HOMEWORK
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## Problem 1: Exercise 16.12

Solve the following linear programs using simplex method:
(a) Maximize $-4x_1 - 3x_2$ subject to

$$\begin{aligned} 5x_1 + x_2 &\geq 11 \\ -2x_1 - x_2 &\leq -8 \\ x_1 + 2x_2 &\geq 7 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Introduce slack variables $x_3, x_4, x_5$, we will have:

$$\begin{aligned} 5x_1 + x_2 - x_3 &= 11 \\ -2x_1 - x_2 + x_4 &= -8 \\ x_1 + 2x_2 - x_5 &= 7 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

We are trying to minimize $4x_1 + 3x_2$

$$C = [4, 3, 0, 0, 0]$$

Then

$$\begin{bmatrix} 5 & 1 & -1 & 0 & 0 & 11 \\ 2 & 1 & 0 & -1 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 7 \end{bmatrix}$$

Now we are using Two-Phase simplex method to solve this problem.

$$\begin{bmatrix} 5 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 11 \\ 2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

We must update the last row :

$$\begin{bmatrix} 5 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 11 \\ 2 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 8 \\ 1 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 7 \\ -8 & -4 & 1 & 1 & 1 & 0 & 0 & 0 & -26 \end{bmatrix}$$

Take $\alpha_{11}$ as pivot:

$$\begin{bmatrix} 1 & {}^{1}/_{5} & -{}^{1}/_{5} & 0 & 0 & {}^{1}/_{5} & 0 & 0 & {}^{11}/_{5} \\ 0 & {}^{3}/_{5} & {}^{2}/_{5} & -1 & 0 & {}^{2}/_{5} & 1 & 0 & {}^{18}/_{5} \\ 0 & {}^{9}/_{5} & {}^{1}/_{5} & 0 & -1 & -{}^{1}/_{5} & 0 & 1 & {}^{24}/_{5} \\ 0 & -{}^{12}/_{5} & -{}^{3}/_{5} & 1 & 1 & {}^{8}/_{5} & 0 & 0 & {}^{42}/_{5} \end{bmatrix}$$

Take $\alpha_{31}$ as pivot:

$$\begin{bmatrix} 1 & 0 & {}^{2}/_{9} & 0 & {}^{1}/_{9} & {}^{2}/_{9} & 0 & {}^{1}/_{9} & {}^{5}/_{3} \\ 0 & 0 & {}^{1}/_{3} & -1 & {}^{1}/_{3} & -{}^{1}/_{3} & 1 & -{}^{1}/_{3} & 2 \\ 0 & 1 & {}^{1}/_{9} & 0 & -{}^{5}/_{9} & -{}^{1}/_{9} & 0 & {}^{5}/_{9} & {}^{8}/_{3} \\ 0 & 0 & {}^{1}/_{3} & 1 & -{}^{1}/_{3} & {}^{4}/_{3} & 0 & {}^{4}/_{3} & -2 \end{bmatrix}$$

Take $\alpha_{23}$ as pivot:

$$\begin{bmatrix} 1 & 0 & 0 & {}^{2}/_{3} & {}^{1}/_{3} & 0 & {}^{2}/_{3} & -{}^{1}/_{3} & 3 \\ 0 & 0 & 1 & -3 & 1 & -1 & 3 & -1 & 6 \\ 0 & 1 & 0 & {}^{1}/_{3} & -{}^{2}/_{3} & 0 & -3 & {}^{2}/_{3} & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Now we can remove column 6-8.

$$\begin{bmatrix} 1 & 0 & 0 & -{}^{2}/_{3} & {}^{1}/_{3} & 3 \\ 0 & 0 & 1 & -3 & 1 & 6 \\ 0 & 1 & 0 & {}^{1}/_{3} & -{}^{2}/_{3} & 2 \\ 4 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Updating the last row, we can have:

$$\begin{bmatrix} 1 & 0 & 0 & -{}^{2}/_{3} & {}^{1}/_{3} & 3 \\ 0 & 0 & 1 & -3 & 1 & 6 \\ 0 & 1 & 0 & {}^{1}/_{3} & -{}^{2}/_{3} & 2 \\ 0 & 0 & 0 & {}^{5}/_{3} & 0 & -18 \end{bmatrix}$$

All the reduced cost coefficients are nonnegative, hence the optimal solution is

$$x = \begin{bmatrix} 3 \\ 2 \\ 6 \\ 0 \\ 0 \end{bmatrix}$$

and the optimal value is 18.

## Problem 2: Exercise 16.12

The dual problem: Maximize $11\lambda_1 + 8\lambda_2 + 7\lambda_3$ subject to:

$$\begin{aligned} 5\lambda_1 + 2\lambda_2 + \lambda_3 &\leq 4 \\ \lambda_1 + \lambda_2 + 2\lambda_3 &\leq 3 \\ \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

If we introduce slack variables $\lambda_4$ and $\lambda_5$, we will have :

$$\begin{aligned} 5\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_4 &= 4 \\ \lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_5 &= 3 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 &\geq 0 \end{aligned}$$

Then

$$\begin{bmatrix} 5 & 2 & 1 & 1 & 0 & 4 \\ 1 & 1 & 2 & 0 & 1 & 3 \\ -11 & -8 & -7 & 0 & 0 & 0 \end{bmatrix}$$

Take $\alpha_{11}$ as pivot to get :

$$\begin{bmatrix} 1 & {}^{2}/_{5} & {}^{1}/_{5} & {}^{1}/_{5} & 0 & {}^{4}/_{5} \\ 0 & {}^{3}/_{5} & {}^{9}/_{5} & -{}^{1}/_{5} & 1 & {}^{11}/_{5} \\ 0 & -{}^{18}/_{5} & -{}^{24}/_{5} & {}^{11}/_{5} & 0 & {}^{44}/_{5} \end{bmatrix}$$

Take $\alpha_{23}$ as pivot to get :

$$\begin{bmatrix} 1 & {}^{1}/_{3} & 0 & {}^{2}/_{9} & -{}^{1}/_{9} & {}^{5}/_{9} \\ 0 & {}^{1}/_{3} & 1 & -{}^{1}/_{9} & {}^{5}/_{9} & {}^{11}/_{9} \\ 0 & -2 & 0 & {}^{5}/_{3} & {}^{8}/_{3} & {}^{44}/_{3} \end{bmatrix}$$

Take $\alpha_{12}$ as pivot to get :

$$\begin{bmatrix} 3 & 1 & 0 & {}^{2}/_{3} & -{}^{1}/_{3} & {}^{5}/_{3} \\ -1 & 0 & 1 & -{}^{1}/_{3} & {}^{2}/_{3} & {}^{2}/_{3} \\ 6 & 0 & 0 & 3 & 2 & 18 \end{bmatrix}$$

All the reduced cost coefficients are nonnegative, hence the optimal solution is

$$\lambda = \begin{bmatrix} 0 \\ {}^{5}/_{3} \\ {}^{2}/_{3} \\ 0 \\ 0 \end{bmatrix}$$

and the optimal value is 18.

## Problem 3: Exercise 20.2 b.

Find local extremizers for the following optimization problem:
Minimize: $4x_1 + x_2^2$
subject to : $x_1^2 + x_2^2 = 9$

$$\begin{aligned} f(x_1, x_2) &= 4x_1 + x_2^2 \\ h(x_1, x_2) &= x_1^2 + x_2^2 - 9 \\ \nabla f &= \begin{bmatrix} 4 & 2x_2 \end{bmatrix} \\ \nabla h &= \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \end{aligned}$$

By the Lagrange condition:

$$\begin{aligned} 4 - \lambda \times 2x_1 &= 0 \\ 2x_2 - \lambda \times 2x_2 &= 0 \\ x_1^2 + x_2^2 - 9 &= 0 \end{aligned}$$

If $\lambda = -1$, we have:

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 2 & \sqrt{5} \end{bmatrix} \\ \mathbf{x} &= \begin{bmatrix} 2 & -\sqrt{5} \end{bmatrix} \end{aligned}$$

If $\lambda = \frac{2}{3}$, we have: $\mathbf{x} = \begin{bmatrix} -3 & 0 \end{bmatrix}$
If $\lambda = -\frac{2}{3}$, we have: $\mathbf{x} = \begin{bmatrix} 3 & 0 \end{bmatrix}$
Since we want maximizers only, we keep $\lambda = -1$. Now we want to show that

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 2 & \sqrt{5} \end{bmatrix} \\ \mathbf{x} &= \begin{bmatrix} 2 & -\sqrt{5} \end{bmatrix} \end{aligned}$$

are indeed the maximizers. The Hessian matrix is :

$$\begin{aligned} H(x_1, x_2) &= \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + \lambda \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

To find the tangent space we have :

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & \pm\sqrt{5} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

So the tangent space is $\begin{bmatrix} \frac{\sqrt{5}}{2}a, & a \end{bmatrix}$ and $\begin{bmatrix} -\frac{\sqrt{5}}{2}a, & a \end{bmatrix}$ Then,

$$\begin{aligned} \begin{bmatrix} \frac{\sqrt{5}}{2}a, & a \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{5}}{2}a \\ a \end{bmatrix} &= -\frac{5}{2}a^2 < 0 \\ \begin{bmatrix} \frac{-\sqrt{5}}{2}a, & a \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-\sqrt{5}}{2}a \\ a \end{bmatrix} &= -\frac{5}{2}a^2 < 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 2 & \sqrt{5} \end{bmatrix} \\ \mathbf{x} &= \begin{bmatrix} 2 & -\sqrt{5} \end{bmatrix} \end{aligned}$$

are indeed the maximizers.

## Problem 4: Exercise 20.9

Find all maximizers of the function: $f(x_1, x_2) = \frac{18x_1^2 - 8x_1x_2 + 12x_2^2}{2x_1^2 + 2x_2^2}$ In this problem we have

$$\begin{aligned} \boldsymbol{P} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ \boldsymbol{Q} &= \begin{bmatrix} 18 & -4 \\ -4 & 12 \end{bmatrix} \end{aligned}$$

Then we can have :

$$\boldsymbol{P}^{-1}\boldsymbol{Q} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

The eigenvalues are $\lambda = 10$ and $\lambda = 5$, Since we are looking for maximizer, we keep $\lambda = 10$ only. Now we will show that the $\left[-\frac{2}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right]$ and $\left[\frac{2}{\sqrt{10}}, -\frac{1}{\sqrt{10}}\right]$and is indeed the maximizer.
The Hessian is

$$H(x_1, x_2) = 2\boldsymbol{Q} - 2\lambda\boldsymbol{P} = \begin{bmatrix} -8 & -8 \\ -8 & -16 \end{bmatrix}$$

To find the tangent space we have:

$$\begin{aligned} \begin{bmatrix} 4x_1^*, 4x_2^* \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= 0 \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} a \\ 2a \end{bmatrix} \end{aligned}$$

Then

$$\begin{bmatrix} a & 2a \end{bmatrix} \begin{bmatrix} -8 & -8 \\ -8 & -16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} a \\ 2a \end{bmatrix} = -104a^2 < 0$$

Therefore,

$$\mathbf{x} = \left[-\tfrac{2}{\sqrt{10}}, \tfrac{1}{\sqrt{10}}\right]$$
$$\mathbf{x} = \left[\tfrac{2}{\sqrt{10}}, -\tfrac{1}{\sqrt{10}}\right]$$

are indeed the maximizers.

## Problem 5: Exercise 21.2

Find local extremizers for :
a. $x_1^2 + x_2^2 - 2x_1 - 10x_2 + 26$, subject to ${}^1\!/_5 x_2 - x_1^2 \le 0, 5x_1 + {}^1\!/_2 x_2 \le 5$
b. $x_1^2 + x_2^2$, subject to $x_1 \ge 0, x_2 \ge 0, x_1 + x_2 \ge 5$
c. $x_1^2 + 6x_1x_2 - 4x_1 - 2x_2$, subject to $x_1^2 + 2x_2 \le 1, 2x_1 - 2x_2 \le 1$

(a)

$$\begin{aligned} f(\mathbf{x}) &= x_1^2 + x_2^2 - x_1 - 10x_2 + 26 \\ g_1(\mathbf{x}) &= {}^1\!/_5 x_2 - x_1^2 \le 0 \\ g_2(\mathbf{x}) &= 5x_1 + {}^1\!/_2 x_2 - 5 \le 0 \\ \nabla f(\mathbf{x}) &= \begin{bmatrix} 2x_1 - 2, & 2x_2 - 10 \end{bmatrix} \\ \nabla g_1(\mathbf{x}) &= \begin{bmatrix} -2x_1, & {}^1\!/_5 \end{bmatrix} \\ \nabla g_2(\mathbf{x}) &= \begin{bmatrix} 5, & {}^1\!/_2 \end{bmatrix} \end{aligned}$$

Write the KKT condition:

$$\begin{aligned} (2x_1 - 2) - 2\mu_1 x_1 + 5\mu_2 &= 0 \\ (2x_2 - 10) + \frac{1}{5}\mu_1 + \frac{1}{2}\mu_2 &= 0 \\ \mu_1(\frac{1}{5}x_2 - x_1^2) + \mu_2(5x_1 - \frac{1}{2} - 5) &= 0 \end{aligned}$$

If $\mu_1 = 0$ and $\mu_2 = 0$, we have

$$\mathbf{x} = \begin{bmatrix} 1, & 5 \end{bmatrix}$$

which does not satisfy $g_2(x) \le 0$, so no feasible solutions.
If $\mu_1 \ne 0$ and $\mu_2 = 0$, we have only one feasible solution:

$$\mathbf{x} = \begin{bmatrix} -1.02, & 5.2 \end{bmatrix}$$

If $\mu_1 = 0$ and $\mu_2 \ne 0$, we do not have only one feasible solution.
If $\mu_1 \ne 0$ and $\mu_2 \ne 0$, we have only one feasible solution:

$$\mathbf{x} = \begin{bmatrix} -1 + \sqrt{2}, & 3 - 2\sqrt{2} \end{bmatrix}$$

(b)

$$
\begin{aligned}
f(\mathbf{x}) &= x_1^2 + x_2^2 \\
g_1(\mathbf{x}) &= x_1 \geq 0 \\
g_2(\mathbf{x}) &= x_2 \geq 0 \\
g_3(\mathbf{x}) &= x_1 + x_2 - 5 \geq 0 \\
\nabla f(\mathbf{x}) &= \begin{bmatrix} 2x_1, & 2x_2 \end{bmatrix} \\
\nabla g_1(\mathbf{x}) &= \begin{bmatrix} 1, & 0 \end{bmatrix} \\
\nabla g_2(\mathbf{x}) &= \begin{bmatrix} 0, & 1 \end{bmatrix} \\
\nabla g_3(\mathbf{x}) &= \begin{bmatrix} 1, & 1 \end{bmatrix}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
2x_1 + \mu_1 + \mu_3 &= 0 \\
2x_2 + \mu_2 + \mu_3 &= 0 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
\mu_1 x_1 + \mu_2 x_2 + \mu_3(x_1 + x_2 - 5) &= 0
\end{aligned}
$$

If $\mu_1 = \mu_2 = 0$ and $\mu_3 \neq 0$ then $x_1 = x_2 = \frac{5}{2}$
After trying different combination of $\mu_1$, $\mu_2$ and $\mu_3$ The only extreminzer is $(\frac{5}{2}, \frac{5}{2})$

(c)

$$
\begin{aligned}
f(\mathbf{x}) &= x_1^2 + 6x_1x_2 - 4x_1 - 2x_2 \\
g_1(\mathbf{x}) &= x_1^2 + 2x_2 - 1 \leq 0 \\
g_2(\mathbf{x}) &= 2x_1 - 2x_2 - 1 \leq 0 \\
\nabla f(\mathbf{x}) &= \begin{bmatrix} 2x_1 + 6x_2 - 4, & 6x_1 - 2 \end{bmatrix} \\
\nabla g_1(\mathbf{x}) &= \begin{bmatrix} 2x_1, & 2 \end{bmatrix} \\
\nabla g_2(\mathbf{x}) &= \begin{bmatrix} 2, & -2 \end{bmatrix}
\end{aligned}
$$

Then we have

$$
\begin{aligned}
(2x_1 + 6x_2 - 4) + \mu_1(2x_1) + 2\mu_2 &= 0 \\
(6x_1 - 2) + 2\mu_1 - 2\mu_2 &= 0 \\
x_1^2 + 2x_2 - 1 &\leq 0 \\
2x_1 - 2x_2 - 1 &\leq 0 \\
\mu_1(x_1^2 + 2x_2 - 1) + \mu_2(2x_1 - 2x_2 - 1) &= 0
\end{aligned}
$$

If $\mu_1 = 0$ and $\mu_2 = 0$: we can get $x_1 = \frac{1}{3}$ and $x_2 = \frac{5}{9}$, which will make $x_1^2 + 2x_2 - 1 = \frac{2}{9} > 0$, not feasible.
If $\mu_1 = 0$ and $\mu_2 \neq 0$: we can get $\begin{bmatrix} {}^1/_7, & x_2 = {}^9/_{14} \end{bmatrix}$
If $\mu_1 \neq 0$ and $\mu_2 = 0$: we can not get any solutions.
If $\mu_1 \neq 0$ and $\mu_2 \neq 0$: we can get only one feasible point $\begin{bmatrix} -1 - \sqrt{2}, 3 + 2\sqrt{2} \end{bmatrix}$.