Solution to Homework 6

Problem 1. Compute the subdifferentials of the following functions

(a) $f(\mathbf{x}) = \|\mathbf{x}\|_2$

(b) Given a closed convex set $\mathcal{C}$, define

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}$$

Solution.

(a)

$$\partial f(\mathbf{x}) = \begin{cases} \frac{\mathbf{x}}{||\mathbf{x}||_2} & \text{if } \mathbf{x} \neq 0 \\ \{\mathbf{g}| \ ||\mathbf{g}||_2 \leq 1\} & \text{if } \mathbf{x} = 0 \end{cases}$$

(b)

$$\partial f(\mathbf{x}) = \begin{cases} \emptyset & \text{if } \mathbf{x} \notin \mathcal{C} \\ \{\mathbf{g}| \ \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle \leq 0, \forall \mathbf{y} \in \mathcal{C}\} & \text{if } \mathbf{x} \in \partial\mathcal{C} \\ 0 & \text{if } \mathbf{x} \in \mathcal{C}^\circ \end{cases}$$

Note: Problem 2 does not count towards the score because the original problem has a mistake: "$\mathbf{x} \in$ *dom* f" should be "$\mathbf{x} \in$ int(*dom* f)".

Problem 2. If function f is convex, Show that $\partial f(\mathbf{x}) \neq \emptyset$ for all $\mathbf{x} \in$ *dom* f.

Solution. Notice that $(\mathbf{x}, f(\mathbf{x}))$ is on the boundary of epi f. The hyperplane supporting theorem say there exists $(\mathbf{a}, b)$ with $\mathbf{a} \neq \mathbf{0}$ such that

$$\left\langle \begin{bmatrix} \mathbf{a} \\ b \end{bmatrix}, \begin{bmatrix} \mathbf{y} - \mathbf{x} \\ t - f(\mathbf{x}) \end{bmatrix} \right\rangle \leq 0$$

for any $(\mathbf{y}, t) \in$ epi f, which means

$$S \triangleq \langle \mathbf{a}, \mathbf{y} - \mathbf{x} \rangle + b(t - f(\mathbf{x})) \leq 0.$$

We can conclude $b \leq 0$, otherwise, let $t \to +\infty$, then S goes to $+\infty$.

Since $\mathbf{x}$ is in the interior, we can find some $\epsilon > 0$ such that $\mathbf{y} = \mathbf{x} + \epsilon\mathbf{a} \in$ dom f, which leads to $S = \epsilon \|\mathbf{a}\|_2^2 + b(t - f(\mathbf{x}))$. Let $t > f(\mathbf{x})$, then we know $b \neq 0$. Hence we can say $b < 0$. Thus, $\langle \mathbf{a}/b, \mathbf{y} - \mathbf{x} \rangle + (t - f(\mathbf{x})) \geq 0$, i.e., $t \geq f(\mathbf{x}) + \langle -\mathbf{a}/b, \mathbf{y} - \mathbf{x} \rangle$.

Take $t = f(\mathbf{y})$ means $\mathbf{g} = -\mathbf{a}/b$ is a subgradient at $\mathbf{x}$.

Remark. $\partial f(\mathbf{x})$ may be empty if $\mathbf{x}$ is on the boundary of dom f. For example, suppose the function is $f(\mathbf{x}) = -\sqrt{\mathbf{x}}$ for $\mathbf{x} \geq 0$, then $\partial f(\mathbf{0}) = \emptyset$.

Problem 3. If function f is μ-strongly convex, and $\mathbf{g}$ is a subgradient of f at $\mathbf{x}$. Show that for any $\mathbf{y} \in \mathit{dom}\ f$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \left\| \mathbf{y} - \mathbf{x} \right\|_2^2.$$

Solution. Let $h(\mathbf{x}) = f(\mathbf{x}) - \frac{\mu}{2} \left\| \mathbf{x} \right\|_2^2$. Then $h(\mathbf{x})$ is convex and $\mathbf{g} - \mu\mathbf{x}$ is a subgradient of h at $\mathbf{x}$. Thus we have

$$h(\mathbf{y}) \geq h(\mathbf{x}) + \langle \mathbf{g} - \mu\mathbf{x}, \mathbf{y} - \mathbf{x} \rangle,$$

which means

$$\begin{aligned} f(\mathbf{y}) \geq & f(\mathbf{x}) + \frac{\mu}{2}(\|\mathbf{y}\|_2^2 - \|\mathbf{x}\|_2^2) + \langle \mathbf{g} - \mu\mathbf{x}, \mathbf{y} - \mathbf{x} \rangle \\ = & f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \left\| \mathbf{y} - \mathbf{x} \right\|_2^2. \end{aligned}$$