

# Optimization for Machine Learning

## 机器学习中的优化方法

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# Review

condition	stepsize	convergence rate	iteration complexity
convex & smooth	$\eta_t = \frac{1}{L}$	$O\left(\frac{1}{t}\right)$	$O\left(\frac{1}{\varepsilon}\right)$
strongly convex & smooth	$\eta_t = \frac{1}{L}$	$O\left(\left(1 - \frac{1}{\kappa}\right)^t\right)$	$O(\kappa \log \frac{1}{\varepsilon})$

Table: Convergence Properties of smooth convex optimization

	stepsize	convergence rate	iteration complexity
convex	$\eta_t \approx \frac{1}{\sqrt{t}}$	$O\left(\frac{1}{\sqrt{t}}\right)$	$O\left(\frac{1}{\varepsilon^2}\right)$
strongly convex	$\eta_t \approx \frac{1}{t}$	$O\left(\frac{1}{t}\right)$	$O\left(\frac{1}{\varepsilon}\right)$

Table: Convergence Properties of nonsmooth convex optimization

# Outline

1 Proximal gradient descent

2 Proximal Operator

3 Convergence Analysis

## Composite problems

Consider  $\ell_1$  regularized minimization with smooth function  $f$ :

$$\min_{\mathbf{x}} f(\mathbf{x}) + \lambda \|\mathbf{x}\|_1.$$

The nonsmooth term  $\lambda \|\mathbf{x}\|_1$  is simple and easy to optimize!

This week, we consider optimization on composite problems:

$$\min_{\mathbf{x}} F(\mathbf{x}) = f(\mathbf{x}) + h(\mathbf{x})$$

- $f$  is convex and smooth
- $h$  is convex (may not be differentiable)
- Let  $F^* = \min_{\mathbf{x}} F(\mathbf{x})$  be the optimal value

# A proximal view of gradient descent

We first revisit gradient descent

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t)$$

$\Updownarrow$

$$\mathbf{x}_{t+1} = \arg \min_{\mathbf{x}} \left\{ \underbrace{f(\mathbf{x}_t) + \langle \nabla f(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_t \rangle}_{\text{first-order approximation}} + \underbrace{\frac{1}{2\eta_t} \|\mathbf{x} - \mathbf{x}_t\|_2^2}_{\text{proximal term}} \right\}$$

Then we define  $\hat{f}(\mathbf{x}) = f(\mathbf{x}_t) + \langle \nabla f(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_t \rangle + \frac{1}{2\eta_t} \|\mathbf{x} - \mathbf{x}_t\|_2^2$ , we can approximate the minimum of  $F(\mathbf{x}) = f(\mathbf{x}) + h(\mathbf{x})$  by

$$\mathbf{x}_{t+1} = \arg \min_{\mathbf{x}} \hat{f}(\mathbf{x}) + h(\mathbf{x}).$$

## How about projected gradient descent?

$$\mathbf{x}_{t+1} = \mathcal{P}_{\mathcal{C}}(\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t))$$

$\Updownarrow$

$$\begin{aligned}\mathbf{x}_{t+1} &= \arg \min_{\mathbf{x}} \left\{ f(\mathbf{x}_t) + \langle \nabla f(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_t \rangle + \frac{1}{2\eta_t} \|\mathbf{x} - \mathbf{x}_t\|_2^2 + \mathbb{1}_{\mathcal{C}}(\mathbf{x}) \right\} \\ &= \arg \min_{\mathbf{x}} \left\{ \frac{1}{2} \|\mathbf{x} - (\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t))\|_2^2 + \eta_t \mathbb{1}_{\mathcal{C}}(\mathbf{x}) \right\}\end{aligned}$$

where

$$\mathbb{1}_{\mathcal{C}}(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \in \mathcal{C} \\ +\infty, & \text{otherwise} \end{cases}$$

# Proximal operator (近端算子)

Define the proximal operator

$$\text{prox}_h(\mathbf{x}) \triangleq \arg \min_{\mathbf{z}} \left\{ \frac{1}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + h(\mathbf{z}) \right\}$$

for any convex function  $h$ .

Then, the update of projected gradient descent is

$$\mathbf{x}_{t+1} = \text{prox}_{\eta_t \mathbb{1}_C}(\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t))$$

# Proximal gradient descent (近端梯度下降法)

In each iteration, the proximal gradient descent method for composite objective function  $F(\mathbf{x}) = f(\mathbf{x}) + h(\mathbf{x})$  computes

$$\mathbf{x}_{t+1} = \text{prox}_{\eta_t h}(\mathbf{x}_t - \eta_t \nabla f(\mathbf{x}_t)).$$

- alternates between gradient updates on  $f$  and proximal minimization on  $h$
- useful if the  $\text{prox}_h$  can be efficiently computed

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# Proximal operator

$$\text{prox}_h(\mathbf{x}) \triangleq \arg \min_{\mathbf{z}} \left\{ \frac{1}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + h(\mathbf{z}) \right\}$$

- well-defined under very general conditions (including nonsmooth convex functions)
- can be evaluated efficiently for many widely used functions (in particular, regularizers)
- this abstraction is mathematically simple but covers many well-known optimization algorithms

## Example: indicator functions

If  $h(\mathbf{x}) = \mathbb{1}_{\mathcal{C}}$  is the “indicator” function

$$h(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \in \mathcal{C} \\ +\infty, & \text{otherwise} \end{cases}$$

then

$$\text{prox}_h(\mathbf{x}) = \arg \min_{\mathbf{z} \in \mathcal{C}} \|\mathbf{z} - \mathbf{x}\|_2^2 \quad (\text{Euclidean projection})$$

## Example: $\ell_1$ Norm

If  $h(\mathbf{x}) = \lambda \|\mathbf{x}\|_1$ , then

$$(\text{prox}_{\lambda h}(\mathbf{x}))_i = \psi_{st}(x_i; \lambda) \quad \text{soft-thresholding}$$

where

$$\psi_{st}(x) = \begin{cases} x - \lambda, & \text{if } x > \lambda \\ x + \lambda, & \text{if } x < -\lambda \\ 0, & \text{otherwise} \end{cases}$$

## Basic rules of proximal operator

- **affine addition:** if  $f(\mathbf{x}) = g(\mathbf{x}) + \mathbf{a}^\top \mathbf{x} + b$ , then

$$\text{prox}_f(\mathbf{x}) = \text{prox}_g(\mathbf{x} - \mathbf{a})$$

- **quadratic addition:** if  $f(\mathbf{x}) = g(\mathbf{x}) + \frac{\rho}{2} \|\mathbf{x} - \mathbf{a}\|_2^2$ , then

$$\text{prox}_f(\mathbf{x}) = \text{prox}_{\frac{1}{1+\rho}g} \left( \frac{1}{1+\rho} \mathbf{x} - \frac{\rho}{1+\rho} \mathbf{a} \right)$$

- **scaling and translation:** if  $f(\mathbf{x}) = g(a\mathbf{x} + b)$ , then

$$\text{prox}_f(\mathbf{x}) = \frac{1}{a} (\text{prox}_{a^2 g}(a\mathbf{x} + b) - b)$$

## Basic rules of proximal operator

- **norm composition:** if  $f(\mathbf{x}) = g(\|\mathbf{x}\|_2)$  with  $\text{dom } g = [0, \infty)$ , then

$$\text{prox}_f(\mathbf{x}) = \text{prox}_g(\|\mathbf{x}\|_2) \frac{\mathbf{x}}{\|\mathbf{x}\|_2}, \quad \forall \mathbf{x} \neq \mathbf{0}$$

# Nonexpansiveness of proximal operators

- **(firm nonexpansiveness)**

$$\langle \text{prox}_h(\mathbf{x}_1) - \text{prox}_h(\mathbf{x}_2), \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq \|\text{prox}_h(\mathbf{x}_1) - \text{prox}_h(\mathbf{x}_2)\|_2^2$$

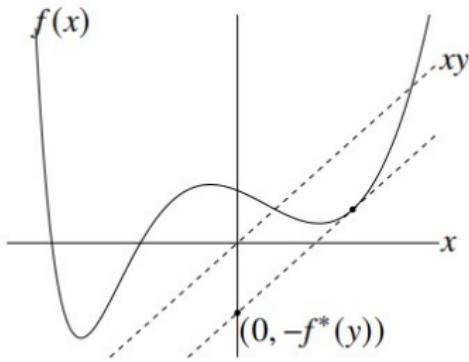
- **(nonexpansiveness)**

$$\|\text{prox}_h(\mathbf{x}_1) - \text{prox}_h(\mathbf{x}_2)\|_2 \leq \|\mathbf{x}_1 - \mathbf{x}_2\|_2$$

# Conjugate functions (共轭函数)

The **conjugate** of a function  $f$  is

$$f^*(\mathbf{y}) = \sup_{\mathbf{x} \in \text{dom } f} \{ \langle \mathbf{y}, \mathbf{x} \rangle - f(\mathbf{x}) \}$$



**Fenchel's inequality:**  $f(\mathbf{x}) + f^*(\mathbf{y}) \geq \langle \mathbf{y}, \mathbf{x} \rangle$

# Conjugate Functions

**Property:** If  $f$  is convex and closed. Then

- $\mathbf{y} \in \partial f(\mathbf{x}) \iff \mathbf{x} \in \partial f^*(\mathbf{y})$
- $f^{**} = f$

**Examples:**

- **Indicator function:**

$$f(\mathbf{x}) = \mathbb{1}_{\mathcal{C}}(\mathbf{x}), \quad f^*(\mathbf{y}) = \sup_{\mathbf{x} \in \mathcal{C}} \langle \mathbf{x}, \mathbf{y} \rangle$$

- **Norm:**

$$f(\mathbf{x}) = \|\mathbf{x}\|, \quad f^*(\mathbf{y}) = \begin{cases} 0, & \|\mathbf{y}\|_* \leq 1 \\ +\infty, & \|\mathbf{y}\|_* > 1 \end{cases}$$

where  $\|\mathbf{y}\|_* = \sup_{\|\mathbf{x}\| \leq 1} \langle \mathbf{x}, \mathbf{y} \rangle$  is the dual norm.

# Moreau Decomposition

Suppose  $f$  is closed and convex. Then

$$\mathbf{x} = \text{prox}_f(\mathbf{x}) + \text{prox}_{f^*}(\mathbf{x})$$

## Example: prox of support function

For any closed and convex set  $\mathcal{C}$ , the support function is defined as  
 $S_{\mathcal{C}}(\mathbf{x}) = \sup_{\mathbf{z} \in \mathcal{C}} \langle \mathbf{x}, \mathbf{z} \rangle$ . Then

$$\text{prox}_{S_{\mathcal{C}}}(\mathbf{x}) = \mathbf{x} - \mathcal{P}_{\mathcal{C}}(\mathbf{x})$$

## Examples

- $\ell_\infty$  norm:

$$\text{prox}_{\|\cdot\|_\infty}(\mathbf{x}) = \mathbf{x} - \mathcal{P}_{\mathcal{B}_{\|\cdot\|_1}}(\mathbf{x})$$

where  $\mathcal{B}_{\|\cdot\|_1} = \{\mathbf{z} | \|\mathbf{z}\|_1 \leq 1\}$  is unit  $\ell_1$  ball.

- max function: Let  $g(\mathbf{x}) = \{x_1, \dots, x_n\}$ , then

$$\text{prox}_g(\mathbf{x}) = \mathbf{x} - \mathcal{P}_\Delta(\mathbf{x})$$

where  $\Delta = \{\mathbf{z} \in \mathbb{R}_+^n | \mathbf{1}^\top \mathbf{z} = 1\}$  is probability simplex.

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# Convergence analysis

## Lemma

Let  $\mathbf{y}^+ = \text{prox}_{\frac{1}{L}h}(\mathbf{y} - \frac{1}{L}\nabla f(\mathbf{y}))$ , then

$$F(\mathbf{y}^+) - F(\mathbf{x}) \leq \frac{L}{2} \|\mathbf{x} - \mathbf{y}\|_2^2 - \frac{L}{2} \|\mathbf{x} - \mathbf{y}^+\|_2^2 - g(\mathbf{x}, \mathbf{y})$$

where  $g(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle$ .

- Take  $\mathbf{x} = \mathbf{y} = \mathbf{x}_t$ , we get  $F(\mathbf{x}_{t+1}) \leq F(\mathbf{x}_t)$ .
- Take  $\mathbf{x} = \mathbf{x}^*$ ,  $\mathbf{y} = \mathbf{x}_t$ , we get  $\|\mathbf{x}_{t+1} - \mathbf{x}^*\|_2 \leq \|\mathbf{x}_t - \mathbf{x}^*\|_2$ .

## Proof

Define  $\phi(\mathbf{z}) = f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \frac{L}{2} \|\mathbf{z} - \mathbf{y}\|_2^2 + h(\mathbf{z})$ .

Three facts:

- ①  $\mathbf{y}^+ = \arg \min_{\mathbf{z}} \phi(\mathbf{z})$ ;
- ②  $\phi(\mathbf{z})$  is  $L$ -strongly convex, thus

$$\phi(\mathbf{x}) \geq \phi(\mathbf{y}^+) + \langle \nabla \phi(\mathbf{y}^+), \mathbf{x} - \mathbf{y}^+ \rangle + \frac{L}{2} \|\mathbf{x} - \mathbf{y}^+\|_2^2;$$

- ③  $f$  is  $L$ -smooth, thus

$$\begin{aligned}\phi(\mathbf{y}^+) &= f(\mathbf{y}) + \langle \nabla f(\mathbf{y}), \mathbf{y}^+ - \mathbf{y} \rangle + \frac{L}{2} \|\mathbf{y}^+ - \mathbf{y}\|_2^2 + h(\mathbf{y}^+) \\ &\geq f(\mathbf{y}^+) + h(\mathbf{y}^+) = F(\mathbf{y}^+)\end{aligned}$$

# Convergence for Convex Problems

Suppose  $f$  is convex and  $L$ -smooth. The proximal gradient descent with stepsize  $\eta_t = 1/L$  obeys

$$F(\mathbf{x}_t) - F(\mathbf{x}^*) \leq \frac{L \|\mathbf{x}_0 - \mathbf{x}^*\|_2^2}{2t}.$$

- Achieves better iteration complexity  $O(1/\varepsilon)$  than subgradient method  $O(1/\varepsilon^2)$ .

# Convergence for Strongly Convex Problems

Suppose  $f$  is  $\mu$ -strongly convex and  $L$ -smooth. The proximal gradient descent with stepsize  $\eta_t = 1/L$  obeys

$$\|\mathbf{x}_t - \mathbf{x}^*\|_2^2 \leq \left(1 - \frac{\mu}{L}\right)^t \|\mathbf{x}_0 - \mathbf{x}^*\|_2^2.$$

- Achieves linear convergence  $O(\kappa \log \frac{1}{\varepsilon})$ .

## Summary

condition	stepsize	convergence rate	iteration complexity
convex	$\eta_t = \frac{1}{L}$	$O\left(\frac{1}{t}\right)$	$O\left(\frac{1}{\varepsilon}\right)$
strongly convex	$\eta_t = \frac{1}{L}$	$O\left(\left(1 - \frac{1}{\kappa}\right)^t\right)$	$O(\kappa \log \frac{1}{\varepsilon})$

Table: Convergence Properties of Proximal Gradient Descent

condition	stepsize	convergence rate	iteration complexity
convex	$\eta_t \approx \frac{1}{\sqrt{t}}$	$O\left(\frac{1}{\sqrt{t}}\right)$	$O\left(\frac{1}{\varepsilon^2}\right)$
strongly convex	$\eta_t \approx \frac{1}{t}$	$O\left(\frac{1}{t}\right)$	$O\left(\frac{1}{\varepsilon}\right)$

Table: Convergence Properties of Subgradient Descent