Provable Sparse Tensor Decomposition for Personalized Recommendation and High-dimensional Latent Variable Models

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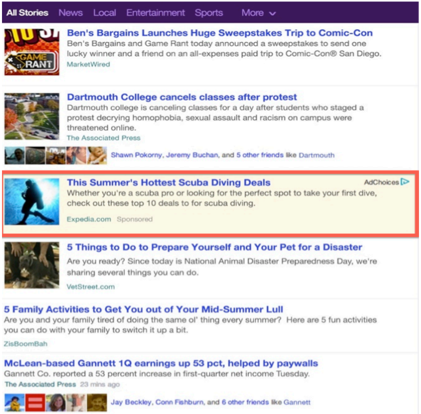
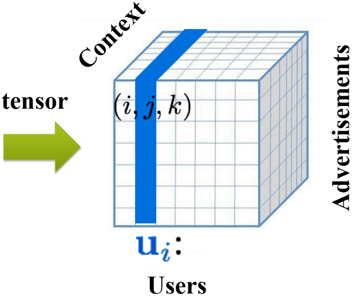
Outline

- Motivation Examples
- Sparse Tensor Decomposition
- Local and Global Convergence Analysis
- Experiments
- Future Work on Statistical-and-Computational Tradeoffs

Motivation: Personalized Recommendation

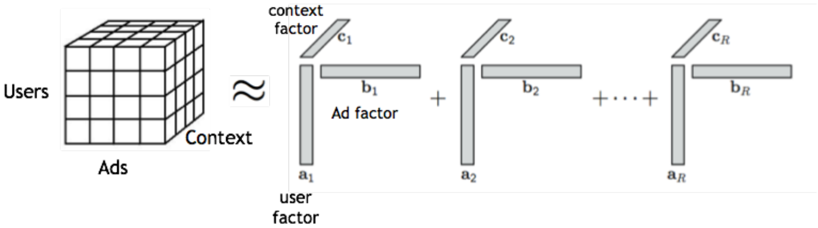




$(i,\ j,\ k)$

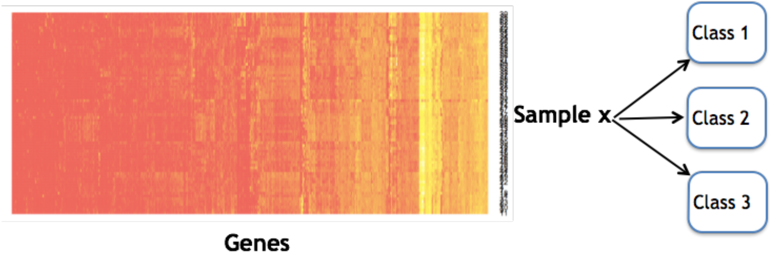
Whether *user* i, under *context* j will click *ad.* k ?

Motivation: Personalized Recommendation



- Goal: Given the observed tensor, compute the factors to recover the whole tensor.
- Difficulty: the tensor is sparse and the factors are sparse.

Motivation: High-dimensional Latent Variable Model



- Gaussian mixture: $\boldsymbol{x} \sim \sum_{k=1}^{K} w_k N(\boldsymbol{\mu}_k, \sigma^2 \mathbb{1})$

Motivation: High-dimensional Latent Variable Model

- (Hsu and Kakade, 2013) Define

$$\mathcal{M} := \mathbb{E}[\boldsymbol{x} \otimes \boldsymbol{x} \otimes \boldsymbol{x}] - \sigma^2 f(\mathbb{E}[\boldsymbol{x}]),$$

  then

$$\mathcal{M} = \sum_{k=1}^{K} w_k \boldsymbol{\mu}_k \otimes \boldsymbol{\mu}_k \otimes \boldsymbol{\mu}_k.$$

- Goal: Recover w_k and $\boldsymbol{\mu}_k$ from empirical tensor $\widehat{\mathcal{M}}$.
- Difficulty: many genes contain no information about clustering structure. Require sparse $\boldsymbol{\mu}_k$'s!

Sparse Tensor Decomposition

- Assume tensor $\mathcal{T} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ to be sparse and have rank K,

$$\mathcal{T} = \sum_{i=1}^{K} w_i \mathbf{a}_i \otimes \mathbf{b}_i \otimes \mathbf{c}_i,$$

  where $w_i \in \mathbb{R}$, $\mathbf{a}_i \in \mathbb{R}^{d_1}$, $\mathbf{b}_i \in \mathbb{R}^{d_2}$, $\mathbf{c}_i \in \mathbb{R}^{d_3}$, and $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i \in \mathcal{S}_{d_0} := \{\mathbf{v} : \|\mathbf{v}\|_2 = 1, \|\mathbf{v}\|_0 \leq d_0\}$ for any i.
- It generalizes matrix SVD to tensor. For a matrix A

$$A = UDV = \sum_i \sigma_i \boldsymbol{u}_i \otimes \boldsymbol{v}_i.$$

Existing Tensor Decomposition Methods

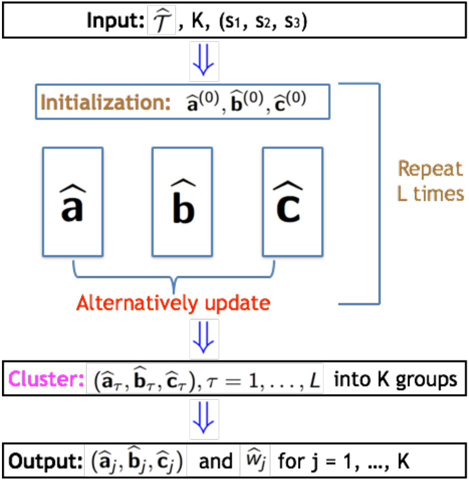
- Allen (2012) imposed an lasso penalty on $\mathbf{a}, \mathbf{b}, \mathbf{c}$ for rank-1 tensor recovery, but without theoretical guarantees,

$$\min_{\|\mathbf{a}\|=\|\mathbf{b}\|=\|\mathbf{c}\|=1} \|\mathcal{T} - w\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}\|_F + \lambda_1 \|\mathbf{a}\|_1 + \lambda_2 \|\mathbf{b}\|_1 + \lambda_3 \|\mathbf{c}\|_1.$$

- Anandkumar et al. (2014) proposed a non-sparse tensor decomposition method with guaranteed rates of convergence.

Our focus: propose a sparse tensor decomposition via l_0 optimization with theoretical guarantees of estimation accuracy.

Main Algorithm: Outline



Tensor Operations

- For $\mathcal{T} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ and $\mathbf{u} \in \mathbb{R}^{d_1}, \mathbf{v} \in \mathbb{R}^{d_2}, \mathbf{w} \in \mathbb{R}^{d_3}$, define

$$\mathcal{T} \times_2 \mathbf{v} \times_3 \mathbf{w} := \sum_{j,l} \mathbf{v}_j \mathbf{w}_l [\mathcal{T}]_{:,j,l}$$

$$\mathcal{T} \times_1 \mathbf{u} \times_2 \mathbf{v} \times_3 \mathbf{w} := \sum_{i,j,l} \mathbf{u}_i \mathbf{v}_j \mathbf{w}_l [\mathcal{T}]_{i,j,l}$$

- Define $\mathrm{Norm}(\mathbf{v}) = \mathbf{v}/\|\mathbf{v}\|$.
- Define a truncation operator as

$$[\mathrm{Truncate}(\mathbf{v}, s)]_i = \begin{cases} \mathbf{v}_i, & \text{if } i \in \mathrm{supp}(\mathbf{v}, s) \\ 0, & \text{otherwise} \end{cases}.$$

- $\mathrm{Truncate}\big(\underbrace{(0.1, 0.3, -0.2, -0.6)}_{\mathbf{v}}, 2\big) = (0, 0.2, 0, -0.6).$

Main Algorithm: Continued

- Key: alternative update steps

$$\bar{\mathbf{a}} = \mathrm{Norm}\left(\widehat{\mathcal{T}} \times_2 \widehat{\mathbf{b}} \times_3 \widehat{\mathbf{c}}\right); \ \check{\mathbf{a}} = \mathrm{Truncate}\left(\bar{\mathbf{a}}, s_1\right); \ \widehat{\mathbf{a}} = \mathrm{Norm}\left(\check{\mathbf{a}}\right)$$

$$\bar{\mathbf{b}} = \mathrm{Norm}\left(\widehat{\mathcal{T}} \times_1 \widehat{\mathbf{a}} \times_3 \widehat{\mathbf{c}}\right); \ \check{\mathbf{b}} = \mathrm{Truncate}\left(\bar{\mathbf{b}}, s_2\right); \ \widehat{\mathbf{b}} = \mathrm{Norm}\left(\check{\mathbf{b}}\right)$$

$$\bar{\mathbf{c}} = \mathrm{Norm}\left(\widehat{\mathcal{T}} \times_1 \widehat{\mathbf{a}} \times_2 \widehat{\mathbf{b}}\right); \ \check{\mathbf{c}} = \mathrm{Truncate}\left(\bar{\mathbf{c}}, s_3\right); \ \widehat{\mathbf{c}} = \mathrm{Norm}\left(\check{\mathbf{c}}\right)$$

- Initialization: Random (fast) or via sparse SVD (provable)

Tuning Procedure

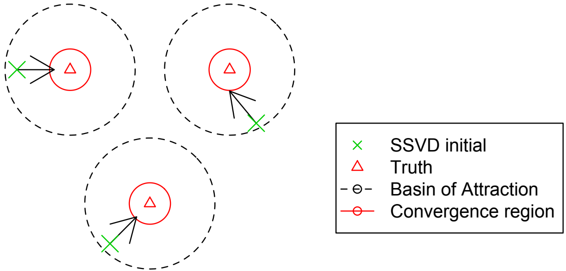
- Find exact tensor rank is an NP hard problem (Kolda, 2009).
- Tune (K, s_1, s_2, s_3) by minimizing BIC (Allen, 2012),

$$\mathrm{BIC} := \underbrace{\log\left(\frac{\|\widehat{\mathcal{E}}\|_F^2}{d_1 d_2 d_3}\right)}_{Model \;\; fitting} + \underbrace{\frac{\log(d_1 d_2 d_3)}{d_1 d_2 d_3}\left[K(s_1 + s_2 + s_3)\right]}_{Sparsity \;\; control}$$

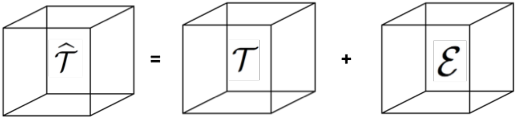
with $\widehat{\mathcal{E}} = \widehat{\mathcal{T}} - \sum_{i=1}^{K} \widehat{w}_i \widehat{\mathbf{a}}_i \circ \widehat{\mathbf{b}}_i \circ \widehat{\mathbf{c}}_i$.

Theoretical Analysis: Local and Global Convergence

- Goal: Quantify the rates of convergence of the estimators $\widehat{\mathbf{a}}_j$, $\widehat{\mathbf{b}}_j$, $\widehat{\mathbf{c}}_j$, and $\widehat{w}_j$ for each $j = 1, \ldots, K$.



Theoretical Analysis: Noisy Tensor Decomposition



- Observe the noisy tensor $\widehat{\mathcal{T}} = \mathcal{T} + \mathcal{E}$ where

$$\mathcal{T} = \sum_{i=1}^{K} w_i \mathbf{a}_i \otimes \mathbf{b}_i \otimes \mathbf{c}_i$$

- Require assumptions on true tensor $\mathcal{T}$ and error tensor $\mathcal{E}$.

Theoretical Analysis: Key Assumptions

(A1) **Incoherence:** The decomposition components are incoherent s.t.

$$\max_{i \neq j} \{ |\langle \mathbf{a}_i, \mathbf{a}_j \rangle|, |\langle \mathbf{b}_i, \mathbf{b}_j \rangle|, |\langle \mathbf{c}_i, \mathbf{c}_j \rangle| \} \leq \frac{C}{\sqrt{d_0}},$$

for some constant C.

(A2) **Bounded error:** Define the sparse norm of $\mathcal{E}$ as

$$\rho(\mathcal{E}, m) := \sup_{\substack{\|\mathbf{u}\|=\|\mathbf{v}\|=\|\mathbf{w}\|=1 \\ \|\mathbf{u}\|_0 \leq m, \|\mathbf{v}\|_0 \leq m, \|\mathbf{w}\|_0 \leq m}} \left| \mathcal{E} \times_1 \mathbf{u} \times_2 \mathbf{v} \times_3 \mathbf{w} \right|.$$

Let $s = \max\{s_1, s_2, s_3\}$. For some constant C_0, assume

$$\rho(\mathcal{E}, d_0 + s) \leq \min \left\{ \frac{w_{\min}}{6}, \frac{w_{\min}\sqrt{\log K}}{C_0 \sqrt{d_0}} \right\}.$$

Theoretical Analysis: Local Convergence Analysis

$$\epsilon_R := \underbrace{C_1 \rho(\mathcal{E}, d_0 + s)}_{Sample\ \ error} + \underbrace{C_2 \frac{\sqrt{K}}{d_0}}_{Model\ \ error}$$

Theorem

If the initializations $\widehat{a}^{(0)}, \widehat{b}^{(0)}, \widehat{c}^{(0)}$ satisfy

$$\max\left\{ dist(\widehat{a}^{(0)}, a_j), dist(\widehat{b}^{(0)}, b_j) \right\} = O\left(\frac{w_{\min}}{w_{\max}}\right),$$

then our algorithm with $s \geq d_0$ satisfies w.h.p., for some $j \in [K]$,

$$\begin{aligned} \max\left\{ dist(\widehat{a}, a_j), dist(\widehat{b}, b_j), dist(\widehat{c}, c_j) \right\} &\leq O(\epsilon_R) \\ |\widehat{w} - w_j| &\leq O(\epsilon_R). \end{aligned}$$

Theoretical Analysis: Global Convergence Analysis

$$\epsilon_R := \underbrace{C_1 \rho(\mathcal{E}, d_0 + s)}_{Sample\ \ error} + \underbrace{C_2 \frac{\sqrt{K}}{d_0}}_{Model\ \ error}$$

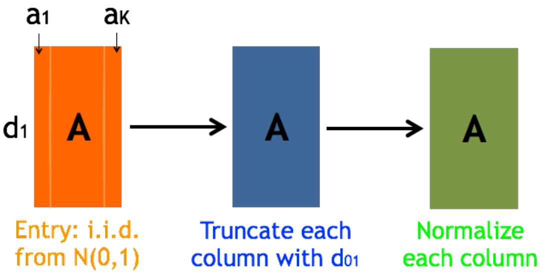
Theorem

For any $j \in [k]$, the output of our algorithm with $s \geq d_0$ using sparse SVD initialization satisfies, w.h.p.,

$$\begin{aligned} \max\left\{ dist(\widehat{a}_j, a_j), dist(\widehat{b}_j, b_j), dist(\widehat{c}_j, c_j) \right\} &\leq O(\epsilon_R), \\ |\widehat{w} - w_j| &\leq O(\epsilon_R). \end{aligned}$$

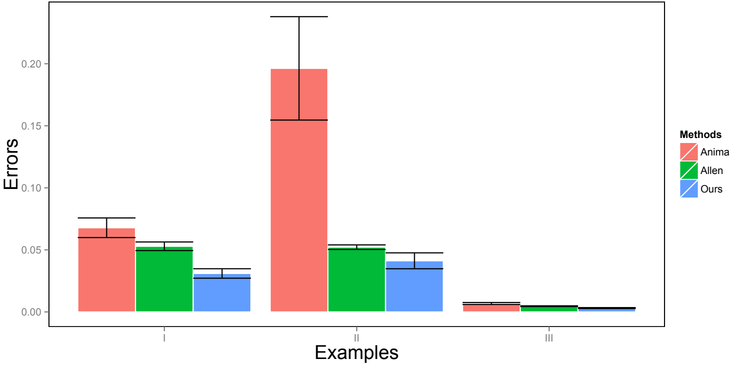
- Non-sparse tensor decomposition (Anandkumar et al., 2014) obtained an estimation error $O(\rho(\mathcal{E}, d) + \sqrt{K}/d)$.
- In high-dim regime, it is slower than ours.

Simulation 1: Sparse Tensor Recovery



- Generate $\widehat{\mathcal{T}} = \mathcal{T} + \mathcal{E}$
- $(d_1, d_2, d_3) = (1000, 100, 10)$ and $d_{0j} = 0.2 * d_j$.
  Example I: $[\mathcal{E}]_{i,j,k} \sim N(0,1), \quad K = 1;$
  Example II: $[\mathcal{E}]_{i,j,k} \sim N(0,1), \quad K = 2;$
  Example III: $[\mathcal{E}]_{i,j,k} \sim N(0,0.1), K = 1.$

Simulation 1: Estimation Accuracy



$$\epsilon_R := \underbrace{C_1 \rho(\mathcal{E}, d_0 + s)}_{Sample\ \ error} + \underbrace{C_2 \sqrt{K} / d_0}_{Model\ \ error}$$

Simulation 1: Variable Selection

Examples	Methods	TPR	FPR
I	Anima	1_0	1_0
	Allen	1_0	$0.003_{0.0022}$
	Ours	1_0	$0.016_{0.0130}$
II	Anima	1_0	1_0
	Allen	1_0	$0.002_{0.0016}$
	Ours	1_0	$0.067_{0.0311}$
III	Anima	1_0	1_0
	Allen	1_0	$0.002_{0.0022}$
	Ours	1_0	0_0

Simulation 2: Sparse Gaussian Mixture Model

- $\boldsymbol{x}_i \sim \sum_k w_k N(\boldsymbol{\mu}_k, 0.1 * \mathbb{1}) : n = 1000, d = 10, K = 4, w_k = \frac{1}{4}$
  $\boldsymbol{\mu}_1 = \mathbf{e}_1 + 0.2\mathbf{e}_2, \boldsymbol{\mu}_2 = \mathbf{e}_2 + 0.2\mathbf{e}_3$
  $\boldsymbol{\mu}_3 = \mathbf{e}_3 + 0.2\mathbf{e}_4, \boldsymbol{\mu}_4 = \mathbf{e}_4 + 0.2\mathbf{e}_1$

- **Step 1:** Estimate $\mathbb{E}[\boldsymbol{x} \otimes \boldsymbol{x} \otimes \boldsymbol{x}]$ and $\mathbb{E}[\boldsymbol{x}]$ to obtain $\widehat{\mathcal{M}}$ for

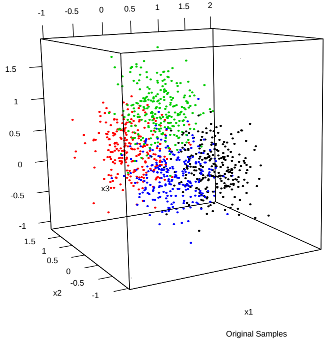
$$\mathcal{M} = \mathbb{E}[\boldsymbol{x} \otimes \boldsymbol{x} \otimes \boldsymbol{x}] - \sigma^2 f(\mathbb{E}[\boldsymbol{x}])$$

- **Step 2:** Apply sparse tensor decomposition on $\widehat{\mathcal{M}}$ to solve

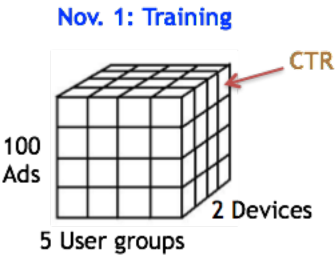
$$\widehat{\mathcal{M}} \approx \sum_{k=1}^{K} \widehat{w}_k \widehat{\boldsymbol{\mu}}_k \otimes \widehat{\boldsymbol{\mu}}_k \otimes \widehat{\boldsymbol{\mu}}_k.$$

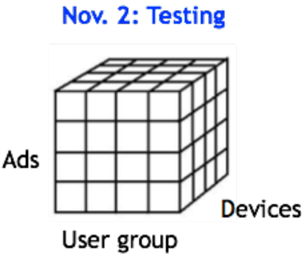
Simulation 2: Reconstruction Performance

- Left: original samples; Right: reconstructed samples.



Real Application 1: Click-through Rate Prediction

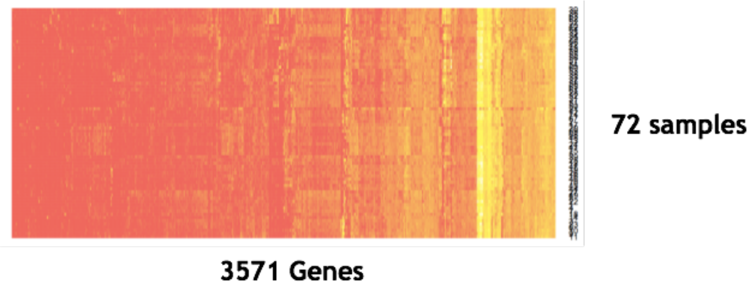




Methods	Training error	Testing error
Linear regression	0.189	0.534
Gradient boosting machine	0.190	0.533
Ours	**0.141**	**0.511**

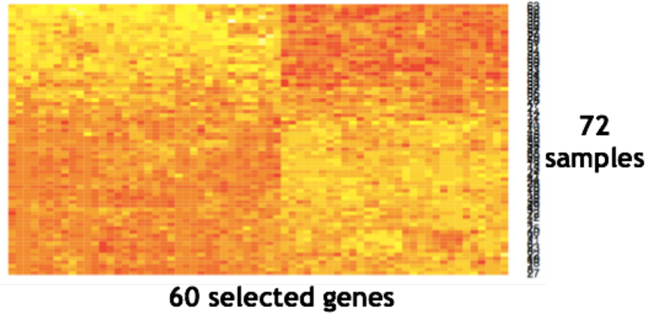
Real Application 2: High-dim Gene Clustering

- Leukemia data: cluster samples into 2 groups.

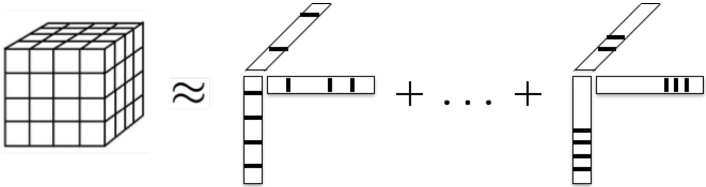


Real Application 2: High-dim Gene Clustering

Methods	No. genes	cluster error
K-means	3571	2/72
Reg. k-means (S. et al., 2012)	211	2/72
Ours	**60**	2/72



Summary

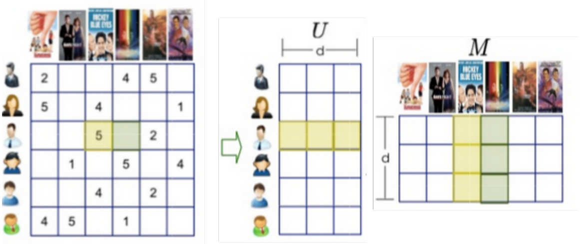


- new sparse tensor decomposition algorithm via ℓ_0 truncation
- local/global rates of convergence, faster than non-sparse one
- personalized recommendation, high-dim latent variable models

Future Work: Statistical-and-Computational Tradeoffs

- A function $g(\boldsymbol{a}, \mathbf{b}) : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ is biconvex if $g(\boldsymbol{a}, \mathbf{b})$ is convex in $\boldsymbol{a}$ for fixed $\mathbf{b} \in \mathcal{B}$, and convex in $\mathbf{b}$ for fixed $\boldsymbol{a} \in \mathcal{A}$.
- Biconvex optimization:

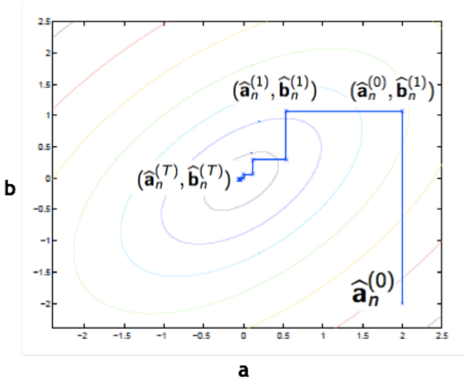
$$
\begin{aligned}
&\min \;\; g(\boldsymbol{a}, \mathbf{b}) \\
&s.t. \;\; \boldsymbol{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}
\end{aligned}
$$



Source: A. Karatzoglou, ESSIR 2013 Recommender Systems tutorial

Statistical-and-Computational Tradeoffs: Problem

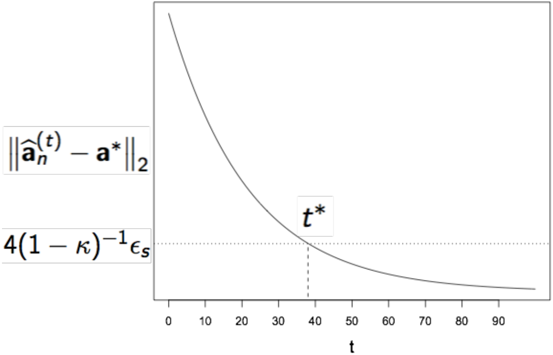
- Population version: $(\mathbf{a}^*, \mathbf{b}^*) = \arg\min_{\mathbf{a}\in\mathcal{A}, \mathbf{b}\in\mathcal{B}} \; g(\mathbf{a}, \mathbf{b})$
- Goal: Find $(\widehat{\mathbf{a}}, \widehat{\mathbf{b}})$ via sample $g_n(\mathbf{a}, \mathbf{b})$ s.t. $\|\widehat{\mathbf{a}} - \mathbf{a}^*\|_2$ and $\|\widehat{\mathbf{b}} - \mathbf{b}^*\|_2$ are small given limited computational resources.



Statistical-and-Computational Tradeoffs: Main Result

$$\left\|\widehat{\mathbf{a}}_n^{(t)} - \mathbf{a}^*\right\|_2 \le \underbrace{2(1-\kappa)^{-1}\epsilon_s}_{\text{Statistical Error}} + \underbrace{\kappa^t \epsilon_0}_{\text{Optimization Error}}$$

- ϵ_s: error due to sample function; ϵ_0: initialization error.
- Constant $\kappa < 1$.



Sparse Tensor Graphical Model (S. et al., 2015, NIPS)

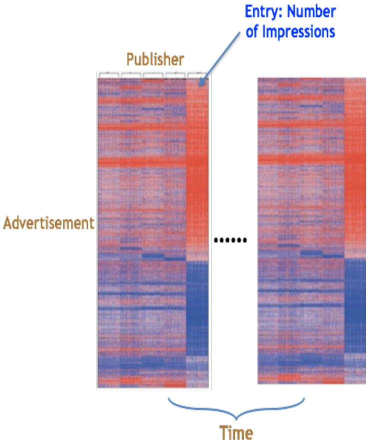


Figure: Tensor data

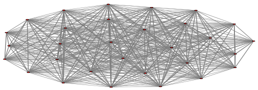


Figure: Advertisement network

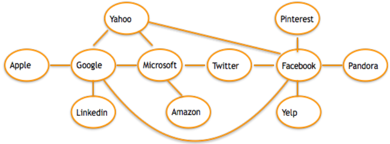


Figure: Publisher network

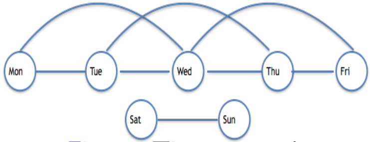


Figure: Time network

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Backup Slides

- Backup slides start from here!

Initalization via Sparse SVD

Input: tensor $\widehat{\mathcal{T}}$, cardinality parameter (s_1, s_2, s_3)

Step 1: Generate a d_3-dim standard Gaussian vector $\boldsymbol{\theta}$.

Step 2: $\check{\boldsymbol{\theta}} = \mathrm{Truncate}(\boldsymbol{\theta}, \max\{s_1, s_2, s_3\})$.

Step 3: Calculate top left (right) singular vectors $\mathbf{u}_1$ $(\mathbf{v}_1)$ of $\widehat{\mathcal{T}} \times_3 \check{\boldsymbol{\theta}}$.

Step 4: $\check{\mathbf{u}}_1 = \mathrm{Truncate}(\mathbf{u}_1, s_1)$ and $\check{\mathbf{v}}_1 = \mathrm{Truncate}(\mathbf{v}_1, s_2)$.

Step 5: $\widehat{\mathbf{a}}_\tau^{(0)} = \mathrm{Norm}(\check{\mathbf{u}}_1)$, $\widehat{\mathbf{b}}_\tau^{(0)} = \mathrm{Norm}(\check{\mathbf{v}}_1)$, and update $\widehat{\mathbf{c}}_\tau^{(0)}$.

Output: $(\widehat{\mathbf{a}}_\tau^{(0)}, \widehat{\mathbf{b}}_\tau^{(0)}, \widehat{\mathbf{c}}_\tau^{(0)})$.

- Intuition:

$$\mathcal{T} \times_3 \check{\boldsymbol{\theta}} = \sum_{i \in [K]} \underbrace{w_i(\mathbf{c}_i^\top \check{\boldsymbol{\theta}})}_{singular\ \ value} \underbrace{\mathbf{a}_i \mathbf{b}_i^\top}_{singular\ \ vectors} \in \mathbb{R}^{d_1 \times d_2}$$

Clustering Procedure

Input: tensor $\widehat{\mathcal{T}}$, set $\left\{(\widehat{\mathbf{a}}_\tau, \widehat{\mathbf{b}}_\tau, \widehat{\mathbf{c}}_\tau), \tau \in [L]\right\}$.

For $j = 1$ **to** K **Do**

Step 1: Find $(\widehat{\mathbf{a}}, \widehat{\mathbf{b}}, \widehat{\mathbf{c}}) = \arg\max_{(\mathbf{a},\mathbf{b},\mathbf{c}) \in S} |\widehat{\mathcal{T}} \times_1 \mathbf{a} \times_2 \mathbf{b} \times_3 \mathbf{c}|$.

Step 2: Perform alternative update steps with initialization $(\widehat{\mathbf{a}}, \widehat{\mathbf{b}}, \widehat{\mathbf{c}})$.

Step 3: Output the cluster center as the final update in Step 2.

Step 4: Remove tupes with $\min\{\|\widehat{\mathbf{a}}_\tau \pm \widehat{\mathbf{a}}\|, \|\widehat{\mathbf{b}}_\tau \pm \widehat{\mathbf{b}}\|, \|\widehat{\mathbf{c}}_\tau \pm \widehat{\mathbf{c}}\|\} \leq 0.5$.

End For

Output: $\{(\widehat{\mathbf{a}}_j, \widehat{\mathbf{b}}_j, \widehat{\mathbf{c}}_j), j \in [K]\}$.

- Intuition 1: if $|\widehat{\mathcal{T}} \times_1 \mathbf{a} \times_2 \mathbf{b} \times_3 \mathbf{c}|$ is large for some $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, then it is close to some $(\mathbf{a}_j, \mathbf{b}_j, \mathbf{c}_j)$.
- Intuition 2: if $(\widehat{\mathbf{a}}_\tau, \widehat{\mathbf{b}}_\tau, \widehat{\mathbf{c}}_\tau)$ is close to $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, then their distance is *very* small; otherwise their distance is *very* large.

Illustration of Clustering Procedure

- $d_1 = d_2 = d_3 = 100, d_{01} = d_{02} = d_{03} = 50, K = 5.$
- Distance: $\min\{\|\widehat{\mathbf{a}}_\tau \pm \widehat{\mathbf{a}}\|, \|\widehat{\mathbf{b}}_\tau \pm \widehat{\mathbf{b}}\|, \|\widehat{\mathbf{c}}_\tau \pm \widehat{\mathbf{c}}\|\}.$

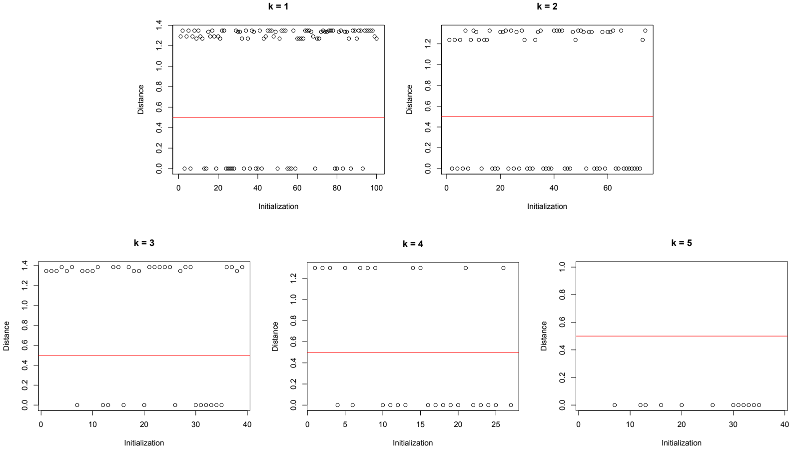

Illustration of Tuning Procedure

- $(d_1, d_2, d_3) = (40, 30, 20)$, $d_{0j} = 0.2 * d_j$, and $K = 3$