Stat 598W: HW5/Final exam

Due: Friday May 8th at 11:59 pm (sveinno@purdue.edu).

This homework is to be done individually and it accounts for 25% of the final grade. Your solution should be in the form of a typed report containing your C++ code and the sample runs used to generate your output. Write clear and readable code with relevant comments.

Problem 1: Rebalancing a portfolio (25 points)

We would like to compare two investment strategies for a stock:

- *Buy-and-hold strategy*: At the beginning of Day 1, buy $1 worth of the stock and hold the position until the beginning of Day 20. By the end of each day, the stock's price has either doubled or halved, with equal probabilities.
- *Rebalancing strategy*: Buy or sell the stock each day to maintain a 50/50 balance between stock and cash. For example, at the beginning of Day 1, buy 50 cents worth of stock and keep the remaining 50 cents in cash. If the stock doubles during the day, you have $1.00 in stock and 50 cents in cash, and therefore sell 25 cents of stock so that, at the beginning of Day 2, you have 75 cents in stock and 75 cents in cash (i.e. if the stock does well, sell some of it and, if the stock does poorly, buy more of it.)

Denote by U and V the value of the buy-and-hold and rebalancing portfolios, respectively, at the beginning of Day 20.

(a) Use Monte Carlo to compute a point estimate and 95% confidence interval for $E(U)$, based on 3000 replications. Do the same for $E(V)$, using a separate stream of random numbers so that your point and interval estimates for $E(V)$ are independent of those for $E(U)$.

(b) Combine your two confidence intervals from Part (a) to obtain an approximate 95% confidence interval for $E(V) - E(U) = E(V - U)$ [Recall that if X is approximately $\mathcal{N}(\mu_x, \sigma_x^2)$ and Y is independent of X and approximately $\mathcal{N}(\mu_y, \sigma_y^2)$, then $aX + bY$ is approximately $\mathcal{N}(a\mu_x + b\mu_y, a^2\sigma_x^2 + b^2\sigma_y^2)$, for any real a and b.]

(c) Compute a 95% confidence interval for $E(V - U)$ based on 3000 simulation replications. That is, for each replication, compute both V and U and then subtract U from V. Is this confidence interval wider or narrower than that of Part (b)? Give an intuitive explanation of the difference in widths.

(d) Now use a "logarithmic utility function", i.e. suppose that the degree of satisfaction from having $\$x$ is $\log_{10}(x)$. As in Part (c), compute a point estimate and 95% confidence interval for $E(\log_{10}(V) - \log_{10}(U))$ based on 3000 replications. Compare the result to the result of Part (c). Do you think using utility functions gives a better comparison of investment alternatives?

Problem 2 (25 points)

A. Black-Scholes delta hedging

We would like to use simulation to measure the hedging error resulting from discrete rebalancing of a hedge. You sell a 3-month European call option at the Black-Scholes price and try to hedge it by holding "delta" shares of the stock.

- You can deposit and borrow money from the bank at constant interest rate.
- At initiation of the contract, you get the premium from the client and buy delta shares of the stock. You may need to borrow extra money.
- At each time step, the stock has evolved from the previous step and the delta must be adjusted. Depending on how it has changed, you need either to buy or sell shares. You also pay or earn interest on any money borrowed or deposited over the previous period.

- At maturity, you close your position. This means: selling all shares you own, reimburse the bank for the money you owe or get what is left in your account, and paying your client the amount $(S_T - K)^+$. How much cash is left after that is your profit or loss.

Assume the underlying S is modeled by geometric Brownian motion and use the following parameters: Initial price $S_0 = 50$, rate of return $\mu = 10\%$, volatility $\sigma = 30\%$, interest rate $r = 5\%$, strike $K = 50$, expiration $T = 0.25$. Black-Scholes theory says that as the number of times you rebalance goes to infinity (with T fixed), your hedging error (profit or loss) goes to zero on each path. Since in practice continuous trading is impossible, the hedge is imperfect and we want to study this imperfection.

(a) Find the mean and standard deviation of the hedging error with daily and weekly rebalancing. Make a histogram of the distribution of hedging errors for both cases.

(b) Take different values for μ. How does that change the results? Is this surprising?

(c) Let Δt denote the rebalancing interval. We know that as $\Delta t \to 0$ the hedging error goes to 0. What does your simulation suggest about the rate of convergence? Does the hedging error appear to be $O((\Delta t)^\alpha)$ for some α? If so, what α? (You may want to draw a log-log plot).

B. A stop-loss strategy

Here we consider another hedging scheme for call options. This strategy consists first of charging $(S_0 - Ke^{-rT})^+$ for the option, and then hedging it by holding one share when $S_t > Ke^{-r(T-t)}$ and no share at all when $S_t \le Ke^{-r(T-t)}$. It is based on the simple observation that the seller will need one share at expiry if $S_T > K$ and none if $S_T \le K$.

Modify your code from Part A and find the mean and standard deviation of the hedging error with daily and weekly rebalancing. Make a histogram of the distribution of hedging errors for both cases. Compare with the delta hedging strategy and explain the difference. What is the main disadvantage of this strategy?

Problem 3

A. Barrier options in the Black-Scholes model

The purpose of this problem is to compute the price of an up-and-out put option. An up-and-out put option (UOP for short) with strike K and barrier level H has the same payoff at time T as a vanilla put option, $(K - S_T)^+$, unless the stock goes above the barrier level H during the life of the option, in which case the holder receives nothing. Is such an option cheaper or more expensive than a put option without a barrier? By discretizing at times $t_i = i\frac{T}{m}$ for $i = 1, \dots, m$, an estimator for the price is

$$\hat{C}_{n,m} = e^{-rT}\frac{1}{n}\sum_{k=1}^{n}(K - S_T^{(k)})^+ \mathbf{1}_{\{\max_{1\le i\le m} S_{t_i}^{(k)} < H\}}$$

Use the parameters $S_0 = 50$, $K = 60$, $\sigma = 30\%$, $T = 0.25$, $r = 5\%$, and $H = 55$, to compute an estimate of the price. Take $m = 0.25 \cdot 252 = 63$. Take n large enough so that your 95% confidence interval is smaller than 5 cents. Is the estimate biased? Is it biased low or high?

B. Reduction of the bias

There exists an explicit formula for the probability of geometric Brownian motion crossing the barrier. Namely,

$$q\big(t_i, x, t_{i+1}, y\big) = \mathbb{Q}\left(\sup_{t_i \le u \le t_{i+1}} S_u \ge H \,\middle|\, S_{t_i} = x, S_{t_{i+1}} = y\right) = \exp\left(\frac{2/\sigma^2}{t_{i+1} - t_i}\ln\left(\frac{H}{x}\right)\ln\left(\frac{y}{H}\right)\right), \tag{0.1}$$

where $x, y < H$. This can be used to improve the estimator: A new estimator for the price is

$$\widetilde{C}_n = e^{-rT}\frac{1}{n}\sum_{k=1}^{n}\big(K - S_T^{(k)}\big)^+\big(1 - q\big(0, S_0, T, S_T^{(k)}\big)\big).$$

Show that this estimator is unbiased. Give an estimate of the UOP option with a 95% confidence interval. Is it different from the estimate in Part A?

C. Non-constant volatility

Suppose now that the stock price follows a CEV (constant elasticity of variance) process:

$$\frac{dS_t}{S_t} = \mu dt + \alpha S_t^{\beta} dW_t.$$

The estimator of Part A (and Euler discretization) can be used to price an UOP. However, one way to reduce the bias is saying that between discretization times, the CEV process can be approximated by a geometric Brownian motion. Then, the formula (0.1) can be used to compute an approximate value for the probability that the barrier is crossed between two discretization times. Using that, explain precisely how you would construct a Monte Carlo estimate for the price of an UOP (you don't need to implement it).

Problem 4: Pricing of basket options

We want to price a call on a basket of assets. Suppose that we have $(S_t^i), i = 1, \dots, d$ stocks with dynamics under the risk neutral measure $\mathbb{Q}$:

$$\frac{dS_t^i}{S_t^i} = rdt + \sigma_i dW_t^i, \qquad S_0^i = s_0^i.$$

Suppose that $(W^1, \dots, W^d)$ is a d-dimensional Brownian motion with $\mathbb{E}(W_t^i W_t^j) = \rho t$ for $i \neq j$. We want to price a call option on this basket with payoff $(S_T^1 + \cdots + S_T^d - K)^+$ where $K = s_0^1 + \cdots + s_0^d$. Consider $d = 10$, $r = 0.04$, $s_0^i = 100$, $\sigma_i = 0.15$, $T = 1$, and $\rho = 0.3$.

(a) Compute the price of this option by Monte Carlo.

(b) What can you say about the convergence of the Monte Carlo estimate?

(c) Explain why we could use the control variate $(d(S_T^1 \dots S_T^d)^{1/d} - K)^+$ to reduce the variance.

(d) Implement this variance reduction technique. In particular, you will have to compute

$$\mathbb{E}^{\mathbb{Q}}[(d(S_T^1 \dots S_T^d)^{1/d} - K)^+]$$

in terms of the Black-Scholes formula. What can you conclude?