PURDUE UNIVERSITY · CS 51500
MATRIX COMPUTATIONS

HOMEWORK
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Problem 0: Homework checklist

✓I didn't talk with any one about this homework.
✓Source-code are included at the end of this document.

Problem 1: The Cholesky Factorization

1. Here is my implementation of Cholesky decomposition in Matlab:

```
function L = cholesky(A)
n = size(A);
n = n(1);
L = zeros(n,n);
%L(1,1) = sqrt(A(1,1));
for i=(1:n)
    sum1 =0;
    for j=(1:i)

        if i==j
            L(j,j) = sqrt(A(j,j)-sum1);
        else
            sum2 = 0;
            for k = (1:j-1)
                sum2 = sum2 + L(i,k)*L(j,k);
            end

            L(i,j) = 1/L(j,j)*(A(i,j)-sum2);
        end
        sum1= sum1 + L(i,j)^2;
    end
end
end
```

2. Cholesky factorization is unique if $\boldsymbol{A}$ is positive definite and the decomposition need not be unique when $\boldsymbol{A}$ is positive semidefinite.

3. Both have the same result.

4. For each size of the matrices I repeated 10 times. The comparison result is shown. Obviously the the Cholesky factorization has better performance as the growth of matrix size.

5. Since $\boldsymbol{A}$ is positive definite, and assume an vector $\mathbf{v} = \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix}$,

   then

$$\mathbf{v}_T \boldsymbol{A} \mathbf{v} > 0$$

$$\begin{bmatrix} x_0 & \mathbf{x} \end{bmatrix} \begin{bmatrix} \alpha & \mathbf{b}_T \\ \mathbf{b} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix} = \alpha x_0^2 + x_0(\mathbf{x}^T\mathbf{b} + \mathbf{b}^T\mathbf{x}) + \mathbf{x}^T\boldsymbol{C}\mathbf{x} > 0$$

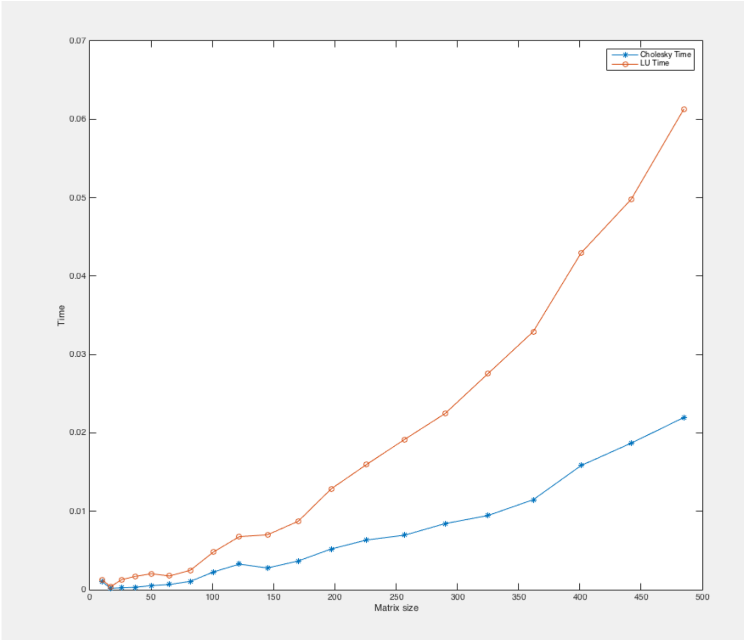

Figure 1: Performance comparison between LU decomposition and Cholesky factorization

Then

$$\begin{aligned}
\mathbf{x}^T(\boldsymbol{C} - \frac{\mathbf{b}\mathbf{b}^T}{\alpha})\mathbf{x} = \mathbf{x}^T\boldsymbol{C}\mathbf{x} - \frac{\mathbf{x}_T\mathbf{b}\mathbf{b}^T\mathbf{x}}{\alpha} \\
&> -\alpha x_0^2 - x_0(\mathbf{x}^T\mathbf{b} + \mathbf{b}^T\mathbf{x}) - \frac{\mathbf{x}^T\mathbf{b}\mathbf{b}^T\mathbf{x}}{\alpha} \\
&= -\frac{1}{\alpha}\left[x_0^2 + \frac{\mathbf{x}^T\mathbf{b} + \mathbf{b}^T\mathbf{x})x_0}{\alpha} + \frac{\mathbf{x}^T\mathbf{b}\mathbf{b}^T\mathbf{x}}{\alpha^2}\right] \\
&= -\frac{1}{\alpha}\left(x_0 + \frac{\mathbf{x}^T\mathbf{b}}{\alpha}\right)\left(x_0 + \frac{\mathbf{b}^T\mathbf{x}}{\alpha}\right) \\
&= -\frac{1}{\alpha}\left\|x_0 + \frac{\mathbf{b}^T\mathbf{x}}{\alpha}\right\|^2
\end{aligned}$$

Because $\mathbf{x}^T(\boldsymbol{C} - \frac{\mathbf{b}\mathbf{b}^T}{\alpha})\mathbf{x} > -\frac{1}{\alpha}\left\|x_0 + \frac{\mathbf{b}^T\mathbf{x}}{\alpha}\right\|^2$ holds for any vector $\mathbf{x}$. Therefore

$$\mathbf{x}^T(\boldsymbol{C} - \frac{\mathbf{b}\mathbf{b}^T}{\alpha})\mathbf{x} > 0 \tag{1}$$

So $\boldsymbol{C} - \frac{\mathbf{b}\mathbf{b}^T}{\alpha}$ is positive definite.

6. The base case is that C is a 1×1.

7.

Problem 2: Stability analysis

1.

2.

Problem 3: Backwards stability

$$
\begin{aligned}
fl(x_i) &= x_i(1+\epsilon_{ix}) \\
fl(y_i) &= y_i(1+\epsilon_{iy})
\end{aligned}
$$

where $|\epsilon_{ix}|, |\epsilon_{iy}| \leq \epsilon_{machine}$

$$
\begin{aligned}
\alpha &= \mathbf{x}^T\mathbf{y} \\
&= \begin{bmatrix} x_1 & x_2 & .. & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ .. \\ y_n \end{bmatrix} \\
&= \sum_{i=1}^{n} x_i y_i
\end{aligned}
$$

Then

$$
\begin{aligned}
fl(\mathbf{x}^T\mathbf{y}) &= \sum_{i=1}^{n} fl(x_i) \times fl(y_i) \\
&= \left[fl(x_1) \times fl(y_1) + fl(x_2) \times fl(y_2) + ... + fl(x_n) \times fl(y_n)\right](1+\epsilon_{addition}) \\
&= \left[x_1y_1(1+\epsilon_{1x})(1+\epsilon_{1y}) + ... + x_ny_n(1+\epsilon_{nx})(1+\epsilon_{ny})\right](1+\epsilon_{addition}) \\
&= \left[x_1y_1 + x_2y_2 + ... + x_ny_n\right](1+\epsilon_1)(1+\epsilon_2)(1+\epsilon_3) \\
&= \left[x_1y_1 + x_2y_2 + ... + x_ny_n\right](1+\mathrm{O}(\epsilon_{machine}^2))
\end{aligned}
$$

Therefore the inner product is backwards stable.