1. Boltzmann Equation

Boltzmann equation read as

$$\frac{\partial f}{\partial t} + v_\alpha \frac{\partial f}{\partial x_\alpha} - \frac{\partial U}{\partial x_\alpha}\frac{\partial f}{\partial v_\alpha} = \left(\frac{\partial f}{\partial t}\right)_c . \tag{1}$$

Here we use Cartesian coordinates(x_α, v_α), $U(\mathbf{x})$ is the gravitational potential and $(\partial f/\partial t)_c$ is the ratio of change of f due to collisions.

2. Moment Equations

2.1. Defination

Number density $n(\mathbf{x})$:

$$n = \int f \, d\mathbf{v} \tag{2}$$

Mean velocity vector $\mathbf{u}(\mathbf{x})$:

$$n u_\alpha = \int f v_\alpha \, d\mathbf{v} \tag{3}$$

Pressure tensor $p_{\alpha\beta\gamma\ldots}$:

$$p_{\alpha\beta\gamma\ldots} = \int f \left(v_\alpha - u_\alpha\right)\left(v_\beta - u_\beta\right)\left(v_\gamma - u_\gamma\right)\ldots d\mathbf{v} \tag{4}$$

2.2. Three Sets of Moment Equations

We obtain three sets of moment equations by multiplying equation 1 successively by **1**, v_α and $v_\alpha v_\beta$ and integrating over $d\mathbf{v}$

$$\frac{\partial n}{\partial t} + \frac{\partial \left(n u_\alpha\right)}{\partial x_\alpha} = \left(\frac{\partial n}{\partial t}\right)_c \tag{5}$$

$$\frac{\partial\left(nu_\alpha\right)}{\partial t}+\frac{\partial}{\partial x_\beta}\left(p_{\alpha\beta}+nu_\alpha u_\beta\right)+n\frac{\partial U}{\partial x_\alpha}=\left(\frac{\partial}{\partial t}nu_\alpha\right)_c \tag{6}$$

$$\frac{\partial}{\partial t}\left(p_{\alpha\beta}+nu_\alpha u_\beta\right)+\frac{\partial}{\partial x_\gamma}\left(p_{\alpha\beta\gamma}+u_\beta p_{\alpha\gamma}+u_\gamma p_{\alpha\beta}+u_\alpha p_{\beta\gamma}+nu_\alpha u_\beta u_\gamma\right)$$
$$+n\left(u_\alpha\frac{\partial U}{\partial x_\beta}+u_\beta\frac{\partial U}{\partial x_\alpha}\right)=\left[\frac{\partial}{\partial t}\left(p_{\alpha\beta}+nu_\alpha u_\beta\right)\right]_c \tag{7}$$

2.3. Simplify

Assumption:

- identical
- indestructible
- imperfectly elastic spheres
- neglect particle spins

This model ensures that

- $\left(\partial n/\partial t\right)_c=0$
- $\left(\partial nu_\alpha/\partial t\right)_c=0$

ASSUME: Random motions are much smaller than the mean orbital motions, so that the term $\partial p_{\alpha\beta\gamma}/\partial x_\gamma$ can be neglected. After some algebra, equations 5, 6, 7 becomes

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x_\alpha}\left(nu_\alpha\right)=0 \tag{8}$$

$$\frac{\partial u_\alpha}{\partial t}+u_\beta\frac{\partial u_\alpha}{\partial x_\beta}=-\frac{\partial U}{\partial x_\alpha}-\frac{1}{n}\frac{\partial p_{\alpha\beta}}{\partial x_\beta} \tag{9}$$

$$\frac{\partial p_{\alpha\beta}}{\partial t} + p_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma} + p_{\beta\gamma}\frac{\partial u_\alpha}{\partial x_\gamma} + \frac{\partial}{\partial x_\gamma}\left(p_{\alpha\beta}u_\gamma\right) = \left(\frac{\partial p_{\alpha\beta}}{\partial t}\right)_c \tag{10}$$

These are the continuity equation, Euler's equation, and the viscous-stress equation. We focus our attention on the last equation.

3. Viscous-stress Equation

3.1. Symmetrical Simplify

- Azimuthal symmetry

then we can rewrite equation 10 into cylindrical coordinates (r, θ, z) where the disk is symmetric about $z = 0$.

- In the limit of small random motions.

  $\hookrightarrow$ we can replace $\mathbf{u}(\mathbf{x})$ by $r\Omega(r)\mathbf{e}_\theta = (rdU/dr)^{1/2}\mathbf{e}_\theta$

The resulting equations are

$$\frac{\partial p_{rr}}{\partial t} - 4\Omega p_{r\theta} = \left(\frac{\partial p_{rr}}{\partial t}\right)_c \tag{11}$$

$$\frac{\partial p_{r\theta}}{\partial t} + \frac{p_{rr}}{r}\frac{d}{d\,r}\left(r^2\Omega\right) - 2\Omega p_{\theta\theta} = \left(\frac{\partial p_{r\theta}}{\partial t}\right)_c \tag{12}$$

$$\frac{\partial p_{\theta\theta}}{\partial t} + \frac{2p_{r\theta}}{r}\frac{d}{d\,r}\left(r^2\Omega\right) = \left(\frac{\partial p_{\theta\theta}}{\partial t}\right)_c \tag{13}$$

$$\frac{\partial p_{zz}}{\partial t} = \left(\frac{\partial p_{zz}}{\partial t}\right)_c \tag{14}$$

symmetry about $z = 0 \Longrightarrow p_{rz} = p_{\theta z} = 0$

We can go futher if $\Omega \sim r^{\alpha}$, here we adopt $\Omega \sim r^{-\frac{3}{2}}$

$$\frac{1}{r}\frac{d(r^2\Omega)}{dr} = \Omega\frac{d\ln(r^2\Omega)}{d\ln r} = \frac{\Omega}{2} \tag{15}$$

so

$$\frac{\partial p_{rr}}{\partial t} - 4\Omega p_{r\theta} = \left(\frac{\partial p_{rr}}{\partial t}\right)_c \tag{16}$$

$$\frac{\partial p_{r\theta}}{\partial t} + \frac{\Omega}{2}p_{rr} - 2\Omega p_{\theta\theta} = \left(\frac{\partial p_{r\theta}}{\partial t}\right)_c \tag{17}$$

$$\frac{\partial p_{\theta\theta}}{\partial t} + \Omega p_{r\theta} = \left(\frac{\partial p_{\theta\theta}}{\partial t}\right)_c \tag{18}$$

$$\frac{\partial p_{zz}}{\partial t} = \left(\frac{\partial p_{zz}}{\partial t}\right)_c \tag{19}$$

3.2. Principal Axes Simplify

For any given position, we can find the principal axes $\mathbf{e}_i$, $i = 1, 2, 3$ of the pressure tensor. We put $\mathbf{e}_3 = \mathbf{e}_z$ and define an angle δ by

$$\mathbf{e}_z \sin\delta = \mathbf{e}_r \times \mathbf{e}_1$$

- Principal Axes Transformation

$$p_{11} = \frac{p_{rr} + p_{\theta\theta} + \sqrt{(p_{rr} - p_{\theta\theta})^2 + 4p_{r\theta}^2}}{2} \tag{20}$$

$$p_{22} = \frac{p_{rr} + p_{\theta\theta} - \sqrt{(p_{rr} - p_{\theta\theta})^2 + 4p_{r\theta}^2}}{2} \tag{21}$$

$$\tan\delta = \frac{p_{r\theta}}{p_{11} - p_{\theta\theta}} = \frac{2p_{r\theta}}{(p_{rr} - p_{\theta\theta}) + \sqrt{(p_{rr} - p_{\theta\theta})^2 + 4p_{r\theta}^2}} \tag{22}$$

Without loss of generality, here we require $|\delta| \leq \pi/4$

and in the principal axis basis, the viscous stress equations read

$$\frac{\partial p_{11}}{\partial t} - S\Omega \sin 2\delta p_{11} = \left(\frac{\partial p_{11}}{\partial t}\right)_c \tag{23}$$

$$\frac{\partial p_{22}}{\partial t} + S\Omega \sin 2\delta p_{22} = \left(\frac{\partial p_{22}}{\partial t}\right)_c \tag{24}$$

$$p_{11}\left(2 - S\cos^2\delta\right) - p_{22}\left(2 - S\sin^2\delta\right) = 0 \tag{25}$$

$$\frac{\partial p_{33}}{\partial t} = \left(\frac{\partial p_{33}}{\partial t}\right)_c \tag{26}$$

where $S = -\frac{d\,ln\Omega}{d\,lnr}$

- Specialize to a Keplerian rotation law, $\Omega \propto r^{-3/2}$, so $S = -\frac{d\,ln\Omega}{d\,lnr} = \frac{3}{2}$

then the viscous stress equations finally re-casted as

$$\frac{\partial p_{11}}{\partial t} - \frac{3\Omega}{2} \sin 2\delta p_{11} = \left(\frac{\partial p_{11}}{\partial t}\right)_c \tag{27}$$

$$\frac{\partial p_{22}}{\partial t} + \frac{3\Omega}{2} \sin 2\delta p_{22} = \left(\frac{\partial p_{22}}{\partial t}\right)_c \tag{28}$$

$$p_{11}\left(1 + 3\sin^2\delta\right) - p_{22}\left(1 + 3\cos^2\delta\right) = 0 \tag{29}$$

$$\frac{\partial p_{33}}{\partial t} = \left(\frac{\partial p_{33}}{\partial t}\right)_c \tag{30}$$

Detail derivation is displayed in appendix

4. Collision Terms

Now we evaluate the terms $(\partial p_{ii}/\partial t)_c$. Consider a collision between two particles with vlocities $\mathbf{v}_1$ and $\mathbf{v}_2$ which changes the velocities to $\mathbf{v}_1'$ and $\mathbf{v}_2'$. Relative velocities before and after the collision are $\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_r' = \mathbf{v}_1' - \mathbf{v}_2'$. The center-of-mass velocity $\mathbf{v}_c$ is conserved and the relative motion of the two particles is found by assuming that one acts as a fixed center of force while the other has the reduced mass $\mu = m/2$.

Assumption about energy dissipation

- The impact conserves the relatives tangential velocity
- The impact reduce the absolute value of the relative normal velocity by a factor of ϵ

We have

$$\mathbf{v}_r' = \mathbf{v}_r - \lambda\left(1+\epsilon\right)\mathbf{v}_r \cdot \lambda \tag{31}$$

$$\lambda \cdot \mathbf{v}_r = |v_r|\left(1 - b^2/4a^2\right)^{1/2} \tag{32}$$

The collision dynamics are conveniently described in a frame $(X, Y, Z) \equiv (r, \theta, \phi)$ whose Z axis $(\theta = 0)$ is the direction of $\mathbf{v}_r$. This means we treat the collision as what we usually do in statistical physics.

A collision is completely specified by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\lambda = (\theta_\lambda, \phi_\lambda)$ or, alternatively by $\mathbf{v}_1$, $\mathbf{v}_2$, b, and ϕ_λ since $\lambda \cdot \mathbf{v}_r = |v_r| \cos\theta_\lambda$ is given in terms of b.

Collision rate per unit volume in the interval $\mathbf{v}_1 \to \mathbf{v}_1 + d\mathbf{v}_1$, $\mathbf{v}_2 \to \mathbf{v}_2 + d\mathbf{v}_2$, $b \to b + db$, $\phi_\lambda \to \phi_\lambda + d\phi_\lambda$ is

$$f\left(\mathbf{v}_1\right) f\left(\mathbf{v}_2\right) d\mathbf{v}_1 d\mathbf{v}_2 |\mathbf{v}_r| b db d\phi_\lambda, \tag{33}$$

Expression $\left(\frac{\partial p_{ii}}{\partial t}\right)_c$ is not differential with time but the alteration of p_{ii} in an time interval, for each collision,

$$\left(\frac{\Delta p_{ii}}{\Delta t}\right)_c = \frac{p'_{ii} - p_{ii}}{\Delta t} \tag{34}$$

Definition of p_{ii} is

$$p_{ii} = \int f(v_i - \langle v_i \rangle)^2 d\mathbf{v} = \int f(v_i^2 - 2v_i \langle v_i \rangle + \langle v_i \rangle^2) d\mathbf{v} \tag{35}$$

$$= \int f v_i^2 d\mathbf{v} - \int f 2v_i \langle v_i \rangle \, d\mathbf{v} + \int f \langle v_i \rangle^2 \, d\mathbf{v} \tag{36}$$

$$= \int f v_i^2 d\mathbf{v} - \int f \langle v_i \rangle^2 \, d\mathbf{v} \tag{37}$$

with this formula, which could easily be simplified as $p_{ii} = n \langle v_i^2 \rangle - n \langle v_i \rangle^2 = n\sigma_{ii}^2$, the collision term could be expressed as

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{1}{2} \int f(\mathbf{v}_1) f(\mathbf{v}_2) d\mathbf{v}_1 d\mathbf{v}_2 |v_r| \tag{38}$$

$$\Big[(\mathbf{e}_i \cdot \mathbf{v}'_1)^2 - \langle \mathbf{e}_i \cdot \mathbf{v}'_1 \rangle^2 \tag{39}$$

$$+ (\mathbf{e}_i \cdot \mathbf{v}'_2)^2 - \langle \mathbf{e}_i \cdot \mathbf{v}'_2 \rangle^2 \tag{40}$$

$$- (\mathbf{e}_i \cdot \mathbf{v}_1)^2 + \langle \mathbf{e}_i \cdot \mathbf{v}_1 \rangle^2 \tag{41}$$

$$- (\mathbf{e}_i \cdot \mathbf{v}_2)^2 + \langle \mathbf{e}_i \cdot \mathbf{v}_2 \rangle^2\Big] \tag{42}$$

$$b db d\phi_\lambda \tag{43}$$

where $\langle \mathbf{e}_i \cdot \mathbf{v}'_1 \rangle = \langle \mathbf{e}_i \cdot \mathbf{v}'_2 \rangle = \langle \mathbf{e}_i \cdot \mathbf{v}_1 \rangle = \langle \mathbf{e}_i \cdot \mathbf{v}_2 \rangle = \langle v_i \rangle$

thus the collision term is

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{1}{2} \int f(\mathbf{v}_1) f(\mathbf{v}_2) d\mathbf{v}_1 d\mathbf{v}_2 |v_r| \left[(\mathbf{e}_i \cdot \mathbf{v}'_1)^2 + (\mathbf{e}_i \cdot \mathbf{v}'_2)^2 - (\mathbf{e}_i \cdot \mathbf{v}_1)^2 - (\mathbf{e}_i \cdot \mathbf{v}_2)^2\right] b db d\phi_\lambda \tag{44}$$

The factor $\frac{1}{2}$ has been inserted so that each collision is counted only once. We express $\mathbf{v}_1$, $\mathbf{v}_2$,$\mathbf{v}_1'$, $\mathbf{v}_2'$, in terms of $\mathbf{v}_c$, $\mathbf{v}_r$, $\mathbf{v}_r'$, and obtain

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{1}{4}\int f\left(\mathbf{v}_1\right) f\left(\mathbf{v}_2\right) d\mathbf{v}_1 d\mathbf{v}_2 |v_r| \left[\left(\mathbf{e}_i\cdot\mathbf{v}_r'\right)^2 - \left(\mathbf{e}_i\cdot\mathbf{v}_r\right)^2\right] bdbd\phi_\lambda \tag{45}$$

From geometry relation between $\mathbf{v}_c$, $\mathbf{v}_r$, $\mathbf{v}_r'$ and $\mathbf{v}_1$, $\mathbf{v}_2$,$\mathbf{v}_1'$, $\mathbf{v}_2'$, we have

$$\left(\mathbf{e}_i\cdot\mathbf{v}_r'\right)^2 - \left(\mathbf{e}_i\cdot\mathbf{v}_r\right)^2 = \left(\mathbf{e}_i\cdot\lambda\right)^2\left(\mathbf{v}_r\cdot\lambda\right)^2(1+\epsilon)^2 - 2\left(\mathbf{e}_i\cdot\lambda\right)\left(\mathbf{e}_i\cdot\mathbf{v}_r\right)\left(\mathbf{v}_r\cdot\lambda\right)(1+\epsilon) \tag{46}$$

To evaluate the eight-dimensional integral in equation 45, we write

$$\lambda = \left(\sin\theta_\lambda\cos\phi_\lambda, \sin\theta_\lambda\sin\phi_\lambda, \cos\theta_\lambda\right) \tag{47}$$

and

$$\mathbf{e}_i = \left(e_{iX}, e_{iY}, e_{iZ}\right) \tag{48}$$

then

$$\int_0^{2\pi} d\phi_\lambda\left(\mathbf{e}_i\cdot\lambda\right) = 2\pi e_{iZ}\cos\theta_\lambda \tag{49}$$

$$\int_0^{2\pi} d\phi_\lambda\left(\mathbf{e}_i\cdot\lambda\right)^2 = \pi\left(e_{iX}^2+e_{iY}^2\right)\sin^2\theta_\lambda + 2\pi e_{iZ}^2\cos^2\theta_\lambda = \pi\left(1-e_{iZ}^2\right)\sin^2\theta_\lambda + 2\pi e_{iZ}^2\cos^2\theta_\lambda \tag{50}$$

These are the only factors in equation 46 which depend on ϕ_λ . The integral of 46 over ϕ_λ yields

$$\int_0^{2\pi} d\phi_\lambda\left[\left(\mathbf{e}_i\cdot\mathbf{v}_r'\right)^2 - \left(\mathbf{e}_i\cdot\mathbf{v}_r\right)^2\right] = -4\pi e_{iZ}^2 v_r^2\cos^2\theta_\lambda(1+\epsilon)$$
$$+\pi\left[\left(1-e_{iZ}^2\right)\sin^2\theta_\lambda + 2e_{iZ}^2\cos^2\theta_\lambda\right] v_r^2\cos^2\theta_\lambda(1+\epsilon)^2 \tag{51}$$

then for θ_λ could be expressed in b, we obtain

$$\int_0^{2a} bdb\int_0^{2\pi} d\phi_\lambda\left[\left(\mathbf{e}_i\cdot\mathbf{v}_r'\right)^2 - \left(\mathbf{e}_i\cdot\mathbf{v}_r\right)^2\right] = -4\pi a^2 v_r^2 e_{iZ}^2(1+\epsilon) + \frac{1}{3}\pi a^2 v_r^2\left(1+3e_{iZ}^2\right)(1+\epsilon)^2. \tag{52}$$

Replace e_{iZ} by $v_{ri}/|v_r|$, then $\left(\frac{\partial p_{ii}}{\partial t}\right)_c$ could rewrite as

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \pi a^2 (1+\epsilon) \int f(\mathbf{v}_1) f(\mathbf{v}_2) d\mathbf{v}_1 d\mathbf{v}_2 |v_r| \left[\frac{1}{4}(1+\epsilon)\left(v_{ri}^2 + \frac{1}{3}|v_r|^2\right) - v_{ri}^2\right] \tag{53}$$

Assumption:

- f is triaxial Gaussian in velocity space

$$f(\mathbf{v}) = \frac{n}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3} \exp\left(-\sum_{j=1}^{3} \frac{v_j^2}{2\sigma_j^2}\right) \tag{54}$$

where $\sigma_j^2 = p_{jj}/n$

$$\mathbf{v}_c = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2) \tag{55}$$

$$\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2 \tag{56}$$

$$f(\mathbf{v}_1)f(\mathbf{v}_2)d\mathbf{v}_1 d\mathbf{v}_2 = f(\mathbf{v}_r)f(\mathbf{v}_c)d\mathbf{v}_r d\mathbf{v}_c \tag{57}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \pi a^2 (1+\epsilon) \int f(\mathbf{v}_r) f(\mathbf{v}_c) d\mathbf{v}_r d\mathbf{v}_c |v_r| \left[\frac{1}{4}(1+\epsilon)\left(v_{ri}^2 + \frac{1}{3}|v_r|^2\right) - v_{ri}^2\right] \tag{58}$$

$$f(\mathbf{v}_c) = \frac{n}{(2\pi)^{3/2}\sigma_{c1}\sigma_{c2}\sigma_{c3}} \exp\left(-\sum_{j=1}^{3} \frac{v_{cj}^2}{2\sigma_{cj}^2}\right) \tag{59}$$

$$f(\mathbf{v}_r) = \frac{n}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}} \exp\left(-\sum_{j=1}^{3} \frac{v_{rj}^2}{2\sigma_{rj}^2}\right) \tag{60}$$

terms in [] have no relation with $\mathbf{v}_c$, so we can do the integration individually.

$$\int d\mathbf{v}_c f(\mathbf{v}_c) = n \tag{61}$$

After integration over $\mathbf{v}_c$,

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = n\pi a^2 (1+\epsilon) \int \frac{n}{(2\pi)^{3/2}\,\sigma_{r1}\sigma_{r2}\sigma_{r3}} \exp\left(-\sum_{j=1}^{3}\frac{v_{rj}^2}{2\sigma_{rj}^2}\right) d\mathbf{v}_r |v_r| \left[\frac{1}{4}(1+\epsilon)\left(v_{ri}^2+\frac{1}{3}|v_r|^2\right)-v_{ri}^2\right] \tag{62}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\,\sigma_{r1}\sigma_{r2}\sigma_{r3}} \int d\mathbf{v}_r \exp\left(-\sum_{j=1}^{3}\frac{v_{rj}^2}{2\sigma_{rj}^2}\right) |v_r| \left[\frac{1}{4}(1+\epsilon)\left(v_{ri}^2+\frac{1}{3}|v_r|^2\right)-v_{ri}^2\right] \tag{63}$$

Schematic diagram

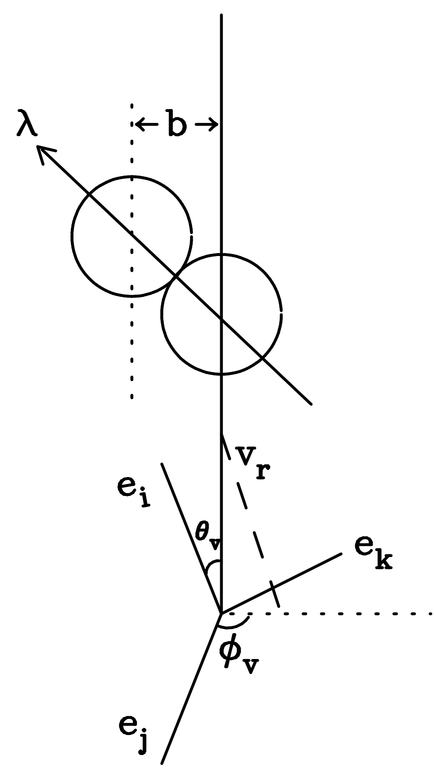

Fig. 1.— schematic

Then we change to polar coordinates in $\mathbf{v}_r$, $(|v_r|, \theta_v, \phi_v)$, with $\theta_v = 0$ along the $\mathbf{e}_i$ axis.

Integrals over $|v_r|$,θ_v are easily done leaving only a single integral over $\mu = \cos\theta_v$

We define two principal axes normal to $\mathbf{e}_i$ by $\mathbf{e}_j$ and $\mathbf{e}_k$

$$d\mathbf{v}_r = |v_r|^2 \sin\theta_v dv_r d\theta_v d\phi_v \tag{64}$$

$$|v_r| = v_r \tag{65}$$

$$\mathbf{e}_i \cdot \mathbf{v}_r = v_{ri} = v_r \cos\theta_v \tag{66}$$

$$v_{ri} = v_r \cos\theta_v \tag{67}$$

is the component of $\mathbf{v}$ in $\mathbf{e}_i$ direction and as $\mathbf{e}_j$ and $\mathbf{e}_k$ vertical to $\mathbf{e}_i$

$$v_{rj} = v_r \sin\theta_v \cos\phi_v \tag{68}$$

and

$$v_{rk} = v_r \sin\theta_v \sin\phi_v \tag{69}$$

so

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2 \pi a^2 (1+\epsilon)}{(2\pi)^{3/2} \sigma_{r1}\sigma_{r2}\sigma_{r3}} \int |v_r|^2 \sin\theta_r dv_r d\theta_v d\phi_v \exp\left(-\sum_{j=1}^{3} \frac{v_{rj}^2}{2\sigma_{rj}^2}\right) |v_r| \left[\frac{1}{4}(1+\epsilon)\left(v_{ri}^2 + \frac{1}{3}|v_r|^2\right) - v_{ri}^2\right] \tag{70}$$

and

$$\begin{aligned} \exp\left(-\sum_{j=1}^{3} \frac{v_{rj}^2}{2\sigma_{rj}^2}\right) &= \exp\left(-\frac{v_r^2 \cos^2\theta_v}{2\sigma_{ri}^2} - \frac{v_r^2 \sin^2\theta_v \cos^2\phi_v}{2\sigma_{rj}^2} - \frac{v_r^2 \sin^2\theta_v \sin^2\phi_v}{2\sigma_{rk}^2}\right) && (71) \\ &= \exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2} + \frac{\sin^2\theta_v \cos^2\phi_v}{2\sigma_{rj}^2} + \frac{\sin^2\theta_v \sin^2\phi_v}{2\sigma_{rk}^2}\right) v_r^2\right] && (72) \end{aligned}$$

then

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2 \pi a^2 (1+\epsilon)}{(2\pi)^{3/2} \sigma_{r1}\sigma_{r2}\sigma_{r3}} \int v_r^2 \sin\theta_v dv_r d\theta_v d\phi_v \tag{73}$$

$$\exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2} + \frac{\sin^2\theta_v \cos^2\phi_v}{2\sigma_{rj}^2} + \frac{\sin^2\theta_v \sin^2\phi_v}{2\sigma_{rk}^2}\right) v_r^2\right] \tag{74}$$

$$v_r \left[\frac{1}{4}(1+\epsilon)\left(v_r^2 \cos^2\theta_v + \frac{1}{3} v_r^2\right) - v_r^2 \cos^2\theta_v\right] \tag{75}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2 \pi a^2 (1+\epsilon)}{(2\pi)^{3/2} \sigma_{r1}\sigma_{r2}\sigma_{r3}} \int v_r^2 \sin\theta_v dv_r d\theta_v d\phi_v \tag{76}$$

$$\exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2} + \frac{\sin^2\theta_v \cos^2\phi_v}{2\sigma_{rj}^2} + \frac{\sin^2\theta_v \sin^2\phi_v}{2\sigma_{rk}^2}\right) v_r^2\right] \tag{77}$$

$$v_r^3 \left[\frac{1}{4}(1+\epsilon)\left(\cos^2\theta_v + \frac{1}{3}\right) - \cos^2\theta_v\right] \tag{78}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2 \pi a^2 (1+\epsilon)}{(2\pi)^{3/2} \sigma_{r1}\sigma_{r2}\sigma_{r3}} \int \sin\theta_v d\theta_v \left[\frac{1}{4}(1+\epsilon)\left(\cos^2\theta_v + \frac{1}{3}\right) - \cos^2\theta_v\right] \tag{79}$$

$$\int d\phi_v \int v_r^5 dv_r \exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2} + \frac{\sin^2\theta_v \cos^2\phi_v}{2\sigma_{rj}^2} + \frac{\sin^2\theta_v \sin^2\phi_v}{2\sigma_{rk}^2}\right) v_r^2\right] \tag{80}$$

$$\tag{81}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2 \pi a^2 (1+\epsilon)}{(2\pi)^{3/2} \sigma_{r1}\sigma_{r2}\sigma_{r3}} \int_0^\pi \sin\theta_v d\theta_v \left[\frac{1}{4}(1+\epsilon)\left(\cos^2\theta_v + \frac{1}{3}\right) - \cos^2\theta_v\right] \tag{82}$$

$$\int_0^{2\pi} d\phi_v \int_0^\infty v_r^5 dv_r \exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2} + \frac{\sin^2\theta_v \cos^2\phi_v}{2\sigma_{rj}^2} + \frac{\sin^2\theta_v \sin^2\phi_v}{2\sigma_{rk}^2}\right) v_r^2\right] \tag{83}$$

$$\tag{84}$$

$$\int_0^\infty x^5 \exp\left(-ax^2\right) dx = \frac{1}{a^3} \tag{85}$$

so the last integration over v_r is

$$\int_0^\infty v_r^5 dv_r \exp\left[-\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)v_r^2\right]=\frac{1}{\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)^3} \tag{86}$$

then

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c=\frac{n^2\pi a^2(1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}}\int_0^\pi \sin\theta_v d\theta_v\left[\frac{1}{4}(1+\epsilon)\left(\cos^2\theta_v+\frac{1}{3}\right)-\cos^2\theta_v\right] \tag{87}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\frac{\cos^2\theta_v}{2\sigma_{ri}^2}+\frac{\sin^2\theta_v\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{\sin^2\theta_v\sin^2\phi_v}{2\sigma_{rk}^2}\right)^3} \tag{88}$$

denote $\mu=\cos\theta_v$ the upper expression will be

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c=\frac{n^2\pi a^2(1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}}\int_{-1}^1 d\mu\left[\frac{1}{4}(1+\epsilon)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{89}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\frac{\mu^2}{2\sigma_{ri}^2}+\frac{(1-\mu^2)\cos^2\phi_v}{2\sigma_{rj}^2}+\frac{(1-\mu^2)\sin^2\phi_v}{2\sigma_{rk}^2}\right)^3} \tag{90}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c=\frac{n^2\pi a^2(1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}}\int_{-1}^1 d\mu\left[\frac{1}{4}(1+\epsilon)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{91}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\frac{\mu^2}{2\sigma_{ri}^2}+\frac{(1-\mu^2)}{2\sigma_{rj}^2}\cos^2\phi_v+\frac{(1-\mu^2)}{2\sigma_{rk}^2}\sin^2\phi_v\right)^3} \tag{92}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c=\frac{n^2\pi a^2(1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}}8\sigma_{ri}^6\int_{-1}^1 d\mu\left[\frac{1}{4}(1+\epsilon)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{93}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\mu^2+\frac{(1-\mu^2)\sigma_{ri}^2}{\sigma_{rj}^2}\cos^2\phi_v+\frac{(1-\mu^2)\sigma_{ri}^2}{\sigma_{rk}^2}\sin^2\phi_v\right)^3} \tag{94}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c=\frac{n^2\pi a^2(1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}}8\sigma_{ri}^6\int_{-1}^1 d\mu\left[\frac{1}{4}(1+\epsilon)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{95}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\mu^2\left(\cos^2\phi_v+\sin^2\phi_v\right)+\frac{(1-\mu^2)\sigma_{ri}^2}{\sigma_{rj}^2}\cos^2\phi_v+\frac{(1-\mu^2)\sigma_{ri}^2}{\sigma_{rk}^2}\sin^2\phi_v\right)^3} \tag{96}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}} 8\sigma_{ri}^6 \int_{-1}^{1} d\mu \left[\frac{1}{4}(1+\epsilon)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{97}$$

$$\int_0^{2\pi} d\phi_v \frac{1}{\left(\left(\mu^2+(1-\mu^2)\frac{\sigma_{ri}^2}{\sigma_{rj}^2}\right)\cos^2\phi_v + \left(\mu^2+(1-\mu^2)\frac{\sigma_{ri}^2}{\sigma_{rk}^2}\right)\sin^2\phi_v\right)^3} \tag{98}$$

$$\int_0^{2\pi} \frac{1}{\left(A\cos^2\phi_v + B\sin^2\phi_v\right)^3} d\phi_v = \frac{\pi}{4}\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}} + 2A^{-\frac{3}{2}}B^{-\frac{3}{2}} + 3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{99}$$

where

$$A = \mu^2 + \left(1-\mu^2\right)\frac{\sigma_{ri}^2}{\sigma_{rj}^2} \tag{100}$$

$$B = \mu^2 + \left(1-\mu^2\right)\frac{\sigma_{ri}^2}{\sigma_{rk}^2} \tag{101}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{2n^2\pi a^2 (1+\epsilon)}{(2\pi)^{3/2}\sigma_{r1}\sigma_{r2}\sigma_{r3}} 8\sigma_{ri}^6 \int_{0}^{1} d\mu \left[\frac{1}{4}(1+\epsilon)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{102}$$

$$\times\frac{\pi}{4}\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}} + 2A^{-\frac{3}{2}}B^{-\frac{3}{2}} + 3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{103}$$

define:

$$f_p(a,b) = \left(a^2 - a^2b^2 + b^2\right)^{-p/2} \tag{104}$$

then

$$f_p\left(\mu, \frac{\sigma_{ri}}{\sigma_{rj}}\right) = A^{-\frac{p}{2}} \tag{105}$$

$$f_p\left(\mu, \frac{\sigma_{ri}}{\sigma_{rk}}\right) = B^{-\frac{p}{2}} \tag{106}$$

$$\sigma_{r1}\sigma_{r2}\sigma_{r3} = \sigma_{ri}\sigma_{rj}\sigma_{rk} \tag{107}$$

as

$$v_{ri} = v_{1i} - v_{2i} \tag{108}$$

$$\begin{aligned}
\sigma_{ri}^2 &= \left\langle v_{ri}^2 \right\rangle - \left\langle v_{ri} \right\rangle^2 && (109)\\
&= \left\langle \left(v_{1i} - v_{2i}\right)^2 \right\rangle - \left\langle v_{1i} - v_{2i} \right\rangle^2 && (110)\\
&= \left\langle v_{1i}^2 - 2v_{1i}v_{2i} + v_{2i}^2 \right\rangle - \left\langle v_{1i} - v_{2i} \right\rangle^2 && (111)\\
&= \left\langle v_{1i}^2 - 2v_{1i}v_{2i} + v_{2i}^2 \right\rangle - \left(\left\langle v_{1i} \right\rangle - \left\langle v_{2i} \right\rangle\right)^2 && (112)\\
&= \left\langle v_{1i}^2 \right\rangle - \left\langle 2v_{1i}v_{2i} \right\rangle + \left\langle v_{2i}^2 \right\rangle - \left(\left\langle v_{1i} \right\rangle^2 - 2\left\langle v_{1i} \right\rangle \left\langle v_{2i} \right\rangle + \left\langle v_{2i} \right\rangle^2\right) && (113)\\
&= \left\langle v_{1i}^2 \right\rangle - \left\langle 2v_{1i}v_{2i} \right\rangle + \left\langle v_{2i}^2 \right\rangle - \left\langle v_{1i} \right\rangle^2 + 2\left\langle v_{1i} \right\rangle \left\langle v_{2i} \right\rangle - \left\langle v_{2i} \right\rangle^2 && (114)\\
&= \sigma_{1i}^2 + \sigma_{2i}^2 - 2\left\langle v_{1i}v_{2i} \right\rangle + 2\left\langle v_{1i} \right\rangle \left\langle v_{2i} \right\rangle && (115)
\end{aligned}$$

with individual $\mathbf{v}_1$, $\mathbf{v}_2$,

$$\left\langle v_{1i}v_{2i} \right\rangle = \left\langle v_{1i} \right\rangle \left\langle v_{2i} \right\rangle \tag{116}$$

and

$$\sigma_{1i} = \sigma_{2i} \tag{117}$$

so

$$\sigma_{ri}^2 = 2\sigma_i^2 \tag{118}$$

$$\frac{\sigma_{ri}^2}{\sigma_{rj}^2} = \frac{\sigma_i^2}{\sigma_j^2} \tag{119}$$

$$\frac{\sigma_{ri}^2}{\sigma_{rk}^2} = \frac{\sigma_i^2}{\sigma_k^2} \tag{120}$$

$$\begin{aligned}
\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{2n^2\pi a^2\left(1+\epsilon\right)}{\left(2\pi\right)^{3/2}} 8\frac{\pi}{4}\frac{\sigma_{ri}^6}{\sigma_{r1}\sigma_{r2}\sigma_{r3}}\int_0^1 d\mu\left[\frac{1}{4}\left(1+\epsilon\right)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] && (121)\\
\times\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}} + 2A^{-\frac{3}{2}}B^{-\frac{3}{2}} + 3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) && (122)
\end{aligned}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{2n^2\pi a^2(1+\epsilon)}{(2\pi)^{3/2}} 8\frac{\pi}{4}\frac{8\sigma_i^6}{\sqrt{2}\sigma_1\sqrt{2}\sigma_2\sqrt{2}\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}(1+\epsilon)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{123}$$

$$\times\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}}+2A^{-\frac{3}{2}}B^{-\frac{3}{2}}+3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{124}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = 4\pi^{1/2}n^2a^2(1+\epsilon)\frac{\sigma_i^6}{\sigma_1\sigma_2\sigma_3}\int_0^1 d\mu\left[\frac{1}{4}(1+\epsilon)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{125}$$

$$\times\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}}+2A^{-\frac{3}{2}}B^{-\frac{3}{2}}+3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{126}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = 4\pi^{1/2}n^2a^2(1+\epsilon)\frac{\sigma_i^5}{\sigma_j\sigma_k}\int_0^1 d\mu\left[\frac{1}{4}(1+\epsilon)\left(\mu^2+\frac{1}{3}\right)-\mu^2\right] \tag{127}$$

$$\times\left(3A^{-\frac{1}{2}}B^{-\frac{5}{2}}+2A^{-\frac{3}{2}}B^{-\frac{3}{2}}+3A^{-\frac{5}{2}}B^{-\frac{1}{2}}\right) \tag{128}$$

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = 4\pi^{1/2}n^2a^2(1+\epsilon)\frac{\sigma_i^5}{\sigma_j\sigma_k}\left[(1+\epsilon)J_P^i+J_Q^i\right] \tag{129}$$

5. Velocity dispersion in z direction

The density distribution in z direction is determined dynamically by

$$\frac{\partial p}{\partial z} = -n\frac{\partial U}{\partial z} = -n\Omega_k^2 z \tag{130}$$

the second = valid when the disk is thin, recall that $p_{ii} = n\sigma_{ii}^2$, so

$$n(z) = n(0)\exp\left[-\frac{U(r,z)}{\sigma_3^2}\right] = n(0)\exp\left[-\frac{z^2}{2H^2}\right] \tag{131}$$

$H^2 = \frac{\sigma_3^2}{\Omega^2}$ is the height of the disk. Integrating over z using $p_{ii} = n(z)\sigma_i^2$, We obtain

$$\sin 2\delta = -\frac{2(2\pi)^{1/2}n(0)a^2}{3\Omega\sigma_2\sigma_3}\sigma_1^3(1+\epsilon)\left[(1+\epsilon)J_P^1+J_Q^1\right] \tag{132}$$

$$\sin 2\delta = +\frac{2(2\pi)^{1/2}n(0)a^2}{3\Omega\sigma_1\sigma_3}\sigma_2^3(1+\epsilon)\left[(1+\epsilon)J_P^2+J_Q^2\right] \tag{133}$$

$$\sigma_1^2\left(1+3\sin^2\delta\right)=\sigma_2^2\left(1+3\cos^2\delta\right) \tag{134}$$

$$\left(1+\epsilon\right)J_P^3+J_Q^3=0 \tag{135}$$

In the limit of $a\to 0$ we can see $\delta=0$ and

$$\sigma_{11}^2=4\sigma_{22}^2 \tag{136}$$

or,

$$p_{rr}=4p_{\theta\theta} \tag{137}$$

this is the well-known(?) result for a collisionless disk in a Keplerian force field

We involve the velocity dispersion in z direction as

$$\tau=\pi a^2\int_{-\infty}^{+\infty}n\left(z\right)dz=\left(2\pi^3\right)^{1/2}n(0)a^2\sigma_3\Omega^{-1} \tag{138}$$

6. Final form of equations

$$\sin 2\delta=-\frac{4}{3}\frac{\tau}{\pi}\frac{\sigma_1^3}{\sigma_2\sigma_3^2}(1+\epsilon)\left[\left(1+\epsilon\right)J_P^1+J_Q^1\right] \tag{139}$$

$$\sin 2\delta=+\frac{4}{3}\frac{\tau}{\pi}\frac{\sigma_2^3}{\sigma_1\sigma_3^2}(1+\epsilon)\left[\left(1+\epsilon\right)J_P^2+J_Q^2\right] \tag{140}$$

$$\sigma_1^2\left(1+3\sin^2\delta\right)=\sigma_2^2\left(1+3\cos^2\delta\right) \tag{141}$$

$$\left(1+\epsilon\right)J_P^3+J_Q^3=0 \tag{142}$$

For any assumed value of the optical depth τ , we can calculate $\{\epsilon, \delta, \sigma_2/\sigma_1, \sigma_3/\sigma_1\}$ as function of τ.

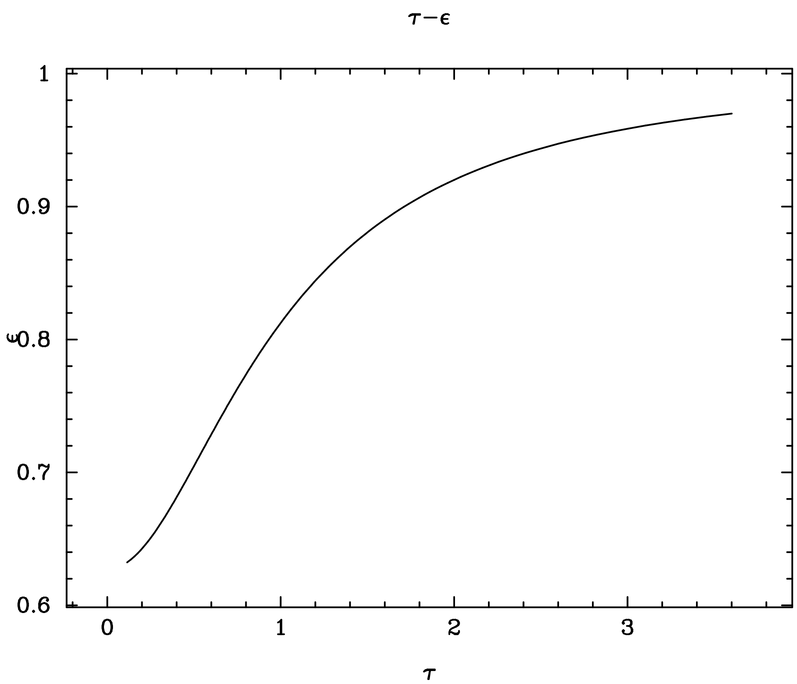

Fig. 2.— The $\epsilon(\tau)$ relation

7. Viscousity

In any differentially rotating system, one of the most important effects of collisions is the radial transport of angular momentum.

The momentum information is contained in Euler's equation

$$n\frac{D\mathbf{u}}{D\,t} = -n\nabla U - \nabla\cdot\mathbf{p} \tag{143}$$

where D/Dt is a Lagrangian derivative and $\mathbf{p}$ is the pressure tensor. The rate of gain of angular momentum per unit mass is

$$\frac{D\mathbf{J}}{D\,t} = \frac{D}{D\,t}\left(\mathbf{r}\times\mathbf{u}\right) = -\mathbf{r}\times\nabla U = \frac{1}{n}\mathbf{r}\times\nabla\cdot\mathbf{p} \tag{144}$$

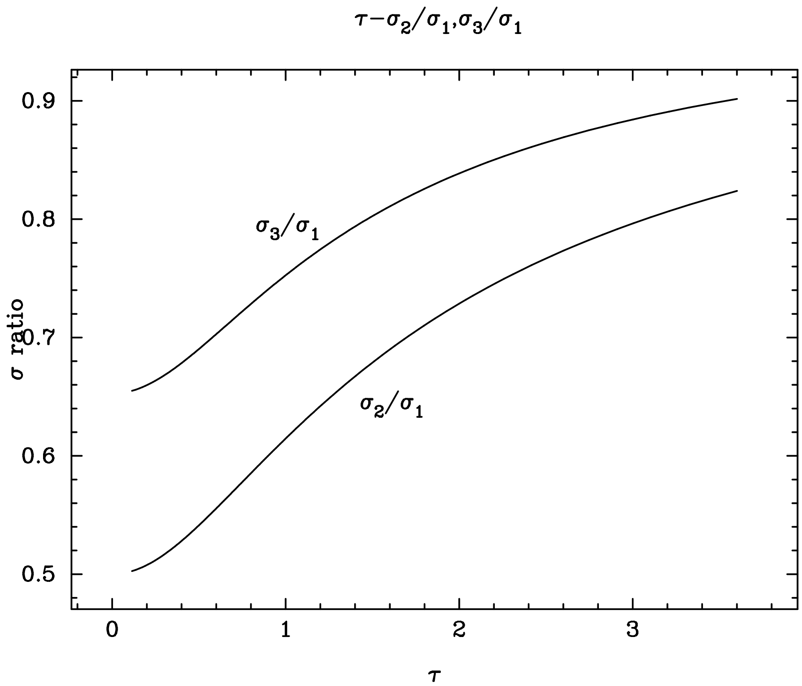

Fig. 3.— Axis ratio σ_2/σ_1 and σ_3/σ_1 as a function of optical depth

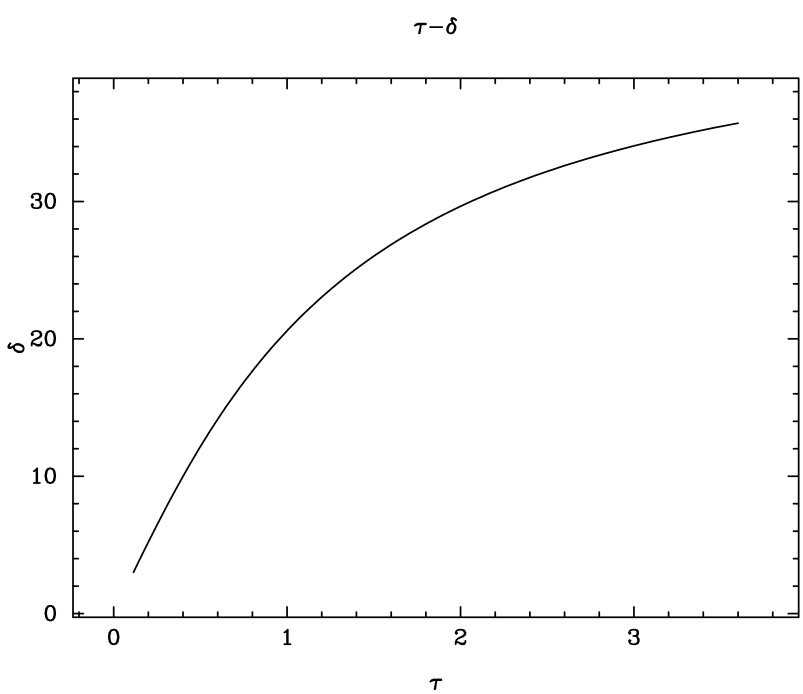

Fig. 4.— The orientation angle δ V.S. τ

$$= \begin{vmatrix} i & j & k \\ r & 0 & 0 \\ \mathbf{e}_r \cdot (\nabla \cdot \mathbf{p}) & \mathbf{e}_\theta \cdot (\nabla \cdot \mathbf{p}) & \mathbf{e}_z \cdot (\nabla \cdot \mathbf{p}) \end{vmatrix} \tag{145}$$

where

$$\nabla \cdot \mathbf{p} = \frac{\mathbf{e}_r}{r}\left[\frac{\partial(p_{rr}r)}{\partial r} + \frac{\partial p_{r\theta}}{\partial \theta} + \frac{\partial(p_{rz}r)}{\partial z} - p_{\theta\theta}\right] \tag{146}$$

$$+\frac{\mathbf{e}_\theta}{r}\left[\frac{\partial(p_{r\theta}r)}{\partial r} + \frac{\partial p_{\theta\theta}}{\partial \theta} + \frac{\partial(p_{z\theta}r)}{\partial z} + p_{\theta r}\right] \tag{147}$$

$$+\frac{\mathbf{e}_z}{r}\left[\frac{\partial(p_{rz}r)}{\partial r} + \frac{\partial p_{\theta z}}{\partial \theta} + \frac{\partial(p_{zz}r)}{\partial z}\right] \tag{148}$$

$$\tag{149}$$

for an azimuthally symmetric disk with reflection symmetry about $z = 0$: $p_{rz} = p_{\theta z} = 0$ and $\partial/\partial\theta = 0$, so

$$\frac{DJ}{Dt} = -\frac{1}{nr}\frac{\partial}{\partial r}\left(r^2 p_{r\theta}\right) \tag{150}$$

Euler equation for a viscous fluid disk reads

$$\rho\frac{D\mathbf{u}}{Dt} = -\rho\nabla U + \nabla \cdot \mathbf{s} \tag{151}$$

$\mathbf{s}$ is the stress tensor

$$s_{\alpha\beta} = -p\delta_{\alpha\beta} + \eta\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial x_\gamma}\right) + \zeta\delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial x_\gamma} \tag{152}$$

η is shear viscosity and $zeta$ is the bulk viscosity. So for the viscous fluid disk,

$$\frac{DJ}{D\,t} = \frac{1}{\rho r}\frac{\partial}{\partial r}\left(r^2 s_{\theta r}\right) \tag{153}$$

$$= \frac{1}{\rho r}\frac{\partial}{\partial r}\left(\eta r^3 \frac{d\Omega}{d\,r}\right) \tag{154}$$

Comparison of this two equations shows that the radial transport of angular momentum in the particle disk is exactly the same as in the fluid disk if the kinematic viscosity

$$\nu = \frac{\eta}{\rho}$$

is assigned the value

$$\nu = \frac{1}{n} p_{\theta r} \left(-r \frac{d\Omega}{dr} \right)^{-1} \tag{155}$$

for

$$p_{r\theta} = \frac{\tan\delta}{1+\tan^2\delta} \left(p_{11} - p_{22} \right) = \frac{\sin 2\delta}{2} n \left(\sigma_1^2 - \sigma_2^2 \right) \tag{156}$$

so

$$\nu = \frac{1}{n} \frac{\sin 2\delta}{2} n \left(\sigma_1^2 - \sigma_2^2 \right) \left(-\Omega \frac{d \ln \Omega}{d \ln r} \right)^{-1} = \frac{\sin 2\delta}{2} \left(\sigma_1^2 - \sigma_2^2 \right) \frac{2}{3\Omega} = \frac{1}{3\Omega} \left(\sigma_1^2 - \sigma_2^2 \right) \sin 2\delta \tag{157}$$

Define:

$$3\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \tag{158}$$

then the kinematic viscosity is

$$\nu = \frac{\sigma^2}{\Omega} \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} \sin 2\delta \tag{159}$$

%%

A. Appendix

A.1. DERIVATION: $\{p_{rr},\, p_{r\theta},\, p_{\theta\theta}\}$ into $\{p_{11},\, p_{22},\, \delta\}$

definition:

$$p_{11} = \frac{p_{rr} + p_{\theta\theta} + \sqrt{\left(p_{rr} - p_{\theta\theta}\right)^2 + 4p_{r\theta}^2}}{2} \tag{A1}$$

$$p_{22} = \frac{p_{rr} + p_{\theta\theta} - \sqrt{\left(p_{rr} - p_{\theta\theta}\right)^2 + 4p_{r\theta}^2}}{2} \tag{A2}$$

$$\tan\delta = \frac{p_{r\theta}}{p_{11} - p_{\theta\theta}} = \frac{2p_{r\theta}}{(p_{rr} - p_{\theta\theta}) + \sqrt{(p_{rr} - p_{\theta\theta})^2 + 4p_{r\theta}^2}} \tag{A3}$$

the third equation:

$$\tan\delta\,(p_{rr} - p_{\theta\theta}) + \tan\delta\sqrt{(p_{rr} - p_{\theta\theta})^2 + 4p_{r\theta}^2} = 2p_{r\theta} \tag{A4}$$

$$\tan\delta\sqrt{(p_{rr} - p_{\theta\theta})^2 + 4p_{r\theta}^2} = 2p_{r\theta} - \tan\delta\,(p_{rr} - p_{\theta\theta}) \tag{A5}$$

$$\tan^2\delta\,(p_{rr} - p_{\theta\theta})^2 + \tan^2\delta 4p_{r\theta}^2 = (2p_{r\theta} - \tan\delta\,(p_{rr} - p_{\theta\theta}))^2 \tag{A6}$$

$$\tan^2\delta\,(p_{rr} - p_{\theta\theta})^2 + \tan^2\delta 4p_{r\theta}^2 = 4p_{r\theta}^2 - 4p_{r\theta}\tan\delta\,(p_{rr} - p_{\theta\theta}) + \tan^2\delta\,(p_{rr} - p_{\theta\theta})^2 \tag{A7}$$

$$\tan^2\delta 4p_{r\theta}^2 = 4p_{r\theta}^2 - 4p_{r\theta}\tan\delta\,(p_{rr} - p_{\theta\theta}) \tag{A8}$$

$$\tan^2\delta p_{r\theta} = p_{r\theta} - \tan\delta\,(p_{rr} - p_{\theta\theta}) \tag{A9}$$

$$\tan\delta\,(p_{rr} - p_{\theta\theta}) = \left(1 - \tan^2\delta\right) p_{r\theta} \tag{A10}$$

$$p_{r\theta} = \frac{\tan\delta}{1 - \tan^2\delta}\,(p_{rr} - p_{\theta\theta}) \tag{A11}$$

suppose $p_{rr} > p_{\theta\theta}$ then the first equation

$$p_{11} = \frac{p_{rr} + p_{\theta\theta} + \sqrt{(p_{rr} - p_{\theta\theta})^2 + 4p_{r\theta}^2}}{2} \tag{A12}$$

$$2p_{11} = p_{rr} + p_{\theta\theta} + \sqrt{(p_{rr} - p_{\theta\theta})^2 + 4p_{r\theta}^2} \tag{A13}$$

$$2p_{11} = p_{rr} + p_{\theta\theta} + \sqrt{(p_{rr} - p_{\theta\theta})^2 + 4\left(\frac{\tan\delta}{1-\tan^2\delta}\,(p_{rr} - p_{\theta\theta})\right)^2} \tag{A14}$$

$$2p_{11} = p_{rr} + p_{\theta\theta} + \sqrt{1 + 4\left(\frac{\tan\delta}{1-\tan^2\delta}\right)^2}\,(p_{rr} - p_{\theta\theta}) \tag{A15}$$

$$2p_{11} = p_{rr} + p_{\theta\theta} + \sqrt{\frac{(1-\tan^2\delta)^2}{(1-\tan^2\delta)^2} + \frac{4\tan^2\delta}{(1-\tan^2\delta)^2}}\,(p_{rr} - p_{\theta\theta}) \tag{A16}$$

$$2p_{11} = p_{rr} + p_{\theta\theta} + \sqrt{\frac{(1+\tan^2\delta)^2}{(1-\tan^2\delta)^2}}\,(p_{rr} - p_{\theta\theta}) \tag{A17}$$

$$2p_{11} = p_{rr} + p_{\theta\theta} + \frac{(1+\tan^2\delta)}{(1-\tan^2\delta)}\,(p_{rr} - p_{\theta\theta}) \tag{A18}$$

$$2p_{22} = p_{rr} + p_{\theta\theta} - \frac{(1+\tan^2\delta)}{(1-\tan^2\delta)}\,(p_{rr} - p_{\theta\theta}) \tag{A19}$$

$$2p_{11} = \frac{2}{(1-\tan^2\delta)}p_{rr} - \frac{2\tan^2\delta}{(1-\tan^2\delta)}p_{\theta\theta} \tag{A20}$$

$$p_{11} = \frac{1}{(1-\tan^2\delta)}p_{rr} - \frac{\tan^2\delta}{(1-\tan^2\delta)}p_{\theta\theta} \tag{A21}$$

$$p_{22} = -\frac{\tan^2\delta}{(1-\tan^2\delta)}p_{rr} + \frac{1}{(1-\tan^2\delta)}p_{\theta\theta} \tag{A22}$$

so

$$p_{rr} = \frac{1}{(1+\tan^2\delta)}p_{11} + \frac{\tan^2\delta}{(1+\tan^2\delta)}p_{22} \tag{A23}$$

$$p_{\theta\theta} = \frac{\tan^2\delta}{(1+\tan^2\delta)}p_{11} + \frac{1}{(1+\tan^2\delta)}p_{22} \tag{A24}$$

$$p_{r\theta} = \frac{\tan\delta}{1-\tan^2\delta}\left(p_{rr}-p_{\theta\theta}\right) = \frac{\tan\delta}{1-\tan^2\delta}\frac{1-\tan^2\delta}{1+\tan^2\delta}\left(p_{11}-p_{22}\right) = \frac{\tan\delta}{1+\tan^2\delta}\left(p_{11}-p_{22}\right) \tag{A25}$$

eq:

$$-4\Omega p_{r\theta} = \left(\frac{\partial p_{rr}}{\partial t}\right)_c \tag{A26}$$

$$+\frac{\Omega}{2}p_{rr} - 2\Omega p_{\theta\theta} = \left(\frac{\partial p_{r\theta}}{\partial t}\right)_c \tag{A27}$$

$$\Omega p_{r\theta} = \left(\frac{\partial p_{\theta\theta}}{\partial t}\right)_c \tag{A28}$$

the middle equation:

$$+\frac{\Omega}{2}p_{rr} - 2\Omega p_{\theta\theta} = \frac{\tan\delta}{1-\tan^2\delta}\left(\frac{\partial}{\partial t}\left(p_{rr}-p_{\theta\theta}\right)\right)_c = \frac{\tan\delta}{1-\tan^2\delta}\left(-5\Omega p_{r\theta}\right) \tag{A29}$$

$$+\frac{1}{2}\left(\frac{1}{(1+\tan^2\delta)}p_{11} + \frac{\tan^2\delta}{(1+\tan^2\delta)}p_{22}\right) - 2\left(\frac{\tan^2\delta}{(1+\tan^2\delta)}p_{11} + \frac{1}{(1+\tan^2\delta)}p_{22}\right) = \tag{A30}$$

$$\frac{\tan\delta}{1-\tan^2\delta}\left(-5\right)\left(\frac{\tan\delta}{1+\tan^2\delta}\left(p_{11}-p_{22}\right)\right) \tag{A31}$$

$$+\frac{1}{2}\left(\frac{1}{1}p_{11} + \frac{\tan^2\delta}{1}p_{22}\right) - 2\left(\frac{\tan^2\delta}{1}p_{11} + \frac{1}{1}p_{22}\right) = \tag{A32}$$

$$\frac{\tan\delta}{1-\tan^2\delta}\left(-5\right)\left(\frac{\tan\delta}{1}\left(p_{11}-p_{22}\right)\right) \tag{A33}$$

$$+\frac{1}{2}\left(p_{11} + \tan^2\delta p_{22}\right) - 2\left(\tan^2\delta p_{11} + p_{22}\right) = -\frac{5\tan^2\delta}{1-\tan^2\delta}\left(p_{11}-p_{22}\right) \tag{A34}$$

$$+\left(p_{11} + \tan^2\delta p_{22}\right) - 4\left(\tan^2\delta p_{11} + p_{22}\right) = -\frac{10\tan^2\delta}{1-\tan^2\delta}\left(p_{11}-p_{22}\right) \tag{A35}$$

$$\left(1-4\tan^2\delta\right)p_{11} + \left(\tan^2\delta - 4\right)p_{22} = -\frac{10\tan^2\delta}{1-\tan^2\delta}\left(p_{11}-p_{22}\right) \tag{A36}$$

$$\left(1-\tan^2\delta\right)\left(1-4\tan^2\delta\right)p_{11}+\left(1-\tan^2\delta\right)\left(\tan^2\delta-4\right)p_{22}=-10\tan^2\delta\left(p_{11}-p_{22}\right) \quad \text{(A37)}$$

$$\left(1-\tan^2\delta\right)\left(1-4\tan^2\delta\right)p_{11}+\left(1-\tan^2\delta\right)\left(\tan^2\delta-4\right)p_{22}+10\tan^2\delta\left(p_{11}-p_{22}\right)=0 \quad \text{(A38)}$$

$$\left(1+5\tan^2\delta+4\tan^4\delta\right)p_{11}+\left(-4-5\tan^2\delta-\tan^4\delta\right)p_{22}=0 \quad \text{(A39)}$$

$$\left(1+5\tan^2\delta+4\tan^4\delta\right)p_{11}=\left(4+5\tan^2\delta+\tan^4\delta\right)p_{22} \quad \text{(A40)}$$

$$\left(1+\tan^2\delta\right)\left(1+4\tan^2\delta\right)p_{11}=\left(1+\tan^2\delta\right)\left(\tan^2\delta+4\right)p_{22} \quad \text{(A41)}$$

$$\left(1+4\tan^2\delta\right)p_{11}=\left(\tan^2\delta+4\right)p_{22} \quad \text{(A42)}$$

$$\frac{p_{11}}{p_{22}}=\frac{\left(\tan^2\delta+4\right)}{\left(1+4\tan^2\delta\right)}=\frac{1+3\cos^2\delta}{1+3\sin^2\delta} \quad \text{(A43)}$$

which is exhibited as equation 29.

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c=\frac{1}{\left(1-\tan^2\delta\right)}\left(\frac{\partial p_{rr}}{\partial t}\right)_c-\frac{\tan^2\delta}{\left(1-\tan^2\delta\right)}\left(\frac{\partial p_{\theta\theta}}{\partial t}\right)_c \quad \text{(A44)}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c=\frac{1}{\left(1-\tan^2\delta\right)}\left(-4\right)\Omega p_{r\theta}-\frac{\tan^2\delta}{\left(1-\tan^2\delta\right)}\Omega p_{r\theta} \quad \text{(A45)}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c=\Omega\frac{-4-\tan^2\delta}{\left(1-\tan^2\delta\right)}p_{r\theta} \quad \text{(A46)}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c=-\Omega\frac{4+\tan^2\delta}{1-\tan^2\delta}p_{r\theta} \quad \text{(A47)}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c = -\Omega\frac{4+\tan^2\delta}{1-\tan^2\delta}\frac{\tan\delta}{1+\tan^2\delta}\left(p_{11}-p_{22}\right) \tag{A48}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c = -\Omega\frac{4+\tan^2\delta}{1-\tan^2\delta}\frac{\tan\delta}{1+\tan^2\delta}p_{11}\left(1-\frac{(1+4\tan^2\delta)}{(\tan^2\delta+4)}\right) \tag{A49}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c = -\Omega\frac{4+\tan^2\delta}{1-\tan^2\delta}\frac{\tan\delta}{1+\tan^2\delta}p_{11}\left(\frac{\tan^2\delta+4-1-4\tan^2\delta}{\tan^2\delta+4}\right) \tag{A50}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c = -\Omega\frac{1}{1-\tan^2\delta}\frac{\tan\delta}{1+\tan^2\delta}p_{11}\left(\frac{+3-3\tan^2\delta}{1}\right) \tag{A51}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c = -3\Omega\frac{\tan\delta}{1+\tan^2\delta}p_{11} \tag{A52}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c = -\frac{3}{2}\Omega\frac{2\tan\delta}{1+\tan^2\delta}p_{11} \tag{A53}$$

$$\left(\frac{\partial p_{11}}{\partial t}\right)_c = -\frac{3}{2}\Omega\sin 2\delta p_{11} \tag{A54}$$

and

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = -\frac{\tan^2\delta}{(1-\tan^2\delta)}\left(\frac{\partial p_{rr}}{\partial t}\right)_c + \frac{1}{(1-\tan^2\delta)}\left(\frac{\partial p_{\theta\theta}}{\partial t}\right)_c \tag{A55}$$

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = -\frac{\tan^2\delta}{(1-\tan^2\delta)}\left(-4\right)\Omega p_{r\theta} + \frac{1}{(1-\tan^2\delta)}\Omega p_{r\theta} \tag{A56}$$

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = \Omega\frac{4\tan^2\delta+1}{1-\tan^2\delta}p_{r\theta} \tag{A57}$$

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = \Omega\frac{4\tan^2\delta+1}{1-\tan^2\delta}\frac{\tan\delta}{1+\tan^2\delta}\left(p_{11}-p_{22}\right) \tag{A58}$$

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = \Omega\frac{4\tan^2\delta+1}{1-\tan^2\delta}\frac{\tan\delta}{1+\tan^2\delta}\left(\frac{\tan^2\delta+4}{1+4\tan^2\delta}-1\right)p_{22} \tag{A59}$$

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = \Omega\frac{4\tan^2\delta+1}{1-\tan^2\delta}\frac{\tan\delta}{1+\tan^2\delta}\left(\frac{\tan^2\delta+4-1-4\tan^2\delta}{1+4\tan^2\delta}\right)p_{22} \tag{A60}$$

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = \Omega\frac{4\tan^2\delta+1}{1-\tan^2\delta}\frac{\tan\delta}{1+\tan^2\delta}\left(\frac{+3-3\tan^2\delta}{1+4\tan^2\delta}\right)p_{22} \tag{A61}$$

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = 3\Omega\frac{\tan\delta}{1+\tan^2\delta}p_{22} \tag{A62}$$

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = \frac{3}{2}\Omega\frac{2\tan\delta}{1+\tan^2\delta}p_{22} \tag{A63}$$

$$\left(\frac{\partial p_{22}}{\partial t}\right)_c = \frac{3}{2}\Omega\sin 2\delta p_{22} \tag{A64}$$

A.2. DERIVATION: From Cartesian to Cylindrical

Here we collapse the equation 10:

$$\frac{\partial p_{\alpha\beta}}{\partial t} + p_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma} + p_{\beta\gamma}\frac{\partial u_\alpha}{\partial x_\gamma} + \frac{\partial}{\partial x_\gamma}(p_{\alpha\beta}u_\gamma) = \left(\frac{\partial p_{\alpha\beta}}{\partial t}\right)_c \tag{A65}$$

into cylindrical coordinate.

The first term $\frac{\partial p_{\alpha\beta}}{\partial t}$ is the coefficient of $e_\alpha e_\beta$

The second term is a partial derivative about vector u_β.

$$P_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma} = \mathbf{e}_\alpha P_{\alpha\gamma}\frac{\partial u_\beta \mathbf{e}_\beta}{\partial x_\gamma} = \mathbf{e}_\alpha P_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma}\mathbf{e}_\beta + \mathbf{e}_\alpha P_{\alpha\gamma}u_\beta\frac{\partial \mathbf{e}_\beta}{\partial x_\gamma} \tag{A66}$$

term $\frac{\partial \mathbf{e}_\beta}{\partial x_\gamma}$ is taken into account for the gradient of the base of cylindrical coordinate is non-trivial for

$$\frac{\partial \mathbf{e}_R}{\partial \theta} = \mathbf{e}_\phi, \quad \frac{\partial \mathbf{e}_\phi}{\partial \theta} = -\mathbf{e}_R \tag{A67}$$

$$\mathbf{e}_\alpha P_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma}\mathbf{e}_\beta = \mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_\beta}{\partial R}\mathbf{e}_\beta + \mathbf{e}_\alpha P_{\alpha\theta}\frac{\partial u_\beta}{R\partial\theta}\mathbf{e}_\beta + \mathbf{e}_\alpha P_{\alpha z}\frac{\partial u_\beta}{\partial z}\mathbf{e}_\beta \tag{A68}$$

If we take $\mathbf{u}$ only the function of R, so the only non-zero differential is ∂_R, so

$$\begin{aligned}
\mathbf{e}_\alpha P_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma}\mathbf{e}_\beta &= \mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_\beta}{\partial R}\mathbf{e}_\beta + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha\theta}\frac{\partial u_\beta}{R\partial\theta}\mathbf{e}_\beta} + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha z}\frac{\partial u_\beta}{\partial z}\mathbf{e}_\beta} && \text{(A69)}\\
&= \mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_\beta}{\partial R}\mathbf{e}_\beta && \text{(A70)}\\
&= \cancelto{0}{\mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_R}{\partial R}}\mathbf{e}_R + \mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha R}\frac{\partial u_z}{\partial R}}\mathbf{e}_z && \text{(A71)}\\
&= \mathbf{e}_R P_{RR}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta + \mathbf{e}_\theta P_{\theta R}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta + \cancelto{0}{\mathbf{e}_z P_{zR}\frac{\partial u_\theta}{\partial R}}\mathbf{e}_\theta && \text{(A72)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_\alpha P_{\alpha\gamma}u_\beta\frac{\partial \mathbf{e}_\beta}{\partial x_\gamma} &= \cancelto{0}{\mathbf{e}_\alpha P_{\alpha\gamma}u_R\frac{\partial \mathbf{e}_R}{\partial x_\gamma}} + \mathbf{e}_\alpha P_{\alpha\gamma}u_\theta\frac{\partial \mathbf{e}_\theta}{\partial x_\gamma} + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha\gamma}u_z\frac{\partial \mathbf{e}_z}{\partial x_\gamma}} && \text{(A73)}\\
&= \cancelto{0}{\mathbf{e}_\alpha P_{\alpha R}u_\theta\frac{\partial \mathbf{e}_\theta}{\partial R}} + \mathbf{e}_\alpha P_{\alpha\theta}u_\theta\frac{\partial \mathbf{e}_\theta}{R\partial\theta} + \cancelto{0}{\mathbf{e}_\alpha P_{\alpha z}u_\theta\frac{\partial \mathbf{e}_\theta}{\partial z}} && \text{(A74)}\\
&= -\mathbf{e}_\alpha P_{\alpha\theta}\Omega\mathbf{e}_R && \text{(A75)}\\
&= -\mathbf{e}_R P_{R\theta}\Omega\mathbf{e}_R - \mathbf{e}_\theta P_{\theta\theta}\Omega\mathbf{e}_R - \cancelto{0}{\mathbf{e}_z P_{z\theta}\Omega\mathbf{e}_R} && \text{(A76)}
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_\alpha P_{\alpha\gamma}\frac{\partial u_\beta\mathbf{e}_\beta}{\partial x_\gamma} &= \mathbf{e}_R P_{RR}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta + \mathbf{e}_\theta P_{\theta R}\frac{\partial u_\theta}{\partial R}\mathbf{e}_\theta - \mathbf{e}_R P_{R\theta}\Omega\mathbf{e}_R - \mathbf{e}_\theta P_{\theta\theta}\Omega\mathbf{e}_R && \text{(A77)}\\
&= \begin{pmatrix} -P_{R\theta}\Omega & P_{RR}\frac{\partial u_\theta}{\partial R} & 0 \\ -P_{\theta\theta}\Omega & P_{\theta R}\frac{\partial u_\theta}{\partial R} & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{(A78)}
\end{aligned}$$

The third and second term of equation 10 is transposed to each other. So

$$P_{\beta\gamma}\frac{\partial u_\alpha}{\partial x_\gamma} = \begin{pmatrix} -P_{R\theta}\Omega & -P_{\theta\theta}\Omega & 0 \\ P_{RR}\frac{\partial u_\theta}{\partial R} & P_{\theta R}\frac{\partial u_\theta}{\partial R} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A79}$$

The forth term in equation 10

$$\begin{aligned}
\frac{\partial}{\partial x_\gamma}\left(p_{\alpha\beta}u_\gamma\right) &= \frac{\partial}{\partial x_\gamma}\left(p_{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta u_\gamma\right) = \cancelto{0}{\frac{\partial p_{\alpha\beta}u_\gamma}{\partial x_\gamma}}\mathbf{e}_\alpha\mathbf{e}_\beta + p_{\alpha\beta}u_\gamma\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{\partial x_\gamma} && \text{(A80)}\\
&= \cancelto{0}{p_{\alpha\beta}u_R\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{\partial R}} + p_{\alpha\beta}u_\theta\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{R\partial\theta} + \cancelto{0}{p_{\alpha\beta}u_z\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{\partial z}} && \text{(A81)}\\
&= p_{\alpha\beta}\Omega\frac{\partial \mathbf{e}_\alpha\mathbf{e}_\beta}{\partial\theta} && \text{(A82)}\\
&= p_{\alpha\beta}\Omega\frac{\partial \mathbf{e}_\alpha}{\partial\theta}\mathbf{e}_\beta + p_{\alpha\beta}\Omega\mathbf{e}_\alpha\frac{\partial \mathbf{e}_\beta}{\partial\theta} && \text{(A83)}\\
&= p_{R\beta}\Omega\frac{\partial \mathbf{e}_R}{\partial\theta}\mathbf{e}_\beta + p_{\theta\beta}\Omega\frac{\partial \mathbf{e}_\theta}{\partial\theta}\mathbf{e}_\beta + p_{\alpha R}\Omega\mathbf{e}_\alpha\frac{\partial \mathbf{e}_R}{\partial\theta} + p_{\alpha\theta}\Omega\mathbf{e}_\alpha\frac{\partial \mathbf{e}_\theta}{\partial\theta} && \text{(A84)}\\
&= p_{R\beta}\Omega\mathbf{e}_\theta\mathbf{e}_\beta - p_{\theta\beta}\Omega\mathbf{e}_R\mathbf{e}_\beta + p_{\alpha R}\Omega\mathbf{e}_\alpha\mathbf{e}_\theta - p_{\alpha\theta}\Omega\mathbf{e}_\alpha\mathbf{e}_R && \text{(A85)}\\
&= p_{RR}\Omega\mathbf{e}_\theta\mathbf{e}_R + p_{R\theta}\Omega\mathbf{e}_\theta\mathbf{e}_\theta - p_{\theta R}\Omega\mathbf{e}_R\mathbf{e}_R - p_{\theta\theta}\Omega\mathbf{e}_R\mathbf{e}_\theta && \text{(A86)}\\
&\quad + p_{RR}\Omega\mathbf{e}_R\mathbf{e}_\theta + p_{\theta R}\Omega\mathbf{e}_\theta\mathbf{e}_\theta - p_{R\theta}\Omega\mathbf{e}_R\mathbf{e}_R - p_{\theta\theta}\Omega\mathbf{e}_\theta\mathbf{e}_R && \text{(A87)}\\
&= \begin{pmatrix} -2P_{R\theta}\Omega & P_{RR}\Omega - P_{\theta\theta}\Omega & 0 \\ P_{RR}\Omega - P_{\theta\theta}\Omega & 2P_{\theta R}\Omega & 0 \\ 0 & 0 & 0 \end{pmatrix} && \text{(A88)}
\end{aligned}$$

For stabile case, i.e. $\partial_t = 0$, we have the left side of equation 10

$$p_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma} + p_{\beta\gamma}\frac{\partial u_\alpha}{\partial x_\gamma} + \frac{\partial}{\partial x_\gamma}\left(p_{\alpha\beta}u_\gamma\right) \tag{A89}$$

$$= \begin{pmatrix} -P_{R\theta}\Omega & P_{RR}\frac{\partial u_\theta}{\partial R} & 0 \\ -P_{\theta\theta}\Omega & P_{\theta R}\frac{\partial u_\theta}{\partial R} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -P_{R\theta}\Omega & -P_{\theta\theta}\Omega & 0 \\ P_{RR}\frac{\partial u_\theta}{\partial R} & P_{\theta R}\frac{\partial u_\theta}{\partial R} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -2P_{R\theta}\Omega & P_{RR}\Omega - P_{\theta\theta}\Omega & 0 \\ P_{RR}\Omega - P_{\theta\theta}\Omega & 2P_{\theta R}\Omega & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A90}$$

$$= \begin{pmatrix} -4P_{R\theta}\Omega, & P_{RR}\frac{\partial u_\theta}{\partial R} + P_{RR}\Omega - 2P_{\theta\theta}\Omega & 0 \\ P_{RR}\frac{\partial u_\theta}{\partial R} + P_{RR}\Omega - 2P_{\theta\theta}\Omega & 2P_{\theta R}\frac{\partial u_\theta}{\partial R} + 2P_{\theta R}\Omega & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A91}$$

$$= \begin{pmatrix} -4P_{R\theta}\Omega, & \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta}, & 0 \\ \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta}, & \frac{2P_{R\theta}}{R}\frac{dR^2\Omega}{dR}, & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A92}$$

equal above matrix to the right side of equation 10

$$= \begin{pmatrix} \left(\frac{\partial P_{RR}}{\partial t}\right)_c, & \left(\frac{\partial P_{R\theta}}{\partial t}\right)_c, & 0 \\ \left(\frac{\partial P_{\theta R}}{\partial t}\right)_c, & \left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c, & 0 \\ 0 & 0 & \left(\frac{\partial P_{zz}}{\partial t}\right)_c \end{pmatrix} \tag{A93}$$

So the full expression about equation 10 is

$$\begin{pmatrix} \frac{\partial P_{RR}}{\partial t} - 4P_{R\theta}\Omega, & \frac{\partial P_{R\theta}}{\partial t} + \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta}, & 0 \\ \frac{\partial P_{\theta R}}{\partial t} + \frac{P_{RR}}{R}\frac{dR^2\Omega}{dR} - 2\Omega P_{\theta\theta}, & \frac{\partial P_{\theta\theta}}{\partial t} + \frac{2P_{R\theta}}{R}\frac{dR^2\Omega}{dR}, & 0 \\ 0 & 0 & \frac{\partial P_{zz}}{\partial t} \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial P_{RR}}{\partial t}\right)_c, & \left(\frac{\partial P_{R\theta}}{\partial t}\right)_c, & 0 \\ \left(\frac{\partial P_{\theta R}}{\partial t}\right)_c, & \left(\frac{\partial P_{\theta\theta}}{\partial t}\right)_c, & 0 \\ 0 & 0 & \left(\frac{\partial P_{zz}}{\partial t}\right)_c \end{pmatrix} \tag{A94}$$

A.3. DERIVATION: From Equation 5, 6, 7 to Equation 8 9,10

Equation 9 could be arrived as

$$\frac{\partial n u_i}{\partial t} + \frac{\partial P_{ij} + n u_i u_j}{\partial x_j} + n\frac{\partial \Phi}{\partial x_i} = \left(\frac{\partial n \langle v_i \rangle}{\partial t}\right)_c \tag{A95}$$

$$\frac{\partial n u_i}{\partial t} + \frac{\partial P_{ij}}{\partial x_j} + \frac{\partial n u_i u_j}{\partial x_j} + n\frac{\partial \Phi}{\partial x_i} = \left(\frac{\partial n \langle v_i \rangle}{\partial t}\right)_c \tag{A96}$$

$$n\frac{\partial u_i}{\partial t} + \frac{\partial n}{\partial t}u_i + \frac{\partial P_{ij}}{\partial x_j} + \frac{\partial n u_j}{\partial x_j}u_i + n u_j\frac{\partial u_i}{\partial x_j} + n\frac{\partial \Phi}{\partial x_i} = \left(\frac{\partial n \langle v_i \rangle}{\partial t}\right)_c \tag{A97}$$

$$n\frac{\partial u_i}{\partial t}+\frac{\partial P_{ij}}{\partial x_j}+\left(\frac{\partial n}{\partial t}\right)_c u_i+nu_j\frac{\partial u_i}{\partial x_j}+n\frac{\partial \Phi}{\partial x_i}=\left(\frac{\partial n\left\langle v_i\right\rangle}{\partial t}\right)_c \tag{A98}$$

$$n\frac{\partial u_i}{\partial t}+\frac{\partial P_{ij}}{\partial x_j}+nu_j\frac{\partial u_i}{\partial x_j}+n\frac{\partial \Phi}{\partial x_i}=\left(\frac{\partial nu_i}{\partial t}\right)_c-\left(\frac{\partial n}{\partial t}\right)_c u_i \tag{A99}$$

Then the equation 7:

$$\frac{\partial\left(P_{ij}+nu_iu_j\right)}{\partial t}+\frac{\partial\left(P_{ijk}+u_iP_{jk}+u_jP_{ki}+u_kP_{ij}+nu_iu_ju_k\right)}{\partial x_k} \tag{A100}$$

$$+nu_i\frac{\partial \Phi}{\partial x_j}+nu_j\frac{\partial \Phi}{\partial x_i}=\left(\frac{\partial n\left\langle v_iv_j\right\rangle}{\partial t}\right)_c \tag{A101}$$

where

$$P_{ijk}+u_iP_{jk}+u_jP_{ki}+u_kP_{ij}+nu_iu_ju_k=n\left\langle v_iv_jv_k\right\rangle \tag{A102}$$

could be checked easily.

$$\begin{aligned}\frac{\partial P_{ij}}{\partial t}+\frac{\partial P_{ijk}}{\partial x_k}+u_iu_j\frac{\partial n}{\partial t} &+ u_iu_j\frac{\partial nu_k}{\partial x_k}+P_{ik}\frac{\partial u_j}{\partial x_k}+P_{jk}\frac{\partial u_i}{\partial x_k}+P_{ij}\frac{\partial u_k}{\partial x_k}+u_k\frac{\partial P_{ij}}{\partial x_k}\\ &+ u_i\left(\left(\frac{\partial nu_j}{\partial t}\right)_c-\left(\frac{\partial n}{\partial t}\right)_c u_j\right) &\text{(A103)}\\ &+ u_j\left(\left(\frac{\partial nu_i}{\partial t}\right)_c-\left(\frac{\partial n}{\partial t}\right)_c u_i\right) &\text{(A104)}\\ &= \left(\frac{\partial P_{ij}+nu_iu_j}{\partial t}\right)_c &\text{(A105)}\end{aligned}$$

$$\begin{aligned}\frac{\partial P_{ij}}{\partial t}+\frac{\partial P_{ijk}}{\partial x_k} &+ u_iu_j\left(\frac{\partial n}{\partial t}\right)_c+P_{ik}\frac{\partial u_j}{\partial x_k}+P_{jk}\frac{\partial u_i}{\partial x_k}+P_{ij}\frac{\partial u_k}{\partial x_k}+u_k\frac{\partial P_{ij}}{\partial x_k}\\ &+ u_i\left(\left(\frac{\partial nu_j}{\partial t}\right)_c-\left(\frac{\partial n}{\partial t}\right)_c u_j\right) &\text{(A106)}\\ &+ u_j\left(\left(\frac{\partial nu_i}{\partial t}\right)_c-\left(\frac{\partial n}{\partial t}\right)_c u_i\right) &\text{(A107)}\\ &= \left(\frac{\partial P_{ij}+nu_iu_j}{\partial t}\right)_c &\text{(A108)}\end{aligned}$$

$$\begin{aligned}
\frac{\partial P_{ij}}{\partial t} + \frac{\partial P_{ijk}}{\partial x_k} \quad &+ \quad P_{ik}\frac{\partial u_j}{\partial x_k} + P_{jk}\frac{\partial u_i}{\partial x_k} + P_{ij}\frac{\partial u_k}{\partial x_k} + u_k\frac{\partial P_{ij}}{\partial x_k} \\
&= \quad \left(\frac{\partial P_{ij} + nu_iu_j}{\partial t}\right)_c - u_iu_j\left(\frac{\partial n}{\partial t}\right)_c - u_i\left(\frac{\partial nu_j}{\partial t}\right)_c \qquad \text{(A109)} \\
&\quad -u_j\left(\frac{\partial nu_i}{\partial t}\right)_c + u_i\left(\frac{\partial n}{\partial t}\right)_c u_j + u_j\left(\frac{\partial n}{\partial t}\right)_c u_i \qquad \text{(A110)}
\end{aligned}$$

for the right side is

$$\begin{aligned}
\left(\frac{\partial P_{ij} + nu_iu_j}{\partial t}\right)_c \quad &- \quad u_iu_j\left(\frac{\partial n}{\partial t}\right)_c - u_i\left(\frac{\partial nu_j}{\partial t}\right)_c - u_j\left(\frac{\partial nu_i}{\partial t}\right)_c \qquad \text{(A111)} \\
&\quad +u_i\left(\frac{\partial n}{\partial t}\right)_c u_j + u_j\left(\frac{\partial n}{\partial t}\right)_c u_i \qquad \text{(A112)} \\
&= \quad \left(\frac{\partial P_{ij}}{\partial t}\right)_c + \left(\frac{\partial nu_iu_j}{\partial t}\right)_c - u_iu_j\left(\frac{\partial n}{\partial t}\right)_c - u_i\left(\frac{\partial nu_j}{\partial t}\right)_c \qquad \text{(A113)} \\
&\quad -u_j\left(\frac{\partial nu_i}{\partial t}\right)_c + u_i\left(\frac{\partial n}{\partial t}\right)_c u_j + u_j\left(\frac{\partial n}{\partial t}\right)_c u_i \qquad \text{(A114)} \\
&= \quad \left(\frac{\partial P_{ij}}{\partial t}\right)_c \qquad \text{(A115)}
\end{aligned}$$

$\therefore$

$$\frac{\partial P_{ij}}{\partial t} + \frac{\partial P_{ijk}}{\partial x_k} + P_{ik}\frac{\partial u_j}{\partial x_k} + P_{jk}\frac{\partial u_i}{\partial x_k} + P_{ij}\frac{\partial u_k}{\partial x_k} + u_k\frac{\partial P_{ij}}{\partial x_k} = \left(\frac{\partial P_{ij}}{\partial t}\right)_c$$

and we can see this is equation 10 after assuming $\frac{\partial P_{ijk}}{\partial x_k} = 0$