PURDUE UNIVERSITY · CS 51500
MATRIX COMPUTATIONS

HOMEWORK
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Problem 0: Homework checklist

✓I didn't talk with any one about this homework.
✓Source-code are included at the end of this document.

Problem 1: The Cholesky Factorization

1. Here is my implementation of Cholesky decomposition in Matlab:

```
function L = cholesky(A)
n = size(A);
n = n(1);
L = zeros(n,n);
%L(1,1) = sqrt(A(1,1));
for i=(1:n)
    sum1 =0;
    for j=(1:i)

        if i==j
            L(j,j) = sqrt(A(j,j)-sum1);
        else
            sum2 = 0;
            for k = (1:j-1)
                sum2 = sum2 + L(i,k)*L(j,k);
            end

            L(i,j) = 1/L(j,j)*(A(i,j)-sum2);
        end
        sum1= sum1 + L(i,j)^2;
    end
end
end
```

2. Cholesky factorization is unique if $\boldsymbol{A}$ is positive definite and the decomposition need not be unique when $\boldsymbol{A}$ is positive semidefinite.
Proof:
Get the LU decomposition of $\boldsymbol{A} = \boldsymbol{LU}$, and $u_{11} > 0, u_{22} > 0, ..., u_{mm} > 0$

$$\boldsymbol{U} = \begin{bmatrix} u_{11} & u_{12} & ... & u_{1m} \\ 0 & u_{22} & ... & u_{2m} \\ 0 & 0 & ... & ... \\ 0 & 0 & ... & u_{mm} \end{bmatrix}$$
$$= \begin{bmatrix} u_{11} & 0 & ... & 0 \\ 0 & u_{22} & ... & 0 \\ 0 & 0 & ... & ... \\ 0 & 0 & ... & u_{mm} \end{bmatrix} \begin{bmatrix} 1 & v_{12} & ... & v_{1m} \\ 0 & 1 & ... & v_{2m} \\ 0 & 0 & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix} = \boldsymbol{TW}$$

Since $u_{11} > 0, ..., u_{mm} > 0$, we can get $\boldsymbol{T} = \boldsymbol{D}^2$, where

$$\boldsymbol{D} = \begin{bmatrix} \sqrt{u_{11}} & 0 & ... & 0 \\ 0 & \sqrt{u_{22}} & ... & 0 \\ 0 & 0 & ... & ... \\ 0 & 0 & ... & \sqrt{u_{mm}} \end{bmatrix}$$
$$\boldsymbol{A} = \boldsymbol{LU} = \boldsymbol{LD}_2\boldsymbol{W}$$

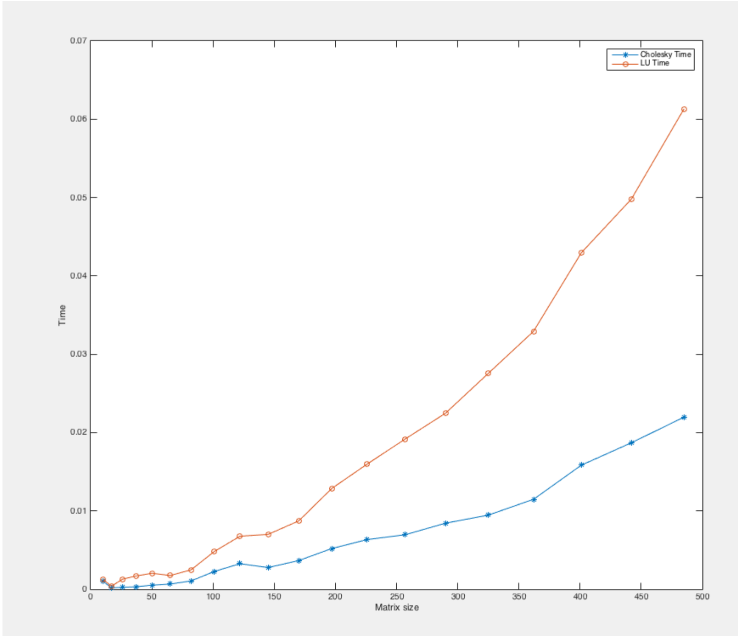

Figure 1: Performance comparison between LU decomposition and Cholesky factorization

Because $\boldsymbol{A}_* = \boldsymbol{A}$ and the LU factorization is unique, then $\boldsymbol{L} = \boldsymbol{W}^*$. Therefore the Cholesky factorization is unique if $\boldsymbol{A}$ is positive definite.
Reference: http://math.utoledo.edu/ mtsui/4350sp08/homework/Lec23.pdf

3. I computed the norm of difference of my Cholesky result and Matlab *chlo*:

$$1.9110e - 13$$

4. For each size of the matrices I repeated 10 times. The comparison result is shown. Obviously the the Cholesky factorization has better performance as the growth of matrix size.

5. Since $\boldsymbol{A}$ is positive definite, and assume an vector $\mathbf{v} = \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix}$,

   then

$$\mathbf{v}_T \boldsymbol{A} \mathbf{v} > 0$$

$$\begin{bmatrix} x_0 & \mathbf{x} \end{bmatrix} \begin{bmatrix} \alpha & \mathbf{b}_T \\ \mathbf{b} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix} = \alpha x_0^2 + x_0(\mathbf{x}^T\mathbf{b} + \mathbf{b}^T\mathbf{x}) + \mathbf{x}^T\boldsymbol{C}\mathbf{x} > 0$$

Then

$$
\begin{aligned}
\mathbf{x}^T(\boldsymbol{C} - \frac{\mathbf{b}\mathbf{b}^T}{\alpha})\mathbf{x} &= \mathbf{x}^T\boldsymbol{C}\mathbf{x} - \frac{\mathbf{x}_T\mathbf{b}\mathbf{b}^T\mathbf{x}}{\alpha} \\
&> -\alpha x_0^2 - x_0(\mathbf{x}^T\mathbf{b} + \mathbf{b}^T\mathbf{x}) - \frac{\mathbf{x}^T\mathbf{b}\mathbf{b}^T\mathbf{x}}{\alpha} \\
&= -\frac{1}{\alpha}\left[x_0^2 + \frac{\mathbf{x}^T\mathbf{b} + \mathbf{b}^T\mathbf{x})x_0}{\alpha} + \frac{\mathbf{x}^T\mathbf{b}\mathbf{b}^T\mathbf{x}}{\alpha^2}\right] \\
&= -\frac{1}{\alpha}\left(x_0 + \frac{\mathbf{x}^T\mathbf{b}}{\alpha}\right)\left(x_0 + \frac{\mathbf{b}^T\mathbf{x}}{\alpha}\right) \\
&= -\frac{1}{\alpha}\left\|x_0 + \frac{\mathbf{b}^T\mathbf{x}}{\alpha}\right\|^2
\end{aligned}
$$

Because $\mathbf{x}^T(\boldsymbol{C} - \frac{\mathbf{b}\mathbf{b}^T}{\alpha})\mathbf{x} > -\frac{1}{\alpha}\left\|x_0 + \frac{\mathbf{b}^T\mathbf{x}}{\alpha}\right\|^2$ holds for any vector $\mathbf{x}$. Therefore

$$\mathbf{x}^T(\boldsymbol{C} - \frac{\mathbf{b}\mathbf{b}^T}{\alpha})\mathbf{x} > 0 \tag{1}$$

So $\boldsymbol{C} - \frac{\mathbf{b}\mathbf{b}^T}{\alpha}$ is positive definite.

6. The base case is that C is a 1×1.

7. In the Matlab code, we need check positiveness before take square root. The modified code is here:

```
function L = cholesky(A)
n = size(A);
n = n(1);
L = zeros(n,n);
%L(1,1) = sqrt(A(1,1));
for i=(1:n)
    sum1 =0;
    for j=(1:i)

        if i==j
            if A(j,j)-sum1≤0
                return       % check whether A is positive ...
                    definite.
            else
                L(j,j) = sqrt(A(j,j)-sum1);
            end
        else
            sum2 = 0;
            for k = (1:j-1)
                sum2 = sum2 + L(i,k)*L(j,k);
            end

            L(i,j) = 1/L(j,j)*(A(i,j)-sum2);
        end
        sum1= sum1 + L(i,j)^2;
    end
end
end
```

After testing, the matrix for Poisson's equation is not positive definite or negative definite.

Problem 2: Stability analysis

1.

$$V = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \frac{1}{n}x_i)^2$$
$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2\right]$$

The condition number is defined as :

$$\kappa = lim_{\epsilon\to 0}sup\left[\frac{|V(x)-V(x+\Delta x)|}{\epsilon V(x)} : \|\Delta x\|_2 \leq \epsilon\|x\|_2\right]$$

Then

$$\kappa = 2\frac{\|x\|_2}{\sqrt{(n-1)V(x)}}$$

2. This is not appropriate. I will use $q = 100ands = 2$ to do a test.

```
q = 500;
s = 7;
z = zeta(s);
h = z-sum((1:(q-1)).^(-s))
```

Hurwitz zeta function:

$$H(s,q) = \sum_{n=0}^{\infty}\frac{1}{(q+n)^s}$$
$$= \frac{1}{q^s} + \frac{1}{(q+1)^s} + ... + \frac{1}{n^s}$$

Riemann zeta function:

$$R(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$
$$= \frac{1}{1^s} + \frac{1}{2^s} + ... + \frac{1}{n^s}$$
$$= \frac{1}{1^s} + \frac{1}{2^s} + ... + \frac{1}{(q-1)^2} + \frac{1}{q^s} + \frac{1}{(q+1)^s} + ... + \frac{1}{n^s}$$
$$= \frac{1}{1^s} + \frac{1}{2^s} + ... + \frac{1}{(q-1)^2} + H(s,q)$$

Output from Matlab is:

$$2.2204e - 15$$

But the true answer from Wolfram Alpha is

$$1.073081599928320 \times 10^{-17}$$

Here is the explanation: when you calculate the Riemann zeta function, the terms with larger n will round off because they are relative smaller than terms with smaller n. The leading terms

$$\frac{1}{1^s} + \frac{1}{2^s} + ... + \frac{1}{(q-1)^2}$$

is dominant in $R(s)$. But those are the terms we need for Hurwitz zeta with large q. Once you round off them, there would be large errors.

Problem 3: Backwards stability

$$\begin{aligned} fl(x_i) &= x_i(1+\epsilon_{ix}) \\ fl(y_i) &= y_i(1+\epsilon_{iy}) \end{aligned}$$

where $|\epsilon_{ix}|, |\epsilon_{iy}| \leq \epsilon_{machine}$

$$\begin{aligned} \alpha &= \mathbf{x}^T\mathbf{y} \\ &= \begin{bmatrix} x_1 & x_2 & .. & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{bmatrix} \\ &= \sum_{i=1}^{n} x_i y_i \end{aligned}$$

Then

$$\begin{aligned} fl(\mathbf{x}^T\mathbf{y}) &= \sum_{i=1}^{n} fl(x_i) \times fl(y_i) \\ &= [fl(x_1) \times fl(y_1) + fl(x_2) \times fl(y_2) + ... + fl(x_n) \times fl(y_n)]\,(1+\epsilon_{addition}) \\ &= [x_1y_1(1+\epsilon_{1x})(1+\epsilon_{1y}) + ... + x_ny_n(1+\epsilon_{nx})(1+\epsilon_{ny})]\,(1+\epsilon_{addition}) \\ &= [x_1y_1 + x_2y_2 + ... + x_ny_n]\,(1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3) \\ &= [x_1y_1 + x_2y_2 + ... + x_ny_n]\,(1+\mathrm{O}\big(\epsilon_{machine}\big)) \end{aligned}$$

Therefore the inner product is backwards stable.