UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 13
Fall 2023

1. **Orthogonal LLSE**

   a. Consider zero-mean random variables X, Y, Z with Y, Z orthogonal. Show that

   $$\mathbb{L}(X \mid Y, Z) = \mathbb{L}(X \mid Y) + \mathbb{L}(X \mid Z).$$

   b. Now, for *any* zero-mean random variables X, Y, Z, explain why it holds that

   $$\mathbb{L}(X \mid Y, Z) = \mathbb{L}(X \mid Y) + \mathbb{L}[X \mid (Z - \mathbb{L}(Z \mid Y))].$$

2. **Hypothesis Testing for Bernoulli Random Variables**

   Suppose that

   - The null hypothesis is $X = 0$: $Y \sim \text{Bernoulli}(\frac{1}{4})$, and
   - The alternative hypothesis is $X = 1$: $Y \sim \text{Bernoulli}(\frac{3}{4})$.

   Using the Neyman–Pearson formulation of hypothesis testing, find the optimal randomized decision rule $\hat{X}$ with respect to the criterion

   $$\begin{aligned} \min \quad & \mathbb{P}(\hat{X} = 0 \mid X = 1) \\ \text{s.t.} \quad & \mathbb{P}(\hat{X} = 1 \mid X = 0) \leq \beta, \end{aligned}$$

   where $\beta \in [0, 1]$ is a given upper bound on the probability of false alarm (PFA).

   (Note that the Neyman–Pearson decision rule may change depending on the value of β. In particular, consider the two separate cases of $\beta \leq \frac{1}{4}$ and $\beta > \frac{1}{4}$.)

3. **Gaussian LLSE**

   Let X, Y, Z be i.i.d. $\mathcal{N}(0, 1)$.

   a. Find $\mathbb{L}(X^2 + Y^2 \mid X + Y)$.
   b. Find $\mathbb{L}(X + 2Y \mid X + 3Y + 4Z)$.
   c. Find $\mathbb{L}((X + Y)^2 \mid X - Y)$.