PURDUE UNIVERSITY · ECE 580	HOMEWORK
OPTIMIZATION METHODS FOR SYSTEMS AND CONTROL	*Jun Cheng*
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Problem 1:

$$
\begin{aligned}
f(x_1, x_2) &= x_1^3 + 2x_1x_2 - 3x_1^2x_2^2 \\
\frac{df}{dx_1} &= 2x_1^2 + 2x_2 - 6x_1x_2^2 \\
\frac{df}{dx_2} &= 2x_1 - 6x_1^2x_2 \\
f_{x_1x_1} &= 4x_1 - 6x_2^2 \\
f_{x_1x_2} &= 2 - 12x_1x_2 \\
f_{x_2x_1} &= 2 - 12x_1x_2 \\
f_{x_2x_2} &= -6x_1^2
\end{aligned}
$$

Tylor expansion:

$$f(\mathbf{x}) = f(\mathbf{x}^0) + (\mathbf{x} - \mathbf{x}^0)^T \bigtriangledown f(\mathbf{x}^0) + (\mathbf{x} - \mathbf{x}^0)^T H(\mathbf{x})(\mathbf{x} - \mathbf{x}^0)$$

where $\mathbf{x}^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For linear approximation:

$$
\begin{aligned}
l(x_1, x_2) &= f(\mathbf{x}^0) + (\mathbf{x} - \mathbf{x}^0)^T \bigtriangledown f(\mathbf{x}^0) \\
&= f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} x_1 - 1 & x_2 - 1 \end{bmatrix} \begin{bmatrix} -1 \\ -4 \end{bmatrix} \\
&= -x_1 - 4x_2 + 5
\end{aligned}
$$

Hessian:

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{bmatrix}$$

For quadratic approximation:

$$
\begin{aligned}
f(\mathbf{x}) &= f(\mathbf{x}^0) + (\mathbf{x} - \mathbf{x}^0)^T \bigtriangledown f(\mathbf{x}^0) + (\mathbf{x} - \mathbf{x}^0)^T H(\mathbf{x})(\mathbf{x} - \mathbf{x}^0) \\
&= -x_1 - 4x_2 + 5 + \begin{bmatrix} x_1 - 1 & x_2 - 1 \end{bmatrix} \begin{bmatrix} 4x_1 - 6x_2^2 & 2 - 12x_1x_2 \\ 2 - 12x_1x_2 & -6x_1^2 \end{bmatrix} \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \\
&= -x_1 - 4x_2 + 5 + (2x_1 - 6x_1^2x_2)(x_1 - 1)^2 - 6x_1^2(x_2 - 1)^2 + (4 - 24x_1x_2)(x_1 - 1)(x_2 - 1)
\end{aligned}
$$

Problem 2:

1.

$$\begin{aligned}
f(\mathbf{x}) &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 // x_2 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \pi \\
&= x_1^2 + 3x_2^2 + 6x_1x_2 + 2x_1 + x_2 + \pi \\
\frac{df}{dx_1} &= 2x_1 + 6x_2 + 2 \\
\frac{df}{dx_2} &= 6x_2 + 6x_1 + 1 \\
Df(\mathbf{x}) &= \begin{bmatrix} \frac{df}{dx_1} & \frac{df}{dx_2} \end{bmatrix} = \begin{bmatrix} 2x_1 + 6x_2 + 2 & 6x_2 + 6x_1 + 1 \end{bmatrix}
\end{aligned}$$

2.

$$\begin{aligned}
f(\mathbf{x}) &= \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \log 3 \\
&= x_1^2 + \frac{1}{2}x_2^2 + 6x_1x_2 + 2x_1 - 3x_2 + \log 3 \\
\frac{df}{x_1} &= 2x_1 + 6x_2 + 2 \\
\frac{df}{x_2} &= x_2 + 6x_1 - 3 \\
f_{x_1x_1} &= 2 \\
f_{x_1x_2} &= 6 \\
f_{x_2x_1} &= 6 \\
f_{x_2x_2} &= 1
\end{aligned}$$

Hessian:

$$H = \begin{bmatrix} 2 & 6 \\ 6 & 1 \end{bmatrix}$$

Problem 3:

$$f(x_1, x_2) = e^{3x_1x_2^2}$$

1.

$$\begin{aligned}
\frac{df}{x_1} &= 3x_2^2 e^{3x_1x_2^2} \\
\frac{df}{x_2} &= 6x_1x_2 e^{3x_1x_2^2}
\end{aligned}$$

Then the gradient at $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is :

$$\begin{bmatrix} 3e^3 \\ 6e^3 \end{bmatrix}$$

2. Rate of increase of f at the point $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the direction $\mathbf{d}$

$$\frac{Df(\mathbf{x}) \cdot \mathbf{d}}{\|\mathbf{d}\|} = \frac{15e^3}{\sqrt{5}}$$

3. In the direction $\mathbf{d} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, we can have the maximum rate of increase $3\sqrt{5}e^3$

Problem 4:

$$
\begin{aligned}
f(x_1, x_2, x_3) &= -(x_1^2 + 4\epsilon x_2^2 + 5x_3^2 - 2x_1x_3 + 2\epsilon x_1x_2 + 4x_2x_3) \\
&= -\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & \epsilon & -1 \\ \epsilon & 4\epsilon & 2 \\ -1 & 2 & 5 \end{bmatrix}
\end{aligned}
$$

Using principal minor method:

$$
\begin{aligned}
&k = 1: \\
&\quad D_1 = 20\epsilon - 4 > 0 \\
&\quad D_2 = 7 \\
&\quad D_3 = 4\epsilon - \epsilon^2 > 0 \\
&k = 2: \\
&\quad D_1 = 1 \\
&\quad D_2 = 4\epsilon > 0 \\
&\quad D_3 = 5 \\
&k = 3: \\
&\quad D = 12\epsilon - 4 - 5\epsilon^2 > 0
\end{aligned}
$$

To make the function be negative semi-definite, the range of ϵ is $0.4 \le \epsilon \le 2$

Problem 5:

1.

$$
\begin{aligned}
f(x_1, x_2) &= \frac{1}{3}x_2^3 + \frac{1}{2}x_2^2 + 2x_1x_2 + \frac{1}{2}x_1^2 - x_1 + 10 \\
\frac{df}{dx_1} &= 2x_2 + x_1 - 1 = 0 \\
\frac{df}{dx_2} &= x_2^2 + x_2 + 2x_1 = 0
\end{aligned}
$$

We can get two points $(-1, 1)$ and $(-3, 2)$ which satisfy the first-order necessary conditions for the extremum.

2.

$$
\begin{aligned}
\frac{df}{dx_1dx_1} &= 1 \\
\frac{df}{dx_1dx_2} &= 2 \\
\frac{df}{dx_2dx_1} &= 2 \\
\frac{df}{dx_2dx_2} &= 2x_2 + 1
\end{aligned}
$$

Then Hessian is :

$$\begin{bmatrix} 1 & 2 \\ 2 & 2x_2+1 \end{bmatrix}$$

For point $(-1, 1)$,
$H = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is indefinite For point $(-3, 2)$,
$H = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is positive definite Then the point $(-3, 2)$ is a strict local minimizer.

Problem 6: