purdue university · CS 580
Introduction to the Analysis of Algorithms

HOMEWORK
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January 26, 2016

Problem 1:

Using master theorem:

$$\begin{aligned} a &= 4 \\ b &= 4 \\ f(n) &= n \log n \end{aligned}$$

Then

$$\begin{aligned} \log_b^a &= 1 \\ f(n) &= \Omega(n^{\log_b^a - \epsilon}) \end{aligned}$$

Therefore,

$$T(n) = \Theta(n \log n)$$

Problem 2:

We guess $\Theta(n)$:

$$\begin{aligned} T(n) &= T(\lfloor \frac{n}{7} \rfloor) + 5T(\lfloor \frac{n}{6} \rfloor) + cn \\ &= \left(\frac{1}{7} + \frac{5}{6} + 1 \right) nc \\ &= \frac{83}{42} nc \end{aligned}$$

Problem 3:

For the SoSoSplotchy numbers, we know that :

$$\begin{aligned} S(0) &= 1 \\ S(1) &= 2 \\ S(n) &= 2S(n-1) + S(n-2) \end{aligned}$$

(a) To prove $S(n) = S(a+1)S(n-a-1) + S(a)S(n-2-a)$
Basis: $(a = 0)$ and $n \geq 2$

$$\begin{aligned} S(n) &= S(a+1)S(n-a-1) + S(a)S(n-2-a) \\ &= S(1)S(n-1) + S(0)S(n-2) \\ &= 2S(n-1) + S(n-2) \end{aligned}$$

Induction step: $(a > 0)$
If we know:

$$\begin{aligned} S(n) &= S(a+1)S(n-a-1) + S(a)S(n-a-2) \\ &= S(a+1)[2S(n-a-2) + S(n-a-3)] + S(a)S(n-2-a) \\ &= [2S(a+1) + S(a)]S(n-a-2) + S(a+1)S(n-a-3) \\ &= S(a+2)S(n-a-2) + S(a+1)S(n-a-3)\checkmark \end{aligned}$$

Then for $a+1$, we will have

$$\begin{aligned} S(n) &= S(a+2)S(n-a-2) + S(a+1)S(n-a-3) \\ &= S(a+2)S(n-a-2) + S(a+1)S(n-a-3) \end{aligned}$$

(b) Assuming the fact from part a, and let $n = 2k$ and $a = k-1$, we will have:

$$\begin{aligned} S(n) &= S(a+1)S(n-a-1) + S(a)S(n-2-a) \\ S(2k) &= S(k)S(k) + S(k-1)S(k-1) \end{aligned}$$

If we let $n = 2k+1$ and $a+1 = k+1$, then we will have:

$$\begin{aligned} S(2k+1) &= S(k+1)S(k) + S(k)S(k-1) \\ &= [2S(k) + S(k-1)]S(k) + S(k)S(k-1) \\ &= 2S(k)S(k) + 2S(k-1)S(k) \end{aligned}$$

(c) We know $S(2k+1) = 2S(k)S(k) + 2S(k-1)S(k)$
Replace k with $k-1$, then we will have:

$$\begin{aligned} S(2k-1) &= 2S(k-1)S(k-1) + 2S(k-1)S(k-2) \\ &= 2S(k-1)S(k-1) + 2S(k-1)[S(k) - 2S(k-1)] \\ &= 2S(k-1)[S(k) - S(k-1)] \end{aligned}$$

(d) If n is an odd number, let $2k+1 = n$ and $k = \frac{n-1}{2}$, then

$$\begin{aligned} S(n) &= S(2k+1) \\ &= 2S(k)S(k) + 2S(k-1)S(k) \\ &= 2S(\frac{n-1}{2})S(\frac{n-1}{2}) + 2S(\frac{n-3}{2})S(\frac{n-1}{2}) \\ S(n-1) &= S(2k) \\ &= S(k)S(k) + S(k-1)S(k-1) \\ &= S(\frac{n-1}{2})S(\frac{n-1}{2}) + S(\frac{n-3}{2})S(\frac{n-3}{2}) \end{aligned}$$

If n is an even number, let $2k = n$ and $k = \frac{n}{2}$, then

$$\begin{aligned} S(n) &= S(2k) \\ &= S(k)S(k) + S(k-1)S(k-1) \\ &= S(\frac{n}{2})S(\frac{n}{2}) + S(\frac{n-2}{2}S(\frac{n-2}{2}) \\ S(n-1) &= S(2k-1) \\ &= 2S(k-1)S(k) - 2S(k-1)S(k-1) \\ &= 2S(\frac{n-2}{2})S(\frac{n}{2}) - 2S(\frac{n-2}{2})S(\frac{n-2}{2}) \end{aligned}$$

Pseudocode:

Function $SoSoSplotchy(n)$:

if $n = 1$ **then**

return (2, 1)

else

if $n = odd$ **then**

$$S(\frac{n-1}{2}), S(\frac{n-3}{2}) = SoSoplotchy(\frac{n-1}{2})$$

$$S(n) = 2S(\frac{n-1}{2})S(\frac{n-1}{2}) + 2S(\frac{n-3}{2})S(\frac{n-1}{2})$$

$$S(n-1) = S(\frac{n-1}{2})S(\frac{n-1}{2}) + S(\frac{n-3}{2})S(\frac{n-3}{2})$$

if $n = even$ **then**

$$S(\frac{n}{2}), S(\frac{n-2}{2}) = SoSoplotchy(\frac{n}{2})$$

$$S(n) = S(\frac{n}{2})S(\frac{n}{2}) + S(\frac{n-2}{2}S(\frac{n-2}{2})$$

$$S(n-1) = 2S(\frac{n-2}{2})S(\frac{n}{2}) - 2S(\frac{n-2}{2})S(\frac{n-2}{2})$$

end

return $S(n), S(n-1)$

(e)