Homework 2

Total 40 points

Problem 1. (12 points) Which of the following sets are convex?

(a) A slab, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n | \alpha \leq \mathbf{a}^{\mathrm{T}}\mathbf{x} \leq \beta\}$.

(b) The set of points closer to a given point than a given set, i.e., $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 \text{ for all } \mathbf{y} \in S\}$ where $S \subseteq \mathbb{R}^n$.

(c) The set of points closer to one set than another, i.e., $\{\mathbf{x} | \mathbf{dist}(\mathbf{x}, S) \leq \mathbf{dist}(\mathbf{x}, T)\}$ where $S, T \subseteq \mathbb{R}^n$, and $\mathbf{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{z}\|_2 | \mathbf{z} \in S\}$.

(d) The set of points whose distance to $\mathbf{a}$ does not exceed a fixed fraction θ of the distance to $\mathbf{b}$, i.e., the set $\{\mathbf{x} | \|\mathbf{x} - \mathbf{a}\|_2 \leq \theta \|\mathbf{x} - \mathbf{b}\|_2\}$ ($\mathbf{a} \neq \mathbf{b}$ and $0 \leq \theta \leq 1$).

Problem 2. (12 points) Judge which of the following functions are (strict) convex.

(a) $f(x) = e^x - 1$.

(b) $f(x_1, x_2) = x_1 x_2$, $x_1 > 0, x_2 > 0$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$, $x_1 > 0, x_2 > 0$.

(d) $f(x_1, x_2) = x_1^2 / x_2$, $x_2 > 0$.

Problem 3. (8 points) Prove that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only of for every $\mathbf{x} \neq \mathbf{y} \in \mathrm{dom} f$, the function $g(t) = f(t\mathbf{x} + (1-t)\mathbf{y})$ is a convex function on $[0, 1]$.

Problem 4. (8 points) Prove that if f is a convex function, then for all $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$, and a_1, a_2 and $a_3 \in (0, 1)$ such that $a_1 + a_2 + a_3 = 1$, we have

$$\langle \nabla f(\mathbf{x}_3), a_1\mathbf{x}_1 + a_2\mathbf{x}_2 - (1 - a_3)\mathbf{x}_3 \rangle \leq a_1 f(\mathbf{x}_1) + a_2 f(\mathbf{x}_2) - (1 - a_3) f(\mathbf{x}_3).$$