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On the distributivity of fuzzy implications over continuous Archimedean t-conorms and continuous t-conorms given as ordinal sums ☆

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Abstract

In this paper, we investigate the distributive functional equation $I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$, where $I : [0, 1]^2 \to [0, 1]$ is an unknown function, S_2 a continuous Archimedean t-conorm and S_1 a continuous t-conorm given as an ordinal sum. First, based on the special case with one summand in the ordinal sum of S_1, all the sufficient and necessary conditions of solutions to the distributive equation above are given and the characterization of its continuous solutions is derived. It is shown that the distributive equation does not have continuous fuzzy implication solutions. Subsequently, we characterize its non-continuous fuzzy implication solutions. Finally, it is pointed out that the case with finite summands in the ordinal sum of S_1 is equivalent to the one with one summand.

Keywords: T-conorms; Fuzzy implications; Distributive equations; Functional equations

1. Introduction

Rule explosion in fuzzy systems is an important problem which hinders the further development of the fuzzy logical applications. Combs and Andrews [6–8] introduced the following classical tautology:

$$(p \wedge q) \to r = (p \to r) \vee (q \to r)$$

to avoid combinational rule explosion by reducing the complexity of fuzzy "IF–THEN" rules. And then, in the standard fuzzy sets theory, Trillas and Alsina [16] first transformed the above tautology into

$$I(T(x, y), z) = S(I(x, z), I(y, z)), \quad x, y, z \in [0, 1], \tag{1}$$

where I is a fuzzy implication, T a t-norm and S a t-conorm. Also they found out solutions T and S to Eq. (1) when I is an R-implication, an S-implication or a QL-implication. Later, some authors [13–15] generalized Eq. (1) into the case where T and S are replaced by uninorms and I is a fuzzy implication derived from uninorms.

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Balasubramaniam and Rao [5] posed the following dual equation of Eq. (1):

$$I(x, S_1(y, z)) = S_2(I(x, y), I(x, z)), \quad x, y, z \in [0, 1], \tag{2}$$

where I denotes a fuzzy implication and S_1, S_2 the t-conorms. They obtained that $S_1 = S_2 = \max$ when I is an S-implication or an R-implication derived from a nilpotent t-norm.

Baczyński and Jayaram [4] studied Eq. (2) based on two t-conorms which are both strict or both nilpoten. And they obtained the fuzzy implication solutions to Eq. (2). Baczyński [3] discussed Eq. (2) with a strict t-conorm and a nilpotent t-conorm. He [2] also studied the following distributive equation:

$$I(x, T(y, z)) = T(I(x, y), I(x, z)), \quad x, y, z \in [0, 1] \tag{3}$$

in the case that T is a strict t-norm. Following Baczyński's work, Qin and Yang [12] characterized the fuzzy implication solutions to Eq. (3) when T is a nilpotent t-norm. In the literature, Eq. (2) has not yet been discussed in the case that not both S_1 and S_2 are continuous Archimedean. Now this paper aims to find out the general fuzzy implication solutions to Eq. (2) under the condition that S_2 is a continuous Archimedean t-conorm and S_1 a continuous t-conorm given as an ordinal sum.

The rest of this work is organized as follows. Some basic definitions and theorems are reviewed in Section 2. In Section 3, we divide this section into two parts to characterize better the solutions to Eq. (2). One part studies Eq. (2) when t-conorm S_2 is strict and the other part discusses Eq. (2) when S_2 is nilpotent. We end this article with Conclusions, Acknowledgments and References.

2. Preliminaries

In this section, we briefly recall some of the concepts and results used in the sequel.

Definition 2.1 (*Klement et al. [10]*). A binary function $S : [0, 1]^2 \to [0, 1]$ is called a *triangular conorm* (t-conorm for short), if for all $x, y, z \in [0, 1]$, the following four axioms are satisfied:

(i) $S(x, y) = S(y, x)$ (commutativity),
(ii) $S(S(x, y), z) = S(x, S(y, z))$ (associativity),
(iii) $S(x, y) \le S(x, z)$, whenever $y \le z$ (monotonicity),
(iv) $S(x, 0) = x$ (boundary condition).

Example 2.1 (*Klement et al. [10]*). The following are the four basic t-conorms S_M, S_P, S_L and S_D given by, respectively:

$$S_M(x, y) = \max(x, y), \quad S_P(x, y) = x + y - xy,$$

$$S_L(x, y) = \min(x + y, 1), \quad S_D(x, y) = \begin{cases} \max(x, y), & x = 0 \text{ or } y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Definition 2.2 (*Klement et al. [10]*). A t-conorm S is called *continuous Archimedean* if it is continuous and satisfies $S(x, x) > x$ for all $x \in (0, 1)$.

A continuous t-conorm S is said to be *strict* if it holds that $S(x, y) < S(x, z)$ for all $x < 1$ and $y < z$. A t-conorm S is said to be *nilpotent* if it is continuous and for any $x \in (0, 1)$, there exists some $n \in \mathcal{N}$ such that $x_S^n = 1$, where $x_S^n = \underbrace{S(x, x, \ldots, x)}_{n \text{ times}}$.

We all know that any continuous Archimedean t-conorm is ether strict or nilpotent.

Theorem 2.1 (*Klement et al. [10] and Ling [11]*). *For a binary function $S : [0, 1]^2 \to [0, 1]$, the following statements are equivalent*:

(i) *S is a continuous Archimedean t-conorm.*

(ii) *S has a continuous additive generator, i.e., there exists a continuous and strictly increasing function* $s: [0,1] \to [0,\infty]$ *with* $s(0) = 0$, *which is uniquely determined up to a positive multiplicative constant, such that* $S(x, y) = s^{(-1)}(s(x) + s(y))$ *for all* $x, y \in [0, 1]$, *where* $s^{(-1)}$ *is the pseudo-inverse of S, given by*

$$s^{(-1)}(x) = \begin{cases} s^{-1}(x), & x \in [0, s(1)], \\ 1, & x \in [s(1), \infty]. \end{cases}$$

Remark 2.1.

(i) Without the pseudo-inverse, the representation of a t-conorm S in Theorem 2.1 can be rewritten as $S(x, y) = s^{-1}(\min(s(x) + s(y), s(1)))$ for all $x, y \in [0, 1]$.
(ii) A t-conorm S is strict if and only if each continuous additive generator s of S satisfies $s(1) = \infty$.
(iii) A t-conorm S is nilpotent if and only if each continuous additive generator s of S satisfies $s(1) < \infty$.

Definition 2.3 (*Klement et al. [10]*). Let $\{S_m\}_{m\in A}$ be a family of t-conorms and $\{(a_m, b_m)\}_{m\in A}$ be a family of nonempty, pairwise disjoint and open subintervals of [0, 1], where A is a finite or countable infinite index set. Then the function $S: [0,1]^2 \to [0,1]$ defined by

$$S(x, y) = \begin{cases} a_m + (b_m - a_m)S_m\left(\dfrac{x - a_m}{b_m - a_m}, \dfrac{y - a_m}{b_m - a_m}\right), & (x, y) \in [a_m, b_m]^2, \\ \max(x, y) & \text{otherwise} \end{cases}$$

is a t-conorm which is called the *ordinal sum* of summands $\langle a_m, b_m, S_m\rangle_{m\in A}$, and we write $S = (\langle a_m, b_m, S_m\rangle)_{m\in A}$.

Remark 2.2 (*Klement et al. [10]*). A t-conorm S is continuous if and only if S is either S_M, a continuous Archimedean t-conorm or an ordinal sum $(\langle a_m, b_m, S_m\rangle)_{m\in A}$ with S_m being continuous Archimedean.

Definition 2.4 (*Fodor and Roubens [9]*). A binary function $I: [0,1]^2 \to [0,1]$ is said to be a *fuzzy implication* if it satisfies the following:

(I1) the first place antitonicity,
(I2) the second place isotonicity, and
(I3) $I(0, 0) = I(1, 1) = I(0, 1) = 1$ and $I(1, 0) = 0$.

From the definition above, it is clear that each fuzzy implication I satisfies $I(0, x) = I(x, 1) = 1$ for all $x \in [0, 1]$.

Now, let us recall some facts about the additive Cauchy functional equation (see [1]) and equations similar to the additive Cauchy functional equation.

Theorem 2.2 (*Baczyński and Jayaram [4]*). *For a function* $f: [0,\infty] \to [0,\infty]$, *the following statements are equivalent*:

(i) *f satisfies the additive Cauchy functional equation*

$$f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in [0, \infty].$$

(ii) *Either* $f = \infty$, *or* $f = 0$, *or*

$$f(x) = \begin{cases} 0, & x = 0, \\ \infty, & x \in (0, \infty], \end{cases}$$

or

$$f(x) = \begin{cases} 0, & x \in [0, \infty), \\ \infty, & x = \infty, \end{cases}$$

or there exists a unique constant $c \in (0, \infty)$ *such that* $f(x) = cx$ *for all* $x \in [0, \infty]$.

Clearly, if f is continuous, then $f = \infty$, *or* $f = 0$, *or* $f(x) = cx$ *with a unique constant* $c \in (0, \infty)$.

Theorem 2.3 (*Baczyński [3]*). *Fix real $a > 0$. For a function $f : [0, a] \to [0, \infty]$, the following are equivalent*:

(i) *f satisfies the functional equation*

$$f(\min(x + y, a)) = f(x) + f(y) \quad for\ all\ x, y \in [0, a].$$

(ii) *Either $f = \infty$, or $f = 0$, or*

$$f(x) = \begin{cases} 0, & x = 0, \\ \infty, & x \in (0, a]. \end{cases}$$

Theorem 2.4 (*Baczyński [3]*). *Fix real $b > 0$. For a function $f : [0, \infty] \to [0, b]$, the following are equivalent*:

(i) *f satisfies the functional equation $f(x + y) = \min(f(x) + f(y), b)$ for all $x, y \in [0, \infty]$.*
(ii) *Either $f = b$, or $f = 0$, or*

$$f(x) = \begin{cases} 0, & x = 0, \\ b, & x \in (0, \infty], \end{cases}$$

or

$$f(x) = \begin{cases} 0, & x \in [0, \infty), \\ b, & x = \infty, \end{cases}$$

or there exists a unique constant $c \in (0, \infty)$ such that $f(x) = \min(cx, b)$ for all $x \in [0, \infty]$.

Theorem 2.5 (*Baczyński and Jayaram [4]*). *Fix real $a, b > 0$. For a function $f : [0, a] \to [0, b]$, the following statements are equivalent*:

(i) *f satisfies the functional equation $f(\min(x + y, a)) = \min(f(x) + f(y), b)$ for all $x, y \in [0, a]$.*
(ii) *Either $f = b$, or $f = 0$, or*

$$f(x) = \begin{cases} 0, & x = 0, \\ b, & x \in (0, a], \end{cases}$$

or there exists a unique constant $c \in [b/a, \infty)$ such that $f(x) = \min(cx, b)$ for all $x \in [0, a]$.

3. Solutions to Eq. (2)

In this section, we study Eq. (2) in detail in the case that S_2 is a continuous Archimedean t-conorm and S_1 is a continuous t-conorm given as an ordinal sum with only one summand in its sum. The case with more summands in the sum of S_1 is investigated later in Remark 3.2.9. Based on whether S_2 is strict or nilpotent, we divide this section into two parts.

3.1. Case of that S_2 is strict

Theorem 3.1.1. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ $(0 < a < b < 1)$, where S is a strict t-conorm with an additive generator s. Let S_2 be a strict t-conorm with an additive generator s_2 and $I : [0, 1]^2 \to [0, 1]$ be a binary function. Then the triple of functions S_1, S_2 and I satisfies Eq.* (2) *if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied*:

(i) $I(x, y) = 1$ *for all* $y \in [0, 1]$,
(ii) $I(x, y) = 0$ *for all* $y \in [0, 1]$,
(iii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x], \\ 1, & y \in (d_x, 1], \end{cases}$ *for some* $d_x \in [0, a)$,

(iv) $I(x,y)=\begin{cases}0, & y\in[0,d_x),\\ 1, & y\in[d_x,1],\end{cases}$ *for some* $d_x\in(0,a)$,

(v) $I(x,y)=\begin{cases}0, & y\in[0,e_x],\\ 1, & y\in(e_x,1],\end{cases}$ *for some* $e_x\in(b,1)$,

(vi) $I(x,y)=\begin{cases}0, & y\in[0,e_x),\\ 1, & y\in[e_x,1],\end{cases}$ *for some* $e_x\in(b,1]$,

(vii) $I(x,y)=0$ *for* $y\in[0,a)$, $I(x,y)=1$ *for* $y\in(b,1]$, *and for* $y\in[a,b]$, *the vertical section* $I(x,\cdot)$ *has one of the following representations*:

$$I(x,y)=0,$$

$$I(x,y)=1,$$

$$I(x,y)=\begin{cases}0, & y=a,\\ 1, & a<y\le b,\end{cases}$$

$$I(x,y)=\begin{cases}0, & a\le y<b,\\ 1, & y=b,\end{cases}$$

$$I(x,y)=s_2^{-1}\left(c_x s\left(\frac{y-a}{b-a}\right)\right), \tag{4}$$

where $c_x\in(0,\infty)$ *is some constant related to* x, *uniquely determined up to a positive multiplicative constant depending on constants for* s_2 *and* s.

Proof. ($\Leftarrow$) It is easy to know that the result is true if the conditions in (i) and (ii) are satisfied.

Let us fix arbitrarily $x\in[0,1]$ and suppose that

$$I(x,y)=\begin{cases}0, & y\in[0,e_x),\\ 1, & y\in[e_x,1],\end{cases}\quad e_x\in(b,1].$$

In this case, if $\max(y,z)\in[e_x,1]$, without loss of generality, we suppose $y\in[e_x,1]$. Then $S_1(y,z)=\max(y,z)\in[e_x,1]$. So we have that $I(x,S_1(y,z))=I(x,\max(y,z))=1$ and $S_2(I(x,y),I(x,z))=S_2(1,I(x,z))=1$.

If $\max(y,z)\in[0,e_x)$, i.e., $y,z\in[0,e_x)$, then $S_2(I(x,y),I(x,z))=S_2(0,0)=0$. In this situation, if $\max(y,z)\in[0,b]$, i.e., $y,z\le b$, then $S_1(y,z)\le S_1(b,b)=b<e_x$, which implies that $I(x,S_1(y,z))=0$. If $\max(y,z)\in(b,e_x)$, then $S_1(y,z)=\max(y,z)<e_x$, which indicates that $I(x,S_1(y,z))=0$.

Therefore, we have proven $I(x,S_1(y,z))=S_2(I(x,y),I(x,z))$ under the conditions in (vi). Similarly, we can prove the result if the conditions in (iii)–(v) are satisfied.

In the following we will discuss Eq. (2) when the conditions in (vii) are satisfied. We get from (vii) that

$$I(x,y)=\begin{cases}0, & y\le b,\\ 1, & y>b,\end{cases}$$

$$I(x,y)=\begin{cases}0, & y<a,\\ 1, & y\ge a,\end{cases}$$

$$I(x,y)=\begin{cases}0, & y\le a,\\ 1, & y>a,\end{cases}$$

$$I(x,y)=\begin{cases}0, & y<b,\\ 1, & y\ge b,\end{cases}$$

and

$$I(x,y)=\begin{cases}0, & y\in[0,a),\\ s_2^{-1}\left(c_x s\left(\dfrac{y-a}{b-a}\right)\right), & y\in[a,b],\\ 1, & y\in(b,1],\end{cases}$$

with some constant $c_x \in (0,\infty)$.

Now suppose that for the arbitrarily fixed x,

$$I(x,y)=\begin{cases}0, & y<b,\\ 1, & y\geq b.\end{cases}$$

If $\max(y,z)\geq b$, without loss of generality, we suppose that $y\geq b$. Then $S_1(y,z)=\max(y,z)\geq b$. This means that $I(x,S_1(y,z))=1$ and $S_2(I(x,y),I(x,z))=S_2(1,I(x,z))=1$.

If $\max(y,z)<b$, i.e., $y,z<b$, then $S_2(I(x,y),I(x,z))=S_2(0,0)=0$. In this case, when $y\in[0,a)$ and $z\in[0,b)$, we have that $S_1(y,z)=\max(y,z)<b$, which shows that $I(x,S_1(y,z))=0$. When $y\in[a,b)$ and $z\in[0,a)$, we get that $S_1(y,z)=\max(y,z)=y$ and $I(x,S_1(y,z))=I(x,y)=0$. When $y\in[a,b)$ and $z\in[a,b)$, it holds that $S_1(y,z)=a+(b-a)S((y-a)/(b-a),(z-a)/(b-a))$. Again noticing that t-conorm S is strict, we have that S has no zero elements in $[0,1)^2$, i.e., $S((y-a)/(b-a),(z-a)/(b-a))<1$ for all $y,z\in[a,b)$. Hence, $S_1(y,z)=a+(b-a)S((y-a)/(b-a),(z-a)/(b-a))<b$ and then $I(x,S_1(y,z))=0$. Thus, if $\max(y,z)<b$, then $I(x,S_1(y,z))=S_2(I(x,y),I(x,z))$.

As a consequence, if

$$I(x,y)=\begin{cases}0, & y<b,\\ 1, & y\geq b,\end{cases}$$

then $I(x,S_1(y,z))=S_2(I(x,y),I(x,z))$ always holds. In a similar way, we can verify that Eq. (2) holds if for the arbitrarily fixed x,

$$I(x,y)=\begin{cases}0, & y\leq b,\\ 1, & y>b,\end{cases}$$

or

$$I(x,y)=\begin{cases}0, & y<a,\\ 1, & y\geq a,\end{cases}$$

or

$$I(x,y)=\begin{cases}0, & y\leq a,\\ 1, & y>a.\end{cases}$$

Let us suppose that for the arbitrarily fixed x, the vertical section $I(x,\cdot)$ admits form

$$I(x,y)=\begin{cases}0, & y\in[0,a),\\ s_2^{-1}\left(c_x s\left(\dfrac{y-a}{b-a}\right)\right), & y\in[a,b],\\ 1, & y\in(b,1].\end{cases}$$

If $y\in[0,a)$ and $z\in[0,a)$, then we get that $I(x,S_1(y,z))=I(x,\max(y,z))=0$ and $S_2(I(x,y),I(x,z))=S_2(0,0)=0$. If $y\in[0,a)$ and $z\in[a,1]$, then $I(x,S_1(y,z))=I(x,\max(y,z))=I(x,z)$ and $S_2(I(x,y),I(x,z))=S_2(0,I(x,z))=I(x,z)$.

If $y\in(b,1]$ and $z\in[0,b]$, then we have that $I(x,S_1(y,z))=I(x,\max(y,z))=I(x,y)=1$ and $S_2(I(x,y),I(x,z))=S_2(1,I(x,z))=1$. If $y\in(b,1]$ and $z\in(b,1]$, it holds that $I(x,S_1(y,z))=I(x,\max(y,z))=1$ and $S_2(I(x,y),I(x,z))=S_2(1,1)=1$.

If $y \in [a, b]$ and $z \in [0, a)$, it is true that $I(x, S_1(y, z)) = I(x, \max(y, z)) = I(x, y)$ and $S_2(I(x, y), I(x, z)) = S_2(I(x, y), 0) = I(x, y)$. If $y \in [a, b]$ and $z \in (b, 1]$, it follows that $I(x, S_1(y, z)) = I(x, \max(y, z)) = I(x, z) = 1$ and $S_2(I(x, y), I(x, z)) = S_2(I(x, y), 1) = 1$. If $y \in [a, b]$ and $z \in [a, b]$, then the left side of Eq. (2) is equal to

$$\begin{aligned} I(x, S_1(y, z)) &= I\left(x, a + (b-a)S\left(\frac{y-a}{b-a}, \frac{z-a}{b-a}\right)\right) \\ &= s_2^{-1}\left(c_x s\left(S\left(\frac{y-a}{b-a}, \frac{z-a}{b-a}\right)\right)\right) \\ &= s_2^{-1}\left(c_x\left(s\left(\frac{y-a}{b-a}\right) + s\left(\frac{z-a}{b-a}\right)\right)\right), \end{aligned}$$

since $S_1(y, z) \in [a, b]$. And the right one of it is

$$\begin{aligned} S_2(I(x, y), I(x, z)) &= S_2\left(s_2^{-1}\left(c_x s\left(\frac{y-a}{b-a}\right)\right), s_2^{-1}\left(c_x s\left(\frac{z-a}{b-a}\right)\right)\right) \\ &= s_2^{-1}\left(s_2\left(s_2^{-1}\left(c_x s\left(\frac{y-a}{b-a}\right)\right)\right) + s_2\left(s_2^{-1}\left(c_x s\left(\frac{z-a}{b-a}\right)\right)\right)\right) \\ &= s_2^{-1}\left(c_x\left(s\left(\frac{y-a}{b-a}\right) + s\left(\frac{z-a}{b-a}\right)\right)\right). \end{aligned}$$

Therefore, for the x mentioned above, if the vertical section $I(x, \cdot)$ has representation

$$I(x, y) = \begin{cases} 0, & y \in [0, a), \\ s_2^{-1}\left(c_x s\left(\frac{y-a}{b-a}\right)\right), & y \in [a, b], \\ 1, & y \in (b, 1], \end{cases}$$

then $I(x, S_1(y, z)) = S_2(I(x, y), I(x, z))$ holds.

($\Rightarrow$) Assume that the triple of functions S_1, S_2 and I is a solution to Eq. (2) satisfying the required properties. For every $x \in [0, 1]$, if $y \in [0, a]$ or $y \in [b, 1]$, we get from Eq. (2) that $I(x, S_1(y, y)) = S_2(I(x, y), I(x, y))$. This yields that $I(x, y) = S_2(I(x, y), I(x, y))$, which indicates that $I(x, y) = 0$ or 1 for $y \in [0, a]$ or $y \in [b, 1]$.

We will show that for every $x \in [0, 1]$, if there exists some $y_0 \in [0, a]$ or $y_0 \in [b, 1]$ such that $I(x, y_0) = 1$, then $I(x, y) = 1$ for all $y \in [y_0, 1]$. In fact, for any $y \in (y_0, 1]$, $I(x, y) = I(x, \max(y, y_0)) = I(x, S_1(y, y_0)) = S_2(I(x, y), I(x, y_0)) = S_2(I(x, y), 1) = 1$. Thus $I(x, y) = 1$ for all $y \in [y_0, 1]$. Similarly, $I(x, y) = 0$ for all $y \in [0, y_0]$ if $I(x, y_0) = 0$.

Fix arbitrarily $x \in [0, 1]$. Let $d_x = \inf_{y\in[0,a]}\{y | I(x, y) = 1\}$ with the convention $\inf \emptyset = a$ and let $e_x = \sup_{y\in[b,1]}\{y | I(x, y) = 0\}$ with the convention $\sup \emptyset = b$. Clearly, $d_x \in [0, a]$ and $e_x \in [b, 1]$.

Case 1. If $d_x = 0$ and $0 \in \{y \in [0, a] | I(x, y) = 1\}$, i.e., $I(x, 0) = 1$, then $I(x, y) = 1$ for all $y \in [0, 1]$. Thus we get (i).

Case 2. If $d_x = 0$ but $0 \notin \{y \in [0, a] | I(x, y) = 1\}$, then

$$I(x, y) = \begin{cases} 0, & y = 0, \\ 1, & y \in (0, 1]. \end{cases}$$

Case 3. If $d_x \in (0, a)$, then we have from the definition of d_x that $I(x, y) = 0$ for $y \in [0, d_x]$ (or $y \in [0, d_x)$) and $I(x, y) = 1$ for $y \in (d_x, a)$ (or $y \in [d_x, a)$), which implies that $I(x, y) = 0$ for $y \in [0, d_x]$ (or $y \in [0, d_x)$) and $I(x, y) = 1$ for $y \in (d_x, 1]$ (or $y \in [d_x, 1]$), i.e.,

$$I(x, y) = \begin{cases} 0, & y \in [0, d_x], \\ 1, & y \in (d_x, 1], \end{cases}$$

or

$$I(x, y) = \begin{cases} 0, & y \in [0, d_x), \\ 1, & y \in [d_x, 1]. \end{cases}$$

From Cases 2 and 3, we obtain (iii) and (iv).

Case 4. If $d_x = a$, then we have $e_x \in [b, 1]$. For convenience, this case is divided into four subcases again.

Case 4.1. If $d_x = a$, $e_x = 1$ and $1 \in \{y \in [b, 1] | I(x, y) = 0\}$, i.e., $I(x, 1) = 0$, then $I(x, y) = 0$ for all $y \in [0, 1]$. So (ii) is derived.

Case 4.2. If $d_x = a$, $e_x = 1$ but $1 \notin \{y \in [b, 1] | I(x, y) = 0\}$, then we get that

$$I(x, y) = \begin{cases} 0, & y \in [0, 1), \\ 1, & y = 1. \end{cases}$$

Case 4.3. If $d_x = a$ and $e_x \in (b, 1)$, then, similar to Case 3,

$$I(x, y) = \begin{cases} 0, & y \in [0, e_x], \\ 1, & y \in (e_x, 1], \end{cases}$$

or

$$I(x, y) = \begin{cases} 0, & y \in [0, e_x), \\ 1, & y \in [e_x, 1]. \end{cases}$$

This subcase with Case 4.2 produces (v) and (vi).

Case 4.4. If $d_x = a$ and $e_x = b$, then $I(x, y) = 0$ for $y \in [0, a)$ and $I(x, y) = 1$ for $y \in (b, 1]$. So we only need to consider the vertical section $I(x, \cdot)$ on the interval $[a, b]$.

Let $y, z \in [a, b]$. Then we have $S_1(y, z) = a + (b - a)S((y - a)/(b - a), (z - a)/(b - a))$. Define a function $\varphi : [a, b] \to [0, 1]$ by $\varphi(x) = (x - a)/(b - a)$. Thus the previous equation can be rewritten as $S_1(y, z) = \varphi^{-1}(S(\varphi(y), \varphi(z)))$ and Eq. (2) also can be rewritten as

$$I(x, \varphi^{-1}(s^{-1}(s(\varphi(y)) + s(\varphi(z))))) = s_2^{-1}(s_2(I(x, y)) + s_2(I(x, z))), \quad y, z \in [a, b]. \tag{5}$$

For the arbitrarily fixed $x \in [0, 1]$, define a function $I_x : [0, 1] \to [0, 1]$ given by $I_x(y) = I(x, y)$. By routine substitutions, $h_x = s_2 \circ I_x \circ \varphi^{-1} \circ s^{-1}$, $u = s(\varphi(y))$, $v = s(\varphi(z))$ for all $y, z \in [a, b]$, from Eq. (5) we obtain the following functional equation:

$$h_x(u + v) = h_x(u) + h_x(v), \quad u, v \in [0, \infty],$$

where $h_x : [0, \infty] \to [0, \infty]$. In line with Theorem 2.2, we get that either $h_x = \infty$, or $h_x = 0$, or

$$h_x(u) = \begin{cases} 0, & u = 0, \\ \infty, & u > 0, \end{cases}$$

or

$$h_x(u) = \begin{cases} 0, & u < \infty, \\ \infty, & u = \infty, \end{cases}$$

or $h_x(u) = c_x u$ with some constant $c_x \in (0, \infty)$. According to the definition of h_x, we have that for all $y \in [a, b]$, either $I(x, y) = 1$, or $I(x, y) = 0$, or

$$I(x, y) = \begin{cases} 0, & y = a, \\ 1, & a < y \le b, \end{cases}$$

or

$$I(x, y) = \begin{cases} 0, & a \le y < b, \\ 1, & y = b, \end{cases}$$

or $I(x, y) = s_2^{-1}(c_x s((y - a)/(b - a)))$ with some constant $c_x \in (0, \infty)$.

In addition, it can be proved that the constant c_x is uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s. Actually, let $s_2'(x) = ks_2(x)$ and $s'(x) = ls(x)$ for all $x \in [0, 1]$ and some

$k, l \in (0, \infty)$. And let c'_x be a constant for s'_2, s'. If $s_2^{-1}(c_x s((y-a)/(b-a))) = s_2'^{-1}(c'_x s'((y-a)/(b-a)))$, then we get $s_2^{-1}(c_x s((y-a)/(b-a))) = s_2^{-1}(c'_x s'((y-a)/(b-a))/k)$, which leads to $c_x s((y-a)/(b-a)) = c'_x s'((y-a)/(b-a))/k$. Thus $c'_x = (k/l)c_x$. $\square$

Remark 3.1.1. If $a = 0$ or $b = 1$ and other conditions stay the same in Theorem 3.1.1, we get similarly the following statements:

(1) If $a > 0$ and $b = 1$, then Eq. (2) holds if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied:

(i) $I(x, y) = 1$ for all $y \in [0, 1]$,

(ii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x], \\ 1, & y \in (d_x, 1], \end{cases}$ for some $d_x \in [0, a)$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x), \\ 1, & y \in [d_x, 1], \end{cases}$ for some $d_x \in (0, a)$,

(iv) $I(x, y) = 0$ for $y \in [0, a)$, and for $y \in [a, 1]$, the vertical section $I(x, \cdot)$ has one of the following representations:

$$I(x, y) = 0;\ \ I(x, y) = 1;\ \ I(x, y) = \begin{cases} 1, & y = 1, \\ 0, & y \in [a, 1); \end{cases}$$

$$I(x, y) = \begin{cases} 1, & y \in (a, 1], \\ 0, & y = a; \end{cases}$$

$$I(x, y) = s_2^{-1}\left(c_x s\left(\frac{y-a}{1-a}\right)\right),$$

where $c_x \in (0, \infty)$ is some constant related to x, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s.

(2) If $a = 0$ and $b < 1$, then Eq. (2) holds if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied:

(i) $I(x, y) = 0$ for all $y \in [0, 1]$,

(ii) $I(x, y) = \begin{cases} 0, & y \in [0, e_x], \\ 1, & y \in (e_x, 1], \end{cases}$ for some $e_x \in (b, 1)$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, e_x), \\ 1, & y \in [e_x, 1], \end{cases}$ for some $e_x \in (b, 1]$,

(iv) $I(x, y) = 1$ for $y \in (b, 1]$, and for $y \in [0, b]$, the vertical section $I(x, \cdot)$ has one of the following forms:

$$I(x, y) = 0; \quad I(x, y) = 1; \quad I(x, y) = \begin{cases} 1, & y = b, \\ 0, & y \in [0, b); \end{cases}$$

$$I(x, y) = \begin{cases} 1, & y \in (0, b], \\ 0, & y = 0; \end{cases}$$

$$I(x, y) = s_2^{-1}\left(c_x s\left(\frac{y}{b}\right)\right),$$

where $c_x \in (0, \infty)$ is a certain constant related to x, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s.

Now we will describe the continuous solutions to Eq. (2).

Theorem 3.1.2. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ $(0 < a < b < 1)$, where S is a strict t-conorm with an additive generator s. Let S_2 be a strict t-conorm with an additive generator s_2 and $I : [0, 1]^2 \to [0, 1]$ be a continuous binary function. Then the triple of functions S_1, S_2 and I satisfies Eq.* (2) *if and only if for all $x, y \in [0, 1]$,* *either $I=1$, or $I=0$, or there exists a continuous function $c:[0,1]\to(0,\infty)$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that*

$$I(x,y)=\begin{cases}0, & y\in[0,a),\\ s_2^{-1}\left(c(x)s\left(\dfrac{y-a}{b-a}\right)\right), & y\in[a,b],\\ 1, & y\in(b,1],\end{cases}\quad x,y\in[0,1]. \tag{6}$$

Proof. ($\Leftarrow$) If either $I=0$, or $I=1$, or I is defined by form (6), then I is obviously continuous. Just as the proof of Theorem 3.1.1, we can prove that the triple of S_1, S_2 and I satisfies Eq. (2).

($\Rightarrow$) We know that for every fixed $x\in[0,1]$, the vertical section $I(x,\cdot)$ may have one of the forms in (i)–(vii) in Theorem 3.1.1. Because I is continuous, the vertical sections are also continuous. Consequently, the vertical sections in (i), (ii) and (vii) with form (4) in Theorem 3.1.1 are possible.

Suppose that there exists some $x_0\in[0,1]$ such that $I(x_0,y)=0$ for all $y\in[0,1]$. In particular, $I(x_0,1)=0$. But for the other two possible vertical sections, $I(x,1)=1$ always holds for every $x\in[0,1]$. Again considering the continuity of I on the first variable, we have that the only possibility in this case is $I=0$.

Suppose that there exists some $x_0\in[0,1]$ such that $I(x_0,y)=1$ for all $y\in[0,1]$. Similarly, we can obtain that $I=1$.

Finally, suppose that $I_x\neq 0$ and $I_x\neq 1$ for all $x\in[0,1]$. Then we get that for all $x,y\in[0,1]$, there exists a function $c:[0,1]\to(0,\infty)$ such that

$$I(x,y)=\begin{cases}0, & y\in[0,a),\\ s_2^{-1}\left(c(x)s\left(\dfrac{y-a}{b-a}\right)\right), & y\in[a,b],\\ 1, & y\in(b,1].\end{cases}$$

Clearly, function c is continuous because for any fixed $y\in(a,b)$, it is a composition of continuous functions: $c(x)=s_2(I(x,y))/s((y-a)/(b-a))$. And from the previous formula, we know easily that function c is uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s. □

Remark 3.1.2. Here, it is necessary to point out the difference between Theorems 3.1.1 and 3.1.2. In the former, only the forms of the vertical section $I(x,\cdot)$ are given, while in the latter, the whole structure of I on $[0,1]^2$ is characterized.

Example 3.1.1. Let $S_2=S_P$ with an additive generator $s_2(x)=\ln(1/(1-x))$, $c(x)=x+1$ and $S_1=\langle\frac14,\frac34,S_P\rangle$. Then by (6) we get

$$I(x,y)=\begin{cases}0, & y\in\left[0,\frac14\right),\\ 1-\left(\frac32-2y\right)^{x+1}, & y\in\left[\frac14,\frac34\right],\\ 1, & y\in\left(\frac34,1\right],\end{cases}\quad x,y\in[0,1].$$

Its plot is presented in Fig. 1.

Remark 3.1.3. Especially, if $a=0$ or $b=1$ with other conditions unchanged in Theorem 3.1.2, we get similarly the following results.

(i) If $a>0$ and $b=1$, then I is a continuous solution to Eq. (2) if and only if for all $x,y\in[0,1]$, either $I=1$, or $I=0$, or

$$I(x,y)=\begin{cases}0, & y\in[0,a),\\ s_2^{-1}\left(c(x)s\left(\dfrac{y-a}{1-a}\right)\right), & y\in[a,1],\end{cases}\quad x,y\in[0,1],$$

where $c:[0,1]\to(0,\infty)$ is a continuous function, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s.

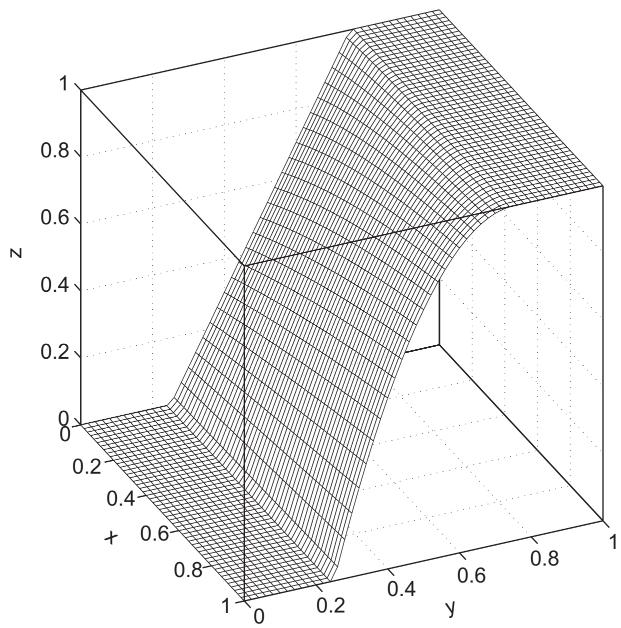

Fig. 1. The plot of continuous function I in Example 3.1.1.

(ii) If $a = 0$ and $b < 1$, then I is a continuous solution to Eq. (2) if and only if for all $x, y \in [0, 1]$, either $I = 1$, or $I = 0$, or

$$I(x, y) = \begin{cases} 1, & y \in (b, 1], \\ s_2^{-1}\left(c(x)s\left(\dfrac{y}{b}\right)\right), & y \in [0, b], \end{cases} \quad x, y \in [0, 1],$$

where $c : [0, 1] \to (0, \infty)$ is a continuous function, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s.

Corollary 3.1.1. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ with a strict t-conorm S and S_2 be a strict t-conorm. Then there are no continuous solutions to Eq.* (2) *which satisfy* I3.

Proof. Suppose that $0 < a < b < 1$ and $I : [0, 1]^2 \to [0, 1]$ is a continuous binary function satisfying I3. Let s and s_2 be the additive generators of S, S_2, respectively. From Theorem 3.1.2 and the assumption above, we have that I must admit representation

$$I(x, y) = \begin{cases} 0, & y \in [0, a), \\ s_2^{-1}\left(c(x)s\left(\dfrac{y-a}{b-a}\right)\right), & y \in [a, b], \\ 1, & y \in (b, 1], \end{cases} \quad x, y \in [0, 1].$$

Consequently, we get $I(0, 0) = 0$, which shows that I does not satisfy I3.

Suppose $a = 0$ or $b = 1$. Then we can prove similarly that there are no continuous solutions to Eq. (2) which satisfy I3, either. □

Theorem 3.1.3. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ $(0 < a < b < 1)$, where S is a strict t-conorm with an additive generator s. Let S_2 be a strict t-conorm with an additive generator s_2 and $I : [0, 1]^2 \to [0, 1]$ be a binary function, which is continuous except the vertical section $I(0, y)$ for $y \in [0, a]$ and satisfies* I3. *Then the triple of functions S_1, S_2 and I satisfies Eq.* (2) *if and only if there exists a continuous function $c : [0, 1] \to (0, \infty]$ with*

$c(0)=\infty$ and $c(x)\in(0,\infty)$ for $x\in(0,1]$, uniquely determined up to a positive multiplicative constant, such that I is given by

$$I(x,y)=\begin{cases}1, & x=0,\ y\in[0,1],\\ 1, & x\neq 0,\ y\in(b,1],\\ s_2^{-1}\left(c(x)s\left(\dfrac{y-a}{b-a}\right)\right), & x\neq 0,\ y\in[a,b],\\ 0, & x\neq 0,\ y\in[0,a),\end{cases}\quad x,y\in[0,1]. \tag{7}$$

Proof. ($\Rightarrow$) Suppose that the triple of functions S_1, S_2 and I is a solution to Eq. (2) satisfying the required conditions. From Theorem 3.1.1, we know that for each x, the possible vertical sections are $I_x=0$, $I_x=1$ and

$$I_x(y)=\begin{cases}1, & y\in(b,1],\\ s_2^{-1}\left(c(x)s\left(\dfrac{y-a}{b-a}\right)\right), & y\in[a,b],\\ 0, & y\in[0,a).\end{cases}$$

We will show that $I_x\neq 0$ and $I_x\neq 1$. In fact, if we take $y=1$, then there are only two possibilities: either $I_x(1)=0$ or $I_x(1)=1$. But $I_1(1)=1$. We have that $I_x(1)=1$ because of the continuity of I on the first variable (for $x\in(0,1]$, $y=1$). So $I_x\neq 0$ for every $x\in(0,1]$. Moreover, by putting $y=0$, we also obtain two possibilities, either $I_x(0)=0$ or $I_x(0)=1$. From $I_1(0)=0$, we get similarly that $I_x(0)=0$, which implies that $I_x\neq 1$ for every $x\in(0,1]$.

As a consequence, there exists a function $c:\ (0,1]\to(0,\infty)$, uniquely determined up to a positive multiplicative constant, such that

$$I(x,y)=\begin{cases}1, & y\in(b,1],\\ s_2^{-1}\left(c(x)s\left(\dfrac{y-a}{b-a}\right)\right), & y\in[a,b],\\ 0, & y\in[0,a),\end{cases}\quad x\in(0,1],\ y\in[0,1].$$

It is not difficult to check that function c is continuous on $(0,1]$, since for any fixed $y\in(a,b)$, $c(x)=s_2(I(x,y))/s((y-a)/(b-a))$. Namely, it is a composition of continuous functions on $(0,1]$.

Putting $x=0=z$ in Eq. (2), we obtain that $I(0,S_1(y,0))=S_2(I(0,y),I(0,0))$, which means $I(0,y)=1$ for all $y\in[0,1]$.

The above steps indicate that function I must admit representation (7).

On the other hand, if $x=0$ and $y,z\in(a,b]$, then using similar steps as in the proof of Case 4.4 in Theorem 3.1.1, we get that $h_0(u+v)=h_0(u)+h_0(v)$, $u,v\in(0,\infty]$, where $h_0:\ (0,\infty]\to[0,\infty]$ defined by $h_0=s_2\circ I_0\circ\varphi^{-1}\circ s^{-1}$ and $u=s(\varphi(y))$, $v=s(\varphi(z))$. Again observing that function I is continuous except the vertical section $I(0,y)$ $(y\in[0,a])$, we get that h_0 is continuous on $(0,\infty]$ as well. Hence, from Theorem 2.2, we have that either $h_0=0$, or $h_0=\infty$, or $h_0(u)=c(0)u$ with some constant $c(0)\in(0,\infty)$. But $h_0(\infty)=s_2\circ I_0\circ\varphi^{-1}\circ s^{-1}(\infty)=s_2\circ I_0\circ(\varphi^{-1}(1))=s_2(I_0(b))=s_2(1)=\infty$, then it holds that $h_0\neq 0$. As for solution $h_0=\infty$, it can be seen as $c(0)=\infty$ in $h_0(u)=c(0)u$. Thus, $h_0(u)=c(0)u$ with $u\in(0,\infty]$ and $c(0)\in(0,\infty]$, from which we obtain that for all $y\in(a,b]$, $I_0(y)=s_2^{-1}(c(0)s((y-a)/(b-a)))$ with $c(0)\in(0,\infty]$. Next we will show that $c(0)=\infty$.

In fact, for any fixed $y\in(a,b)$, we get that $1=I(0,y)=I_0(y)=s_2^{-1}(c(0)s((y-a)/(b-a)))$. So $s_2(1)=c(0)s((y-a)/(b-a))$ and then $c(0)=\infty$.

To complete the proof, we need to check that function c is continuous at point 0. Fix $y\in(a,b)$. Since $I(x,y)$ is continuous when $y\in(a,b]$, then $1=I(0,y)=\lim_{x\to 0^+}I(x,y)=\lim_{x\to 0^+}s_2^{-1}(c(x)s((y-a)/(b-a)))$. This yields that $\infty=s_2(1)=\lim_{x\to 0^+}c(x)s((y-a)/(b-a))$. So $\lim_{x\to 0^+}c(x)=\infty=c(0)$.

($\Leftarrow$) It is easy to test that function I defined by (7) is continuous except the vertical section $I(0,y)$ $(y\in[0,a])$ and satisfies I3. By previously general solution, we can prove that it satisfies Eq. (2), too. □

Remark 3.1.4.

(i) If $a>0$ and $b=1$ and other conditions do not change in Theorem 3.1.3, then Eq. (2) holds if and only if there exists a continuous function $c:[0,1]\to(0,\infty]$ with $c(0)=\infty$ and $c(x)\in(0,\infty)$ for $x\in(0,1]$, such that I is given by

$$I(x, y) = \begin{cases} 1, & x = 0,\ y \in [0, 1], \\ 0, & x \neq 0,\ y \in [0, a), \\ s_2^{-1}\left(c(x)s\left(\dfrac{y-a}{1-a}\right)\right) & x \neq 0,\ y \in [a, 1], \end{cases} \quad x, y \in [0, 1]. \tag{8}$$

(ii) If $a = 0$ and $b < 1$ in Theorem 3.1.3, then function I, which is continuous except at the point (0, 0) and satisfies I3, is a solution to Eq. (2) if and only if there exists a continuous function $c : [0, 1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, such that I is given by

$$I(x, y) = \begin{cases} 1, & x = 0,\ y \in [0, 1], \\ 1, & x \neq 0,\ y \in (b, 1], \\ s_2^{-1}\left(c(x)s\left(\dfrac{y}{b}\right)\right), & x \neq 0,\ y \in [0, b], \end{cases} \quad x, y \in [0, 1]. \tag{9}$$

To ensure I is a fuzzy implication, function c should be monotonic. The following corollary is immediately obtained.

Corollary 3.1.2. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ $(0 < a < b < 1)$, where S is a strict t-conorm with an additive generator s. Let S_2 be a strict t-conorm with an additive generator s_2 and I be a fuzzy implication, which is continuous except the vertical section $I(0, y)$ for $y \in [0, a]$. Then the triple of functions S_1, S_2 and I satisfies Eq.* (2) *if and only if there exists a continuous and decreasing function $c : [0, 1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, uniquely determined up to a positive multiplicative constant, such that I has form* (7) *for all $x, y \in [0, 1]$.*

Example 3.1.2. Let $S_2 = S_P$ with an additive generator $s_2(x) = \ln(1/(1 - x))$, $c(x) = 1/x$ (with the assumption $\frac{1}{0} = \infty$) and $S_1 = \langle \frac{1}{4}, \frac{3}{4}, S_P\rangle$. Then fuzzy implication I is given by

$$I(x, y) = \begin{cases} 1, & x = 0,\ y \in [0, 1], \\ 0, & x \neq 0,\ y \in \left[0, \frac{1}{4}\right), \\ 1 - \left(\frac{3}{2} - 2y\right)^{1/x}, & x \neq 0,\ y \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 1, & x \neq 0,\ y \in \left(\frac{3}{4}, 1\right], \end{cases} \quad x, y \in [0, 1].$$

The plot of I is presented in Fig. 2.

Remark 3.1.5. If $a = 0$ or $b = 1$ in Corollary 3.1.2, the sufficient conditions will be changed as follows.

(i) If $a > 0$ and $b = 1$, then there exists a continuous and decreasing function $c : [0, 1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, such that for all $x, y \in [0, 1]$, I has form (8).
(ii) If $a = 0$ and $b < 1$, then there exists a continuous and decreasing function $c : [0, 1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, such that for all $x, y \in [0, 1]$, I has form (9).

In the following we will study the case in which t-conorm S is nilpotent.

Theorem 3.1.5. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ $(0 < a < b < 1)$ with a nilpotent t-conorm S, S_2 be a strict t-conorm and $I : [0, 1]^2 \to [0, 1]$ be a binary function. Then the triple of functions S_1, S_2 and I satisfies Eq.* (2) *if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied*:

(i) $I(x, y) = 1$ *for all* $y \in [0, 1]$,
(ii) $I(x, y) = 0$ *for all* $y \in [0, 1]$,
(iii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x], \\ 1, & y \in (d_x, 1], \end{cases}$ *for some* $d_x \in [0, a)$,
(iv) $I(x, y) = \begin{cases} 0, & y \in [0, d_x), \\ 1, & y \in [d_x, 1], \end{cases}$ *for some* $d_x \in (0, a)$,

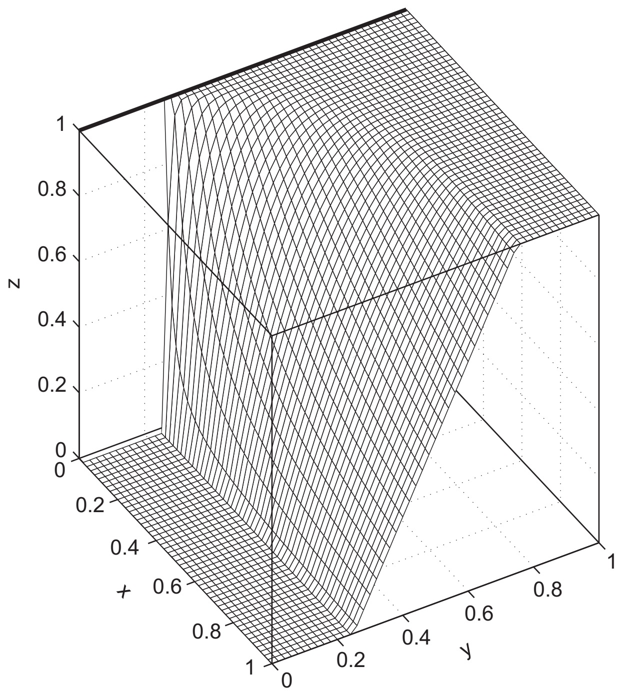

Fig. 2. The plot of fuzzy implication I in Example 3.1.2 (the bolded line is the line $I(0, y) = 1$).

(v) $I(x, y) = \begin{cases} 0, & y \in [0, e_x], \\ 1, & y \in (e_x, 1], \end{cases}$ *for some* $e_x \in (b, 1)$,

(vi) $I(x, y) = \begin{cases} 0, & y \in [0, e_x), \\ 1, & y \in [e_x, 1], \end{cases}$ *for some* $e_x \in (b, 1]$,

(vii) $I(x, y) = 0$ *for* $y \in [0, a)$, $I(x, y) = 1$ *for* $y \in (b, 1]$, *and for* $y \in [a, b]$, *the vertical section* $I(x, \cdot)$ *has one of the following representations*:

$$I(x, y) = 0,$$

$$I(x, y) = 1,$$

$$I(x, y) = \begin{cases} 0, & y = a, \\ 1, & a < y \leq b. \end{cases}$$

Proof. Applying Theorem 2.3, we can get the result just as the proof in Theorem 3.1.1. □

Remark 3.1.6. If $a = 0$ or $b = 1$ and other conditions in Theorem 3.1.5 do not change, then it is easy to obtain the following:

(1) If $a > 0$ and $b = 1$, then Eq. (2) holds if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied:

(i) $I(x, y) = 1$ for all $y \in [0, 1]$,

(ii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x], \\ 1, & y \in (d_x, 1], \end{cases}$ for some $d_x \in [0, a)$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x), \\ 1, & y \in [d_x, 1], \end{cases}$ for some $d_x \in (0, a)$,

(iv) $I(x, y) = 0$ for $y \in [0, a)$, and for $y \in [a, 1]$, the vertical section $I(x, \cdot)$ has one of the following representations:

$$I(x, y) = 0; \quad I(x, y) = 1; \quad I(x, y) = \begin{cases} 0, & y = a, \\ 1, & a < y \leq 1. \end{cases}$$

(2) If $a = 0$ and $b < 1$, then Eq. (2) holds if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied:

(i) $I(x, y) = 0$ for all $y \in [0, 1]$,

(ii) $I(x, y) = \begin{cases} 0, & y \in [0, e_x], \\ 1, & y \in (e_x, 1], \end{cases}$ for some $e_x \in (b, 1)$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, e_x), \\ 1, & y \in [e_x, 1], \end{cases}$ for some $e_x \in (b, 1]$,

(iv) $I(x, y) = 1$ for $y \in (b, 1]$, and for $y \in [0, b]$, the vertical section $I(x, \cdot)$ has one of the following representations:

$$I(x, y) = 0; \quad I(x, y) = 1; \quad I(x, y) = \begin{cases} 0, & y = 0, \\ 1, & 0 < y \leq b. \end{cases}$$

Theorem 3.1.6. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ with a nilpotent t-conorm S, S_2 be a strict t-conorm and $I : [0, 1]^2 \to [0, 1]$ be a continuous binary function. Then the triple of functions S_1, S_2 and I satisfies Eq. (2) if and only if either $I = 0$ or $I = 1$.*

Remark 3.1.7. In this case we have only trivial continuous solutions $I = 0$ and $I = 1$, which are not fuzzy implications.

3.2. Case of that S_2 is nilpotent

Theorem 3.2.1. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ $(0 < a < b < 1)$, where S is a strict t-conorm with an additive generator s. Let S_2 be a nilpotent t-conorm with an additive generator s_2 and $I : [0, 1]^2 \to [0, 1]$ be a binary function. Then the triple of functions S_1, S_2 and I satisfies Eq. (2) if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied*:

(i) $I(x, y) = 1$ *for all* $y \in [0, 1]$,

(ii) $I(x, y) = 0$ *for all* $y \in [0, 1]$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x], \\ 1, & y \in (d_x, 1], \end{cases}$ *for some* $d_x \in [0, a)$,

(iv) $I(x, y) = \begin{cases} 0, & y \in [0, d_x), \\ 1, & y \in [d_x, 1], \end{cases}$ *for some* $d_x \in (0, a)$,

(v) $I(x, y) = \begin{cases} 0, & y \in [0, e_x], \\ 1, & y \in (e_x, 1], \end{cases}$ *for some* $e_x \in (b, 1)$,

(vi) $I(x, y) = \begin{cases} 0, & y \in [0, e_x), \\ 1, & y \in [e_x, 1], \end{cases}$ *for some* $e_x \in (b, 1]$,

(vii) $I(x, y) = 0$ *for* $y \in [0, a)$, $I(x, y) = 1$ *for* $y \in (b, 1]$, *and for* $y \in [a, b]$, *the vertical section* $I(x, \cdot)$ *has one of the following forms*:

$$I(x, y) = 0,$$

$$I(x, y) = 1,$$

$$I(x, y) = \begin{cases} 0, & y = a, \\ 1, & a < y \leq b, \end{cases}$$

$$I(x, y) = \begin{cases} 0, & a \leq y < b, \\ 1, & y = b, \end{cases}$$

$$I(x, y) = s_2^{-1}\left(\min\left(c_x s\left(\frac{y-a}{b-a}\right), s_2(1)\right)\right),$$

where $c_x \in (0, \infty)$ is a certain constant, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s.

Proof. By using Theorem 2.4, we can get the result similarly to Theorem 3.1.1. □

Remark 3.2.1.

(1) Especially, if $a > 0$ and $b = 1$ in the theorem above, then Eq. (2) holds if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied:

(i) $I(x, y) = 1$ for all $y \in [0, 1]$,

(ii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x], \\ 1, & y \in (d_x, 1], \end{cases}$ for some $d_x \in [0, a)$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x), \\ 1, & y \in [d_x, 1], \end{cases}$ for some $d_x \in (0, a)$,

(iv) $I(x, y) = 0$ for $y \in [0, a)$, and for $y \in [a, 1]$, the vertical section $I(x, \cdot)$ has one of the following representations:

$$I(x, y) = 0; \quad I(x, y) = 1; \quad I(x, y) = \begin{cases} 0, & y = a, \\ 1, & a < y \leq 1; \end{cases}$$

$$I(x, y) = \begin{cases} 0, & a \leq y < 1, \\ 1, & y = 1; \end{cases}$$

$I(x, y) = s_2^{-1}(\min(c_x s((y - a)/(1 - a)), s_2(1)))$, where $c_x \in (0, \infty)$ is some constant, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s.

(2) If $a = 0$ and $b < 1$ in the theorem above, then Eq. (2) holds if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied:

(i) $I(x, y) = 0$ for all $y \in [0, 1]$,

(ii) $I(x, y) = \begin{cases} 0, & y \in [0, e_x], \\ 1, & y \in (e_x, 1], \end{cases}$ for some $e_x \in (b, 1)$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, e_x), \\ 1, & y \in [e_x, 1], \end{cases}$ for some $e_x \in (b, 1]$,

(iv) $I(x, y) = 1$ for $y \in (b, 1]$, and for $y \in [0, b]$, the vertical section $I(x, \cdot)$ has one of the following representations:

$$I(x, y) = 0; \quad I(x, y) = 1; \quad I(x, y) = \begin{cases} 0, & y = 0, \\ 1, & 0 < y \leq b; \end{cases}$$

$$I(x, y) = \begin{cases} 0, & 0 \leq y < b, \\ 1, & y = b; \end{cases}$$

$I(x, y) = s_2^{-1}(\min(c_x s(y/b), s_2(1)))$, where $c_x \in (0, \infty)$ is some constant, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s.

Theorem 3.2.2. *Let S_1 be a continuous t-conorm given by $\langle a, b, S \rangle$ $(0 < a < b < 1)$, where S is a strict t-conorm with an additive generator s. Let S_2 be a nilpotent t-conorm with an additive generator s_2 and $I : [0, 1]^2 \to [0, 1]$ be a continuous binary function. Then the triple of functions S_1, S_2 and I satisfies Eq. (2) if and only if for all $x, y \in [0, 1]$, either $I = 0$, or $I = 1$, or there exists a continuous function $c : [0, 1] \to (0, \infty)$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that*

$$I(x, y) = \begin{cases} 0, & y \in [0, a), \\ 1, & y \in (b, 1], \\ s_2^{-1}\left(\min\left(c(x)s\left(\dfrac{y - a}{b - a}\right), s_2(1)\right)\right), & y \in [a, b], \end{cases} \quad x, y \in [0, 1]. \tag{10}$$

Proof. The proof is similar to the one of Theorem 3.1.2. □

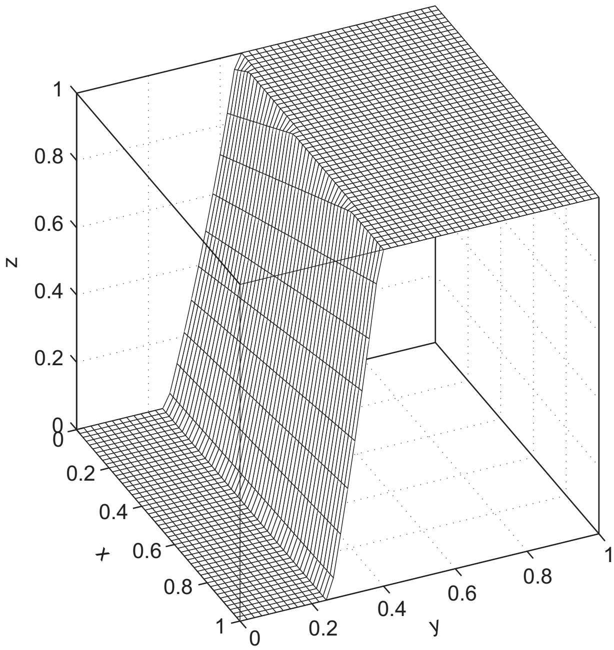

Fig. 3. The plot of continuous function I in Example 3.2.1.

Example 3.2.1. Let $S_2 = S_L$ with an additive generator $s_2(x) = x$, $c(x) = x + 2$ and $S_1 = \langle \frac{1}{4}, \frac{3}{4}, S_P \rangle$ with $s(x) = \ln(1/(1-x))$ being an additive generator of S_P. Then the continuous solution I is given by

$$I(x, y) = \begin{cases} 0, & y \in [0, \frac{1}{4}), \\ \min\left(-(2+x)\ln\left(\frac{3}{2} - 2y\right), 1\right), & y \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 1, & y \in \left(\frac{3}{4}, 1\right], \end{cases} \quad x, y \in [0, 1].$$

Its plot can be seen in Fig. 3.

Remark 3.2.2. If $a = 0$ or $b = 1$ in Theorem 3.2.2, then the sufficient conditions will be changed.

(i) If $a > 0$ and $b = 1$, then for all $x, y \in [0, 1]$, either $I = 0$, or $I = 1$, or there exists a continuous function $c : [0, 1] \to (0, \infty)$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that

$$I(x, y) = \begin{cases} 0, & y \in [0, a), \\ s_2^{-1}\left(\min\left(c(x)s\left(\dfrac{y-a}{1-a}\right), s_2(1)\right)\right), & y \in [a, 1], \end{cases} \quad x, y \in [0, 1].$$

(ii) If $a = 0$ and $b < 1$, then for all $x, y \in [0, 1]$, either $I = 0$, or $I = 1$, or there exists a continuous function $c : [0, 1] \to (0, \infty)$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that

$$I(x, y) = \begin{cases} 1, & y \in (b, 1], \\ s_2^{-1}\left(\min\left(c(x)s\left(\dfrac{y}{b}\right), s_2(1)\right)\right), & y \in [0, b], \end{cases} \quad x, y \in [0, 1].$$

Similar to Corollary 3.1.1, we have the following corollary.

Corollary 3.2.1. *Let S_1 be a continuous t-conorm given by $\langle a, b, S \rangle$ with a strict t-conorm S and S_2 be a nilpotent t-conorm. Then there are no continuous solutions to Eq.* (2) *which satisfy* I3.

Theorem 3.2.3. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ $(0 < a < b < 1)$, where S is a strict t-conorm with an additive generator s. Let S_2 be a nilpotent t-conorm with an additive generator s_2 and $I: [0,1]^2 \to [0,1]$ be a binary function, which is continuous except the vertical section $I(0, y)$ for $y \in [0, a]$ and satisfies* I3. *Then the triple of functions S_1, S_2 and I satisfies Eq.* (2) *if and only if there exists a continuous function $c: [0,1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that I is given by*

$$I(x,y) = \begin{cases} 1, & x = 0,\ y \in [0,1], \\ 0, & x \neq 0,\ y \in [0,a), \\ s_2^{-1}\left(\min\left(c(x)s\left(\dfrac{y-a}{b-a}\right), s_2(1)\right)\right), & x \neq 0,\ y \in [a,b], \\ 1, & x \neq 0,\ y \in (b,1], \end{cases} \quad x, y \in [0,1]. \tag{11}$$

Proof. Similar to Theorem 3.1.3. □

Remark 3.2.3.

(1) If $a > 0$ and $b = 1$, then Eq. (2) holds if and only if there exists a continuous function $c: [0,1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that for all $x, y \in [0,1]$,

$$I(x,y) = \begin{cases} 1, & x = 0,\ y \in [0,1], \\ 0, & x \neq 0,\ y \in [0,a), \\ s_2^{-1}\left(\min\left(c(x)s\left(\dfrac{y-a}{1-a}\right), s_2(1)\right)\right), & x \neq 0,\ y \in [a,1], \end{cases} \quad x, y \in [0,1]. \tag{12}$$

(2) If $a = 0$ and $b < 1$, then Eq. (2) holds if and only if there exists a continuous function $c: [0,1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that for all $x, y \in [0,1]$,

$$I(x,y) = \begin{cases} 1, & x = 0,\ y \in [0,1], \\ s_2^{-1}\left(\min\left(c(x)s\left(\dfrac{y}{b}\right), s_2(1)\right)\right), & x \neq 0,\ y \in [0,b], \\ 1 & x \neq 0,\ y \in (b,1], \end{cases} \quad x, y \in [0,1]. \tag{13}$$

Similar to Corollary 3.1.2, we have the following corollary.

Corollary 3.2.2. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ $(0 < a < b < 1)$, where S is a strict t-conorm with an additive generator s. Let S_2 be a nilpotent t-conorm with an additive generator s_2 and I be a fuzzy implication, which is continuous except the vertical section $I(0, y)$ for $y \in [0, a]$. Then the triple of functions S_1, S_2 and I satisfies Eq.* (2) *if and only if there exists a continuous and decreasing function $c: [0,1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that for all $x, y \in [0,1]$, I has form* (11).

Example 3.2.2. Let $S_2 = S_L$ with an additive generator $s_2(x) = x$, $c(x) = (2-x)/x$ and $S_1 = \langle \frac{1}{4}, \frac{3}{4}, S_P\rangle$ with $s(x) = \ln(1/(1-x))$ being an additive generator of S_P. Then fuzzy implication I is given by

$$I(x,y) = \begin{cases} 1, & x = 0,\ y \in [0,1], \\ 0, & x \neq 0,\ y \in \left[0, \frac{1}{4}\right) \\ \min\left(\dfrac{x-2}{x}\ln\left(\dfrac{3}{2} - 2y\right), 1\right), & x \neq 0,\ y \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 1, & x \neq 0,\ y \in \left(\frac{3}{4}, 1\right], \end{cases} \quad x, y \in [0,1].$$

Its plot can be seen in Fig. 4.

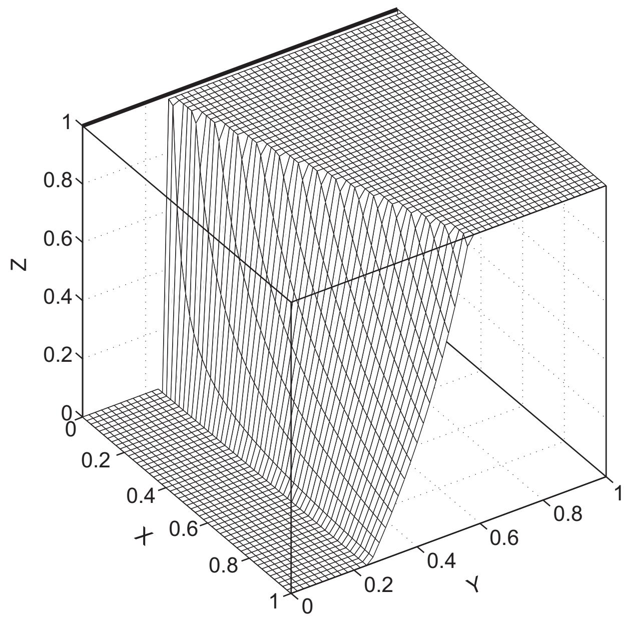

Fig. 4. The plot of fuzzy implication I in Example 3.2.1 (the bolded line is the line $I(0, y) = 1$).

Remark 3.2.4.

(i) If $a > 0$ and $b = 1$ in Corollary 3.2.2, then the triple of functions S_1, S_2 and I satisfies Eq. (2) if and only if there exists a continuous and decreasing function $c : [0, 1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that for all $x, y \in [0, 1]$, I has form (12).

(ii) If $a = 0$ and $b < 1$ in Corollary 3.2.2, then the triple of functions S_1, S_2 and I satisfies Eq. (2) if and only if there exists a continuous and decreasing function $c : [0, 1] \to (0, \infty]$ with $c(0) = \infty$ and $c(x) \in (0, \infty)$ for $x \in (0, 1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that for all $x, y \in [0, 1]$, I has form (13).

The following will investigate the case in which t-conorm S is nilpotent.

Theorem 3.2.4. *Let S_1 be a continuous t-conorm given by $\langle a, b, S\rangle$ $(0 < a < b < 1)$, where S is a nilpotent t-conorm with an additive generator s. Let S_2 be a nilpotent t-conorm with an additive generator s_2 and $I : [0, 1]^2 \to [0, 1]$ be a binary function. Then the triple of functions S_1, S_2 and I satisfies Eq.* (2) *if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied*:

(i) $I(x, y) = 1$ *for all* $y \in [0, 1]$,

(ii) $I(x, y) = 0$ *for all* $y \in [0, 1]$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x], \\ 1, & y \in (d_x, 1], \end{cases}$ *for some* $d_x \in [0, a)$,

(iv) $I(x, y) = \begin{cases} 0, & y \in [0, d_x), \\ 1, & y \in [d_x, 1], \end{cases}$ *for some* $d_x \in (0, a)$,

(v) $I(x, y) = \begin{cases} 0, & y \in [0, e_x], \\ 1, & y \in (e_x, 1], \end{cases}$ *for some* $e_x \in (b, 1)$,

(vi) $I(x, y) = \begin{cases} 0, & y \in [0, e_x), \\ 1, & y \in [e_x, 1], \end{cases}$ *for some* $e_x \in (b, 1]$,

(vii) $I(x, y) = 0$ *for* $y \in [0, a)$, $I(x, y) = 1$ *for* $y \in (b, 1]$, *and for* $y \in [a, b]$, *the vertical section* $I(x, \cdot)$ *has one of the following representations*:

$$I(x, y) = 0,$$

$$I(x, y) = 1,$$

$$I(x, y) = \begin{cases} 0, & y = a, \\ 1, & a < y \le b, \end{cases}$$

$$I(x, y) = s_2^{-1}\left(\min\left(c_x s\left(\frac{y-a}{b-a}\right), s_2(1)\right)\right),$$

where $c_x \in [s_2(1)/s(1), \infty)$ *is some constant, uniquely determined up to a positive multiplicative constant depending on constants for* s_2 *and* s.

Proof. According to Theorem 2.5, we can get the result just as in Theorem 3.1.1. □

Remark 3.2.5.

(1) If $a > 0$ and $b = 1$ with other conditions unchanged in the above theorem, then Eq. (2) holds if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied:

(i) $I(x, y) = 1$ for all $y \in [0, 1]$,

(ii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x], \\ 1, & y \in (d_x, 1], \end{cases}$ for some $d_x \in [0, a)$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, d_x), \\ 1, & y \in [d_x, 1], \end{cases}$ for some $d_x \in (0, a)$,

(iv) $I(x, y) = 0$ for $y \in [0, a)$, and for $y \in [a, 1]$, the vertical section $I(x, \cdot)$ has one of the following representations:

$$I(x, y) = 0; \quad I(x, y) = 1; \quad I(x, y) = \begin{cases} 0, & y = a, \\ 1, & y \in (a, 1]; \end{cases}$$

$I(x, y) = s_2^{-1}(\min(c_x s((y-a)/(1-a)), s_2(1)))$, where $c_x \in [s_2(1)/s(1), \infty)$ is some constant, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s.

(2) If $a = 0$ and $b < 1$ with other conditions unchanged in the above theorem, then Eq. (2) holds if and only if for every fixed $x \in [0, 1]$, one of the following is satisfied:

(i) $I(x, y) = 0$ for all $y \in [0, 1]$,

(ii) $I(x, y) = \begin{cases} 0, & y \in [0, e_x], \\ 1, & y \in (e_x, 1], \end{cases}$ for some $e_x \in (b, 1)$,

(iii) $I(x, y) = \begin{cases} 0, & y \in [0, e_x), \\ 1, & y \in [e_x, 1], \end{cases}$ for some $e_x \in (b, 1]$,

(iv) $I(x, y) = 1$ for $y \in (b, 1]$, and for $y \in [0, b]$, the vertical section $I(x, \cdot)$ has one of the following representations:

$$I(x, y) = 0; \quad I(x, y) = 1; \quad I(x, y) = \begin{cases} 0, & y = 0, \\ 1, & y \in (0, b]; \end{cases}$$

$I(x, y) = s_2^{-1}(\min(c_x s(y/b), s_2(1)))$, where $c_x \in [s_2(1)/s(1), \infty)$ is a certain constant, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s.

Theorem 3.2.5. *Let* S_1 *be a continuous t-conorm given by* $\langle a, b, S\rangle$ $(0 < a < b < 1)$, *where S is a nilpotent t-conorm with an additive generator s. Let* S_2 *be a nilpotent t-conorm with an additive generator* s_2 *and* $I : [0, 1]^2 \to [0, 1]$ *be a continuous binary function. Then the triple of functions* S_1, S_2 *and I satisfies Eq.* (2) *if and only if for all* $x, y \in [0, 1]$,

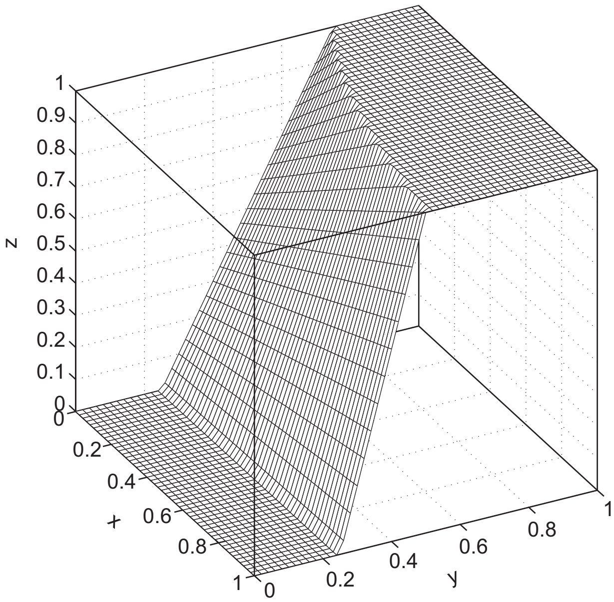

Fig. 5. The plot of continuous function I in Example 3.2.3.

either $I = 0$, *or* $I = 1$, *or there exists a continuous function* $c: [0,1] \to [s_2(1)/s(1), \infty)$, *uniquely determined up to a positive multiplicative constant depending on constants for* s_2 *and* s, *such that*

$$I(x,y) = \begin{cases} 0, & y \in [0,a), \\ 1, & y \in (b,1], \\ s_2^{-1}\left(\min\left(c(x)s\left(\dfrac{y-a}{b-a}\right), s_2(1)\right)\right), & y \in [a,b], \end{cases} \quad x, y \in [0,1].$$

Proof. Similar to Theorem 3.1.2. □

Example 3.2.3. Let $S_2 = S_L$ with an additive generator $s_2(x) = x$, $c(x) = x + 1$ and $S_1 = \langle \frac{1}{4}, \frac{3}{4}, S_L \rangle$. Then by simple calculation we get that the continuous solution I is given by

$$I(x,y) = \begin{cases} 0, & y \in \left[0, \frac{1}{4}\right), \\ \min\left((x+1)\left(2y - \frac{1}{2}\right), 1\right), & y \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 1, & y \in \left(\frac{3}{4}, 1\right], \end{cases} \quad x, y \in [0,1].$$

The plot of this function I is given in Fig. 5.

Remark 3.2.6.

(i) If $a > 0$ and $b = 1$ in this situation, then the sufficient conditions become that for all $x, y \in [0,1]$, either $I = 0$, or $I = 1$, or there exists a continuous function $c: [0,1] \to [s_2(1)/s(1), \infty)$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that

$$I(x,y) = \begin{cases} 0, & y \in [0,a), \\ s_2^{-1}\left(\min\left(c(x)s\left(\dfrac{y-a}{1-a}\right), s_2(1)\right)\right), & y \in [a,1], \end{cases} \quad x, y \in [0,1].$$

(ii) If $a = 0$ and $b < 1$ in this situation, then the sufficient conditions become that for all $x, y \in [0,1]$, either $I = 0$, or $I = 1$, or there exists a continuous function $c: [0,1] \to [s_2(1)/s(1), \infty)$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that

$$I(x,y)=\begin{cases}1, & y\in(b,1],\\ s_2^{-1}\left(\min\left(c(x)s\left(\frac{y}{b}\right),s_2(1)\right)\right), & y\in[0,b],\end{cases}\quad x,y\in[0,1].$$

Corollary 3.2.3. *Let S_1 be a continuous t-conorm given by $\langle a,b,S\rangle$ with a nilpotent t-conorm S and S_2 be a nilpotent t-conorm. Then there are no continuous solutions to Eq.* (2) *which satisfy* I3.

Proof. This proof is similar to the one of Corollary 3.1.1. □

Theorem 3.2.6. *Let S_1 be a continuous t-conorm given by $\langle a,b,S\rangle$ $(0<a<b<1)$, where S is a nilpotent t-conorm with an additive generator s. Let S_2 be a nilpotent t-conorm with an additive generator s_2 and $I:[0,1]^2\to[0,1]$ be a binary function, which is continuous except the vertical section $I(0,y)$ for $y\in[0,a]$ and satisfies* I3. *Then the triple of functions S_1, S_2 and I satisfies Eq.* (2) *if and only if there exists a continuous function $c:[0,1]\to[s_2(1)/s(1),\infty]$ with $c(0)=\infty$ and $c(x)\in[s_2(1)/s(1),\infty)$ for $x\in(0,1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that I is given by*

$$I(x,y)=\begin{cases}1, & x=0,\ y\in[0,1],\\ 0, & x\neq 0,\ y\in[0,a),\\ s_2^{-1}\left(\min\left(c(x)s\left(\frac{y-a}{b-a}\right),s_2(1)\right)\right), & x\neq 0,\ y\in[a,b],\\ 1, & x\neq 0,\ y\in(b,1],\end{cases}\quad x,y\in[0,1]. \tag{14}$$

Proof. Similar to Theorem 3.1.3. □

Remark 3.2.7. If $a=0$ or $b=1$ in the theorem above, then it is obvious to obtain the following sufficient conditions:

(i) If $a>0$ and $b=1$, then for all $x,y\in[0,1]$, there exists a continuous function $c:[0,1]\to[s_2(1)/s(1),\infty]$ with $c(0)=\infty$ and $c(x)\in[s_2(1)/s(1),\infty)$ for $x\in(0,1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that

$$I(x,y)=\begin{cases}1, & x=0,\ y\in[0,1],\\ 0, & x\neq 0,\ y\in[0,a),\\ s_2^{-1}\left(\min\left(c(x)s\left(\frac{y-a}{1-a}\right),s_2(1)\right)\right), & x\neq 0,\ y\in[a,1],\end{cases}\quad x,y\in[0,1]. \tag{15}$$

(ii) If $a=0$ and $b<1$, then for all $x,y\in[0,1]$, there exists a continuous function $c:[0,1]\to[s_2(1)/s(1),\infty]$ with $c(0)=\infty$ and $c(x)\in[s_2(1)/s(1),\infty)$ for $x\in(0,1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that

$$I(x,y)=\begin{cases}1, & x=0,\ y\in[0,1],\\ s_2^{-1}\left(\min\left(c(x)s\left(\frac{y}{b}\right),s_2(1)\right)\right), & x\neq 0,\ y\in[0,b],\\ 1, & x\neq 0,\ y\in(b,1],\end{cases}\quad x,y\in[0,1]. \tag{16}$$

Corollary 3.2.4. *Let S_1 be a continuous t-conorm given by $\langle a,b,S\rangle$ $(0<a<b<1)$, where S is a nilpotent t-conorm with an additive generator s. Let S_2 be a nilpotent t-conorm with an additive generator s_2 and I be a fuzzy implication, which is continuous except the vertical section $I(0,y)$ for $y\in[0,a]$. Then the triple of functions S_1, S_2* and *I satisfies Eq.* (2) *if and only if there exists a continuous and decreasing function $c:[0,1]\to[s_2(1)/s(1),\infty]$* with $c(0)=\infty$ *and $c(x)\in[s_2(1)/s(1),\infty)$ for $x\in(0,1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that for all $x,y\in[0,1]$, I has representation* (14).

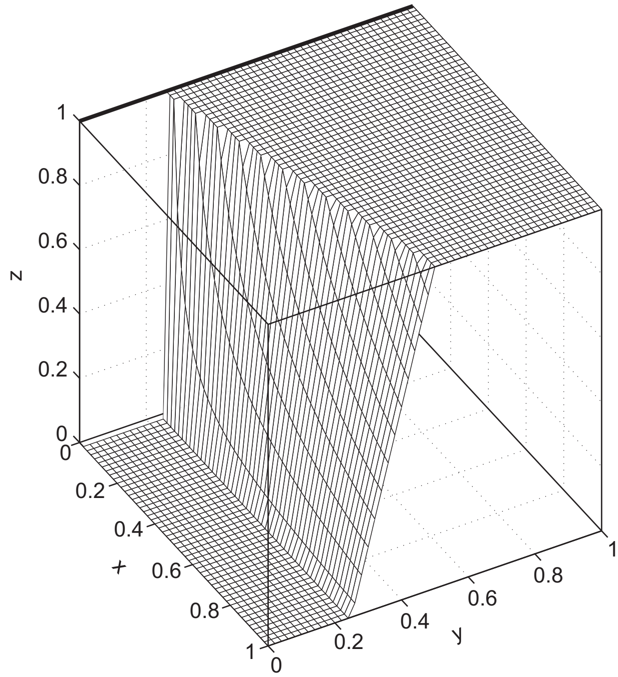

Fig. 6. The plot of fuzzy implication I in Example 3.2.4 (the bolded line is the line $I(0, y) = 1$).

Example 3.2.4. Let $S_2 = S_L$ with an additive generator $s_2(x) = x$, $c(x) = 2/x$ and $S_1 = \langle \frac{1}{4}, \frac{3}{4}, S_L\rangle$. Then by simple calculation we get that fuzzy implication I is given by

$$I(x, y) = \begin{cases} 1, & x = 0,\ y \in [0, 1], \\ 0, & x \neq 0,\ y \in \left[0, \frac{1}{4}\right), \\ \min\left(\dfrac{4y-1}{x}, 1\right), & x \neq 0,\ y \in \left[\frac{1}{4}, \frac{3}{4}\right], \\ 1, & x \neq 0,\ y \in \left(\frac{3}{4}, 1\right]. \end{cases}$$

Its plot is presented in Fig. 6.

Remark 3.2.8.

(i) If $a > 0$ and $b = 1$ in Corollary 3.2.4, then Eq. (2) holds if and only if for all $x, y \in [0, 1]$, there exists a continuous and decreasing function $c: [0, 1] \to [s_2(1)/s(1), \infty]$ with $c(0) = \infty$ and $c(x) \in [s_2(1)/s(1), \infty)$ for $x \in (0, 1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that I has representation (15).

(ii) If $a = 0$ and $b < 1$ in Corollary 3.2.4, then Eq. (2) holds if and only if for all $x, y \in [0, 1]$, there exists a continuous and decreasing function $c: [0, 1] \to [s_2(1)/s(1), \infty]$ with $c(0) = \infty$ and $c(x) \in [s_2(1)/s(1), \infty)$ for $x \in (0, 1]$, uniquely determined up to a positive multiplicative constant depending on constants for s_2 and s, such that I has representation (16).

Remark 3.2.9. In the above, we have only considered the case with one summand in the ordinal sum of S_1. We will show that the case with finite summands will be turned into the one with one summand.

Let $S_1 = (\langle a_i, b_i, S_i\rangle)_{i\in A=\{1,2,\ldots,n\}}$ and $0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n \le 1$.

We first suppose that the triple of functions S_1, S_2 and I satisfies Eq. (2).

For any $i \in A$, similar to the necessary proof of Theorem 3.1.1, we can get that for every $x \in [0, 1]$, $I(x, y) = 0$ or 1 if $y \in [b_i, a_{i+1}]$. Also we have that for every $x \in [0, 1]$, if there exists some $y_0 \in [b_i, a_{i+1}]$ such that $I(x, y_0) = 0$, then $I(x, y) = 0$ for $y \in [0, y_0]$, and $I(x, y) = 1$ for $y \in [y_0, 1]$ if $I(x, y_0) = 1$.

Fix arbitrarily $x \in [0, 1]$ and let $d_{x,i} = \sup_{y\in[b_i,a_{i+1}]}\{y|I(x, y) = 0, 1\le i\le n-1\}$ with the convention $\sup\emptyset = b_i$ and $\inf\emptyset = a_{i+1}$. Obviously, $d_{x,i} \in [b_i, a_{i+1}]$. If $b_i = a_{i+1}$, then $d_{x,i} = b_i = a_{i+1}$ whenever $I(x, b_i) = 1$ or $I(x, b_i) = 0$. For convenience, we denote $d_{x,i} = b_i$ (not a_{i+1}) if $b_i = a_{i+1}$ and $I(x, b_i) = 1$, and denote $d_{x,i} = a_{i+1}$ (not b_i) if $b_i = a_{i+1}$ and $I(x, b_i) = 0$.

(i) Assume that there exists some $i_0 \in A$ such that $d_{x,i_0} \in (b_{i_0}, a_{i_0+1})$. Then the vertical section $I(x, \cdot)$ is obviously given by

$$I(x, y) = \begin{cases} 0, & y\le d_{x,i_0}, \\ 1, & y > d_{x,i_0}, \end{cases} \quad \text{for some } d_{x,i_0} \in (b_{i_0}, a_{i_0+1}),$$

or

$$I(x, y) = \begin{cases} 0, & y < d_{x,i_0}, \\ 1, & y\ge d_{x,i_0}, \end{cases} \quad \text{for some } d_{x,i_0} \in (b_{i_0}, a_{i_0+1}).$$

(ii) Assume that $d_{x,i} = b_i$ for all $i \in A$. Then we get that $I(x, y) = 1$ for all $y \in (b_1, 1]$. Consequently, for the arbitrarily fixed x, we only need to discuss Eq. (2) with summand $\langle a_1, b_1, S_1\rangle$.

(iii) Assume that $d_{x,i} = a_{i+1}$ for all $i \in A$ ($i\le n-1$). Then for the arbitrarily fixed x, $I(x, y) = 0$ for $y \in [0, a_n)$ and thus only the case with summand $\langle a_n, b_n, S_n\rangle$ ought to be discussed.

(iv) Assume that there exists some $i_1 \in A$ ($2\le i_1\le n-1$) such that $d_{x,i_1-1} = a_{i_1}$ and $d_{x,i_1} = b_{i_1}$. Then $I(x, y) = 0$ for $y \in [0, a_{i_1})$ and $I(x, y) = 1$ for $y \in (b_{i_1}, 1]$. Hence, for the arbitrarily fixed x, we need to investigate Eq. (2) with summand $\langle a_{i_1}, b_{i_1}, S_{i_1}\rangle$.

From the above, we know that if Eq. (2) holds, then for every fixed $x \in [0, 1]$, there is only one possible corresponding summand. Therefore, we can deal with Eq. (2) similarly to Theorem 3.1.1 and Remark 3.1.1.

It should be pointed out that different x may correspond to different summand. However, if I is continuous, one of the necessary conditions of Eq. (2) is that all the corresponding summands equal each other for all $x \in [0, 1]$.

In fact, let us suppose that Eq. (2) holds and I is continuous. We should notice that the result in (i) is not possible since I is continuous. Suppose that there exist x_1, x_2 ($x_1 \ne x_2$) such that $\langle a_k, b_k, S_k\rangle$ and $\langle a_l, b_l, S_l\rangle$ are their corresponding summands, respectively. Let us assume $a_k\ge b_l$ because the case that $b_k\le a_l$ is similar. Then for arbitrarily fixed $y \in [b_l, a_k]$, we have that $I(x_1, y) = 0$ but $I(x_2, y) = 1$, which contradicts the continuity of I over the first place. So all the corresponding summands are equal to each other for all $x \in [0, 1]$.

Hence, if I is continuous, then we only need to investigate the continuous solutions to Eq. (2) with a summand of S_1. Similarly to Theorem 3.1.2 and Remark 3.1.2, we can get the sufficient and necessary conditions under which the triple of functions S_1, S_2 and I satisfies Eq. (2).

Let I be continuous on $(0, 1] \times [0, 1]$. Similar to the above, we can get that one of the necessary conditions of Eq. (2) is that all the summands equal a summand of S_1 for all $x \in (0, 1]$. Suppose that the summand is $\langle a_{i'}, b_{i'}, S_{i'}\rangle$ ($i' \in A$) and I is continuous except the vertical section $I(0, y)$ for $y \in [0, a_{i'}]$. Then just as the proof of Theorem 3.1.3 and Remark 3.1.4, we can characterize the non-continuous solutions to Eq. (2) and then we can get its non-continuous fuzzy implication solutions.

Therefore, the case with finite summands in S_1 can be turned into the one with one summand in S_1.

4. Conclusions

The authors in [3,4] investigated Eq. (2) in the case that both S_1 and S_2 are continuous Archimedean t-conorms. Also Baczyński and Jayaram [4] pointed out that their future work is to discuss Eq. (2) with two t-conorms which are not both continuous Archimedean. In this work, we studied Eq. (2) under the condition that S_1 is a continuous t-conorm given as an ordinal sum and S_2 is a continuous Archimedean t-conorm. Moreover, we characterized the continuous solutions to Eq. (2) and found out its non-continuous fuzzy implication solutions. To some extent, this work partly resolved Baczyński and Jayaram's problem [4].

In our future work, we will concentrate on the case in which both S_1 and S_2 are continuous t-conorms given as ordinal sums.

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