UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Homework 05
Fall 2023

1. **Convergence in Probability**

   Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables distributed uniformly in $[-1,1]$. Show that the following sequences $(Y_n)_{n\in\mathbb{N}}$ converge in probability to some limit.

   a. $Y_n = \prod_{i=1}^n X_i$.
   b. $Y_n = \max\{X_1, \ldots, X_n\}$.
   c. $Y_n = (X_1^2 + \cdots + X_n^2)/n$.

2. **Bernoulli Convergence**

Consider an independent sequence of random variables $X_n \sim \text{Bernoulli}(\frac{1}{n})$.

a. Show that X_n converges to 0 in probability.

b. Argue that

$$\mathbb{P}\left(\left\{\lim_{n\to\infty} X_n = 0\right\}\right) = \mathbb{P}\left(\bigcup_{N=1}^{\infty} \{X_n = 0 \text{ for all } n \geq N\}\right).$$

c. Using part b, show that X_n does **not** converge almost surely to 0.
Hint: Consider applying the union bound and the independence of the X_n.

3. **Mean Square Convergence**

   A sequnce of random variables $\{X_n\}_{n\geq 0}$, each satisfying $\mathbb{E}[X_n^2] < \infty$, is said to converge in *mean square* to a random variable X if

   $$\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0.$$

   a. Show that convergence in mean square implies convergence in probability.
   b. Consider the sequence of random variables $\{X_n\}_{n\geq 1}$, where each $X_n \sim \text{Bernoulli}(1/n)$. Show that this sequence converges to 0 in mean square.
   c. Does it converge almost surely?