PURDUE UNIVERSITY · ECE 580
OPTIMIZATION METHODS FOR SYSTEMS AND CONTROL

HOMEWORK
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Problem 1:

(Contraposition): Suppose $x, y \in \mathbf{R}$, If $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.
To prove by contraposition, assume $y > x$, and y and x can not be 0 at the same time. Then

$$x^2 + y^2 > 0$$
$$y(y^2 + x^2) > x(x^2 + y^2)$$
$$y^3 + yx^2 > x^3 + xy^2$$

So $y^3 + yx^2 \leq x^3 + xy^2$ is not true. Therefore the statement is true.

Source: http://www.people.vcu.edu/ rhammack/BookOfProof/Contrapositive.pdf

Problem 2:

(Contradiction): There is no rational number solution to the equation$x^5 + x^4 + x^3 + x^2 + 1 = 0$
Suppose there is a rational solution to this equation $\frac{p}{q}$where both p and q are integers and $\frac{p}{q}$ is in a reduced form. Then

$$p^5 + p^4q + p^3q^2 + p^2q^3 + q^5 = 0$$

Case 1: p and q are both odd, then the left hand side of the equation above is odd. But the right hand side zero is even which leaves us a contradiction.
Case 2: p is even and q is odd, then the left hand side is still odd which leaves us a contradiction.
Case 3: p is odd and q is even, then the left hand side is still odd which leaves us a contradiction.
Overall, it is impossible for the equation above to have a rational solution.
Source: http://zimmer.csufresno.edu/ larryc/proofs/proofs.contradict.html

Problem 3:

(Induction) : For all positive integers, $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{1}{6}n(n+1)(2n+1)$
Base case: $n = 1$

$$\begin{aligned} LHS &= 1 \\ RHS &= \frac{1}{6} \times 1 \times 2 \times 3 \\ &= 1 \\ LHS &= RHS \end{aligned}$$

By induction hypothesis, if $1^2+2^2+3^2+...+k^2=\frac{1}{6}k(k+1)(2k+1)$ is known, then for $n=k+1$, we have

$$\begin{aligned}
LHS &= 1^2+2^2+3^2+...+k^2+(k+1)^2\\
&= \frac{1}{6}k(k+1)(2k+1)+(k+1)^2\\
&= [\frac{1}{6}k(2k+1)+(k+1)](k+1)\\
&= \frac{1}{6}(k+1)(2k^2+7k+6)\\
&= \frac{1}{6}(k+1)(k+2)(2k+3)\\
RHS &= \frac{1}{6}(k+1)(k+2)(2k+3)\\
LHS &= RHS
\end{aligned}$$

Therefore, the statement is true.
Source: http://zimmer.csufresno.edu/ larryc/proofs/proofs.mathinduction.html

Problem 4:

1.

$$\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} 1 & 1 & 2 & 1\\ 1 & 2 & 4 & -2 \end{bmatrix}\\
\mathbf{b} &= \begin{bmatrix} 1\\ 0 \end{bmatrix}\\
rank(\boldsymbol{A}) &= 2\\
rank(\boldsymbol{A},\mathbf{b}) &= 2
\end{aligned}$$

Based on Theorem 2.1, $rank(\boldsymbol{A}) = rank(\boldsymbol{A},\mathbf{b})$, then the system has a solution.

$$\begin{aligned}
\begin{bmatrix} 1 & 1\\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1\\ x_2 \end{bmatrix} &= \begin{bmatrix} 1-2d_3-d_4\\ -4x_3-2x_4 \end{bmatrix}\\
\begin{bmatrix} x_1\\ x_2 \end{bmatrix} &= \begin{bmatrix} 2 & -1\\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1-2d_3-d_4\\ -4x_3-2x_4 \end{bmatrix}\\
&= \begin{bmatrix} 2\\ -1-2d_3 \end{bmatrix}
\end{aligned}$$

2.

$$\begin{aligned}
\boldsymbol{A} &= \begin{bmatrix} 2 & 1 & 2 & 1\\ 6 & 3 & 6 & 3 \end{bmatrix}\\
\mathbf{b} &= \begin{bmatrix} 0\\ 1 \end{bmatrix}\\
rank(\boldsymbol{A}) &= 1\\
rank(\boldsymbol{A},\mathbf{b}) &= 2
\end{aligned}$$

Based on Theorem 2.1, $rank(\boldsymbol{A}) \neq rank(\boldsymbol{A},\mathbf{b})$, then the system has no solutions.

Problem 5:

$$\boldsymbol{A} = \begin{bmatrix} c & 0 & a \\ 0 & c & b \\ b & a & 0 \end{bmatrix}$$
$$\mathbf{b} = \begin{bmatrix} b \\ a \\ c \end{bmatrix}$$
$$\boldsymbol{A}|\mathbf{b} = \begin{bmatrix} c & 0 & a \\ 0 & c & b \\ 0 & 0 & -\frac{2ab}{c} \end{bmatrix} \mid \begin{bmatrix} b \\ a \\ \frac{c^2-a^2-b^2}{c} \end{bmatrix}$$

The one unique solution for this linear system is:

$$a \neq 0$$
$$b \neq 0$$
$$c \neq 0$$

The solution is :

$$x_1 = \frac{b^2 + c^2 - a^2}{2bc}$$
$$x_2 = \frac{a^2 + c^2 - b^2}{2ac}$$
$$x_3 = \frac{a^2 + b^2 - c^2}{2bc}$$

Problem 6:

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{bmatrix}$$
$$= \begin{bmatrix} \cos x \cos y - \sin x \sin y & \cos x \sin y + \sin x \cos y \\ -\sin x \cos y - \sin y \cos x & \cos x \cos y - \sin x \sin y \end{bmatrix}$$
$$= \begin{bmatrix} \cos(x+y) & \sin(x+y) \\ -\sin(x+y) & \cos(x+y) \end{bmatrix}$$

Then

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}^2 = \begin{bmatrix} \cos 2x & \sin 2x \\ -\sin 2x & \cos 2x \end{bmatrix}$$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}^{57} = \begin{bmatrix} \cos 57x & \sin 57x \\ -\sin 57x & \cos 57x \end{bmatrix}$$

Problem 7:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$
$$f_1(x) = x_2 - 2x + 5$$
$$f_2(x) = 7x + 5$$

Then

$$
\begin{aligned}
f_1(\boldsymbol{A}) &= \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 10 & -2 \\ -3 & 11 \end{bmatrix} \\
f_2(\boldsymbol{A}) &= 7 \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \\
&= \begin{bmatrix} 12 & 14 \\ 21 & 5 \end{bmatrix} \\
5f_1(\boldsymbol{A}) - 3f_2(\boldsymbol{A}) &= \begin{bmatrix} 14 & -62 \\ 78 & 40 \end{bmatrix}
\end{aligned}
$$

Problem 8:

1. $f(x_1, x_2) = x_1^2 + x_2^2 + 4x_1x_2 + \frac{2}{3}x_2^3 - 2x_2 + 7$

$$
\begin{aligned}
&\frac{\partial f}{\partial x_1} = 2x_1 + 4x_2 = 0 \\
&\frac{\partial f}{\partial x_2} = 2x_2 + 4x_1 + 2x_2^2 - 2 = 0 \\
&x_1 = -6.6 \\
&x_2 = 3.3 \\
&\mathit{or} \\
&x_1 = 0.6 \\
&x_2 = -0.3
\end{aligned}
$$

So there are two extremum points $(-6.6, 3.3)$ and $(0.6, -0.3)$

2.

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x_1^2} &= 2 > 0 \\
\frac{\partial^2 f}{\partial x_2^2} &= 2 + 4x_2 > 0
\end{aligned}
$$

Then we need $x_2 > -0.5$, and both points satisfies these conditions. Therefore both points are local minimizers.

Problem 9:

1. $f(x_1, x_2, x_3, x_4) = 7x_1^2 + x_3^2 - 2x_1x_3 + x_1x_4$

$$
\begin{aligned}
f &= \frac{1}{2}\mathbf{x}^T\boldsymbol{Q}\mathbf{x} \\
&= \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 14 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
\end{aligned}
$$

2. $f(x_1, x_2, x_3) = x_2^2 - 3x_1x_2$

$$
\begin{aligned}
f &= \frac{1}{2}\mathbf{x}^T \boldsymbol{Q} \mathbf{x} \\
&= \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & -3 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$

3. $f(x_1, x_2, x_3) = 2x_1^2 - 5x_2^2 + 2x_1x_2$

$$
\begin{aligned}
f &= \frac{1}{2}\mathbf{x}^T \boldsymbol{Q} \mathbf{x} \\
&= \frac{1}{2}\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{aligned}
$$