

## Problem 0: Homework checklist

- ✓I didn't talk with any one about this homework.
- ✓Source-code are included at the end of this document.

## Problem 1: Prove or disprove

For the proof below, I will assume all the matrices are  $n \times n$  square matrices.

1. The product of two diagonal matrices is diagonal.

$\mathbf{A}$  and  $\mathbf{B}$  are diagonal matrices, so

$$\mathbf{A}_{ij} = 0, \text{ if } i \neq j$$

$$\mathbf{B}_{ij} = 0, \text{ if } i \neq j$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C}_{ij} = \sum_{k=1}^n \mathbf{A}_{ik} \mathbf{B}_{kj}$$

Only if  $i = j = k$ ,  $\mathbf{C}_{ij}$  will not be zero, which mean  $\mathbf{C}$  is also a diagonal matrix.

2. The product of two upper triangular matrices is upper triangular  $\mathbf{A}$  and  $\mathbf{B}$  are two upper triangular matrices, so

$$\mathbf{A}_{ij} = 0, \text{ if } i > j$$

$$\mathbf{B}_{ij} = 0, \text{ if } i > j$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C}_{ij} = \sum_{k=1}^n \mathbf{A}_{ik} \mathbf{B}_{kj}$$

When  $i > j$ ,  $\mathbf{C}_{ij} = 0$  because one of  $\mathbf{A}_{ik}$  and  $\mathbf{B}_{kj}$  must be zero.

3. The product of two symmetric matrices is symmetric.  $\mathbf{A}$  and  $\mathbf{B}$  are two symmetric matrices, so

$$\mathbf{A}_{ij} = \mathbf{A}_{ji}, \text{ if } i = j$$

$$\mathbf{B}_{ij} = \mathbf{B}_{ji}, \text{ if } i = j$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C}_{ij} = \sum_{k=1}^n \mathbf{A}_{ik} \mathbf{B}_{kj}$$

Then  $\mathbf{C}_{ij}$  is not necessary to be equal to  $\mathbf{C}_{ji}$

Counter-example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ is a symmetric matrix;}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ is also a symmetric matrix.}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \text{ which is not a symmetric matrices.}$$

4. The product of two orthogonal matrices is orthogonal.
5. The product of two square, full rank matrices is full rank This statement is incorrect.

Counter-example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ is a full rank matrix;}$$

$$\mathbf{B} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix} \text{ is also a full rank matrix.}$$

$$C = A \times B = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix} \text{ which is not a full matrices.}$$

## Problem 2

We know  $U$  and  $V$  are square orthogonal matrices, then

$$U^T U = I, V^T V = I \quad (1)$$

$$\begin{aligned} \|UAV\|_2^2 &= V^T A^T U^T U A V \\ &= V^T A^T A V \\ &= V^T \|A\|_2^2 V \\ &= V^T V \|A\|_2^2 \\ &= \|A\|_2^2 \end{aligned}$$

Therefore  $\|A\|_2$  is orthogonally invariant.

## Problem 3

$$f(A) = \max_{i,j} |A_{ij}| \quad (2)$$

1.
  - $f(A) > 0$ , when  $A \neq 0$ ,  
 $f(A) = 0$  iff  $A = 0$
  - For and scalar  $k$ ,  
 $f(kA)$   
 $= \max_{i,j} |kA_{ij}|$   
 $= k \max_{i,j} |A_{ij}|$   
 $= kf(A)$
  - $f(A+B)$   
 $= \max_{i,j} |A_{ij} + B_{ij}|$   
 $\leq \max_{i,j} |A_{ij}| + \max_{i,j} |B_{ij}|$   
 $= f(A) + f(B)$

Therefore  $f(A)$  is a matrix norm.

2.  $f(AB)$   
 $= \max_{i,j} |A_{ij} B_{ij}|$   
 $\leq \max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}|$   
 $= f(A)f(B)$

The  $\leq$  is not necessary to be. A counter example:

$$A = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 18 & 18 \\ 18 & 18 \end{bmatrix}$$

$$f(A) = 3$$

$$f(B) = 3$$

$$f(AB) = 18$$

$$\text{Then } f(AB) > f(A)f(B)$$

So  $f$  does not satisfy the sub-multiplicative property.

- 3.

$$g(A) = \sigma f(A) \quad (3)$$

$$g(\mathbf{AB}) = \sigma \max_{i,j} |A_{ij}B_{ij}|$$

$$g(\mathbf{A})g(\mathbf{B}) = \sigma^2 \max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}|$$

To make

$$g(\mathbf{AB}) \leq g(\mathbf{A})g(\mathbf{B}) \quad (4)$$

$$\sigma \max_{i,j} |A_{ij}B_{ij}| \leq \sigma^2 \max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}| \quad (5)$$

$$\sigma \geq \frac{\max_{i,j} |A_{ij}B_{ij}|}{\max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}|} \quad (6)$$

$$(7)$$

## Problem 4

$$f(x) = \|x\mathbf{k}^T\| \quad (8)$$

1.
  - $f(x) > 0$ , when  $\mathbf{A} \neq 0$ ,  
 $f(x) = 0$  iff  $\mathbf{A} = 0$  because  $\|\mathbf{A}\|$  is a matrix norm.
  - $f(mx) = \|m x \mathbf{k}^T\| = m \|x \mathbf{k}^T\| = m f(x)$
  - $f(x+y) = \|(x+y)\mathbf{k}^T\| = \|x\mathbf{k}^T + y\mathbf{k}^T\| \leq \|x\mathbf{k}^T\| + \|y\mathbf{k}^T\| = f(x) + f(y)$

Therefore  $f$  is a vector norm.

2. If  $\|\mathbf{A}\|$  is a sub-multiplicative matrix norm,

$$f(\mathbf{A}x) = \|\mathbf{A}x\mathbf{k}^T\| \leq \|\mathbf{A}\| \|x\mathbf{k}^T\| = \|\mathbf{A}\| f(x) \quad (9)$$

## Problem 5 (Choice 2)

$f(x)$  is a vector norm, it satisfies those conditions:

- $f(x) > 0$ , when  $\mathbf{A} \neq 0$ ,  
 $f(x) = 0$ , iff  $\mathbf{A} = 0$
- $f(mx) = m f(x)$
- $f(x+y) \leq f(x) + f(y)$