Homework 5 report

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Problem 1

Part (a)

Buy-and-hold strategy:

Rebalancing strategy: Parameters:

- S_0: initial stock price.
- S_t: stock price at time t.
- x_t: number of stock shares at time t.
- C_t: cash held at time t in dollar.

Use the Monte Carlo to simulate the stock price to get all S_t, and then update all all C_i and x_i:

$$C_{t+1} = \frac{x_t S_{t+1}}{2}$$
$$x_{t+1} = \frac{C_1}{S_1}$$

Monte Carlo simulations:

- $u = 2$
- $d = 0.5$
- $p_u = p_d = 0.5$

Running the attached code, we can get,

$$E(U) = 56.7275 \quad var(U) = 171812$$
$$E(V) = 8.76893 \quad var(V) = 366.343$$

To get 95% confidence interval, we should $\delta = 0.05$, $z_{1-\delta/2} = 1.96$, then the confidence interval

$$\left[E(U) - 1.96\frac{\sigma_u}{\sqrt{n}}, E(U) + 1.96\frac{\sigma_u}{\sqrt{n}}\right] : [41.8947, 71.5603]$$
$$\left[E(V) - 1.96\frac{\sigma_v}{\sqrt{n}}, E(V) + 1.96\frac{\sigma_v}{\sqrt{n}}\right] : [8.08401, 9.45385]$$

Part (b)

Now we have new random variable $T = V - U$. Since V and U are independent, then we can have

$$E(T) = E(V) - E(U) = -47.9586$$
$$var(T) = var(T) + var(E) = 172178$$

then the confidence interval

$$\left[E(T) - 1.96\frac{\sigma_u}{\sqrt{n}}, E(T) + 1.96\frac{\sigma_u}{\sqrt{n}}\right] : [-62.8071, -33.11]$$

Part(c)

If we use the same stream of random numbers then the confidence interval is

$$E(V-U) = -37.2872$$
$$var(V-U) = 125996$$
$$Confidenceinterval : [-49.9892, -24.5851]$$

This confidence is wider than that of Part(b). Because when we use the same stream of random numbers to simulate both U and V, we need to consider about the correlation between them. Obviously U and V are positively correlated. This is kind of variate control method which can reduce the variance of Monte Carlo simulation.

Part(d)

If we use the same stream of random numbers then the confidence interval is

$$E(log_{10}V - log_{10}U) = 0.491468$$
$$var(log_{10}V - log_{10}U) = 0.439653$$
$$Confidenceinterval : [0.46774, 0.515195]$$

Compare with Part(c) using utility functions gives a better quatitative comparison of investment alternatives, because the variance range is narrow and then it is more easy to quantify the risk.

Problem 2

Part(a)

For daily rebalancing:

$$E(error) = -0.0056658$$

$$var(error) = 0.272379$$

$$\sigma = 0.52$$

For weekly rebalancing::

$$E(error) = -0.02113$$

$$var(error) = 1.88433$$

$$\sigma = 1.37$$

Part(b)

Changing values of μ doesn't change the result. That makes sense because the value of the option and the delta both are not related to μ.

Part(c)

Yes, the hedging error is $O((\Delta T)^{\alpha})$.

dt	$abs(error)$
$1e-06$	$7.17567e-05$
$1e-05$	0.000637571
0.0001	0.000753965
0.001	0.00321909
0.01	0.0187996

The linear regression equation is: $y = 0.554x - 0.7201$. Therefore, the α is 0.554 as show in Figure 3.

Stop-loss strategy

For daily rebalancing:

$$E(error) = 0.947117$$

$$var(error) = 37.2515$$

$$\sigma = 6.103$$

For weekly rebalancing::

$$E(error) = 0.91152$$

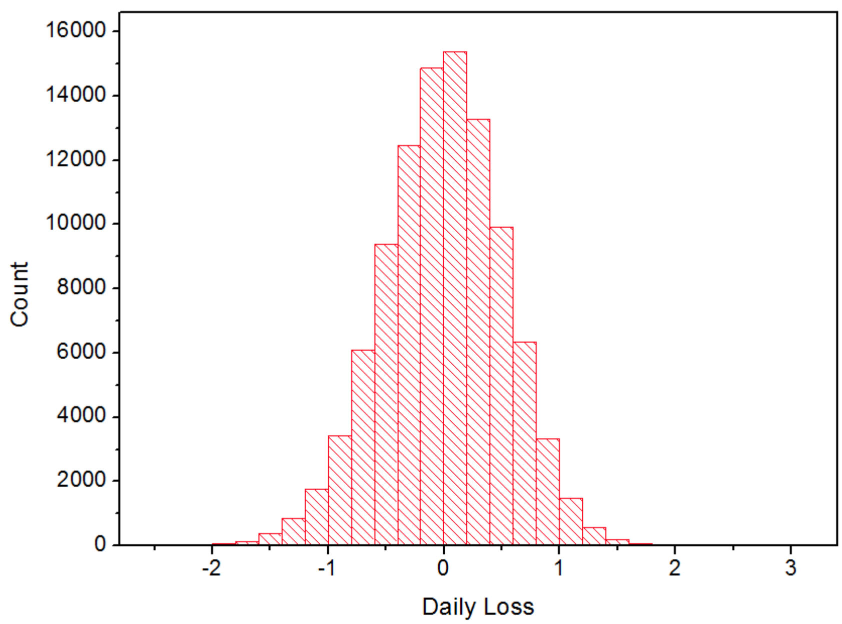


Figure 1: Delta hedging loss for daily reblancing

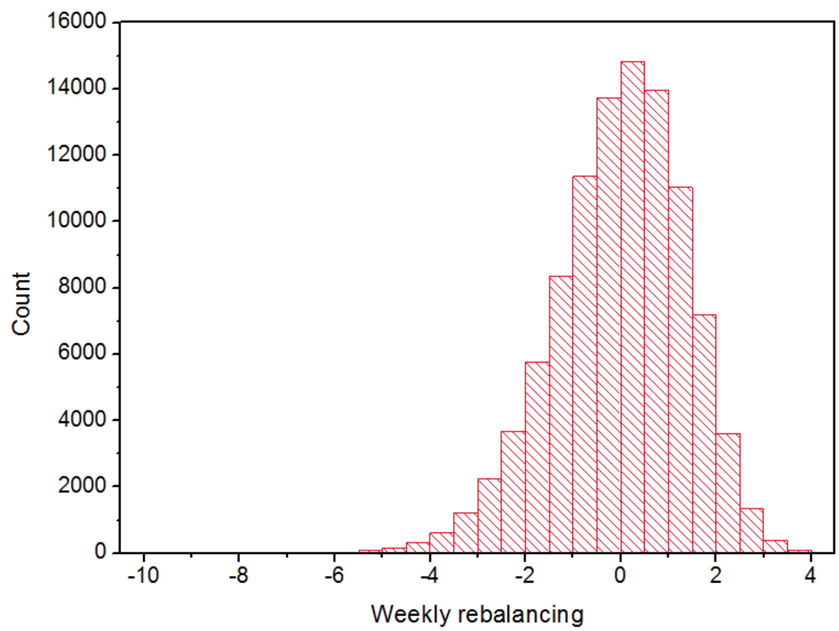
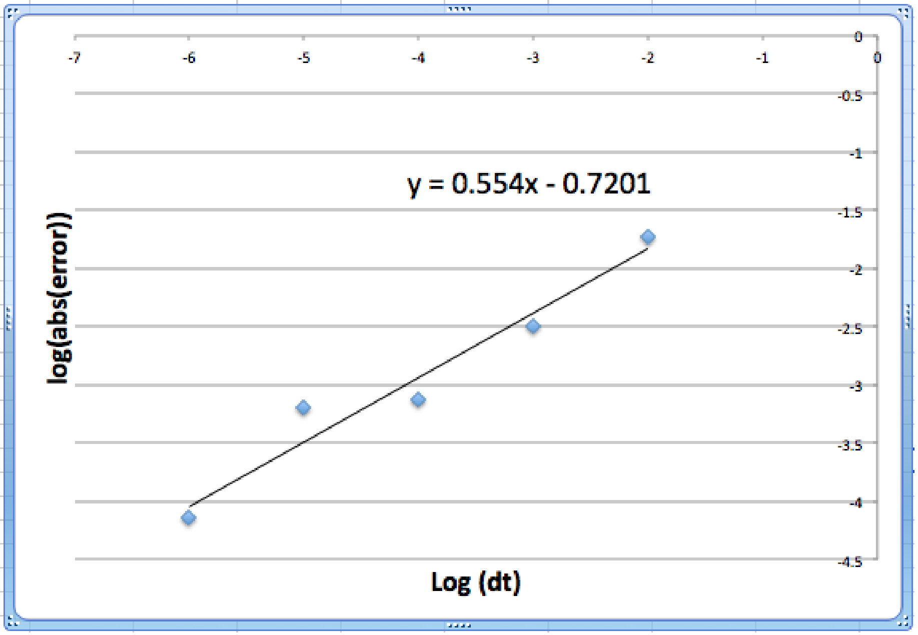


Figure 2: Delta hedging loss for weekly reblancing

Figure 3: Plot of $log(error)$ vs $log(dt)$



$$var(error) = 35.9063$$

$$\sigma = 5.992$$

Compared with delta hedging, you don't need to frequently buy or sell underlying as delta hedging. But disadvantage is that this method introduce large variance for the final loss/gain, which is shown in Figure and Figure 4.

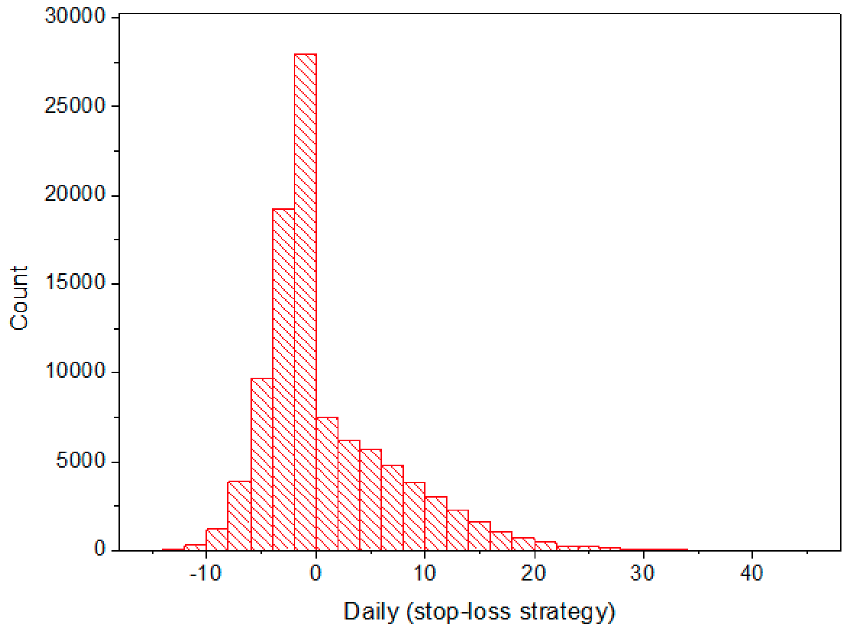


Figure 4: Stop-loss strategy loss for daily reblancing

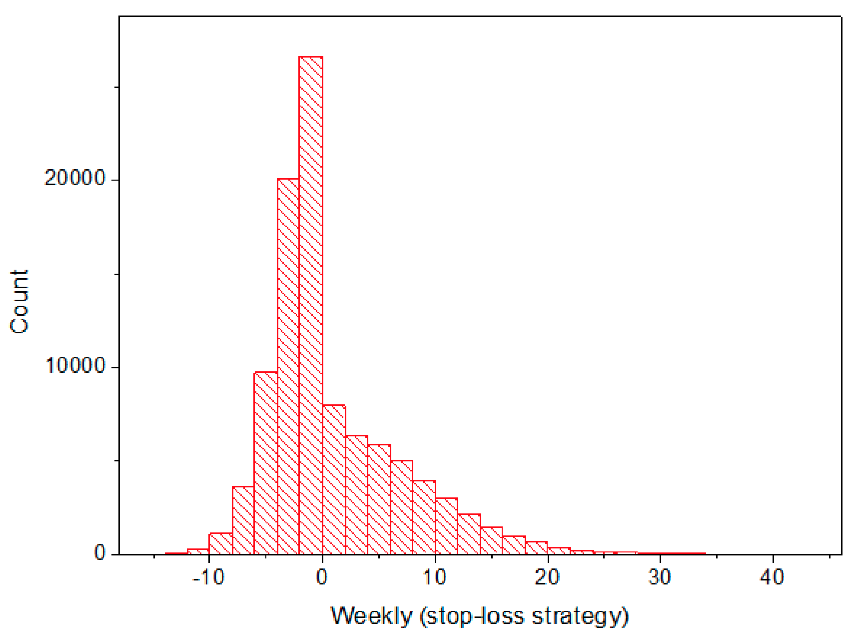


Figure 5: Stop-loss strategy loss for weekly reblancing

Problem 3

Part(a)

$$E(X) = 7.23497$$
$$std(X) = 7.9105$$
$$95\% confidence interval : 0.0310091$$

I used 1million replications to get this 95% interval. This estimator has bias, which is than real value.

Part(b)

$$E(X) = 7.22314$$
$$std(X) = 6.43481$$
$$95\% confidence interval : 0.0252244$$

This is an unbiased estimator of UOP option. The standard deviation is smaller.

Problem 4

Part(a)

Implementation of Cholesky algorithm:

$$a_{i,j} = (\sum_{i,j} - \sum_{k=1}^{j-1} a_{i,k}a_{j,k})/a_{i,j}, j < i$$
$$a_{i,i} = \sqrt{\sum_{i,i} - \sum_{k=1}^{i-1} a_{i,k}^2}$$

Parameters setting:

- Number of steps: 100
- Number of replication: 30000
- ====================
- Price of the option: ***58.53***

Part(b)

The convergence of Monte Carlo estimate is slow, which is $\frac{1}{\sqrt{n}}$.

Part(c)

Take Euler Scheme for example, for one stock:

$$S_T^i = S_0^i exp\left(\left(r - \frac{1}{2}\sigma^2\right)dt + \sigma\sqrt{dt}\sum_{t=1}^{T} Z_t^i\right)$$

Then:

$$d(S_T^1 ... S_T^d)^{1/d} = dS_0^i exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \frac{\sigma\sqrt{dt}}{d}\sum_{t=1}^{T}\sum_{i=1}^{d} Z_t^i\right)$$

And then the expectation of $\left(d(S_T^1 ... S_T^d)^{1/d} - K\right)^+$ is known using Block Schole formula. Also, we know that $\left(d(S_T^1 ... S_T^d)^{1/d} - K\right)^+$ is correlated with $\left(S_T^1 + ... + S_T - K\right)^+$. Therefore, we can use it as a control variate to reduce the variance.

Part(d)

$$var(\sum_{i=1}^{d} Z_i) = 10 + 90 * \rho_{ij}\sigma_i\sigma_j = 37$$

The option with payoff function $\left(d(S_T^1...S_T^d)^{1/d} - K\right)^+$ has this feature:

- Stock price at time 0: $dS_0 = 1000$
- Maturity time: $T = 1$
- Adjusted sigma: $\sigma_{new} = \frac{\sqrt{37}\sigma_{old}}{d} = 0.091$
- Adjusted risk free rate: $r = r_{old} - \frac{\sigma_{old}^2}{2} + \frac{\sigma_{new}^2}{2} = 0.03$
- Strike price: $K = 1000$

Using the Black Schole formula, we can have the expected value: 52.45
The optimal b is :

$$b^* = \frac{\sum_{i=1}^{n}(X_i - \hat{X})(Y_i - \hat{Y})}{\sum_{i=1}^{n}(X_i - \hat{X})^2} = 1.037$$

Then the new payoff function becomes:

$$\hat{Y_n} = \frac{1}{n}\sum_{i=1}^{n}(Y_i - b(X_i - E(X))),$$

The implementation is attached. And we can conclude that the variance is reduced which is shown in Figure and Figure 6.

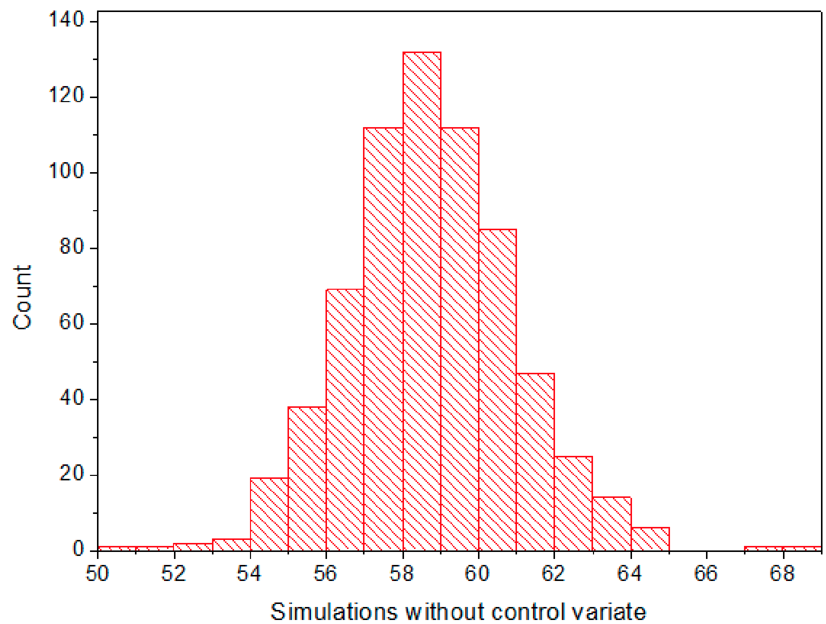


Figure 6: Histgram for the naive Monte Carlo simulation of the price. The variance is relatively large.

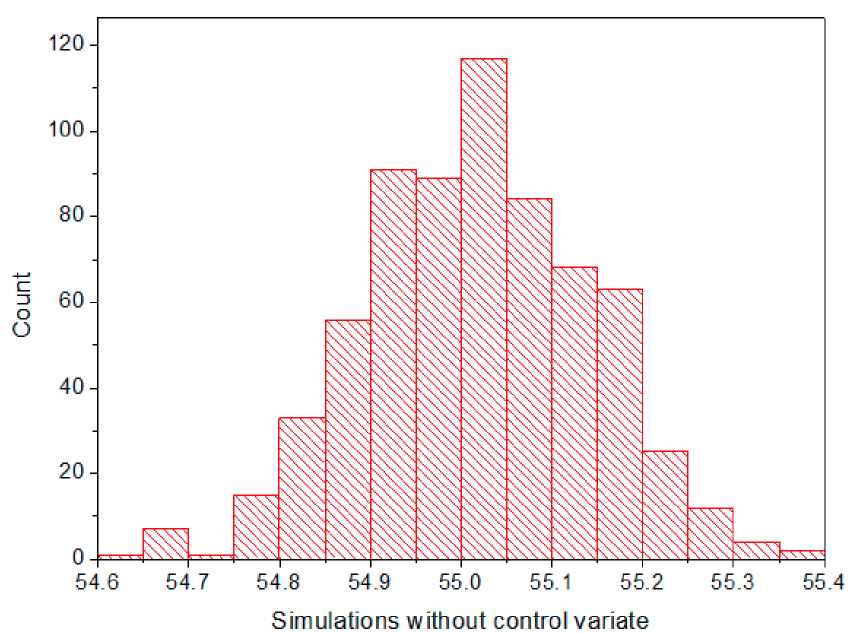


Figure 7: Histgram for the Monte Carlo simulation of the price using control variate to reduce the variance. The variance is relatively small.