Solution to Homework 6

Total 40 points

Problem 1. (8 points) Compute the subdifferentials of the following functions

(a) $f(\mathbf{x}) = \|\mathbf{x}\|_2$

(b) Given a closed convex set $\mathcal{C}$, define

$$f(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}$$

Problem 2. (8 points) If function f is convex, Show that $\partial f(\mathbf{x}) \neq \emptyset$ for all $\mathbf{x} \in \text{int}(dom\ f)$.

Problem 3. (6 points) If function f is μ-strongly convex, and $\mathbf{g}$ is a subgradient of f at $\mathbf{x}$. Show that for any $\mathbf{y} \in dom\ f$,

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \|\mathbf{y} - \mathbf{x}\|_2^2.$$

Problem 4. (6 points) Suppose f is convex and G-Lipschitz continuous over the constraint $\mathcal{C}$, which is bounded and convex with diameter $D > 0$. If we run projected subgradient descent method for T rounds with $\eta_t = \frac{D}{G\sqrt{T}}$, then we have

$$f(\bar{\mathbf{x}}_t) - f^* \leq \frac{DG}{\sqrt{T}},$$

where $\bar{\mathbf{x}}_t = \frac{1}{T}\sum_{t=1}^{T} \mathbf{x}_t$.

Problem 5. (12 points) Let f be μ-strongly convex and G-Lipschitz continuous over the constraint $\mathcal{C}$. Let $\eta_t = \frac{2}{\mu(t+1)}$ and $\bar{\mathbf{x}}_t = \sum_{k=1}^{t} \frac{2k}{t(t+1)} \mathbf{x}_k$. Prove that the projected subgradient descent obeys

(a)

$$f(\bar{\mathbf{x}}_t) - f^* \leq \frac{2G^2}{\mu(t+1)};$$

(b)

$$\|\bar{\mathbf{x}}_t - \mathbf{x}^*\|_2 \leq \frac{2G}{\mu\sqrt{t+1}}.$$