UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Homework 03
Fall 2023

1. **Expected Norm**

   Pick two points $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ independently and uniformly in $[0, 1]^2$. Calculate $\mathbb{E}(\|X - Y\|_2^2)$.

2. **Joint Density for Exponential Distribution**

   a. If $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$ are independent, compute $\mathbb{P}(X < Y)$.

   b. If $X_1, \ldots, X_n$ are independent and Exponentially distributed with parameters $\lambda_1, \ldots, \lambda_n$, show that $\min_{1\le k\le n} X_k \sim \text{Exponential}(\sum_{j=1}^n \lambda_j)$.

   c. Deduce that
$$\mathbb{P}\left(X_i = \min_{1\le k\le n} X_k\right) = \frac{\lambda_i}{\sum_{j=1}^n \lambda_j}.$$

3. **Change of Variables**

   Let X be a continuous random variable with cdf F_X and pdf $f_X > 0$ everywhere, and let $Y = g(X)$, where g is a differentiable function.

   a. Suppose that g is also invertible. Find the pdf of Y, f_Y, in terms of g and f_X.
   b. Let $U \sim \text{Uniform}([0, 1])$. Using the conclusion from part a, show that $F_X^{-1}(U)$ has the same distribution as X. (This allows us to generate a given random variable given only a uniform random number generator.)
   c. Now suppose that $g(x) = x^2$. Find the pdf of Y in terms of the pdf of X. Also find the pdf of Y when X is a standard normal random variable in particular.

      (Note that this g is not invertible, unlike in part a.)

4. **Really Random Binomial**

Consider the random variables $U \sim \text{Uniform}([0,1])$ and $X|U \sim \text{Binomial}(n, U)$, where X is a binomial random variable with a random success probability. Given that $X = k$, we wish to find the conditional distribution of U, $f_{U|X}(u \mid k)$ using the steps below.

a. Write $f_{U|X}(u \mid k)$ in terms of the distributions of X, U, and $X \mid U$ using Bayes' Rule. Plug in any distribution given in the setup.

b. You may realize that the denominator $\mathbb{P}(X = k)$ of your expression above is hard to evaluate. It requires integrating over values of U and iterative integration by parts. Instead, we resort to an approach based on moment generating functions. Write the mgf of X as a summation in terms of $\mathbb{P}(X = k)$. Then, write $\mathbb{P}(X = k)$ as an integral over values of U and exchange the summation and integration. Use the binomial theorem to absorb the summation so we are left with an integral.

c. Carry out the evaluation of the integral. Use the identity $\frac{1-s^{n+1}}{1-s} = \sum_{i=0}^{n} s^i$ to leave your answer as a summation. Does this expression look like the mgf of some discrete random variable, and which one?

d. Conclude the distribution of X is the distribution of the discrete random variable you found above. Use this to find $\mathbb{P}(X = k)$, then find $f_{U|X}(u \mid k)$.

5. **Poisson Practice**

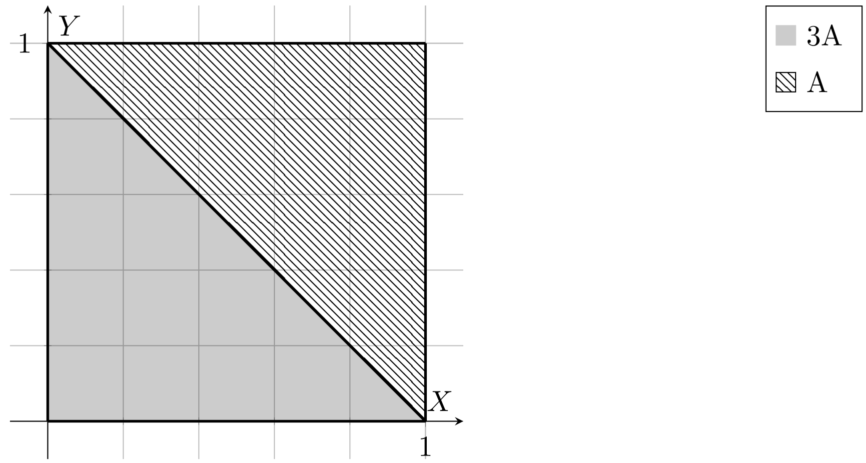
   Suppose X is a Poisson random variable with parameter λ. Find the following:

   a. $\mathbb{E}(X^2)$.

   b. $\mathbb{P}(X \text{ is even})$. (*Hint*: Use the Taylor series expansion of e^x.)

6. **Graphical Density**

The following figure depicts the joint density $f_{X,Y}$ of X and Y.



a. Are X and Y independent? Remember to justify your answer.

b. What is the value of A?

c. Compute $f_X(x)$.

d. Compute $\mathbb{E}(Y \mid X = x)$. You may leave your answer as a fraction of terms containing x, but you may not have an integral.

e. What is $\mathbb{E}(X - Y \mid X + Y)$?