

Problem 0: Homework checklist

- ✓I didn't talk with any one about this homework.
- ✓Source-code are included at the end of this document.

Problem 1: Prove or disprove

For the proof below, I will assume all the matrices are $n \times n$ square matrices.

1. The product of two diagonal matrices is diagonal.

\mathbf{A} and \mathbf{B} are diagonal matrices, so

$$\mathbf{A}_{ij} = 0, \text{ if } i \neq j$$

$$\mathbf{B}_{ij} = 0, \text{ if } i \neq j$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C}_{ij} = \sum_{k=1}^n \mathbf{A}_{ik} \mathbf{B}_{kj}$$

Only if $i = j = k$, \mathbf{C}_{ij} will not be zero, which mean \mathbf{C} is also a diagonal matrix.

2. The product of two upper triangular matrices is upper triangular \mathbf{A} and \mathbf{B} are two upper triangular matrices, so

$$\mathbf{A}_{ij} = 0, \text{ if } i > j$$

$$\mathbf{B}_{ij} = 0, \text{ if } i > j$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C}_{ij} = \sum_{k=1}^n \mathbf{A}_{ik} \mathbf{B}_{kj}$$

When $i > j$, $\mathbf{C}_{ij} = 0$ because one of \mathbf{A}_{ik} and \mathbf{B}_{kj} must be zero.

3. The product of two symmetric matrices is symmetric. \mathbf{A} and \mathbf{B} are two symmetric matrices, so

$$\mathbf{A}_{ij} = \mathbf{A}_{ji}$$

$$\mathbf{B}_{ij} = \mathbf{B}_{ji}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C}_{ij} = \sum_{k=1}^n \mathbf{A}_{ik} \mathbf{B}_{kj}$$

Then \mathbf{C}_{ij} is not necessary to be equal to \mathbf{C}_{ji}

Counter-example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ is a symmetric matrix;}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ is also a symmetric matrix.}$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \text{ which is not a symmetric matrices.}$$

4. The product of two orthogonal matrices is orthogonal.
5. The product of two square, full rank matrices is full rank This statement is incorrect.

Counter-example:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ is a full rank matrix;}$$

$$\mathbf{B} = \begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix} \text{ is also a full rank matrix.}$$

$$C = A \times B = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix} \text{ which is not a full matrices.}$$

Problem 2

We know U and V are square orthogonal matrices, then

$$U^T U = I, V^T V = I \quad (1)$$

$$\begin{aligned} \|UAV\|_2^2 &= V^T A^T U^T U A V \\ &= V^T A^T A V \\ &= V^T \|A\|_2^2 V \\ &= V^T V \|A\|_2^2 \\ &= \|A\|_2^2 \end{aligned}$$

Therefore $\|A\|_2$ is orthogonally invariant.

Problem 3

$$f(A) = \max_{i,j} |A_{ij}| \quad (2)$$

1.
 - $f(A) > 0$, when $A \neq 0$,
 $f(A) = 0$ iff $A = 0$
 - For and scalar k ,
 $f(kA)$
 $= \max_{i,j} |kA_{ij}|$
 $= k \max_{i,j} |A_{ij}|$
 $= kf(A)$
 - $f(A+B)$
 $= \max_{i,j} |A_{ij} + B_{ij}|$
 $\leq \max_{i,j} |A_{ij}| + \max_{i,j} |B_{ij}|$
 $= f(A) + f(B)$

Therefore $f(A)$ is a matrix norm.

2. $f(AB)$
 $= \max_{i,j} |A_{ij} B_{ij}|$
 $\leq \max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}|$
 $= f(A)f(B)$

The \leq is not necessary to be. A counter example:

$$A = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 18 & 18 \\ 18 & 18 \end{bmatrix}$$

$$f(A) = 3$$

$$f(B) = 3$$

$$f(AB) = 18$$

$$\text{Then } f(AB) > f(A)f(B)$$

So f does not satisfy the sub-multiplicative property.

- 3.

$$g(A) = \sigma f(A) \quad (3)$$

$$g(\mathbf{AB}) = \sigma \max_{i,j} |A_{ij}B_{ij}|$$

$$g(\mathbf{A})g(\mathbf{B}) = \sigma^2 \max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}|$$

To make

$$g(\mathbf{AB}) \leq g(\mathbf{A})g(\mathbf{B}) \quad (4)$$

$$\sigma \max_{i,j} |A_{ij}B_{ij}| \leq \sigma^2 \max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}| \quad (5)$$

$$\sigma \geq \frac{\max_{i,j} |A_{ij}B_{ij}|}{\max_{i,j} |A_{ij}| \max_{i,j} |B_{ij}|} \quad (6)$$

Therefore there exists $\sigma > 0$ such that: $g(\mathbf{A}) = \sigma f(\mathbf{A})$

Problem 4

$$f(x) = \|x\mathbf{k}^T\| \quad (7)$$

1.
 - $f(x) > 0$, when $\mathbf{A} \neq 0$,
 $f(x) = 0$ iff $\mathbf{A} = 0$ because $\|\mathbf{A}\|$ is a matrix norm.
 - $f(mx) = \|mx\mathbf{k}^T\| = m\|x\mathbf{k}^T\| = mf(x)$
 - $f(x+y) = \|(x+y)\mathbf{k}^T\| = \|x\mathbf{k}^T + y\mathbf{k}^T\| \leq \|x\mathbf{k}^T\| + \|y\mathbf{k}^T\| = f(x) + f(y)$

Therefore f is a vector norm.

2. If $\|\mathbf{A}\|$ is a sub-multiplicative matrix norm,

$$f(\mathbf{Ax}) = \|\mathbf{Ax}\mathbf{k}^T\| \leq \|\mathbf{A}\| \|x\mathbf{k}^T\| = \|\mathbf{A}\| f(x) \quad (8)$$

Problem 5 (Choice 1)

$f(\mathbf{x})$ = sum of two largest entries in \mathbf{x} by absolute value is a vector form.
 Suppose

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

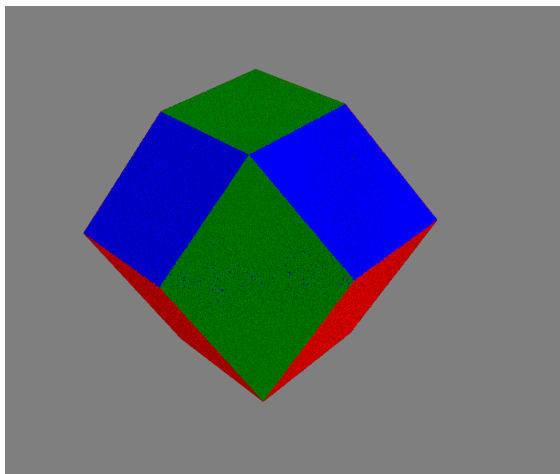
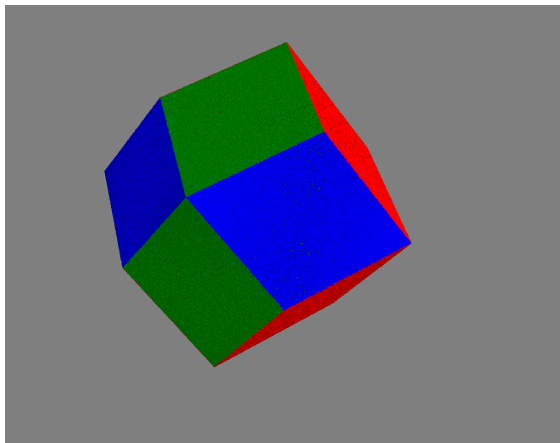
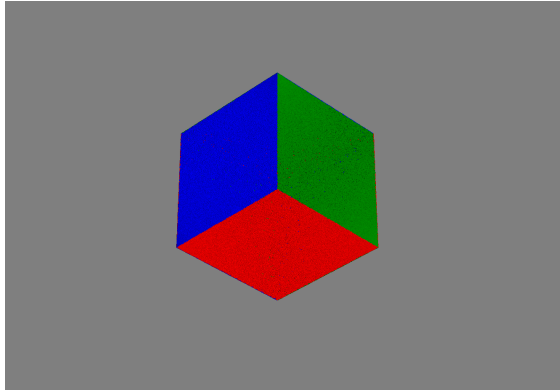
Then the unit-ball should include all points which satisfy conditions:

$$|x_1| + |x_2| = 1, |x_3| \leq |x_1|, |x_3| \leq |x_2|$$

$$|x_2| + |x_3| = 1, |x_1| \leq |x_2|, |x_3| \leq |x_3|$$

$$|x_3| + |x_1| = 1, |x_2| \leq |x_3|, |x_3| \leq |x_1|$$

I used Monte Carlo method to generate $N = 300000$ points which satisfy those conditions, as shown in Figure 1.



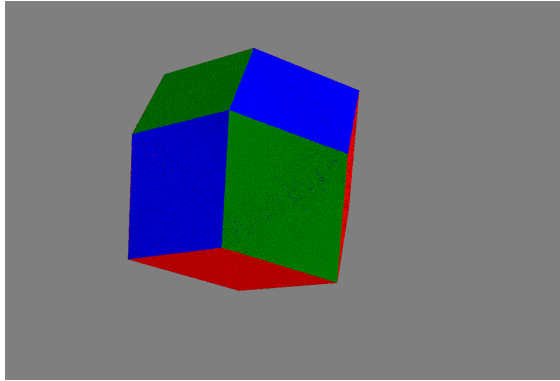


Figure 1: The unit-ball surface from different views. Color images are in electronic version.