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On decomposable measures constructed by using stationary fuzzy pseudo-ultrametrics

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We prove that for a given stationary fuzzy ultrametric space (in the sense of Kramosil & Michalek) it can induce a σ-$\vee$-superdecomposable measure, by constructing a Hausdorff fuzzy pseudo-metric on its power set. We also prove that the restriction of the σ-$\vee$-superdecomposable measure to the σ-algebra of all measurable sets is a σ-$\vee$-decomposable measure. Finally we conclude this paper with two open problems.

Keywords: non-additive measures; decomposable measures; fuzzy metric spaces; *t*-norm; *t*-conorm

1. Introduction

The classical measure theory is one of the most important theories in mathematics, and it was extended, generalized, and deeply examined in many directions. For an exhaustive state-of-the-art overview we recommend the handbook edited by Pap (2002). Fuzzy measure (Sugeno, 1974) is an extension of the measure in the sense that the additivity of the measure is replaced with a weaker condition, the monotonicity. The non-additivity is the main characteristic of the fuzzy measure. So fuzzy measure is also called a non-additive measure or a monotone measure (Pap, 2002; Wang & Klir, 2009). There are many kinds of fuzzy measures (Pap, 2002, Wang & Klir, 2009): the Choquet capacity, the decomposable measure, the λ-additive measure, the belief measure, the plausibility measure, etc. We discuss mainly the decomposable measure among them.

The decomposable measure and has been independently introduced by Dubois & Prade (1982) and Weber (1984). Further developments of decomposable measures and related integrals have been done by many mathematicians (Ban, 2006; Ban & Gal, 2001; Dubois & Prade, 1996; Dubois, Prade, & Sabbadin, 2001; Hadžić & Pap, 2002; Klement, Mesiar, & Pap, 2000a; Pap, 1990a, 1990b, 1994, 1997, 2001; Saminger & Mesiar, 2003; Zhang & Guo, 1996). Decomposable measures include several well-known fuzzy measures such as λ-additive measure and probability and possibility measures, and they appear as a natural setting for relaxing the probabilistic assumptions regarding the modelling of uncertainty (Dubois & Prade, 1996; Dubois *et al.*, 2001). Decomposable measures and the corresponding integrals are very useful in decision theory (Ban, 2006, Dubois & Prade, 1996, Dubois *et al.*, 2001, Ichihashi & Tanaka, 1986, Saminger & Mesiar, 2003) and the theory of nonlinear differential and integral equations (Hadžić & Pap, 2002; Pap, 1997, 2001).

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The problem of constructing general measures in various application contexts is not one of generalized measure theory per se. It is rather a problem of knowledge acquisition (Wang & Klir, 2009). In this paper, we will give a method for inducing a σ-$\vee$-superdecomposable measure from a given stationary fuzzy ultrametric space (in the sense of Kramosil & Michalek (1975)) on its power set, by defining a Hausdorff fuzzy pseudo-metric on the power set. Furthermore we will prove that the restriction of the σ-$\vee$-superdecomposable measure to the σ-algebra of all measurable sets is a σ-$\vee$-decomposable measure. It should be noted that in our definition of the Hausdorff fuzzy pseudo-metric we remove the compactness condition which is necessary in the previous literatures (Repovš, Savchenko, & Zarichnyi, 2011, Rodríguez-López & Romaguera, 2004, Savchenko & Zarichnyi, 2009).

This paper is divided into four sections. In Section 2, we present some preliminary notions on t-norms and t-conorms, fuzzy metric spaces, and decomposable measures. In Section 3, we give the main results. Finally in Section 4, we conclude this paper with two open problems.

2. Preliminaries

Definition 2.1. (Klement, Mesiar, & Pap, 2000b) A triangular norm (t-norm for short) is a binary operation $\top$ on the unit interval [0,1], i.e. a function $\top:[0,1]^2 \to [0,1]$, such that for all $a,b,c,d \in [0,1]$ the following four axioms are satisfied:

(T-1) $a\top 1 = a$. (boundary condition)
(T-2) $a\top b \le c\top d$ whenever $a \le c$ and $b \le d$. (monotonicity)
(T-3) $a\top b = b\top a$. (commutativity)
(T-4) $a\top(b\top c) = (a\top b)\top c$. (associativity)

A t-norm $\top$ is said to be continuous if it is a continuous function in $[0,1]^2$. The following are examples of t-norms: $a\top_P b = a \cdot b$; $a \wedge b = \min(a,b)$.

Definition 2.2. (Klement *et al.*, 2000b) A triangular conorm (t-conorm for short) is a binary operation $\perp$ on the unit interval [0,1], i.e. a function $\perp:[0,1]^2 \to [0,1]$, such that for all $a,b,c,d \in [0,1]$ it satisfies (T-2)–(T-4) and

(S-1) $a \perp 0 = a$. (boundary condition).

A t-conorm $\perp$ is said to be continuous if it is a continuous function in $[0,1]^2$. The following are examples of t-conorms: $a \perp_P b = a + b - ab$; $a \vee b = \max(a,b)$.

Definition 2.3. (Weber, 1984) For any t-conorm $\perp$, the t-norm $\perp^*$ defined by

$$a \perp^* b = 1 - (1-a) \perp (1-b)$$

is called the dual t-norm of $\perp$. For any t-norm $\top$ by analogy $\top^*$.

It is easy to see that $\top_P^* = \perp_P$ and $\vee^* = \wedge$. Because of the associative property, the t-conorm $\perp$ can be extended by induction to n-ary operation by setting

$$\mathop{\perp}_{i=1}^{n} x_i = \left(\mathop{\perp}_{i=1}^{n-1} x_i\right) \perp x_n.$$

Due to monotonicity, for each sequence $(x_i)_{i\in\mathbb{N}}$ of elements of [0,1], the following limit can be considered:

$$\overset{\infty}{\underset{i=1}{\perp}} x_i = \lim_{n\to\infty} \overset{n}{\underset{i=1}{\perp}} x_i.$$

Definition 2.4. *(Kramosil & Michalek, 1975)* The 3-tuple $(X, M, \top)$ is said to be a fuzzy metric space if X is an arbitrary non-empty set, $\top$ is a t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $t, s > 0$:

(KM-1) $M(x, y, 0) = 0$,
(KM-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
(KM-3) $M(x, y, t) = M(y, x, t)$,
(KM-4) $M(x, z, t + s) \geq M(x, y, t) \top M(y, z, s)$,
(KM-5) $M(x, y, \cdot):[0, \infty) \to [0, 1]$ is left continuous.

Definition 2.5. The 3-tuple $(X, M, \top)$ is said to be a fuzzy pseudo-metric space if X is an arbitrary non-empty set, $\top$ is a t-norm, and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $t, s > 0$:

(KMp-2) $M(x, x, t) = 1$ for all $x \in X$,
(KM-3) $M(x, y, t) = M(y, x, t)$,
(KM-4) $M(x, z, t + s) \geq M(x, y, t) \top M(y, z, s)$,
(KM-5) $M(x, y, \cdot):[0, \infty) \to [0, 1]$ is left continuous.

If $(X, M, \top)$ is a fuzzy metric space, we say that $(M, \top)$, or M (if it is not necessary to mention $\top$), is a fuzzy metric on X. Also, we say that (X, M) or, simply, X is a fuzzy metric space.

Definition 2.6. (George & Veeramani, 1997) A sequence $(x_i)_{i\in\mathbb{N}}$ in a fuzzy metric space (X, M) is said to be Cauchy if $\lim_{i,j\to\infty} M(x_i, x_j, t) = 1$ for all $t > 0$; a sequence $(x_i)_{i\in\mathbb{N}}$ in X converges to x iff $\lim_{i\to\infty} M(x_i, x, t) = 1$ for all $t > 0$.

Definition 2.7. (Gregori, Morillas, & Sapena, 2011, Gregori & Romaguera, 2004) A fuzzy metric M on X is said to be stationary if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,\,y}(t) = M(x, y, t)$ is constant.

If M is a stationary fuzzy metric, we will simply write $M(x, y)$ instead of $M(x, y, t)$ if no confusion arises.

Definition 2.8. (Gregori, Morillas, & Sapena, 2010) Let $(X, M, \top)$ be a fuzzy metric space. The fuzzy metric M (or the fuzzy metric space $(X, M, \top)$) is said to be strong if it satisfies for each $x, y, z \in X$ and each $t > 0$

$$M(x, z, t) \geq M(x, y, t) \top M(y, z, t).$$

Obviously, stationary fuzzy metrics are strong. As a particular case, if $\top = \wedge$ and $(M, \top)$ is a strong fuzzy metric, we obtain the notion of fuzzy ultrametric (Gregori *et al.*, 2010; Savchenko & Zarichnyi, 2009):

$$M(x, z, t) \geq M(x, y, t) \wedge M(y, z, t).$$

We denoted the power set of X by $P(X)$. A non-empty subset R of $P(X)$ is called an algebra if for every $E, F \in R$,

$$E \cup F \in R \quad \text{and} \quad E^C \in R,$$

where E^C is the complement of E. A σ-algebra is an algebra which is closed under the formation of countable unions (Wang & Klir, 2009).

DEFINITION 2.9. Let $\perp$ be a t-conorm and R an algebra. A set function $\mu{:}R \to [0,1]$ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$ is

1. a $\perp$-superdecomposable measure iff $\mu(A \cup B) \geq \mu(A) \perp \mu(B)$;
2. a $\perp$-decomposable measure iff $\mu(A \cup B) = \mu(A) \perp \mu(B)$, for each pair (A, B) of disjoint elements of R (Weber, 1984).

   Furthermore, if R is a σ-algebra, then the set function $\mu{:}R \to [0,1]$ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$ is
3. a σ-$\perp$-superdecomposable measure iff

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) \geq \mathop{\perp}_{i=1}^{\infty} \mu(A_i);$$

4. a σ-$\perp$-decomposable measure if

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \mathop{\perp}_{i=1}^{\infty} \mu(A_i),$$

   for each sequence $(A_i)_{i\in\mathbb{N}}$ of disjoint elements of R (Weber, 1984).

3. Main results

Let B be a non-empty subset of a stationary fuzzy ultrametric space $(X, M, \wedge)$. For $x \in X$, let (Rodríguez-López & Romaguera 2004)

$$M(B, x) = M(x, B) := \sup_{y\in B} M(x, y).$$

For the empty index set $\emptyset$, we will make the convention that for $a_x \in [0,1]$,

$$\sup_{x\in\emptyset} a_x = 0 \quad \text{and} \quad \inf_{x\in\emptyset} a_x = 1,$$

which implies $M(x, \emptyset) = M(\emptyset, x) = 0$. We define a function H_M on $P(X) \times P(X)$ by

$$H_M(A, B) = \inf_{x\in A} M(x, B) \wedge \inf_{y\in B} M(A, y),$$

for all A, $B \in P(X)$ (Rodríguez-López & Romaguera 2004).

THEOREM 3.1. Let $(X, M, \wedge)$ be a stationary fuzzy ultrametric space. Then $(P(X), H_M, \wedge)$ is a stationary fuzzy pseudo-ultrametric space.

Proof. By the above convention, we have that

$$H_M(\emptyset, \emptyset) = \inf_{x\in\emptyset} M(x, \emptyset) \wedge \inf_{y\in\emptyset} M(\emptyset, y) = 1 \wedge 1 = 1,$$

and

$$H_M(\emptyset, A) = \inf_{x\in\emptyset} M(x, A) \wedge \inf_{y\in A} M(\emptyset, y) = 1 \wedge 0 = 0,$$

for all non-empty subsets $A \in P(X)$. In addition, it is clear that $H_M(A, A) = 1$ and $H_M(A, B) = H_M(B,A)$ for all $A, B \in P(X)$.

Let $A, B, C \in P(X)$. If at least one of the three sets is empty, then one can easily prove the triangle inequality. Thus, without loss of generality, suppose that the three sets are not empty. For any three points $x_0 \in A$, $y_0 \in B$, and $z_0 \in C$, we have that

$$M(x_0, y_0) \wedge M(y_0, z_0) \leq M(x_0, z_0),$$

which implies that

$$M(A, y_0) \wedge M(y_0, z_0) = \sup_{x\in A} M(x, y_0) \wedge M(y_0, z_0) \leq \sup_{x\in A} M(x, z_0) = M(A, z_0).$$

Consequently, we get that

$$\inf_{y\in B} M(A, y) \wedge M(y_0, z_0) \leq M(A, y_0) \wedge M(y_0, z_0) \leq M(A, z_0).$$

By the arbitrariness of y_0, we have that

$$\inf_{y\in B} M(A, y) \wedge M(B, z_0) = \inf_{y\in B} M(A, y) \wedge \sup_{y_0\in B} M(y_0, z_0) \leq M(A, z_0).$$

Then we have that

$$\inf_{y\in B} M(A, y) \wedge \inf_{z\in C} M(B, z) \leq \inf_{y\in B} M(A, y) \wedge M(B, z_0) \leq M(A, z_0),$$

which implies that

$$\inf_{y\in B} M(A, y) \wedge \inf_{z\in C} M(B, z) \leq \inf_{z_0\in C} M(A, z_0).$$

Similarly, we can get that

$$\inf_{y\in B} M(y, C) \wedge \inf_{x\in A} M(x, B) \leq \inf_{x_0\in A} M(x_0, C).$$

It follows that

$$\begin{aligned} H_M(A, B) \wedge H_M(B, C) &= \inf_{y\in B} M(A, y) \wedge \inf_{z\in C} M(B, z) \wedge \inf_{y\in B} M(y, C) \wedge \inf_{x\in A} M(x, B) \\ &\leq \inf_{z_0\in C} M(A, z_0) \wedge \inf_{x_0\in A} M(x_0, C) = H_M(A, C). \end{aligned}$$

We conclude that $(P(X), H_M, \wedge)$ is a stationary fuzzy pseudo-ultrametric space. □

Now we define a set function μ on $P(X)$ by

$$\mu(A) = 1 - H_M(X, A^C),$$

for all $A \in P(X)$.

THEOREM 3.2. The set function μ is a $\vee$-superdecomposable measure on $P(X)$.

Proof. It is easy to see that $\mu(\emptyset) = 0$ and $\mu(X) = 1$. Let $A, B \in P(X)$ with $A \subseteq B$. By the definition of μ, we have that

$$\mu(A) = 1 - H_M(X, A^C) = 1 - \inf_{x \in X} M(x, A^C) \wedge \inf_{y \in A^C} M(X, y) = 1 - \inf_{x \in X} M(x, A^C) \wedge 1$$

$$= 1 - \inf_{x \in X} (\sup_{y \in A^C} M(x, y)) \leq 1 - \inf_{x \in X} (\sup_{y \in B^C} M(x, y)) = 1 - \inf_{x \in X} M(x, B^C) \wedge 1$$

$$= 1 - \inf_{x \in X} M(x, B^C) \wedge \inf_{y \in B^C} M(X, y) = 1 - H_M(X, B^C) = \mu(B),$$

which shows that the set function μ is monotonous. Thus, for any two sets $A, B \in P(X)$, we have

$$\mu(A \cup B) \geq \mu(A) \vee \mu(B).$$

□

THEOREM 3.3. The set function μ is a σ-$\vee$-superdecomposable measure on $P(X)$.

Proof. Due to the monotonicity of μ, for each sequence $(A_i)_{i \in \mathbb{N}}$ of elements of $P(X)$ and every positive integer n, by mathematical induction we have that

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) \geq \bigvee_{i=1}^{n} \mu(A_i),$$

which implies that

$$\mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) \geq \bigvee_{i=1}^{\infty} \mu(A_i).$$

□

LEMMA 3.1. If $(A_i)_{i \in \mathbb{N}}$ is a sequence in $P(X)$ such that $\lim_{i \to \infty} A_i = A$, then $\lim_{i \to \infty} \mu(A_i) = \mu(A)$.

Proof. By the definition of μ, we have that

$$\lim_{i \to \infty} \mu(A_i) = 1 - \lim_{i \to \infty} H_M(X, A_i^C) \leq 1 - \lim_{i \to \infty} (H_M(X, A^C) \wedge H_M(A^C, A_i^C))$$

$$= 1 - H_M(X, A^C) \wedge \lim_{i \to \infty} H_M(A^C, A_i^C) = 1 - H_M(X, A^C) = \mu(A),$$

and analogously $\mu(A) \leq \lim_{i \to \infty} \mu(A_i)$. Thus we get $\lim_{i \to \infty} \mu(A_i) = \mu(A)$. □

DEFINITION 3.1. A set E in $P(X)$ is μ-measurable if, for every set A in $P(X)$,

$$\mu(A) = \mu(A \cap E) \vee \mu(A \cap E^C).$$

THEOREM 3.4. If $\mathbb{S}$ is the class of all μ-measurable sets, then $\mathbb{S}$ is an algebra.

Proof. It is easy to see that $\emptyset, X \in \mathbb{S}$, and that if $E \in \mathbb{S}$ then $E^C \in \mathbb{S}$. Let $E, F \in \mathbb{S}$ and $A \in P(X)$. It follows that

$$\begin{aligned}\mu(A \cap (E \cup F)) &= \mu(A \cap (E \cup F) \cap F) \vee \mu(A \cap (E \cup F) \cap F^C) \\ &= \mu(A \cap F) \vee \mu(A \cap E \cap F^C),\end{aligned}$$

which implies that

$$\begin{aligned}\mu(A \cap (E \cup F)) \vee \mu(A \cap (E \cup F)^C) &= \mu(A \cap F) \vee \mu(A \cap E \cap F^C) \vee \mu(A \cap (E \cup F)^C) \\ &= \mu(A \cap F) \vee \mu(A \cap F^C \cap E) \vee \mu(A \cap F^C \cap E^C) \\ &= \mu(A \cap F) \vee \mu(A \cap F^C) = \mu(A).\end{aligned}$$

Thus $\mathbb{S}$ is closed under the formation of union. □

THEOREM 3.5. $\mathbb{S}$ is a σ-algebra.

Proof. Let $A \in P(X)$ and $\{E_i\}_{i=1}^{\infty}$ be a disjoint sequence set in $\mathbb{S}$ with $\bigcup_{i=1}^{\infty} E_i = E$. It follows that

$$\begin{aligned}\mu(A \cap (E_1 \cup E_2)) &= \mu(A \cap (E_1 \cup E_2) \cap E_2) \vee \mu(A \cap (E_1 \cup E_2) \cap E_2^C) \\ &= \mu(A \cap E_2) \vee \mu(A \cap E_1 \cap E_2^C) = \mu(A \cap E_1) \vee \mu(A \cap E_2).\end{aligned}$$

By mathematical induction we can get that

$$\mu\left(A \cap \bigcup_{n}^{i=1} E_i\right) = \bigvee_{i=1}^{n} \mu(A \cap E_i),$$

for every positive integer n. It follows from the monotonicity of $\vee$ that

$$\begin{aligned}\mu(A) &= \mu\left(A \cap \bigcup_{i=1}^{n} E_i\right) \vee \mu\left(A \cap \left(\bigcup_{n}^{i=1} E_i\right)^C\right) \\ &= \left(\bigvee_{n}^{i=1} \mu(A \cap E_i)\right) \vee \mu\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)^C\right) \\ &\leq \left(\bigvee_{i=1}^{\infty} \mu(A \cap E_i)\right) \vee \mu\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)^C\right).\end{aligned}$$

By Lemma 3.1 and Theorem 3.3 we have that

$$\mu(A) \leq \left(\bigvee_{i=1}^{\infty} \mu(A \cap E_i)\right) \vee \mu\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)^C\right) \leq \mu(A \cap E) \vee \mu(A \cap E^C).$$

Moreover, it is noted that the reverse inequality, $\mu(A) \geq \mu(A \cap E) \vee \mu(A \cap E^C)$, is an automatic consequence of the superdecomposability of μ. It follows that $\mathbb{S}$ is closed under the formation of disjoint countable unions. Since every countable union of sets in an algebra may be written as a disjoint countable union of sets in the algebra, we see also that $\mathbb{S}$ is a σ-algebra. □

THEOREM 3.6. The restriction of set function μ to $\mathbb{S}$, $\mu|_{\mathbb{S}}$, is a σ-$\vee$-decomposable measure.

Proof. Let $A \in P(X)$ and $\{E_i\}_{i=1}^{\infty}$ be a disjoint sequence set in $\mathbb{S}$ with $\bigcup_{i=1}^{\infty} E_i = E$. From the proof of Theorem 3.5, we can get that

$$\left(\bigvee_{i=1}^{\infty} \mu(A \cap E_i)\right) \vee \mu(A \cap E^C) = \mu(A \cap E) \vee \mu(A \cap E^C).$$

Replacing A by E in the above equality, we have that

$$\left(\bigvee_{i=1}^{\infty} \mu(E_i)\right) = \mu(E).$$

Thus, $\mu|_{\mathbb{S}}$ is a σ-$\vee$-decomposable measure. □

4. Concluding remarks

For any given stationary fuzzy ultrametric space, we have proved that it can induce a σ-$\vee$-superdecomposable measure, by constructing a Hausdorff fuzzy pseudo-metric on its power set. We have also proved that the restriction of the σ-$\vee$-superdecomposable measure to the σ-algebra of all measurable sets is a σ-$\vee$-decomposable measure.

Let $(X, M, \top)$ be a stationary fuzzy strong metric space with a continuous t-norm $\top$. If we define a function H_M on $P(X) \times P(X)$ by

$$H_M(A, B) = \inf_{x \in A} M(x, B) \top \inf_{y \in B} M(A, y),$$

for all $A, B \in P(X)$, then by a similar proof of Theorem 3.1, we can get that $(P(X), H_M, \top)$ is a stationary fuzzy strong pseudo-metric space. We can also define a set function μ on $P(X)$ by

$$\mu(A) = 1 - H_M(X, A^C),$$

for all $A \in P(X)$. However, the following problems remain open.

Problem 1. Is μ a σ-$\top^*$-superdecomposable measure on $P(X)$?

Problem 2. Is the class of all μ-measurable sets with respect to $\top^*$ a σ-algebra?

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