Distributive Equations of Fuzzy Implications Based on Continuous Triangular Conorms Given as Ordinal Sums

Aifang Xie, Cheng Li, and Huawen Liu

***Abstract*—Recently, the distributive equations of fuzzy implications based on *t*-norms or *t*-conorms have become a focus of research. The solutions to these equations can help people design the structures of fuzzy systems in such a way that the number of rules is largely reduced. This paper studies the distributive functional equation $I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$, where S_1 and S_2 are two continuous *t*-conorms given as ordinal sums, and $I : [0, 1]^2 \to [0, 1]$ is a binary function which is increasing with respect to the second place. If there is no summand of S_2 in the interval $[I(1, 0), I(1, 1)]$, we get its continuous solutions directly. If there are summands of S_2 in the interval $[I(1, 0), I(1, 1)]$, by defining a new concept called feasible correspondence and using this concept, we describe the solvability of the distributive equation above and characterize its general continuous solutions. When I is restricted to fuzzy implications, it is showed that there is no continuous solution to this equation. We characterize its fuzzy implication solutions, which are continuous on $(0, 1] \times [0, 1]$.**

***Index Terms*—Continuous *t*-conorms, distributive functional equations, fuzzy implications, ordinal sums.**

I. Introduction

DURING the past 30 years, fuzzy systems have been successfully used in many fields, such as in control systems [1]–[4], decision making, and signal processing. It is well known that fuzzy systems can approximate any continuous function to any desired accuracy (see [5]–[7]). However, a large rule base is required usually to achieve high accuracy. In many situations, the problem of rule explosion has become a major drawback which hinders the successful application of fuzzy systems.

To construct fuzzy systems using as less as possible fuzzy rules with guaranteed performance, some researchers have done a lot of work and have obtained encouraging results [8]–[11]. For example, in order to avoid combinational rule explosion, Combs and Andrews [9] introduced the classical tautology $(p \wedge q) \to r \equiv (p \to r) \vee (q \to r)$. They referred to the left-hand side of this equivalence as an intersection rule configuration (IRC) and to its right-hand side as a union rule configuration (URC). Later, some discussions appeared [12]–[15], and most of them believed that it is necessary to theoretically investigate this tautology before employing it. Dick and Kandel [14] wrote, “Future work on this issue will require an examination of the properties of various combinations of fuzzy unions, intersections and implications.” Mendel and Liang [15] also said, “We think that what this all means is that we have to look past the mathematics of IRC⇔ URC and inquire whether what we are doing when we replace IRC by URC makes sense.” Trillas and Alsina [16], in the standard fuzzy sets theory, first transformed the above tautology into the following functional equation

$$I(T(x, y), z) = S(I(x, z), I(y, z)), \qquad x, y, z \in [0, 1] \quad (1)$$

where I is a fuzzy implication, T is a *t*-norm, and S is a *t*-conorm. They also found out all the solutions T and S to (1) when I is an *R*-implication, an *S*-implication, or a *QL*-implication, respectively. In [17]–[19], T, S, and I in (1) are generalized into uninorms and the implications derived from uninorms, respectively.

Balasubramaniam and Rao [20] proposed three dual equations of (1). One of those dual equations is

$$I(x, S_1(y, z)) = S_2(I(x, y), I(x, z)), \qquad x, y, z \in [0, 1] \quad (2)$$

where I denotes a fuzzy implication, and S_1 and S_2 are *t*-conorms. Obviously, the above equation is a generalization of the classical tautology $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$. In [20], the authors also proved that if I is an *R*-implication derived from a nilpotent *t*-norm or is an *S*-implication, then $S_1 = S_2 = \max$.

Baczyński [21], [22] investigated (2) in the case that S_1 and S_2 are continuous Archimedean *t*-conorms and obtained its general fuzzy implication solutions. Following his work, Xie *et al.* [23] studied (2) under the condition that S_2 is a continuous Archimedean *t*-conorm and S_1 is a continuous *t*-conorm given as an ordinal sum. It is well known that if S is a continuous *t*-conorm, then either $S = S_M$, or S is a continuous Archimedean *t*-conorm, or S is an ordinal sum of continuous Archimedean *t*-conorms [24]. By far, (2) has not been considered when both S_1 and S_2 are continuous *t*-conorms given as ordinal sums.

In this paper, we will study (2) with two continuous *t*-conorms given as ordinal sums. This paper generalizes the results in [21]–[23] since a continuous Archimedean *t*-conorm is an ordinal sum with one summand. In addition, it should be pointed out that the method in this paper is different from

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the previous ones in [21]–[23]. For example, in [21]–[23], the form of the vertical section $I(x,\cdot)$ was given for every fixed $x \in [0,1]$, and then, these forms of vertical sections are used to derive the general continuous solutions to (2). However, in this paper, there is no need to get the form of the vertical section $I(x,\cdot)$ for every fixed $x \in [0,1]$. In fact, if there is no summand of S_2 in the interval $[I(1,0), I(1,1)]$, we get directly its general continuous solutions; if there are summands of S_2 in the interval $[I(1,0), I(1,1)]$, by introducing a new concept called feasible correspondence, we describe the solvability of (2) and characterize its general continuous solutions.

The rest of this paper is organized as follows. Some basic definitions and theorems are reviewed in Section II. In Section III, we discuss (2) and obtain its general continuous solutions and fuzzy implication solutions. We end this paper with the conclusions.

II. Preliminaries

In this section, we briefly recall some of the concepts and results used in the sequel.

Definition 1 [24], [25]: An associative, commutative, and increasing operation $S : [0,1]^2 \to [0,1]$ is called a *triangular conorm* (*t*-conorm for short) if it has the neutral element 0, i.e., $S(x,0) = x$ for any $x \in [0,1]$.

Example 1 [24]: The following are the four basic *t*-conorms S_M, S_P, S_L, and S_D given by, respectively

$$S_M(x,y) = \max(x,y), S_P(x,y) = x + y - xy$$
$$S_L(x,y) = \min(x+y,1), \text{ and}$$
$$S_D(x,y) = \begin{cases} \max(x,y), & x = 0 \text{ or } y = 0 \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2 [24]: A *t*-conorm S is called a *continuous Archimedean t-conorm* if it is continuous and satisfies $S(x,x) > x$ for all $x \in (0,1)$.

A continuous *t*-conorm S is said to be *strict* if it holds that $S(x,y) < S(x,z)$ for all $x < 1$ and $y < z$. A *t*-conorm S is said to be *nilpotent* if it is continuous, and for any $x \in (0,1)$, there exists some $n \in \mathcal{N}$ such that $x_S^n = 1$, where $x_S^n = \underbrace{S(x,x,\ldots,x)}_{n \text{ times}}$.

For example, S_P is strict and S_L is nilpotent. We also know that any continuous Archimedean *t*-conorm is either strict or nilpotent.

Theorem 1 [24], [26]: A function $S : [0,1]^2 \to [0,1]$ is a continuous Archimedean *t*-conorm if and only if S has a continuous additive generator, i.e., there exists a continuous and strictly increasing function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely determined up to a positive multiplicative constant, such that $S(x,y) = s^{(-1)}(s(x) + s(y))$ for all $x, y \in [0,1]$, where $s^{(-1)}$ is the pseudoinverse of s, given by

$$s^{(-1)}(x) = \begin{cases} s^{-1}(x), & x \in [0, s(1)] \\ 1, & x \in [s(1), \infty]. \end{cases}$$

Remark 1: 1) Without the pseudoinverse, the representation of a *t*-conorm S in Theorem 1 can be rewritten as $S(x,y) = s^{-1}(\min(s(x) + s(y), s(1)))$ for all $x, y \in [0,1]$.

2) A *t*-conorm S is strict if and only if each continuous additive generator s of S satisfies $s(1) = \infty$.

3) A *t*-conorm S is nilpotent if and only if each continuous additive generator s of S satisfies $s(1) < \infty$.

Proposition 1 [24]: Let $(S_m)_{m \in J}$ be a family of *t*-conorms and $((a_m, b_m))_{m \in J}$ be a family of nonempty, nonoverlapping, and open subintervals of $[0,1]$, where J is a finite or countable infinite index set. Then the function $S : [0,1]^2 \to [0,1]$ that is defined by

$$S(x,y) = \begin{cases} a_m + (b_m - a_m) S_m\left(\dfrac{x - a_m}{b_m - a_m}, \dfrac{y - a_m}{b_m - a_m}\right) \\ \qquad\qquad\qquad\qquad (x,y) \in [a_m, b_m]^2 \\ \max(x,y), \qquad\qquad \text{otherwise} \end{cases}$$

is a *t*-conorm which is called the *ordinal sum* of summands $\langle a_m, b_m, S_m \rangle_{m \in J}$, and we write $S = (\langle a_m, b_m, S_m \rangle)_{m \in J}$.

Remark 2 [24]: A *t*-conorm S is continuous if and only if S is either S_M, a continuous Archimedean *t*-conorm, or an ordinal sum $(\langle a_m, b_m, S_m \rangle)_{m \in J}$ with each S_m being a continuous Archimedean *t*-conorm.

Definition 3 [27]: A function $I : [0,1]^2 \to [0,1]$ is said to be a *fuzzy implication* if it satisfies the following:

I1) the first place antitonicity;

I2) the second place isotonicity;

I3) $I(0,0) = I(1,1) = I(0,1) = 1$ and $I(1,0) = 0$.

From the definition above, it is clear that for any fuzzy implication I, it holds that $I(0,x) = I(x,1) = 1$ for all $x \in [0,1]$.

Now, let us recall some facts about the additive Cauchy functional equation (see [28]) and some other similar equations.

Theorem 2 [22]: For a function $f : [0,\infty] \to [0,\infty]$, the following statements are equivalent:

1) f satisfies the additive Cauchy functional equation

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in [0,\infty].$$

2) Either

$$f = \infty, \text{ or } f = 0, \text{ or } f(x) = \begin{cases} 0, & x = 0 \\ \infty, & x \in (0,\infty] \end{cases} \text{ or}$$
$$f(x) = \begin{cases} 0, & x \in [0,\infty) \\ \infty, & x = \infty \end{cases}$$

or there exists a unique constant $p \in (0,\infty)$ such that $f(x) = px$ for all $x \in [0,\infty]$.

Clearly, if f is continuous, then $f = \infty$, or $f = 0$, or $f(x) = px$ with a unique constant $p \in (0,\infty)$.

Theorem 3 [21]: Fix real $b > 0$. For a function $f : [0,\infty] \to [0,b]$, the following are equivalent:

1) f satisfies the functional equation

$$f(x+y) = \min(f(x) + f(y), b) \text{ for all } x, y \in [0,\infty].$$

2) Either

$$f=b,\ \text{or } f=0,\ \text{or } f(x)=\begin{cases}0, & x=0\\ b, & x\in(0,\infty]\end{cases}\quad\text{or}$$

$$f(x)=\begin{cases}0 & x\in[0,\infty)\\ b & x=\infty\end{cases}$$

or there exists a unique constant $p\in(0,\infty)$ such that $f(x)=\min(px,b)$ for all $x\in[0,\infty]$.

Theorem 4 [22]: Fix real $a,b>0$. For a function $f:[0,a]\to[0,b]$, the following statements are equivalent:

1) f satisfies the functional equation

$$f(\min(x+y,a))=\min(f(x)+f(y),b)\ \text{for all } x,y\in[0,a].$$

2) Either

$$f=b,\ \text{or } f=0,\ \text{or } f(x)=\begin{cases}0, & x=0\\ b, & x\in(0,a]\end{cases}$$

or there exists a unique constant $p\in[\frac{b}{a},\infty)$ such that $f(x)=\min(px,b)$ for all $x\in[0,a]$.

III. Solutions to (2)

In this section, we discuss (2) with two continuous *t*-conorms given as ordinal sums. This section is divided into two sections. In Section III-A, we discuss the general continuous solutions to (2). In Section III-B, we discuss its fuzzy implication solutions which are continuous on $(0,1]\times[0,1]$.

A. Continuous Solutions to (2)

Lemma 1: Let S_1 and S_2 be two *t*-conorms and $I:[0,1]^2\to[0,1]$ be a binary function. If the triple (S_1,S_2,I) satisfies (2) and y is an idempotent element of S_1, then for every $x\in[0,1]$, $I(x,y)$ is an idempotent element of S_2.

Proof: Suppose y is an idempotent element of S_1. For every $x\in[0,1]$, we get from (2) that $I(x,y)=I(x,S_1(y,y))=S_2(I(x,y),I(x,y))$. Therefore, $I(x,y)$ is an idempotent element of S_2. □

Lemma 2: Let S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m,b_m,S'_m\rangle)_{m\in J_1}$ and $(\langle c_n,d_n,S''_n\rangle)_{n\in J_2}$, respectively, and $I:[0,1]^2\to[0,1]$ be a binary function which is increasing in its second place. Arbitrarily fix $x\in[0,1]$. Suppose the triple (S_1,S_2,I) satisfies (2). If there exist an $m_0\in J_1$ and a $y_0\in(a_{m_0},b_{m_0})$ such that $I(x,y_0)$ is an idempotent element of S_2, then $I(x,y)=I(x,y_0)$ for all $y\in[y_0,b_{m_0}]$.

Proof: Case 1. Suppose the corresponding *t*-conorm S'_{m_0} is a nilpotent *t*-conorm with an additive generator s'_{m_0}.

Because $I(x,y_0)$ is an idempotent element of S_2, then for any $y\in[y_0,b_{m_0}]$, $I(x,S_1(y,y_0))=S_2(I(x,y),I(x,y_0))=\max(I(x,y),I(x,y_0))=I(x,y)$, namely,

$$I(x,S_1(y,y_0))=I(x,y)$$

for any $y\in[y_0,b_{m_0}]$.

Let us define function $\varphi:[a_{m_0},b_{m_0}]\to[0,1]$ by $\varphi(x)=\frac{x-a_{m_0}}{b_{m_0}-a_{m_0}}$ and function $I_x:[a_{m_0},b_{m_0}]\to[0,1]$ by $I_x(y)=I(x,y)$.

Then the above equation can be rewritten as $I_x\circ\varphi^{-1}\circ s'^{-1}_{m_0}\circ\min(s'_{m_0}\circ\varphi(y)+s'_{m_0}\circ\varphi(y_0),s'_{m_0}(1))=I_x(y)$ for any $y\in[y_0,b_{m_0}]$. By routine substitutions, $h_x=I_x\circ\varphi^{-1}\circ s'^{-1}_{m_0}$, $u=s'_{m_0}\circ\varphi(y)$, and $v_0=s'_{m_0}\circ\varphi(y_0)$, we get the following equation:

$$h_x(\min(u+v_0,s'_{m_0}(1))=h_x(u)\tag{3}$$

for any $u\in[v_0,s'_{m_0}(1)]$, where $h_x:[0,s'_{m_0}(1)]\to[0,1]$ is an increasing function.

Let $a=\varphi^{-1}\circ s'^{-1}_{m_0}(s'_{m_0}(1)-s'_{m_0}(\varphi(y_0)))$. Clearly, it holds that $a\in(a_{m_0},b_{m_0})$.

Case 1.1. Assume that $y_0\le a$, i.e., $2v_0\le s'_{m_0}(1)$.

For any $y\in[y_0,a]$, we have $u=s'_{m_0}\circ\varphi(y)\in[v_0,s'_{m_0}(1)-v_0]$ and $u+v_0\in[2v_0,s'_{m_0}(1)]$. Thus, we get from (3) that

$$h_x(u+v_0)=h_x(u)\tag{4}$$

for any $u\in[v_0,s'_{m_0}(1)-v_0]$.

Let $P=\{n|n\ \text{is an integer and}\ nv_0\in[v_0,s'_{m_0}(1)-v_0]\}$. Since $1\cdot v_0\in[v_0,s'_{m_0}(1)-v_0]$, it holds that $P\neq\emptyset$. Because $s'_{m_0}(1)-v_0$ is a real number, there exists an $n_0=\max\{n|n$ is an integer and $nv_0\in[v_0,s'_{m_0}(1)-v_0]\}$. Therefore, $n_0v_0\in[v_0,s'_{m_0}(1)-v_0]$ and $(n_0+1)v_0\in(s'_{m_0}(1)-v_0,s'_{m_0}(1)]$.

From (4), we get that $h_x(v_0)=h_x(2v_0)=\cdots=h_x(n_0v_0)=h_x((n_0+1)v_0)$. Observing the monotonicity of function h_x, we have $h_x(t)=h_x(v_0)$ for any $t\in[v_0,(n_0+1)v_0]$ and then $h_x((n_0+1)v_0)=h_x(v_0)$.

On the other hand, we can obtain from (4) that $h_x(s'_{m_0}(1)-v_0)=h_x(s'_{m_0}(1))$. Hence, $h_x(t)=h_x(s'_{m_0}(1))$, $t\in[s'_{m_0}(1)-v_0,s'_{m_0}(1)]$. Since $(n_0+1)v_0\in(s'_{m_0}(1)-v_0,s'_{m_0}(1)]$, we have $h_x((n_0+1)v_0)=h_x(s'_{m_0}(1))$.

From the above steps, we obtain that $h_x(v_0)=h_x(s'_{m_0}(1))$. Consequently, $h_x(t)=h_x(v_0),t\in[v_0,s'_{m_0}(1)]$. Therefore, $I(x,y)=I(x,y_0),y\in[y_0,b_{m_0}]$.

Case 1.2. Assume that $y_0>a$, i.e., $2v_0>s'_{m_0}(1)$. For any $y\in[y_0,b_{m_0}]$, we get that $u=s'_{m_0}\circ\varphi(y)\in[v_0,s'_{m_0}(1)]$. Therefore, $u+v_0\in[2v_0,s'_{m_0}(1)+v_0]$. Because $2v_0>s'_{m_0}(1)$, we obtain from (3) that $h_x(s'_{m_0}(1))=h_x(u)$, namely, $I(x,y)=I(x,b_{m_0})$ for any $y\in[y_0,b_{m_0}]$. Especially, $I(x,y_0)=I(x,b_{m_0})$. Hence, $I(x,y)=I(x,y_0)$ for all $y\in[y_0,b_{m_0}]$.

Case 2. Suppose the corresponding *t*-conorm S'_{m_0} is a strict *t*-conorm with an additive generator s'_{m_0}. It follows from (2) that for any $y\in[y_0,b_{m_0}]$, $I(x,S_1(y,y_0))=S_2(I(x,y),I(x,y_0))=\max(I(x,y),I(x,y_0))$, i.e.,

$$I(x,S_1(y,y_0))=I(x,y)$$

for any $y\in[y_0,b_{m_0}]$.

Just as the proof in Case 1, we get the following equation:

$$h_x(u+v_0)=h_x(u),\ \text{for any } u\in[v_0,\infty]$$

where $h_x:[0,\infty]\to[0,1]$. This equation shows that h_x is a periodic function on $[v_0,\infty]$. The increasing property of function h_x implies that h_x must be a constant on $[v_0,\infty]$. Hence, $h_x(u)=h_x(v_0)$ for all $u\in[v_0,\infty]$. Therefore, $I(x,y)=I(x,y_0),y\in[y_0,b_{m_0}]$. □

Corollary 1: Let S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m, b_m, S'_m\rangle)_{m\in J_1}$ and $(\langle c_n, d_n, S''_n\rangle)_{n\in J_2}$, respectively, and $I:[0,1]^2 \to [0,1]$ be a binary function which is increasing with respect to the second place. Arbitrarily fix $x \in [0,1]$. Suppose the triple (S_1, S_2, I) satisfies (2). If $I(x,\cdot)$ is continuous on $[a_{m_0}, b_{m_0}]$ (some $m_0 \in J_1$) and $I(x,y)$ is an idempotent element of S_2 for any $y \in (a_{m_0}, b_{m_0})$, then there exists an idempotent element e_{x,m_0} of S_2 such that $I(x,y) = e_{x,m_0}$ for all $y \in [a_{m_0}, b_{m_0}]$.

Proof: It can be proved that $I(x,y)$ equals each other for any $y \in (a_{m_0}, b_{m_0})$. Otherwise, suppose that there exist y_1 and y_2 $(y_1 < y_2)$ in the interval (a_{m_0}, b_{m_0}) such that $I(x,y_1) \neq I(x,y_2)$. Since $y_1 < y_2$ and $I(x,y_1)$ is an idempotent element of S_2, we obtain from Lemma 2 that $I(x,y_2) = I(x,y_1)$. This produces a contradiction. Consequently, there exists an idempotent element e_{x,m_0} of S_2 such that $I(x,y) = e_{x,m_0}$ for all $y \in (a_{m_0}, b_{m_0})$. By the continuity of $I(x,\cdot)$ on $[a_{m_0}, b_{m_0}]$, we get the result. □

In this paper, for any binary function f and any set G, we denote $\{f(x,y) | y \in G\}$ as $f(x,G)$.

Lemma 3: Let S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m, b_m, S'_m\rangle)_{m\in J_1}$ and $(\langle c_n, d_n, S''_n\rangle)_{n\in J_2}$, respectively, and $I:[0,1]^2 \to [0,1]$ be a binary function which is increasing in its second place. Arbitrarily fix $x \in [0,1]$. Suppose the triple (S_1, S_2, I) satisfies (2) and $I(x,\cdot)$ is continuous on $[a_{m_0}, b_{m_0}]$ (some $m_0 \in J_1$). If there exists a $y_0 \in (a_{m_0}, b_{m_0})$ such that $I(x,y_0)$ is a non-idempotent element of S_2, then there exists an $n_{x,m_0} \in J_2$ such that $I(x,[a_{m_0}, b_{m_0}]) = [c_{n_{x,m_0}}, d_{n_{x,m_0}}]$. Especially,

1) if there exists a $y_1 \in (a_{m_0}, b_{m_0})$ such that $I(x,y_1)$ is an idempotent element of S_2, then the corresponding *t*-conorm $S''_{n_{x,m_0}}$ must be nilpotent;

2) if for any $y \in (a_{m_0}, b_{m_0})$, $I(x,y)$ is a non-idempotent element of S_2, then the corresponding *t*-conorm $S''_{n_{x,m_0}}$ is strict if S'_{m_0} is strict, and $S''_{n_{x,m_0}}$ is nilpotent if S'_{m_0} is nilpotent.

Proof: We get from Lemma 2 that $I(x,y)$ is not an idempotent element of S_2 for any $y \in (a_{m_0}, y_0)$. Let us define $k_{x,m_0} = \inf\{y \in (a_{m_0}, b_{m_0}) | I(x,y) \text{ is an idempotent element of } S_2\}$ (with the convention that $\inf \emptyset = b_{m_0}$). Clearly, $a_{m_0} < y_0 \leq k_{x,m_0} \leq b_{m_0}$.

Denote $T = \{y \in (a_{m_0}, b_{m_0}) | I(x,y)$ is an idempotent element of $S_2\}$. In the following, we will prove that $I(x, k_{x,m_0})$ is an idempotent element of S_2.

In fact, according to the continuity of S_2 and the continuity of $I(x,\cdot)$ on $[a_{m_0}, b_{m_0}]$, we have

$$\begin{aligned}
&S_2(I(x,k_{x,m_0}), I(x,k_{x,m_0}))\\
&= S_2(I(x,\inf T), I(x,\inf T))\\
&= S_2(\inf\{I(x,y)|y\in T\}, \inf\{I(x,y)|y\in T\})\\
&= \inf\{S_2(I(x,y), I(x,y))|y\in T\}\\
&= \inf\{I(x,y)|y\in T\}\\
&= I(x,\inf\{y|y\in T\}) = I(x,k_{x,m_0}).
\end{aligned}$$

Therefore, $I(x,k_{x,m_0})$ is an idempotent element of S_2. According to Lemma 2, we get that $I(x,y) = I(x,k_{x,m_0})$ for any $y \in [k_{x,m_0}, b_{m_0}]$. In addition, it is easy to see that $I(x,y) \in [I(x,a_{m_0}), I(x,k_{x,m_0})]$ for any $y \in (a_{m_0}, k_{x,m_0})$.

Considering that $I(x,y)$ is not an idempotent element for any $y \in (a_{m_0}, k_{x,m_0})$, $I(x,a_{m_0})$ and $I(x,k_{x,m_0})$ are idempotent elements of S_2, and $I(x,\cdot)$ is continuous on $[a_{m_0}, b_{m_0}]$, we obtain that there exists an $n_{x,m_0} \in J_2$ such that $I(x,y) \in (c_{n_{x,m_0}}, d_{n_{x,m_0}})$ for all $y \in (a_{m_0}, k_{x,m_0})$, $I(x,a_{m_0}) = c_{n_{x,m_0}}$, and $I(x,y) = d_{n_{x,m_0}}$ for all $y \in [k_{x,m_0}, b_{m_0}]$. Therefore, $I(x,[a_{m_0}, b_{m_0}]) = [c_{n_{x,m_0}}, d_{n_{x,m_0}}]$.

Next we will prove 1). Clearly, it holds that $k_{x,m_0} \leq y_1 < b_{m_0}$.

Suppose $S''_{n_{x,m_0}}$ is strict. Because k_{x,m_0} is a non-idempotent element of S_1, it holds that $S_1(k_{x,m_0}, k_{x,m_0}) > k_{x,m_0}$. Since S_1 is continuous, there must exist a $y' \in (a_{m_0}, k_{x,m_0})$ such that $S_1(y',y') \in (k_{x,m_0}, S_1(k_{x,m_0}, k_{x,m_0}))$, which implies that $I(x, S_1(y',y')) = d_{n_{x,m_0}}$. However, $S_2(I(x,y'), I(x,y')) < d_{n_{x,m_0}}$ because $I(x,y') \in (c_{n_{x,m_0}}, d_{n_{x,m_0}})$ and $S''_{n_{x,m_0}}$ is strict. This contradicts (2). Consequently, $S''_{n_{x,m_0}}$ must be nilpotent.

Finally, we will prove 2). Suppose the corresponding *t*-conorm S'_{m_0} is strict. In the following, we will show that $S''_{n_{x,m_0}}$ is strict. Otherwise, suppose $S''_{n_{x,m_0}}$ is nilpotent. Then there exists a $z^* \in (c_{n_{x,m_0}}, d_{n_{x,m_0}})$ such that $S_2(z^*, z^*) = d_{n_{x,m_0}}$. Because of the continuity of $I(x,\cdot)$ on $[a_{m_0}, b_{m_0}]$, there exists a $y'' \in (a_{m_0}, b_{m_0})$ such that $I(x,y'') = z^*$. Since S'_{m_0} is strict, we have $S_1(y'',y'') \in (a_{m_0}, b_{m_0})$, and then $I(x, S_1(y'',y'')) \in (c_{n_{x,m_0}}, d_{n_{x,m_0}})$. This contradicts the fact that $S_2(I(x,y''), I(x,y'')) = S_2(z^*, z^*) = d_{n_{x,m_0}}$.

In a similar way, we can prove that if S'_{m_0} is nilpotent, then $S''_{n_{x,m_0}}$ is nilpotent. □

Lemma 4: Let S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m, b_m, S'_m\rangle)_{m\in J_1}$ and $(\langle c_n, d_n, S''_n\rangle)_{n\in J_2}$, respectively, and $I:[0,1]^2 \to [0,1]$ be a binary function which is increasing in its second place. Arbitrarily fix $x \in [0,1]$. Suppose the triple (S_1, S_2, I) satisfies (2) and $I(x,\cdot)$ is continuous on $[a_m, b_m]$ (any $m \in J_1$). If there exist $m_1, m_2 \in J_1$ such that $a_{m_1} < a_{m_2}$, $I(x,[a_{m_1}, b_{m_1}]) = [c_{n_{x,m_1}}, d_{n_{x,m_1}}]$ and $I(x,[a_{m_2}, b_{m_2}]) = [c_{n_{x,m_2}}, d_{n_{x,m_2}}]$ (some $n_{x,m_1}, n_{x,m_2} \in J_2$), then $c_{n_{x,m_1}} < c_{n_{x,m_2}}$.

Proof: First, we prove that $c_{n_{x,m_1}} \neq c_{n_{x,m_2}}$. Otherwise, suppose $c_{n_{x,m_1}} = c_{n_{x,m_2}}$. Then $I(x,a_{m_2}) = c_{n_{x,m_1}}$ and $I(x,b_{m_1}) = d_{n_{x,m_1}}$. Hence, $I(x,a_{m_2}) < I(x,b_{m_1})$ because $c_{n_{x,m_1}} < d_{n_{x,m_1}}$. Obviously, this contradicts the increasing property of $I(x,\cdot)$. Therefore, $c_{n_{x,m_1}} \neq c_{n_{x,m_2}}$. The increasing property of $I(x,\cdot)$ immediately implies that $c_{n_{x,m_1}} < c_{n_{x,m_2}}$. □

Lemma 5: Let S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m, b_m, S'_m\rangle)_{m\in J_1}$ and $(\langle c_n, d_n, S''_n\rangle)_{n\in J_2}$, respectively, and $I:[0,1]^2 \to [0,1]$ be a continuous binary function which is increasing in its second place. Suppose the triple (S_1, S_2, I) satisfies (2). If there exist an $x_1 \in [0,1]$ and some $m_0 \in J_1, n_0 \in J_2$ such that $I(x_1,[a_{m_0}, b_{m_0}]) = [c_{n_0}, d_{n_0}]$, then $I(x,[a_{m_0}, b_{m_0}]) = [c_{n_0}, d_{n_0}]$ for any $x \in [0,1]$.

Proof: Suppose there exists an $x_2 \in [0,1]$ $(x_1 \neq x_2)$ such that $I(x_2,[a_{m_0}, b_{m_0}]) \neq [c_{n_0}, d_{n_0}]$. According to Corollary 1 and Lemma 3, we have $I(x_2,[a_{m_0}, b_{m_0}]) = e_{x_2,m_0}$ (e_{x_2,m_0} is an idempotent element of S_2), or $I(x_2,[a_{m_0}, b_{m_0}]) = [c_{n'_0}, d_{n'_0}]$. Let us suppose $x_1 < x_2$ and $c_{n'_0} < c_{n_0}$.

Case 1. Suppose $I(x_2, [a_{m_0}, b_{m_0}]) = e_{x_2,m_0}$. Then $I(x_1, a_{m_0}) = c_{n_0}$ and $I(x_2, a_{m_0}) = e_{x_2,m_0}$. In this case, if $e_{x_2,m_0} \geq d_{n_0}$, we have from the continuity of $I(\cdot, a_{m_0})$ that for any $y_0 \in (c_{n_0}, d_{n_0}) \subset [c_{n_0}, e_{x_2,m_0}]$, there exists some $x_0 \in (x_1, x_2)$ such that $I(x_0, a_{m_0}) = y_0$. This shows that $I(x_0, a_{m_0})$ is not an idempotent element of S_2, which contradicts Lemma 1.

If $e_{x_2,m_0} \leq c_{n_0}$, then we have $I(x_1, b_{m_0}) = d_{n_0}$ and $I(x_2, b_{m_0}) = e_{x_2,m_0}$. Similarly to the above, we can get a contradiction.

Case 2. Suppose $I(x_2, [a_{m_0}, b_{m_0}]) = [c_{n_0'}, d_{n_0'}]$. Then we obtain that $I(x_1, a_{m_0}) = c_{n_0}$ and $I(x_2, a_{m_0}) = c_{n_0'}$. For any $y' \in (c_{n_0'}, d_{n_0'}) \subset (c_{n_0'}, c_{n_0}]$, there exists some $x' \in (x_1, x_2)$ satisfying that $I(x', a_{m_0}) = y'$. This produces a contradiction with Lemma 1. □

Theorem 5: Let S_1 and S_2 be two continuous *t*-conorms given by ordinal sums and $I : [0,1]^2 \to [0,1]$ be a binary function. Fix arbitrarily $x \in [0,1]$. Suppose $\langle a_{m_0}, b_{m_0}, S'_{m_0} \rangle$ and $\langle c_{n_0}, d_{n_0}, S''_{n_0} \rangle$ are two summands of S_1 and S_2, respectively. Then $I(x, \cdot)$ is increasing and continuous on $[a_{m_0}, b_{m_0}]$, $I(x, [a_{m_0}, b_{m_0}]) = [c_{n_0}, d_{n_0}]$, and $I(x, S_1(y,z)) = S_2(I(x,y), I(x,z))$ for all $y, z \in [a_{m_0}, b_{m_0}]$ if and only if one of the following statements holds.

1) If S'_{m_0} is a strict *t*-conorm with an additive generator s'_{m_0} and S''_{n_0} is a nilpotent *t*-conorm with an additive generator s''_{n_0}, then

$$I(x,y) = \begin{cases} E_x(y), & y \in [a_{m_0}, k_{x,m_0}) \\ d_{n_0}, & y \in [k_{x,m_0}, b_{m_0}] \end{cases}$$

for some $k_{x,m_0} \in (a_{m_0}, b_{m_0})$, where $E_x(y) = c_{n_0} + (d_{n_0} - c_{n_0}) s''^{-1}_{n_0}\left(\frac{s''_{n_0}(1)}{s'_{m_0}\left(\frac{k_{x,m_0} - a_{m_0}}{b_{m_0} - a_{m_0}}\right)} s'_{m_0}\left(\frac{y - a_{m_0}}{b_{m_0} - a_{m_0}}\right)\right)$.

2) If S'_{m_0} and S''_{n_0} are strict *t*-conorms with s'_{m_0} and s''_{n_0} being their respective additive generators, then for any $y \in [a_{m_0}, b_{m_0}]$,

$$I(x,y) = c_{n_0} + (d_{n_0} - c_{n_0}) s''^{-1}_{n_0}\left(p_{x,m_0} s'_{m_0}\left(\frac{y - a_{m_0}}{b_{m_0} - a_{m_0}}\right)\right)$$

where $p_{x,m_0} \in (0, \infty)$ is a certain constant related to x and m_0, uniquely determined up to a positive multiplicative constant depending on constants for s'_{m_0} and s''_{n_0}.

3) If S'_{m_0} and S''_{n_0} are nilpotent *t*-conorms with s'_{m_0} and s''_{n_0} being their respective additive generators, then

$$I(x,y) = \begin{cases} E_x(y), & y \in [a_{m_0}, k_{x,m_0}) \\ d_{n_0}, & y \in [k_{x,m_0}, b_{m_0}] \end{cases}$$

for some $k_{x,m_0} \in (a_{m_0}, b_{m_0}]$, where $E_x(y)$ is the same as in statement 1).

Proof: ($\Rightarrow$) Since $I(x, [a_{m_0}, b_{m_0}]) = [c_{n_0}, d_{n_0}]$, then we get from Lemma 3 that there are only three possible cases.

N1) S'_{m_0} is a strict *t*-conorm, and S''_{n_0} is a nilpotent *t*-conorm.

N2) S'_{m_0} and S''_{n_0} are strict *t*-conorms.

N3) S'_{m_0} and S''_{n_0} are nilpotent *t*-conorms.

The proof of 1). Suppose that S'_{m_0} is strict and S''_{n_0} is nilpotent, i.e., N1 is satisfied. In line with Lemma 3, there exist $y_1, y_2 \in (a_{m_0}, b_{m_0})$ such that $I(x, y_1)$ is an idempotent element of S_2 and $I(x, y_2)$ is not an idempotent element of S_2. Let $k_{x,m_0} = \inf\{y \in (a_{m_0}, b_{m_0}) | I(x,y)$ is an idempotent element of $S_2\}$. Clearly, $a_{m_0} < y_2 < k_{x,m_0} \leq y_1 < b_{m_0}$. By the proof of Lemma 3, we get the fact that $I(x, [k_{x,m_0}, b_{m_0}]) = d_{n_0}$, $I(x, a_{m_0}) = c_{n_0}$, and $I(x,y) \in [c_{n_0}, d_{n_0}]$ if $y \in (a_{m_0}, k_{x,m_0})$.

Define functions $\varphi : [a_{m_0}, b_{m_0}] \to [0,1]$ by $\varphi(x) = \frac{x - a_{m_0}}{b_{m_0} - a_{m_0}}$ and $\psi : [c_{n_0}, d_{n_0}] \to [0,1]$ by $\psi(x) = \frac{x - c_{n_0}}{d_{n_0} - c_{n_0}}$. It follows from (2) that for any $y, z \in (a_{m_0}, k_{x,m_0})$,

$$I(x, \varphi^{-1}(s'^{-1}_{m_0}(s'_{m_0}(\varphi(y)) + s'_{m_0}(\varphi(z)))))$$
$$= \psi^{-1}(s''^{-1}_{n_0}(\min(s''_{n_0}(\psi(I(x,y))) + s''_{n_0}(\psi(I(x,z))), s''_{n_0}(1)))).$$

For the arbitrarily fixed $x \in [0,1]$, define a function $I_x : [a_{m_0}, b_{m_0}] \to [c_{n_0}, d_{n_0}]$ given by $I_x(y) = I(x,y)$. By routine substitutions, $h_x = s''_{n_0} \circ \psi \circ I_x \circ \varphi^{-1} \circ s'^{-1}_{m_0}$, $u = s'_{m_0} \circ \varphi(y)$, and $v = s'_{m_0} \circ \varphi(z)$, from the above equation, we obtain that

$$h_x(u + v) = \min(h_x(u) + h_x(v), s''_{n_0}(1)) \tag{5}$$

for any $u, v \in (0, s'_{m_0}(\varphi(k_{x,m_0})))$, where $h_x : [0, \infty] \to [0, s''_{n_0}(1)]$ is a continuous and increasing function.

By simple computation, we have that $h_x(0) = 0$ and $h_x(s'_{m_0}(\varphi(k_{x,m_0}))) = s''_{n_0}(1)$. Then $h_x(t) = s''_{n_0}(1)$ for any $t \geq s'_{m_0}(\varphi(k_{x,m_0}))$ since h_x is increasing. It is easy to check that h_x satisfies the following functional equation:

$$h_x(t_1 + t_2) = \min(h_x(t_1) + h_x(t_2), s''_{n_0}(1))$$

for all $t_1, t_2 \in [0, \infty]$.

From Theorem 3, we get that for any $t \in [0, \infty]$, $h_x(t) = s''_{n_0}(1)$; or $h_x(t) = 0$; or $h_x(t) = \begin{cases} 0, & t = 0 \\ s''_{n_0}(1), & t \in (0, \infty]; \end{cases}$ or $h_x(t) = \begin{cases} 0, & t \in (0, \infty] \\ s''_{n_0}(1), & t = \infty; \end{cases}$ or $h_x(t) = \min(p_{x,m_0} t, s''_{n_0}(1))$ with some constant $p_{x,m_0} \in (0, \infty)$. Because of the continuity of h_x in $(0, s'_{m_0}(\varphi(k_{x,m_0})))$, then for any $t \in (0, s'_{m_0}(\varphi(k_{x,m_0})))$, $h_x(t) = s''_{n_0}(1)$, $h_x(t) = 0$, or $h_x(t) = \min(p_{x,m_0} t, s''_{n_0}(1))$. This indicates that for any $y \in (a_{m_0}, k_{x,m_0})$, $I(x,y) = d_{n_0}$, $I(x,y) = c_{n_0}$, or $I(x,y) = \psi^{-1} \circ s''^{-1}_{n_0}(\min(p_{x,m_0} s'_{m_0} \circ \varphi(y), s''_{n_0}(1)))$.

Again observing that $I(x,y)$ is not an idempotent element of S_2 for any $y \in (a_{m_0}, k_{x,m_0})$, we have $I(x,y) = \psi^{-1} \circ s''^{-1}_{n_0}(p_{x,m_0} s'_{m_0} \circ \varphi(y))$ and $p_{x,m_0} s'_{m_0} \circ \varphi(y) < s''_{n_0}(1)$ for any $y \in (a_{m_0}, k_{x,m_0})$. Therefore, $p_{x,m_0} \leq \frac{s''_{n_0}(1)}{s'_{m_0} \circ \varphi(k_{x,m_0})}$. In the following, we will prove that $p_{x,m_0} = \frac{s''_{n_0}(1)}{s'_{m_0} \circ \varphi(k_{x,m_0})}$.

In fact, according to the continuity of $I(x, \cdot)$ on $[a_{m_0}, b_{m_0}]$, we get that $\lim_{y \to k_{x,m_0}^-} I(x,y) = I(x, k_{x,m_0})$, which shows that $\psi^{-1} \circ s''^{-1}_{n_0}(p_{x,m_0} s'_{m_0} \circ \varphi(k_{x,m_0})) = d_{n_0}$. Thus, $p_{x,m_0} = \frac{s''_{n_0}(1)}{s'_{m_0} \circ \varphi(k_{x,m_0})}$. Therefore, 1) is derived.

The proof of 2). Suppose both S'_{m_0} and S''_{n_0} are strict, i.e., N2 is satisfied. We get from Lemma 3 that $I(x,y)$ is a non-idempotent element of S_2 for any $y \in (a_{m_0}, b_{m_0})$. Obviously, $I(x, a_{m_0}) = c_{n_0}$ and $I(x, b_{m_0}) = d_{n_0}$.

For any $y, z \in (a_{m_0}, b_{m_0})$, similarly to the proof of 1), we get the equation

$$h_x(u + v) = h_x(u) + h_x(v), \quad u, v \in (0, \infty)$$

where $h_x : [0, \infty] \to [0, \infty]$ is given by $h_x = s''_{n_0} \circ \psi \circ I_x \circ \varphi^{-1} \circ s'^{-1}_{m_0}$, $u = s'_{m_0} \circ \varphi(y)$, and $v = s'_{m_0} \circ \varphi(z)$. Similarly to the proof of 1), we can use Theorem 2 to solve the above equation and get that $I(x,y) = c_{n_0} + (d_{n_0} - c_{n_0}) s''^{-1}_{n_0}(p_{x,m_0} s'_{m_0}(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}))$ with some constant $p_{x,m_0} \in (0,\infty)$.

In addition, similar to the proof of Theorem 10 in [22], it can be proved that the constant p_{x,m_0} is uniquely determined up to a positive multiplicative constant depending on constants for s'_{m_0} and s''_{n_0}.

The proof of 3). Suppose S'_{m_0} and S''_{n_0} are nilpotent, i.e., N3 is satisfied. According to Lemma 3, there exist two possible cases: One case is that $I(x,y)$ is not an idempotent element of S_2 for any $y \in (a_{m_0}, b_{m_0})$, and the other case is that there exist some $y,z \in (a_{m_0}, b_{m_0})$ such that $I(x,y)$ is a non-idempotent element of S_2 and $I(x,z)$ is an idempotent element of S_2.

If the case is the former, then just like the proof of 2), we can use Theorem 4 to solve the corresponding equation and then get $I(x,y) = c_{n_0} + (d_{n_0} - c_{n_0}) s''^{-1}_{n_0}(p_{x,m_0} s'_{m_0}(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}))$, where $p_{x,m_0} = \frac{s''_{n_0}(1)}{s'_{m_0}(1)}$.

If the case is the latter, we also can obtain, similarly to the proof of 1), that

$$I(x,y) = \begin{cases} E_x(y), & y \in [a_{m_0}, k_{x,m_0}) \\ d_{n_0}, & y \in [k_{x,m_0}, b_{m_0}] \end{cases}$$

for some $k_{x,m_0} \in (a_{m_0}, b_{m_0})$.

By combining the two cases, we get 3).

($\Leftarrow$) Suppose the conditions in 1) are satisfied. It is clear to get that $I(x,\cdot)$ is increasing and continuous on $[a_{m_0}, b_{m_0}]$ and $I(x,[a_{m_0}, b_{m_0}]) = [c_{n_0}, d_{n_0}]$. In the following, we will show that $I(x, S_1(y,z)) = S_2(I(x,y), I(x,z))$ for all $y,z \in [a_{m_0}, b_{m_0}]$.

Case 1. If at least one of y,z is an idempotent element of S_1, without loss of generality, we suppose $y = a_{m_0}$. Therefore, $I(x,y) = c_{n_0}$ is an idempotent element of S_2. Thus, we obtain that $I(x, S_1(y,z)) = I(x, \max(y,z)) = \max(I(x,y), I(x,z)) = S_2(I(x,y), I(x,z))$.

Case 2. If $y,z \in (a_{m_0}, b_{m_0})$, then it holds that $S_1(y,z) \in (a_{m_0}, b_{m_0})$ since S'_{m_0} is strict.

Case 2.1. If $S_1(y,z) \in (a_{m_0}, k_{x,m_0})$, then we obtain that $k_{x,m_0} > S_1(y,z) \geq \max(y,z)$, and consequently, it holds that $y,z < k_{x,m_0}$. Therefore, $I(x,y) = c_{n_0} + (d_{n_0} - c_{n_0}) s''^{-1}_{n_0}(\frac{s''_{n_0}(1)}{s'_{m_0}(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}})} s'_{m_0}(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}})) \in [c_{n_0}, d_{n_0}]$ and $I(x,z) = c_{n_0} + (d_{n_0} - c_{n_0}) s''^{-1}_{n_0}(\frac{s''_{n_0}(1)}{s'_{m_0}(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}})} s'_{m_0}(\frac{z-a_{m_0}}{b_{m_0}-a_{m_0}})) \in [c_{n_0}, d_{n_0}]$. Then we can have forms (6) and (7), shown at the bottom of the page.

Moreover, we get from $S_1(y,z) < k_{x,m_0}$ that $a_{m_0} + (b_{m_0} - a_{m_0})(s'^{-1}_{m_0}(s'_{m_0}(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}) + s'_{m_0}(\frac{z-a_{m_0}}{b_{m_0}-a_{m_0}})) < k_{x,m_0}$, which shows that

$$s'_{m_0}\left(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}\right) + s'_{m_0}\left(\frac{z-a_{m_0}}{b_{m_0}-a_{m_0}}\right) < s'_{m_0}\left(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}}\right)$$

and then that

$$\begin{aligned} &\frac{s''_{n_0}(1)}{s'_{m_0}\left(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}}\right)} \left(s'_{m_0}\left(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}\right) + s'_{m_0}\left(\frac{z-a_{m_0}}{b_{m_0}-a_{m_0}}\right)\right) \\ &\quad < \frac{s''_{n_0}(1)}{s'_{m_0}\left(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}}\right)} s'_{m_0}\left(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}}\right) = s''_{n_0}(1). \end{aligned}$$

Therefore, $S_2(I(x,y), I(x,z)) = c_{n_0} + (d_{n_0} - c_{n_0})(s''^{-1}_{n_0}(\frac{s''_{n_0}(1)}{s'_{m_0}(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}})}(s'_{m_0}(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}) + s'_{m_0}(\frac{z-a_{m_0}}{b_{m_0}-a_{m_0}}))))$. Thus, $S_2(I(x,y), I(x,z)) = I(x, S_1(y,z))$ holds.

Case 2.2. If $S_1(y,z) \in [k_{x,m_0}, b_{m_0})$, it follows that $I(x, S_1(y,z)) = d_{n_0}$. In this case, if $y,z \in (a_{m_0}, k_{x,m_0})$, then $S_2(I(x,y), I(x,z))$ has form (7). Again taking into account

$$\begin{aligned} I(x, S_1(y,z)) &= I\left(x, a_{m_0} + (b_{m_0} - a_{m_0})\left(s'^{-1}_{m_0}\left(s'_{m_0}\left(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}\right) + s'_{m_0}\left(\frac{z-a_{m_0}}{b_{m_0}-a_{m_0}}\right)\right)\right)\right) \\ &= c_{n_0} + (d_{n_0} - c_{n_0})(s''^{-1}_{n_0}\left(\frac{s''_{n_0}(1)}{s'_{m_0}\left(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}}\right)}\left(s'_{m_0}\left(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}\right) + s'_{m_0}\left(\frac{z-a_{m_0}}{b_{m_0}-a_{m_0}}\right)\right)\right) \end{aligned} \tag{6}$$

$$\begin{aligned} S_2(I(x,y), I(x,z)) &= c_{n_0} + (d_{n_0} - c_{n_0}) S''_{n_0}\left(\frac{I(x,y)-c_{n_0}}{d_{n_0}-c_{n_0}}, \frac{I(x,z)-c_{n_0}}{d_{n_0}-c_{n_0}}\right) \\ &= c_{n_0} + (d_{n_0} - c_{n_0})\left(s''^{-1}_{n_0}\left(\min\left(s''_{n_0}\left(\frac{I(x,y)-c_{n_0}}{d_{n_0}-c_{n_0}}\right) + s''_{n_0}\left(\frac{I(x,z)-c_{n_0}}{d_{n_0}-c_{n_0}}\right), s''_{n_0}(1)\right)\right)\right) \\ &= c_{n_0} + (d_{n_0} - c_{n_0}) \\ &\quad \times \left(s''^{-1}_{n_0}\left(\min\left(\frac{s''_{n_0}(1)}{s'_{m_0}\left(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}}\right)}\left(s'_{m_0}\left(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}\right) + s'_{m_0}\left(\frac{z-a_{m_0}}{b_{m_0}-a_{m_0}}\right)\right), s''_{n_0}(1)\right)\right)\right). \end{aligned} \tag{7}$$

that $S_1(y,z) \geq k_{x,m_0}$, we can easily obtain

$$\frac{s''_{n_0}(1)}{s'_{m_0}\left(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}}\right)}\left(s'_{m_0}\left(\frac{y-a_{m_0}}{b_{m_0}-a_{m_0}}\right)+s'_{m_0}\left(\frac{z-a_{m_0}}{b_{m_0}-a_{m_0}}\right)\right)$$
$$\geq \frac{s''_{n_0}(1)}{s'_{m_0}\left(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}}\right)} s'_{m_0}\left(\frac{k_{x,m_0}-a_{m_0}}{b_{m_0}-a_{m_0}}\right) = s''_{n_0}(1).$$

Therefore, it holds that $S_2(I(x,y), I(x,z)) = d_{n_0} = I(x, S_1(y,z))$. In this case, if at least one of y and z is greater than k_{x,m_0}, without loss of generality, we suppose that $y \geq k_{x,m_0}$ and then $I(x,y) = d_{n_0}$. Hence, $S_2(I(x,y), I(x,z)) = S_2(d_{n_0}, I(x,z)) = d_{n_0}$. As a consequence, $S_2(I(x,y), I(x,z)) = d_{n_0} = I(x, S_1(y,z))$.

If the conditions in 2) or 3) are satisfied, we can similarly get the result. □

Definition 4: Let $I : [0,1]^2 \to [0,1]$ be a binary function, and S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m, b_m, S'_m\rangle)_{m\in J_1}$ and $(\langle c_n, d_n, S''_n\rangle)_{n\in J_2}$, respectively. c, d ($c \leq d$) are two idempotent elements of S_2. Define Condition M as follows.

M1) I is continuous on $[0,1]^2$.

M2) I is increasing with respect to the second place.

M3) $I(x, S_1(y,z)) = S_2(I(x,y), I(x,z))$ for any $x, y, z \in [0,1]$.

M4) $I(1,0) = c, I(1,1) = d$.

Theorem 6: Let I, S_1, S_2, and M be the same as above. Suppose that there is not any summand of S_2 in $[c,d]$. Then I satisfies Condition M if and only if for any $x, y \in [0,1], I(x,y) = f(x,y)$, where $f : [0,1]^2 \to [\sup\{d_n|d_n \leq c, n \in J_2\}, \inf\{c_n|c_n \geq d, n \in J_2\}]$ is a continuous function, which is increasing in its second place and satisfies $f(1,0) = c$ and $f(1,1) = d$. In addition, for any $y \in [a_m, b_m]$ $(m \in J_1), f(x,y) = e_m(x), e_m : [0,1] \to [\sup\{d_n|d_n \leq c, n \in J_2\}, \inf\{c_n|c_n \geq d, n \in J_2\}]$ is a continuous function.

Proof: ($\Rightarrow$) We will show that Range(I)$\subseteq [\sup\{d_n|d_n \leq c, n \in J_2\}, \inf\{c_n|c_n \geq d, n \in J_2\}]$. In fact, it can be proved that for any $x \in [0,1]$ and $y \in [a_m, b_m]$ (any $m \in J_1$), $I(x,y)$ is an idempotent element of S_2. Otherwise, suppose that there exist some $x_0 \in [0,1], y_0 \in [a_{m_0}, b_{m_0}]$ (some $m_0 \in J_1$) such that $I(x_0, y_0)$ is a non-idempotent element of S_2. From Lemma 3, we can get that there exists some $n_0 \in J_2$ such that $I(x_0, [a_{m_0}, b_{m_0}]) = [c_{n_0}, d_{n_0}]$. Considering the continuity of I and using the result of Lemma 5, we have that $I(x, [a_{m_0}, b_{m_0}]) = [c_{n_0}, d_{n_0}]$ for any $x \in [0,1]$. Especially, $I(1, [a_{m_0}, b_{m_0}]) = [c_{n_0}, d_{n_0}]$. Therefore, $[c_{n_0}, d_{n_0}] \subseteq [c,d]$, which is a contradiction with the premise in the theorem. Because function I is continuous, we get that Range(I)$\subseteq [\sup\{d_n|d_n \leq c, n \in J_2\}, \inf\{c_n|c_n \geq d, n \in J_2\}]$.

Denote $f = I$. f is continuous because I is continuous. $f(1,0) = c$ and $f(1,1) = d$ since $I(1,0) = c$ and $I(1,1) = d$. In line with Corollary 1, we can get that $f(x,y) = e_m(x)$ for any $y \in [a_m, b_m]$ $(m \in J_1)$, where $e_m : [0,1] \to [\sup\{d_n|d_n \leq c, n \in J_2\}, \inf\{c_n|c_n \geq d, n \in J_2\}]$.

($\Leftarrow$) It is obvious that I is continuous and is increasing in its second place and satisfies $I(1,0) = c, I(1,1) = d$. In the following, we will check (2). Clearly, the range of I only contains idempotent elements of S_2. Therefore, $S_2(I(x,y), I(x,z)) = \max(I(x,y), I(x,z))$.

If at least one of y and z is an idempotent element of S_1, then it holds that $I(x, S_1(y,z)) = I(x, \max(y,z)) = \max(I(x,y), I(x,z)) = S_2(I(x,y), I(x,z))$.

If y, z are non-idempotent elements of S_1 such that $y \in [a_{m_1}, b_{m_1}], z \in [a_{m_2}, b_{m_2}]$ $(m_1 \neq m_2$ and $m_1, m_2 \in J_1)$, then we have $I(x, S_1(y,z)) = I(x, \max(y,z)) = \max(I(x,y), I(x,z)) = S_2(I(x,y), I(x,z))$.

If y, z are non-idempotent elements of S_1 such that $y, z \in [a_{m_3}, b_{m_3}]$ (some $m_3 \in J_1$), then $S_1(y,z) \in [a_{m_3}, b_{m_3}]$. Consequently, $I(x, S_1(y,z)) = e_{m_3}(x)$ and $S_2(I(x,y), I(x,z)) = S_2(e_{m_3}(x), e_{m_3}(x)) = e_{m_3}(x)$.

Thus, $I(x, S_1(y,z))= S_2(I(x,y), I(x,z))$ for any $x, y, z \in [0,1]$. □

If $c = d$ in the above theorem, then $I = c$ is a particular solution to (2). The following corollary is immediately obtained.

Corollary 2: If $c = d$, then $I(x,y) = c$ $(x, y \in [0,1])$ is a solution to (2).

Theorem 6 only considers the case in which there is no summand of S_2 in the interval $[I(1,0), I(1,1)]$. In the following, we will discuss the case in which there exist summands of S_2 in the interval $[I(1,0), I(1,1)]$. First, we introduce a concept of feasible correspondence.

Definition 5: Let S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m, b_m, S'_m\rangle)_{m\in J_1}$ and $(\langle c_n, d_n, S''_n\rangle)_{n\in J_2}$, respectively. Denote $A = \{\langle a_m, b_m, S'_m\rangle | m \in J_1\}$ and $B = \{\langle c_n, d_n, S''_n\rangle | n \in J_2\}$.

Define a partial order "$\leq_1$" on A as: $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle \leq_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ if and only if $a_{m_1} \leq a_{m_2}$ $(m_1, m_2 \in J_1)$.

Define a partial order "$\leq_2$" on B as: $\langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle \leq_2 \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$ if and only if $c_{n_1} \leq c_{n_2}$ $(n_1, n_2 \in J_2)$.

Denote $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ if $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle \leq_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ and $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle \neq \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$. Denote $\langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle <_2 \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$ if $\langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle \leq_2 \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$ and $\langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle \neq \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$.

Remark 3: Because (a_m, b_m) $(m \in J_1)$ are nonoverlapping open intervals of $[0,1]$, $(A, \leq_1)$ is a linearly ordered set. Similarly, $(B, \leq_2)$ is a linearly ordered set.

Definition 6: Suppose that A' is a subset of A and B' is a subset of B. Two summands $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle, \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle \in A$ satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ are called consecutive summands in A if there is no summand $\langle a_{m_3}, b_{m_3}, S'_{m_3}\rangle \in A$ such that $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_3}, b_{m_3}, S'_{m_3}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$.

Two summands $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle$, $\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle \in A'$ satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ are called consecutive summands in A' if there is no summand $\langle a_{m_3}, b_{m_3}, S'_{m_3}\rangle \in A'$ such that $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_3}, b_{m_3}, S'_{m_3}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$.

The consecutive summands in B and B' are defined similarly.

Remark 4: Two consecutive summands in A' may be not consecutive in A.

Now, we introduce the concept of feasible correspondence which plays a central role in describing the solvability of (2).

Definition 7: Suppose that c and d $(c < d)$ are two idempotent elements of S_2. Let $B' = \{\langle c_n, d_n, S''_n\rangle | [c_n, d_n] \subseteq [c, d]\} \neq \emptyset$ be a subset of B and $A' = \{\langle a_m, b_m, S'_m\rangle | m \in J'_1 \subseteq J_1\}$ be a subset of A satisfying card(A')=card(B'). Then a mapping F: $A' \to B'$ is called a feasible correspondence from A' to B' if the following conditions are satisfied.

C1) F is a one to one correspondence from A' to B'.

C2) $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ if and only if $F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) <_2 F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle)$ $(m_1, m_2 \in J'_1)$.

C3) Suppose $\langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$ and $\langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$ are two consecutive summands in B' satisfying $\langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle <_2 \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle, F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) = \langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$ and $F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle) = \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$ $(m_1, m_2 \in J'_1)$. If $d_{n_1} < c_{n_2}$, then $b_{m_1} < a_{m_2}$ and $\bigcup_{\langle a_m, b_m, S'_m\rangle \in M} [a_m, b_m] \neq [b_{m_1}, a_{m_2}]$, where $M = \{\langle a_m, b_m, S'_m\rangle \in A | \langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_m, b_m, S'_m\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle\}$ and $\bigcup_{\langle a_m, b_m, S'_m\rangle \in M} [a_m, b_m]$ is the union of the set $[a_m, b_m]_{m \in J_1}$ satisfying $\langle a_m, b_m, S'_m\rangle \in M$.

C4) If $F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle$ and S''_n is a strict *t*-conorm, then S'_m is a strict *t*-conorm $(m \in J'_1, \langle c_n, d_n, S''_n\rangle \in B')$.

Example 2:

1) Suppose that $S_1 = (\langle \frac{1}{5}, \frac{2}{5}, S_P\rangle, \langle \frac{2}{5}, \frac{3}{5}, S_P\rangle, \langle \frac{2}{3}, \frac{3}{4}, S_P\rangle, \langle \frac{3}{4}, \frac{5}{6}, S_L\rangle)$ and $S_2 = (\langle \frac{1}{8}, \frac{1}{4}, S_P\rangle, \langle \frac{1}{3}, \frac{1}{2}, S_P\rangle, \langle \frac{1}{2}, \frac{2}{3}, S_L\rangle)$. Choose $c = \frac{1}{16}$ and $d = \frac{5}{6}$. Then there are only two feasible correspondences F_1 and F_2.

1.1)

$$A' = \left\{\left\langle \frac{1}{5}, \frac{2}{5}, S_P\right\rangle, \left\langle \frac{2}{3}, \frac{3}{4}, S_P\right\rangle, \left\langle \frac{3}{4}, \frac{5}{6}, S_L\right\rangle\right\}$$

$$F_1\left(\left\langle \frac{1}{5}, \frac{2}{5}, S_P\right\rangle\right) = \left\langle \frac{1}{8}, \frac{1}{4}, S_P\right\rangle$$

$$F_1\left(\left\langle \frac{2}{3}, \frac{3}{4}, S_P\right\rangle\right) = \left\langle \frac{1}{3}, \frac{1}{2}, S_P\right\rangle$$

$$\text{and } F_1\left(\left\langle \frac{3}{4}, \frac{5}{6}, S_L\right\rangle\right) = \left\langle \frac{1}{2}, \frac{2}{3}, S_L\right\rangle.$$

1.2)

$$A' = \left\{\left\langle \frac{2}{5}, \frac{3}{5}, S_P\right\rangle, \left\langle \frac{2}{3}, \frac{3}{4}, S_P\right\rangle, \left\langle \frac{3}{4}, \frac{5}{6}, S_L\right\rangle\right\}$$

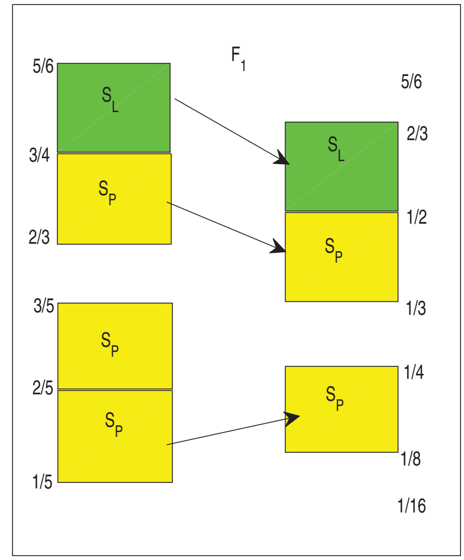
$$F_2\left(\left\langle \frac{2}{5}, \frac{3}{5}, S_P\right\rangle\right) = \left\langle \frac{1}{8}, \frac{1}{4}, S_P\right\rangle$$

$$F_2\left(\left\langle \frac{2}{3}, \frac{3}{4}, S_P\right\rangle\right) = \left\langle \frac{1}{3}, \frac{1}{2}, S_P\right\rangle$$

$$\text{and } F_2\left(\left\langle \frac{3}{4}, \frac{5}{6}, S_L\right\rangle\right) = \left\langle \frac{1}{2}, \frac{2}{3}, S_L\right\rangle.$$

The plots of F_1 and F_2 are presented in Fig. 1.

2) If $S_1 = (\langle \frac{1}{6}, \frac{1}{3}, S_L\rangle)$ and $S_2 = (\langle \frac{1}{3}, \frac{1}{2}, S_P\rangle)$. Choose $c = 0$ and $d = 1$. Obviously, $B' = B = \{\langle \frac{1}{3}, \frac{1}{2}, S_P\rangle\}$ and $A' = A = \{\langle \frac{1}{6}, \frac{1}{3}, S_L\rangle\}$. There does not exist any feasible correspondence F from A to B since C4 cannot be satisfied for F.



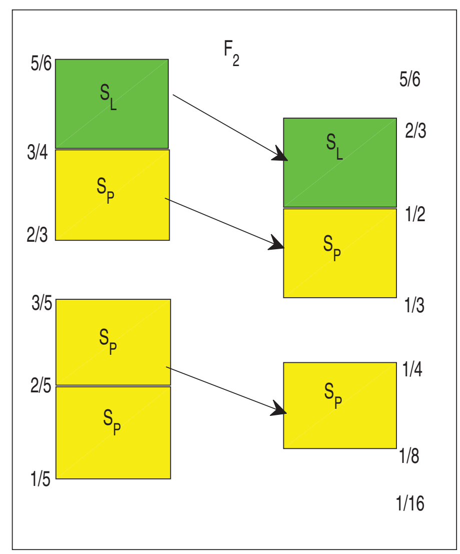


Fig. 1. Plots of F_1 and F_2.

The following theorem shows that the existence of feasible correspondences is necessary for the solvability of (2).

Theorem 7: Let S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m, b_m, S'_m\rangle)_{m \in J_1}$ and $(\langle c_n, d_n, S''_n\rangle)_{n \in J_2}$, respectively, and $I : [0,1]^2 \to [0,1]$ be a binary function with $I(x, \cdot)$ being increasing and continuous for each $x \in [0,1]$. Suppose c and d $(c < d)$ are two idempotent elements of S_2 and $B' = \{\langle c_n, d_n, S''_n\rangle | [c_n, d_n] \subseteq [c, d]\} \neq \emptyset$. If I satisfies $I(1, 0) = c$ and $I(1, 1) = d$ and (2) holds, then there exist a subset A' of A satisfying card(A') = card(B') and a feasible correspondence F from A' to B'.

Proof: Because $I(1, 0) = c, I(1, 1) = d$ and $I(1, \cdot)$ is continuous, then for any $\langle c_n, d_n, S''_n\rangle \in B'$, there exists a $y \in [0, 1]$ such that $I(1, y) \in (c_n, d_n)$. It follows from Lemma 1 that y is not an idempotent element of S_2. Therefore, there exists an (a_m, b_m) (some $m \in J_1$) such that $y \in (a_m, b_m)$. From Lemma 3, we obtain that $I(1, [a_m, b_m]) = [c_n, d_n]$. Just like the proof of Lemma 4, it is easy to prove that $[a_{m'}, b_{m'}] = [a_m, b_m]$ if $I(1, [a_{m'}, b_{m'}]) = [c_n, d_n]$ $(m' \in J_1)$. Therefore, for any $\langle c_n, d_n, S''_n\rangle \in B'$, there is a unique $[a_m, b_m](m \in J_1)$ such that $I(1, [a_m, b_m]) = [c_n, d_n]$.

Let us define $A' = \{\langle a_m, b_m, S'_m\rangle \in A|$ there exists a $\langle c_n, d_n, S''_n\rangle \in B'$ such that $I(1, [a_m, b_m]) = [c_n, d_n]\}$. Clearly, $A' \subseteq A$ and card(A')=card(B').

Define a mapping $F : A' \to B'$ by

$$F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle \text{ if } I(1, [a_m, b_m]) = [c_n, d_n].$$

It is obvious that F is a one-to-one correspondence from A' to B'. Therefore, C1 is satisfied.

If $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle$ and $\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ are two summands in A' satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle, F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) = \langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$ and $F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle) = \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$, then $a_{m_1} < a_{m_2}, I(1, [a_{m_1}, b_{m_1}]) = [c_{n_1}, d_{n_1}]$ and $I(1, [a_{m_2}, b_{m_2}]) = [c_{n_2}, d_{n_2}]$. Using Lemma 4, we have $c_{n_1} < c_{n_2}$. Hence, $F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) <_2 F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle)$. Because F is a bijection, we have $F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) <_2 F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle)$ if and only if $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$. Thus, C2 is satisfied.

Now, we will prove C3. Suppose that $\langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$ and $\langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$ are two consecutive summands in B' with $d_{n_1} < c_{n_2}, F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) = \langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$, and $F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle) = \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$. It holds that $b_{m_1} \le a_{m_2}$ because $I(1, b_{m_1}) = d_{n_1} < c_{n_2} = I(1, a_{m_2})$ and $I(1, \cdot)$ is increasing. Suppose $b_{m_1} = a_{m_2}$. Then $d_{n_1} = I(1, b_{m_1}) = I(1, a_{m_2}) = c_{n_2}$. This contradicts the assumption that $d_{n_1} < c_{n_2}$. Therefore, $b_{m_1} < a_{m_2}$.

Moreover, if $\bigcup_{\langle a_m, b_m, S'_m\rangle \in M} [a_m, b_m] = [b_{m_1}, a_{m_2}]$, then for each $\langle a_m, b_m, S'_m\rangle \in M$, we have from Lemma 1 and Corollary 1 that there exists a unique $e_{1,m} \in [d_{n_1}, c_{n_2}]$ such that $I(1, [a_m, b_m]) = e_{1,m}$. Therefore, $I(1, \bigcup_{\langle a_m, b_m, S'_m\rangle \in M} [a_m, b_m]) = \bigcup_{\langle a_m, b_m, S'_m\rangle \in M} e_{1,m}$ is a finite or countable infinite set. Then we have $I(1, [b_{m_1}, a_{m_2}]) \ne [d_{n_1}, c_{n_2}]$. This contradicts the continuity of $I(1, \cdot)$. Hence, we obtain that $\bigcup_{\langle a_m, b_m, S'_m\rangle \in M} [a_m, b_m] \ne [b_{m_1}, a_{m_2}]$.

Lemma 3 implies that C4 is satisfied. □

Definition 8: Let $I : [0, 1]^2 \to [0, 1]$ be a binary function, and S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m, b_m, S'_m\rangle)_{m \in J_1}$ and $(\langle c_n, d_n, S''_n\rangle)_{n \in J_2}$, respectively. Suppose $A' = \{\langle a_m, b_m, S'_m\rangle | m \in J'_1 \subseteq J_1\}$ is a subset of A, $B' = \{\langle c_n, d_n, S''_n\rangle | [c_n, d_n] \subseteq [c, d]\} \ne \emptyset$ (c and d are two idempotent elements of S_2 with $c < d$) and F is a feasible correspondence from A' to B'. Define Condition L as follows:

L1) I is continuous on $[0, 1]^2$.

L2) I is increasing with respect to the second place.

L3) $I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$ for any $x, y, z \in [0, 1]$.

L4) $I(1, [a_m, b_m]) = [c_n, d_n]$ if and only if $F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle$ $(m \in J'_1, \langle c_n, d_n, S''_n\rangle \in B')$.

L5) $I(1, 0) = c, I(1, 1) = d$.

The following proposition shows that the existence of feasible correspondences is also sufficient for the solvability of (2).

Proposition 2: Let I, S_1, S_2, and L be the same as in Definition 8. If I satisfies Condition L, then we have the following statements.

1) For each $\langle a_m, b_m, S'_m\rangle \in A'$ with $F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle, I$ has one of the following forms on $[0, 1] \times (a_m, b_m)$:

1.1) If S'_m is a strict *t*-conorm and S''_n is a nilpotent *t*-conorm with s'_m and s''_n being their respective additive generators, then

$$I(x, y) = \begin{cases} E(x, y), & y \in (a_m, k_m(x)) \\ d_n, & y \in [k_m(x), b_m) \end{cases} \tag{8}$$

where $k_m : [0, 1] \to (a_m, b_m)$ is a continuous function, and $E(x, y) = c_n + (d_n - c_n)s''^{-1}_n(\frac{s''_n(1)}{s'_m(\frac{k_m(x) - a_m}{b_m - a_m})} s'_m(\frac{y - a_m}{b_m - a_m}))$.

1.2) If S'_m and S''_n are nilpotent *t*-conorms with s'_m and s''_n being their respective additive generators, then

$$I(x, y) = \begin{cases} E(x, y), & y \in (a_m, k_m(x)) \\ d_n, & y \in [k_m(x), b_m) \end{cases} \tag{9}$$

where $k_m : [0, 1] \to (a_m, b_m]$ is a continuous function, and $E(x, y)$ is the same as in 1.1).

1.3) If S'_m and S''_n are strict *t*-conorms with s'_m and s''_n being their respective additive generators, then

$$I(x, y) = c_n + (d_n - c_n)s''^{-1}_n\left(p_m(x)s'_m\left(\frac{y - a_m}{b_m - a_m}\right)\right) \tag{10}$$

where $p_m : [0, 1] \to (0, \infty)$ is a continuous function.

2) Suppose that $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle$ and $\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ are two consecutive summands in A' satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle, F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) = \langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$ and $F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle) = \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$. Then I has one of the following forms on $[0, 1] \times [b_{m_1}, a_{m_2}]$.

2.1) If $d_{n_1} = c_{n_2}$, then $I(x, y) = d_{n_1}$.

2.2) If $d_{n_1} < c_{n_2}$, then $I(x, y) = f_{m_1,m_2}(x, y)$, where $f_{m_1,m_2} : [0, 1] \times [b_{m_1}, a_{m_2}] \to [d_{n_1}, c_{n_2}]$ is a continuous function. It is increasing in its second place and satisfies $f_{m_1,m_2}(x, b_{m_1}) = d_{n_1}$ and $f_{m_1,m_2}(x, a_{m_2}) = c_{n_2}$ for any $x \in [0, 1]$. In addition, for any $[a_m, b_m]$ $(m \in J_1)$ satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_m, b_m, S'_m\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$, it holds that $f_{m_1,m_2}(x, y) = e_m(x)$ on $[0, 1] \times [a_m, b_m]$, where $e_m : [0, 1] \to [d_{n_1}, c_{n_2}]$ is a continuous function.

3) Define $i_a = \inf\{a_m | \langle a_m, b_m, S'_m\rangle \in A'\}$ and $i_c = \inf\{c_n | \langle c_n, d_n, S''_n\rangle \in B'\}$. Then for any $(x, y) \in [0, 1] \times [0, i_a], I(x, y) = g(x, y)$, where $g : [0, 1] \times [0, i_a] \to [\sup\{d_n | [c_n, d_n] \subseteq [0, c], \langle c_n, d_n, S''_n\rangle \in B\}, i_c]$ is a continuous function. It is increasing in its second place and satisfies $g(1, 0) = c$ and $g(x, i_a) = i_c$ for any $x \in [0, 1]$. In addition, for any $[a_m, b_m] \subseteq [0, i_a]$ $(\langle a_m, b_m, S'_m\rangle \in A)$, it holds that $g(x, y) = e'_m(x)$ on $[0, 1] \times [a_m, b_m]$, where $e'_m : [0, 1] \to [\sup\{d_n | [c_n, d_n] \subseteq [0, c], \langle c_n, d_n, S''_n\rangle \in B\}, i_c]$ is a continuous function.

4) Define $s_b = \sup\{b_m | \langle a_m, b_m, S'_m\rangle \in A'\}$ and $s_d = \sup\{d_n | \langle c_n, d_n, S''_n\rangle \in B'\}$. Then for any $(x, y) \in [0, 1] \times [s_b, 1], I(x, y) = h(x, y)$, where $h : [0, 1] \times [s_b, 1] \to [s_d, \inf\{c_n | [c_n, d_n] \subseteq [d, 1], \langle c_n, d_n, S''_n\rangle \in B\}]$ is a continuous function. It is increasing in its second place and

satisfies $h(1,1) = d$ and $h(x, s_b) = s_d$ for any $x \in [0,1]$. In addition, for any $[a_m, b_m] \subseteq [s_b, 1]$ ($\langle a_m, b_m, S'_m\rangle \in A$), it holds that $h(x,y) = e''_m(x)$ on $[0,1] \times [a_m, b_m]$, where $e''_m : [0,1] \to [s_d, \inf\{c_n | [c_n, d_n] \subseteq [d,1], \langle c_n, d_n, S''_n\rangle \in B\}]$ is a continuous function.

Proof: The proof of 1). $I(1, [a_m, b_m]) = [c_n, d_n]$ since $F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle$. By Lemma 5, we obtain that $I(x, [a_m, b_m]) = [c_n, d_n]$ for any $x \in [0,1]$. Theorem 5 and the continuity of function I immediately imply 1).

The proof of 2). Suppose that $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle$ and $\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle \in A'$ are two consecutive summands satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$, $F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) = \langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$, and $F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle) = \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$. Then we get that $I(1, [a_{m_1}, b_{m_1}]) = [c_{n_1}, d_{n_1}]$ and $I(1, [a_{m_2}, b_{m_2}]) = [c_{n_2}, d_{n_2}]$, which together with Lemma 5 indicates that $I(x, [a_{m_1}, b_{m_1}]) = [c_{n_1}, d_{n_1}]$ and $I(x, [a_{m_2}, b_{m_2}]) = [c_{n_2}, d_{n_2}]$ for any $x \in [0,1]$. As a consequence, $I(x, a_{m_2}) = c_{n_2}$ and $I(x, b_{m_1}) = d_{n_1}$ for any $x \in [0,1]$. By the continuity of I, we have $I(x,y) \in [d_{n_1}, c_{n_2}]$ for any $x \in [0,1]$ and $y \in [b_{m_1}, a_{m_2}]$.

In this case, if $d_{n_1} = c_{n_2}$, then clearly it holds that $I(x,y) = d_{n_1}, x \in [0,1], y \in [b_{m_1}, a_{m_2}]$.

If $d_{n_1} < c_{n_2}$, then denote $f_{m_1, m_2} = I |_{[0,1]\times[b_{m_1}, a_{m_2}]}$. The properties of I imply that $f_{m_1,m_2} : [0,1] \times [b_{m_1}, a_{m_2}] \to [d_{n_1}, c_{n_2}]$ is a continuous function which is increasing in its second place and satisfies $f_{m_1,m_2}(x, b_{m_1}) = d_{n_1}$ and $f_{m_1,m_2}(x, a_{m_2}) = c_{n_2}$ for any $x \in [0,1]$.

We can prove that $[d_{n_1}, c_{n_2}]$ only contains idempotent elements of S_2. In fact, according to C1 and C2, we know that $\langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$ and $\langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$ are two consecutive summands in B'. By the definition of B', we have $\langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$ and $\langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$ are also two consecutive summands in B. Therefore, $[d_{n_1}, c_{n_2}]$ contains only idempotent elements of S_2. Then for any $[a_m, b_m]$ ($m \in J_1$) satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_m, b_m, S'_m\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$, we have from Corollary 1 that $I(x, [a_m, b_m]) = e_m(x) \in [d_{n_1}, c_{n_2}]$. Therefore, $f_{m_1,m_2}(x,y) = e_m(x)$ on $[0,1] \times [a_m, b_m]$, where $e_m : [0,1] \to [d_{n_1}, c_{n_2}]$ is a continuous function. Thus, 2) is derived.

The proof of 3). It is clear that $i_c \geq c \geq \sup\{d_n | [c_n, d_n] \subseteq [0,c], \langle c_n, d_n, S''_n\rangle \in B\}$.

For any $\langle a_m, b_m, S'_m\rangle \in A'$, there exists some $\langle c_n, d_n, S''_n\rangle \in B'$ such that $F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle$. Thus, $I(1, [a_m, b_m]) = [c_n, d_n]$. Lemma 5 indicates that $I(x, [a_m, b_m]) = [c_n, d_n]$ and $I(x, a_m) = c_n$ for any $x \in [0,1]$. Because I is continuous and F is a one to one mapping, we have that

$$\begin{aligned} I(x, i_a) &= I(x, \inf\{a_m | \langle a_m, b_m, S'_m\rangle \in A'\}) \\ &= \inf\{I(x, a_m) | \langle a_m, b_m, S'_m\rangle \in A'\} \\ &= \inf\{c_n | \langle c_n, d_n, S''_n\rangle \in B'\} = i_c \end{aligned}$$

which indicates that $I(x, i_a) = i_c$ and $I(x,y) \leq I(x, i_a) = i_c$ for any $x \in [0,1]$ and $y \in [0, i_a]$.

We will show that if $x \in [0,1]$ and $y \in [0, i_a]$, then $I(x,y) \geq \sup\{d_n | [c_n, d_n] \subseteq [0,c], \langle c_n, d_n, S''_n\rangle \in B\}$. First, let us prove that for any $x \in [0,1]$ and $y \in [0, i_a]$, $I(x,y)$ is an idempotent element of S_2.

Otherwise, suppose that there exist some $x_0 \in [0,1]$ and $y_0 \in [0, i_a]$ such that $I(x_0, y_0)$ is not an idempotent element of S_2. Obviously, there exists some $[a_{m'}, b_{m'}]$ ($m' \in J_1$) such that $y_0 \in (a_{m'}, b_{m'})$. According to Lemma 3, there exists a unique $[c_{n'}, d_{n'}]$ such that $I(x_0, [a_{m'}, b_{m'}]) = [c_{n'}, d_{n'}]$. It follows from Lemma 5 and the continuity of I that $I(x, [a_{m'}, b_{m'}]) = [c_{n'}, d_{n'}]$ for any $x \in [0,1]$. Especially, $I(1, [a_{m'}, b_{m'}]) = [c_{n'}, d_{n'}]$. Considering that $I(1,0) = c$ and $I(1,1) = d$, we get that $[c_{n'}, d_{n'}] \subseteq [c,d]$ and then $\langle c_{n'}, d_{n'}, S''_{n'}\rangle \in B'$, which means that $c_{n'} \geq i_c$. Then $I(x_0, y_0) > c_{n'} \geq i_c$ since $I(x_0, y_0) \in [c_{n'}, d_{n'}]$ is not an idempotent element of S_2. However, we have proved that $I(x_0, y) \leq I(x_0, i_a) = i_c$ for any $y \in [0, i_a]$. This produces a contradiction. Therefore, $I(x,y)$ is an idempotent element of S_2. Therefore, $I(x,y) \geq \sup\{d_n | [c_n, d_n] \subseteq [0,c], \langle c_n, d_n, S''_n\rangle \in B\}$ if $x \in [0,1]$ and $y \in [0, i_a]$.

From the above steps, one can see that for any $x \in [0,1]$ and any $y \in [0, i_a]$, $I(x,y) \in [\sup\{d_n | [c_n, d_n] \subseteq [0,c], \langle c_n, d_n, S''_n\rangle \in B\}, i_c]$. Denote $g = I |_{[0,1]\times[0,i_a]}$. Then $g : [0,1] \times [0, i_a] \to [\sup\{d_n | [c_n, d_n] \subseteq [0,c], \langle c_n, d_n, S''_n\rangle \in B\}, i_c]$ is a continuous function which is increasing in its second place. $g(1,0) = c$ and $g(x, i_a) = i_c$ ($x \in [0,1]$) since $I(1,0) = c$ and $I(x, i_a) = i_c$. According to Corollary 1, it holds that for any $[a_m, b_m] \subseteq [0, i_a]$ ($\langle a_m, b_m, S'_m\rangle \in A$), $g(x,y) = e'_m(x)$ on $[0,1] \times [a_m, b_m]$, where $e'_m : [0,1] \to [\sup\{d_n | [c_n, d_n] \subseteq [0,c], \langle c_n, d_n, S''_n\rangle \in B\}, i_c]$ is a continuous function. Thus, we have proved 3).

The proof of 4) is similar to that of 3). □

Theorem 8: Let I, S_1, S_2, and L be the same as in Definition 8. Then I satisfies Condition L if and only if I has the forms as in 1)–4) of Proposition 2.

Proof: The necessary proof is the one of Proposition 2. Only the sufficient proof remains to be considered.

First, we can prove that I is well defined on $[0,1]^2$. In fact, if $y \leq i_a$ or $y \geq s_b$, then $I(x,y)$ can be calculated by 3) or 4) in Proposition 2. If $i_a < y < s_b$, by the definition of i_a and s_b, there exist a_{m_1}, b_{m_2} ($m_1, m_2 \in J'_1$), such that $a_{m_1} < y < b_{m_2}$, then $I(x,y)$ can be calculated by 1) or 2) in Proposition 2.

It is easy to check L1, L2, and L5. In order to prove L3 and L4, we first prove that the ranges of f_{m_1,m_2}, g and h in 2)–4) of Proposition 2 only contain idempotent elements of S_2.

Suppose $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle$ and $\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ are two consecutive summands in A' satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$, $F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) = \langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$, $F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle) = \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$, and $d_{n_1} < c_{n_2}$. Just as the proof of 2) in Proposition 2, we can get that $[d_{n_1}, c_{n_2}]$ contains only idempotent elements of S_2. Therefore, the range of f_{m_1,m_2} contains only idempotent elements of S_2. In addition, it is easy to check that the ranges of g and h contain only idempotent elements of S_2. Consequently, if $I(x,y)$ has one of the forms in 2)–4), then $I(x,y)$ is an idempotent element of S_2.

In the following, we will verify L4. Suppose $F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle$ $(m \in J_1)$. Then I has form (8), form (9), or form (10) on $[0,1] \times (a_m, b_m)$. While from forms (8)–(10), we get that $I(1, (a_m, b_m)) = (c_n, d_n]$ (or (c_n, d_n)). Considering the continuity of I, we can have that $I(1, [a_m, b_m]) = [c_n, d_n]$.

Conversely, if $I(1, [a_m, b_m]) = [c_n, d_n]$, then it can proved that there exists some $n' \in J_2$ such that $F(\langle a_m, b_m, S'_m\rangle) = \langle c_{n'}, d_{n'}, S''_{n'}\rangle$. Otherwise, $I(1, y)$ $(y \in (a_m, b_m))$ has one of the forms in 2)–4). Therefore, $I(1, y)$ is an idempotent element of S_2 for any $y \in (a_m, b_m)$, which contradicts $I(1, [a_m, b_m]) = [c_n, d_n]$. Similarly to the previous proof of $F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle \Rightarrow I(1, [a_m, b_m]) = [c_n, d_n]$, we can obtain that $I(1, [a_m, b_m]) = [c_{n'}, d_{n'}]$. Therefore, $c_n = c_{n'}$ and $d_n = d_{n'}$. Thus, L4 has been proved.

Now, we will focus on the proof of L3, i.e., the proof of (2).

Case 1. Suppose at least one of y and z is an idempotent element of S_1. Without loss of generality, assume that y is an idempotent element of S_1. Thus, $y \notin (a_m, b_m)$ for any $\langle a_m, b_m, S'_m\rangle \in A'$. Therefore, for any $x \in [0,1], I(x, y)$ has one of the forms in 2)–4), and then $I(x, y)$ is an idempotent element of S_2. Consequently, $I(x, S_1(y, z)) = I(x, \max(y, z))$ $= \max(I(x, y), I(x, z)) = S_2(I(x, y), I(x, z))$.

Case 2. Suppose both y and z are non-idempotent elements of S_1 with $y \in (a_{m'}, b_{m'})$ and $z \in (a_{m''}, b_{m''})$ $(m' \neq m'')$. Then $S_1(y, z) = \max(y, z)$ and $I(x, S_1(y, z)) = I(x, \max(y, z)) = \max(I(x, y), I(x, z))$.

If $\langle a_{m'}, b_{m'}, S'_{m'}\rangle, \langle a_{m''}, b_{m''}, S'_{m''}\rangle \in A'$, then $F(\langle a_{m'}, b_{m'}, S'_{m'}\rangle) = \langle c_{n'}, d_{n'}, S''_{n'}\rangle \neq F(\langle a_{m''}, b_{m''}, S'_{m''}\rangle) = \langle c_{n''}, d_{n''}, S''_{n''}\rangle$. Hence, $I(x, y) \in [c_{n'}, d_{n'}]$ and $I(x, z) \in [c_{n''}, d_{n''}]$. As a consequence, $S_2(I(x, y), I(x, z)) = \max(I(x, y), I(x, z))$.

If $\langle a_{m'}, b_{m'}, S'_{m'}\rangle \notin A'$ or $\langle a_{m''}, b_{m''}, S'_{m''}\rangle \notin A'$, without loss of generality, then we suppose that $\langle a_{m'}, b_{m'}, S'_{m'}\rangle \notin A'$. This shows that $I(x, y) = e_{m'}(x)$ (or $e'_{m'}(x)$, or $e''_{m'}(x)$). Then $I(x, y)$ is an idempotent element of S_2, which implies that $S_2(I(x, y), I(x, z)) = \max(I(x, y), I(x, z)) = I(x, S_1(y, z))$.

Case 3. Suppose there exists some $m_0 \in J_1$ such that $y, z \in (a_{m_0}, b_{m_0})$. It is easy to get that $S_1(y, z) \in [a_{m_0}, b_{m_0}]$. If $\langle a_{m_0}, b_{m_0}, S'_{m_0}\rangle \in A'$, then just as the proof of Theorem 5, we can get the result. If $\langle a_{m_0}, b_{m_0}, S'_{m_0}\rangle \notin A'$, then $I(x, y) = I(x, z) = I(x, S_1(y, z)) = e_{m_0}(x)$ (or $e'_{m_0}(x)$, or $e''_{m_0}(x)$). Therefore, we have $S_2(I(x, y), I(x, z)) = I(x, S_1(y, z))$ since $e_{m_0}(x), e'_{m_0}(x)$ and $e''_{m_0}(x)$ are idempotent elements of S_2. $\square$

Remark 5: In the theorem above, by fixing $I(1, 0) = c$ and $I(1, 1) = d$, we get the continuous solutions to (2). This approach is different from the previous ones in [21]–[23], where the authors first derived the form of the vertical section $I(x, \cdot)$ for each fixed x, and then used the forms of the vertical sections to get its continuous solutions. However, the theorem above shows that we can get the continuous solutions to (2) by just fixing $I(1, 0)$ and $I(1, 1)$. Therefore, the method of finding the form of the vertical section $I(x, \cdot)$ for each fixed x in [21]–[23] is unnecessary in this study.

B. Fuzzy Implication Solutions to (2)

The general continuous solutions to (2) have been fully characterized in Theorems 6 and 8. Now, we turn to the fuzzy implication solutions.

Proposition 3: Let S_1 and S_2 be two *t*-conorms and I be a continuous fuzzy implication. If (2) holds, then $S_2 = \max$.

Proof: According to Lemma 1 and the fact that 0 is an idempotent element of S_1, we have that $I(x, 0)$ is an idempotent element of S_2 for any $x \in [0,1]$. Considering $I(0,0) = 1, I(1,0) = 0$ and the continuity of I, we obtain that for any $z \in [0,1]$, there exists some $x \in [0,1]$ such that $I(x, 0) = z$. Therefore, it holds that $S_2(z, z) = S_2(I(x, 0), I(x, 0)) = I(x, 0) = z$, namely, $S_2(z, z) = z$ for any $z \in [0,1]$. Then we obtain that $S_2 = \max$. $\square$

From Proposition 3, we see that there is no continuous fuzzy implication solution to (2) when S_2 is an ordinal sum. Therefore, if one wants to maintain the boundary conditions I1–I3, the continuity of implications on $[0,1] \times [0,1]$ should be dropped. Usually, we wish the continuous interval of implications to be as large as possible since the continuity can provide us with some convenience and robustness in practical applications. Therefore, in the sequel, we will focus on finding fuzzy implication solutions which are continuous on $(0,1] \times [0,1]$.

Letting $c = 0, d = 1$ (and thus $B' = B$) in Theorem 8, using the following definition of Condition R and considering the antitonicity of I with respect to the first place, we will obtain the fuzzy implication solutions to (2) (its fuzzy implication solutions can not be obtained from Theorem 6 because S_2 is an ordinal sum and there exist summands of S_2 in $[0,1]$).

Definition 9: Let I be a fuzzy implication, and S_1 and S_2 be two continuous *t*-conorms given by $(\langle a_m, b_m, S'_m\rangle)_{m \in J_1}$ and $(\langle c_n, d_n, S''_n\rangle)_{n \in J_2}$, respectively. Suppose $A' = \{\langle a_m, b_m, S'_m\rangle | m \in J'_1 \subseteq J_1\}$ is a subset of A, and F is a feasible correspondence from A' to B. Define Condition R as follows.

R1) I is continuous on $(0,1] \times [0,1]$.

R2) $I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$ for any $x, y, z \in [0,1]$.

R3) $I(1, [a_m, b_m]) = [c_n, d_n]$ if and only if $F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle$ $(m \in J'_1, \langle c_n, d_n, S''_n\rangle \in B)$.

Theorem 9: Let I, S_1, S_2, and R be the same as in Definition 9. Then fuzzy implication I satisfies Condition R if and only if I has forms as follows.

1) For any $\langle a_m, b_m, S'_m\rangle \in A'$ with $F(\langle a_m, b_m, S'_m\rangle) = \langle c_n, d_n, S''_n\rangle, I$ has one of the following forms on $(0,1] \times (a_m, b_m)$:

1.1) If S'_m is a strict *t*-conorm and S''_n is a nilpotent *t*-conorm with s'_m and s''_n being their respective additive generators, then

$$I(x, y) = \begin{cases} E(x, y), & y \in (a_m, k_m(x)) \\ d_n, & y \in [k_m(x), b_m) \end{cases} \tag{11}$$

where $k_m : (0,1] \to (a_m, b_m)$ is a continuous and increasing function, and $E(x, y) = c_n + (d_n - c_n) s_n''^{-1}\left(\frac{s''_n(1)}{s'_m\left(\frac{k_m(x) - a_m}{b_m - a_m}\right)} s'_m\left(\frac{y - a_m}{b_m - a_m}\right)\right)$.

1.2) If S'_m and S''_n are nilpotent *t*-conorms with s'_m and s''_n being their respective additive generators, then

$$I(x,y) = \begin{cases} E(x,y), & y \in (a_m, k_m(x)) \\ d_n, & y \in [k_m(x), b_m) \end{cases} \tag{12}$$

where $k_m : (0,1] \to (a_m, b_m]$ is a continuous and increasing function, and $E(x,y)$ is the same as in 1.1).

1.3) If S'_m and S''_n are strict *t*-conorms with s'_m and s''_n being their respective additive generators, then

$$I(x,y) = c_n + (d_n - c_n) s_n''^{-1}\left(p_m(x) s'_m\left(\frac{y-a_m}{b_m-a_m}\right)\right) \tag{13}$$

where $p_m : (0,1] \to (0,\infty)$ is a continuous and decreasing function.

2) Suppose that $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle$ and $\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$ are two consecutive summands in A' satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$, $F(\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle) = \langle c_{n_1}, d_{n_1}, S''_{n_1}\rangle$ and $F(\langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle) = \langle c_{n_2}, d_{n_2}, S''_{n_2}\rangle$. Then I has one of the following forms on $(0,1] \times [b_{m_1}, a_{m_2}]$.

2.1) If $d_{n_1} = c_{n_2}$, then $I(x,y) = d_{n_1}$.

2.2) If $d_{n_1} < c_{n_2}$, then $I(x,y) = f_{m_1,m_2}(x,y)$, where $f_{m_1,m_2} : (0,1] \times [b_{m_1}, a_{m_2}] \to [d_{n_1}, c_{n_2}]$ is a continuous function. It is decreasing in its first place and increasing in its second place and satisfies $f_{m_1,m_2}(x, b_{m_1}) = d_{n_1}$ and $f_{m_1,m_2}(x, a_{m_2}) = c_{n_2}$ for any $x \in (0,1]$. For any $[a_m, b_m]$ ($m \in J_1$) satisfying $\langle a_{m_1}, b_{m_1}, S'_{m_1}\rangle <_1 \langle a_m, b_m, S'_m\rangle <_1 \langle a_{m_2}, b_{m_2}, S'_{m_2}\rangle$, it holds that $f_{m_1,m_2}(x,y) = e_m(x)$ on $(0,1] \times [a_m, b_m]$, where $e_m : (0,1] \to [d_{n_1}, c_{n_2}]$ is a continuous and deceasing function.

3) Define $i_a = \inf\{a_m | \langle a_m, b_m, S'_m\rangle \in A'\}$ and $i_c = \inf\{c_n | \langle c_n, d_n, S''_n\rangle \in B\}$. Then for any $(x,y) \in (0,1] \times [0, i_a]$, $I(x,y) = g(x,y)$, where $g : (0,1] \times [0,i_a] \to [0,i_c]$ is a continuous function. It is deceasing in its first place and increasing in its second place and satisfies $g(1,0) = 0$ and $g(x, i_a) = i_c$ for any $x \in (0,1]$. In addition, for any $[a_m, b_m] \subseteq [0, i_a]$ ($\langle a_m, b_m, S'_m\rangle \in A$), it holds that $g(x,y) = e'_m(x)$ on $(0,1] \times [a_m, b_m]$, where $e'_m : (0,1] \to [0, i_c]$ is a continuous and deceasing function.

4) Define $s_b = \sup\{b_m | \langle a_m, b_m, S'_m\rangle \in A'\}$ and $s_d = \sup\{d_n | \langle c_n, d_n, S''_n\rangle \in B\}$. Then for any $(x,y) \in (0,1] \times [s_b, 1]$, $I(x,y) = h(x,y)$, where $h : (0,1] \times [s_b,1] \to [s_d,1]$ is a continuous function. It is deceasing in its first place and increasing in its second place and satisfies $h(1,1) = 1$ and $h(x, s_b) = s_d$ for any $x \in (0,1]$. In addition, for any $[a_m, b_m] \subseteq [s_b, 1]$ ($\langle a_m, b_m, S'_m\rangle \in A$), it holds that $h(x,y) = e''_m(x)$ on $(0,1] \times [a_m, b_m]$, where $e''_m : (0,1] \to [s_d, 1]$ is a continuous and deceasing function.

5) $I(0,y) = 1$ for any $y \in [0,1]$.

Proof: ($\Rightarrow$) The proofs of 1)–4) are similar to the ones of Proposition 2. Property 5) can be got directly from Definition 3.

($\Leftarrow$) If $(x,y) \in (0,1] \times [0,1]$, then the corresponding proofs are similar to the ones of Theorem 8. If $x = 0$ and $y \in [0,1]$, then (2) clearly holds since $I(0,y) = 1$. □

Algorithm of finding fuzzy implication solutions to (2): According to Theorems 7 and 9, we can summarize the method of finding fuzzy implication solutions to (2).

1) Find $A' \subseteq A$ and the corresponding $F : A' \to B$ such that F is a feasible correspondence from A' to B. If there is no such feasible correspondence, then, according to Theorem 7, there is no fuzzy implication solution to (2).

2) For each F, according to Theorem 9, find the fuzzy implication solutions with parameters to (2).

3) (optional) Choose proper parameters to construct the desired particular fuzzy implication solutions.

Example 3: Suppose that $S_1 = (\langle \frac{1}{6}, \frac{1}{3}, S_L\rangle)$ and $S_2 = (\langle \frac{1}{3}, \frac{1}{2}, S_P\rangle)$. According to 2) of Example 2, there is no feasible correspondence. Hence, (2) has no fuzzy implication solutions.

Example 4: Suppose that $S_1 = (\langle \frac{1}{8}, \frac{1}{4}, S_L\rangle, \langle \frac{1}{4}, \frac{1}{2}, S_P\rangle, \langle \frac{1}{2}, \frac{2}{3}, S_L\rangle, \langle \frac{5}{6}, \frac{11}{12}, S_P\rangle)$ and $S_2 = (\langle 0, \frac{1}{3}, S_P\rangle, \langle \frac{1}{2}, \frac{2}{3}, S_L\rangle)$ with $s(x) = -\ln(1-x)$ and $s(x) = x$ being two additive generators of S_P and S_L, respectively.

1) We can find A' and the feasible correspondence F_3 from A' to B.

$A' = \{\langle \frac{1}{4}, \frac{1}{2}, S_P\rangle, \langle \frac{5}{6}, \frac{11}{12}, S_P\rangle\}$, $F_3(\langle \frac{1}{4}, \frac{1}{2}, S_P\rangle) = \langle 0, \frac{1}{3}, S_P\rangle$ and $F_3(\langle \frac{5}{6}, \frac{11}{12}, S_P\rangle) = \langle \frac{1}{2}, \frac{2}{3}, S_L\rangle$.

The plot of F_3 can be seen in Fig. 2.

2) Now, we find fuzzy implication solutions with parameters.

For feasible correspondence F_3, we can get the representation of fuzzy implication I

$$I(x,y) = \begin{cases} 1, & x = 0, y \in [0,1] \\ 0, & x \in (0,1], y \in \left[0, \frac{1}{4}\right] \\ \frac{1}{3} - \frac{1}{3}(2-4y)^{p_2(x)}, & x \in (0,1], y \in \left(\frac{1}{4}, \frac{1}{2}\right) \\ f_{2,4}(x,y), & x \in (0,1], y \in \left[\frac{1}{2}, \frac{5}{6}\right] \\ \frac{1}{2} + \frac{1}{6}\frac{\ln(11-12y)}{\ln(11-12k_4(x))}, & x \in (0,1], y \in \left(\frac{5}{6}, k_4(x)\right) \\ \frac{2}{3}, & x \in (0,1], y \in \left[k_4(x), \frac{11}{12}\right) \\ h(x,y), & x \in (0,1], y \in \left[\frac{11}{12}, 1\right] \end{cases} \tag{14}$$

where $h : (0,1] \times [\frac{11}{12}, 1] \to [\frac{2}{3}, 1]$ and $f_{2,4} : (0,1] \times [\frac{1}{2}, \frac{5}{6}] \to [\frac{1}{3}, \frac{1}{2}]$ are continuous functions which are increasing in their second places and decreasing in their first places and satisfy $h(x, \frac{11}{12}) = \frac{2}{3}$, $h(1,1) = 1$, $f_{2,4}(x, \frac{1}{2}) = \frac{1}{3}$, $f_{2,4}(x, \frac{5}{6}) = \frac{1}{2}$, and $f_{2,4}(x,y) = e_3(x)$ on $(0,1] \times [\frac{1}{2}, \frac{2}{3}]$ with $e_3 : (0,1] \to [\frac{1}{3}, \frac{1}{2}]$ being a continuous and decreasing function; $p_2 : (0,1] \to (0,\infty)$ is a decreasing and continuous functions; $k_4 : (0,1] \to (\frac{5}{6}, \frac{11}{12})$ is an increasing and continuous function.

3) We can choose some desired parameters to get some particular solutions.

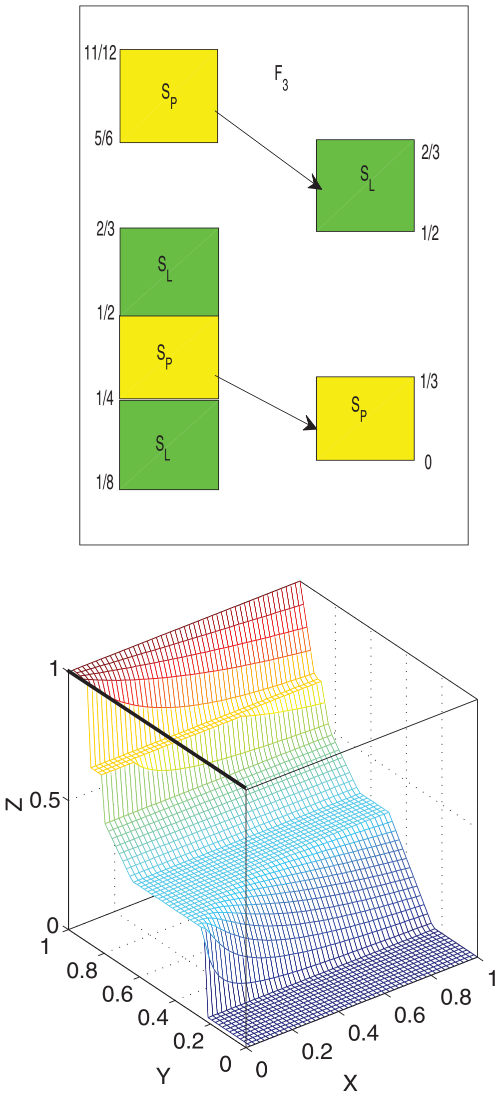


Fig. 2. Plots of F_3 and the particular fuzzy implication solution I in Example 4. The bold line is the line $I(0, y) = 1$.

For feasible correspondence F_3, take

$$f_{2,4}(x,y) = \begin{cases} \dfrac{1}{3}, & y \in \left[\dfrac{1}{2}, \dfrac{2}{3}\right] \\ y - \dfrac{1}{3}, & y \in \left(\dfrac{2}{3}, \dfrac{5}{6}\right] \end{cases} \quad (x,y) \in (0,1] \times \left[\dfrac{1}{2}, \dfrac{5}{6}\right]$$

$$h(x,y) = \frac{1}{3}(12y-11)^x + \frac{2}{3}, \quad (x,y) \in (0,1] \times \left[\frac{11}{12}, 1\right]$$

$$p_2(x) = \frac{1}{x}, \quad x \in (0,1]$$

$$k_4(x) = \frac{1}{24}x + \frac{5}{6}, \; x \in (0,1].$$

Then we get a particular solution which is presented in form (15)

$$I(x,y) = \begin{cases} 1, & x = 0, y \in [0,1] \\ 0, & x \in (0,1], y \in \left[0, \dfrac{1}{4}\right] \\ \dfrac{1}{3} - \dfrac{1}{3}(2-4y)^{\frac{1}{x}}, & x \in (0,1], y \in \left(\dfrac{1}{4}, \dfrac{1}{2}\right) \\ \dfrac{1}{3}, & x \in (0,1], y \in \left[\dfrac{1}{2}, \dfrac{2}{3}\right] \\ y - \dfrac{1}{3}, & x \in (0,1], y \in \left[\dfrac{2}{3}, \dfrac{5}{6}\right] \\ \dfrac{1}{2} + \dfrac{1}{6}\dfrac{\ln(11-12y)}{\ln\left(1-\frac{1}{2}x\right)}, & x \in (0,1], y \in \left(\dfrac{5}{6}, \dfrac{1}{24}x + \dfrac{5}{6}\right) \\ \dfrac{2}{3}, & x \in (0,1], y \in \left[\dfrac{1}{24}x + \dfrac{5}{6}, \dfrac{11}{12}\right) \\ \dfrac{1}{3}(12y-11)^x + \dfrac{2}{3}, & x \in (0,1], y \in \left[\dfrac{11}{12}, 1\right]. \end{cases} \tag{15}$$

Its plot is presented in Fig. 2.

IV. Conclusion

Baczyński and Jayaram [22] investigated (2) in the case where S_1 and S_2 are continuous Archimedean t-conorms and pointed out that, "In our future works, we will try to concentrate on some cases that are not considered in this paper, for example, when S_1 is a strict t-conorm and S_2 is a nilpotent t-conorm, and vice versa. Also, the situation when S_1 and S_2 are continuous t-conorms is still unsolved." In [21], Baczyński studied (2) in the case that S_1 is a strict t-conorm and S_2 is a nilpotent t-conorm, and *vice versa*. Xie *et al.* [23] solved (2) under the condition that S_2 is a continuous Archimedean t-conorm and S_1 is a continuous t-conorm given as an ordinal sum. In this paper, we discussed (2) with two continuous t-conorms given as ordinal sums. Clearly, this paper together with [21]–[23] completely resolved the problems in [22].

If there is no summand of S_2 in the interval $[I(1,0), I(1,1)]$, we get directly its general continuous solutions. In order to characterize the solvability of (2) if there are summands of S_2 in the interval $[I(1,0), I(1,1)]$, we introduced a new concept called feasible correspondence. Using feasible correspondences, we obtained the general continuous solutions to (2) and then got its fuzzy implication solutions which are continuous on $(0,1] \times [0,1]$. It should be pointed out that the method used in this paper can also be applied to the other three distributive equations which are closely related to (2).

The approach introduced in the paper can help reduce the number of rules in fuzzy systems. However, the approach seems to be a bit complicated, because it only gives the algorithm of finding fuzzy implication solutions to (2) but not its concrete fuzzy implication solutions. Our future work will focus on finding simpler methods to reduce the number of rules in fuzzy systems.

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