HOMEWORK
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Problem 0: Homework checklist

✓I didn't talk with any one about this homework.
✓Source-code are included at the end of this document.

Problem 1

1. $\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 5 & 0 \\ 2 & 0 \end{bmatrix}$
3. $\begin{bmatrix} 5 & -5 \\ 2 & -2 \\ 0 & 0 \end{bmatrix}$
4. $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

Problem 2:

1. For example, The matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \tag{1}$$

has two different decomposition. But in both cases they have the same singular value.

2.

3. Suppose $\boldsymbol{A}$ is a matrix with eigenvalues $\{\sigma_i\}$
The SVD of $\boldsymbol{A}$ is $\boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^T$ where $\boldsymbol{U}$ and $\boldsymbol{V}$ are both orthogonal matrix.

$$\begin{aligned} \boldsymbol{A}^{-1} &= (\boldsymbol{U}\Sigma\boldsymbol{V}^T)^{-1} && (2) \\ &= (\boldsymbol{V}^T)^{-1}\Sigma^{-1}\boldsymbol{U}^{-1} && (3) \\ &= (\boldsymbol{V}^T)^T\Sigma^{-1}\boldsymbol{U}^T && (4) \\ &= \boldsymbol{V}\Sigma^{-1}\boldsymbol{U}^T && (5) \end{aligned}$$

Then $\boldsymbol{A}^{-1}$ is a matrix with eigenvalues $\{\frac{1}{\sigma_i}\}$

$$\begin{aligned} \|\boldsymbol{A}^{-1}\| &= max\{\frac{1}{\sigma_i}\} && (6) \\ &= \frac{1}{\sigma_{min}} && (7) \\ & && (8) \end{aligned}$$

4. The SVD of $\boldsymbol{Q}$ is $\boldsymbol{Q} = \boldsymbol{U}\Sigma\boldsymbol{V}^T = \boldsymbol{Q}\boldsymbol{I}\boldsymbol{I}^T$ where $\boldsymbol{I}$ is an identity matrix and it is also an orthogonal matrix.
In particular, the singular value are all 1.

Problem 3:

1. If $m < n$, we can use $\boldsymbol{A}^T = (\boldsymbol{U}\Sigma\boldsymbol{V}^T)^T = \boldsymbol{V}\Sigma\boldsymbol{U}^T$ is still SVD.

2. $\boldsymbol{A} = diag(\sigma_1, \sigma_2, ...\sigma_n, 0, ...)$
   Then the best diagonal rank k approximation to $\boldsymbol{A}$ is
   $\boldsymbol{A}_k = diag(\sigma_1, \sigma_2, ...\sigma_k, 0, ...)$
   We aussume $k < n$ since we want to do the lower rank approximation.
   Then

$$
\begin{aligned}
\|\boldsymbol{A} - \boldsymbol{A}_k\|_2 &= \|diag(\sigma_1, \sigma_2, ...\sigma_n, 0, ...) - diag(\sigma_1, \sigma_2, ...\sigma_k, 0, ...)\|_2 && (9)\\
&= \|diag(0, .., \sigma_{k+1}, ...\sigma_n, 0, ...)\|_2 && (10)\\
&= max\{\sigma_{k+1}, ...\sigma_n\} && (11)
\end{aligned}
$$

   The small least answer could be σ_n which is the smallest singular value of $\boldsymbol{A}$.

3.

$$
\begin{aligned}
\|\boldsymbol{A} - \boldsymbol{A}_n\| &= \|\sum_{i=1}^{n}\sigma_i \mathbf{u}_i \mathbf{v}_i^T - \sum_{i=1}^{n}\sigma_i \mathbf{u}_i \mathbf{v}_i^T\| && (12)\\
&= 0 && (13)\\
& && (14)
\end{aligned}
$$

4.

$$
\begin{aligned}
\|\boldsymbol{A} - \boldsymbol{A}_R\| &= \|\sum_{i=1}^{n}\sigma_i \mathbf{u}_i \mathbf{v}_i^T - \sum_{i=1}^{R}\sigma_i \mathbf{u}_i \mathbf{v}_i^T\| && (15)\\
&= \|\sum_{i=R+1}^{n}\sigma_i \mathbf{u}_i \mathbf{v}_i^T\| && (16)\\
&= max\{\sigma_i\} && (17)\\
& && (18)
\end{aligned}
$$

   When they have the same set of singular values, Then

$$
\begin{aligned}
\|\boldsymbol{A} - \boldsymbol{A}_R\| = 0 && (19)\\
&& (20)
\end{aligned}
$$

5.

$$
\begin{aligned}
\|\boldsymbol{A} - \boldsymbol{A}_k\| &= \|\sum_{i=1}^{n}\sigma_i \mathbf{u}_i \mathbf{v}_i^T - \sum_{i=1}^{k}\sigma_i \mathbf{u}_i \mathbf{v}_i^T\| && (21)\\
&= \|\sum_{i=k+1}^{n}\sigma_i \mathbf{u}_i \mathbf{v}_i^T\| && (22)\\
&= max\{\sigma_{k+1}, ..., \sigma_n\} && (23)\\
&= \sigma_{k+1} && (24)
\end{aligned}
$$

   σ_{k+1} is the largest singular value of $\boldsymbol{A} - \boldsymbol{A}_k$

Problem 4:

1. The data is stored as a 3 dimension array $256 \times 1100 \times 10$.
   For each digit, there are 1100 images. Those images include 256 pixels with shape $16 by 16$
   And the sample images are shown as below.

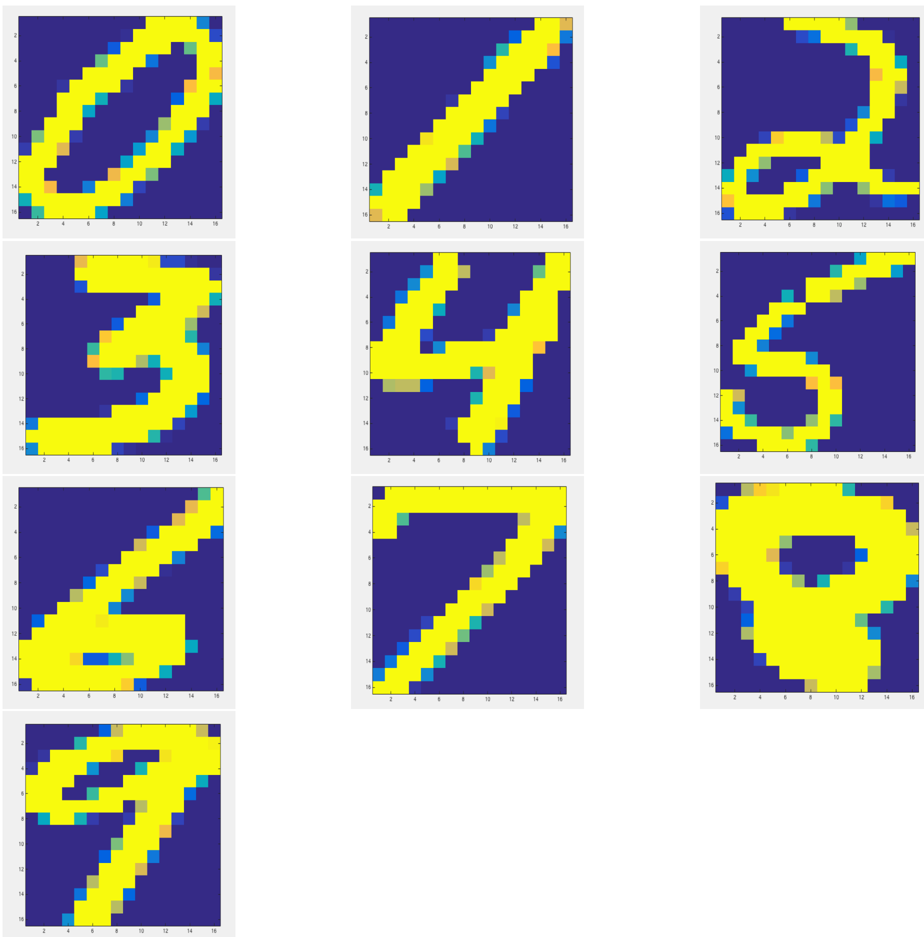

Figure 1: Sample plots for each individual digits from 0 to 9.

2.

3.

4.