EECS 126 Probability and Random Processes — UC Berkeley, Fall 2023
Jiantao Jiao — September 26, 2023

Midterm 1

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

Rules.

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- **All work you want graded should be on the fronts of the sheets in the space provided. Back sides may be used for scratch work, but will not be scanned/graded.**
- You have 10 minutes to read the exam and 70 minutes to complete the exam. (DSP students with $X\,\%$ time accommodation should spend $10 \cdot X\,\%$ time on reading and $70 \cdot X\,\%$ time on completing the exam).
- This exam is closed-book. You may reference one double-sided handwritten sheet of paper. No calculator or phones allowed.
- Collaboration with others is strictly prohibited. If you are caught cheating, you may fail the course and face disciplinary consequences.

Problem	out of
SID	1
Problem 1	66
Problem 2	19
Problem 3	15
Problem 4	20
Problem 5	5
Total	126

1 Medley of 126 Basics [66 points]

Important instructions (Please read before solving Problem 1):
Problem 1 consists of six subparts. However, you only need to choose **four** of them to solve. For the other two subparts you do not choose, we will automatically give you **full marks** of those two subparts.
To receive scores of subparts you choose not to solve, please write down "FULL MARK" under that subpart **without any other contents**. For the subparts you choose to solve, please write down the solution as usual. You may choose at most two subparts and write down "FULL MARK". If you choose more than two subparts and write down "FULL MARK", you will receive 0 point for the whole Problem 1.
To summarize, the grading policy of Problem 1 is that for each subpart:

- As long as we see "FULL MARK" (even if you also write something else), we will give you the full mark of that subpart and won't look at your solution;
- otherwise, we will grade the subpart as usual;
- However, if you write down "FULL MARK" for more than two subparts, you will get 0 for the whole Problem 1.

1.1 Oski needs Help [11 points]

Oski is rolling a 4-sided dice and wants to compute the probability of some event. In order to do so, they first want to define the probability space $(\Omega, \mathcal{F}, P)$. Naturally, Oski decides $\Omega = \{1, 2, 3, 4\}$ corresponding to the outcome of the roll of the die. Oski considers $\mathcal{F} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}, \{1\}, \{2, 3, 4\}\}$. Is this a valid sigma-algebra? Explain your answer in brief.

No, it is not a valid field/sigma-algebra.
If we define $A_1 = \{3, 4\}$ and $A_2 = \{1\}$, then $A_1 \cup A_2 \notin \mathcal{F}$.

1.2 Battle of the Bears [11 points]

Oski and Winnie-the-Pooh are playing a game. They toss a biased coin n times and the probability of observing heads is p each time. Oski wins the game if the total number of heads X is in the interval $(np-\sqrt{n}, np+\sqrt{n})$. Using Chebyshev inequality, lower bound Oski's probability of winning in this game in terms of p.

Note that $X \sim Bin(n,p)$.
Calculation (or just writing down) of variance
Using Chebyshev, $P(|X-np| < \sqrt{n}) = 1 - P(|X-np| \geq \sqrt{n}) \geq 1 - p(1-p)$

1.3 Conditional Independence [11 points]

On some probability space $(\Omega, \mathcal{F}, P)$, consider three events $A, B, C \in \mathcal{F}$ such that $P(B) > 0$. Paddington Bear notices that $P(C|A, B) = P(C|B)$ and concludes that $P(AC|B) = P(C|B)P(A|B)$. Oski is confused how Paddington Bear conlcudes this, prove it to Oski that $P(C|A, B) = P(C|B)$ implies $P(AC|B) = P(C|B)P(A|B)$.

$$\begin{aligned}
P(A, C|B) =& \frac{P(A, B, C)}{P(B)} \\
=& \frac{P(C|A, B)P(A, B)}{P(B)} \\
=& P(C|B)\frac{P(A, B)}{P(B)} \\
=& P(C|B)P(A|B).
\end{aligned}$$

1.4 Honey Hunting [11 points]

Winnie-the-Pooh and Oski are competing for honey! When Oski finds honey in Berkeley Hills, there is less honey left for Winnie. The probability that Winnie finds honey given that Oski has found honey is 0.2. The probability that Winnie finds honey given that Oski has NOT found honey is 0.7. Oski can find honey with probability 0.4. Paddington Bear sees that Winnie has successfully found honey today and wants to calculate the probability that Oski also found honey today. Find the probability that Oski found honey today given that Winnie also found honey today.

Let O be the event Oski finds honey and let W be the event Winnie finds honey.

$$\begin{aligned}
P(O|W) =& \frac{P(W|O)P(O)}{P(W)} \\
=& \frac{P(W|O)P(O)}{P(W|O)P(O) + P(W|O^c)P(O^c)} \\
=& \frac{0.2 \times 0.4}{0.2 \times 0.4 + 0.7 \times 0.6} \\
=& 0.16
\end{aligned}$$

1.5 Intercepted [11 points]

Grizzly and Panda from We Bare Bears are trying to communicate but Ice Bear is attempting to disrupt their communication. Grizzly sends a random variable $X \sim Exp(\lambda)$. Ice Bear deletes numbers after the decimal point in X and let this random variable be called Y. Ice Bear then sends Y to Panda so Panda observes Y. Find the distribution of Y.

For example, if the realization of X is 92.3408 then Y is 92. Note that this operation is known as the floor function or the greatest integer function and one can represent $Y = \lfloor X \rfloor$. Some more examples: $\lfloor 1.34 \rfloor = 1$, $\lfloor 4 \rfloor = 4$, $\lfloor 8.9999 \rfloor = 8$.

Hint: Y is a discrete random variable so specifying the pmf of Y is sufficient.

For $t \in \mathbb{N}_0$,

$$\begin{aligned} P(Y = t) =& P(X \in [t, t+1)) \\ =& \int_t^{t+1} \lambda e^{-\lambda s} ds \\ =& \left(e^{-\lambda}\right)^t (1 - e^{-\lambda}). \end{aligned}$$

This is a shifted Geo$(1 - e^{-\lambda})$ distribution.

1.6 Min and Max [11 points]

Oski and Joe Bruins are playing a game. Joe Bruins draws a random variable X_J distributed as Uniform$[0,1]$. Oski, trying to outsmart Joe, first looks at Joe's random variable X_J and then draws a random variable X_O distributed Uniformly in $[X_J - 0.5, X_J + 0.5]$. The bear with the higher score wins. Kevin, the bear from Zootopia, looks at the point obtained by the winner, given by $X_W = \max\{X_O, X_J\}$ and the points obtained by the loser $X_L = \min\{X_O, X_J\}$. Find $\mathbb{E}[X_W]$ and $\mathbb{E}[X_L]$.

Note that $X_O = X_J + U$ where U is Uniform$[-0.5, 0.5]$ and independent of X_J.

$$\begin{aligned}
\mathbb{E}[X_W] &= \mathbb{E}[X_W|X_J < X_O]P(X_J < X_O) + \mathbb{E}[X_W|X_J \geq X_O]P(X_J \geq X_O) \\
&= \mathbb{E}[X_J + U|U > 0]P(U > 0) + \mathbb{E}[X_J|U \leq 0]P(U \leq 0) \\
&= \left(\mathbb{E}[X_J|U > 0] + \mathbb{E}[U|U > 0]\right) P(U > 0) + \mathbb{E}[X_J|U \leq 0]P(U \leq 0) \\
&= \left(\mathbb{E}[X_J] + \mathbb{E}[U|U > 0]\right) P(U > 0) + \mathbb{E}[X_J]P(U \leq 0) \\
&= \left(\frac{1}{2} + \frac{1}{4}\right)\frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \\
&= \frac{5}{8}.
\end{aligned}$$

Now, $\mathbb{E}[X_W] + \mathbb{E}[X_L] = \mathbb{E}[X_O + X_J] = 2\mathbb{E}[X_J] = 1$. Therefore, $\mathbb{E}[X_L] = \frac{3}{8}$.

2 Joint Gaussian random variables [19 points]

Consider two Gaussian random variables, η and ξ, both with mean $\mathbb{E}[\xi] = \mathbb{E}[\eta] = 0$ and variance $\mathrm{Var}(\xi) = \mathrm{Var}(\eta) = 1$. Additionally, their covariance is given by $\mathrm{Cov}(\eta, \xi) = \rho$.

(a) Are the random variables $\eta - \xi$ and $\eta + \xi$ independent? Provide a proof or counterexample to support your claim. (8 points)

(b) Show that $\mathbb{E}[\max(\xi, \eta)] = \sqrt{\frac{1-\rho}{\pi}}$. (11 points)

Hint: 1. What is the relationship between $\mathbb{E}[|\xi - \eta|]$ *and* $\mathbb{E}[\max(\xi, \eta)]$*? 2. You can use this indefinite integral:* $\int xe^{-\alpha x^2} dx = -\frac{e^{-\alpha x^2}}{2\alpha} + c$.

(a) They are independent.

By property of Gaussian, $\begin{pmatrix} \eta - \xi \\ \eta + \xi \end{pmatrix}$ is still Gaussian and its covariance matrix is given by

$$\begin{aligned} &\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^\top \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ = &\begin{pmatrix} 2(1+\rho) & 0 \\ 0 & 2(1-\rho) \end{pmatrix}. \end{aligned}$$

It follows from property of Gaussian that $\eta - \xi$ and $\eta + \xi$ are independent.

(b) Notice that $\max(\xi, \eta) = \frac{\xi + \eta + |\xi - \eta|}{2}$. It follows that

$$\mathbb{E}[\max(\xi, \eta)] = \mathbb{E}[(\xi + \eta + |\xi - \eta|)/2] = \mathbb{E}[|\xi - \eta|]/2.$$

For $\mathbb{E}[|\xi - \eta|]$, we calculate the integral

$$\begin{aligned} \mathbb{E}[|\xi - \eta|] &= \frac{1}{\sqrt{2\pi}\sqrt{2(1-\rho)}} \int_{-\infty}^{+\infty} |x| e^{-\frac{x^2}{4(1-\rho)}} dx \\ &= \frac{2}{\sqrt{2\pi}\sqrt{2(1-\rho)}} \int_0^{+\infty} x e^{-\frac{x^2}{4(1-\rho)}} dx \\ &= -\frac{1}{\sqrt{\pi(1-\rho)}} \cdot \frac{4(1-\rho)}{2} \cdot e^{-\frac{x^2}{4(1-\rho)}} \Big|_0^\infty \\ &= 2\sqrt{\frac{1-\rho}{\pi}}. \end{aligned}$$

3 Self-grading [15 points]

In EECS126, there are n students who have submitted their homework solutions along with their corresponding self-grades. In the spirit of probability theory, each self-grade is assigned to a homework submission uniformly at random (without replacement).
In this problem, we assume that every homework submission and self-grade is distinct from each others.

(a) Alice and Bob are two of the students. What is the probability that Alice's homework submission is correctly matched with her own self-grade? What is the probability that their homework submissions are both correctly matched with their corresponding self-grades? (5 points)

(b) Let μ represent the number of correct matches between the homework submission and its corresponding self-grade. Find the expectation and variance of μ. (9 points)

(c) Now Alice notices that her homework submission is correctly matched with her self-grade. Answer 'Yes' or 'No' for the following question.

- Does the expectation of μ change given this new observation? (1 point)

(a) Define A as the event that Alice gets her correct self-grade and B as the event that Bob gets his correct self-grade. Obviously $\Pr[A] = 1/n$.

$$\begin{aligned}&\Pr[\text{Alice and Bob both correctly get their corresponding self-grades}]\\ =\ &\Pr[A] \cdot \Pr[B|A]\\ =\ &\frac{1}{n(n-1)}.\end{aligned}$$

(b) Define the following random variable

$$\xi_i = \mathbb{1}(\text{the i-th student gets their correct self-grade}).$$

Then $\mu = \sum_{i=1}^n \xi_i$. By (a), $\mathbb{E}[\xi_i] = \frac{1}{n}$ and $\mathbb{E}[\xi_i \xi_j] = \frac{1}{n(n-1)}$. Furthermore, $\mathrm{Var}(\xi_i) = \frac{1}{n} - \frac{1}{n^2}$. It follows by linearity that

$$\begin{aligned}\mathbb{E}[\mu] &= \sum_{i=1}^n \mathbb{E}[\xi_i] = 1.\\ \mathrm{Var}(\mu) &= \sum_{i=1}^n \mathrm{Var}(\xi_i) + 2\sum_{i<j} \mathrm{Cov}(\xi_i, \xi_j)\\ &= n \cdot \left(\frac{1}{n} - \frac{1}{n^2}\right) + 2 \cdot \binom{n}{2} \cdot \left(\frac{1}{n(n-1)} - \frac{1}{n^2}\right)\\ &= 1.\end{aligned}$$

(c) Yes.

4 Bear Transit [20 points]

An empty Bear Transit bus embarks on its journey from the parking lot at time $t = 0$. For every $i \in \mathbb{Z}_+$, the bus arrives at the i-th stop at time $t = i$. At each stop, the number of passengers boarding the bus follows a Poisson distribution with mean λ. Furthermore, for each passenger boarding at the i-th stop, they will disembark at the j-th stop with probability $p_{i,j}$, independently from each other.

For all the following questions, your answer should be expressions in terms of $\lambda, p_{i,j} (i < j \in \mathbb{Z}_+)$.

(a) What is the probability distribution of the time when the bus encounters its first passenger? (6 points)

(b) Conditioned on that there are a total of k passengers who board the bus at either of the first two stops, what is the probability distribution of the number of passengers who board the bus at the first stop? (7 points)

(c) What is the probability distribution of the number of passengers who disembark from the bus at the j-th stop? (7 points)

Hint: Does the number of people who board at the i-th stop and disembark at the j-th stop follow Poisson distribution?

(a) Define the random variable

$$T = \text{the time when the bus encounters its first passenger.}$$

Then for any $t \in \mathbb{N}$,

$$\begin{aligned}\Pr[T > t] &= \Pr[\text{no passenger board at the 1, 2, ..., t-th stop}] \\ &= \prod_{i=1}^{t} \Pr[\text{no passenger board at the i-th stop}] \\ &= \prod_{i=1}^{t} e^{-\lambda} = e^{-\lambda t}.\end{aligned}$$

Thus $T \sim \text{Geometric}(1 - e^{-\lambda})$.

(b) Define the random variables

$$\begin{aligned}N &= \text{the number of passengers who board the bus at the first stop,} \\ M &= \text{the number of passengers who board the bus at the second stop.}\end{aligned}$$

Then N and M are independent Poisson random variables with parameter λ . Then by property of Poisson distribution $N + M \sim \text{Poisson}(2\lambda)$. Notice that

$$\begin{aligned}
\Pr[N = n | N + M = k] &= \frac{\Pr[N = n, M = k - n]}{\Pr[N + M = k]} \\
&= \frac{\frac{\lambda^n e^{-\lambda}}{n!} \cdot \frac{\lambda^{k-n} e^{-\lambda}}{(k-n)!}}{\frac{(2\lambda)^k e^{-2\lambda}}{k!}} \\
&= \binom{k}{n} 2^{-k}.
\end{aligned}$$

This means $N | M + N \sim \text{Binom}(k, 1/2)$.

(c) Define the following random variables

$$\begin{aligned}
X_j &= \text{the number of passengers who disembark from the bus at the } j\text{-th stop,} \\
X_{i,j} &= \text{the number of people who board at the } i\text{-th stop and disembark at the } j\text{-th stop} \\
Y_i &= \text{the number of people who board at the } i\text{-th stop.}
\end{aligned}$$

Then from the following calculation (abbreviate $p = p_{i,j}$)

$$\begin{aligned}
\Pr[X_{i,j} = k] &= \sum_{n \geq k} \Pr[Y_i = n] \cdot \Pr[\text{k out of n people disembark}] \\
&= \sum_{n \geq k} \frac{\lambda^n e^{-\lambda}}{n!} \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= \frac{(\lambda p)^k e^{-\lambda}}{k!} \cdot \sum_{n \geq k} \frac{(1-p)^{n-k} \lambda^{n-k}}{(n-k)!} \\
&= \frac{(\lambda p)^k e^{-\lambda}}{k!} \cdot e^{\lambda(1-p)} \\
&= \frac{(\lambda p)^k e^{-\lambda p}}{k!}
\end{aligned}$$

we know that $X_{i,j} \sim \text{Poisson}(\lambda p_{i,j})$. It follows that

$$X_j = \sum_{i=1}^{j-1} X_{i,j} \sim \text{Poisson}(\lambda \sum_{i=1}^{j-1} p_{i,j})$$

5 Distinct sums [5 points]

In this question, we will prove the following result step by step.

> If S is a k-element subset of $\{1, 2, \ldots, n\}$ such that all 2^k subset sums of S are distinct, then $n \geq \frac{2^k}{126\sqrt{k}}$.

(a) Let $S = \{x_1, \ldots, x_k\}$ be a k-element subset of $\{1, 2, \ldots, n\}$ with distinct subset sums. Define a random variable $X = \sum_{i=1}^{k} \epsilon_i x_i$ where ϵ_i are chosen uniformly at random from $\{0, 1\}$, independently. Use concentration inequalities to find a lower bound of $\Pr\left[\left|X - \frac{\sum_{i=1}^{k} x_i}{2}\right| < n\sqrt{k}\right]$. (3 points)

(b) Use union bound to find an upper bound of $\Pr\left[\left|X - \frac{\sum_{i=1}^{k} x_i}{2}\right| < n\sqrt{k}\right]$. (1 point)

(c) Prove the result by combining what you derived in (a) and (b). (1 point)

(a) We have

$$\mathbb{E}[X] = \frac{\sum_{i=1}^{k} x_i}{2}$$
$$\mathrm{Var}[X] = \frac{\sum_{i=1}^{k} x_i^2}{4} \leq n^2 k/4.$$

By Chebyshev's inequality, $\Pr\left[\left|X - \frac{\sum_{i=1}^{k} x_i}{2}\right| < n\sqrt{k}\right] \geq 3/4$.

(b) Notice $\Pr[X = x] \leq 2^{-k}$ any any x. By union bound, $\Pr\left[\left|X - \frac{\sum_{i=1}^{k} x_i}{2}\right| < n\sqrt{k}\right] \leq 2n\sqrt{k}2^{-k}$.

(c) Combining (a) and (b), $2n\sqrt{k}2^{-k} \geq 3/4$. This established the statement.