

Problem 0: Homework checklist

- ✓ I didn't talk with any one about this homework.
- ✓ Source-code are included at the end of this document.

Problem 1: Prove or disprove

1. False. The eigenvalues of an $n \times n$ real-valued matrix are not always real.
For example,

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$\lambda_1 = \cos\theta + i \sin\theta$$
$$\lambda_2 = \cos\theta - i \sin\theta$$

2. True.

$$\det(\mathbf{A}^T \mathbf{A} + \gamma \mathbf{I}) \neq 0$$

So the solution to $(\mathbf{A}^T \mathbf{A} + \gamma \mathbf{I})\mathbf{x} = \mathbf{b}$ is unique for any $\gamma > 0$.

3. True. Suppose that $\alpha \neq 0$ is not an eigenvalue of \mathbf{A} . Then

$$(\mathbf{A} - \alpha \mathbf{I})\mathbf{v} \neq 0$$

So $\mathbf{A} - \alpha \mathbf{I}$ is non-singular.

Then $\alpha \mathbf{I}$ and $\mathbf{A} = (\mathbf{A} - \alpha \mathbf{I}) + \alpha \mathbf{I}$ are non-singular matrices.

4. False. An symmetric matrix has an orthogonal set of eigenvector.

$$\mathbf{A} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$$
$$\lambda_1 = 3$$
$$\lambda_2 = -2$$
$$\mathbf{v}_1 = \begin{bmatrix} 0.9701 \\ -0.2425 \end{bmatrix}$$
$$\mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

Those two eigenvectors are not orthogonal.

Problem 2: The power method, and beyond!

1. Based on the definition of eigenvalue:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$
$$\mathbf{A}\mathbf{x}^{(i)} = \lambda^{(i)}\mathbf{x}^{(i)}$$

Then

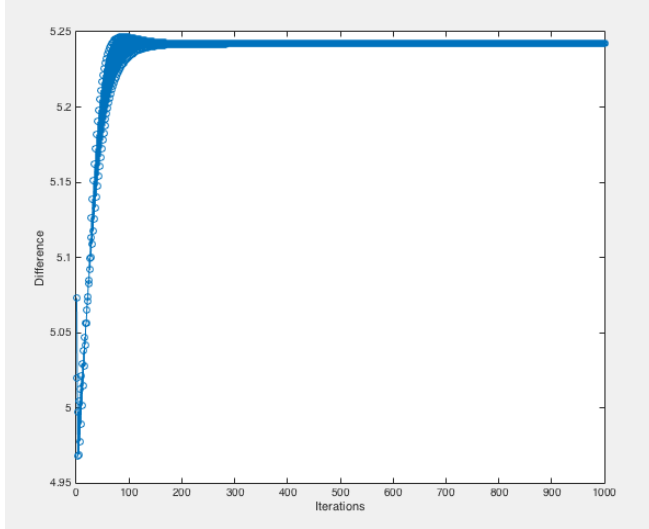
$$\begin{aligned}
\lambda &= \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \\
\lambda^{(i)} &= \frac{\mathbf{x}^{(i)T} \mathbf{A} \mathbf{x}^{(i)}}{\mathbf{x}^{(i)T} \mathbf{x}^{(i)}} \\
|\lambda - \lambda^{(i)}| &= \left| \frac{\mathbf{x}^T \mathbf{A} \mathbf{x}}{\mathbf{x}^T \mathbf{x}} - \frac{\mathbf{x}^{(i)T} \mathbf{A} \mathbf{x}^{(i)}}{\mathbf{x}^{(i)T} \mathbf{x}^{(i)}} \right| = |(\mathbf{x} - \mathbf{x}^{(i)})^T (\mathbf{x} - \mathbf{x}^{(i)})| + O(\epsilon^3) \\
&= \|\mathbf{x} - \mathbf{x}^{(i)}\|^2 + O(\epsilon^3) \\
&= |\epsilon^2 + O(\epsilon^3)| \\
&= O(\epsilon^2)
\end{aligned}$$

2.

$$\begin{aligned}
\mathbf{A} \mathbf{v} &= \lambda \mathbf{v} \\
\mathbf{A}^{-1} \mathbf{A} \mathbf{v} &= \lambda \mathbf{A}^{-1} \mathbf{v} \\
\mathbf{A}^{-1} \mathbf{v} &= \frac{1}{\lambda} \mathbf{v}
\end{aligned}$$

Therefore the eigenvalues are $\frac{1}{\lambda}$.

3. It doesn't converge. The 'difference vs iterations' plot shows it does not converge to 0.



4. If $\mathbf{v}^{(k)}$ is close to an eigenvector, then

$$\begin{aligned}
\|\mathbf{v}^{(k)} - (\pm \mathbf{q}_J)\| &\leq \epsilon \\
|\lambda^{(k)} - \lambda_J| &= O(\epsilon^2)
\end{aligned}$$

Then one step of inverse iteration then gives:

$$\|\mathbf{v}^{(k+1)} - \mathbf{q}_J\| = O(|\lambda^{(k)} - \lambda_J| \|\mathbf{v}^{(k)} - \mathbf{q}_J\|) = O(\epsilon^3)$$

5. The code is list below:

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maxit = 20;
dim = 6;

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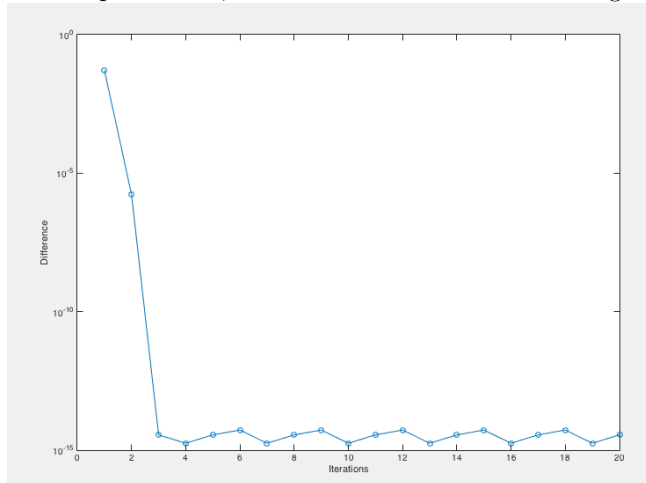
```

A = rand(dim);
A = A*A;
e = abs(eig(A));
lamda = max(e);
[V, D] = eig(A);
true_X = V(:,1);
true_X = true_X/norm(true_X);

x= rand(dim,1);
x= x/norm(x);
lam = x'*A*x;
diff = zeros(maxit,1);
iter = zeros(maxit,1);
eps = zeros(maxit,1);
for i=1:maxit
    x=(A-lam*eye(size(A,1)))\x;
    x = x/norm(x,2);
    lam = x'*A*x;
    diff(i) = norm(lam - lamda);
    iter(i) = i;
end
semilogy(iter, diff, 'o-') ;
xlabel('Iterations');
ylabel('Difference');

```

The result is plot below, and we can see the cubic convergence.



Problem 3: PageRank and the power method

1.

$$M = I - \alpha P$$

$$M^T = I - \alpha P^T$$

Because the P is a column-stochastic matrix, and then P^T is a row-stochastic

matrix.

$$\begin{aligned}
1 &= \sum |P_{ij}^T| \\
P_{ii} &= \sum_{i \neq j} |P_{ij}^T| \\
M_{ii} &= 1 - \alpha P_{ii} \\
&= 1 - \alpha \sum_{i \neq j} |P_{ij}^T|
\end{aligned}$$

Because $\alpha < 1$, then

$$|M_{ii}^T| > \sum_{i \neq j} |M_{ij}^T|$$

Then M_T is strictly diagonally dominant. We know that a strictly diagonally dominant matrix is nonsingular.

2. Because \mathbf{v} is a non-negative vector whose elements sum to 1.

$$\begin{aligned}
\mathbf{v} \mathbf{e}^T \mathbf{v} &= \begin{bmatrix} v_1(v_1 + v_2 + \dots) \\ v_2(v_1 + v_2 + \dots) \\ \dots \\ \dots \\ v_n(v_1 + v_2 + \dots) \end{bmatrix} \\
&= \mathbf{v} \\
M\mathbf{v} &= [\alpha P + (1 - \alpha)\mathbf{v} \mathbf{e}^T] \mathbf{v} \\
&= \alpha P\mathbf{v} + (1 - \alpha)\mathbf{v} \mathbf{e}^T \mathbf{v} \\
&= \alpha P\mathbf{v} + (1 - \alpha)\mathbf{v} \\
&= \alpha \begin{bmatrix} \sum P_{1j} v_j \\ \sum P_{2j} v_j \\ \dots \\ \dots \\ \sum P_{nj} v_j \end{bmatrix} + (1 - \alpha)\mathbf{v} \\
\mathbf{x}^{(1)} &= M\mathbf{v} \\
\|\mathbf{v}\|_1 &= \|M\mathbf{v}\|_1 \\
&= \alpha \sum_i v_i [\sum_j P_{ji}] + (1 - \alpha) \\
&= 1
\end{aligned}$$

For $\mathbf{x}^{(k+1)} = M\mathbf{x}^{(k)}$, we always have $\|\mathbf{x}^{(k+1)}\|_1 = 1$

3. Suppose \mathbf{x} is the dominant eigenvector of matrix M

$$\begin{aligned}
M\mathbf{x} &= \mathbf{x} = [\alpha P + (1 - \alpha)\mathbf{v} \mathbf{e}^T] \mathbf{x} \\
[I - (\alpha P + (1 - \alpha)\mathbf{v} \mathbf{e}^T)] \mathbf{x} &= 0
\end{aligned}$$

We know that $\mathbf{v} \mathbf{e}^T \mathbf{x} = \mathbf{x}$

Then

$$\begin{aligned}
(I - \alpha P^T) \mathbf{x} - (1 - \alpha)\mathbf{v} &= 0 \\
(I - \alpha P^T) \mathbf{x} &= (1 - \alpha)\mathbf{v}
\end{aligned}$$

4. The simplified iterations:

$$\begin{aligned}
\mathbf{x}^{(k+1)} &= \alpha \mathbf{P} \mathbf{x}^{(k)} + (1 - \alpha) \mathbf{v} \\
\mathbf{x} &= \alpha \mathbf{P} \mathbf{x} + (1 - \alpha) \mathbf{v} \\
\mathbf{x} - \mathbf{x}^{(k+1)} &= \alpha \mathbf{P} \mathbf{x} - \alpha \mathbf{P} \mathbf{x}^{(k)} \\
\|\mathbf{x} - \mathbf{x}^{(k+1)}\| &= \|\alpha \mathbf{P} (\mathbf{x} - \mathbf{x}^{(k)})\| \\
&= \|\alpha \mathbf{P}\| \|\mathbf{x} - \mathbf{x}^{(k)}\| \\
&\leq \|\alpha \mathbf{P}\| \|\alpha \mathbf{P}\| \dots \|\mathbf{x} - \mathbf{x}^{(k-1)}\| \\
&\leq \|\alpha \mathbf{P}\|^{k+1} \|\mathbf{x} - \mathbf{x}^{(0)}\|
\end{aligned}$$

Therefore, the power method will converge to the solution of the linear system $(\mathbf{I} - \alpha \mathbf{P}^T) \mathbf{x} = (1 - \alpha) \mathbf{v}$.

5. The first url shown is : 'http://aae.www.ecn.purdue.edu/'

6. $\mathbf{x}(1)$ is $\mathbf{v}(1) = [\frac{1}{n}, \frac{1}{n}, \frac{1}{n} \dots]$

Then the top 27 entries are :

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'http://aae.www.ecn.purdue.edu/'
'http://aae.www.ecn.purdue.edu/AAE/'
'http://aae.www.ecn.purdue.edu/AAE/Academic/'
'http://aae.www.ecn.purdue.edu/AAE/Academic/Index.html'
'http://aae.www.ecn.purdue.edu/AAE/Academic/New_Index.html'
'http://aae.www.ecn.purdue.edu/AAE/Alumni/'
'http://aae.www.ecn.purdue.edu/AAE/Alumni/97DEAPic.html'
'http://aae.www.ecn.purdue.edu/AAE/Alumni/AlumniEvents.html'
'http://aae.www.ecn.purdue.edu/AAE/Alumni/DEA.html'
'http://aae.www.ecn.purdue.edu/AAE/Alumni/HonDoc.html'
'http://aae.www.ecn.purdue.edu/AAE/Alumni/IndAdvisory.html'
'http://aae.www.ecn.purdue.edu/AAE/Alumni/Index.html'
'http://aae.www.ecn.purdue.edu/AAE/Alumni/New_Index.html'
'http://aae.www.ecn.purdue.edu/AAE/Alumni/YourGift.html'
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'http://aae.www.ecn.purdue.edu/AAE/Alumni/outstandingeng.html'
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'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Covey.html'
'http://aae.www.ecn.purdue.edu/AAE/Astronaut/Gardner.html'

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