}}$ is $\vee$-closed, since $\vee$ is continuous. $\square$

Theorem 3.3. *Let $(\mathcal{S}, \perp)$ be a $\vee$-closed subspace with respect to a continuous t-conorm $\perp$. Then $(\hat{\mathcal{S}}, \perp)$ is a subspace which is upper-complete and $\vee$-closed. Moreover, if $\mathcal{S}$ is closed under a continuous operation $\star$ on $\mathcal{F}(X)$ then so is $\hat{\mathcal{S}}$.*

Proof. This follows from Theorems 3.1 and 3.2. $(\hat{\mathcal{S}}, \perp)$ is a subspace since for all $f, g \in \hat{\mathcal{S}}$ the operation $(f, g) \mapsto (a \otimes f)\perp(b \otimes g)$ is continuous for all $a, b \in [0, +\infty)$. □

Theorem 3.4. *An upper-complete regular subspace $(\mathcal{S}, \perp)$ with respect to a continuous t-conorm $\perp$ is also lower-complete.*

Proof. Let $\{f_n\}_{n\geq 1}$ be a decreasing sequence from $\mathcal{S}$ with $f = \lim_n f_n$. For each $n \geq 1$ let $g_n = \sup_{h\in f_1 -_\perp f_n} h$; then $f_1 = f_n \perp g_n$ since $\perp$ is continuous and $f_1 = f_n \perp h$ for all $h \in f_1 -_\perp f_n$. Now we show that $\{g_n\}_{n\geq 1}$ is an increasing sequence from $\mathcal{S}$. If otherwise, then there are $x_0 \in X$ and m, n with $m > n \geq 1$ such that $g_m(x_0) < g_n(x_0)$. Since $\perp$ is monotonous, it follows that

$$f_1(x_0) = f_n(x_0)\perp g_n(x_0) \geq f_m(x_0)\perp g_n(x_0) \geq f_m(x_0)\perp g_m(x_0) = f_1(x_0)$$

which implies $f_m(x_0)\perp g_n(x_0) = f_1(x_0)$. This contradicts the definition of $g_m(x_0)$. Then $\{g_n\}_{n\geq 1}$ is an increasing sequence from $\mathcal{S}$ and thus $g = \lim_n g_n \in \mathcal{S}$. But $f_1 = f \perp g$ and so $f \in f_1 -_\perp g \subseteq \mathcal{S}$. This shows that $\mathcal{S}$ is lower-complete. □

Note that $(f \vee g)\perp(f \wedge g) = f \perp g$ for all $f, g \in \mathcal{F}(X)$ and thus a para-complemented subspace of $\mathcal{F}(X)$ is $\wedge$-closed if and only if it is $\vee$-closed. Note also that a subspace always contains the constant mapping 0 since $0 = (0 \otimes f)\perp(0 \otimes g)$. Thus if a subspace $(\mathcal{S}, \perp)$ contains the constant 1 then $a \in \mathcal{S}$ for all $a \in [0, 1]$ since $a = (a \otimes 1)\perp(0 \otimes 1)$. Let $(\mathcal{S}, \perp)$ be a regular subspace. Then by Theorem 3.3 $\hat{\mathcal{S}}$ is closed under both $\vee$ and $\wedge$ and of course $1 \in \hat{\mathcal{S}}$. Therefore $(\hat{\mathcal{S}}, \perp)$ is regular if and only if $\hat{\mathcal{S}}$ is para-complemented. However, in general this will fail to be the case, and the best partial results are perhaps the following:

Theorem 3.5. *Let $(\mathcal{S}, \perp)$ be a $\vee$-closed para-complemented subspace with respect to a continuous t-conorm $\perp$ and let $g \in \mathcal{S}$ and $f \in \hat{\mathcal{S}}$ with $g \leq f$. Then $\hat{\mathcal{S}} \cap (f -_\perp g) \neq \emptyset$.*

Proof. Let $\{f_n\}_{n\geq 1}$ be an increasing sequence from $\mathcal{S}$ with $f = \lim_n f_n$; then $\{f_n \vee g\}_{n\geq 1}$ is an increasing sequence from $\mathcal{S}$, also with $\lim_n (f_n \vee g) = f \vee g = f$. Now $g \leq f_n \vee g$ and so there exists $k_n \in \mathcal{S}$ with $k_n = \sup_{k\in((f_n\vee g)-_\perp g)} k$ and $f_n \vee g = k_n \perp g$. Let $h_n = k_1 \vee \cdots \vee k_n$, and $\{h_n\}_{n\geq 1}$ is an increasing sequence from $\mathcal{S}$ such that

$$\begin{aligned} f_n \vee g &= (f_1 \vee g) \vee \cdots \vee (f_n \vee g) \\ &= (g\perp k_1) \vee \cdots \vee (g\perp k_n) \\ &= g\perp(k_1 \vee \cdots \vee k_n) \\ &= g\perp h_n. \end{aligned}$$

Let $h = \lim_n h_n$; then $h \in \hat{\mathcal{S}}$ and $h \in (f -_\perp g)$. □

Theorem 3.6. *Let $(\mathcal{S}, \perp)$ be a regular subspace with respect to a continuous t-conorm $\perp$. If $(\hat{\mathcal{S}}, \perp)$ is lower-complete, then $\hat{\mathcal{S}} \cap (f -_\perp g) \neq \emptyset$ for all $g, f \in \hat{\mathcal{S}}$ with $g \leq f$.*

Proof. Suppose $\hat{\mathcal{S}}$ is lower-complete. Let $f, g \in \hat{\mathcal{S}}$ with $g \leq f$ and let $\{g_n\}_{n\geq 1}$ be an increasing sequence from $\mathcal{S}$ with $\lim_n g_n = g$. By Theorem 3.5 there exists $h_n = \inf_{w\in(f-_\perp g_n)} w \in \hat{\mathcal{S}}$ such that $f = g_n \perp h_n$. Now we show that $\{h_n\}_{n\geq 1}$ is a decreasing sequence from $\hat{\mathcal{S}}$. If on the contrary, then there are $x_0 \in X$ and m, n with $m > n \geq 1$ such that $h_m(x_0) > h_n(x_0)$. Since $\perp$ is monotonous, it follows that

$$f(x_0) = g_n(x_0)\perp h_n(x_0) \leq g_m(x_0)\perp h_n(x_0) \leq g_m(x_0)\perp h_m(x_0) = f(x_0)$$

which implies $g_m(x_0)\perp h_n(x_0) = f(x_0)$. This contradicts the definition of $h_m(x_0)$. Then $f = g_n \perp h_n$ and $\{h_n\}_{n\geq 1}$ is a decreasing sequence from $\hat{\mathcal{S}}$. Hence, since $\hat{\mathcal{S}}$ is lower-complete, $h = \lim_n h_n \in \hat{\mathcal{S}}$, and $f = g \perp h$. □

Since the above theorems also hold for the upper-closures and the normal subspaces with respect to a strict t-conorm, particularly we can get the following corollary.

Corollary 3.7. *Let $(\mathcal{S}, \perp)$ be a normal subspace with respect to a strict t-conorm $\perp$. Then $(\hat{\mathcal{S}}, \perp)$ is normal if and only if it is lower-complete.*

Proof. We can obtain a similar result with Theorem 3.4 that an upper-complete normal subspace is also lower-complete. In the meantime, by some similar discussions with the notes before Theorem 3.5, we can get that $(\hat{\mathcal{S}}, \perp)$ is normal if and only if it is complemented. It thus remains to show that $(\hat{\mathcal{S}}, \perp)$ is complemented when it is lower-complete. By Theorem 3.6, $\hat{\mathcal{S}} \cap (f -_\perp g) \neq \emptyset$ for all $g, f \in \hat{\mathcal{S}}$ with $g \leq f$ which exactly implies $f -'_\perp g \in \hat{\mathcal{S}}$ for all $g, f \in \hat{\mathcal{S}}$ with $g \leq f$. □

Theorem 3.8. (1) *$(\mathcal{E}(X), \perp)$ is a subspace of $(\mathcal{F}(X), \perp)$ with respect to any continuous t-conorm $\perp$. Moreover if $\perp$ is a strict t-conorm, then $(\mathcal{E}(X), \perp)$ is a normal subspace.*

(2) *If $(\mathcal{S}, \perp)$ is a subspace with $I_A \in \mathcal{S}$ for all $A \subseteq X$ then $(\mathcal{E}(X), \perp) \subseteq (\mathcal{S}, \perp)$, which means $(\mathcal{E}(X), \perp)$ is the smallest subspace containing the mappings $I_A, A \subseteq X$.*

(3) $(\mathcal{F}(X), \perp) = (\widehat{\mathcal{E}(X)}, \perp)$.

Proof. (1) For any $a, b \in [0, +\infty)$ and $f, g \in \mathcal{E}(X)$,

$$(a \otimes f)\perp(b \otimes g)(X) \subseteq A = \{(a \otimes c)\perp(b \otimes d) : c \in f(X), d \in g(X)\}$$

and A is a finite subset of $[0, 1]$ whenever both $f(X)$ and $g(X)$ are. Thus $\mathcal{E}(X)$ is a subspace. Moreover, if $\perp$ is strict, then for any $f, g \in \mathcal{E}(X)$ with $f \geq g$, $(f -'_{\perp} g)(X)$ is a subset of the finite set $\{a -'_{\perp} b : a \in f(X), b \in g(X), a \geq b\}$. It is easy to see $(f \vee g)(X)$ and $(f \wedge g)(X)$ are finite sets for all $f, g \in \mathcal{E}(X)$. Finally, it is clear that $1 \in \mathcal{E}(X)$.

(2) If $f \in \mathcal{E}(X)$ then $\perp_{a \in f(X)}(a \otimes I_{f^{-1}(a)})$, where $f^{-1}(a) = \{x \in X : f(x) = a\}$. Thus if $I_A \in \mathcal{S}$ for each $A \subseteq X$ then $\mathcal{E}(X) \subseteq \mathcal{S}$.

(3) Let $f \in \mathcal{F}(X)$ and for each $n \geq 1$ define $f_n \in \mathcal{E}(X)$ by

$$f_n = \mathop{\perp}_{m=1}^{2^n}((m-1)2^{-n} \otimes I_{A_{m,n}}),$$

where $A_{m,n} = \{x \in X : (m-1)2^{-n} < f(x) \leq m2^{-n}\}$. Then $f_n \geq f_{n+1} \geq f$ for each $n \geq 1$ and $f(x) \leq 2^{-n} + f_n(x)$ for all $x \in X$ with $f(x) \leq 1$. Thus $\{f_n\}_{n\geq 1}$ is an increasing sequence from $\mathcal{E}(X)$ with $\lim_n f_n = f$. □

4. Nonlinear functional spaces based on strict *t*-conorms

Those theorems in the previous section also hold for the upper-closures and the normal subspaces with respect to a strict t-conorm. Furthermore, we can get some deeper results for the functional space based on a strict t-conorm in this section.

Let $(X, \mathscr{S})$ be a measurable space and $M(\mathscr{S})$ the set of all measurable mappings from $(X, \mathscr{S})$ to $([0, 1], \mathscr{B}_{[0,1]})$, i.e.,

$$\mathcal{M}(\mathscr{S}) = \left\{f \in \mathcal{F}(X) : f^{-1}(\mathscr{B}_{[0,1]}) \subseteq \mathscr{S}\right\}.$$

If $\mathscr{A}$ is an algebra of subsets of X then $\mathcal{E}(\mathscr{A})$ will denote the set of those elements $f \in \mathcal{E}(X)$ for which $f^{-1}(a) = \{x \in X : f(x) = a\} \in \mathscr{A}$ for each $a \in f(X)$. In particular this means that $\mathcal{E}(\mathscr{S}) = \mathcal{M}(\mathscr{S}) \cap \mathcal{E}(X)$.

Theorem 4.1. *Let $\mathscr{A}$ be an algebra of subsets of X and $\perp$ a strict t-conorm. Then $(\mathcal{E}(\mathscr{A}), \perp)$ is a normal subspace of $(\mathcal{F}(X), \perp)$.*

Proof. For any $a, b \in [0, 1]$ and $f, g \in \mathcal{E}(X)$,

$$((a \otimes f)\perp(b \otimes g))^{-1}(c) = \bigcup_{(a\otimes d)\perp(b\otimes e)=c} f^{-1}(d) \cap g^{-1}(e),$$

where

$$c \in C = \{(a \otimes d)\perp(b \otimes e) : d \in f(X), e \in g(X)\}$$

and C is a finite subset of $[0, 1]$ whenever both $f(X)$ and $g(X)$ are. Thus

$$((a \otimes f)\perp(b \otimes g))^{-1}(c) \in \mathscr{A}$$

for each $c \in C$ and therefore $(a \otimes f)\perp(b \otimes g) \in \mathcal{E}(\mathscr{A})$. Thus $\mathcal{E}(\mathscr{A})$ is a subspace. Similarly, we can get that $(f \vee g) \in \mathcal{E}(\mathscr{A})$ and $(f \wedge g) \in \mathcal{E}(\mathscr{A})$ for all $f, g \in \mathcal{E}(\mathscr{A})$.

Moreover, for any $f, g \in \mathcal{E}(\mathscr{A})$ with $f \geq g$, $(f -'_{\perp} g)(X)$ is a subset of the finite set $\{a -'_{\perp} b : a \in f(X), b \in g(X), a \geq b\}$. Thus

$$(f -'_{\perp} g)^{-1}(c) = \bigcup_{d -'_{\perp} e = c} f^{-1}(d) \cap g^{-1}(e),$$

where $c \in (f -'_{\perp} g)(X)$. Thus $(f -'_{\perp} g)^{-1}(c) \in \mathscr{A}$ for each $c \in (f -'_{\perp} g)(X)$ and therefore $f -'_{\perp} g \in \mathcal{E}(\mathscr{A})$. Finally, it is clear that $1 \in \mathcal{E}(\mathscr{A})$. □

Theorem 4.2. *Let $\mathscr{A}$ be an algebra of subsets of X and $\perp$ a strict t-conorm. Then the functional I_A is in $(\mathcal{E}(\mathscr{A}), \perp)$ for all $A \in \mathscr{A}$, and any subspace $(\mathcal{S}, \perp)$ with $I_A \in S$ for each $A \in \mathscr{A}$ contains $(\mathcal{E}(\mathscr{A}), \perp)$. Therefore $(\mathcal{E}(\mathscr{A}), \perp)$ is the smallest subspace containing the functionals I_A, $A \in \mathscr{A}$.*

Proof. It follows from Theorem 3.8(2) and the proof of Theorem 4.1. □

Theorem 4.3. *Let $\mathscr{S}$ be an σ-algebra of subsets of X and $\perp$ a strict t-conorm. Then $(\mathcal{M}(\mathscr{S}), \perp) = (\widehat{\mathcal{E}(\mathscr{S}), \perp})$.*

Proof. Let $f \in \mathcal{M}(\mathscr{S})$ and for each $n \geq 1$ define $f_n \in \mathcal{E}(\mathscr{S})$ by

$$f_n = \mathop{\perp}_{m=1}^{2^n}((m-1)2^{-n} \otimes I_{A_{m,n}}),$$

where $A_{m,n} = \{x \in X : (m-1)2^{-n} < f(x) \leq m2^{-n}\}$. Then $\{f_n\}_{n\geq 1}$ is an increasing sequence from $\mathcal{E}(X)$ with $\lim_n f_n = f$. But $A_{m,n} \in \mathscr{S}$ for all $n \geq 1$, $1 \leq m \leq 2^n$, and therefore $f_n \in \mathcal{E}(\mathscr{S})$. □

Theorem 4.4. *Let $\mathscr{S}$ be an σ-algebra of subsets of X and $\perp$ a strict t-conorm. Then $(\mathcal{M}(\mathscr{S}), \perp)$ are both upper-complete and lower-complete.*

Proof. By Theorem 3.1, $\mathcal{M}(\mathscr{S})$ is upper-complete since $\mathcal{M}(\mathscr{S}) = \widehat{\mathcal{E}(\mathscr{S})}$. Let $\{f_n\}_{n\geq 1}$ be a decreasing sequence from $\mathcal{M}(\mathscr{S})$ with $\lim_n f_n = f$. Then for all $a \in [0, 1]$

$$\{x \in X : f(x) < a\} = \bigcup_{n\geq 1} \{x \in X : f_n(x) < a\}$$

is an element of $\mathscr{S}$ and therefore again $f \in \mathcal{M}(\mathscr{S})$. $\square$

Theorem 4.5. *Let $\mathscr{S}$ be a σ-algebra of subsets of X and $\perp$ a strict t-conorm. Then $(\mathcal{M}(\mathscr{S}), \perp)$ is an upper-complete normal subspace of $(\mathcal{F}(X), \perp)$.*

Proof. By Theorem 4.1 $\mathcal{E}(\mathscr{S})$ is a normal subspace of $\mathcal{F}(X)$ and by Theorem 4.3 $\mathcal{M}(\mathscr{S}) = \widehat{\mathcal{E}(\mathscr{S})}$. Moreover, by Theorem 4.4 $\mathcal{M}(\mathscr{S})$ is both upper-complete and lower-complete. Therefore by Corollary 3.7 $\mathcal{M}(\mathscr{S})$ is an upper-complete normal subspace of $\mathcal{F}(X)$. $\square$

Theorem 4.6. *Let $\mathscr{S}$ be an σ-algebra of subsets of X and $\perp$ a strict t-conorm, and let $(\mathcal{S}, \perp)$ be an upper-complete subspace of $(\mathcal{M}(\mathscr{S}), \perp)$ with $I_A \in \mathcal{S}$ for each $A \in \mathscr{S}$. Then $(\mathcal{S}, \perp) = (\mathcal{M}(\mathscr{S}), \perp)$.*

Proof. Since $I_A \in \mathcal{S}$ for each $A \in \mathscr{S}$, Theorem 4.2 implies that $\mathcal{E}(\mathscr{S}) \subseteq \mathcal{S}$. Therefore by Theorem 4.3 $\mathcal{S} = \mathcal{M}(\mathscr{S})$, since S is upper-complete. $\square$

Theorem 4.7. *Let $\perp$ be a strict t-conorm and let $(\mathcal{S}, \perp)$ be an upper-complete normal subspace of $(\mathcal{F}(X), \perp)$. Then*

$$\mathscr{S} = \{A \subseteq X : I_A \in \mathcal{S}\}$$

is a σ-algebra and $(\mathcal{S}, \perp) = (\mathcal{M}(\mathscr{S}), \perp)$.

Proof. If $A \in \mathscr{S}$ then $I_A \in S$ and I_{X-A} is the element of $\mathcal{F}(X)$ with $1 = I_A \perp I_{X-A}$. Thus $I_{X-A} \in \mathcal{S}$, since $\mathcal{S}$ is complemented, i.e., $X - A \in \mathcal{S}$ for all $A \in \mathcal{S}$. Let $A, B \in \mathscr{S}$; then $I_{A\cap B} = I_A \wedge I_B \in \mathcal{S}$, since $\mathcal{S}$ is $\wedge$-closed, and hence$A \cap B \in \mathscr{S}$. This shows $\mathscr{S}$ is an algebra, since $I_X = 1 \in \mathcal{S}$.

Now let $\{A_n\}_{n\geq 1}$ be an increasing sequence from $\mathscr{S}$ and put $A = \bigcup_{n\geq 1} A_n$. Then $\{I_{A_n}\}_{n\geq 1}$ is an increasing sequence from $\mathcal{S}$ with $\lim_n I_{A_n} = I_A$ and thus $I_A \in \mathcal{S}$. This shows $A \in \mathscr{S}$ and therefore $\mathscr{S}$ is a σ-algebra.

Let $f \in \mathcal{S}$ and $a \in [0, 1]$ with $a < 1$. Define the functional g as

$$g(x) = (f -'_{\perp} (f \wedge a))(x) = \begin{cases} f(x) -'_{\perp} a, & \text{if } x \in f^{-1}((a, 1]), \\ 0, & \text{otherwise.} \end{cases}$$

Then $g \in \mathcal{S}$, since $\mathcal{S}$ is complemented and $\wedge$-closed and $a \in \mathcal{S}$. For each $n \geq 1$ let $f_n = n \otimes g$; then $\{f_n\}_{n\geq 1}$ is an increasing sequence from $\mathcal{S}$ with $\lim_n f_n = I_{f^{-1}((a,1])}$. Thus $I_{f^{-1}((a,1])} \in \mathcal{S}$, since $\mathcal{S}$ is upper-complete, i.e., $f^{-1}((a, 1]) \in \mathscr{S}$ which implies that $f \in \mathcal{M}(\mathscr{S})$. This shows $\mathcal{S} \subseteq \mathcal{M}(\mathscr{S})$. But by definition $I_A \in \mathcal{S}$ for all $A \in \mathscr{S}$ and therefore by Theorem 4.6 $\mathcal{S} = \mathcal{M}(\mathscr{S})$. $\square$

By Theorem 4.5 $\mathcal{M}(\mathscr{S})$ is an upper-complete normal subspace of $\mathcal{F}(X)$ and it is clear that $\mathcal{S} = \{A \subseteq X : I_A \in \mathcal{M}(\mathscr{S})\}$. Thus the following result holds.

Theorem 4.8. *Let $\perp$ a strict t-conorm. Then $(\mathcal{S}, \perp)$ is a upper-complete normal subspace if and only if*

$$\mathscr{S} = \{A \subseteq X : I_A \in \mathcal{S}\}$$

is a σ-algebra and $(\mathcal{S}, \perp) = (\mathcal{M}(\mathscr{S}), \perp)$.

5. Conclusions

In this work we have discussed the properties of the upper-closures of the regular subspaces of the nonlinear functional space $(\mathcal{F}(X), \perp)$ based on a continuous t-conorm. Furthermore, we have obtained that with respect to a strict t-conorm, a subset $\mathcal{S}$ of $(\mathcal{F}(X), \perp)$ is an upper-complete normal subspace if and only if the family of all sets whose characteristic functionals are contained in $\mathcal{S}$ is a σ-algebra. In further work, we will try to lay bare more properties of those nonlinear functional spaces based on t-conorms. In addition, because the concepts of t-conorm-decomposable measures [24–26], t-conorm-decomposable integrals [27,28] are very useful in the theory of nonlinear differential and integral equations [19–21], the relationships between nonlinear functional spaces based on t-conorms and those concepts will also be explored in our future research.

Acknowledgments

The authors thank the anonymous reviewers for their valuable comments. This work was supported by The Mathematical Tianyuan Foundation of China (Grant no. 11126087) and The National Natural Science Foundation of China (Grant no. 11201512).

References

[1] M. Bakry, H.M. Abu-Donia, Fixed-point theorems for a probabilistic 2-metric space, J. King Saud Univ.-Sci. 22 (2010) 217–221.
[2] L. Ćirić, Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces, Nonlinear Anal. TMA 72 (2010) 2009–2018.
[3] O. Hadžić, E. Pap, Probabilistic multi-valued contractions and decomposable measures, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 10 (Suppl.) (2002) 59–74.
[4] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North-Holland, New York, 1983.
[5] K. Menger, Statistical metrics, Proc. Natl. Acad. Sci. USA 8 (1942) 535–537.
[6] D. Dubois, H. Prade, Aggregation of decomposable measures with application to utility theory, Theory and Decision 41 (1996) 59–95.
[7] J.C. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer, Dordrecht, 1994.
[8] M. Grabisch, H.T. Nguyen, E.A. Walker, Fundamentals of Uncertainty Calculi with Applications to Fuzzy Inference, Kluwer, Dordrecht, 1995.
[9] A. Pankowska, M. Wygralak, General IF-sets with triangular norms and their applications to group decisionmaking, Inform. Sci. 176 (2006) 2713–2754.
[10] D. Dubois, H. Prade, R. Sabbadin, Decision-theoretic foundations of possibility theory, European J. Oper. Res. 128 (2001) 459–478.
[11] R.B. Nelsen, An Introduction to Copulas, in: Lecture Notes in Statistics, vol. 139, Springer, New York, 1999.
[12] Z. Wang, G.J. Klir, Generalized Measure Theory, Springer, New York, 2009.
[13] D. Butnariu, E.P. Klement, Triangular Norm-Based Measures and Games with Fuzzy Coalitions, Kluwer, Dordrecht, 1993.
[14] R. McCain, Cooperative games and cooperative organizations, J. Socio-Econ. 37 (2010) 2155–2167.
[15] E.P. Klement, R. Mesiar, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms, Elsevier, Amsterdam, 2005.
[16] P. Cintula, P. Hájek, Triangular norm based predicate fuzzy logics, Fuzzy Sets and Systems 161 (2010) 311–346.
[17] K. Lin, M. Wu, K. Hung, Y. Kuo, Developing a T_ω (the weakest t-norm) fuzzy GERT for evaluating uncertain process reliability in semiconductor manufacturing, Appl. Soft Comput. 11 (2011) 5165–5180.
[18] B. Mihailović, E. Pap, Asymmetric general Choquet integrals, Acta Polytech. Hungarica 6 (2009) 161–173.
[19] E. Pap, Decomposable measures and nonlinear equations, Fuzzy Sets and Systems 92 (1997) 205–221.
[20] E. Pap, Applications of decomposable measures on nonlinear differential equations, Novi Sad J. Math. 31 (2001) 89–98.
[21] E. Pap, Handbook of Measure Theory, Elsevier, Amsterdam, 2002.
[22] D.L. Cohn, Measure Theory, Birkhäuser, Boston, 1980.
[23] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer, Dordrecht, 2000.
[24] S. Weber, $\perp$-decomposable measures and integrals for Archimedean t-conorm, J. Math. Anal. Appl. 101 (1984) 114–138.
[25] D. Dubois, M. Prade, A class of fuzzy measures based on triangular norms, Int. J. Gen. Syst. 8 (1982) 43–61.
[26] S. Saminger, R. Mesiar, A general approach to decomposable bi-capacities, Kybernetika 39 (2003) 631–642.
[27] E.P. Klement, R. Mesiar, E. Pap, Integration with respect to decomposable measures, based on a conditionally distributive semiring on the unit interval, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 8 (2000) 701–717.
[28] D. Zhang, C. Guo, Integrals of set-valued functions for $\perp$-decomposable measures, Fuzzy Sets and Systems 78 (1996) 341–346.