Homework 5

Total 30 points

Problem 1. (8 points) Suppose f is a convex and differentiable function, $\mathcal{C}$ is a closed convex set. Show that

$$\mathbf{x}^* \in \arg\min_{\mathbf{x} \in \mathcal{C}} f(\mathbf{x}) \iff \langle -\nabla f(\mathbf{x}^*), \mathbf{z} - \mathbf{x}^* \rangle \leq 0, \ \forall \, \mathbf{z} \in \mathcal{C}.$$

Problem 2. (5 points) Let $\mathcal{C} \in \mathbb{R}^d$ be a nonempty closed and convex set, and let f be a strongly convex function over $\mathcal{C}$. Prove that f has a unique minimizer $\mathbf{x}^*$ over $\mathcal{C}$.

Problem 3. (5 points) Consider the projected gradient descent algorithm introduced in the class. Suppose that for some iteration t, $\mathbf{x}_{t+1} = \mathbf{x}_t$. Prove that in this case, $\mathbf{x}_t$ is a minimizer of the convex objective function f over the closed and convex set $\mathcal{C}$.

Problem 4. (12 points) Suppose f is a L-smooth and μ-strongly convex function, $\mathcal{C}$ is a closed and convex set.

- Let $T(\mathbf{x}) = \mathcal{P}_{\mathcal{C}}(\mathbf{x} - \frac{1}{L}\nabla f(\mathbf{x}))$ and $g_{\mathcal{C}}(\mathbf{x}) = L(\mathbf{x} - T(\mathbf{x}))$, show that

$$\langle g_{\mathcal{C}}(\mathbf{x}), \mathbf{x} - \mathbf{x}^* \rangle \geq \frac{\mu}{2} \|\mathbf{x} - \mathbf{x}^*\|_2^2 + \frac{1}{2L} \|g_{\mathcal{C}}(\mathbf{x})\|_2^2.$$

- Show the projected gradient descent with $\eta = 1/L$ has the following convergence rate:

$$\|\mathbf{x}_t - \mathbf{x}^*\|_2^2 \leq (1 - \frac{\mu}{L})^t \|\mathbf{x}_0 - \mathbf{x}^*\|_2^2.$$