HOMEWORK
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Problem 0: Homework checklist

✓I didn't talk with any one about this homework.
✓Source-code are included at the end of this document.

Problem 1: GERSHGORIN DISKS

1. Suppose $\boldsymbol{A}$ is a complex $n \times n$ matrix, with a_{ij}. Compute all the sum of the absolute values of the non-diagnonal entries in the i-th row $R_i = \sum_{j \neq i} |a_{ij}|$. Then if we draw some closed disc centered at a_{ii} with radius R_i, each eigenvalue of $\boldsymbol{A}$ lies in at least one of those discs. Those discs are called Gershgorin disc.

2. The function for ploting Gershgorin Discs is shown as below:

```
function h= plotDisk(A);
[n,m] = size(A);
R =zeros(n, 1);
lam = eig(A);
for i=1:n
    for j=1:n
        if i≠j
            R(i)=R(i)+abs(A(i,j));
        end
    end
    %diskPlot(A(i,i), 0, R(i));
    th = 0:pi/50:2*pi;
    xunit = R(i) * cos(th) + A(i,i);
    yunit = R(i) * sin(th) + 0;
    h = plot(xunit, yunit);
    scatter(lam(i), 0, 100);
    axis equal;
    hold on;
end
end
```

3. The Gershgorin discs plot is shown in Figure 1.

Problem 2:More eigenvalues and convergence theory

1. To show that the Jacobi iteration converges for 2-by-2 symmetric, positive, definite systems, the matrix $\boldsymbol{A}$ satisfies $\mathbf{x}^T \boldsymbol{A} \mathbf{x} > 0 (\forall x \neq 0)$

$$\boldsymbol{A} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

$$\boldsymbol{A} = \boldsymbol{D} + \boldsymbol{R}$$

$$\boldsymbol{D} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\boldsymbol{R} = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}$$

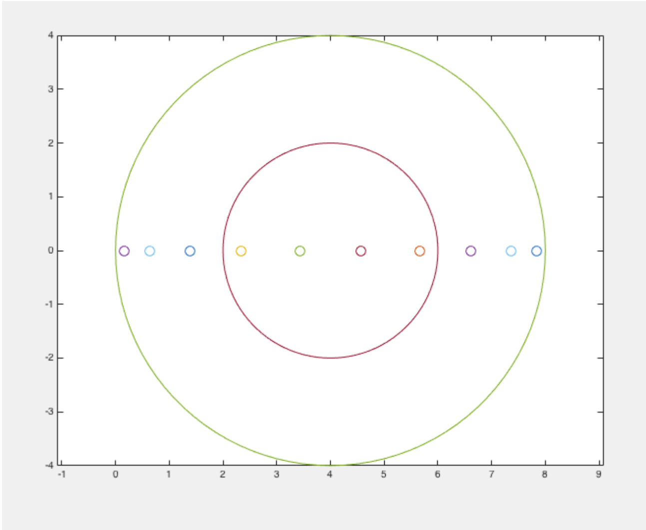

Figure 1: The Gershgorin discs for a matrix in Problme 1.3. All the small circle are the position of true eigenvalues.

Then the Jacobian iteration is $\mathbf{x}^{(k+1)} = \boldsymbol{D}^{-1}(\mathbf{b} - \boldsymbol{R}\mathbf{x}^{(k))}$.
For a symmetric definite matrix, we have $a > 0, b > 0$ and $det(\boldsymbol{A}) > 0$.

$$\begin{aligned} det(\boldsymbol{A}) &= ab - c^2 > 0 \\ \frac{c^2}{ab} &< 1 \end{aligned}$$

Then

$$\begin{aligned} det(\boldsymbol{D}^{-1}\boldsymbol{R} - \lambda\boldsymbol{I}) &= \lambda^2 - \frac{c^2}{ab} \\ |\lambda| &< 1 \end{aligned}$$

Then we know that the absolute value of all eigenvalues for $\boldsymbol{D}^{-1}\boldsymbol{R}$ are smaller than 1.

$$\rho(\boldsymbol{D}^{-1}\boldsymbol{R}) < 1$$

That indicates the Jacobian iteration method converges.

2. If $\rho(\boldsymbol{M}^{-1}\boldsymbol{N}) < 1$ and $e^{(k)} = e^{(k)} - x$

$$\begin{aligned} e^{(k)} &= \boldsymbol{M}^{-1}\boldsymbol{N}e^{(k-1)} \\ &= \boldsymbol{G}e^{(k-1)} \\ &= \boldsymbol{G}^k e^{(0)} \\ \boldsymbol{M}(\mathbf{x}^{(k)} - x) &= \boldsymbol{N}(x^{(k-1)} - x) \end{aligned}$$

This is the solution of equation $\boldsymbol{M}\mathbf{x} = \boldsymbol{N}\mathbf{x} = \mathbf{b}$ and $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ where $\boldsymbol{A} = \boldsymbol{M} + \boldsymbol{N}$. Then the matrix $\boldsymbol{A}$ can not be a singular matrix. Therefore if $\boldsymbol{A}$ is singular, then we can never have $\rho(\boldsymbol{M}^{-1}\boldsymbol{N}) < 1$.