UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 7
Fall 2023

1. **Sum of Rolls**

   You roll a fair 6-sided die 100 times, and you call the sum of the values of all your rolls X. Use the Central Limit Theorem to approximate the probability that $X > 400$. You may use a calculator and Gaussian lookup table.

2. **Entropy of a Sum**

   Let X_1, X_2 be i.i.d. Bernoulli($\frac{1}{2}$). Calculate $H(X_1 + X_2)$ and show that $H(X_1 + X_2) \geq H(X_1)$. Does this make intuitive sense?

3. **Mutual Information and Channel Coding**

The *mutual information* of X and Y is defined as

$$I(X;Y) := H(X) - H(X \mid Y),$$

where $H(X \mid Y)$ is the *conditional entropy* of X given Y,

$$\begin{aligned} H(X \mid Y) &= \sum_{y \in \mathcal{Y}} p_Y(y) \cdot H(X \mid Y = y) \\ &= \sum_{y \in \mathcal{Y}} p_Y(y) \sum_{x \in \mathcal{X}} p_{X|Y}(x \mid y) \log_2 \frac{1}{p_{X|Y}(x \mid y)}. \end{aligned}$$

Conditional entropy can be interpreted as the average amount of uncertainty remaining in the random variable X after observing Y. Then, mutual information is the amount of information about X gained by observing Y.

Now, the channel coding theorem says that the capacity of a channel with input X and output Y is the maximal possible amount of mutual information between them:

$$C = \max_{p_X} I(X;Y) = \max_{p_X} H(X) - H(X \mid Y).$$

a. Let X be the roll of a fair die and $Y = \mathbb{1}_{X \geq 5}$. What is $H(X \mid Y)$?

b. Suppose the channel is a noiseless binary channel, i.e. $X \in \{0, 1\}$ and $Y = X$. Use the theorem above to find its capacity C.