Problem 1: Gauss Seidel Method

$$\boldsymbol{A} = \begin{bmatrix} 5 & -1 & 0 & -1 & 0 & 0 \\ -1 & 5 & -1 & 0 & -1 & 0 \\ 0 & -1 & 5 & 0 & 0 & -1 \\ -1 & 0 & 0 & 5 & -1 & 0 \\ 0 & -1 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 0 & -1 & 5 \end{bmatrix} \tag{1}$$

$$\mathbf{b} = \begin{bmatrix} -5 \\ 1 \\ 1 \\ -2 \\ 1 \\ 2 \end{bmatrix} \tag{2}$$

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

For the first 10 iterations:
residual: 1.27057
residual: 0.189021
residual: 0.0518507
residual: 0.0175355
residual: 0.00437263
residual: 0.00103081
residual: 0.000240774
residual: 5.61515e-05
residual: 1.30917e-05
residual: 3.0522e-06

The residual is defined as $\|x_{n+1} - x_n\|_2$

$$\mathbf{x} = \begin{bmatrix} -1.09909 \\ 0.0840332 \\ 0.317577 \\ -0.579482 \\ 0.20168 \\ 0.503851 \end{bmatrix} \tag{4}$$

Problem 2: Power Method

1.

$$\boldsymbol{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & -1 & 2 \end{bmatrix} \tag{5}$$

When the initial guess is

$$\mathbf{x}_0 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \tag{6}$$

The results of first 21 iterations are:

```
0 [0.5472,0.7295,0.4104]  Eigenvalue: 3.58835758835759
1 [0.591,0.788,0.1724]  Eigenvalue: 3.90661006670709
2 [0.5996,0.7994,0.03747]  Eigenvalue: 3.98970198158839
3 [0.5997,0.7996,-0.03123]  Eigenvalue: 4.00429268292683
4 [0.5987,0.7983,-0.06548]  Eigenvalue: 4.00449237171945
5 [0.598,0.7973,-0.08253]  Eigenvalue: 4.00282729038406
6 [0.5975,0.7967,-0.09103]  Eigenvalue: 4.00155802744283
7 [0.5973,0.7964,-0.09527]  Eigenvalue: 4.00081498421819
8 [0.5971,0.7962,-0.09739]  Eigenvalue: 4.00041646827872
9 [0.5971,0.7961,-0.09845]  Eigenvalue: 4.00021047607401
10 [0.5971,0.7961,-0.09897]  Eigenvalue: 4.00010579825402
11 [0.597,0.7961,-0.09924]  Eigenvalue: 4.00005303914774
12 [0.597,0.796,-0.09937]  Eigenvalue: 4.00002655457485
13 [0.597,0.796,-0.09944]  Eigenvalue: 4.00001328603714
14 [0.597,0.796,-0.09947]  Eigenvalue: 4.00000664520594
15 [0.597,0.796,-0.09949]  Eigenvalue: 4.0000033231498
16 [0.597,0.796,-0.0995]  Eigenvalue: 4.00000166171161
17 [0.597,0.796,-0.0995]  Eigenvalue: 4.00000083088998
18 [0.597,0.796,-0.0995]  Eigenvalue: 4.00000041545353
19 [0.597,0.796,-0.0995]  Eigenvalue: 4.0000002077289
20 [0.597,0.796,-0.0995]  Eigenvalue: 4.00000010386499
```

When the initial guess is

$$\mathbf{x}_0 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \tag{7}$$

The results of first 21 iterations are:

```
0 [0.6114,0.6988,0.3712]  Eigenvalue: 3.69194086790653
1 [0.6485,0.7411,0.1737]  Eigenvalue: 3.92356175405659
2 [0.6571,0.751,0.06454]  Eigenvalue: 3.9856105201556
3 [0.6585,0.7525,0.008819]  Eigenvalue: 3.99901487197656
4 [0.6584,0.7524,-0.0191]  Eigenvalue: 4.00106691546963
5 [0.6581,0.7522,-0.03305]  Eigenvalue: 4.00092262021984
6 [0.658,0.752,-0.04002]  Eigenvalue: 4.00055841300695
7 [0.6579,0.7519,-0.0435]  Eigenvalue: 4.00030345259712
8 [0.6578,0.7518,-0.04524]  Eigenvalue: 4.00015778373466
9 [0.6578,0.7518,-0.04611]  Eigenvalue: 4.00008040569154
10 [0.6578,0.7518,-0.04655]  Eigenvalue: 4.00004058123348
11 [0.6578,0.7518,-0.04677]  Eigenvalue: 4.00002038520503
12 [0.6578,0.7517,-0.04688]  Eigenvalue: 4.0000102162485
13 [0.6578,0.7517,-0.04693]  Eigenvalue: 4.00000511403561
14 [0.6578,0.7517,-0.04696]  Eigenvalue: 4.00000255849563
15 [0.6578,0.7517,-0.04697]  Eigenvalue: 4.00000127961727
16 [0.6578,0.7517,-0.04698]  Eigenvalue: 4.000000639901
17 [0.6578,0.7517,-0.04698]  Eigenvalue: 4.00000031997359
18 [0.6578,0.7517,-0.04698]  Eigenvalue: 4.00000015999257
19 [0.6578,0.7517,-0.04698]  Eigenvalue: 4.00000007999773
20 [0.6578,0.7517,-0.04698]  Eigenvalue: 4.00000003999922
```

2.

$$\boldsymbol{A} = \begin{bmatrix} 25 & -5 & 15 & -5 \\ -57 & 25 & -15 & 5 \\ -5 & 5 & 5 & 5 \\ 5 & -5 & 15 & 15 \end{bmatrix} \tag{8}$$

When the initial guess is

$$\mathbf{x}_0 = \begin{bmatrix} 7 \\ 8 \\ 9 \\ 6 \end{bmatrix} \tag{9}$$

The results of first 21 iterations are:

```
0 [0.7047,0.1762,0.2349,0.6459]  Eigenvalue: 22.7586206896552
1 [0.7296,0.02516,0.07547,0.6793]  Eigenvalue: 21.6455696202532
2 [0.7294,-0.03473,0.01158,0.6831]  Eigenvalue: 20.8310991957105
3 [0.7276,-0.06109,-0.01666,0.6831]  Eigenvalue: 20.4133251079581
4 [0.7263,-0.07345,-0.02992,0.6828]  Eigenvalue: 20.2057125943466
5 [0.7256,-0.07943,-0.03635,0.6825]  Eigenvalue: 20.1025771140672
6 [0.7253,-0.08237,-0.03951,0.6824]  Eigenvalue: 20.0512139775591
7 [0.7251,-0.08383,-0.04108,0.6823]  Eigenvalue: 20.02558777141
8 [0.725,-0.08456,-0.04186,0.6823]  Eigenvalue: 20.0127890113521
9 [0.7249,-0.08492,-0.04225,0.6823]  Eigenvalue: 20.0063932784323
10 [0.7249,-0.0851,-0.04245,0.6823]  Eigenvalue: 20.0031963313293
11 [0.7249,-0.08519,-0.04254,0.6822]  Eigenvalue: 20.0015980885588
12 [0.7249,-0.08524,-0.04259,0.6822]  Eigenvalue: 20.0007990249862
13 [0.7249,-0.08526,-0.04262,0.6822]  Eigenvalue: 20.0003995076677
14 [0.7249,-0.08527,-0.04263,0.6822]  Eigenvalue: 20.0001997526272
15 [0.7249,-0.08527,-0.04263,0.6822]  Eigenvalue: 20.0000998760119
16 [0.7249,-0.08528,-0.04264,0.6822]  Eigenvalue: 20.0000499379305
17 [0.7249,-0.08528,-0.04264,0.6822]  Eigenvalue: 20.0000249689464
18 [0.7249,-0.08528,-0.04264,0.6822]  Eigenvalue: 20.0000124844685
19 [0.7249,-0.08528,-0.04264,0.6822]  Eigenvalue: 20.0000062422331
20 [0.7249,-0.08528,-0.04264,0.6822]  Eigenvalue: 20.0000031211162
```

When the initial guess is

$$\mathbf{x}_0 = \begin{bmatrix} 4 \\ 9 \\ 2 \\ 7 \end{bmatrix} \tag{10}$$

The results of first 21 iterations are:

```
0 [0.1983,0.8327,0.2776,0.4362]  Eigenvalue: 20.5660377358491
1 [0.1341,0.8622,0.3257,0.364]  Eigenvalue: 20.3964757709251
2 [0.1032,0.8724,0.3471,0.3283]  Eigenvalue: 20.2217529039071
3 [0.08811,0.8765,0.3571,0.3107]  Eigenvalue: 20.1161290322581
4 [0.08068,0.8783,0.3619,0.302]  Eigenvalue: 20.0592998788497
5 [0.07699,0.8791,0.3643,0.2976]  Eigenvalue: 20.0299491340105
6 [0.07516,0.8795,0.3655,0.2955]  Eigenvalue: 20.0150481659801
7 [0.07424,0.8797,0.366,0.2944]  Eigenvalue: 20.0075423331937
8 [0.07378,0.8798,0.3663,0.2938]  Eigenvalue: 20.0037757104826
9 [0.07355,0.8798,0.3665,0.2936]  Eigenvalue: 20.0018889888813
10 [0.07344,0.8799,0.3665,0.2934]  Eigenvalue: 20.0009447775593
```

```
11 [0.07338,0.8799,0.3666,0.2934]  Eigenvalue: 20.0004724595229
12 [0.07335,0.8799,0.3666,0.2933]  Eigenvalue: 20.0002362474427
13 [0.07334,0.8799,0.3666,0.2933]  Eigenvalue: 20.0001181281411
14 [0.07333,0.8799,0.3666,0.2933]  Eigenvalue: 20.0000590651754
15 [0.07333,0.8799,0.3666,0.2933]  Eigenvalue: 20.0000295328639
16 [0.07333,0.8799,0.3666,0.2933]  Eigenvalue: 20.000014766501
17 [0.07332,0.8799,0.3666,0.2933]  Eigenvalue: 20.0000073832678
18 [0.07332,0.8799,0.3666,0.2933]  Eigenvalue: 20.0000036916382
19 [0.07332,0.8799,0.3666,0.2933]  Eigenvalue: 20.0000018458202
20 [0.07332,0.8799,0.3666,0.2933]  Eigenvalue: 20.0000009229104
```

Problem 3: Power Method

1.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & -1.5 \\ 0 & 0 & 1 & 2.5 \end{bmatrix} \tag{11}$$

When the initial guess is

$$\mathbf{x}_0 = \begin{bmatrix} 4 \\ 9 \\ 2 \\ 7 \end{bmatrix} \tag{12}$$

The results of first 21 iterations are:

```
0 [0.1648,0.3532,-0.07063,0.9182]  Eigenvalue: 2.41130820399113
1 [0.1787,0.2429,-0.3987,0.8661]  Eigenvalue: 2.17711045779084
2 [0.1977,0.2793,-0.4822,0.8065]  Eigenvalue: 1.93329992603712
3 [0.2084,0.3106,-0.481,0.7929]  Eigenvalue: 1.88369147371446
4 [0.2104,0.3211,-0.4664,0.7969]  Eigenvalue: 1.90310405160878
5 [0.2094,0.3199,-0.4594,0.8017]  Eigenvalue: 1.92327442979667
6 [0.2084,0.3173,-0.4589,0.8033]  Eigenvalue: 1.92916071816977
7 [0.2082,0.3162,-0.4601,0.8031]  Eigenvalue: 1.92802317901948
8 [0.2083,0.3162,-0.4608,0.8027]  Eigenvalue: 1.92632308164457
9 [0.2083,0.3165,-0.4609,0.8025]  Eigenvalue: 1.92572469180958
10 [0.2084,0.3166,-0.4608,0.8025]  Eigenvalue: 1.92576820408011
11 [0.2084,0.3166,-0.4607,0.8026]  Eigenvalue: 1.92590654837988
12 [0.2084,0.3166,-0.4607,0.8026]  Eigenvalue: 1.92596491265444
13 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92596601612052
14 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595511127207
15 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92594962272772
16 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.9259491062206
17 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92594993215116
18 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595043270496
19 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595051284897
20 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595045349808
```

When the initial guess is

$$\mathbf{x}_0 = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 1 \end{bmatrix} \tag{13}$$

The results of first 21 iterations are:

```
0 [0.07454,0.5217,0.5217,0.6708]  Eigenvalue: 1.46111111111111
1 [0.145,0.1772,-0.2095,0.9506]  Eigenvalue: 2.50051921079958
2 [0.1814,0.2367,-0.4765,0.827]  Eigenvalue: 2.01000037808613
3 [0.2051,0.2951,-0.4979,0.7892]  Eigenvalue: 1.86471050679047
4 [0.2115,0.3214,-0.4763,0.7906]  Eigenvalue: 1.87649374009219
5 [0.2106,0.3233,-0.4606,0.7993]  Eigenvalue: 1.91401340973697
6 [0.2088,0.3188,-0.4575,0.8034]  Eigenvalue: 1.93000163879722
7 [0.2081,0.3163,-0.4592,0.8036]  Eigenvalue: 1.93024861702547
8 [0.2082,0.316,-0.4606,0.8029]  Eigenvalue: 1.92722964069017
9 [0.2083,0.3163,-0.461,0.8025]  Eigenvalue: 1.92572794507759
10 [0.2084,0.3165,-0.4609,0.8025]  Eigenvalue: 1.92559210996346
11 [0.2084,0.3166,-0.4607,0.8025]  Eigenvalue: 1.92582074753743
12 [0.2084,0.3166,-0.4607,0.8026]  Eigenvalue: 1.92595794413598
13 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92597946332219
14 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92596297824214
15 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595079093439
16 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92594812925657
17 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92594923085227
18 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595028510293
19 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595057892723
20 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595051437343
```

2.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & -1.5 \\ 0 & 0 & 1 & 2.5 \end{bmatrix} \tag{14}$$

When the initial guess is

$$\mathbf{x}_0 = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 1 \end{bmatrix} \tag{15}$$

The results of first 21 iterations are:

```
0 [0.07454,0.5217,0.5217,0.6708]  Eigenvalue: 1.46111111111111
1 [0.145,0.1772,-0.2095,0.9506]  Eigenvalue: 2.50051921079958
2 [0.1814,0.2367,-0.4765,0.827]  Eigenvalue: 2.01000037808613
3 [0.2051,0.2951,-0.4979,0.7892]  Eigenvalue: 1.86471050679047
4 [0.2115,0.3214,-0.4763,0.7906]  Eigenvalue: 1.87649374009219
5 [0.2106,0.3233,-0.4606,0.7993]  Eigenvalue: 1.91401340973697
6 [0.2088,0.3188,-0.4575,0.8034]  Eigenvalue: 1.93000163879722
7 [0.2081,0.3163,-0.4592,0.8036]  Eigenvalue: 1.93024861702547
8 [0.2082,0.316,-0.4606,0.8029]  Eigenvalue: 1.92722964069017
9 [0.2083,0.3163,-0.461,0.8025]  Eigenvalue: 1.92572794507759
10 [0.2084,0.3165,-0.4609,0.8025]  Eigenvalue: 1.92559210996346
11 [0.2084,0.3166,-0.4607,0.8025]  Eigenvalue: 1.92582074753743
12 [0.2084,0.3166,-0.4607,0.8026]  Eigenvalue: 1.92595794413598
13 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92597946332219
14 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92596297824214
15 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595079093439
16 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92594812925657
17 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92594923085227
18 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595028510293
19 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595057892723
20 [0.2084,0.3165,-0.4607,0.8026]  Eigenvalue: 1.92595051437343
```

$$\mathbf{x}_0 = \begin{bmatrix} 6 \\ 9 \\ 1 \\ 4 \end{bmatrix} \tag{16}$$

The results of first 21 iterations are:

```
0 [0.8421,0.5263,0.05263,0.1053]  Eigenvalue: 1.50969529085873
1 [0.925,0.3769,0.03426,0.03426]  Eigenvalue: 1.39260563380282
2 [0.9554,0.294,0.0245,0.01225]  Eigenvalue: 1.31140285071268
3 [0.97,0.2425,0.01865,0.004663]  Eigenvalue: 1.25781233010764
4 [0.9781,0.2075,0.01482,0.001853]  Eigenvalue: 1.22051147419464
5 [0.9832,0.1821,0.01214,0.0007586]  Eigenvalue: 1.19316300166967
6 [0.9866,0.1627,0.01017,0.0003179]  Eigenvalue: 1.17225319092724
7 [0.989,0.1475,0.008676,0.0001356]  Eigenvalue: 1.15572782176051
8 [0.9908,0.1351,0.007506,5.864e-05]  Eigenvalue: 1.1423178748547
9 [0.9922,0.1248,0.006571,2.567e-05]  Eigenvalue: 1.13120062126532
10 [0.9932,0.1162,0.005808,1.134e-05]  Eigenvalue: 1.12182038990716
11 [0.9941,0.1087,0.005177,5.056e-06]  Eigenvalue: 1.11378866668672
12 [0.9947,0.1023,0.004648,2.27e-06]  Eigenvalue: 1.10682569520657
13 [0.9953,0.09659,0.0042,1.025e-06]  Eigenvalue: 1.10072488050872
14 [0.9958,0.09157,0.003815,4.657e-07]  Eigenvalue: 1.09533028617996
15 [0.9962,0.08708,0.003483,2.126e-07]  Eigenvalue: 1.09052194802356
16 [0.9965,0.08305,0.003194,9.747e-08]  Eigenvalue: 1.08620601707797
17 [0.9968,0.07939,0.002941,4.487e-08]  Eigenvalue: 1.08230797831406
18 [0.9971,0.07607,0.002717,2.073e-08]  Eigenvalue: 1.07876788165995
19 [0.9973,0.07304,0.002519,9.607e-09]  Eigenvalue: 1.07553692169126
20 [0.9975,0.07025,0.002342,4.466e-09]  Eigenvalue: 1.07257494091909
```