Corollary (Matrix determinant lemma) Consider a general partitioned matrix $\mathbf{M} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$, where E and H are assumed *invertible*, then:

$$\det(E - FH^{-1}G) = \det(H - GE^{-1}F)\det(H^{-1})\det(E)$$

proof: recall the following two properties of matrix determinant:

$$\begin{aligned} \det(AB) &= \det A \det B \\ \det(A^{-1}) &= \frac{1}{\det A} \end{aligned}$$

(1), We can rearrange the matrix in the following way:

$$\begin{aligned} \begin{pmatrix} E & F \\ G & H \end{pmatrix} &= \begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & E^{-1}F \\ G & H \end{pmatrix} \\ &= \begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ G & I \end{pmatrix} \begin{pmatrix} I & E^{-1}F \\ 0 & H - GE^{-1}F \end{pmatrix} \end{aligned}$$

taking determinant on both sides:

$$\begin{aligned} \det \begin{pmatrix} E & F \\ G & H \end{pmatrix} &= \det \begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix} \det \begin{pmatrix} I & 0 \\ G & I \end{pmatrix} \det \begin{pmatrix} I & E^{-1}F \\ 0 & H - GE^{-1}F \end{pmatrix} \\ &= \det(E)\det(I)\det(H - GE^{-1}F) \\ &= \det(E)\det(H - GE^{-1}F) \qquad (1) \end{aligned}$$

(2), we can pre-multiply a matrix on M in order to zero-out its upper-right block:

$$\begin{pmatrix} I & -FH^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} E - FH^{-1}G & 0 \\ G & H \end{pmatrix}$$

taking determinant on both sides:

$$\det \begin{pmatrix} I & -FH^{-1} \\ 0 & I \end{pmatrix} \det \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \det \begin{pmatrix} E - FH^{-1}G & 0 \\ G & H \end{pmatrix}$$

$\Rightarrow$

$$\det(I) \det \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \det\left(E - FH^{-1}G\right) \det H$$

$\Rightarrow$

$$\begin{aligned} \det\left(E - FH^{-1}G\right) &= \frac{1}{\det H} \det \begin{pmatrix} E & F \\ G & H \end{pmatrix} \\ &= = \det(H^{-1}) \det \begin{pmatrix} E & F \\ G & H \end{pmatrix} \qquad (2) \end{aligned}$$

Insert Eq. (1) into Eq. (2),

$$\det\left(E - FH^{-1}G\right) = \det(H^{-1})\det(E)\det(H - GE^{-1}F) \qquad \# \qquad (3)$$