UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 2
Fall 2023

1. **Numbered Balls**

   A bin contains balls numbered $1, \dots, n$. You reach in and select $k \in \{1, \dots, n\}$ balls at random, sampling *without* replacement, so you do not put the balls back into the bin after each draw. Let T be the sum of the numbers on the balls you picked.

   a. If $k = 1$, what is $\mathbb{E}(T)$?

   b. Find $\mathbb{E}(T)$ for any value of $k \in \{1, \dots, n\}$.

   c. What is $\mathrm{var}(T)$ for any value of $k \in \{1, \dots, n\}$? You may leave your answer in terms of summations.

2. **Upperclassmen**

   You meet two students in the library. Assume that each student is an upperclassman or underclassman with equal probability, and each student takes EECS 126 with probability $\frac{1}{10}$, independent of each other and independent of their class standing. What is the probability that both students are upperclassmen, given at least one of them is an upperclassman currently taking EECS 126?

3. **Law of the Unconscious Statistician**

a. Prove the *Law of the Unconscious Statistician* (LOTUS): Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X : \Omega \to \mathbb{Z}$ and $F : \mathbb{Z} \to \mathbb{Z}$ be random variables. Note that the composition $Y = F(X) : \Omega \to \mathbb{Z}$ is another random variable. If $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$, and $\mathbb{E}_{\mathcal{L}_X}$ is expectation with respect to the *law* of X on $\mathbb{Z}$, then

$$\mathbb{E}(F(X)) = \mathbb{E}_{\mathcal{L}_X}(F).$$

You should assume that Ω is **discrete** for the sake of simplicity, although LOTUS holds more generally.

b. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the space of all sequences of independent fair coin tosses. Formulate N, the minimum number of tosses needed until we see heads, as a random variable on Ω.

c. Find $\mathbb{E}(N^2)$.

Hint: By the linearity of expectation, $\mathbb{E}(N^2) = \mathbb{E}(N(N-1)) + \mathbb{E}(N)$. You may use the Law of the Unconscious Statistician from part a, and the following identity:

$$\sum_{k=1}^{\infty} k(k-1)x^{k-2} = \frac{d}{dx} \sum_{k=1}^{\infty} kx^{k-1}.$$