HOMEWORK

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Problem 1:

(Contraposition): Suppose $x, y \in \mathbf{R}$, If $y^3 + yx^2 \le x^3 + xy^2$, then $y \le x$.
To prove by contraposition, assume $y > x$, and y and x can not be 0 at the same time. Then

$$\begin{aligned} x^2 + y^2 &> 0 \\ y(y^2 + x^2) &> x(x^2 + y^2) \\ y^3 + yx^2 &> x^3 + xy^2 \end{aligned}$$

So $y^3 + yx^2 \le x^3 + xy^2$ is not true. Therefore the statement is true.

Problem 2:

Problem 3:

Problem 4:

1.

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 4 & -2 \end{bmatrix} \\ \mathbf{b} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ rank(\boldsymbol{A}) &= 2 \\ rank(\boldsymbol{A}, \mathbf{b}) &= 2 \end{aligned}$$

Based on Theorem 2.1, $rank(\boldsymbol{A}) = rank(\boldsymbol{A}, \mathbf{b})$, then the system has a solution.

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 - 2d_3 - d_4 \\ -4x_3 - 2x_4 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 - 2d_3 - d_4 \\ -4x_3 - 2x_4 \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ -1 - 2d_3 \end{bmatrix} \end{aligned}$$

2.

$$\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & 3 & 6 & 3 \end{bmatrix} \\ \mathbf{b} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ rank(\boldsymbol{A}) &= 1 \\ rank(\boldsymbol{A}, \mathbf{b}) &= 2 \end{aligned}$$

Based on Theorem 2.1, $rank(\boldsymbol{A}) \neq rank(\boldsymbol{A}, \mathbf{b})$, then the system has no solutions.

Problem 5:

$$\boldsymbol{A} = \begin{bmatrix} c & 0 & a \\ 0 & c & b \\ b & a & 0 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b \\ a \\ c \end{bmatrix}$$

$$\boldsymbol{A}|\mathbf{b} = \begin{bmatrix} c & 0 & a \\ 0 & c & b \\ 0 & 0 & -\frac{2ab}{c} \end{bmatrix} \mid \begin{bmatrix} b \\ a \\ \frac{c^2-a^2-b^2}{c} \end{bmatrix}$$

The one unique solution for this linear system is:

$$a \neq 0$$
$$b \neq 0$$
$$c \neq 0$$

The solution is :

$$x_1 = \frac{b^2 + c^2 - a^2}{2bc}$$
$$x_2 = \frac{a^2 + c^2 - b^2}{2ac}$$
$$x_3 = \frac{a^2 + b^2 - c^2}{2bc}$$

Problem 6:

$$\begin{aligned} & \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix} \\ &= \begin{bmatrix} \cos x \cos y - \sin x \sin y & \cos x \sin y + \sin x \cos y \\ -\sin x \cos y - \sin y \cos x & \cos x \cos y - \sin x \sin y \end{bmatrix} \\ &= \begin{bmatrix} \cos(x+y) & \sin(x+y) \\ -\sin(x+y) & \cos(x+y) \end{bmatrix} \end{aligned}$$

Then

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}^2 = \begin{bmatrix} \cos 2x & \sin 2x \\ -\sin 2x & \cos 2x \end{bmatrix}$$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}^{57} = \begin{bmatrix} \cos 57x & \sin 57x \\ -\sin 57x & \cos 57x \end{bmatrix}$$

Problem 7:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$
$$f_1(x) = x_2 - 2x + 5$$
$$f_2(x) = 7x + 5$$

Then

$$
\begin{aligned}
f_1(\boldsymbol{A}) &= \begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix} - 2\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix} + \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}\\
&= \begin{bmatrix}10 & -2\\ -3 & 11\end{bmatrix}\\
f_2(\boldsymbol{A}) &= 7\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix} + \begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}\\
&= \begin{bmatrix}12 & 14\\ 21 & 5\end{bmatrix}\\
5f_1(\boldsymbol{A}) - 3f_2(\boldsymbol{A}) &= \begin{bmatrix}14 & -62\\ 78 & 40\end{bmatrix}
\end{aligned}
$$

Problem 8:

1. $f(x_1, x_2) = x_1^2 + x_2^2 + 4x_1x_2 + \frac{2}{3}x_2^3 - 2x_2 + 7$

$$
\begin{aligned}
\frac{\partial f}{\partial x_1} &= 2x_1 + 4x_2 = 0\\
\frac{\partial f}{\partial x_2} &= 2x_2 + 4x_1 + 2x_2^2 - 2 = 0\\
x_1 &= -6.6\\
x_2 &= 3.3\\
&or\\
x_1 &= 0.6\\
x_2 &= -0.3
\end{aligned}
$$

So there are two extremum points $(-6.6, 3.3)$ and $(0.6, -0.3)$

2.

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_1^2} &= 2 > 0\\
\frac{\partial^2 f}{\partial x_2^2} &= 2 + 4x_2 > 0
\end{aligned}
$$

Then we need $x_2 > -0.5$, and both points satisfies these conditions. Therefore both points are local minimizers.

Problem 9:

1. $f(x_1, x_2, x_3, x_4) = 7x_1^2 + x_3^2 - 2x_1x_3 + x_1x_4$

$$
\begin{aligned}
f &= \frac{1}{2}\mathbf{x}^T\boldsymbol{Q}\mathbf{x}\\
&= \frac{1}{2}\begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}\begin{bmatrix}14 & 0 & -2 & 1\\ 0 & 0 & 0 & 0\\ -2 & 0 & 2 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}
\end{aligned}
$$

2. $f(x_1, x_2, x_3) = x_2^2 - 3x_1x_2$

$$
\begin{aligned}
f &= \frac{1}{2}\mathbf{x}^T\boldsymbol{Q}\mathbf{x} \\
&= \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 0 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$

3. $f(x_1, x_2, x_3) = 2x_1^2 - 5x_2^2 + 2x_1x_2$

$$
\begin{aligned}
f &= \frac{1}{2}\mathbf{x}^T\boldsymbol{Q}\mathbf{x} \\
&= \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 4 & 2 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$