UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Homework 12
Fall 2023

1. **Balls in Bins Estimation**

   You throw n balls into m bins, where $n \geq 1$ and $m \geq 2$. Each ball lands in each bin with the same probability, independently of all other events. Let X and Y be the number of balls in bin 1 and 2 respectively.

   a. What is $\mathbb{E}(Y \mid X)$?

   b. Define $\mathbb{Q}(Y \mid X)$ to be the best quadratic function in X that minimizes mean squared error when used to estimate Y. Without doing any mathematical work, what are $\mathbb{L}(Y \mid X)$ and $\mathbb{Q}(Y \mid X)$? Justify your answer.

   c. Your friend from UCLA who hasn't learned about the Hilbert space of random variables isn't convinced by your explanation. Use the formula

   $$\mathbb{L}(Y \mid X) = \mathbb{E}(Y) + \frac{\operatorname{cov}(X, Y)}{\operatorname{var}(X)}(X - \mathbb{E}(X))$$

   to calculate the LLSE and verify your claim.

2. **Gaussian Random Vector MMSE**

Consider the Gaussian random vector

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right),$$

and define the sign of Y to be the random variable

$$W = \begin{cases} 1 & \text{if } Y > 0 \\ 0 & \text{if } Y = 0 \\ -1 & \text{if } Y < 0 \end{cases}.$$

a. Find $\mathbb{E}(WX \mid Y)$.

b. Is the LLSE $\mathbb{L}(WX \mid Y)$ the same as the MMSE you found in part a?

c. Are WX and Y jointly Gaussian?

3. **Geometric MMSE**

   Let N be a geometric random variable with parameter $1-p$, and $(X_i)_{i\in\mathbb{N}}$ be i.i.d. exponential random variables with parameter λ. Let $T = X_1 + \cdots + X_N$. Compute the LLSE and MMSE of N given T.

   Hint: Compute the MMSE first.

4. **Exam Difficulty**

   The difficulty of an EECS 126 exam, Θ, is uniformly distributed on $[0, 100]$ (continuously). Alice gets a score X that is uniformly distributed on $[0, \Theta]$, and she wants to estimate the difficulty of the exam given her score.

   a. What is the MLE of Θ? What is the MAP of Θ?

   b. What is the LLSE for Θ?

5. **Even-Times Kalman Filter**

Consider a random process $(X_n)_{n\in\mathbb{N}}$ with state space model

$$\begin{aligned} X_{n+1} &= aX_n + V_n, \quad V_n \sim_{\text{i.i.d.}} \mathcal{N}(0, \sigma_V^2) \\ Y_n &= X_n + W_n, \quad W_n \sim_{\text{i.i.d.}} \mathcal{N}(0, \sigma_W^2) \end{aligned}$$

where $(V_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$ are independent. We can only observe the process at even times, i.e. we observe the random variables $Y_0, Y_2, Y_4, \ldots$.

a. Derive a recurrence relation for the estimator $\hat{X}_{2n|2n} := \mathbb{L}(X_{2n} \mid Y_0, Y_2, \ldots, Y_{2n})$ in terms of $\hat{X}_{2n-2|2n-2}$.

b. Derive a recurrence relation for $\hat{X}_{2n+1|2n}$ in terms of $\hat{X}_{2n|2n}$.

6. **Kalman Filter with Correlated Noise**

Consider the state space model

$$\begin{aligned} X_n &= aX_{n-1} + V_n \\ Y_n &= X_n + V_n, \end{aligned}$$

with $X_0 = 0$ and $(V_n)_{n\geq 0} \sim_{\text{i.i.d.}} \mathcal{N}(0,1)$. Note that the observation noise is the same as the process noise V_n, not independent of it, so this is different from the usual Kalman filter model. Derive recursive update equations for $\hat{X}_{n|n} := \mathbb{L}(X_n \mid Y_0, \ldots, Y_n)$.

Hint: You may use the fact that the equations will be of the form

$$\begin{aligned} \hat{X}_{n|n} &= a\hat{X}_{n-1|n-1} + K_n\tilde{Y}_n \\ \tilde{Y}_n &= Y_n - a\hat{X}_{n-1|n-1}, \end{aligned}$$

where you should find the Kalman gain and the estimator covariance recurrence relation

$$\begin{aligned} K_n &= ? \\ \sigma^2_{n|n-1} &= a^2\sigma^2_{n-1|n-1} + 1 \\ \sigma^2_{n|n} &= ?(\sigma^2_{n|n-1}). \end{aligned}$$