HOMEWORK
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Problem 0: Homework checklist

✓I didn't talk with any one about this homework.
✓Source-code are included at the end of this document.

Problem 1: Prove or disprove

For the proof below, I will assume all the matrices are $n \times n$ square matrices.

1. The product of two diagonal matrices is diagonal.
   $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonal matrices, so
   $\boldsymbol{A}_{ij} = 0, if\ i \neq j$
   $\boldsymbol{B}_{ij} = 0, if\ i \neq j$
   $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$
   $\boldsymbol{C}_{ij} = \sum_{k=1}^{n} \boldsymbol{A}_{ik}\boldsymbol{B}_{kj}$
   Only if $i = j = k$, $\boldsymbol{C}_{ij}$ will not be zero, which mean $\boldsymbol{C}$ is also a diagonal matrix.

2. The product of two upper triangular matrices is upper triangular $\boldsymbol{A}$ and $\boldsymbol{B}$ are two upper triangular matrices, so

   $\boldsymbol{A}_{ij} = 0, if\ i \geqslant j$
   $\boldsymbol{B}_{ij} = 0, if\ i \geqslant j$
   $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$
   $\boldsymbol{C}_{ij} = \sum_{k=1}^{n} \boldsymbol{A}_{ik}\boldsymbol{B}_{kj}$
   When $i \geqslant j$, $\boldsymbol{C}_{ij} = 0$ because one of $\boldsymbol{A}_{ik}$ and $\boldsymbol{B}_{kj}$ must be zero.

3. The product of two symmetric matrices is symmetric. $\boldsymbol{A}$ and $\boldsymbol{B}$ are two symmetric matrices, so

   $\boldsymbol{A}_{ij} = \boldsymbol{A}_{ji}, if\ i = j$
   $\boldsymbol{B}_{ij} = \boldsymbol{B}_{ji}, if\ i = j$
   $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$
   $\boldsymbol{C}_{ij} = \sum_{k=1}^{n} \boldsymbol{A}_{ik}\boldsymbol{B}_{kj}$
   Then $\boldsymbol{C}_{ij}$ is not necessary to be equal to $\boldsymbol{C}_{ji}$
   Counter-example:
   $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is a symmetric matrix;
   $\boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is also a symmetric matrix.
   $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ which is not a symmetric matrices.

4. The product of two orthogonal matrices is orthogonal.

5. The product of two square, full rank matrices is full rank This statement is incorrect.
   Counter-example:
   $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is a full rank matrix;
   $\boldsymbol{B} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix}$ is also a full rank matrix.

$\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B} = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ which is not a full matrices.

Problem 2

We know $\boldsymbol{U}$ and $\boldsymbol{V}$ are square orthogonal matrices, then

$$\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}, \boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I} \tag{1}$$

$\|\boldsymbol{UAV}\|_2^2 = \boldsymbol{V}^T\boldsymbol{A}^T\boldsymbol{U}^T\boldsymbol{UAV}$
$= \boldsymbol{V}^T\boldsymbol{A}^T\boldsymbol{AV}$
$= \boldsymbol{V}^T\|\boldsymbol{A}\|_2^2\boldsymbol{V}$
$= \boldsymbol{V}^T\boldsymbol{V}\|\boldsymbol{A}\|_2^2$
$= \|\boldsymbol{A}\|_2^2$
Therefore $\|\boldsymbol{A}\|_2$ is orthogonally invariant.

Problem 3

$$f(\boldsymbol{A}) = \max_{i,j} |A_{ij}| \tag{2}$$

1. - $f(A) > 0, when \boldsymbol{A} \neq 0,$
     $f(A) = 0 \ iff \boldsymbol{A} = 0$
   - For and scalar k,
     $f(k\boldsymbol{A})$
     $= \max_{i,j} |kA_{ij}|$
     $= k \max_{i,j} |A_{ij}|$
     $= kf(\boldsymbol{A})$
   - $f(\boldsymbol{A} + \boldsymbol{B})$
     $= \max_{i,j} |A_{ij} + B_{ij}|$
     $\leq \max_{i,j} |A_{ij}| + \max_{i,j} |B_{ij}|$
     $= f(\boldsymbol{A}) + f(\boldsymbol{B})$

   Therefore $f(\boldsymbol{A})$is a matrix norm.

2. $f(\boldsymbol{AB})$
   $= \max_{i,j} |A_{ij}B_{ij}|$
   $\leq \max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}|$
   $= f(\boldsymbol{A})f(\boldsymbol{B})$

   The $\leq$ is not necessary to be. A counter example:
   $\boldsymbol{A} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$
   $\boldsymbol{B} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$
   $\boldsymbol{AB} = \begin{bmatrix} 18 & 18 \\ 18 & 18 \end{bmatrix}$
   $f(\boldsymbol{A}) = 3$
   $f(\boldsymbol{B}) = 3$
   f($\boldsymbol{AB}$) = 18
   Then $f(\boldsymbol{AB}) > f(\boldsymbol{A})f(\boldsymbol{B})$
   So f does not satisfy the sub-multiplicative property.

3. $$g(\boldsymbol{A}) = \sigma f(\boldsymbol{A}) \tag{3}$$

$g(\boldsymbol{AB}) = \sigma \max_{i,j} |A_{ij}B_{ij}|$
$g(\boldsymbol{A})g(\boldsymbol{B}) = \sigma^2 \max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}|$

To make

$$g(\boldsymbol{AB}) \leq g(\boldsymbol{A})g(\boldsymbol{B}) \tag{4}$$

$$\sigma \max_{i,j} |A_{ij}B_{ij}| \leq \sigma^2 \max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}| \tag{5}$$

$$\sigma \geq \frac{\max_{i,j} |A_{ij}B_{ij}|}{\max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}|} \tag{6}$$

Therefore there exists $\sigma > 0$ such that: $g(\boldsymbol{A}) = \sigma f(\boldsymbol{A})$

Problem 4

$$f(x) = \|x\mathbf{k}^T\| \tag{7}$$

1. - $f(x) > 0, when \boldsymbol{A} \neq 0,$
     $f(x) = 0 \ iff \boldsymbol{A} = 0$ because $\|\boldsymbol{A}\|$is a matrix norm.
   - $f(mx) = \|mx\mathbf{k}^T\| = m\|x\mathbf{k}^T\| = mf(x)$
   - $f(x+y) = \|(x+y)\mathbf{k}^T\| = \|x\mathbf{k}^T + y\mathbf{k}^T\| \leq \|x\mathbf{k}^T\| + \|y\mathbf{k}^T\| = f(x) + f(y)$

   Therefore f is a vector norm.

2. If $\|\boldsymbol{A}\|$ is a sub-multiplicative matrix norm,

$$f(\boldsymbol{A}x) = \|\boldsymbol{A}x\mathbf{k}^T\| \leq \|\boldsymbol{A}\|\|x\mathbf{k}^T\| = \|\boldsymbol{A}\|f(x) \tag{8}$$

Problem 5 (Choice 1)

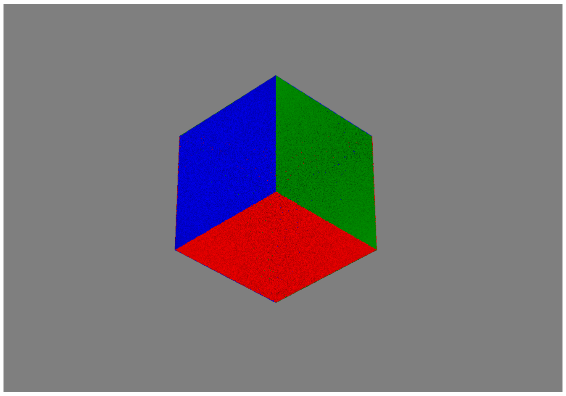
$f(\mathbf{x})$ = sum of two largest entries in $\mathbf{x}$ by absolute value is a vector form.
Suppose

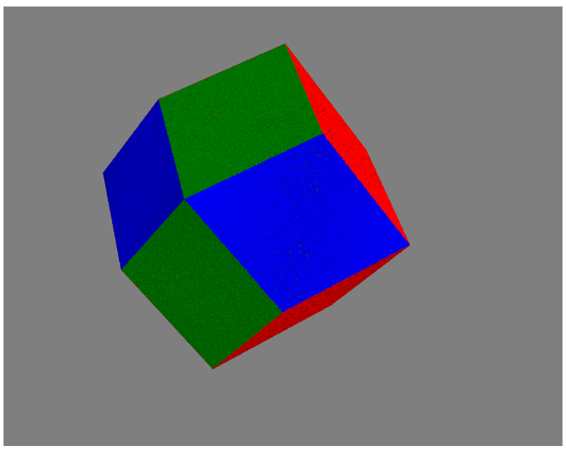
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

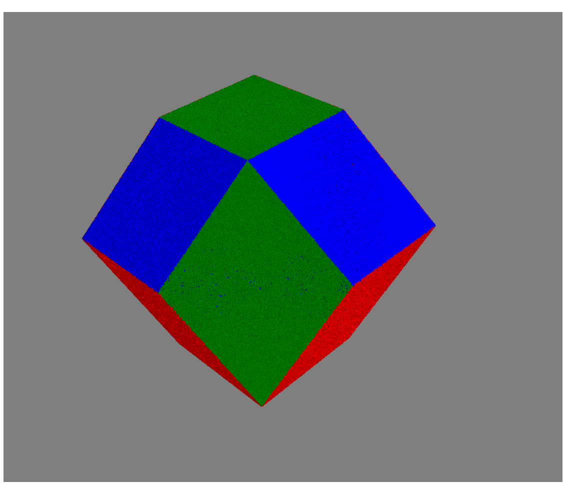
Then the unit-ball should include all points which satisfy conditions:

$$|x_1| + |x_2| = 1, |x_3| \leq |x_1|, \ |x_3| \leq |x_2|$$
$$|x_2| + |x_3| = 1, |x_1| \leq |x_2|, \ |x_3| \leq |x_3|$$
$$|x_3| + |x_1| = 1, |x_2| \leq |x_3|, \ |x_3| \leq |x_1|$$

I used Monte Carlo method to generate $N = 300000$ points which satisfy those conditions, as shown ind Figure 1.







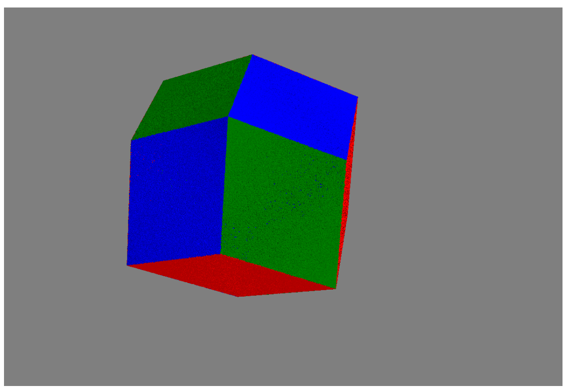


Figure 1: The unit-ball surface from different views. Color images are in electronic version.