Solution to Homework 5

Problem 1. Suppose f is a convex and differentiable function, $\mathcal{C}$ is a closed convex set. Show that

$$\mathbf{x}^* \in \arg\min_{\mathbf{x}\in\mathcal{C}} f(\mathbf{x}) \iff \langle -\nabla f(\mathbf{x}^*), \mathbf{z}-\mathbf{x}^*\rangle \le 0,\ \forall\ \mathbf{z}\in\mathcal{C}.$$

Solution.

$\Rightarrow$: We prove by contradiction. Suppose there exists $\mathbf{y}\in\mathcal{C}$ such that $\langle \nabla f(\mathbf{x}^*), \mathbf{y}-\mathbf{x}^*\rangle < 0$. Consider $\mathbf{z}(t) = t\mathbf{y} + (1-t)\mathbf{x}^*$ where $t\in[0,1]$ is a parameter. Since $\mathbf{x}^*, \mathbf{y}\in\mathcal{C}$, we know $\mathbf{z}(t)\in\mathcal{C}$. Since

$$\frac{\mathrm{d}}{\mathrm{d}\,t} f(\mathbf{z}(t))\bigg|_{t=0} = \langle \nabla f(\mathbf{x}^* + 0\cdot(\mathbf{y}-\mathbf{x}^*)), \mathbf{y}-\mathbf{x}^*\rangle = \langle \nabla f(\mathbf{x}^*), \mathbf{y}-\mathbf{x}^*\rangle < 0,$$

we can get for small positive t, $f(\mathbf{z}(t)) < f(\mathbf{x}^*)$.

$\Leftarrow$: Since $f(\mathbf{x})$ is convex, we have

$$f(\mathbf{x}^*) \le f(\mathbf{z}) + \langle -\nabla f(\mathbf{x}^*), \mathbf{z}-\mathbf{x}^*\rangle \le f(\mathbf{z}),\ \forall\ \mathbf{z}\in\mathcal{C}.$$

Problem 2. Consider the projected gradient descent algorithm introduced in the class. Suppose that for some iteration t, $\mathbf{x}_{t+1} = \mathbf{x}_t$. Prove that in this case, $\mathbf{x}_t$ is a minimizer of the convex objective function f over the closed and convex set $\mathcal{C}$.

Solution. Let $\mathbf{y}_{t+1} = \mathbf{x}_t - \eta_t\nabla f(\mathbf{x}_t)$, then we know $\mathbf{x}_{t+1} = \mathcal{P}_{\mathcal{C}}(\mathbf{y}_{t+1})$. According to property 1 in page 8 of the slides, we have

$$\langle \mathbf{x}-\mathbf{x}_{t+1}, \mathbf{y}_{t+1}-\mathbf{x}_{t+1}\rangle \le 0,\ \forall\ \mathbf{x}\in\mathcal{C}.$$

Since $\mathbf{x}_{t+1} = \mathbf{x}_t$ and $\mathbf{y}_{t+1} - \mathbf{x}_t = -\eta_t f(\mathbf{x}_t)$, we have

$$\langle \mathbf{x}-\mathbf{x}_t, -\eta_t f(\mathbf{x}_t)\rangle \le 0,\ \forall\ \mathbf{x}\in\mathcal{C}.$$

According to the problem 1, we know $\mathbf{x}_t \in \arg\min_{\mathbf{x}\in\mathcal{C}} f(\mathbf{x})$.

Problem 3. Let $\mathcal{C}\in\mathbb{R}^d$ be a nonempty closed and convex set, and let f be a strongly convex function over $\mathcal{C}$. Prove that f has a unique minimizer $\mathbf{x}^*$ over $\mathcal{C}$.

Solution. Suppose there are two minimizer $\mathbf{x}^*$ and $\mathbf{y}^*$. Since f is strongly convex, we have

$$f(\mathbf{y}^*) \ge f(\mathbf{x}^*) + \langle \nabla f(\mathbf{x}^*), \mathbf{y}^*-\mathbf{x}^*\rangle + \frac{\mu}{2}\left\|\mathbf{x}^*-\mathbf{y}^*\right\|_2^2.$$

Since $\mathbf{x}^*$ is a minimizer, according to the problem 1, we know $\langle -\nabla f(\mathbf{x}^*), \mathbf{y}^*-\mathbf{x}^*\rangle \le 0$. Thus

$$\frac{\mu}{2}\left\|\mathbf{x}^*-\mathbf{y}^*\right\|_2^2 \le \langle -\nabla f(\mathbf{x}^*), \mathbf{y}^*-\mathbf{x}^*\rangle + f(\mathbf{y}^*) - f(\mathbf{x}^*) \le 0.$$

which means $\mathbf{x}^* = \mathbf{y}^*$.