HOMEWORK

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November 7, 2015

Problem 0: Homework checklist

✓I didn't talk with any one about this homework.
✓Source-code are included at the end of this document.

Problem 1: Prove or disprove

1. False. The eigenvalues of an $n \times n$ real-valued matrix are not always real. For example,

$$\boldsymbol{A} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$
$$\lambda_1 = cos\theta + i\sin\theta$$
$$\lambda_2 = cos\theta - i\sin\theta$$

2. True.

$$det(\boldsymbol{A}^T\boldsymbol{A} + \gamma\boldsymbol{I}) \neq 0$$

So the solution to $(\boldsymbol{A}^T\boldsymbol{A} + \gamma\boldsymbol{I})\mathbf{x} = \mathbf{b}$ is unique for any $\gamma > 0$.

3. True. Suppose that $\alpha \neq 0$ is not an eigenvalue of $\boldsymbol{A}$. Then

$$(\boldsymbol{A} - \alpha\boldsymbol{I})\mathbf{v} \neq 0$$

So $\boldsymbol{A} - \alpha\boldsymbol{I}$ is non-singular.
Then $\alpha\boldsymbol{I}$ and $\boldsymbol{A} = (\boldsymbol{A} - \alpha\boldsymbol{I}) + \alpha\boldsymbol{I}$ are non-singular matrices.

4. False. An symmetric matrix has an orthogonal set of eigenvector.

$$\boldsymbol{A} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$
$$\lambda_1 = 3$$
$$\lambda_2 = -2$$
$$\mathbf{v}_1 = \begin{bmatrix} 0.9701 \\ -0.2425 \end{bmatrix}$$
$$\mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

Those two eigenvectors are not orthogonal.

Problem 2: The power method, and beyond!

1. Based on the definition of eigenvalue:

$$\boldsymbol{A}\mathbf{x} = \lambda\mathbf{x}$$
$$\boldsymbol{A}\mathbf{x}^{(i)} = \lambda^{(i)}\mathbf{x}^{(i)}$$

Then

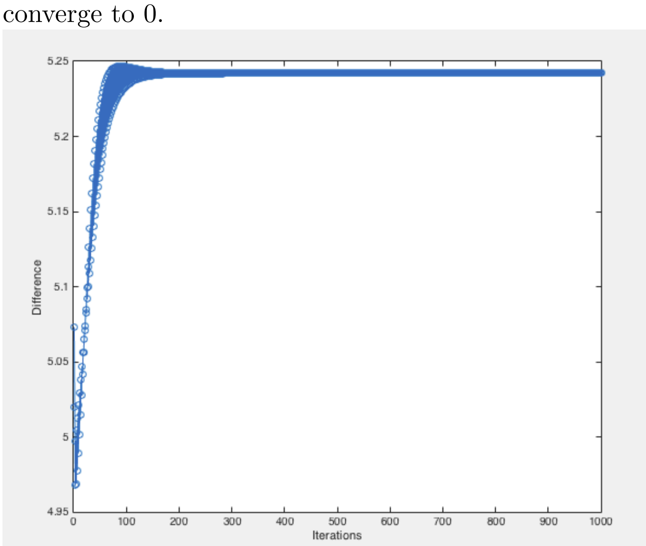
$$
\begin{aligned}
\lambda &= \frac{\mathbf{x}^T \boldsymbol{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \\
\lambda^{(i)} &= \frac{\mathbf{x}^{(i)T} \boldsymbol{A}\mathbf{x}^{(i)}}{\mathbf{x}^{(i)T}\mathbf{x}^{(i)}} \\
|\lambda - \lambda^{(i)}| &= |\frac{\mathbf{x}^T \boldsymbol{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} - \frac{\mathbf{x}^{(i)T} \boldsymbol{A}\mathbf{x}^{(i)}}{\mathbf{x}^{(i)T}\mathbf{x}^{(i)}}| \quad = |(\mathbf{x}-\mathbf{x}^{(i)})^T(\mathbf{x}-\mathbf{x}^{(i)})| + O(\epsilon^3)| \\
&= |\|\mathbf{x}-\mathbf{x}^{(i)}\|^2 + O(\epsilon^3)| \\
&= |\epsilon^2 + O(\epsilon^3)| \\
&= O(\epsilon^2)
\end{aligned}
$$

2.

$$
\begin{aligned}
\boldsymbol{A}\mathbf{v} &= \lambda\mathbf{v} \\
\boldsymbol{A}^{-1}\boldsymbol{A}\mathbf{v} &= \lambda\boldsymbol{A}^{-1}\mathbf{v} \\
\boldsymbol{A}^{-1}\mathbf{v} &= \frac{1}{\lambda}\mathbf{v}
\end{aligned}
$$

Therefore the eigenvalues are $\frac{1}{\lambda}$.

3. It doesn't converge. The 'difference vs iterations' plot shows it does not converge to 0.



4. If $\mathbf{v}^{(k)}$is close to an eigenvector, then

$$
\begin{aligned}
\|\mathbf{v}^{(k)} - (\pm q_J)\| &\leq \epsilon \\
|\lambda^{(k)} - \lambda_J| &= O(\epsilon^2)
\end{aligned}
$$

Then one step of inverse iteration then gives:

$$
\|\mathbf{v}^{(k+1)} - \mathbf{q}_J\| = O(|\lambda^{(k)} - \lambda_J|\|\mathbf{v}^{(k)} - \mathbf{q}_J\|) = O(\epsilon^3)
$$

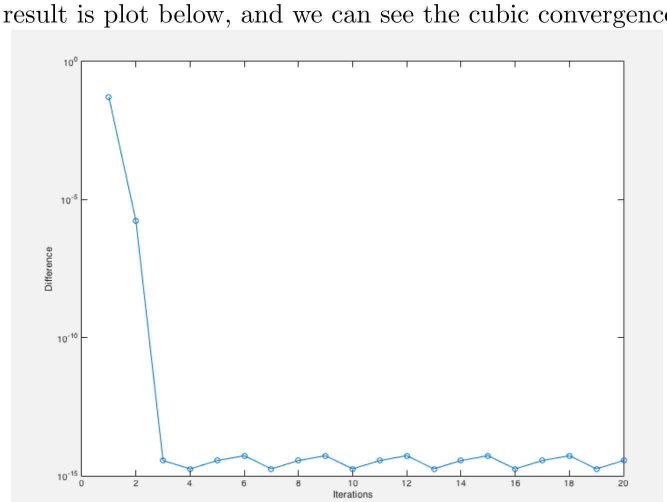
5. The code is list below:

```
maxit = 20;
dim = 6;
```

```
A = rand(dim);
A = A*A;
e = abs(eig(A));
lamda = max(e);
[V, D]  = eig(A);
true_X = V(:,1) ;
true_X = true_X/norm(true_X);

x= rand(dim,1);
x= x/norm(x);
lam = x'*A*x;
diff = zeros(maxit,1);
iter = zeros(maxit,1);
eps = zeros(maxit,1);
for i=1:maxit
    x=(A-lam*eye(size(A,1)))\x;
    x = x/norm(x,2);
    lam = x'*A*x;
    diff(i) = norm(lam - lamda);
    iter(i) = i;
end
semilogy(iter, diff, 'o-') ;
xlabel('Iterations');
ylabel('Difference');
```

The result is plot below, and we can see the cubic convergence.



Problem 3: PageRank and the power method

1.

$$M = I - \alpha P$$
$$M^T = I - \alpha P^T$$

Because the $\boldsymbol{P}$ is a column-stochastic matrix, and then $\boldsymbol{P}$ is a row-stochastic matrix.

$$\begin{aligned}
1 &= \sum |\boldsymbol{P}^T_{ij}| \\
\boldsymbol{P}_{ii} &= \sum_{i\neq} |\boldsymbol{P}^T_{ij}| \\
\boldsymbol{M}_{ii} &= 1 - \alpha \boldsymbol{P}_{ii} \\
&= 1 - \alpha \sum_{i\neq j} |\boldsymbol{P}^T_{ij}|
\end{aligned}$$

Because $\alpha < 1$, then

$$|\boldsymbol{M}^T_{ii}| > \sum_{i\neq j} |\boldsymbol{M}^T_{ij}|$$

Then$\boldsymbol{M}_T$ is strictly diagonally dominant. We know that a strictly diagonally dominant matrix is nonsingular.

2. Because $\mathbf{v}$ is a non-negative vector whose elements sum to 1.

$$\begin{aligned}
\mathbf{v}\mathbf{e}^T\mathbf{v} &= \begin{bmatrix} v_1(v_1 + v_2 + ...) \\ v_2(v_1 + v_2 + ...) \\ ... \\ ... \\ v_n(v_1 + v_2 + ...) \end{bmatrix} \\
&= \mathbf{v} \\
\boldsymbol{M}\mathbf{v} &= [\alpha \boldsymbol{P} + (1-\alpha)\mathbf{v}\mathbf{e}^T]\mathbf{v} \\
&= \alpha \boldsymbol{P}\mathbf{v} + (1-\alpha)\mathbf{v}\mathbf{e}^T\mathbf{v} \\
&= \alpha \boldsymbol{P}\mathbf{v} + (1-\alpha)\mathbf{v} \\
&= \alpha \begin{bmatrix} \sum P_{1j}v_j \\ \sum P_{2j}v_j \\ ... \\ ... \\ \sum P_{nj}v_j \end{bmatrix} + (1-\alpha)\mathbf{v} \\
\mathbf{x}^1 &= \boldsymbol{M}\mathbf{v} \\
\|\mathbf{v}\|_1 &= \|\boldsymbol{M}\mathbf{v}\|_1 = \alpha \sum_i v_i [\sum_j P_{ji}] + (1-\alpha) \\
&= 1
\end{aligned}$$

For $\mathbf{x}^{(k+1)} = \boldsymbol{M}\mathbf{x}^{(k)}$, we always have $\|x^{(k+1)}\|_1 = 1$

3. Suppose $\mathbf{x}$ is the dominant eigenvector of matrix $\boldsymbol{M}$

$$\begin{aligned}
\boldsymbol{M}\mathbf{x} &= \mathbf{x} \\
&= [\alpha \boldsymbol{P} + (1-\alpha)\mathbf{v}\mathbf{e}^T]\mathbf{x}
\end{aligned}$$

$$[\boldsymbol{I} - (\alpha \boldsymbol{P} + (1-\alpha)\mathbf{v}\mathbf{e}^T)]\mathbf{x} = 0$$

We know that $\mathbf{v}\mathbf{e}^T\mathbf{x} = \mathbf{x}$
Then

$$
\begin{aligned}
(\boldsymbol{I} - \alpha \boldsymbol{P}^T)\mathbf{x} - (1-\alpha)\mathbf{v} = 0 \\
(\boldsymbol{I} - \alpha \boldsymbol{P}^T)\mathbf{x} = (1-\alpha)\mathbf{v}
\end{aligned}
$$

4. The simplified iterations:

$$
\begin{aligned}
\mathbf{x}^{(k+1)} &= \alpha \boldsymbol{P}\mathbf{x}^{(k)} + (1-\alpha)\mathbf{v} \\
\mathbf{x} &= \alpha \boldsymbol{P}\mathbf{x} + (1-\alpha)\mathbf{v} \\
\mathbf{x} - \mathbf{x}^{(k+1)} &= \alpha \boldsymbol{P}\mathbf{x} - \alpha \boldsymbol{P}\mathbf{x}^{(k)} \\
\|\mathbf{x} - \mathbf{x}^{(k+1)}\| &= \|\alpha \boldsymbol{P}(\mathbf{x} - \mathbf{x}^{(k)}\| \\
&= \|\alpha \boldsymbol{P}\|\|\mathbf{x} - \mathbf{x}^{(k+1)}\| \\
&\leq \|\alpha \boldsymbol{P}\|\|\alpha \boldsymbol{P}\|...\|\mathbf{x} - \mathbf{x}^{(k-1)}\| \\
&\leq \|\alpha \boldsymbol{P}\|^{k+1}\|\mathbf{x} - \mathbf{x}^{(0)}\|
\end{aligned}
$$

Therefore, the power method will converge to the solution of the linear system $(\boldsymbol{I} - \alpha \boldsymbol{P}^T)\mathbf{x} = (1-\alpha)\mathbf{v}$.

5. The first url shown is : 'http://aae.www.ecn.purdue.edu/'

6. $\mathbf{x}(1)$ is $\mathbf{v}(1) = [\frac{1}{n}, \frac{1}{n}, \frac{1}{n}...]$
Then the top 27 entries are :

```
'http://aae.www.ecn.purdue.edu/'
    'http://aae.www.ecn.purdue.edu/AAE/'
    'http://aae.www.ecn.purdue.edu/AAE/Academic/'
    'http://aae.www.ecn.purdue.edu/AAE/Academic/Index.html'
    'http://aae.www.ecn.purdue.edu/AAE/Academic/New_Index.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/97DEAPic.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/AlumniEvents.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/DEA.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/HonDoc.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/IndAdvisory.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/Index.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/New_Index.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/YourGift.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/industrialaffiliatesprogram.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/outstandingeng.html'
    'http://aae.www.ecn.purdue.edu/AAE/Alumni/williameboeing.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Armstrong.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Astronaut.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Blaha.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Bridges.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Brown.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Casper.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Cernan.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Chaffee.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Covey.html'
    'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Gardner.html'
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