UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 5
Fall 2023

1. **On Almost Sure Convergence**

   a. Suppose that, with probability 1, the sequence $(X_n)_{n\in\mathbb{N}}$ oscillates between two values $a \neq b$ infinitely often. Is this enough to prove that $(X_n)_{n\in\mathbb{N}}$ does *not* converge almost surely? Justify your answer.

   b. Suppose that Y is uniform on $[-1, 1]$, and X_n has distribution

   $$\mathbb{P}\big(X_n = (y + n^{-1})^{-1} \mid Y = y\big) = 1.$$

   Does $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

   c. Define random variables $(X_n)_{n\in\mathbb{N}}$ in the following way: first, set each X_n to 0. Then, for each $k \in \mathbb{N}$, pick j uniformly randomly in $\{2^k, \ldots, 2^{k+1} - 1\}$, and set $X_j = 2^k$. Does the sequence $(X_n)_{n\in\mathbb{N}}$ converge a.s.?

   d. Does the sequence $(X_n)_{n\in\mathbb{N}}$ from the previous part converge in probability to some X? If so, is it true that $\mathbb{E}(X_n) \to \mathbb{E}(X)$ as $n \to \infty$?

2. **Convergence in Probability**

   Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables distributed uniformly in $[-1, 1]$. Show that the following sequence $(Y_n)_{n\in\mathbb{N}}$ converges in probability to some limit where $Y_n = (X_n)^n$.

3. **Convergence in L^p**

   Let $p \geq 1$. A sequence of random variables $(X_n)_{n\geq 1}$ is said to **converge in** L^p (norm) to a random variable X if

   $$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^p) = 0.$$

   Prove that if $X_n \to X$ in L^p, then $X_n \to X$ in probability.