## 1. Boltzmann Equation

Boltzmann equation read as

$$\frac{\partial \mathscr{F}}{\partial t}+\sum_{i=1}^{3}\left(v_i\frac{\partial \mathscr{F}}{\partial x_i}+\dot{v}_i\frac{\partial \mathscr{F}}{\partial v_i}+\mathscr{F}\frac{\partial \dot{v}_i}{\partial v_i}\right)=\left(\frac{\partial \mathscr{F}}{\partial t}\right)_c. \tag{1}$$

$$\dot{v}_i=-\frac{\partial \Phi}{\partial x_i}+F_i \tag{2}$$

$$F_i=f\left(v_i-V_i\right)^s \tag{3}$$

$$\frac{\partial \mathscr{F}}{\partial t}+\sum_{i=1}^{3}\left(v_i\frac{\partial \mathscr{F}}{\partial x_i}+\left(-\frac{\partial \Phi}{\partial x_i}+f\left(v_i-V_i\right)^s\right)\frac{\partial \mathscr{F}}{\partial v_i}+\mathscr{F}fs\left(v_i-V_i\right)^{s-1}\right)=\left(\frac{\partial \mathscr{F}}{\partial t}\right)_c. \tag{4}$$

## 2. Moment Equation

$\int d\mathbf{v}$:

$$\frac{\partial \mathscr{N}}{\partial t}+\frac{\partial \mathscr{N}u_i}{\partial x_i}=\left(\frac{\partial \mathscr{N}}{\partial t}\right)_c \tag{5}$$

where$\mathscr{N}=\int fd\mathbf{v}$, $\mathscr{N}u_i=\int v_i\mathscr{F}d\mathbf{v}$

$\int v_i d\mathbf{v}$:

$$\frac{\partial \mathscr{N}u_i}{\partial t}+\frac{\partial \mathscr{N}\left\langle v_iv_j\right\rangle}{\partial x_j}+\mathscr{N}\frac{\partial \Phi}{\partial x_i}-f\mathscr{N}\left\langle\left(v_i-V_i\right)^s\right\rangle=\left(\frac{\partial \mathscr{N}\left\langle v_i\right\rangle}{\partial t}\right)_c \tag{6}$$

$\int v_iv_j d\mathbf{v}$ :

$$\frac{\partial \mathscr{N}\left\langle v_iv_j\right\rangle}{\partial t}+\frac{\partial \mathscr{N}\left\langle v_iv_jv_k\right\rangle}{\partial x_k}+\mathscr{N}u_j\frac{\partial \Phi}{\partial x_i}+\mathscr{N}u_i\frac{\partial \Phi}{\partial x_j}-\mathscr{N}f\left\langle\left(v_j-V_j\right)^s v_i\right\rangle-\mathscr{N}f\left\langle\left(v_i-V_i\right)^s v_j\right\rangle=\left(\frac{\partial \mathscr{N}\left\langle v_iv_j\right\rangle}{\partial t}\right)_c \tag{7}$$

where $\left\langle v_i\right\rangle=u_i$

Define: $P_{ij}=\mathscr{N}\left\langle v_iv_j\right\rangle-\mathscr{N}u_iu_j$ then the equation 6 could be written as

$$\frac{\partial \mathscr{N}u_i}{\partial t}+\frac{\partial P_{ij}+\mathscr{N}u_iu_j}{\partial x_j}+\mathscr{N}\frac{\partial \Phi}{\partial x_i}-f\mathscr{N}\left\langle\left(v_i-V_i\right)^s\right\rangle=\left(\frac{\partial \mathscr{N}\left\langle v_i\right\rangle}{\partial t}\right)_c \tag{8}$$

$$\frac{\partial \mathscr{N}u_i}{\partial t}+\frac{\partial P_{ij}}{\partial x_j}+\frac{\partial \mathscr{N}u_iu_j}{\partial x_j}+\mathscr{N}\frac{\partial \Phi}{\partial x_i}-f\mathscr{N}\left\langle\left(v_i-V_i\right)^s\right\rangle=\left(\frac{\partial \mathscr{N}\left\langle v_i\right\rangle}{\partial t}\right)_c \tag{9}$$

$$\mathscr{N}\frac{\partial u_i}{\partial t}+\frac{\partial \mathscr{N}}{\partial t}u_i+\frac{\partial P_{ij}}{\partial x_j}+\frac{\partial \mathscr{N}u_j}{\partial x_j}u_i+\mathscr{N}u_j\frac{\partial u_i}{\partial x_j}+\mathscr{N}\frac{\partial \Phi}{\partial x_i}-f\mathscr{N}\left\langle\left(v_i-V_i\right)^s\right\rangle=\left(\frac{\partial \mathscr{N}\left\langle v_i\right\rangle}{\partial t}\right)_c \tag{10}$$

$$\mathscr{N}\frac{\partial u_i}{\partial t}+\frac{\partial P_{ij}}{\partial x_j}+\left(\frac{\partial \mathscr{N}}{\partial t}\right)_c u_i+\mathscr{N}u_j\frac{\partial u_i}{\partial x_j}+\mathscr{N}\frac{\partial \Phi}{\partial x_i}-f\mathscr{N}\left\langle (v_i-V_i)^s\right\rangle=\left(\frac{\partial \mathscr{N}\left\langle v_i\right\rangle}{\partial t}\right)_c \tag{11}$$

$$\mathscr{N}\frac{\partial u_i}{\partial t}+\frac{\partial P_{ij}}{\partial x_j}+\mathscr{N}u_j\frac{\partial u_i}{\partial x_j}+\mathscr{N}\frac{\partial \Phi}{\partial x_i}=\left(\frac{\partial \mathscr{N}u_i}{\partial t}\right)_c+f\mathscr{N}\left\langle (v_i-V_i)^s\right\rangle-\left(\frac{\partial \mathscr{N}}{\partial t}\right)_c u_i \tag{12}$$

Then equation 7:

$$\frac{\partial\left(P_{ij}+\mathscr{N}u_iu_j\right)}{\partial t}+\frac{\partial\left(P_{ijk}+u_iP_{jk}+u_jP_{ki}+u_kP_{ij}+\mathscr{N}u_iu_ju_k\right)}{\partial x_k}+\mathscr{N}u_i\frac{\partial \Phi}{\partial x_j}+\mathscr{N}u_j\frac{\partial \Phi}{\partial x_i} \tag{13}$$

$$-\mathscr{N}f\left\langle (v_i-V_i)^s v_j\right\rangle-\mathscr{N}f\left\langle (v_j-V_j)^s v_i\right\rangle=\left(\frac{\partial \mathscr{N}\left\langle v_iv_j\right\rangle}{\partial t}\right)_c \tag{14}$$

where

$$P_{ijk}+u_iP_{jk}+u_jP_{ki}+u_kP_{ij}+\mathscr{N}u_iu_ju_k=\mathscr{N}\left\langle v_iv_jv_k\right\rangle \tag{15}$$

could be checked easily.

$$\begin{aligned}
\frac{\partial P_{ij}}{\partial t}+\frac{\partial P_{ijk}}{\partial x_k}+u_iu_j\frac{\partial \mathscr{N}}{\partial t} \quad &+ \quad u_iu_j\frac{\partial \mathscr{N}u_k}{\partial x_k}+P_{ik}\frac{\partial u_j}{\partial x_k}+P_{jk}\frac{\partial u_i}{\partial x_k}+P_{ij}\frac{\partial u_k}{\partial x_k}+u_k\frac{\partial P_{ij}}{\partial x_k} \\
&+ \quad u_i\left(\left(\frac{\partial \mathscr{N}u_j}{\partial t}\right)_c+f\mathscr{N}\left\langle (v_j-V_j)^s\right\rangle-\left(\frac{\partial \mathscr{N}}{\partial t}\right)_c u_j\right) \qquad (16)\\
&+ \quad u_j\left(\left(\frac{\partial \mathscr{N}u_i}{\partial t}\right)_c+f\mathscr{N}\left\langle (v_i-V_i)^s\right\rangle-\left(\frac{\partial \mathscr{N}}{\partial t}\right)_c u_i\right) \qquad (17)\\
&- \quad \mathscr{N}f\left\langle (v_i-V_i)^s v_j\right\rangle-\mathscr{N}f\left\langle (v_j-V_j)^s v_i\right\rangle \qquad (18)\\
&= \quad \left(\frac{\partial P_{ij}+\mathscr{N}u_iu_j}{\partial t}\right)_c \qquad (19)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial P_{ij}}{\partial t}+\frac{\partial P_{ijk}}{\partial x_k} \quad &+ \quad u_iu_j\left(\frac{\partial \mathscr{N}}{\partial t}\right)_c+P_{ik}\frac{\partial u_j}{\partial x_k}+P_{jk}\frac{\partial u_i}{\partial x_k}+P_{ij}\frac{\partial u_k}{\partial x_k}+u_k\frac{\partial P_{ij}}{\partial x_k} \\
&+ \quad u_i\left(\left(\frac{\partial \mathscr{N}u_j}{\partial t}\right)_c+f\mathscr{N}\left\langle (v_j-V_j)^s\right\rangle-\left(\frac{\partial \mathscr{N}}{\partial t}\right)_c u_j\right) \qquad (20)\\
&+ \quad u_j\left(\left(\frac{\partial \mathscr{N}u_i}{\partial t}\right)_c+f\mathscr{N}\left\langle (v_i-V_i)^s\right\rangle-\left(\frac{\partial \mathscr{N}}{\partial t}\right)_c u_i\right) \qquad (21)\\
&- \quad \mathscr{N}f\left\langle (v_i-V_i)^s v_j\right\rangle-\mathscr{N}f\left\langle (v_j-V_j)^s v_i\right\rangle \qquad (22)\\
&= \quad \left(\frac{\partial P_{ij}+\mathscr{N}u_iu_j}{\partial t}\right)_c \qquad (23)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial P_{ij}}{\partial t}+\frac{\partial P_{ijk}}{\partial x_k} &+ P_{ik}\frac{\partial u_j}{\partial x_k}+P_{jk}\frac{\partial u_i}{\partial x_k}+P_{ij}\frac{\partial u_k}{\partial x_k}+u_k\frac{\partial P_{ij}}{\partial x_k}+u_i f\mathscr{N}\left\langle (v_j-V_j)^s\right\rangle \\
&+ u_j f\mathscr{N}\left\langle (v_i-V_i)^s\right\rangle-\mathscr{N}f\left\langle (v_i-V_i)^s v_j\right\rangle-\mathscr{N}f\left\langle (v_j-V_j)^s v_i\right\rangle \\
&= \left(\frac{\partial P_{ij}+\mathscr{N}u_iu_j}{\partial t}\right)_c-u_iu_j\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c-u_i\left(\frac{\partial\mathscr{N}u_j}{\partial t}\right)_c \\
&\quad -u_j\left(\frac{\partial\mathscr{N}u_i}{\partial t}\right)_c+u_i\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c u_j+u_j\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c u_i \qquad (24)
\end{aligned}
$$

for the right side is

$$
\begin{aligned}
\left(\frac{\partial P_{ij}+\mathscr{N}u_iu_j}{\partial t}\right)_c &- u_iu_j\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c-u_i\left(\frac{\partial\mathscr{N}u_j}{\partial t}\right)_c-u_j\left(\frac{\partial\mathscr{N}u_i}{\partial t}\right)_c \qquad (25) \\
&+ u_i\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c u_j+u_j\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c u_i \qquad (26) \\
&= \left(\frac{\partial P_{ij}}{\partial t}\right)_c+\left(\frac{\partial\mathscr{N}u_iu_j}{\partial t}\right)_c-u_iu_j\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c-u_i\left(\frac{\partial\mathscr{N}u_j}{\partial t}\right)_c-u_j\left(\frac{\partial\mathscr{N}u_i}{\partial t}\right)_c \qquad (27) \\
&+ u_i\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c u_j+u_j\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c u_i \qquad (28) \\
&= \left(\frac{\partial P_{ij}}{\partial t}\right)_c \qquad (29)
\end{aligned}
$$

$\therefore$

$$
\begin{aligned}
\frac{\partial P_{ij}}{\partial t}+\frac{\partial P_{ijk}}{\partial x_k} &+ P_{ik}\frac{\partial u_j}{\partial x_k}+P_{jk}\frac{\partial u_i}{\partial x_k}+P_{ij}\frac{\partial u_k}{\partial x_k}+u_k\frac{\partial P_{ij}}{\partial x_k}+u_i f\mathscr{N}\left\langle (v_j-V_j)^s\right\rangle \\
&+ u_j f\mathscr{N}\left\langle (v_i-V_i)^s\right\rangle-\mathscr{N}f\left\langle (v_i-V_i)^s v_j\right\rangle-\mathscr{N}f\left\langle (v_j-V_j)^s v_i\right\rangle \qquad (30) \\
&= \left(\frac{\partial P_{ij}}{\partial t}\right)_c \qquad (31)
\end{aligned}
$$

## 3. Final Form of Equation

$$
\frac{\partial\mathscr{N}}{\partial t}+\frac{\partial\mathscr{N}u_i}{\partial x_i}=\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c \tag{32}
$$

$$
\mathscr{N}\frac{\partial u_i}{\partial t}+\frac{\partial P_{ij}}{\partial x_j}+\mathscr{N}u_j\frac{\partial u_i}{\partial x_j}+\mathscr{N}\frac{\partial\Phi}{\partial x_i}=\left(\frac{\partial\mathscr{N}u_i}{\partial t}\right)_c+f\mathscr{N}\left\langle (v_i-V_i)^s\right\rangle-\left(\frac{\partial\mathscr{N}}{\partial t}\right)_c u_i \tag{33}
$$

$$
\begin{aligned}
\frac{\partial P_{ij}}{\partial t}+\frac{\partial P_{ijk}}{\partial x_k} &+ P_{ik}\frac{\partial u_j}{\partial x_k}+P_{jk}\frac{\partial u_i}{\partial x_k}+P_{ij}\frac{\partial u_k}{\partial x_k}+u_k\frac{\partial P_{ij}}{\partial x_k}+u_i f\mathscr{N}\left\langle (v_j-V_j)^s\right\rangle \\
&+ u_j f\mathscr{N}\left\langle (v_i-V_i)^s\right\rangle-\mathscr{N}f\left\langle (v_i-V_i)^s v_j\right\rangle-\mathscr{N}f\left\langle (v_j-V_j)^s v_i\right\rangle \\
&= \left(\frac{\partial P_{ij}}{\partial t}\right)_c \qquad (34)
\end{aligned}
$$

Assumption : $\frac{\partial P_{ijk}}{\partial x_k} = 0$, then

$$\begin{aligned}\frac{\partial P_{ij}}{\partial t} + P_{ik}\frac{\partial u_j}{\partial x_k} &+ P_{jk}\frac{\partial u_i}{\partial x_k} + \frac{\partial (u_k P_{ij})}{\partial x_k} + u_i f \mathscr{N} \langle (v_j - V_j)^s \rangle + u_j f \mathscr{N} \langle (v_i - V_i)^s \rangle \\ &- \mathscr{N} f \langle (v_i - V_i)^s v_j \rangle - \mathscr{N} f \langle (v_j - V_j)^s v_i \rangle = \left(\frac{\partial P_{ij}}{\partial t}\right)_c \end{aligned} \tag{35}$$

## 4. Cylindrical

Drag force on a sphere moving slowly in hot gas could be described by Oseen's formula[1]

$$F_{drag} = -6\pi\mu u a \left(1 + \frac{3}{8} Re\right) \tag{36}$$

where $Re = \rho u a/\mu$ is Reynolds number. We can see in the limit of low Reynolds number fluid, index $s$ in equation 3 could be chosen as 1 while for a higher Reynolds fluid, $s = 2$ is more realistic.

Before we perform our cylindrical description, we give the two choose about index $s$. In the following text, I give the derivation about the case $s = 2$, and leave $s = 1$ in appendix.

### *4.1. For $s = 2$*

$$u_i f \mathscr{N} \langle (v_j - V_j)^s \rangle = u_i f \mathscr{N} \left\langle (v_j - V_j)^2 \right\rangle = u_i f \mathscr{N} \left\langle v_j^2 - 2v_j V_j + V_j^2 \right\rangle \tag{37}$$

$$= u_i f \mathscr{N} \left\langle v_j^2 \right\rangle - u_i f \mathscr{N} \langle 2v_j V_j \rangle + u_i f \mathscr{N} \left\langle V_j^2 \right\rangle = u_i f \mathscr{N} \left\langle v_j^2 \right\rangle - u_i f \mathscr{N} 2 \langle v_j \rangle V_j + u_i f \mathscr{N} V_j^2 \tag{38}$$

$$= u_i f (P_{jj} + \mathscr{N} u_j u_j) - u_i f \mathscr{N} 2u_j V_j + u_i f \mathscr{N} V_j^2 = u_i f P_{jj} + u_i f \mathscr{N} u_j u_j - u_i f \mathscr{N} 2u_j V_j + u_i f \mathscr{N} V_j^2 \tag{39}$$

and

$$\mathscr{N} f \left\langle (v_i - V_i)^2 v_j \right\rangle = \mathscr{N} f \left\langle \left(v_i^2 - 2v_i V_i + V_i^2\right) v_j \right\rangle = \mathscr{N} f \left\langle v_i^2 v_j - 2v_i v_j V_i + v_j V_i^2 \right\rangle \tag{40}$$

$$= \mathscr{N} f \left\langle v_i^2 v_j \right\rangle - \mathscr{N} f \langle 2v_i v_j V_i \rangle + \mathscr{N} f \left\langle v_j V_i^2 \right\rangle = f \mathscr{N} \left\langle v_i^2 v_j \right\rangle - 2f V_i \mathscr{N} \langle v_i v_j \rangle + f V_i^2 \mathscr{N} \langle v_j \rangle \tag{41}$$

$$= f \mathscr{N} \left\langle v_i^2 v_j \right\rangle - 2f V_i \mathscr{N} \langle v_i v_j \rangle + f V_i^2 \mathscr{N} u_j \tag{42}$$

$$= f (P_{iij} + u_i P_{ij} + u_i P_{ji} + u_j P_{ii} + \mathscr{N} u_i u_i u_j) - 2f V_i (P_{ij} + \mathscr{N} u_i u_j) + f V_i^2 \mathscr{N} u_j \tag{43}$$

[1]Landau, Fluid Mechanics, Page 62

where $P_{ijk} + u_i P_{jk} + u_j P_{ki} + u_k P_{ij} + \mathscr{N} u_i u_j u_k = \mathscr{N} \langle v_i v_j v_k \rangle$ is used.

So

$$u_i f \mathscr{N} \langle (v_j - V_j)^s \rangle = u_i f P_{jj} + u_i f \mathscr{N} u_j u_j - u_i f \mathscr{N} 2 u_j V_j + u_i f \mathscr{N} V_j^2 \tag{44}$$

$$u_j f \mathscr{N} \langle (v_i - V_i)^s \rangle = u_j f P_{ii} + u_j f \mathscr{N} u_i u_i - u_j f \mathscr{N} 2 u_i V_i + u_j f \mathscr{N} V_i^2 \tag{45}$$

$$\mathscr{N} f \left\langle (v_i - V_i)^2 v_j \right\rangle = f \left(P_{iij} + u_i P_{ij} + u_i P_{ji} + u_j P_{ii} + \mathscr{N} u_i u_i u_j\right) - 2 f V_i \left(P_{ij} + \mathscr{N} u_i u_j\right) + f V_i^2 \mathscr{N} u_j \tag{46}$$

$$\mathscr{N} f \left\langle (v_j - V_j)^2 v_i \right\rangle = f \left(P_{jji} + u_j P_{ji} + u_j P_{ij} + u_i P_{jj} + \mathscr{N} u_j u_j u_i\right) - 2 f V_j \left(P_{ji} + \mathscr{N} u_j u_i\right) + f V_j^2 \mathscr{N} u_i \tag{47}$$

Then we have:

$$u_i f \mathscr{N} \langle (v_j - V_j)^s \rangle + u_j f \mathscr{N} \langle (v_i - V_i)^s \rangle - \mathscr{N} f \langle (v_i - V_i)^s v_j \rangle - \mathscr{N} f \langle (v_j - V_j)^s v_i \rangle \tag{48}$$

$$\xlongequal{s=2} -f P_{iij} - f u_i P_{ij} - f u_i P_{ji} + 2 f V_i P_{ij} - f P_{jji} - f u_j P_{ji} - f u_j P_{ij} + 2 f V_j P_{ji} \tag{49}$$

$$\xlongequal{P_{ij}=P_{ji}} -f P_{iij} - 2 f u_i P_{ij} + 2 f V_i P_{ij} - f P_{jji} - 2 f u_j P_{ij} + 2 f V_j P_{ij} \tag{50}$$

$$= -f P_{iij} - f P_{jji} + 2 f P_{ij} \left(V_i + V_j - u_i - u_j\right) \tag{51}$$

By this $s = 2$ assumption:

$$\frac{\partial P_{ij}}{\partial t} + P_{ik} \frac{\partial u_j}{\partial x_k} + P_{jk} \frac{\partial u_i}{\partial x_k} + \frac{\partial \left(u_k P_{ij}\right)}{\partial x_k} - f P_{iij} - f P_{jji} + 2 f P_{ij} \left(V_i + V_j - u_i - u_j\right) = \left(\frac{\partial P_{ij}}{\partial t}\right)_c$$

*4.2. DERIVATION: From Cartesian to Cylindrical*

Here we collapse this equation:

$$\frac{\partial p_{\alpha\beta}}{\partial t} + p_{\alpha\gamma} \frac{\partial u_\beta}{\partial x_\gamma} + p_{\beta\gamma} \frac{\partial u_\alpha}{\partial x_\gamma} + \frac{\partial}{\partial x_\gamma} \left(p_{\alpha\beta} u_\gamma\right) = \left(\frac{\partial p_{\alpha\beta}}{\partial t}\right)_c \tag{52}$$

into cylindrical coordinate.

The first term $\frac{\partial p_{\alpha\beta}}{\partial t}$ is the coefficient of $e_\alpha e_\beta$

The second term is a partial derivative about vector $u_\beta$.

$$P_{\alpha\gamma} \frac{\partial u_\beta}{\partial x_\gamma} = \mathbf{e}_\alpha P_{\alpha\gamma} \frac{\partial u_\beta \mathbf{e}_\beta}{\partial x_\gamma} = \mathbf{e}_\alpha P_{\alpha\gamma} \frac{\partial u_\beta}{\partial x_\gamma} \mathbf{e}_\beta + \mathbf{e}_\alpha P_{\alpha\gamma} u_\beta \frac{\partial \mathbf{e}_\beta}{\partial x_\gamma} \tag{53}$$

term $\frac{\partial \mathbf{e}_\beta}{\partial x_\gamma}$ is taken into account for the gradient of the base of cylindrical coordinate is non-trivial for

$$\frac{\partial \mathbf{e}_R}{\partial \theta} = \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \theta} = -\mathbf{e}_R \tag{54}$$

$$\mathbf{e}_\alpha P_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma}\mathbf{e}_\beta = \mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_\beta}{\partial R}\mathbf{e}_\beta + \mathbf{e}_\alpha P_{\alpha\theta}\frac{\partial u_\beta}{R\partial\theta}\mathbf{e}_\beta + \mathbf{e}_\alpha P_{\alpha z}\frac{\partial u_\beta}{\partial z}\mathbf{e}_\beta \tag{55}$$

If we take $\mathbf{u}$ only the function of $R$, so the only non-zero differential is $\partial_R$, so

$$\begin{aligned}
\mathbf{e}_\alpha P_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma}\mathbf{e}_\beta &= \mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_\beta}{\partial R}\mathbf{e}_\beta + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha\theta}\frac{\partial u_\beta}{R\partial\theta}\mathbf{e}_\beta} + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha z}\frac{\partial u_\beta}{\partial z}\mathbf{e}_\beta} && (56)\\
&= \mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_\beta}{\partial R}\mathbf{e}_\beta && (57)\\
&= \mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_R}{\partial R}\mathbf{e}_R + \mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_z}{\partial R}\mathbf{e}_z} && (58)\\
&= \mathbf{e}_R P_{RR}\frac{\partial u_R}{\partial R}\mathbf{e}_R + \mathbf{e}_\theta P_{\theta R}\frac{\partial u_R}{\partial R}\mathbf{e}_R + \cancelto{0}{\mathbf{e}_z P_{zR}\frac{\partial u_R}{\partial R}\mathbf{e}_R} && (59)\\
&\quad + \mathbf{e}_R P_{RR}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta + \mathbf{e}_\theta P_{\theta R}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta + \cancelto{0}{\mathbf{e}_z P_{zR}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta} && (60)
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_\alpha P_{\alpha\gamma} u_\beta \frac{\partial \mathbf{e}_\beta}{\partial x_\gamma} &= \mathbf{e}_\alpha P_{\alpha\gamma} u_R \frac{\partial \mathbf{e}_R}{\partial x_\gamma} + \mathbf{e}_\alpha P_{\alpha\gamma} u_\theta \frac{\partial \mathbf{e}_\theta}{\partial x_\gamma} + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha\gamma} u_z \frac{\partial \mathbf{e}_z}{\partial x_\gamma}} && (61)\\
&= \cancelto{0}{\mathbf{e}_\alpha P_{\alpha R} u_R \frac{\partial \mathbf{e}_R}{\partial R}} + \mathbf{e}_\alpha P_{\alpha\theta} u_R \frac{\partial \mathbf{e}_R}{R\partial\theta} + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha z} u_R \frac{\partial \mathbf{e}_R}{\partial z}} && (62)\\
&\quad + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha R} u_\theta \frac{\partial \mathbf{e}_\theta}{\partial R}} + \mathbf{e}_\alpha P_{\alpha\theta} u_\theta \frac{\partial \mathbf{e}_\theta}{R\partial\theta} + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha z} u_\theta \frac{\partial \mathbf{e}_\theta}{\partial z}} && (63)\\
&= \mathbf{e}_\alpha P_{\alpha\theta}\frac{u_R}{R}\mathbf{e}_\theta - \mathbf{e}_\alpha P_{\alpha\theta}\Omega\mathbf{e}_R && (64)\\
&= \mathbf{e}_R P_{R\theta}\frac{u_R}{R}\mathbf{e}_\theta + \mathbf{e}_\theta P_{\theta\theta}\frac{u_R}{R}\mathbf{e}_\theta + \cancelto{0}{\mathbf{e}_z P_{z\theta}\frac{u_R}{R}\mathbf{e}_\theta} && (65)\\
&\quad - \mathbf{e}_R P_{R\theta}\Omega\mathbf{e}_R - \mathbf{e}_\theta P_{\theta\theta}\Omega\mathbf{e}_R - \cancelto{0}{\mathbf{e}_z P_{z\theta}\Omega\mathbf{e}_R} && (66)
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_\alpha P_{\alpha\gamma}\frac{\partial u_\beta \mathbf{e}_\beta}{\partial x_\gamma} &= \mathbf{e}_R P_{RR}\frac{\partial u_R}{\partial R}\mathbf{e}_R + \mathbf{e}_\theta P_{\theta R}\frac{\partial u_R}{\partial R}\mathbf{e}_R + \mathbf{e}_R P_{R\theta}\frac{u_R}{R}\mathbf{e}_\theta + \mathbf{e}_\theta P_{\theta\theta}\frac{u_R}{R}\mathbf{e}_\theta\\
&\quad + \mathbf{e}_R P_{RR}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta + \mathbf{e}_\theta P_{\theta R}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta - \mathbf{e}_R P_{R\theta}\Omega\mathbf{e}_R - \mathbf{e}_\theta P_{\theta\theta}\Omega\mathbf{e}_R\\
&= \begin{pmatrix} P_{RR}\frac{\partial u_R}{\partial R} - P_{R\theta}\Omega & P_{R\theta}\frac{u_R}{R} + P_{RR}\frac{\partial u_\theta}{\partial R} & 0\\ P_{\theta R}\frac{\partial u_R}{\partial R} - P_{\theta\theta}\Omega & P_{\theta\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_\theta}{\partial R} & 0\\ 0 & 0 & 0 \end{pmatrix} && (67)
\end{aligned}$$

The third and second term of equation 35 is transposed to each other. So

$$P_{\beta\gamma}\frac{\partial u_\alpha}{\partial x_\gamma} = \begin{pmatrix} P_{RR}\frac{\partial u_R}{\partial R} - P_{R\theta}\Omega & P_{\theta R}\frac{\partial u_R}{\partial R} - P_{\theta\theta}\Omega & 0 \\ P_{R\theta}\frac{u_R}{R} + P_{RR}\frac{\partial u_\theta}{\partial R} & P_{\theta\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_\theta}{\partial R} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{68}$$

The forth term in equation 35

$$\begin{aligned}
\frac{\partial}{\partial x_\gamma}\left(p_{\alpha\beta}u_\gamma\right) &= \frac{\partial}{\partial x_\gamma}\left(p_{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta u_\gamma\right) = \frac{\partial p_{\alpha\beta}u_\gamma}{\partial x_\gamma}\mathbf{e}_\alpha\mathbf{e}_\beta + p_{\alpha\beta}u_\gamma\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{\partial x_\gamma} && (69)\\
&= \frac{\partial p_{\alpha\beta}u_R}{\partial R}\mathbf{e}_\alpha\mathbf{e}_\beta + \cancelto{0}{\frac{\partial p_{\alpha\beta}u_\theta}{R\partial\theta}\mathbf{e}_\alpha\mathbf{e}_\beta} + \cancelto{0}{\frac{\partial p_{\alpha\beta}u_z}{\partial z}\mathbf{e}_\alpha\mathbf{e}_\beta} && (70)\\
&\quad + \cancelto{0}{p_{\alpha\beta}u_R\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{\partial R}} + p_{\alpha\beta}u_\theta\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{R\partial\theta} + \cancelto{0}{p_{\alpha\beta}u_z\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{\partial z}} && (71)\\
&= \frac{\partial p_{\alpha\beta}u_R}{\partial R}\mathbf{e}_\alpha\mathbf{e}_\beta + p_{\alpha\beta}\Omega\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{\partial\theta} && (72)\\
&= \frac{\partial p_{\alpha\beta}u_R}{\partial R}\mathbf{e}_\alpha\mathbf{e}_\beta + p_{\alpha\beta}\Omega\frac{\partial \mathbf{e}_\alpha}{\partial\theta}\mathbf{e}_\beta + p_{\alpha\beta}\Omega\mathbf{e}_\alpha\frac{\partial \mathbf{e}_\beta}{\partial\theta} && (73)\\
&= \frac{\partial p_{RR}u_R}{\partial R}\mathbf{e}_R\mathbf{e}_R + \frac{\partial p_{R\theta}u_R}{\partial R}\mathbf{e}_R\mathbf{e}_\theta + \frac{\partial p_{\theta R}u_R}{\partial R}\mathbf{e}_\theta\mathbf{e}_R + \frac{\partial p_{\theta\theta}u_R}{\partial R}\mathbf{e}_\theta\mathbf{e}_\theta + \frac{\partial p_{zz}u_R}{\partial R}\mathbf{e}_z\mathbf{e}_z\\
&\quad + p_{R\beta}\Omega\frac{\partial \mathbf{e}_R}{\partial\theta}\mathbf{e}_\beta + p_{\theta\beta}\Omega\frac{\partial \mathbf{e}_\theta}{\partial\theta}\mathbf{e}_\beta + p_{\alpha R}\Omega\mathbf{e}_\alpha\frac{\partial \mathbf{e}_R}{\partial\theta} + p_{\alpha\theta}\Omega\mathbf{e}_\alpha\frac{\partial \mathbf{e}_\theta}{\partial\theta}\\
&= \frac{\partial p_{RR}u_R}{\partial R}\mathbf{e}_R\mathbf{e}_R + \frac{\partial p_{R\theta}u_R}{\partial R}\mathbf{e}_R\mathbf{e}_\theta + \frac{\partial p_{\theta R}u_R}{\partial R}\mathbf{e}_\theta\mathbf{e}_R + \frac{\partial p_{\theta\theta}u_R}{\partial R}\mathbf{e}_\theta\mathbf{e}_\theta + \frac{\partial p_{zz}u_R}{\partial R}\mathbf{e}_z\mathbf{e}_z\\
&\quad + p_{R\beta}\Omega\mathbf{e}_\theta\mathbf{e}_\beta - p_{\theta\beta}\Omega\mathbf{e}_R\mathbf{e}_\beta + p_{\alpha R}\Omega\mathbf{e}_\alpha\mathbf{e}_\theta - p_{\alpha\theta}\Omega\mathbf{e}_\alpha\mathbf{e}_R\\
&= \frac{\partial p_{RR}u_R}{\partial R}\mathbf{e}_R\mathbf{e}_R + \frac{\partial p_{R\theta}u_R}{\partial R}\mathbf{e}_R\mathbf{e}_\theta + \frac{\partial p_{\theta R}u_R}{\partial R}\mathbf{e}_\theta\mathbf{e}_R + \frac{\partial p_{\theta\theta}u_R}{\partial R}\mathbf{e}_\theta\mathbf{e}_\theta + \frac{\partial p_{zz}u_R}{\partial R}\mathbf{e}_z\mathbf{e}_z\\
&\quad + p_{RR}\Omega\mathbf{e}_\theta\mathbf{e}_R + p_{R\theta}\Omega\mathbf{e}_\theta\mathbf{e}_\theta - p_{\theta R}\Omega\mathbf{e}_R\mathbf{e}_R - p_{\theta\theta}\Omega\mathbf{e}_R\mathbf{e}_\theta\\
&\quad + p_{RR}\Omega\mathbf{e}_R\mathbf{e}_\theta + p_{\theta R}\Omega\mathbf{e}_\theta\mathbf{e}_\theta - p_{R\theta}\Omega\mathbf{e}_R\mathbf{e}_R - p_{\theta\theta}\Omega\mathbf{e}_\theta\mathbf{e}_R\\
&= \begin{pmatrix} \frac{\partial P_{RR}u_R}{\partial R} - 2P_{R\theta}\Omega & \frac{\partial P_{R\theta}u_R}{\partial R} + P_{RR}\Omega - P_{\theta\theta}\Omega & 0 \\ \frac{\partial P_{\theta R}u_R}{\partial R} + P_{RR}\Omega - P_{\theta\theta}\Omega & \frac{\partial P_{\theta\theta}u_R}{\partial R} + 2P_{\theta R}\Omega & 0 \\ 0 & 0 & \frac{\partial P_{zz}u_R}{\partial R} \end{pmatrix}
\end{aligned}$$

For stabile case, i.e. $\partial_t = 0$, we have the left side of equation 35

$$p_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma} + p_{\beta\gamma}\frac{\partial u_\alpha}{\partial x_\gamma} + \frac{\partial}{\partial x_\gamma}\left(p_{\alpha\beta}u_\gamma\right) \tag{74}$$

$$\begin{aligned}
= &\begin{pmatrix} P_{RR}\frac{\partial u_R}{\partial R} - P_{R\theta}\Omega & P_{R\theta}\frac{u_R}{R} + P_{RR}\frac{\partial u_\theta}{\partial R} & 0 \\ P_{\theta R}\frac{\partial u_R}{\partial R} - P_{\theta\theta}\Omega & P_{\theta\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_\theta}{\partial R} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} P_{RR}\frac{\partial u_R}{\partial R} - P_{R\theta}\Omega & P_{\theta R}\frac{\partial u_R}{\partial R} - P_{\theta\theta}\Omega & 0 \\ P_{R\theta}\frac{u_R}{R} + P_{RR}\frac{\partial u_\theta}{\partial R} & P_{\theta\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_\theta}{\partial R} & 0 \\ 0 & 0 & 0 \end{pmatrix}\\
&+ \begin{pmatrix} \frac{\partial P_{RR}u_R}{\partial R} - 2P_{R\theta}\Omega & \frac{\partial P_{R\theta}u_R}{\partial R} + P_{RR}\Omega - P_{\theta\theta}\Omega & 0 \\ \frac{\partial P_{\theta R}u_R}{\partial R} + P_{RR}\Omega - P_{\theta\theta}\Omega & \frac{\partial P_{\theta\theta}u_R}{\partial R} + 2P_{\theta R}\Omega & 0 \\ 0 & 0 & \frac{\partial P_{zz}u_R}{\partial R} \end{pmatrix} =
\end{aligned}$$

$$\left( \begin{array}{cc} 2P_{RR}\frac{\partial u_R}{\partial R} + \frac{\partial P_{RR}u_R}{\partial R} - 4P_{R\theta}\Omega, & P_{R\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_R}{\partial R} + \frac{\partial P_{R\theta}u_R}{\partial R} + P_{RR}\frac{\partial u_\theta}{\partial R} + P_{RR}\Omega - 2P_{\theta\theta}\Omega \\ P_{\theta R}\frac{\partial u_R}{\partial R} + P_{R\theta}\frac{u_R}{R} + P_{RR}\frac{\partial u_\theta}{\partial R} + \frac{\partial P_{\theta R}u_R}{\partial R} + P_{RR}\Omega - 2P_{\theta\theta}\Omega & 2P_{\theta\theta}\frac{u_R}{R} + \frac{\partial P_{\theta\theta}u_R}{\partial R} + 2P_{\theta R}\frac{\partial u_\theta}{\partial R} + 2P_{\theta R}\Omega \\ 0 & 0 \end{array} \right. \tag{75}$$

$$= \left( \begin{array}{ccc} 2P_{RR}\frac{\partial u_R}{\partial R} + \frac{\partial P_{RR}u_R}{\partial R} - 4P_{R\theta}\Omega, & P_{R\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_R}{\partial R} + \frac{\partial P_{R\theta}u_R}{\partial R} + \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta}, & 0 \\ P_{R\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_R}{\partial R} + \frac{\partial P_{R\theta}u_R}{\partial R} + \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta}, & 2P_{\theta\theta}\frac{u_R}{R} + \frac{\partial P_{\theta\theta}u_R}{\partial R} + \frac{2P_{R\theta}}{R}\frac{dR^2\Omega}{dR}, & 0 \\ 0 & 0 & \frac{\partial P_{zz}u_R}{\partial R} \end{array} \right. \tag{76}$$

equal above matrix to the right side of equation 35

$$= \begin{pmatrix} \left(\frac{\partial P_{RR}}{\partial t}\right)_c, & \left(\frac{\partial P_{R\theta}}{\partial t}\right)_c, & 0 \\ \left(\frac{\partial P_{\theta R}}{\partial t}\right)_c, & \left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c, & 0 \\ 0 & 0 & \left(\frac{\partial P_{zz}}{\partial t}\right)_c \end{pmatrix} \tag{77}$$

An easier looking edition for these equations is

$$2P_{RR}\frac{\partial u_R}{\partial R} + \frac{\partial P_{RR}u_R}{\partial R} - 4P_{R\theta}\Omega = \left(\frac{\partial P_{RR}}{\partial t}\right)_c \tag{78}$$

$$P_{R\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_R}{\partial R} + \frac{\partial P_{R\theta}u_R}{\partial R} + \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta} = \left(\frac{\partial P_{R\theta}}{\partial t}\right)_c \tag{79}$$

$$2P_{\theta\theta}\frac{u_R}{R} + \frac{\partial P_{\theta\theta}u_R}{\partial R} + \frac{2P_{R\theta}}{R}\frac{dR^2\Omega}{dR} = \left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c \tag{80}$$

$$\frac{\partial P_{zz}u_R}{\partial R} = \left(\frac{\partial P_{zz}}{\partial t}\right)_c \tag{81}$$

### *4.3. Cylindrical Description*

From appendix section, we can see if we choose the ram pressure index $s = 1$, effect of the friction is only involved with $f$

As has been derived in previous subsection,

$$P_{ik}\frac{\partial u_j}{\partial x_k} + P_{jk}\frac{\partial u_i}{\partial x_k} + \frac{\partial\left(u_k P_{ij}\right)}{\partial x_k} = \tag{82}$$

$$\begin{pmatrix} 2P_{RR}\frac{\partial u_R}{\partial R} + \frac{\partial P_{RR}u_R}{\partial R} - 4P_{R\theta}\Omega, & P_{R\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_R}{\partial R} + \frac{\partial P_{R\theta}u_R}{\partial R} + \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta}, & 0 \\ P_{R\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_R}{\partial R} + \frac{\partial P_{R\theta}u_R}{\partial R} + \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta}, & 2P_{\theta\theta}\frac{u_R}{R} + \frac{\partial P_{\theta\theta}u_R}{\partial R} + \frac{2P_{R\theta}}{R}\frac{dR^2\Omega}{dR}, & 0 \\ 0 & 0 & \frac{\partial P_{zz}u_R}{\partial R} \end{pmatrix} \tag{83}$$

for $s = 2$, follow the paper by Shu, F:

These authors start with the Boltzmann equation, modify it to allow collisions between identical spherical particles to be inelastic, close the hierarchy of moment equations by ignoring third-order moments, and evaluate the effect of inelastic collisions in changing the second-order moments under the assumption that the velocity distribution is a triaxial Gaussian whose principal axes may be tilted with respect to the natural configuration axes of the system.

— The Collisional Dynamics of Particulate Disks by Shu, F 1985

we set $P_{ijk} = 0$

$$\frac{\partial P_{ij}}{\partial t} + P_{ik}\frac{\partial u_j}{\partial x_k} + P_{jk}\frac{\partial u_i}{\partial x_k} + \frac{\partial\left(u_k P_{ij}\right)}{\partial x_k} + 2fP_{ij}\left(V_i + V_j - u_i - u_j\right) = \left(\frac{\partial P_{ij}}{\partial t}\right)_c$$

$$2fP_{ij}\left(V_i + V_j - u_i - u_j\right) = \mathbf{e}_i 2fP_{ij}\left(V_i + V_j - u_i - u_j\right)\mathbf{e}_j \tag{84}$$

$$= \begin{pmatrix} 2fP_{RR}\left(V_R + V_R - u_R - u_R\right), & 2fP_{R\theta}\left(V_R + V_\theta - u_R - u_\theta\right), & 0 \\ 2fP_{\theta R}\left(V_\theta + V_R - u_\theta - u_R\right), & 2fP_{\theta\theta}\left(V_\theta + V_\theta - u_\theta - u_\theta\right), & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{85}$$

where $P_{Rz} = P_{\theta z} = 0$

for a steady state, $\frac{\partial P_{ij}}{\partial t} = 0$

$$P_{ik}\frac{\partial u_j}{\partial x_k} + P_{jk}\frac{\partial u_i}{\partial x_k} + \frac{\partial\left(u_k P_{ij}\right)}{\partial x_k} + 2fP_{ij}\left(V_i + V_j - u_i - u_j\right) = \begin{pmatrix} \left(\frac{\partial P_{RR}}{\partial t}\right)_c, & \left(\frac{\partial P_{R\theta}}{\partial t}\right)_c, & 0 \\ \left(\frac{\partial P_{\theta R}}{\partial t}\right)_c, & \left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c, & 0 \\ 0 & 0 & \left(\frac{\partial P_{zz}}{\partial t}\right)_c \end{pmatrix}$$

then we have equation list:

$$2P_{RR}\frac{\partial u_R}{\partial R} + \frac{\partial P_{RR}u_R}{\partial R} - 4P_{R\theta}\Omega + 2fP_{RR}\left(V_R + V_R - u_R - u_R\right) = \left(\frac{\partial P_{RR}}{\partial t}\right)_c \tag{86}$$

$$P_{R\theta}\frac{u_R}{R} + P_{\theta R}\frac{\partial u_R}{\partial R} + \frac{\partial P_{R\theta}u_R}{\partial R} + \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta} + 2fP_{R\theta}\left(V_R + V_\theta - u_R - u_\theta\right) = \left(\frac{\partial P_{R\theta}}{\partial t}\right)_c \tag{87}$$

$$2P_{\theta\theta}\frac{u_R}{R} + \frac{\partial P_{\theta\theta}u_R}{\partial R} + \frac{2P_{R\theta}}{R}\frac{dR^2\Omega}{dR} + 2fP_{\theta\theta}\left(V_\theta + V_\theta - u_\theta - u_\theta\right) = \left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c \tag{88}$$

$$\frac{\partial P_{zz}u_R}{\partial R} = \left(\frac{\partial P_{zz}}{\partial t}\right)_c \tag{89}$$

## 5. Velocity Distribution

### *5.1. ICE*

Suppose the average velocity in $\theta$ direction is still Keperian.

$$u_\theta = \sqrt{\frac{GM}{R}} \tag{90}$$

Decrease of kinetic energy caused by dissipation force will account for sink in gravitational field, so

$$F_{Diss} v_\theta \simeq F_{Grav} v_R, \tag{91}$$

then we estimate the average velocity in $R$ direction as

$$f u_{\mathrm{Kep}}^2 u_\theta = \frac{GM}{R^2} u_R \tag{92}$$

So we take the form of $u_R$ as

$$u_R = -fR u_\theta = -fR\sqrt{\frac{GM}{R}} \tag{93}$$

To make this assumption consistent, $u_R$ should much less than $u_\theta$.

### *5.2. GAS*

We simply assuming the velocity distribution of gas as Keperian rotation:

$$V_R = 0, \quad V_\theta = \sqrt{\frac{GM}{R}} \tag{94}$$

## 6. Preparing Equations

$$2P_{RR}\frac{\partial u_R}{\partial R} + \frac{\partial P_{RR} u_R}{\partial R} - 4P_{R\theta}\Omega + 2fP_{RR}\left(V_R + V_R - u_R - u_R\right) = \left(\frac{\partial P_{RR}}{\partial t}\right)_c \tag{95}$$

$$3P_{RR}\frac{\partial u_R}{\partial R} + \frac{\partial P_{RR}}{\partial R} u_R - 4P_{R\theta}\Omega + 2fP_{RR}\left(V_R + V_R - u_R - u_R\right) = \left(\frac{\partial P_{RR}}{\partial t}\right)_c \tag{96}$$

$$-3P_{RR} fR\Omega\frac{1}{2} - \frac{\partial P_{RR}}{\partial R} fR^2\Omega - 4P_{R\theta}\Omega + 2fP_{RR}\left(+2fR^2\Omega\right) = \left(\frac{\partial P_{RR}}{\partial t}\right)_c \tag{97}$$

$$-\frac{\partial P_{RR}}{\partial R} fR^2\Omega + P_{RR}\left(-\frac{3}{2} fR\Omega + 4f^2R^2\Omega\right) - 4P_{R\theta}\Omega = \left(\frac{\partial P_{RR}}{\partial t}\right)_c \tag{98}$$

the second equation:

$$P_{R\theta}\frac{u_R}{R}+P_{\theta R}\frac{\partial u_R}{\partial R}+\frac{\partial P_{R\theta}u_R}{\partial R}+\frac{P_{RR}}{R}\frac{dR^2\Omega}{dR}-2\Omega P_{\theta\theta}+2fP_{R\theta}\left(V_R+V_\theta-u_R-u_\theta\right)=\left(\frac{\partial P_{R\theta}}{\partial t}\right)_c \tag{99}$$

$$-P_{R\theta}fR\Omega-P_{\theta R}\frac{1}{2}fR\Omega-P_{R\theta}\frac{1}{2}fR\Omega-\frac{\partial P_{R\theta}}{\partial R}fR^2\Omega+\frac{P_{RR}}{R}\frac{1}{2}R\Omega-2\Omega P_{\theta\theta}+2fP_{R\theta}\left(+fR^2\Omega\right)=\left(\frac{\partial P_{R\theta}}{\partial t}\right)_c \tag{100}$$

$$-\frac{\partial P_{R\theta}}{\partial R}fR^2\Omega+P_{R\theta}\left(-2fR\Omega+2f^2R^2\Omega\right)+\frac{1}{2}P_{RR}\Omega-2\Omega P_{\theta\theta}=\left(\frac{\partial P_{R\theta}}{\partial t}\right)_c \tag{101}$$

Third:

$$-2P_{\theta\theta}fR\Omega-P_{\theta\theta}\frac{1}{2}fR\Omega-\frac{\partial P_{\theta\theta}}{\partial R}fR^2\Omega+P_{R\theta}\Omega=\left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c \tag{102}$$

$$-\frac{\partial P_{\theta\theta}}{\partial R}fR^2\Omega-P_{\theta\theta}\frac{5}{2}fR\Omega+P_{R\theta}\Omega=\left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c \tag{103}$$

$$-\frac{\partial P_{RR}}{\partial R}fR^2\Omega+P_{RR}\left(-\frac{3}{2}fR\Omega+4f^2R^2\Omega\right)-4P_{R\theta}\Omega=\left(\frac{\partial P_{RR}}{\partial t}\right)_c \tag{104}$$

$$-\frac{\partial P_{R\theta}}{\partial R}fR^2\Omega+P_{R\theta}\left(-2fR\Omega+2f^2R^2\Omega\right)+\frac{1}{2}P_{RR}\Omega-2\Omega P_{\theta\theta}=\left(\frac{\partial P_{R\theta}}{\partial t}\right)_c \tag{105}$$

$$-\frac{\partial P_{\theta\theta}}{\partial R}fR^2\Omega-P_{\theta\theta}\frac{5}{2}fR\Omega+P_{R\theta}\Omega=\left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c \tag{106}$$

So finally,

$$-\frac{\partial P_{RR}}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f\right)-P_{R\theta}\frac{4}{fR^2}=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{RR}}{\partial t}\right)_c \tag{107}$$

$$-\frac{\partial P_{R\theta}}{\partial R}+P_{R\theta}\left(-\frac{2}{R}+2f\right)+P_{RR}\frac{1}{2fR^2}-P_{\theta\theta}\frac{2}{fR^2}=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{R\theta}}{\partial t}\right)_c \tag{108}$$

$$-\frac{\partial P_{\theta\theta}}{\partial R}-P_{\theta\theta}\frac{5}{2R}+P_{R\theta}\frac{1}{fR^2}=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c \tag{109}$$

## 7. The Principal Axis Transformation

$$P_{RR}=\frac{1}{\left(1+\tan^2\delta\right)}P_{11}+\frac{\tan^2\delta}{\left(1+\tan^2\delta\right)}P_{22} \tag{110}$$

$$P_{\theta\theta}=\frac{\tan^2\delta}{\left(1+\tan^2\delta\right)}P_{11}+\frac{1}{\left(1+\tan^2\delta\right)}P_{22} \tag{111}$$

$$P_{R\theta} = \frac{\tan\delta}{1+\tan^2\delta}\left(P_{11}-P_{22}\right) \tag{112}$$

$$P_{zz} = P_{33} \tag{113}$$

simplify:

$$P_{RR} = \cos^2\delta P_{11} + \sin^2\delta P_{22} \tag{114}$$

$$P_{\theta\theta} = \sin^2\delta P_{11} + \cos^2\delta P_{22} \tag{115}$$

$$P_{R\theta} = \sin\delta\cos\delta\left(P_{11}-P_{22}\right) \tag{116}$$

$P_{RR}$ :

$$\frac{\partial P_{RR}}{\partial R} = \frac{\partial\cos^2\delta P_{11} + \sin^2\delta P_{22}}{\partial R} = \frac{\partial\cos^2\delta P_{11}}{\partial R} + \frac{\partial\sin^2\delta P_{22}}{\partial R} \tag{117}$$

$$= \frac{\partial\cos^2\delta}{\partial R}P_{11} + \cos^2\delta\frac{\partial P_{11}}{\partial R} + \frac{\partial\sin^2\delta}{\partial R}P_{22} + \sin^2\delta\frac{\partial P_{22}}{\partial R} \tag{118}$$

$$= \frac{\partial\cos^2\delta}{\partial R}P_{11} + \cos^2\delta\frac{\partial P_{11}}{\partial R} + \frac{\partial\sin^2\delta}{\partial R}P_{22} + \sin^2\delta\frac{\partial P_{22}}{\partial R} \tag{119}$$

$$= -2\sin\delta\cos\delta\frac{\partial\delta}{\partial R}P_{11} + \cos^2\delta\frac{\partial P_{11}}{\partial R} + 2\sin\delta\cos\delta\frac{\partial\delta}{\partial R}P_{22} + \sin^2\delta\frac{\partial P_{22}}{\partial R} \tag{120}$$

$$= \cos^2\delta\frac{\partial P_{11}}{\partial R} + \sin^2\delta\frac{\partial P_{22}}{\partial R} + \left(2\sin\delta\cos\delta P_{22} - 2\sin\delta\cos\delta P_{11}\right)\frac{\partial\delta}{\partial R} \tag{121}$$

$P_{\theta\theta}$ :

$$\frac{\partial P_{\theta\theta}}{\partial R} = \frac{\partial\sin^2\delta P_{11} + \cos^2\delta P_{22}}{\partial R} = \frac{\partial\sin^2\delta P_{11}}{\partial R} + \frac{\partial\cos^2\delta P_{22}}{\partial R} \tag{122}$$

$$= \sin^2\delta\frac{\partial P_{11}}{\partial R} + P_{11}\frac{\partial\sin^2\delta}{\partial R} + \cos^2\delta\frac{\partial P_{22}}{\partial R} + P_{22}\frac{\partial\cos^2\delta}{\partial R} \tag{123}$$

$$= \sin^2\delta\frac{\partial P_{11}}{\partial R} + P_{11}2\sin\delta\cos\delta\frac{\partial\delta}{\partial R} + \cos^2\delta\frac{\partial P_{22}}{\partial R} - P_{22}2\cos\delta\sin\delta\frac{\partial\delta}{\partial R} \tag{124}$$

$$= \sin^2\delta\frac{\partial P_{11}}{\partial R} + \cos^2\delta\frac{\partial P_{22}}{\partial R} + \left(2\sin\delta\cos\delta P_{11} - 2\cos\delta\sin\delta P_{22}\right)\frac{\partial\delta}{\partial R} \tag{125}$$

$P_{R\theta}$ :

$$\frac{\partial P_{R\theta}}{\partial R} = \frac{\partial\sin\delta\cos\delta\left(P_{11}-P_{22}\right)}{\partial R} = \sin\delta\cos\delta\frac{\partial\left(P_{11}-P_{22}\right)}{\partial R} + \frac{\partial\sin\delta\cos\delta}{\partial R}\left(P_{11}-P_{22}\right) \tag{126}$$

$$= \sin\delta\cos\delta\frac{\partial P_{11}}{\partial R} - \sin\delta\cos\delta\frac{\partial P_{22}}{\partial R} + \frac{\partial \sin\delta\cos\delta}{\partial R}\left(P_{11} - P_{22}\right) \tag{127}$$

$$= \sin\delta\cos\delta\frac{\partial P_{11}}{\partial R} - \sin\delta\cos\delta\frac{\partial P_{22}}{\partial R} + \frac{\partial \sin\delta\cos\delta}{\partial R}\left(P_{11} - P_{22}\right) \tag{128}$$

$$= \sin\delta\cos\delta\frac{\partial P_{11}}{\partial R} - \sin\delta\cos\delta\frac{\partial P_{22}}{\partial R} + \left(\cos^2\delta - \sin^2\delta\right)\left(P_{11} - P_{22}\right)\frac{\partial\delta}{\partial R} \tag{129}$$

and then the collision terms:

$$\left(\frac{\partial P_{RR}}{\partial t}\right)_c = \left(\frac{\partial \cos^2\delta P_{11} + \sin^2\delta P_{22}}{\partial t}\right)_c \tag{130}$$

$$= \cos^2\delta\left(\frac{\partial P_{11}}{\partial t}\right)_c + \sin^2\delta\left(\frac{\partial P_{22}}{\partial t}\right)_c \tag{131}$$

$$\left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c = \left(\frac{\partial \sin^2\delta P_{11} + \cos^2\delta P_{22}}{\partial t}\right)_c \tag{132}$$

$$= \sin^2\delta\left(\frac{\partial P_{11}}{\partial t}\right)_c + \cos^2\delta\left(\frac{\partial P_{22}}{\partial t}\right)_c \tag{133}$$

$$\left(\frac{\partial P_{R\theta}}{\partial t}\right)_c = \left(\frac{\partial \sin\delta\cos\delta\left(P_{11} - P_{22}\right)}{\partial t}\right)_c \tag{134}$$

$$= \sin\delta\cos\delta\left(\frac{\partial\left(P_{11} - P_{22}\right)}{\partial t}\right)_c \tag{135}$$

$$= \sin\delta\cos\delta\left(\frac{\partial P_{11}}{\partial t}\right)_c - \sin\delta\cos\delta\left(\frac{\partial P_{22}}{\partial t}\right)_c \tag{136}$$

then the equations in principal axes are

$$\begin{aligned} &-\cos^2\delta\frac{\partial P_{11}}{\partial R} - \sin^2\delta\frac{\partial P_{22}}{\partial R} - \left(2\sin\delta\cos\delta P_{22} - 2\sin\delta\cos\delta P_{11}\right)\frac{\partial\delta}{\partial R} + P_{RR}\left(-\frac{3}{2R} + 4f\right) \\ &\qquad -P_{R\theta}\frac{4}{fR^2} = \frac{\cos^2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c + \frac{\sin^2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \end{aligned} \tag{137}$$

$$\begin{aligned} &-\sin^2\delta\frac{\partial P_{11}}{\partial R} - \cos^2\delta\frac{\partial P_{22}}{\partial R} - \left(2\sin\delta\cos\delta P_{11} - 2\cos\delta\sin\delta P_{22}\right)\frac{\partial\delta}{\partial R} - P_{\theta\theta}\frac{5}{2R} \\ &\qquad +P_{R\theta}\frac{1}{fR^2} = \frac{\sin^2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c + \frac{\cos^2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \end{aligned} \tag{138}$$

$$\begin{aligned} &-\sin\delta\cos\delta\frac{\partial P_{11}}{\partial R} + \sin\delta\cos\delta\frac{\partial P_{22}}{\partial R} - \left(\cos^2\delta - \sin^2\delta\right)\left(P_{11} - P_{22}\right)\frac{\partial\delta}{\partial R} + P_{R\theta}\left(-\frac{2}{R} + 2f\right) \\ &\qquad +P_{RR}\frac{1}{2fR^2} - P_{\theta\theta}\frac{2}{fR^2} = \frac{\sin\delta\cos\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c - \frac{\sin\delta\cos\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \end{aligned} \tag{139}$$

*7.1. Simplify*

Combining equation 137 and equation 138, we have

$$-\cos^2\delta\frac{\partial P_{11}}{\partial R}-\sin^2\delta\frac{\partial P_{22}}{\partial R}-\left(2\sin\delta\cos\delta P_{22}-2\sin\delta\cos\delta P_{11}\right)\frac{\partial\delta}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f\right)$$
$$-P_{R\theta}\frac{4}{fR^2}-\sin^2\delta\frac{\partial P_{11}}{\partial R}-\cos^2\delta\frac{\partial P_{22}}{\partial R}-\left(2\sin\delta\cos\delta P_{11}-2\cos\delta\sin\delta P_{22}\right)\frac{\partial\delta}{\partial R}\tag{140}$$
$$-P_{\theta\theta}\frac{5}{2R}+P_{R\theta}\frac{1}{fR^2}=\frac{\cos^2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c+\frac{\sin^2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c+\frac{\sin^2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c+\frac{\cos^2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c\tag{141}$$

$$-\frac{\partial P_{11}}{\partial R}-\frac{\partial P_{22}}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f\right)-P_{R\theta}\frac{3}{fR^2}-P_{\theta\theta}\frac{5}{2R}=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c+\frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c$$

Minus equation 137 and equation 138

$$-\cos^2\delta\frac{\partial P_{11}}{\partial R}-\sin^2\delta\frac{\partial P_{22}}{\partial R}-\left(2\sin\delta\cos\delta P_{22}-2\sin\delta\cos\delta P_{11}\right)\frac{\partial\delta}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f\right)$$
$$-P_{R\theta}\frac{4}{fR^2}+\sin^2\delta\frac{\partial P_{11}}{\partial R}+\cos^2\delta\frac{\partial P_{22}}{\partial R}+\left(2\sin\delta\cos\delta P_{11}-2\cos\delta\sin\delta P_{22}\right)\frac{\partial\delta}{\partial R}+P_{\theta\theta}\frac{5}{2R}\tag{142}$$
$$-P_{R\theta}\frac{1}{fR^2}=\frac{\cos^2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c+\frac{\sin^2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c-\frac{\sin^2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{\cos^2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c\tag{143}$$

$$-\cos 2\delta\frac{\partial P_{11}}{\partial R}+\cos 2\delta\frac{\partial P_{22}}{\partial R}-2\sin 2\delta\left(P_{22}-P_{11}\right)\frac{\partial\delta}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f\right)$$
$$-P_{R\theta}\frac{5}{fR^2}+P_{\theta\theta}\frac{5}{2R}=\frac{\cos 2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{\cos 2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c\tag{144}$$

$$-\frac{\partial P_{11}}{\partial R}+\frac{\partial P_{22}}{\partial R}-2\tan 2\delta\left(P_{22}-P_{11}\right)\frac{\partial\delta}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f\right)/\cos 2\delta-P_{R\theta}\frac{5}{fR^2\cos 2\delta}$$
$$+P_{\theta\theta}\frac{5}{2R\cos 2\delta}=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c\tag{145}$$

$$+\cot 2\delta\frac{\partial P_{11}}{\partial R}+\cot 2\delta\frac{\partial P_{22}}{\partial R}-2\left(P_{22}-P_{11}\right)\frac{\partial\delta}{\partial R}+\cot 2\delta P_{RR}\left(-\frac{3}{2R}+4f\right)/\cos 2\delta-\cot 2\delta P_{R\theta}\frac{5}{fR^2\cos 2\delta}$$
$$+\cot 2\delta P_{\theta\theta}\frac{5}{2R\cos 2\delta}=\frac{\cot 2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{\cot 2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c\tag{146}$$

recall the third equation:

$$-2\sin\delta\cos\delta\frac{\partial P_{11}}{\partial R}+2\sin\delta\cos\delta\frac{\partial P_{22}}{\partial R}-2\left(\cos^2\delta-\sin^2\delta\right)\left(P_{11}-P_{22}\right)\frac{\partial\delta}{\partial R}+P_{R\theta}\left(-\frac{4}{R}+4f\right)$$
$$+P_{RR}\frac{1}{fR^2}-P_{\theta\theta}\frac{4}{fR^2}=\frac{2\sin\delta\cos\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{2\sin\delta\cos\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c\tag{147}$$

$$-\sin 2\delta \frac{\partial P_{11}}{\partial R} + \sin 2\delta \frac{\partial P_{22}}{\partial R} - 2\cos 2\delta \left(P_{11} - P_{22}\right) \frac{\partial \delta}{\partial R} + P_{R\theta}\left(-\frac{4}{R} + 4f\right)$$
$$+P_{RR}\frac{1}{fR^2} - P_{\theta\theta}\frac{4}{fR^2} = \frac{\sin 2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c - \frac{\sin 2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \quad (148)$$

$$-\frac{\partial P_{11}}{\partial R} + \frac{\partial P_{22}}{\partial R} - 2\cot 2\delta \left(P_{11} - P_{22}\right) \frac{\partial \delta}{\partial R} + P_{R\theta}\left(-\frac{4}{R} + 4f\right)/\sin 2\delta$$
$$+P_{RR}\frac{1}{fR^2\sin 2\delta} - P_{\theta\theta}\frac{4}{fR^2\sin 2\delta} = \frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c - \frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \quad (149)$$

$$-\tan 2\delta \frac{\partial P_{11}}{\partial R} + \tan 2\delta \frac{\partial P_{22}}{\partial R} - 2\left(P_{11} - P_{22}\right) \frac{\partial \delta}{\partial R} + \tan 2\delta P_{R\theta}\left(-\frac{4}{R} + 4f\right)/\sin 2\delta$$
$$+\tan 2\delta P_{RR}\frac{1}{fR^2\sin 2\delta} - \tan 2\delta P_{\theta\theta}\frac{4}{fR^2\sin 2\delta} = \frac{\tan 2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c - \frac{\tan 2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \quad (150)$$

$$-\frac{\partial P_{11}}{\partial R} + \frac{\partial P_{22}}{\partial R} - 2\tan 2\delta \left(P_{22} - P_{11}\right) \frac{\partial \delta}{\partial R} + P_{RR}\left(-\frac{3}{2R} + 4f\right)/\cos 2\delta - P_{R\theta}\frac{5}{fR^2\cos 2\delta}$$
$$+P_{\theta\theta}\frac{5}{2R\cos 2\delta} = \frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c - \frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \quad (151)$$

so

$$-2\cot 2\delta \left(P_{11} - P_{22}\right) \frac{\partial \delta}{\partial R} + P_{R\theta}\left(-\frac{4}{R} + 4f\right)/\sin 2\delta + P_{RR}\frac{1}{fR^2\sin 2\delta} - P_{\theta\theta}\frac{4}{fR^2\sin 2\delta} \quad (152)$$
$$= -2\tan 2\delta \left(P_{22} - P_{11}\right) \frac{\partial \delta}{\partial R} + P_{RR}\left(-\frac{3}{2R} + 4f\right)/\cos 2\delta - P_{R\theta}\frac{5}{fR^2\cos 2\delta} + P_{\theta\theta}\frac{5}{2R\cos 2\delta} \quad (153)$$

$$-2\left(\cot 2\delta + \tan 2\delta\right)\left(P_{11} - P_{22}\right) \frac{\partial \delta}{\partial R} + P_{R\theta}\left(-\frac{4}{R} + 4f\right)/\sin 2\delta + P_{RR}\frac{1}{fR^2\sin 2\delta} - P_{\theta\theta}\frac{4}{fR^2\sin 2\delta} \quad (154)$$
$$= +P_{RR}\left(-\frac{3}{2R} + 4f\right)/\cos 2\delta - P_{R\theta}\frac{5}{fR^2\cos 2\delta} + P_{\theta\theta}\frac{5}{2R\cos 2\delta} \quad (155)$$

$$-2\frac{\left(P_{11} - P_{22}\right)}{\sin 2\delta\cos 2\delta} \frac{\partial \delta}{\partial R} + P_{R\theta}\left(-\frac{4}{R} + 4f\right)/\sin 2\delta + P_{RR}\frac{1}{fR^2\sin 2\delta} - P_{\theta\theta}\frac{4}{fR^2\sin 2\delta} \quad (156)$$
$$= +P_{RR}\left(-\frac{3}{2R} + 4f\right)/\cos 2\delta - P_{R\theta}\frac{5}{fR^2\cos 2\delta} + P_{\theta\theta}\frac{5}{2R\cos 2\delta} \quad (157)$$

$$-2\left(P_{11} - P_{22}\right) \frac{\partial \delta}{\partial R} + \cos 2\delta P_{R\theta}\left(-\frac{4}{R} + 4f\right) + P_{RR}\frac{\cos 2\delta}{fR^2} - P_{\theta\theta}\frac{4\cos 2\delta}{fR^2} \quad (158)$$
$$= +\sin 2\delta P_{RR}\left(-\frac{3}{2R} + 4f\right) - \sin 2\delta P_{R\theta}\frac{5}{fR^2} - \sin 2\delta P_{\theta\theta}\frac{5}{2R} \quad (159)$$

$$
-2\left(P_{11}-P_{22}\right)\frac{\partial\delta}{\partial R} = +\sin 2\delta P_{RR}\left(-\frac{3}{2R}+4f\right)-\sin 2\delta P_{R\theta}\frac{5}{fR^2} \tag{160}
$$

$$
-\sin 2\delta P_{\theta\theta}\frac{5}{2R}-\cos 2\delta P_{R\theta}\left(-\frac{4}{R}+4f\right)+P_{\theta\theta}\frac{4\cos 2\delta}{fR^2}-P_{RR}\frac{\cos 2\delta}{fR^2} \tag{161}
$$

$$
-2\left(P_{11}-P_{22}\right)\frac{\partial\delta}{\partial R} = +P_{RR}\left(-\frac{3\sin 2\delta}{2R}+4f\sin 2\delta-\frac{\cos 2\delta}{fR^2}\right)+P_{\theta\theta}\left(\frac{4\cos 2\delta}{fR^2}-\frac{5\sin 2\delta}{2R}\right) \tag{162}
$$

$$
-P_{R\theta}\left(\frac{5\sin 2\delta}{fR^2}-\frac{4\cos 2\delta}{R}+4f\cos 2\delta\right) \tag{163}
$$

$$
-\frac{\partial\delta}{\partial R} = \frac{\cos^2\delta P_{11}+\sin^2\delta P_{22}}{2\left(P_{11}-P_{22}\right)}\left(-\frac{3\sin 2\delta}{2R}+4f\sin 2\delta-\frac{\cos 2\delta}{fR^2}\right)+\frac{\sin^2\delta P_{11}+\cos^2\delta P_{22}}{2\left(P_{11}-P_{22}\right)}\left(\frac{4\cos 2\delta}{fR^2}-\frac{5\sin 2\delta}{2R}\right)
$$

$$
-\frac{\sin\delta\cos\delta\left(P_{11}-P_{22}\right)}{2\left(P_{11}-P_{22}\right)}\left(\frac{5\sin 2\delta}{fR^2}-\frac{4\cos 2\delta}{R}+4f\cos 2\delta\right)
$$

and

$$
-\cot 2\delta\frac{\partial P_{11}}{\partial R}+\cot 2\delta\frac{\partial P_{22}}{\partial R}+\cot 2\delta P_{RR}\left(-\frac{3}{2R}+4f\right)/\cos 2\delta-\cot 2\delta P_{R\theta}\frac{5}{fR^2\cos 2\delta}
$$

$$
+\cot 2\delta P_{\theta\theta}\frac{5}{2R\cos 2\delta}-\tan 2\delta\frac{\partial P_{11}}{\partial R}+\tan 2\delta\frac{\partial P_{22}}{\partial R}+\tan 2\delta P_{R\theta}\left(-\frac{4}{R}+4f\right)/\sin 2\delta \tag{166}
$$

$$
+\tan 2\delta P_{RR}\frac{1}{fR^2\sin 2\delta}-\tan 2\delta P_{\theta\theta}\frac{4}{fR^2\sin 2\delta} \tag{167}
$$

$$
=\frac{\cot 2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{\cot 2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c+\frac{\tan 2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{\tan 2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \tag{168}
$$

$$
-\left(\cot 2\delta+\tan 2\delta\right)\frac{\partial P_{11}}{\partial R}+\left(\cot 2\delta+\tan 2\delta\right)\frac{\partial P_{22}}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f\right)/\sin 2\delta-P_{R\theta}\frac{5}{fR^2\sin 2\delta}
$$

$$
+P_{\theta\theta}\frac{5}{2R\sin 2\delta}+P_{R\theta}\left(-\frac{4}{R}+4f\right)/\cos 2\delta+P_{RR}\frac{1}{fR^2\cos 2\delta}-P_{\theta\theta}\frac{4}{fR^2\cos 2\delta} \tag{169}
$$

$$
=\frac{\cot 2\delta+\tan 2\delta}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{\cot 2\delta+\tan 2\delta}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \tag{170}
$$

$$
-\frac{\partial P_{11}}{\partial R}+\frac{\partial P_{22}}{\partial R}+\frac{P_{RR}}{\left(\cot 2\delta+\tan 2\delta\right)}\left(-\frac{3}{2R}+4f\right)/\sin 2\delta-\frac{P_{R\theta}}{\left(\cot 2\delta+\tan 2\delta\right)}\frac{5}{fR^2\sin 2\delta}
$$

$$
+\frac{P_{\theta\theta}}{\left(\cot 2\delta+\tan 2\delta\right)}\frac{5}{2R\sin 2\delta}+\frac{P_{R\theta}}{\left(\cot 2\delta+\tan 2\delta\right)}\left(-\frac{4}{R}+4f\right)/\cos 2\delta \tag{171}
$$

$$
+\frac{P_{RR}}{\left(\cot 2\delta+\tan 2\delta\right)}\frac{1}{fR^2\cos 2\delta}-\frac{P_{\theta\theta}}{\left(\cot 2\delta+\tan 2\delta\right)}\frac{4}{fR^2\cos 2\delta} \tag{172}
$$

$$
=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \tag{173}
$$

$$-\frac{\partial P_{11}}{\partial R}+\frac{\partial P_{22}}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f\right)\cos 2\delta-P_{R\theta}\frac{5\cos 2\delta}{fR^2}+P_{\theta\theta}\frac{5\cos 2\delta}{2R}+P_{R\theta}\left(-\frac{4}{R}+4f\right)\sin 2\delta$$
$$+P_{RR}\frac{\sin 2\delta}{fR^2}-P_{\theta\theta}\frac{4\sin 2\delta}{fR^2}=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \quad (174)$$

$$-\frac{\partial P_{11}}{\partial R}+\frac{\partial P_{22}}{\partial R}+P_{RR}\left(-\frac{3\cos 2\delta}{2R}+4f\cos 2\delta+\frac{\sin 2\delta}{fR^2}\right)-P_{\theta\theta}\left(\frac{4\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{2R}\right)$$
$$+P_{R\theta}\left(-\frac{4\sin 2\delta}{R}+4f\sin 2\delta-\frac{5\cos 2\delta}{fR^2}\right)=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c-\frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \quad (175)$$

Combine the previous equation:

$$-\frac{\partial P_{11}}{\partial R}-\frac{\partial P_{22}}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f\right)-P_{R\theta}\frac{3}{fR^2}-P_{\theta\theta}\frac{5}{2R}=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c+\frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c$$

$$-2\frac{\partial P_{11}}{\partial R}+P_{RR}\left(-\frac{3\cos 2\delta}{2R}+4f\cos 2\delta+\frac{\sin 2\delta}{fR^2}-\frac{3}{2R}+4f\right)-P_{\theta\theta}\left(\frac{4\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{2R}+\frac{5}{2R}\right) \quad (176)$$
$$+P_{R\theta}\left(-\frac{4\sin 2\delta}{R}+4f\sin 2\delta-\frac{5\cos 2\delta}{fR^2}-\frac{3}{fR^2}\right)=\frac{2}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c \quad (177)$$

$$-2\frac{\partial P_{22}}{\partial R}+P_{RR}\left(-\frac{3}{2R}+4f+\frac{3\cos 2\delta}{2R}-4f\cos 2\delta-\frac{\sin 2\delta}{fR^2}\right)+P_{\theta\theta}\left(-\frac{5}{2R}+\frac{4\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{2R}\right)$$
$$-P_{R\theta}\left(\frac{3}{fR^2}-\frac{4\sin 2\delta}{R}+4f\sin 2\delta-\frac{5\cos 2\delta}{fR^2}\right)=\frac{2}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \quad (178)$$

$$-\frac{\partial P_{11}}{\partial R}+P_{RR}\left(-\frac{3\cos 2\delta}{4R}+2f\cos 2\delta+\frac{\sin 2\delta}{2fR^2}-\frac{3}{4R}+2f\right)-P_{\theta\theta}\left(\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}+\frac{5}{4R}\right) \quad (179)$$
$$+P_{R\theta}\left(-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}-\frac{3}{2fR^2}\right)=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c \quad (180)$$

$$-\frac{\partial P_{22}}{\partial R}+P_{RR}\left(-\frac{3}{4R}+2f+\frac{3\cos 2\delta}{4R}-2f\cos 2\delta-\frac{\sin 2\delta}{2fR^2}\right)+P_{\theta\theta}\left(-\frac{5}{4R}+\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}\right)$$
$$-P_{R\theta}\left(\frac{3}{2fR^2}-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}\right)=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \quad (181)$$

$$P_{RR}=\cos^2\delta P_{11}+\sin^2\delta P_{22} \quad (182)$$

$$P_{\theta\theta}=\sin^2\delta P_{11}+\cos^2\delta P_{22} \quad (183)$$

$$P_{R\theta}=\sin\delta\cos\delta\left(P_{11}-P_{22}\right) \quad (184)$$

$$\frac{\partial P_{33} u_R}{\partial R} = \left(\frac{\partial P_{33}}{\partial t}\right)_c \tag{185}$$

$$-\frac{\partial P_{33} f R^2 \Omega}{\partial R} = \left(\frac{\partial P_{33}}{\partial t}\right)_c \tag{186}$$

$$-\frac{\partial P_{33}}{\partial R} f R^2 \Omega - P_{33} \frac{\partial f R^2 \Omega}{\partial R} = \left(\frac{\partial P_{33}}{\partial t}\right)_c \tag{187}$$

$$-\frac{\partial P_{33}}{\partial R} f R^2 \Omega - P_{33} \frac{f R \Omega}{2} = \left(\frac{\partial P_{33}}{\partial t}\right)_c \tag{188}$$

$$-\frac{\partial P_{33}}{\partial R} - P_{33} \frac{1}{2R} = \frac{1}{f R^2 \Omega} \left(\frac{\partial P_{33}}{\partial t}\right)_c \tag{189}$$

## 8. Collision Terms

Now we evaluate the terms $(\partial p_{ii}/\partial t)_c$. Consider a collision between two particles with vlocities $\mathbf{v}_1$ and $\mathbf{v}_2$ which changes the velocities to $\mathbf{v}_1'$ and $\mathbf{v}_2'$ . Relative velocities before and after the collision are $\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_r' = \mathbf{v}_1' - \mathbf{v}_2'$. The center-of-mass velocity $\mathbf{v}_c$ is conserved and the relative motion of the two particles is found by assuming that one acts as a fixed center of force while the other has the reduced mass $\mu = m/2$.

Assumption about energy dissipation

- The impact conserves the relatives tangential velocity
- The impact reduce the absolute value of the relative normal velocity by a factor of $\epsilon$

We have

$$\mathbf{v}_r' = \mathbf{v}_r - \lambda \left(1 + \epsilon\right) \mathbf{v}_r \cdot \lambda \tag{190}$$

$$\lambda \cdot \mathbf{v}_r = |v_r| \left(1 - b^2/4a^2\right)^{1/2} \tag{191}$$

The collision dynamics are conveniently described in a frame $(X, Y, Z) \equiv (r, \theta, \phi)$ whose $Z$ axis $(\theta = 0)$ is the direction of $\mathbf{v}_r$ . This means we treat the collision as what we usually do in statistical physics.

A collision is completely specified by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\lambda = (\theta_\lambda, \phi_\lambda)$ or, alternatively by $\mathbf{v}_1$, $\mathbf{v}_2$, $b$, and $\phi_\lambda$ since $\lambda \cdot \mathbf{v}_r = |v_r| \cos\theta_\lambda$ is given in terms of $b$.

Collision rate per unit volume in the interval $\mathbf{v}_1 \to \mathbf{v}_1 + d\mathbf{v}_1$, $\mathbf{v}_2 \to \mathbf{v}_2 + d\mathbf{v}_2$, $b \to b + db$, $\phi_\lambda \to \phi_\lambda + d\phi_\lambda$ is

$$f\left(\mathbf{v}_1\right) f\left(\mathbf{v}_2\right) d\mathbf{v}_1 d\mathbf{v}_2 |\mathbf{v}_r| b db d\phi_\lambda, \tag{192}$$

Expression $\left(\frac{\partial p_{ii}}{\partial t}\right)_c$ is not differential with time but the alteration of $p_{ii}$ in an time interval, for each collision,

$$\left(\frac{\Delta p_{ii}}{\Delta t}\right)_c = \frac{p'_{ii} - p_{ii}}{\Delta t} \tag{193}$$

Definition of $p_{ii}$ is

$$p_{ii} = \int f(v_i - \langle v_i \rangle)^2 d\mathbf{v} = \int f(v_i^2 - 2v_i \langle v_i \rangle + \langle v_i \rangle^2) d\mathbf{v} \tag{194}$$

$$= \int f v_i^2 d\mathbf{v} - \int f 2v_i \langle v_i \rangle d\mathbf{v} + \int f \langle v_i \rangle^2 d\mathbf{v} \tag{195}$$

$$= \int f v_i^2 d\mathbf{v} - \int f \langle v_i \rangle^2 d\mathbf{v} \tag{196}$$

with this formula, which could easily be simplified as $p_{ii} = n \langle v_i^2 \rangle - n \langle v_i \rangle^2 = n\sigma_{ii}^2$, the collision term could be expressed as

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{1}{2} \int f(\mathbf{v}_1) f(\mathbf{v}_2) d\mathbf{v}_1 d\mathbf{v}_2 |v_r| \tag{197}$$

$$\Big[(\mathbf{e}_i \cdot \mathbf{v}'_1)^2 - \langle \mathbf{e}_i \cdot \mathbf{v}'_1 \rangle^2 \tag{198}$$

$$+ (\mathbf{e}_i \cdot \mathbf{v}'_2)^2 - \langle \mathbf{e}_i \cdot \mathbf{v}'_2 \rangle^2 \tag{199}$$

$$- (\mathbf{e}_i \cdot \mathbf{v}_1)^2 + \langle \mathbf{e}_i \cdot \mathbf{v}_1 \rangle^2 \tag{200}$$

$$- (\mathbf{e}_i \cdot \mathbf{v}_2)^2 + \langle \mathbf{e}_i \cdot \mathbf{v}_2 \rangle^2\Big] \tag{201}$$

$$b db d\phi_\lambda \tag{202}$$

where $\langle \mathbf{e}_i \cdot \mathbf{v}'_1 \rangle = \langle \mathbf{e}_i \cdot \mathbf{v}'_2 \rangle = \langle \mathbf{e}_i \cdot \mathbf{v}_1 \rangle = \langle \mathbf{e}_i \cdot \mathbf{v}_2 \rangle = \langle v_i \rangle$

thus the collision term is

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{1}{2} \int f(\mathbf{v}_1) f(\mathbf{v}_2) d\mathbf{v}_1 d\mathbf{v}_2 |v_r| \left[(\mathbf{e}_i \cdot \mathbf{v}'_1)^2 + (\mathbf{e}_i \cdot \mathbf{v}'_2)^2 - (\mathbf{e}_i \cdot \mathbf{v}_1)^2 - (\mathbf{e}_i \cdot \mathbf{v}_2)^2\right] b db d\phi_\lambda \tag{203}$$

The factor $\frac{1}{2}$ has been inserted so that each collision is counted only once. We express $\mathbf{v}_1$, $\mathbf{v}_2$,$\mathbf{v}'_1$, $\mathbf{v}'_2$, in terms of $\mathbf{v}_c$, $\mathbf{v}_r$, $\mathbf{v}'_r$, and obtain

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{1}{4} \int f(\mathbf{v}_1) f(\mathbf{v}_2) d\mathbf{v}_1 d\mathbf{v}_2 |v_r| \left[(\mathbf{e}_i \cdot \mathbf{v}'_r)^2 - (\mathbf{e}_i \cdot \mathbf{v}_r)^2\right] b db d\phi_\lambda \tag{204}$$

From geometry relation between $\mathbf{v}_c$, $\mathbf{v}_r$, $\mathbf{v}'_r$ and $\mathbf{v}_1$, $\mathbf{v}_2$,$\mathbf{v}'_1$, $\mathbf{v}'_2$, we have

$$(\mathbf{e}_i \cdot \mathbf{v}'_r)^2 - (\mathbf{e}_i \cdot \mathbf{v}_r)^2 = (\mathbf{e}_i \cdot \lambda)^2 (\mathbf{v}_r \cdot \lambda)^2 (1+\epsilon)^2 - 2(\mathbf{e}_i \cdot \lambda)(\mathbf{e}_i \cdot \mathbf{v}_r)(\mathbf{v}_r \cdot \lambda)(1+\epsilon) \tag{205}$$

To evaluate the eight-dimensional integral in equation 204, we write

$$\lambda = (\sin\theta_\lambda \cos\phi_\lambda, \sin\theta_\lambda \sin\phi_\lambda, \cos\theta_\lambda) \tag{206}$$

and

$$\mathbf{e}_i = (e_{iX}, e_{iY}, e_{iZ}) \tag{207}$$

then

$$\int_0^{2\pi} d\phi_\lambda \left(\mathbf{e}_i \cdot \lambda\right) = 2\pi e_{iZ} \cos\theta_\lambda \tag{208}$$

$$\int_0^{2\pi} d\phi_\lambda \left(\mathbf{e}_i \cdot \lambda\right)^2 = \pi\left(e_{iX}^2 + e_{iY}^2\right)\sin^2\theta_\lambda + 2\pi e_{iZ}^2\cos^2\theta_\lambda = \pi\left(1 - e_{iZ}^2\right)\sin^2\theta_\lambda + 2\pi e_{iZ}^2 \cos^2\theta_\lambda \tag{209}$$

These are the only factors in equation 205 which depend on $\phi_\lambda$ . The integral of 205 over $\phi_\lambda$ yields

$$\begin{aligned}\int_0^{2\pi} d\phi_\lambda \left[\left(\mathbf{e}_i \cdot \mathbf{v}_r'\right)^2 - \left(\mathbf{e}_i \cdot \mathbf{v}_r\right)^2\right] &= -4\pi e_{iZ}^2 v_r^2 \cos^2\theta_\lambda \left(1+\epsilon\right) \\ &+ \pi\left[\left(1 - e_{iZ}^2\right)\sin^2\theta_\lambda + 2e_{iZ}^2\cos^2\theta_\lambda\right] v_r^2 \cos^2\theta_\lambda \left(1+\epsilon\right)^2\end{aligned} \tag{210}$$

then for $\theta_\lambda$ could be expressed in $b$, we obtain

$$\int_0^{2a} bdb \int_0^{2\pi} d\phi_\lambda \left[\left(\mathbf{e}_i \cdot \mathbf{v}_r'\right)^2 - \left(\mathbf{e}_i \cdot \mathbf{v}_r\right)^2\right] = -4\pi a^2 v_r^2 e_{iZ}^2 \left(1+\epsilon\right) + \frac{1}{3}\pi a^2 v_r^2 \left(1 + 3e_{iZ}^2\right)\left(1+\epsilon\right)^2 . \tag{211}$$

Replace $e_{iZ}$ by $v_{ri}/|v_r|$, then $\left(\frac{\partial p_{ii}}{\partial t}\right)_c$ could rewrite as

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \pi a^2 \left(1+\epsilon\right) \int f\left(\mathbf{v}_1\right) f\left(\mathbf{v}_2\right) d\mathbf{v}_1 d\mathbf{v}_2 |v_r| \left[\frac{1}{4}\left(1+\epsilon\right)\left(v_{ri}^2 + \frac{1}{3}|v_r|^2\right) - v_{ri}^2\right] \tag{212}$$

**Assumption:**

- $f$ is triaxial Gaussian in velocity space

$$f\left(\mathbf{v}\right) = \frac{n}{\left(2\pi\right)^{3/2}\sigma_1\sigma_2\sigma_3} \exp\left(-\sum_{j=1}^{3} \frac{v_j^2}{2\sigma_j^2}\right) \tag{213}$$

where $\sigma_j^2 = p_{jj}/n$

$$\mathbf{v}_c = \frac{1}{2}\left(\mathbf{v}_1 + \mathbf{v}_2\right) \tag{214}$$

$$\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2 \tag{215}$$

$$f(\mathbf{v}_1) f(\mathbf{v}_2) d\mathbf{v}_1 d\mathbf{v}_2 = f(\mathbf{v}_r) f(\mathbf{v}_c) d\mathbf{v}_r d\mathbf{v}_c \tag{216}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \pi a^2 (1+\epsilon) \int f(\mathbf{v}_r) f(\mathbf{v}_c) d\mathbf{v}_r d\mathbf{v}_c |v_r| \left[\frac{1}{4}(1+\epsilon)\left(v_{ri}^2 + \frac{1}{3}|v_r|^2\right) - v_{ri}^2\right] \tag{217}$$

$$f(\mathbf{v}_c) = \frac{n}{(2\pi)^{3/2}\,\sigma_{c1}\sigma_{c2}\sigma_{c3}} \exp\left(-\sum_{j=1}^{3} \frac{v_{cj}^2}{2\sigma_{cj}^2}\right) \tag{218}$$

$$f(\mathbf{v}_r) = \frac{n}{(2\pi)^{3/2}\,\sigma_{r1}\sigma_{r2}\sigma_{r3}} \exp\left(-\sum_{j=1}^{3} \frac{v_{rj}^2}{2\sigma_{rj}^2}\right) \tag{219}$$

terms in [] have no relation with $\mathbf{v}_c$, so we can do the integration individually.

$$\int d\mathbf{v}_c f(\mathbf{v}_c) = n \tag{220}$$

After integration over $\mathbf{v}_c$ ,

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = n\pi a^2 (1+\epsilon) \int \frac{n}{(2\pi)^{3/2}\,\sigma_{r1}\sigma_{r2}\sigma_{r3}} \exp\left(-\sum_{j=1}^{3} \frac{v_{rj}^2}{2\sigma_{rj}^2}\right) d\mathbf{v}_r |v_r| \left[\frac{1}{4}(1+\epsilon)\left(v_{ri}^2 + \frac{1}{3}|v_r|^2\right) - v_{ri}^2\right] \tag{221}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2 \pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\,\sigma_{r1}\sigma_{r2}\sigma_{r3}} \int d\mathbf{v}_r \exp\left(-\sum_{j=1}^{3} \frac{v_{rj}^2}{2\sigma_{rj}^2}\right) |v_r| \left[\frac{1}{4}(1+\epsilon)\left(v_{ri}^2 + \frac{1}{3}|v_r|^2\right) - v_{ri}^2\right] \tag{222}$$

Then we change to polar coordinates in $\mathbf{v}_r$, $(|v_r|, \theta_v, \phi_v)$, with $\theta_v = 0$ along the $\mathbf{e}_i$ axis.

Integrals over $|v_r|$,$\theta_v$ are easily done leaving only a single integral over $\mu = \cos\theta_v$

We define two principal axes normal to $\mathbf{e}_i$ by $\mathbf{e}_j$ and $\mathbf{e}_k$

$$d\mathbf{v}_r = |v_r|^2 \sin\theta_v dv_r d\theta_v d\phi_v \tag{223}$$

$$|v_r| = v_r \tag{224}$$

$$\mathbf{e}_i \cdot \mathbf{v}_r = v_{ri} = v_r \cos\theta_v \tag{225}$$

$$v_{ri} = v_r \cos\theta_v \tag{226}$$

is the component of $\mathbf{v}$ in $\mathbf{e}_i$ direction and as $\mathbf{e}_j$ and $\mathbf{e}_k$ vertical to $\mathbf{e}_i$

$$v_{rj} = v_r \sin\theta_v \cos\phi_v \tag{227}$$

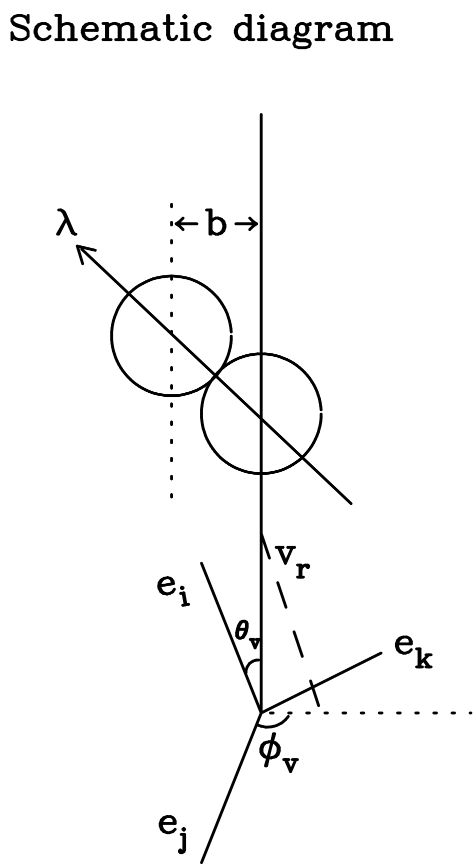


Figure 1: schematic

and

$$v_{rk} = v_r \sin\theta_v \sin\phi_v \tag{228}$$

so

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2 \pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\,\sigma_{r1}\sigma_{r2}\sigma_{r3}} \int |v_r|^2 \sin\theta_r dv_r d\theta_v d\phi_v \exp\left(-\sum_{j=1}^{3}\frac{v_{rj}^2}{2\sigma_{rj}^2}\right)|v_r|\left[\frac{1}{4}(1+\epsilon)\left(v_{ri}^2+\frac{1}{3}|v_r|^2\right)-v_{ri}^2\right] \tag{229}$$

and

$$\exp\left(-\sum_{j=1}^{3}\frac{v_{rj}^2}{2\sigma_{rj}^2}\right) = \exp\left(-\frac{v_r^2\cos^2\theta_v}{2\sigma_{ri}^2}-\frac{v_r^2\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}-\frac{v_r^2\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right) \tag{230}$$

$$= \exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)v_r^2\right] \tag{231}$$

then

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}}\int v_r^2 \sin\theta_v dv_r d\theta_v d\phi_v \tag{232}$$

$$\exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)v_r^2\right] \tag{233}$$

$$v_r\left[\frac{1}{4}(1+\epsilon)\left(v_r^2\cos^2\theta_v+\frac{1}{3}v_r^2\right)-v_r^2\cos^2\theta_v\right] \tag{234}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}}\int v_r^2 \sin\theta_v dv_r d\theta_v d\phi_v \tag{235}$$

$$\exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)v_r^2\right] \tag{236}$$

$$v_r^3\left[\frac{1}{4}(1+\epsilon)\left(\cos^2\theta_v+\frac{1}{3}\right)-\cos^2\theta_v\right] \tag{237}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}}\int \sin\theta_v d\theta_v \left[\frac{1}{4}(1+\epsilon)\left(\cos^2\theta_v+\frac{1}{3}\right)-\cos^2\theta_v\right] \tag{238}$$

$$\int d\phi_v \int v_r^5 dv_r \exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)v_r^2\right] \tag{239}$$

$$\tag{240}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}}\int_0^\pi \sin\theta_v d\theta_v \left[\frac{1}{4}(1+\epsilon)\left(\cos^2\theta_v+\frac{1}{3}\right)-\cos^2\theta_v\right] \tag{241}$$

$$\int_0^{2\pi} d\phi_v \int_0^\infty v_r^5 dv_r \exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)v_r^2\right] \tag{242}$$

$$\tag{243}$$

$$\int_0^\infty x^5 \exp\left(-ax^2\right)dx = \frac{1}{a^3} \tag{244}$$

so the last integration over $v_r$ is

$$\int_0^\infty v_r^5 dv_r \exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)v_r^2\right] = \frac{1}{\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)^3} \tag{245}$$

then

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}} \int_0^\pi \sin\theta_v d\theta_v \left[\frac{1}{4}(1+\epsilon)\left(\cos^2\theta_v + \frac{1}{3}\right) - \cos^2\theta_v\right] \tag{246}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2} + \frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2} + \frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)^3} \tag{247}$$

denote $\mu = \cos\theta_v$ the upper expression will be

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}} \int_0^1 d\mu \left[\frac{1}{4}(1+\epsilon)\left(\mu^2 + \frac{1}{3}\right) - \mu^2\right] \tag{248}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\frac{\mu^2}{2\sigma_{ri}^2} + \frac{(1-\mu^2)\cos^2\phi_v}{2\sigma_{rj}^2} + \frac{(1-\mu^2)\sin^2\phi_v}{2\sigma_{rk}^2}\right)^3} \tag{249}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}} \int_0^1 d\mu \left[\frac{1}{4}(1+\epsilon)\left(\mu^2 + \frac{1}{3}\right) - \mu^2\right] \tag{250}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\frac{\mu^2}{2\sigma_{ri}^2} + \frac{(1-\mu^2)}{2\sigma_{rj}^2}\cos^2\phi_v + \frac{(1-\mu^2)}{2\sigma_{rk}^2}\sin^2\phi_v\right)^3} \tag{251}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}} 8\sigma_{ri}^6 \int_0^1 d\mu \left[\frac{1}{4}(1+\epsilon)\left(\mu^2 + \frac{1}{3}\right) - \mu^2\right] \tag{252}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\mu^2 + \frac{(1-\mu^2)\sigma_{ri}^2}{\sigma_{rj}^2}\cos^2\phi_v + \frac{(1-\mu^2)\sigma_{ri}^2}{\sigma_{rk}^2}\sin^2\phi_v\right)^3} \tag{253}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}} 8\sigma_{ri}^6 \int_0^1 d\mu \left[\frac{1}{4}(1+\epsilon)\left(\mu^2 + \frac{1}{3}\right) - \mu^2\right] \tag{254}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\mu^2\left(\cos^2\phi_v + \sin^2\phi_v\right) + \frac{(1-\mu^2)\sigma_{ri}^2}{\sigma_{rj}^2}\cos^2\phi_v + \frac{(1-\mu^2)\sigma_{ri}^2}{\sigma_{rk}^2}\sin^2\phi_v\right)^3} \tag{255}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}} 8\sigma_{ri}^6 \int_0^1 d\mu \left[\frac{1}{4}(1+\epsilon)\left(\mu^2 + \frac{1}{3}\right) - \mu^2\right] \tag{256}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\left(\mu^2 + (1-\mu^2)\frac{\sigma_{ri}^2}{\sigma_{rj}^2}\right)\cos^2\phi_v + \left(\mu^2 + (1-\mu^2)\frac{\sigma_{ri}^2}{\sigma_{rk}^2}\right)\sin^2\phi_v\right)^3} \tag{257}$$

$$\int_0^{2\pi} \frac{1}{\left(A\cos^2\phi_v + B\sin^2\phi_v\right)^3} d\phi_v = \frac{\pi}{4}\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}} + 2A^{-\frac{3}{2}}B^{-\frac{3}{2}} + 3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{258}$$

where

$$A = \mu^2 + \left(1 - \mu^2\right) \frac{\sigma_{ri}^2}{\sigma_{rj}^2} \tag{259}$$

$$B = \mu^2 + \left(1 - \mu^2\right) \frac{\sigma_{ri}^2}{\sigma_{rk}^2} \tag{260}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2 \pi a^2 (1+\epsilon)}{(2\pi)^{3/2} \sigma_{r1}\sigma_{r2}\sigma_{r3}} 8\sigma_{ri}^6 \int_0^1 d\mu \left[\frac{1}{4}(1+\epsilon)\left(\mu^2 + \frac{1}{3}\right) - \mu^2\right] \tag{261}$$

$$\times \frac{\pi}{4}\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}} + 2A^{-\frac{3}{2}}B^{-\frac{3}{2}} + 3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{262}$$

define:

$$f_p(a,b) = \left(a^2 - a^2 b^2 + b^2\right)^{-p/2} \tag{263}$$

then

$$f_p\left(\mu, \frac{\sigma_{ri}}{\sigma_{rj}}\right) = A^{-\frac{p}{2}} \tag{264}$$

$$f_p\left(\mu, \frac{\sigma_{ri}}{\sigma_{rk}}\right) = B^{-\frac{p}{2}} \tag{265}$$

$$\sigma_{r1}\sigma_{r2}\sigma_{r3} = \sigma_{ri}\sigma_{rj}\sigma_{rk} \tag{266}$$

as

$$v_{ri} = v_{1i} - v_{2i} \tag{267}$$

$$\begin{aligned}
\sigma_{ri}^2 &= \left\langle v_{ri}^2 \right\rangle - \left\langle v_{ri} \right\rangle^2 && (268)\\
&= \left\langle (v_{1i} - v_{2i})^2 \right\rangle - \left\langle v_{1i} - v_{2i} \right\rangle^2 && (269)\\
&= \left\langle v_{1i}^2 - 2v_{1i}v_{2i} + v_{2i}^2 \right\rangle - \left\langle v_{1i} - v_{2i} \right\rangle^2 && (270)\\
&= \left\langle v_{1i}^2 - 2v_{1i}v_{2i} + v_{2i}^2 \right\rangle - \left(\left\langle v_{1i}\right\rangle - \left\langle v_{2i} \right\rangle\right)^2 && (271)\\
&= \left\langle v_{1i}^2 \right\rangle - \left\langle 2v_{1i}v_{2i} \right\rangle + \left\langle v_{2i}^2 \right\rangle - \left(\left\langle v_{1i}\right\rangle^2 - 2\left\langle v_{1i}\right\rangle\left\langle v_{2i}\right\rangle + \left\langle v_{2i} \right\rangle^2\right) && (272)\\
&= \left\langle v_{1i}^2 \right\rangle - \left\langle 2v_{1i}v_{2i} \right\rangle + \left\langle v_{2i}^2 \right\rangle - \left\langle v_{1i}\right\rangle^2 + 2\left\langle v_{1i}\right\rangle\left\langle v_{2i}\right\rangle - \left\langle v_{2i} \right\rangle^2 && (273)\\
&= \sigma_{1i}^2 + \sigma_{2i}^2 - 2\left\langle v_{1i}v_{2i}\right\rangle + 2\left\langle v_{1i}\right\rangle\left\langle v_{2i}\right\rangle && (274)
\end{aligned}$$

with individual $\mathbf{v}_1$, $\mathbf{v}_2$,

$$\left\langle v_{1i}v_{2i}\right\rangle = \left\langle v_{1i}\right\rangle\left\langle v_{2i}\right\rangle \tag{275}$$

and

$$\sigma_{1i} = \sigma_{2i} \tag{276}$$

so

$$\sigma_{ri}^2 = 2\sigma_i^2 \tag{277}$$

$$\frac{\sigma_{ri}^2}{\sigma_{rj}^2} = \frac{\sigma_i^2}{\sigma_j^2} \tag{278}$$

$$\frac{\sigma_{ri}^2}{\sigma_{rk}^2} = \frac{\sigma_i^2}{\sigma_k^2} \tag{279}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2\left(1+\epsilon\right)}{\left(2\pi\right)^{3/2}} 8\frac{\pi}{4}\frac{\sigma_{ri}^6}{\sigma_{r1}\sigma_{r2}\sigma_{r3}}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{280}$$

$$\times\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}}+2A^{-\frac{3}{2}}B^{-\frac{3}{2}}+3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{281}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2\left(1+\epsilon\right)}{\left(2\pi\right)^{3/2}} 8\frac{\pi}{4}\frac{8\sigma_i^6}{\sqrt{2}\sigma_1\sqrt{2}\sigma_2\sqrt{2}\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{282}$$

$$\times\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}}+2A^{-\frac{3}{2}}B^{-\frac{3}{2}}+3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{283}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = 2\pi^{1/2}n^2a^2\left(1+\epsilon\right)\frac{\sigma_i^6}{\sigma_1\sigma_2\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{284}$$

$$\times\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}}+2A^{-\frac{3}{2}}B^{-\frac{3}{2}}+3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{285}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = 2\pi^{1/2}n^2a^2\left(1+\epsilon\right)\frac{\sigma_i^5}{\sigma_j\sigma_k}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{286}$$

$$\times\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}}+2A^{-\frac{3}{2}}B^{-\frac{3}{2}}+3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{287}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = 2\pi^{1/2}n^2a^2\left(1+\epsilon\right)\frac{\sigma_i^5}{\sigma_j\sigma_k}\left[\left(1+\epsilon\right)J_P^i+J_Q^i\right] \tag{288}$$

We can see this equation is 2 times less than the result given by Goldreich 1978, which must be caused by some wrongly derivation.

Then we have these equations as follows:

$$-\frac{\partial P_{11}}{\partial R}+\left(\cos^2\delta P_{11}+\sin^2\delta P_{22}\right)\left(-\frac{3\cos 2\delta}{4R}+2f\cos 2\delta+\frac{\sin 2\delta}{2fR^2}-\frac{3}{4R}+2f\right)-\left(\sin^2\delta P_{11}+\cos^2\delta P_{22}\right)\left(\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}+\frac{5}{4R}\right)$$
$$+\sin\delta\cos\delta\left(P_{11}-P_{22}\right)\left(-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}-\frac{3}{2fR^2}\right)=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{11}}{\partial t}\right)_c \tag{289}$$

$$-\frac{\partial P_{22}}{\partial R}+\left(\cos^2\delta P_{11}+\sin^2\delta P_{22}\right)\left(-\frac{3}{4R}+2f+\frac{3\cos 2\delta}{4R}-2f\cos 2\delta-\frac{\sin 2\delta}{2fR^2}\right)+\left(\sin^2\delta P_{11}+\cos^2\delta P_{22}\right)\left(-\frac{5}{4R}+\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}\right)$$
$$-\sin\delta\cos\delta\left(P_{11}-P_{22}\right)\left(\frac{3}{2fR^2}-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}\right)=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{22}}{\partial t}\right)_c \tag{290}$$

$$-\frac{\partial\delta}{\partial R}=\frac{\cos^2\delta P_{11}+\sin^2\delta P_{22}}{2\left(P_{11}-P_{22}\right)}\left(-\frac{3\sin 2\delta}{2R}+4f\sin 2\delta-\frac{\cos 2\delta}{fR^2}\right)+\frac{\sin^2\delta P_{11}+\cos^2\delta P_{22}}{2\left(P_{11}-P_{22}\right)}\left(\frac{4\cos 2\delta}{fR^2}-\frac{5\sin 2\delta}{2R}\right)-\frac{\sin\delta\cos\delta}{2}\left(\frac{5\sin 2\delta}{fR^2}-\frac{4\cos 2\delta}{R}+4f\cos 2\delta\right)$$

$$-\frac{\partial P_{33}}{\partial R}-P_{33}\frac{1}{2R}=\frac{1}{fR^2\Omega}\left(\frac{\partial P_{33}}{\partial t}\right)_c \tag{291}$$

where $P_{ii}=\mathscr{N}\sigma_i^2$ and

$$\left(\frac{\partial P_{ii}}{\partial t}\right)_c=2\pi^{1/2}\mathscr{N}^2a^2\left(1+\epsilon\right)\frac{\sigma_i^5}{\sigma_j\sigma_k}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{ij}^{-\frac{1}{2}}B_{ik}^{-\frac{5}{2}}+2A_{ij}^{-\frac{3}{2}}B_{ik}^{-\frac{3}{2}}+3A_{ij}^{-\frac{5}{2}}B_{ik}^{-\frac{1}{2}}\right) \tag{292}$$

$$f_p\left(a,b\right)=\left(a^2-a^2b^2+b^2\right)^{-p/2} \tag{293}$$

then

$$A_{ij}^{-\frac{p}{2}}=f_p\left(\mu,\frac{\sigma_i}{\sigma_j}\right),\quad B_{ik}^{-\frac{p}{2}}=f_p\left(\mu,\frac{\sigma_i}{\sigma_k}\right) \tag{294}$$

the derivation of this collision term in appendix.

Then we have these equations as follows:

$$
\begin{aligned}
-\frac{\partial \mathcal{N}\sigma_1^2}{\partial R} &+ \left(\cos^2\delta\mathcal{N}\sigma_1^2 + \sin^2\delta\mathcal{N}\sigma_2^2\right)\left(-\frac{3\cos 2\delta}{4R} + 2f\cos 2\delta + \frac{\sin 2\delta}{2fR^2} - \frac{3}{4R} + 2f\right) - \left(\sin^2\delta\mathcal{N}\sigma_1^2 + \cos^2\delta\mathcal{N}\sigma_2^2\right)\left(\frac{2\sin 2\delta}{fR^2} - \frac{5\cos 2\delta}{4R} + \frac{5}{4R}\right) \\
&+ \sin\delta\cos\delta\left(\mathcal{N}\sigma_1^2 - \mathcal{N}\sigma_2^2\right)\left(\frac{-2\sin 2\delta}{R} + 2f\sin 2\delta - \frac{5\cos 2\delta}{2fR^2} - \frac{3}{2fR^2}\right) \qquad (295) \\
&= \frac{1}{fR^2\Omega}2\pi^{1/2}\mathcal{N}^2a^2\left(1+\epsilon\right)\frac{\sigma_1^5}{\sigma_2\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{12}^{-\frac{1}{2}}B_{13}^{-\frac{5}{2}} + 2A_{12}^{-\frac{3}{2}}B_{13}^{-\frac{3}{2}} + 3A_{12}^{-\frac{5}{2}}B_{13}^{-\frac{1}{2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial \mathcal{N}\sigma_2^2}{\partial R} &+ \left(\cos^2\delta\mathcal{N}\sigma_1^2 + \sin^2\delta\mathcal{N}\sigma_2^2\right)\left(-\frac{3}{4R} + 2f + \frac{3\cos 2\delta}{4R} - 2f\cos 2\delta - \frac{\sin 2\delta}{2fR^2}\right) + \left(\sin^2\delta\mathcal{N}\sigma_1^2 + \cos^2\delta\mathcal{N}\sigma_2^2\right)\left(-\frac{5}{4R} + \frac{2\sin 2\delta}{fR^2} - \frac{5\cos 2\delta}{4R}\right) \\
&- \sin\delta\cos\delta\left(\mathcal{N}\sigma_1^2 - \mathcal{N}\sigma_2^2\right)\left(\frac{3}{2fR^2} + \frac{2\sin 2\delta}{R} - 2f\sin 2\delta - \frac{5\cos 2\delta}{2fR^2}\right) \qquad (296) \\
&= \frac{1}{fR^2\Omega}2\pi^{1/2}\mathcal{N}^2a^2\left(1+\epsilon\right)\frac{\sigma_2^5}{\sigma_1\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{21}^{-\frac{1}{2}}B_{23}^{-\frac{5}{2}} + 2A_{21}^{-\frac{3}{2}}B_{23}^{-\frac{3}{2}} + 3A_{21}^{-\frac{5}{2}}B_{23}^{-\frac{1}{2}}\right)
\end{aligned}
$$

$$
-\frac{\partial\delta}{\partial R} = \frac{\cos^2\delta\sigma_1^2 + \sin^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(-\frac{3\sin 2\delta}{2R} + 4f\sin 2\delta - \frac{\cos 2\delta}{fR^2}\right) + \frac{\sin^2\delta\sigma_1^2 + \cos^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(\frac{4\cos 2\delta}{fR^2} - \frac{5\sin 2\delta}{2R}\right) - \frac{\sin\delta\cos\delta}{2}\left(\frac{5\sin 2\delta}{fR^2} - \frac{4\cos 2\delta}{R} + 4f\cos 2\delta\right)
$$

$$
-\frac{\partial \mathcal{N}\sigma_3^2}{\partial R} - \mathcal{N}\sigma_3^2\frac{1}{2R} = \frac{1}{fR^2\Omega}2\pi^{1/2}\mathcal{N}^2a^2\left(1+\epsilon\right)\frac{\sigma_3^5}{\sigma_1\sigma_2}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{31}^{-\frac{1}{2}}B_{32}^{-\frac{5}{2}} + 2A_{31}^{-\frac{3}{2}}B_{32}^{-\frac{3}{2}} + 3A_{31}^{-\frac{5}{2}}B_{32}^{-\frac{1}{2}}\right) \tag{297}
$$

## 9. Vertical Integration

In the limit of thin disk case, for a gas disk, the density distribution in $z$ direction could be determined dynamically by

$$\frac{\partial p_{\mathrm{gas}}}{\partial z} = -n\frac{\partial \Phi}{\partial z} \simeq -n\Omega_k^2 z, \tag{298}$$

where $p_{\mathrm{gas}} = \rho c_s^2$ , so

$$n\left(z\right) = n\left(0\right)\exp\left[-\frac{\Phi(r,z)}{c_s^2}\right] = n\left(0\right)\exp\left[-\frac{z^2}{2H^2}\right] \tag{299}$$

In our clumpy system, we assume that $\mathscr{N}$ follow an similar distribution:

$$\mathscr{N}\left(z\right) = \mathscr{N}\left(0\right)\exp\left[-\frac{\Phi(r,z)}{\sigma_3^2}\right] = \mathscr{N}\left(0\right)\exp\left[-\frac{z^2}{2H^2}\right], \tag{300}$$

where $H^2 = \frac{\sigma_3^2}{\Omega^2}$ is the scale height of the disk. Integrating over $z$ using $p_{ii} = \mathscr{N}(z)\sigma_i^2$ so

$$\int_{-\infty}^{+\infty} \mathscr{N}\left(z\right)dz = \sqrt{2\pi}H\mathscr{N}\left(0\right) \tag{301}$$

$$\int_{-\infty}^{+\infty} \mathscr{N}\left(z\right)^2 dz = \sqrt{\pi}H\mathscr{N}\left(0\right)^2 \tag{302}$$

here we define collision depth of the icelets,

$$\tau = \pi a^2 \int_{-\infty}^{+\infty} \mathscr{N}\left(z\right)dz = \sqrt{2\pi^3}a^2\mathscr{N}\left(0\right)H = \sqrt{2\pi^3}a^2\mathscr{N}\left(0\right)\sigma_3\Omega^{-1} \tag{303}$$

so

$$\int_{-\infty}^{+\infty} \mathscr{N}\left(z\right)dz = \frac{\tau}{\pi a^2} \tag{304}$$

$$\int_{-\infty}^{+\infty} \mathscr{N}\left(z\right)^2 dz = \frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3} \tag{305}$$

Integral equation 295, 296, 297, 297 in $z$ direction, then we have:

$$-\frac{\partial \frac{\tau}{\pi a^2}\sigma_1^2}{\partial R} + \left(\cos^2\delta \frac{\tau}{\pi a^2}\sigma_1^2 + \sin^2\delta\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(-\frac{3\cos 2\delta}{4R} + 2f\cos 2\delta + \frac{\sin 2\delta}{2fR^2} - \frac{3}{4R} + 2f\right) - \left(\sin^2\delta\frac{\tau}{\pi a^2}\sigma_1^2 + \cos^2\delta\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(\frac{2\sin 2\delta}{fR^2} - \frac{5\cos 2\delta}{4R} + \frac{5}{4R}\right) + \sin\delta\cos\delta\left(\frac{\tau}{\pi a^2}\sigma_1^2 - \frac{\tau}{\pi a^2}\sigma_2^2\right)\left(-\frac{2\sin 2\delta}{R} + 2f\sin 2\delta - \frac{5\cos 2\delta}{2fR^2} - \frac{3}{2fR^2}\right) = \frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_1^5}{\sigma_2\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{12}^{-\frac{1}{2}}B_{13}^{-\frac{5}{2}} + 2A_{12}^{-\frac{3}{2}}B_{13}^{-\frac{3}{2}} + 3A_{12}^{-\frac{5}{2}}B_{13}^{-\frac{1}{2}}\right) \tag{306}$$

$$-\frac{\partial \frac{\tau}{\pi a^2}\sigma_2^2}{\partial R} + \left(\cos^2\delta \frac{\tau}{\pi a^2}\sigma_1^2 + \sin^2\delta\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(-\frac{3}{4R} + 2f + \frac{3\cos 2\delta}{4R} - 2f\cos 2\delta - \frac{\sin 2\delta}{2fR^2}\right) + \left(\sin^2\delta\frac{\tau}{\pi a^2}\sigma_1^2 + \cos^2\delta\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(-\frac{5}{4R} + \frac{2\sin 2\delta}{fR^2} - \frac{5\cos 2\delta}{4R}\right) - \sin\delta\cos\delta\left(\frac{\tau}{\pi a^2}\sigma_1^2 - \frac{\tau}{\pi a^2}\sigma_2^2\right)\left(\frac{3}{2fR^2} - \frac{2\sin 2\delta}{R} + 2f\sin 2\delta - \frac{5\cos 2\delta}{2fR^2}\right) = \frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_2^5}{\sigma_1\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{21}^{-\frac{1}{2}}B_{23}^{-\frac{5}{2}} + 2A_{21}^{-\frac{3}{2}}B_{23}^{-\frac{3}{2}} + 3A_{21}^{-\frac{5}{2}}B_{23}^{-\frac{1}{2}}\right) \tag{307}$$

$$-\frac{\partial\delta}{\partial R} = \frac{\cos^2\delta\sigma_1^2 + \sin^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(-\frac{3\sin 2\delta}{2R} + 4f\sin 2\delta - \frac{\cos 2\delta}{fR^2}\right) + \frac{\sin^2\delta\sigma_1^2 + \cos^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(\frac{4\cos 2\delta}{fR^2} - \frac{5\sin 2\delta}{2R}\right) - \frac{\sin\delta\cos\delta}{2}\left(\frac{5\sin 2\delta}{fR^2} - \frac{4\cos 2\delta}{R} + 4f\cos 2\delta\right)$$

$$-\frac{\partial \frac{\tau}{\pi a^2}\sigma_3^2}{\partial R} - \frac{\tau}{\pi a^2}\sigma_3^2\frac{1}{2R} = \frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_3^5}{\sigma_1\sigma_2}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{31}^{-\frac{1}{2}}B_{32}^{-\frac{5}{2}} + 2A_{31}^{-\frac{3}{2}}B_{32}^{-\frac{3}{2}} + 3A_{31}^{-\frac{5}{2}}B_{32}^{-\frac{1}{2}}\right) \tag{308}$$

Distribution of $\tau$ could be estimated as

$$\dot{M} = -2\pi R u_R \Sigma = -2\pi R u_R \int_{-\infty}^{+\infty} m_c \mathscr{N}\left(z\right) dz = -2\pi R u_R \frac{\tau m_c}{\pi a^2} \tag{309}$$

so

$$\tau\left(R\right)=-\frac{\dot{M}\pi a^{2}}{2\pi Ru_{R}m_{c}}=-\frac{\dot{M}a^{2}}{-2RfR\sqrt{\frac{GM}{R}}m_{c}}=\frac{\dot{M}a^{2}}{2f\sqrt{GM}m_{c}}R^{-3/2} \tag{310}$$

$$\frac{\tau\left(R\right)}{\pi a^{2}}=\frac{\dot{M}}{2\pi f\sqrt{GM}m_{c}}R^{-3/2} \tag{311}$$

and the differential

$$\frac{\partial\frac{\tau}{\pi a^{2}}\sigma_{3}^{2}}{\partial R}=\frac{\tau}{\pi a^{2}}\frac{\partial\sigma_{3}^{2}}{\partial R}+\frac{\partial\frac{\tau}{\pi a^{2}}}{\partial R}\sigma_{3}^{2}=\frac{\dot{M}}{2\pi f\sqrt{GM}m_{c}}R^{-3/2}\frac{\partial\sigma_{3}^{2}}{\partial R}+\left(-\frac{3}{2}\right)\frac{\dot{M}}{2\pi f\sqrt{GM}m_{c}}R^{-5/2}\sigma_{3}^{2}=\frac{\tau}{\pi a^{2}}\frac{\partial\sigma_{3}^{2}}{\partial R}+\left(-\frac{3}{2R}\right)\frac{\tau}{\pi a^{2}}\sigma_{3}^{2} \tag{312}$$

also:

$$\frac{\partial\frac{\tau}{\pi a^{2}}\sigma_{1}^{2}}{\partial R}=\frac{\dot{M}}{2\pi f\sqrt{GM}m_{c}}R^{-3/2}\frac{\partial\sigma_{1}^{2}}{\partial R}+\left(-\frac{3}{2}\right)\frac{\dot{M}}{2\pi f\sqrt{GM}m_{c}}R^{-5/2}\sigma_{1}^{2}=\frac{\tau}{\pi a^{2}}\frac{\partial\sigma_{1}^{2}}{\partial R}+\left(-\frac{3}{2R}\right)\frac{\tau}{\pi a^{2}}\sigma_{1}^{2} \tag{313}$$

$$\frac{\partial\frac{\tau}{\pi a^{2}}\sigma_{2}^{2}}{\partial R}=\frac{\dot{M}}{2\pi f\sqrt{GM}m_{c}}R^{-3/2}\frac{\partial\sigma_{2}^{2}}{\partial R}+\left(-\frac{3}{2}\right)\frac{\dot{M}}{2\pi f\sqrt{GM}m_{c}}R^{-5/2}\sigma_{2}^{2}=\frac{\tau}{\pi a^{2}}\frac{\partial\sigma_{2}^{2}}{\partial R}+\left(-\frac{3}{2R}\right)\frac{\tau}{\pi a^{2}}\sigma_{2}^{2} \tag{314}$$

$$
\begin{aligned}
-\frac{\tau}{\pi a^2}\frac{\partial\sigma_1^2}{\partial R} &= \frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_1^5}{\sigma_2\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{12}^{-\frac{1}{2}}B_{13}^{-\frac{5}{2}}+2A_{12}^{-\frac{3}{2}}B_{13}^{-\frac{3}{2}}+3A_{12}^{-\frac{5}{2}}B_{13}^{-\frac{1}{2}}\right)\\
&-\left(\cos^2\delta\frac{\tau}{\pi a^2}\sigma_1^2+\sin^2\delta\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(-\frac{3\cos 2\delta}{4R}+2f\cos 2\delta+\frac{\sin 2\delta}{2fR^2}-\frac{3}{4R}+2f\right)+\left(\sin^2\delta\frac{\tau}{\pi a^2}\sigma_1^2+\cos^2\delta\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}+\frac{5}{4R}\right)\\
&-\sin\delta\cos\delta\left(\frac{\tau}{\pi a^2}\sigma_1^2-\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}-\frac{3}{2fR^2}\right)+\left(-\frac{3}{2R}\right)\frac{\tau}{\pi a^2}\sigma_1^2
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\tau}{\pi a^2}\frac{\partial\sigma_2^2}{\partial R} &= \frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_2^5}{\sigma_1\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{21}^{-\frac{1}{2}}B_{23}^{-\frac{5}{2}}+2A_{21}^{-\frac{3}{2}}B_{23}^{-\frac{3}{2}}+3A_{21}^{-\frac{5}{2}}B_{23}^{-\frac{1}{2}}\right)\\
&-\left(\cos^2\delta\frac{\tau}{\pi a^2}\sigma_1^2+\sin^2\delta\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(-\frac{3}{4R}+2f+\frac{3\cos 2\delta}{4R}-2f\cos 2\delta-\frac{\sin 2\delta}{2fR^2}\right)-\left(\sin^2\delta\frac{\tau}{\pi a^2}\sigma_1^2+\cos^2\delta\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(-\frac{5}{4R}+\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}\right)\\
&+\sin\delta\cos\delta\left(\frac{\tau}{\pi a^2}\sigma_1^2-\frac{\tau}{\pi a^2}\sigma_2^2\right)\left(\frac{3}{2fR^2}-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}\right)+\left(-\frac{3}{2R}\right)\frac{\tau}{\pi a^2}\sigma_2^2
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\tau}{\pi a^2}\frac{\partial\sigma_3^2}{\partial R} &= \frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_3^5}{\sigma_1\sigma_2}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{31}^{-\frac{1}{2}}B_{32}^{-\frac{5}{2}}+2A_{31}^{-\frac{3}{2}}B_{32}^{-\frac{3}{2}}+3A_{31}^{-\frac{5}{2}}B_{32}^{-\frac{1}{2}}\right)\\
&+\frac{\tau}{\pi a^2}\sigma_3^2\frac{1}{2R}+\left(-\frac{3}{2R}\right)\frac{\tau}{\pi a^2}\sigma_3^2
\end{aligned}
$$

$$
-\frac{\partial\delta}{\partial R}=\frac{\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(-\frac{3\sin 2\delta}{2R}+4f\sin 2\delta-\frac{\cos 2\delta}{fR^2}\right)+\frac{\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(\frac{4\cos 2\delta}{fR^2}-\frac{5\sin 2\delta}{2R}\right)-\frac{\sin\delta\cos\delta}{2}\left(\frac{5\sin 2\delta}{fR^2}-\frac{4\cos 2\delta}{R}+4f\cos 2\delta\right)
$$

$$
-\frac{\partial \sigma_1^2}{\partial R} = \frac{\pi a^2}{\tau}\frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_1^5}{\sigma_2\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{12}^{-\frac{1}{2}}B_{13}^{-\frac{5}{2}}+2A_{12}^{-\frac{3}{2}}B_{13}^{-\frac{3}{2}}+3A_{12}^{-\frac{5}{2}}B_{13}^{-\frac{1}{2}}\right)
$$

$$
-\left(\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2\right)\left(-\frac{3\cos 2\delta}{4R}+2f\cos 2\delta+\frac{\sin 2\delta}{2fR^2}-\frac{3}{4R}+2f\right)+\left(\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2\right)\left(\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}+\frac{5}{4R}\right)
$$

$$
-\sin\delta\cos\delta\left(\sigma_1^2-\sigma_2^2\right)\left(-\frac{2\sin 2\delta}{R}-2f\sin 2\delta+\frac{5\cos 2\delta}{2fR^2}-\frac{3}{2fR^2}\right)+\frac{\pi a^2}{\tau}\left(-\frac{3}{2R}\right)\frac{\tau}{\pi a^2}\sigma_1^2
$$

$$
-\frac{\partial \sigma_2^2}{\partial R} = \frac{\pi a^2}{\tau}\frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_2^5}{\sigma_1\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{21}^{-\frac{1}{2}}B_{23}^{-\frac{5}{2}}+2A_{21}^{-\frac{3}{2}}B_{23}^{-\frac{3}{2}}+3A_{21}^{-\frac{5}{2}}B_{23}^{-\frac{1}{2}}\right)
$$

$$
-\left(\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2\right)\left(-\frac{3}{4R}+2f+\frac{3\cos 2\delta}{4R}-2f\cos 2\delta-\frac{\sin 2\delta}{2fR^2}\right)-\left(\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2\right)\left(-\frac{5}{4R}+\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}\right)
$$

$$
+\sin\delta\cos\delta\left(\sigma_1^2-\sigma_2^2\right)\left(\frac{3}{2fR^2}-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}\right)+\frac{\pi a^2}{\tau}\left(-\frac{3}{2R}\right)\frac{\tau}{\pi a^2}\sigma_2^2
$$

$$
-\frac{\partial \sigma_3^2}{\partial R} = \frac{\pi a^2}{\tau}\frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^2\Omega}{2\pi^{5/2}a^4\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_3^5}{\sigma_1\sigma_2}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{31}^{-\frac{1}{2}}B_{32}^{-\frac{5}{2}}+2A_{31}^{-\frac{3}{2}}B_{32}^{-\frac{3}{2}}+3A_{31}^{-\frac{5}{2}}B_{32}^{-\frac{1}{2}}\right)
$$

$$
+\frac{\pi a^2}{\tau}\frac{\tau}{\pi a^2}\sigma_3^2\frac{1}{2R}+\frac{\pi a^2}{\tau}\left(-\frac{3}{2R}\right)\frac{\tau}{\pi a^2}\sigma_3^2
$$

$$
-\frac{\partial \delta}{\partial R} = \frac{\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(-\frac{3\sin 2\delta}{2R}+4f\sin 2\delta-\frac{\cos 2\delta}{fR^2}\right)+\frac{\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(\frac{4\cos 2\delta}{fR^2}-\frac{5\sin 2\delta}{2R}\right)-\frac{\sin\delta\cos\delta}{2}\left(\frac{5\sin 2\delta}{fR^2}-\frac{4\cos 2\delta}{R}+4f\cos 2\delta\right)
$$

$$
\begin{aligned}
-\frac{\partial\sigma_1^2}{\partial R} = {} & \frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^1\Omega}{2\pi^{3/2}a^2\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_1^5}{\sigma_2\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{12}^{-\frac{1}{2}}B_{13}^{-\frac{5}{2}}+2A_{12}^{-\frac{3}{2}}B_{13}^{-\frac{3}{2}}+3A_{12}^{-\frac{5}{2}}B_{13}^{-\frac{1}{2}}\right) \\
& -\left(\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2\right)\left(-\frac{3\cos 2\delta}{4R}+2f\cos 2\delta+\frac{\sin 2\delta}{2fR^2}-\frac{3}{4R}+2f\right)+\left(\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2\right)\left(\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}+\frac{5}{4R}\right) \\
& -\sin\delta\cos\delta\left(\sigma_1^2-\sigma_2^2\right)\left(-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}-\frac{3}{2fR^2}\right)-\frac{3\sigma_1^2}{2R}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial\sigma_2^2}{\partial R} = {} & \frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^1\Omega}{2\pi^{3/2}a^2\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_2^5}{\sigma_1\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{21}^{-\frac{1}{2}}B_{23}^{-\frac{5}{2}}+2A_{21}^{-\frac{3}{2}}B_{23}^{-\frac{3}{2}}+3A_{21}^{-\frac{5}{2}}B_{23}^{-\frac{1}{2}}\right) \\
& -\left(\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2\right)\left(-\frac{3}{4R}+2f+\frac{3\cos 2\delta}{4R}-2f\cos 2\delta-\frac{\sin 2\delta}{2fR^2}\right)-\left(\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2\right)\left(-\frac{5}{4R}+\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}\right) \\
& +\sin\delta\cos\delta\left(\sigma_1^2-\sigma_2^2\right)\left(\frac{3}{2fR^2}-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}\right)-\frac{3\sigma_2^2}{2R}
\end{aligned}
$$

$$
-\frac{\partial\sigma_3^2}{\partial R} = \frac{1}{fR^2\Omega}2\pi^{1/2}\frac{\tau^1\Omega}{2\pi^{3/2}a^2\sigma_3}a^2\left(1+\epsilon\right)\frac{\sigma_3^5}{\sigma_1\sigma_2}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{31}^{-\frac{1}{2}}B_{32}^{-\frac{5}{2}}+2A_{31}^{-\frac{3}{2}}B_{32}^{-\frac{3}{2}}+3A_{31}^{-\frac{5}{2}}B_{32}^{-\frac{1}{2}}\right)-\frac{\sigma_3^2}{R}
$$

$$
-\frac{\partial\delta}{\partial R} = \frac{\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(-\frac{3\sin 2\delta}{2R}+4f\sin 2\delta-\frac{\cos 2\delta}{fR^2}\right)+\frac{\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(\frac{4\cos 2\delta}{fR^2}-\frac{5\sin 2\delta}{2R}\right)-\frac{\sin\delta\cos\delta}{2}\left(\frac{5\sin 2\delta}{fR^2}-\frac{4\cos 2\delta}{R}+4f\cos 2\delta\right)
$$

$$
\begin{aligned}
-2\sigma_1 \frac{\partial \sigma_1}{\partial R} = -\frac{\partial \sigma_1^2}{\partial R} = \frac{\tau\left(1+\epsilon\right)}{fR^2\pi\sigma_3}\frac{\sigma_1^5}{\sigma_2\sigma_3}\int_0^1 d\mu \left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{12}^{-\frac{1}{2}}B_{13}^{-\frac{5}{2}}+2A_{12}^{-\frac{3}{2}}B_{13}^{-\frac{3}{2}}+3A_{12}^{-\frac{5}{2}}B_{13}^{-\frac{1}{2}}\right) \\
-\left(\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2\right)\left(-\frac{3\cos 2\delta}{4R}+2f\cos 2\delta+\frac{\sin 2\delta}{2fR^2}-\frac{3}{4R}+2f\right)+\left(\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2\right)\left(\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}+\frac{5}{4R}\right) \\
-\sin\delta\cos\delta\left(\sigma_1^2-\sigma_2^2\right)\left(\frac{-2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}-\frac{3}{2fR^2}\right)-\frac{3\sigma_1^2}{2R}
\end{aligned}
$$

$$
\begin{aligned}
-2\sigma_2 \frac{\partial \sigma_2}{\partial R} = -\frac{\partial \sigma_2^2}{\partial R} = \frac{\tau\left(1+\epsilon\right)}{fR^2\pi\sigma_3}\frac{\sigma_2^5}{\sigma_1\sigma_3}\int_0^1 d\mu \left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{21}^{-\frac{1}{2}}B_{23}^{-\frac{5}{2}}+2A_{21}^{-\frac{3}{2}}B_{23}^{-\frac{3}{2}}+3A_{21}^{-\frac{5}{2}}B_{23}^{-\frac{1}{2}}\right) \\
-\left(\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2\right)\left(-\frac{3}{4R}+2f+\frac{3\cos 2\delta}{4R}-2f\cos 2\delta-\frac{\sin 2\delta}{2fR^2}\right)-\left(\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2\right)\left(-\frac{5}{4R}+\frac{2\sin 2\delta}{fR^2}-\frac{5\cos 2\delta}{4R}\right) \\
+\sin\delta\cos\delta\left(\sigma_1^2-\sigma_2^2\right)\left(\frac{3}{2fR^2}-\frac{2\sin 2\delta}{R}+2f\sin 2\delta-\frac{5\cos 2\delta}{2fR^2}\right)-\frac{3\sigma_2^2}{2R}
\end{aligned}
$$

$$
-2\sigma_3 \frac{\partial \sigma_3}{\partial R} = -\frac{\partial \sigma_3^2}{\partial R} = \frac{\tau\left(1+\epsilon\right)}{fR^2\pi\sigma_3}\frac{\sigma_3^5}{\sigma_1\sigma_2}\int_0^1 d\mu \left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right]\left(3A_{31}^{-\frac{1}{2}}B_{32}^{-\frac{5}{2}}+2A_{31}^{-\frac{3}{2}}B_{32}^{-\frac{3}{2}}+3A_{31}^{-\frac{5}{2}}B_{32}^{-\frac{1}{2}}\right)-\frac{\sigma_3^2}{R}
$$

$$
-\frac{\partial \delta}{\partial R} = \frac{\cos^2\delta\sigma_1^2+\sin^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(-\frac{3\sin 2\delta}{2R}+4f\sin 2\delta-\frac{\cos 2\delta}{fR^2}\right)+\frac{\sin^2\delta\sigma_1^2+\cos^2\delta\sigma_2^2}{2\left(\sigma_1^2-\sigma_2^2\right)}\left(\frac{4\cos 2\delta}{fR^2}-\frac{5\sin 2\delta}{2R}\right)-\frac{\sin\delta\cos\delta}{2}\left(\frac{5\sin 2\delta}{fR^2}-\frac{4\cos 2\delta}{R}+4f\cos 2\delta\right)
$$

## Appendix A. For $s = 1$

$$u_i f \mathscr{N} \langle (v_j - V_j)^s \rangle = u_i f \mathscr{N} \langle (v_j - V_j) \rangle = u_i f \mathscr{N} u_j - u_i f \mathscr{N} V_j \tag{A.1}$$

$$\mathscr{N} f \langle (v_i - V_i) v_j \rangle = \mathscr{N} f \langle (v_i v_j - V_i v_j) \rangle = \mathscr{N} f \langle v_i v_j \rangle - \langle v_j V_i \rangle = f P_{ij} + f \mathscr{N} u_i u_j - f \mathscr{N} V_i u_j \tag{A.2}$$

$$u_j f \mathscr{N} \left\langle (v_i - V_i)^1 \right\rangle = u_j f \mathscr{N} \langle (v_i - V_i) \rangle = u_j f \mathscr{N} u_i - u_j f \mathscr{N} V_i \tag{A.3}$$

$$\mathscr{N} f \langle (v_j - V_j) v_i \rangle = \mathscr{N} f \langle (v_j v_i - V_j v_i) \rangle = \mathscr{N} f \langle v_j v_i \rangle - \langle v_i V_j \rangle = f P_{ij} + f \mathscr{N} u_i u_j - f \mathscr{N} V_j u_i \tag{A.4}$$

then we have

$$u_i f \mathscr{N} \langle (v_j - V_j)^s \rangle + u_j f \mathscr{N} \langle (v_i - V_i)^s \rangle - \mathscr{N} f \langle (v_i - V_i)^s v_j \rangle - \mathscr{N} f \langle (v_j - V_j)^s v_i \rangle \tag{A.5}$$

$$= u_i f \mathscr{N} u_j - u_i f \mathscr{N} V_j + u_j f \mathscr{N} u_i - u_j f \mathscr{N} V_i - (f P_{ij} + f \mathscr{N} u_i u_j - f \mathscr{N} V_i u_j) - (f P_{ij} + f \mathscr{N} u_i u_j - f \mathscr{N} V_j u_i) \tag{A.6}$$

$$= u_i f \mathscr{N} u_j - u_i f \mathscr{N} V_j + u_j f \mathscr{N} u_i - u_j f \mathscr{N} V_i - f P_{ij} - f \mathscr{N} u_i u_j + f \mathscr{N} V_i u_j - f P_{ij} - f \mathscr{N} u_i u_j + f \mathscr{N} V_j u_i \tag{A.7}$$

$$= u_i f \mathscr{N} u_j - f \mathscr{N} u_i u_j - u_i f \mathscr{N} V_j + f \mathscr{N} V_j u_i + u_j f \mathscr{N} u_i - f \mathscr{N} u_i u_j - u_j f \mathscr{N} V_i + f \mathscr{N} V_i u_j - f P_{ij} - f P_{ij} \tag{A.8}$$

$$u_i f \mathscr{N} \langle (v_j - V_j)^s \rangle + u_j f \mathscr{N} \langle (v_i - V_i)^s \rangle - \mathscr{N} f \langle (v_i - V_i)^s v_j \rangle - \mathscr{N} f \langle (v_j - V_j)^s v_i \rangle = -2 f P_{ij} \tag{A.9}$$

$$\frac{\partial P_{ij}}{\partial t} + P_{ik} \frac{\partial u_j}{\partial x_k} + P_{jk} \frac{\partial u_i}{\partial x_k} + \frac{\partial (u_k P_{ij})}{\partial x_k} - 2 f P_{ij} = \left( \frac{\partial P_{ij}}{\partial t} \right)_c$$

$$-2 f P_{ij} = -\mathbf{e}_i 2 f P_{ij} \mathbf{e}_j \tag{A.10}$$

$$= \begin{pmatrix} -2 f P_{RR}, & -2 f P_{R\theta}, & 0 \\ -2 f P_{\theta R}, & -2 f P_{\theta\theta}, & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.11}$$

$$\frac{\partial P_{ij}}{\partial t} + P_{ik} \frac{\partial u_j}{\partial x_k} + P_{jk} \frac{\partial u_i}{\partial x_k} + \frac{\partial (u_k P_{ij})}{\partial x_k} - 2 f P_{ij} = \begin{pmatrix} \left(\frac{\partial P_{RR}}{\partial t}\right)_c, & \left(\frac{\partial P_{R\theta}}{\partial t}\right)_c, & 0 \\ \left(\frac{\partial P_{\theta R}}{\partial t}\right)_c, & \left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c, & 0 \\ 0 & 0 & \left(\frac{\partial P_{zz}}{\partial t}\right)_c \end{pmatrix} \tag{A.12}$$

$$2 P_{RR} \frac{\partial u_R}{\partial R} + \frac{\partial P_{RR} u_R}{\partial R} - 4 P_{R\theta} \Omega + 2 f P_{RR} = \left( \frac{\partial P_{RR}}{\partial t} \right)_c \tag{A.13}$$

$$P_{R\theta}\frac{u_R}{R}+P_{\theta R}\frac{\partial u_R}{\partial R}+\frac{\partial P_{R\theta}u_R}{\partial R}+\frac{P_{RR}}{R}\frac{dR^2\Omega}{dR}-2\Omega P_{\theta\theta}-2fP_{R\theta}=\left(\frac{\partial P_{R\theta}}{\partial t}\right)_c \tag{A.14}$$

$$2P_{\theta\theta}\frac{u_R}{R}+\frac{\partial P_{\theta\theta}u_R}{\partial R}+\frac{2P_{R\theta}}{R}\frac{dR^2\Omega}{dR}-2fP_{\theta\theta}=\left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c \tag{A.15}$$

$$\frac{\partial P_{zz}u_R}{\partial R}=\left(\frac{\partial P_{zz}}{\partial t}\right)_c \tag{A.16}$$