Homework 5

Total 40 points

Problem 1. (8 points) Compute the subdifferentials of the following functions

(a) $f(\mathbf{x}) = \|\mathbf{x}\|_2$

(b) Given a closed convex set $\mathcal{C}$, define

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}$$

Solution.

(a)

$$\partial f(\mathbf{x}) = \begin{cases} \frac{\mathbf{x}}{||\mathbf{x}||_2} & \text{if } \mathbf{x} \neq 0 \\ \{\mathbf{g} | \; ||\mathbf{g}||_2 \leq 1\} & \text{if } \mathbf{x} = 0 \end{cases}$$

(b)

$$\partial f(\mathbf{x}) = \begin{cases} \emptyset & \text{if } \mathbf{x} \notin \mathcal{C} \\ \{\mathbf{g} | \; \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle \leq 0, \forall \mathbf{y} \in \mathcal{C}\} & \text{if } \mathbf{x} \in \partial\mathcal{C} \\ 0 & \text{if } \mathbf{x} \in \mathcal{C}^\circ \end{cases}$$

Problem 2. (8 points) If function f is convex, Show that $\partial f(\mathbf{x}) \neq \emptyset$ for all $\mathbf{x} \in \text{int}(dom\ f)$.
Solution. Notice that $(\mathbf{x}, f(\mathbf{x}))$ is on the boundary of epi f. The hyperplane supporting theorem say there exists $(\mathbf{a}, b)$ with $\mathbf{a} \neq \mathbf{0}$ such that

$$\left\langle \begin{bmatrix} \mathbf{a} \\ b \end{bmatrix}, \begin{bmatrix} \mathbf{y} - \mathbf{x} \\ t - f(\mathbf{x}) \end{bmatrix} \right\rangle \leq 0$$

for any $(\mathbf{y}, t) \in$ epi f, which means

$$S \triangleq \langle \mathbf{a}, \mathbf{y} - \mathbf{x} \rangle + b(t - f(\mathbf{x})) \leq 0.$$

We can conclude $b \leq 0$, otherwise, let $t \to +\infty$, then S goes to $+\infty$.

Since $\mathbf{x}$ is in the interior, we can find some $\epsilon > 0$ such that $\mathbf{y} = \mathbf{x} + \epsilon\mathbf{a} \in$ dom f, which leads to $S = \epsilon \|\mathbf{a}\|_2^2 + b(t - f(\mathbf{x}))$. Let $t > f(\mathbf{x})$, then we know $b \neq 0$. Hence we can say $b < 0$. Thus, $\langle \mathbf{a}/b, \mathbf{y} - \mathbf{x} \rangle + (t - f(\mathbf{x})) \geq 0$, i.e., $t \geq f(\mathbf{x}) + \langle -\mathbf{a}/b, \mathbf{y} - \mathbf{x} \rangle$.

Take $t = f(\mathbf{y})$ means $\mathbf{g} = -\mathbf{a}/b$ is a subgradient at $\mathbf{x}$.
Remark. $\partial f(\mathbf{x})$ may be empty if $\mathbf{x}$ is on the boundary of dom f. For example, suppose the function is $f(\mathbf{x}) = -\sqrt{\mathbf{x}}$ for $\mathbf{x} \geq 0$, then $\partial f(\mathbf{0}) = \emptyset$.

Problem 3. (6 points) If function f is μ-strongly convex, and $\mathbf{g}$ is a subgradient of f at $\mathbf{x}$. Show that for any $\mathbf{y} \in dom\ f$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 .$$

Solution. Let $h(\mathbf{x}) = f(\mathbf{x}) - \frac{\mu}{2} \|\mathbf{x}\|_2^2$. Then $h(\mathbf{x})$ is convex and $\mathbf{g} - \mu\mathbf{x}$ is a subgradient of h at $\mathbf{x}$. Thus we have

$$h(\mathbf{y}) \geq h(\mathbf{x}) + \langle \mathbf{g} - \mu\mathbf{x}, \mathbf{y} - \mathbf{x} \rangle,$$

which means

$$\begin{aligned} f(\mathbf{y}) \geq & f(\mathbf{x}) + \frac{\mu}{2}(\|\mathbf{y}\|_2^2 - \|\mathbf{x}\|_2^2) + \langle \mathbf{g} - \mu\mathbf{x}, \mathbf{y} - \mathbf{x} \rangle \\ = & f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \|\mathbf{y} - \mathbf{x}\|_2^2 . \end{aligned}$$

Problem 4. (6 points) Suppose f is convex and G-Lipschitz continuous over the constraint $\mathcal{C}$, which is bounded and convex with diameter $D > 0$. If we run projected subgradient descent method for T rounds with $\eta_t = \frac{D}{G\sqrt{T}}$, then we have

$$f(\bar{\mathbf{x}}_t) - f^* \leq \frac{DG}{\sqrt{T}},$$

where $\bar{\mathbf{x}}_t = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t$.

Solution. In the class, we have shown

$$f(\bar{\mathbf{x}}_t) - f^* \leq \frac{\|\mathbf{x}_0 - \mathbf{x}^*\|_2^2 + \sum_{k=0}^{t} \eta_k^2 \|\mathbf{g}_k\|^2}{2\sum_{k=0}^{t} \eta_k}.$$

Since $\|\mathbf{x}_0 - \mathbf{x}^*\|_2 \leq D$ and $\|\mathbf{g}_k\| \leq G$, we have

$$f(\bar{\mathbf{x}}_t) - f^* \leq \frac{DG}{\sqrt{T}}.$$

Problem 5. (12 points) Let f be μ-strongly convex and G-Lipschitz continuous over the constraint $\mathcal{C}$. Let $\eta_t = \frac{2}{\mu(t+1)}$ and $\bar{\mathbf{x}}_t = \sum_{k=1}^{t} \frac{2k}{t(t+1)} \mathbf{x}_k$. Prove that the projected subgradient descent obeys

(a)

$$f(\bar{\mathbf{x}}_t) - f^* \leq \frac{2G^2}{\mu(t+1)};$$

(b)

$$\|\bar{\mathbf{x}}_t - \mathbf{x}^*\|_2 \leq \frac{2G}{\mu\sqrt{t+1}}.$$

Solution.

(a) In the class, we have shown that

$$\sum_{k=0}^{t} k(f(\mathbf{x}_k) - f^*) \leq \frac{tG^2}{\mu}.$$

By Jensen inequality, we have

$$\sum_{k=0}^{t} k(f(\mathbf{x}_k) - f^*) = \frac{t(t+1)}{2}\left(\sum_{k=1}^{t} \frac{2k}{t(t+1)} f(\mathbf{x}_k) - f^*\right) \geq \frac{t(t+1)}{2}(f(\bar{\mathbf{x}}_t) - f^*).$$

Thus we can get

$$f(\bar{\mathbf{x}}_t) - f^* \leq \frac{2G^2}{\mu(t+1)}.$$

(b) By strong convexity and (a), we have

$$\frac{\mu}{2}\left\|\bar{\mathbf{x}}_t - \mathbf{x}^*\right\|_2^2 \leq \langle \nabla f(\mathbf{x}^*), \bar{\mathbf{x}}_t - \mathbf{x}^* \rangle + \frac{\mu}{2}\left\|\bar{\mathbf{x}}_t - \mathbf{x}^*\right\|_2^2 \leq f(\bar{\mathbf{x}}_t) - f^* \leq \frac{2G^2}{\mu(t+1)}.$$