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Extension of a class of decomposable measures using fuzzy pseudometrics

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Abstract

In this paper, we consider a topological approach to extension of t-conorm-based decomposable measures by introducing a fuzzy pseudometric structure on an algebra of sets. We prove that every non-strict continuous Archimedean t-conorm-based decomposable measure can be extended from an algebra to the completion of this algebra under the fuzzy pseudometric and then to the sigma-algebra generated by this algebra. The existence of such an extension follows very simply from the well-known Carathéodory result. However, our topological proof offers an intuitive interpretation of the extension of decomposable measures.

Keywords: Non-additive measures; Decomposable measures; Fuzzy metric; Space; t-Norm; t-Conorm

1. Introduction

Classical measure theory is one of the most important theories in mathematics and it has been widely extended, generalized and examined in depth. For an exhaustive state-of-the-art overview, we recommend the *Handbook of Measure Theory* [30]. A fuzzy measure [34] is an extension of a measure in the sense that the additivity of the measure is replaced by a weaker condition, monotonicity. Non-additivity is the main characteristic of a fuzzy measure. Therefore, a fuzzy measure is also called a nonadditive measure or a monotone measure [30,41]. There are many types of fuzzy measure [30,41], including the Choquet capacity, the decomposable measure, the λ-additive measure, the belief measure, and the plausibility measure. Among these, we mainly discuss the decomposable measure. The decomposable measure was independently introduced by Dubois and Prade [4] and Weber [42]. Further developments of decomposable measures and related integrals have been extensive [19,22,24,26]. Decomposable measures include several well-known fuzzy measures such as the λ-additive measure and probability and possibility measures, and they are a natural setting for relaxing probabilistic assumptions regarding the modeling of uncertainty [5,6]. Decomposable measures and the corresponding integrals are very useful in decision theory [1,5,17,32] and the theory of nonlinear differential and integral equations [15,27,29]. In classical measure theory, we are not interested in semi-rings, rings

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and algebras themselves, but rather in σ-algebras generated by them. The idea is that it is possible to extend a finitely additive measure on a ring (an algebra) to a countably additive measure on a σ-ring (a σ-algebra) via the Carathéodory extension theorem [2,7]. Similarly, extension of fuzzy measures is an important part of the theory of fuzzy measures. However, Wang and Klir showed that it is impossible to establish a unified extension theorem for all types of fuzzy measure corresponding to the extension theorem in classical measure theory [41]. Hence, extension theorems are only possible for some special classes of fuzzy measure. The issue of extensions of possibility and necessity measures was first addressed by Wang [37–39]. Qiao [31] showed that extensions of possibility and necessity measures can be generalized to monotone sets. Work on extensions of quasi-measures was initiated by Wang [36], who also studied extensions of semi-continuous monotone measures and some other types of monotone measure [40]. A theorem on extensions of null-additive set functions from a ring to the algebra generated by the ring was proved by Pap [28]. Murofushi [23] and Wu and Sun [43] presented further discussions along this line. Pap described some necessary and sufficient conditions for extension of decomposable measures to monotone order continuous $\perp$-subdecomposable set functions [25].

It is natural to ask whether we can extend a σ–$\perp$-decomposable measure to a unique σ–$\perp$-decomposable measure from an algebra R to the σ-algebra $S(R)$ generated by R. If μ is an (NSA)-type decomposable measure, that is, a σ-decomposable measure on R with respect to a non-strict continuous Archimedean t-conorm $\perp$ such that the composition $g \circ \mu$ with an additive generator g of $\perp$ is a finite additive measure [42], then μ can be uniquely extended to $S(R)$. In fact, by the Carathéodory extension theorem [2], $g \circ \mu$ can be uniquely extended to a finite additive measure v on $S(R)$. By the definition of the additive generator g, the composition $g^{-1} \circ v$, where g^{-1} is the inverse function of g, is a σ–$\perp$-decomposable measure on $S(R)$. We can obtain that $g^{-1} \circ v$ is the unique extension of μ on $S(R)$ because $g^{-1} \circ v$ and μ coincide on R and the additive generators g of $\perp$ (the inverse functions g^{-1}) are unique except for multiplication by positive numbers k $(1/k)$.

Although the Carathéodory extension method is very important in measure theory, as pointed out by Halmos, an intuitive understanding of this construction is rather difficult [16, p. 44]. Moreover, much research has focused on investigating the interplay between measure and topology after Carathéodory [8,9,18,35]. In this paper, we present a topological approach to the extension of decomposable measures of (NSA)-type. We show that the σ-algebra arises naturally as the set of all limit points of Cauchy sequences in the original algebra. This approach offers an intuitive understanding of the extension of σ–$\perp$-decomposable measures. It may also be regarded as an attempt to use the theory of fuzzy metric space in fuzzy measure theory. To do so, we consider a fuzzy pseudometric M on the class of subsets of the universe of discourse X. This fuzzy pseudometric M is obtained naturally from the outer set function μ^*, which is induced by the original σ–$\perp$-decomposable measure μ on R. We prove that the fuzzy pseudometric space (R, M) is completable and $(S(R), M)$ is included in the completion of (R, M). Furthermore, we show that completion of (R, M) is a σ-algebra and the σ–$\perp$-decomposable measure μ can be extended from R to the completion of (R, M), which implies the desired result.

2. Preliminaries

Definition 2.1 (*Klement et al. [20]*). A triangular norm (t-norm for short) is a binary operation $\top$ on the unit interval [0, 1], that is, a function $\top : [0,1]^2 \to [0,1]$ such that for all $a, b, c, d \in [0,1]$ the following four axioms are satisfied:

(T-1) $a \top 1 = a$. (boundary condition).
(T-2) $a \top b \le c \top d$ when $a \le c$ and $b \le d$. (monotonicity).
(T-3) $a \top b = b \top a$. (commutativity).
(T-4) $a \top (b \top c) = (a \top b) \top c$. (associativity).

A t-norm $\top$ is said to be continuous if it is a continuous function in $[0,1]^2$; a t-norm $\top$ is called Archimedean if $a \top a < a$ for all $a \in (0,1)$; an Archimedean t-norm $\top$ is called strict if it is strictly increasing in $(0,1)^2$.

Definition 2.2 (*Klement et al. [20]*). A triangular conorm (t-conorm) is a binary operation $\perp$ on the unit interval [0, 1], i.e., a function $\perp : [0,1]^2 \to [0,1]$, such that for all $a, b, c, d \in [0,1]$ satisfies (T-2)–(T-4) and

(S-1) $a \perp 0 = a$. (boundary condition).

A t-conorm $\perp$ is said to be continuous if it is a continuous function in $[0, 1]^2$; a t-conorm $\perp$ is called Archimedean if $a\perp a > a$ for all $a \in (0, 1)$; an Archimedean t-conorm $\perp$ is called strict if it is strictly increasing in $(0, 1)^2$.

Definition 2.3 (*Weber [42]*). For any t-conorm $\perp$, the t-norm $\perp^*$ defined by

$$a\perp^* b = 1 - (1 - a)\perp(1 - b)$$

is called the dual t-norm of $\perp$.

Theorem 2.1 (*Weber [42]*). (*a*) *A function* $\perp : [0, 1]^2 \to [0, 1]$ *is an Archimedean t-conorm iff there exists a continuous, strictly increasing function* $g : [0, 1] \to [0, \infty]$ *with* $g(0) = 0$ *such that*

$$a\perp b = g^{(-1)}(g(a) + g(b)),$$

where $g^{(-1)}$ *is the pseudoinverse of g, defined by*

$$g^{(-1)}(y) = \begin{cases} g^{-1}(y) & if\ y \in [0, g(1)], \\ 1 & if\ y \in [g(1), \infty]. \end{cases}$$

Moreover, $\perp$ *is strict iff* $g(1) = \infty$.

(*b*) *A function* $\top : [0, 1]^2 \to [0, 1]$ *is an Archimedean t-norm iff there exists a continuous, strictly decreasing function* $f : [0, 1] \to [0, \infty]$ *with* $f(1) = 0$ *such that*

$$a\top b = f^{(-1)}(f(a) + f(b)),$$

where $f^{(-1)}$ *is the pseudoinverse of f, defined by*

$$f^{(-1)}(y) = \begin{cases} f^{-1}(y) & if\ y \in [0, f(0)], \\ 0 & if\ y \in [f(0), \infty]. \end{cases}$$

Moreover, $\top$ *is strict iff* $f(0) = \infty$.

Remark 2.1 (*Weber [42]*). The function *g* (*f*) is called an additive generator of $\perp$ ($\top$). It is unique except for multiplication by positive numbers. In the non-strict case we call the additive generator with $g(1) = 1$ ($f(0) = 1$) the normed generator. If $\perp$ is an Archimedean t-conorm with additive generator g, then $\perp^*$ is also Archimedean with additive generator g^*, given by $g^*(x) = g(1 - x)$ with $g^{*(-1)}(y) = 1 - g^{(-1)}(y)$.

Because of the associative property, the t-conorm $\perp$ can be extended by induction to *n*-ary operation by setting

$$\mathop{\perp}_{i=1}^{n} x_i = \left(\mathop{\perp}_{i=1}^{n-1} x_i\right)\perp x_n.$$

Because of monotonicity, for each sequence $(x_i)_{i\in\mathbb{N}}$ of elements of [0, 1], the following limit can be considered:

$$\mathop{\perp}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathop{\perp}_{i=1}^{n} x_i.$$

It follows for Archimedean t-conorms that

$$\mathop{\perp}_{i=1}^{N} x_i = g^{(-1)}\left(\sum_{i=1}^{N} g(x_i)\right) \quad \text{where } N \in \mathbb{N} \cup \{\infty\}.$$

For the t-norm case, analogous statements hold [42].

Definition 2.4 (*Kramosil and Michalek [21]*). The 3-tuple $(X, M, \top)$ is said to be a KM fuzzy metric space if *X* is an arbitrary nonempty set, $\top$ is a *t*-norm, and *M* is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X, t, s > 0$:

(KM-1) $M(x, y, 0) = 0$.

(KM-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$.
(KM-3) $M(x, y, t) = M(y, x, t)$.
(KM-4) $M(x, z, t + s) \geq M(x, y, t) \top M(y, z, s)$.
(KM-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

By strengthening conditions (KM-2) and (KM-5) and modifying some other conditions, George and Veeramani introduced another definition of fuzzy metric space [10].

Definition 2.5 (*George and Veeramani [10]*). The 3-tuple $(X, M, \top)$ is said to be a GV fuzzy metric space if X is an arbitrary nonempty set, $\top$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X, t, s > 0$:

(GV-1) $M(x, y, t) > 0$.
(GV-2) $M(x, y, t) = 1$ iff $x = y$.
(GV-3) $M(x, y, t) = M(y, x, t)$.
(GV-4) $M(x, z, t + s) \geq M(x, y, t) \top M(y, z, s)$.
(GV-5) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

If $(X, M, \top)$ is a KM (GV) fuzzy metric space, we say that $(M, \top)$, or M (if it is not necessary to mention $\top$), is a KM (GV) fuzzy metric on X. We also say that (X, M), or simply X, is a KM (GV) fuzzy metric space.

Definition 2.6 (*George and Veeramani [11]*). A sequence $(x_i)_{i \in \mathbb{N}}$ in a GV fuzzy metric space (X, M) is said to be Cauchy if $\lim_{i,j \to \infty} M(x_i, x_j, t) = 1$ for all $t > 0$; a sequence $(x_i)_{i \in \mathbb{N}}$ in X converges to x iff $\lim_{i \to \infty} M(x_i, x, t) = 1$ for all $t > 0$.

Definition 2.7 (*Gregori and Romaguera [12], Gregori et al. [14]*). A GV fuzzy metric M on X is said to be stationary if M does not depend on t, that is, if, for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant.

Definition 2.8 (*Gregori et al. [13]*). Let $(X, M, \top)$ be a GV fuzzy metric space. The fuzzy metric M (or the fuzzy metric space $(X, M, \top)$) is said to be strong if it satisfies for each $x, y, z \in X$ and each $t > 0$

$$\text{(GV-4)}' \quad M(x, z, t) \geq M(x, y, t) \top M(y, z, t).$$

Obviously, stationary GV fuzzy metrics are strong. As a particular case, if $\top = \min$ and $(M, \top)$ is a strong fuzzy metric, we obtain the notion of a GV fuzzy ultrametric [13,33]

$$M(x, z, t) \geq \min(M(x, y, t), M(y, z, t)).$$

Definition 2.9. The 3-tuple $(X, M, \top)$ is said to be a GV fuzzy pseudometric space if X is an arbitrary nonempty set, $\top$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X, t, s > 0$:

(GVp-2) $M(x, x, t) = 1$ for all $x \in X$.
(GV-3) $M(x, y, t) = M(y, x, t)$.
(GV-4) $M(x, z, t + s) \geq M(x, y, t) \top M(y, z, s)$.
(GV-5) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

This definition is a generalization of that of Yue and Shi [44]. For the KM fuzzy metric case, analogous concepts can be defined.

In this paper, R, $S(R)$ and $P(X)$ denote an algebra of subsets of the given nonempty set X, the σ-algebra generated by this algebra and the power set of X, respectively.

Definition 2.10. Let $\perp$ be a t-conorm. A set function $\mu : R \to [0, 1]$ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$ is:

1. a $\perp$-decomposable measure iff $\mu(A \cup B) = \mu(A) \perp \mu(B)$, for each pair (A, B) of disjoint elements of R [42];

2. a σ-$\perp$-decomposable measure iff

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \mathop{\perp}_{i=1}^{\infty} \mu(A_i)$$

for each sequence $(A_i)_{i\in\mathbb{N}}$ of disjoint elements of R [42];

3. a $\perp$-subdecomposable measure iff $\mu(A\cup B)\le\mu(A)\perp\mu(B)$ [3]; and
4. a σ–$\perp$-subdecomposable measure iff [3]

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \le \mathop{\perp}_{i=1}^{\infty} \mu(A_i).$$

Definition 2.11 (*Weber [42]*). A $\perp$-decomposable measureμ, where $\perp$ is a continuous Archimedean t-conorm with an additive generator g, is said to be an (NSA)-type $\perp$-decomposable measure if $\perp$ is non-strict and $g\circ\mu$ is a finite additive measure.

Definition 2.12. Let μ be a σ–$\perp$-decomposable measure on an algebra $R\subset P(X)$. The set function $\mu^*: P(X)\to[0,1]$ defined by

$$\mu^*(A) = \inf\left\{\mathop{\perp}_{i=1}^{\infty}\mu(A_i) : A_i\in R;\, A\subset\bigcup_{i=1}^{\infty}A_i\right\},\quad A\in P(X)$$

is called the outer set function induced by μ.

Remark 2.2. For any $A\subseteq X$ and any sequence $(A_i)_{i\in\mathbb{N}}$ of sets in R whose union contains A, let $B_i = A_i\cap(\bigcup_{n=1}^{i-1}A_n)^C$ for each $i\in\mathbb{N}$. Then $(B_i)_{i\in\mathbb{N}}$ is a disjoint sequence of sets in R such that

$$B_i\subseteq A_i \quad\text{and}\quad \bigcup_{i=1}^{\infty}A_i = \bigcup_{i=1}^{\infty}B_i.$$

Thus, every sequence $(A_i)_{i\in\mathbb{N}}$ of sets in Definition 2.12 may be replaced by a disjoint sequence $(B_i)_{i\in\mathbb{N}}$ with the same property. Furthermore, if μ is an (NSA)-type $\perp$-decomposable measure with the normed additive generator g of $\perp$, then

$$\sum_{i=1}^{\infty}g(\mu(B_i))\le g(\mu(X)) = 1 \quad\text{and}\quad \mathop{\perp}_{i=1}^{\infty}\mu(B_i) = g^{-1}\left(\sum_{i=1}^{\infty}g(\mu(B_i))\right)$$

because $(B_i)_{i\in\mathbb{N}}$ is a disjoint sequence in R and $g\circ\mu$ is a finite additive measure.

3. Main results

In this section, let μ always be a σ–$\perp$-decomposable measure of (NSA)-type on an algebra R of subsets of X. We extend μ from R to $S(R)$ using a fuzzy pseudometric M.

Lemma 3.1. *The outer set function μ^* induced by μ has the following properties*:

(i) $\mu^*|_R = \mu$;
(ii) $\mu^*(\emptyset) = 0$;
(iii) *μ^* is monotonous, that is, $\mu^*(A)\le\mu^*(B)$ when $A\subset B$*;
(iv) *μ^* is σ–$\perp$-subdecomposable*;
(v) *μ^* is $\perp$-subdecomposable*; *and*
(vi) *for any sets $A, B, C\in P(X)$, $\mu^*(A\triangle C)\le\mu^*(A\triangle B)\perp\mu^*(B\triangle C)$, where $A\triangle B$ denotes the symmetric difference for sets A and B.*

Proof. (i) If $A \in R$, then $A = A \cup \emptyset \cup \emptyset \cup \cdots$ and therefore $\mu^*(A) \le \mu(A) \perp 0 \perp 0 \perp \cdots = \mu(A)$. On the other hand if $A \in R$, $A \subset \bigcup_{n=1}^{\infty} A_i$ and $A_i \in R$, then by the σ-subdecomposability of μ [3, Corollary 3], we have $\mu(A) \le \mu^*(A)$. This proves that $\mu^*|_R = \mu$.

(ii) It follows from (i) that $\mu^*(\emptyset) = 0$.

(iii) If $A, B \in R$, $A \subset B$, and $(A_i)_{i \in \mathbb{N}}$ is a sequence of sets in R that covers B, then $(A_i)_{i \in \mathbb{N}}$ also covers A and therefore $\mu^*(A) \le \mu^*(B)$.

(iv) Suppose that A and A_i are sets in $P(X)$ such that $A \subset \bigcup_{i=1}^{\infty} A_i$ and g is the normed additive generator of $\perp$. For any fixed $k \in \mathbb{N}$ and for all $i \in \mathbb{N}$, by Definition 2.12 and Remark 2.2 we can choose a corresponding disjoint sequence $(A_{ij})_{j \in \mathbb{N}}$ of sets in R such that

$$A_i \subset \bigcup_{j=1}^{\infty} A_{ij} \quad \text{and} \quad \mathop{\perp}_{j=1}^{\infty} \mu(A_{ij}) \le \mu^*(A_i) + \frac{1}{k}.$$

Then, since the sets A_{ij} form a countable class of disjoint sets in R that covers A and $g \circ \mu$ is a finite additive measure, from Remark 2.2 we obtain

$$\begin{aligned}
\mu^*(A) &\le \mathop{\perp}_{i=1}^{\infty} \mathop{\perp}_{j=1}^{\infty} \mu(A_{ij}) \\
&= g^{(-1)}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} g(\mu(A_{ij}))\right) \\
&= g^{(-1)}\left(\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} g(\mu(A_{ij}))\right)\right) \\
&= g^{(-1)}\left(\sum_{i=1}^{\infty} g \circ g^{-1}\left(\sum_{j=1}^{\infty} g(\mu(A_{ij}))\right)\right) \\
&= g^{(-1)}\left(\sum_{i=1}^{\infty} g\left(\mathop{\perp}_{j=1}^{\infty} \mu(A_{ij})\right)\right) \\
&\le g^{(-1)}\left(\sum_{i=1}^{\infty} g\left(\min\left(\mu^*(A_i) + \frac{1}{k}, 1\right)\right)\right).
\end{aligned}$$

Since g is uniformly continuous and $g^{(-1)}$ is continuous, taking $k \to \infty$ on the right-hand side of the above inequality, we obtain

$$\mu^*(A) \le g^{(-1)}\left(\sum_{i=1}^{\infty} g(\mu^*(A_i))\right) = \mathop{\perp}_{i=1}^{\infty} \mu^*(A_i).$$

(v) This follows from (ii) and (iv).

(vi) Since $A \triangle C \subset (A \triangle B) \cup (B \triangle C)$, the claim follows from the monotonicity and $\perp$-subdecomposability of μ^*. $\square$

Theorem 3.1. *If we define the fuzzy set M on $P(X)^2 \times [0, \infty)$ by*

$$M(A, B, t) = 1 - \mu^*(A \triangle B),$$

then M is a strong KM fuzzy pseudometric on $P(X)$ with respect to the dual t-norm $\perp^*$.

Proof. Since M does not depend on t, we can write $M(A, B)$ instead of $M(A, B, t)$ without confusion. From the definition of M, it is obvious that $M(A, A) = 1$ for all $A \in P(X)$ and $M(A, B) = M(B, A)$ for any $A, B \in P(X)$. The only thing we need to prove is the triangular inequality. For any $A, B, C \in P(X)$, we have

$M(A,C)=1-\mu^*(A\triangle C)$, $M(A,B)=1-\mu^*(A\triangle B)$ and $M(B,C)=1-\mu^*(B\triangle C)$. By Condition (vi) of Lemma 3.1, we have

$$\begin{aligned} M(A,B)\perp^* M(B,C) &= (1-\mu^*(A\triangle B))\perp^*(1-\mu^*(B\triangle C)) \\ &= 1-\mu^*(A\triangle B)\perp\mu^*(B\triangle C) \\ &\leq 1-\mu^*(A\triangle C) \\ &= M(A,C). \end{aligned}$$

Thus, M is a stationary KM fuzzy pseudometric and then a strong KM fuzzy pseudometric. □

A KM fuzzy pseudometric M does not have to satisfy Condition (GV-1) of Definition 2.5, so we can obtain a more general conclusion on the completion of a KM fuzzy pseudometric space than those of Gregori and colleagues [12,13].

Theorem 3.2. *Stationary strong KM fuzzy pseudometrics are completable.*

Proof. The proof follows from [12, Proposition 3] and is analogous to [13, Corollary 36]. □

Corollary 3.1. *The strong KM fuzzy pseudometric space (R,M) is completable.*

We denote the completion of R as $\bar{S}$ or, more specifically,

$$\bar{S}=\left\{S\in P(X): \text{there is a Cauchy sequence } (A_i)_{i\in\mathbb{N}}\subset R \text{ such that } \lim_{i\to\infty} M(A_i,S)=1\right\}.$$

Theorem 3.3. *For any $S\in\bar{S}$, if $(A_i)_{i\in\mathbb{N}}\subset R$ is a Cauchy sequence with respect to M such that $\lim_{i\to\infty} M(A_i,S)=1$, then $(\mu(A_i))_{i\in\mathbb{N}}$ is a Cauchy sequence in $[0,1]$ and $\lim_{i\to\infty}\mu(A_i)=\mu^*(S)$.*

Proof. Since

$$\begin{aligned} 1 &= \lim_{i,j\to\infty} M(A_i,A_j) \\ &= 1-\lim_{i,j\to\infty}\mu(A_i\triangle A_j) \\ &= 1-\lim_{i,j\to\infty}\mu((A_i\cap A_j^C)\cup(A_i^C\cap A_j)) \\ &= 1-\lim_{i,j\to\infty}\mu(A_i\cap A_j^C)\perp\mu(A_i^C\cap A_j) \\ &\leq 1-\lim_{i,j\to\infty}\max(\mu(A_i\cap A_j^C),\mu(A_i^C\cap A_j))\leq 1 \end{aligned}$$

we have

$$\lim_{i,j\to\infty}\max(\mu(A_i\cap A_j^C),\mu(A_i^C\cap A_j))=0.$$

By the decomposability of μ and the continuity of $\perp$, we have

$$\lim_{i,j\to\infty}|\mu(A_i)-\mu(A_j)|=\lim_{i,j\to\infty}|\mu(A_i\cap A_j^C)\perp\mu(A_i\cap A_j)-\mu(A_i\cap A_j)\perp\mu(A_i^C\cap A_j)|=0,$$

which implies that $(\mu(A_i))_{i\in\mathbb{N}}$ is a Cauchy sequence in $[0,1]$. In addition, from the inequalities

$$\begin{aligned} 1-\mu^*(S) &= M(S,\emptyset) \\ &\geq \lim_{i\to\infty}(M(S,A_i)\perp^* M(A_i,\emptyset)) \\ &= \lim_{i\to\infty}(M(S,A_i)\perp^*(1-\mu(A_i))) \\ &= 1-\lim_{i\to\infty}\mu(A_i) \end{aligned}$$

and

$$\begin{aligned}1-\lim_{i\to\infty}\mu(A_i)&=\lim_{i\to\infty}M(A_i,\emptyset)\\&\geq\lim_{i\to\infty}(M(A_i,S)\perp^*M(S,\emptyset))\\&=\lim_{i\to\infty}(M(S,A_i)\perp^*(1-\mu^*(S)))\\&=1-\mu^*(S)\end{aligned}$$

we have that $\lim_{i\to\infty}\mu(A_i)=\mu^*(S)$. □

Lemma 3.2. *$M(A,B)\perp^*M(C,D)\leq M(A\cup C,B\cup D)$ for any $A,B,C,D\in P(X)$.*

Proof. By the definitions of M and symmetric difference, and (iii) and (v) of Lemma 3.1, we have that

$$\begin{aligned}M(A,B)\perp^*M(C,D)&=(1-\mu^*(A\triangle B))\perp^*(1-\mu^*(C\triangle D))\\&=1-\mu^*(A\triangle B)\perp\mu^*(C\triangle D)\\&\leq 1-\mu^*((A\triangle B)\cup(C\triangle D))\\&=1-\mu^*((A\cap B^C)\cup(B\cap A^C)\cup(C\cap D^C)\cup(D\cap C^C))\\&\leq 1-\mu^*((A\cap B^C\cap D^C)\cup(B\cap A^C\cap C^C)\cup(C\cap D^C\cap B^C)\cup(D\cap C^C\cap A^C))\\&=1-\mu^*(((A\cup C)\cap(B\cup D)^C)\cup((B\cup D)\cap(A\cup C)^C))\\&=1-\mu^*((A\cup C)\triangle(B\cup D))\\&=M(A\cup C,B\cup D).\quad\square\end{aligned}$$

Theorem 3.4. *$\bar{S}$ is an algebra.*

Proof. For any $S_1,S_2\in\bar{S}$, suppose there exist Cauchy sequences $(A_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ in R that converge to S_1 and S_2, respectively. By Lemma 3.2, we have

$$M(A_i,A_j)\perp^*M(B_i,B_j)\leq M(A_i\cup B_i,A_j\cup B_j),$$

which implies $(A_i\cup B_i)_{i\in\mathbb{N}}$ is a Cauchy sequence. Moreover, by Lemma 3.2, we can obtain that

$$1=\lim_{i\to\infty}M(A_i,S_1)\perp^*M(B_i,S_2)\leq\lim_{i\to\infty}M(A_i\cup B_i,S_1\cup S_2)\leq 1,$$

which implies $S_1\cup S_2\in\bar{S}$. In addition, since

$$M(A_i^C,A_j^C)=1-\mu^*(A_i^C\Delta A_j^C)=1-\mu^*(A_i\Delta A_j)=M(A_i,A_j)$$

and

$$\lim_{i\to\infty}M(A_i^C,S_1^C)=1-\lim_{i\to\infty}\mu^*(A_i^C\Delta S_1^C)=1-\lim_{i\to\infty}\mu^*(A_i\Delta S_1)=\lim_{i\to\infty}M(A_i,S_1)$$

we have $S_1^C\in\bar{S}$. Thus, $\bar{S}$ is an algebra. □

Theorem 3.5. *$\mu^*|_{\bar{S}}$ is a$\perp$-decomposable measure.*

Proof. Let $S_1,S_2\in\bar{S}$ be disjoint. Then there exist Cauchy sequences $(A_i)_{i\in\mathbb{N}}$ and $(B_i)_{i\in\mathbb{N}}$ in R that converge to S_1 and S_2, respectively. As we saw in the proof of Theorem 3.4, $(A_i\cup B_i)_{i\in\mathbb{N}}$ is a Cauchy sequence that converges to $S_1\cup S_2$.

Since S_1 and S_2 are disjoint, we have

$$A_i\cap B_i\subset(A_i\Delta S_1)\cup(B_i\Delta S_2)$$

and

$$\begin{aligned}0 &\le \lim_{i\to\infty} \mu(A_i \cap B_i)\\ &\le \lim_{i\to\infty} \mu^*((A_i\Delta S_1)\cup(B_i\Delta S_2))\\ &\le \lim_{i\to\infty} \mu^*(A_i\Delta S_1)\perp\mu^*(B_i\Delta S_2)\\ &= 1-\lim_{i\to\infty} M(A_i,S_1)\perp^* M(B_i,S_2)=0,\end{aligned}$$

which implies $\lim_{i\to\infty}\mu(A_i\cap B_i)=0$. Thus, by the equality [42, Theorem 3.2(ii)]

$$\mu(A_i\cup B_i)\perp\mu(A_i\cap B_i)=\mu(A_i)\perp\mu(B_i)$$

and the continuity of $\perp$, we have that

$$\begin{aligned}\mu^*(S_1\cup S_2) &= \lim_{i\to\infty}\mu(A_i\cup B_i)\perp\lim_{i\to\infty}\mu(A_i\cap B_i)\\ &= \lim_{i\to\infty}(\mu(A_i\cup B_i)\perp\mu(A_i\cap B_i))\\ &= \lim_{i\to\infty}(\mu(A_i)\perp\mu(B_i))\\ &= \lim_{i\to\infty}\mu(A_i)\perp\lim_{i\to\infty}\mu(B_i)\\ &= \mu^*(S_1)\perp\mu^*(S_2),\end{aligned}$$

which completes the whole proof. □

Theorem 3.6. *$\bar{S}$ is a σ-algebra.*

Proof. Let $S_i \in \bar{S}$, $i\in\mathbb{N}$ be pairwise disjoint. Then there exist Cauchy sequences $(A_{ij})_{j\in\mathbb{N}}$ in R that converge to S_i for every $i\in\mathbb{N}$. Using the same argument as in the proof of Theorem 3.5, we have $\lim_{j\to\infty}\mu(A_{1j}\cap A_{2j})=0$. Thus, we have the following equality:

$$\begin{aligned}\mu^*(S_1) &= \lim_{j\to\infty}\mu(A_{1j})\\ &= \lim_{j\to\infty}\mu((A_{1j}\cap A_{2j}^C)\cup(A_{1j}\cap A_{2j}))\\ &= \lim_{j\to\infty}\mu((A_{1j}\cap A_{2j}^C)\perp\lim_{j\to\infty}\mu(A_{1j}\cap A_{2j}))\\ &= \lim_{j\to\infty}\mu((A_{1j}\cap A_{2j}^C).\end{aligned}$$

Similarly, we have $\mu^*(S_2)=\lim\limits_{j\to\infty}\mu((A_{2j}\cap A_{1j}^C)$. For any fixed $n\in\mathbb{N}$, in general we have that $\mu^*(S_i)=\lim_{j\to\infty}\mu(B_{ij}^{(n)})$ for every $i\in\{1,2,\ldots,n\}$, where

$$B_{ij}^{(n)} = A_{ij}\bigcap\left(\bigcup_{k\in\{1,2,\ldots,n\}-\{i\}} A_{kj}\right)^C.$$

Let g be an additive generator of $\perp$. By the continuity of g, we have $g(\mu^*(S_i)) = \lim_{j\to\infty} g(\mu(B_{ij}^{(n)}))$ for every $i\in\{1,2,\ldots,n\}$. Consequently, for any fixed $n\in\mathbb{N}$, we have

$$\sum_{i=1}^{n} g(\mu^*(S_i)) = \lim_{j\to\infty}\sum_{i=1}^{n} g(\mu(B_{ij}^{(n)}))\le g(\mu(X))=g(1),$$

where the inequality holds because for any fixed $j\in\mathbb{N}$, $\bigcup_{i=1}^{n} B_{ij}^{(n)}$ is a disjoint union and $g\circ\mu$ is a finite additive measure. It follows that

$$\sum_{i=1}^{\infty} g(\mu^*(S_i))\le g(1),$$

which implies that for any $n, m \in \mathbb{N}$ such that $n \le m$,

$$\lim_{n,m\to\infty}\left(\sum_{i=n+1}^{m} g(\mu^*(S_i))\right) = 0 \quad \text{and} \quad \lim_{i\to\infty}\left(\sum_{i=n+1}^{\infty} g(\mu^*(S_i))\right) = 0.$$

Therefore, by the σ–$\perp$-subdecomposability of μ^* we have that

$$\begin{aligned}\lim_{n,m\to\infty} M\left(\bigcup_{i=1}^{n} S_i, \bigcup_{i=1}^{m} S_i\right) &= 1 - \lim_{n,m\to\infty} \mu^*\left(\bigcup_{i=n+1}^{m} S_i\right)\\ &\ge 1 - \lim_{n,m\to\infty}\left(\mathop{\perp}_{i=n+1}^{m} \mu^*(S_i)\right)\\ &= 1 - g^{(-1)}\left(\lim_{n,m\to\infty}\left(\sum_{i=n+1}^{m} g(\mu^*(S_i))\right)\right) = 1\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} M\left(\bigcup_{i=1}^{n} S_i, \bigcup_{i=1}^{\infty} S_i\right) &= 1 - \lim_{n\to\infty} \mu^*\left(\bigcup_{i=n+1}^{\infty} S_i\right)\\ &\ge 1 - \lim_{n\to\infty}\left(\mathop{\perp}_{i=n+1}^{\infty} \mu^*(S_i)\right)\\ &= 1 - g^{(-1)}\left(\lim_{n\to\infty}\left(\sum_{i=n+1}^{\infty} g(\mu^*(S_i))\right)\right) = 1,\end{aligned}$$

which shows that $(\bigcup_{i=1}^{n} S_i)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete fuzzy pseudometric space $(\bar{S}, M)$ and converges to $\bigcup_{i=1}^{\infty} S_i$. Thus, $\bigcup_{i=1}^{\infty} S_i \in \bar{S}$. □

Theorem 3.7. *$\mu^*|_{\bar{S}}$ is a σ–$\perp$-decomposable measure.*

Proof. Let $S_i \in \bar{S}$, $i \in \mathbb{N}$ be pairwise disjoint. By Theorem 3.5 and the proof of Theorem 3.6, we have that

$$\begin{aligned}\mu^*\left(\bigcup_{i=1}^{\infty} S_i\right) &= \mu^*\left(\bigcup_{i=1}^{n} S_i\right) \perp \mu^*\left(\bigcup_{i=n+1}^{\infty} S_i\right)\\ &= \left(\mathop{\perp}_{i=1}^{n} \mu^*(S_i)\right) \perp \mu^*\left(\bigcup_{i=n+1}^{\infty} S_i\right)\end{aligned}$$

and

$$\mu^*\left(\bigcup_{i=n+1}^{\infty} S_i\right) = g^{(-1)}\left(\sum_{i=n+1}^{\infty} g(\mu^*(S_i))\right) \to 0 \quad \text{as } n \to \infty,$$

which implies

$$\begin{aligned}\mu^*\left(\bigcup_{i=1}^{\infty} S_i\right) &= \mu^*\left(\bigcup_{i=1}^{n} S_i\right) \perp \mu^*\left(\bigcup_{i=n+1}^{\infty} S_i\right)\\ &= \lim_{n\to\infty}\left(\left(\mathop{\perp}_{i=1}^{n} \mu^*(S_i)\right) \perp \mu^*\left(\bigcup_{i=n+1}^{\infty} S_i\right)\right)\\ &= \left(\lim_{n\to\infty} \mathop{\perp}_{i=1}^{n} \mu^*(S_i)\right) \perp \left(\lim_{n\to\infty} \mu^*\left(\bigcup_{i=n+1}^{\infty} S_i\right)\right)\\ &= \mathop{\perp}_{i=1}^{\infty} \mu^*(S_i).\end{aligned}$$

Thus, $\mu^*|_{\bar{S}}$ is a σ–$\perp$-decomposable measure. □

Now we present our conclusion on extension of a σ–$\perp$-decomposable measure μ from an algebra R to the σ-algebra $S(R)$ generated.

Theorem 3.8. *Let μ be a σ–$\perp$-decomposable measure of (NSA)-type on an algebra R. Then μ can be uniquely extended to a σ–$\perp$-decomposable measure on $S(R)$.*

Proof. From (i) of Lemma 3.1 and Theorems 3.6 and 3.7, we have $S(R) \subset \bar{S}$ and μ^* is a desired extension of μ to $S(R)$. We only need to prove uniqueness. For each sequence $(A_i)_{i\in\mathbb{N}}$ of disjoint elements of $S(R)$, since

$$\mu^*\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \mathop{\perp}_{i=1}^{\infty} \mu^*(A_i),$$

we can obtain

$$\begin{aligned} g\left(\mu^*\left(\bigcup_{i\in\mathbb{N}} A_i\right)\right) &= g\left(\mathop{\perp}_{i=1}^{\infty} \mu^*(A_i)\right) \\ &= g\left(g^{(-1)}\left(\sum_{i=1}^{\infty} g(\mu^*(A_i))\right)\right) \\ &= g\left(g^{-1}\left(\sum_{i=1}^{\infty} g(\mu^*(A_i))\right)\right) \\ &= \sum_{i=1}^{\infty} g(\mu^*(A_i)), \end{aligned}$$

where the second-last equality holds since

$$\sum_{i=1}^{\infty} g(\mu^*(A_i)) \le g(1)$$

can be proved using the same argument as in the proof of Theorem 3.6. Thus, we obtain that $g\circ\mu^*$ is an extension of $g\circ\mu$ on $S(R)$. According to the discussion in the Introduction, we obtain that $\mu^* = g^{-1}\circ g\circ\mu^*$ is the unique extension of μ on $S(R)$. $\square$

4. Conclusion

The results of this paper enable us to sketch the steps of a topological approach to the extension of σ–$\perp$-decomposable measures of (NSA)-type, and show that the σ-algebra arises naturally as the set of all limit points of Cauchy sequences in the original algebra. This approach offers an intuitive understanding of the extension of σ–$\perp$-decomposable measures. It may also be regarded as an attempt to use the theory of fuzzy metric spaces in fuzzy measure theory. However, for a σ-decomposable measure μ with respect to a continuous t-conorm $\perp$ without any other assumptions, we cannot use this topological approach to extend it since the proof of (iv)–(vi) of Lemma 3.1 depends on the (NSA)-property of μ and all subsequent results depend on Lemma 3.1. Therefore, in general, the following problem is worth further investigation.

Problem 4.1. Can we always extend a σ–$\perp$-decomposable measure μ on an algebra to the σ-algebra generated?

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