HOMEWORK

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Problem 1: Exercise 16.12

Solve the following linear programs using simplex method:
(a) Maximize $-4x_1 - 3x_2$ subject to

$$\begin{aligned} 5x_1 + x_2 &\geq 11 \\ -2x_1 - x_2 &\leq -8 \\ x_1 + 2x_2 &\geq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Introduce slack variables x_3, x_4, x_5, we will have:

$$\begin{aligned} 5x_1 + x_2 - x_3 &= 11 \\ -2x_1 - x_2 + x_4 &= -8 \\ x_1 + 2x_2 - x_5 &= 7 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

We are trying to minimize $4x_1 + 3x_2$

$$C = [4, 3, 0, 0, 0]$$

Then

$$\begin{bmatrix} 5 & 1 & -1 & 0 & 0 & 11 \\ 2 & 1 & 0 & -1 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 7 \end{bmatrix}$$

Now we are using Two-Phase simplex method to solve this problem.

$$\begin{bmatrix} 5 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 11 \\ 2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

We must update the last row :

$$\begin{bmatrix} 5 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 11 \\ 2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 7 \\ -8 & -4 & 1 & 1 & 1 & 0 & 0 & 0 & -26 \end{bmatrix}$$

Take α_{11} as pivot:

$$\begin{bmatrix} 1 & \frac{1}{5} & -\frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{11}{5} \\ 0 & \frac{3}{5} & \frac{2}{5} & -1 & 0 & \frac{2}{5} & 1 & 0 & \frac{18}{5} \\ 0 & \frac{9}{5} & \frac{1}{5} & 0 & -1 & -\frac{1}{5} & 0 & 1 & \frac{24}{5} \\ 0 & -\frac{12}{5} & -\frac{3}{5} & 1 & 1 & \frac{8}{5} & 0 & 0 & \frac{42}{5} \end{bmatrix}$$

Take α_{31} as pivot:

$$\begin{bmatrix} 1 & 0 & \frac{2}{9} & 0 & \frac{1}{9} & \frac{2}{9} & 0 & \frac{1}{9} & \frac{5}{3} \\ 0 & 0 & \frac{1}{3} & -1 & \frac{1}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} & 2 \\ 0 & 1 & \frac{1}{9} & 0 & -\frac{5}{9} & -\frac{1}{9} & 0 & \frac{5}{9} & \frac{8}{3} \\ 0 & 0 & \frac{1}{3} & 1 & -\frac{1}{3} & \frac{4}{3} & 0 & \frac{4}{3} & -2 \end{bmatrix}$$

Take α_{23} as pivot:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{3} & 3 \\ 0 & 0 & 1 & -3 & 1 & -1 & 3 & -1 & 6 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & -3 & \frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Now we can remove column 6-8.

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & 3 \\ 0 & 0 & 1 & -3 & 1 & 6 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{2}{3} & 2 \\ 4 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Updating the last row, we can have:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & 3 \\ 0 & 0 & 1 & -3 & 1 & 6 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & 0 & 0 & \frac{5}{3} & 0 & -18 \end{bmatrix}$$

All the reduced cost coefficients are nonnegative, hence the optimal solution is

$$x = \begin{bmatrix} 3 \\ 2 \\ 6 \\ 0 \\ 0 \end{bmatrix}$$

and the optimal value is 18.

Problem 2: Exercise 16.12

Problem 3: Exercise 20.2 b.

Find local extremizers for the following optimization problem:
Minimize: $4x_1 + x_2^2$
subject to : $x_1^2 + x_2^2 = 9$

$$\begin{aligned} f(x_1, x_2) &= 4x_1 + x_2^2 \\ h(x_1, x_2) &= x_1^2 + x_2^2 - 9 \\ \nabla f &= \begin{bmatrix} 4 & 2x_2 \end{bmatrix} \\ \nabla h &= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \end{aligned}$$

By the Lagrange condition:

$$\begin{aligned} 4 - \lambda \times 2x_1 &= 0 \\ 2x_2 - \lambda \times 2x_2 &= 0 \\ x_1^2 + x_2^2 - 9 &= 0 \end{aligned}$$

Then we can have $\lambda = 1$

$$\mathbf{x} = \begin{bmatrix} 2 & \sqrt{5} \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} 2 & -\sqrt{5} \end{bmatrix}$$

Problem 4: Exercise 20.9

Find all maximizers of the function: $f(x_1, x_2) = \frac{18x_1^2 - 8x_1x_2 + 12x_2^2}{2x_1^2 + 2x_2^2}$ In this problem we have

$$\boldsymbol{P} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\boldsymbol{Q} = \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix}$$

Then we can have :

$$\boldsymbol{P}^{-1}\boldsymbol{Q} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

The eigenvalues are $\lambda = 10$ and $\lambda = 5$, corresponding two extremizers are $\begin{bmatrix} -2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \end{bmatrix}$.
For

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$
$$f(\mathbf{x}) = 10$$
$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$f(\mathbf{x}) = 5$$

Problem 5: Exercise 21.2

Find local extremizers for :
a. $x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26$, subject to $\frac{1}{5}x_2 - x_1^2 \leq 0, 5x_1 + \frac{1}{2}x_2 \leq 5$
b. $x_1^2 + x_2^2$, subject to $x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \geq 5$
c. $x_1^2 + 6x_1x_2 - 4x_1 - 2x_2$, subject to $x_1^2 + 2x_2 \leq 1, 2x_1 - 2x_2 \leq 1$