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The Velocity Dispersion in Saturn's Rings

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The velocity dispersion in a differentially rotating disk of particles such as Saturn's rings is determined by the details of the collision process. Collisions give rise to a viscous stress that converts orbital energy into random motions. Since the collisions are not perfectly elastic, the energy in random motions is dissipated as heat. With increasing velocity dispersion the latter process becomes more important relative to the former because the collisions become less elastic. The velocity dispersion adjusts so that the effects of these two processes balance. The condition for this balance fixes the coefficient of restitution ϵ of the colliding particles as a function of the disk's optical depth τ. We solve the Boltzmann moment equations to determine $\epsilon(\tau)$. If the rings are about as old as the solar system then their radial width implies that the velocity dispersion of the ring particles is less than 0.2 cm sec^{-1}. The corresponding vertical thickness is then less than 10 m. We discuss the effects of collisions on the particles in Saturn's rings. If the particles are made of ice they are eroded by collisions and accrete the collisional debris. The time scale for erosion and accretion is probably shorter than the age of the solar system. Furthermore, for ice particles ϵ is likely to be substantially less than unity even at impact velocities as low as 10^{-3} cm sec^{-1}. Thus, a ring of ice particles would be a monolayer thick.

I. INTRODUCTION

This paper treats the collisional dynamics of a differentially rotating disk of particles. To some extent it is the analytic counterpart of the numerical studies of Brahic (1977 and references therein). Following Brahic, we assume that the particles are identical, indestructible, imperfectly elastic, smooth spheres, and gravitational interactions between the particles are neglected.

The organization of the paper is as follows. The collisional dynamics are described qualitatively in Section II and quantitatively in Section III. In Section IV we discuss the collisional erosion of the particles. Our conclusions are presented in Section V.

II. QUALITATIVE RESULTS

The principal features of the overall energy balance in a differentially rotating disk of particles are described below. The treatment is nonrigorous and quantitatively inaccurate. Nevertheless, it captures the essence of the physics which is modeled in the more mathematical analysis of Section III.

The disk dynamics is characterized by the relative magnitudes of the collision frequency ω_c and the orbital angular velocity Ω. If $\omega_c \gg \Omega$, the disk is a viscous fluid with kinematic viscosity $\nu \simeq \sigma^2/\omega_c$, where σ is the one-dimensional velocity dispersion. If $\omega_c \lesssim \Omega$, the particle paths curve between collisions and the viscosity is not isotropic. For arbitrary values of

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Ω/ω_c, the magnitude of the viscosity is of order

$$\nu \simeq (\sigma^2/\omega_c)(1 + \Omega^2/\omega_c^2)^{-1}. \quad (1)$$

This expression is analogous to that for the viscosity of a plasma in a magnetic field (Spitzer, 1962).

The particles oscillate through the central plane of the disk at a frequency of order Ω. The mean number of collisions suffered by a particle in passing through the disk is of order the normal optical depth τ. Thus, $\omega_c/\Omega \simeq \tau$ and

$$\nu \simeq (\sigma^2/\Omega)(\tau + 1/\tau)^{-1}. \quad (2)$$

The form of the expression for ν is justified in Section III.

The viscous stress converts orbital energy into random kinetic energy at a rate $\nu(rd\Omega/dr)^2$ per unit mass (Landau and Lifshitz, 1959). Thus the velocity dispersion tends to increase as

$$(d\sigma^2/dt)_1 \simeq c_1\nu[r(d\Omega/dr)]^2, \quad (3)$$

where c_1 is of order unity. Since the collisions are imperfectly elastic, the energy in random motions is dissipated as heat.

The rate of damping depends on the details of the collision process. For simplicity, we assume that a collision conserves the relative tangential velocity and reduces the absolute value of the relative normal velocity by a factor ϵ, the coefficient of restitution. The rate of damping is

$$\left(\frac{d\sigma^2}{dt}\right)_2 \simeq -c_2\sigma^2\omega_c(1 - \epsilon^2)$$

$$\simeq -c_2\sigma^2\Omega\tau(1 - \epsilon^2), \quad (4)$$

where c_2 is of order unity.

The condition for a steady state is that $(d\sigma^2/dt)_1 + (d\sigma^2/dt)_2 = 0$. Setting $d\Omega/dr \simeq \Omega/r$ we obtain

$$\epsilon^2 \simeq 1 - c_1/[c_2(1 + \tau^2)]. \quad (5)$$

Unfortunately (5) is not very accurate, although it gives the correct asymptotic behavior $1 - \epsilon \to \tau^{-2}$ as $\tau \to \infty$ and $\epsilon \to$ const as $\tau \to 0$. The main point here is that equilibrium requires a unique value of ϵ for each τ. A more accurate calculation of $\epsilon(\tau)$ is given in Section III.

Cook and Franklin (1964) first derived a form of the $\epsilon(\tau)$ relation. Since the relation is independent of velocity dispersion they concluded that the disk would either expand into a spherical cloud or contract to a monolayer. However, Cook and Franklin overlooked the fact that for realistic materials, ϵ is a monotonically decreasing function of the impact velocity. Thus, equilibrium random motions are possible: the velocity dispersion of a disk adjusts so that the $\epsilon(\tau)$ relation is satisfied. In particular, suppose that ϵ is a monotonically decreasing function of impact velocity and approaches unity at sufficiently small impact velocity. Then consider the evolution of a particle disk in which σ^2 is initially set close to zero. Initially, σ^2 will grow exponentially [cf. Eqs. (2) and (3)] due to the conversion of orbital energy into random motions by the viscous stress. The damping of random motions by inelastic collisions will initially be negligible since $\epsilon \simeq 1$ at very low impact velocity. As σ^2 grows, the typical value of ϵ characterizing collisions decreases and $\sigma^{-2}(d\sigma^2/dt)$ becomes smaller. Ultimately, σ^2 attains the value at which the rate of release of orbital energy by viscous stresses just balances the damping due to inelastic collisions. This is the value of σ^2 for which ϵ satisfies the relation given by (5).

The equilibrium velocity dispersion is established on a time scale comparable to the orbital period. On a much longer time scale the collisions will alter the radial structure of the disk (Lynden-Bell and Pringle, 1974). The collisions conserve angular momentum and dissipate energy. This leads to a radial spreading of the disk on the characteristic time scale

$$T \simeq \Delta r^2/\nu \simeq Re/\Omega, \quad (6)$$

where Δr is the current radial width and

$Re = \Delta r^2\Omega/\nu$ is the Reynolds number. Note that T is just a simple diffusion time. It is the time it takes a particle to random walk a distance Δr. An earlier but incorrect estimate of the evolution time was made by Jeffreys (1947), who erroneously used $Re^{1/2}$ instead of Re in (6). Substituting from (2), we find

$$T \simeq (\Delta r^2\Omega/\sigma^2)(\tau + 1/\tau). \tag{7}$$

To make a crude application to Saturn's rings, we put $\Delta r = 5 \times 10^9$ cm (roughly the distance between the inner edge of the B ring and the outer edge of the A ring), $\tau = 1$ and $T = 5 \times 10^9$ yr. Equation (7) then yields

$$\sigma \lesssim 0.2 \text{ cm sec}^{-1}. \tag{8}$$

Since the velocity dispersion is related to the vertical thickness by $h \simeq \sigma/\Omega$, the corresponding upper limit to the thickness is $h \lesssim 10$ m. This is fairly close to the limit $h \lesssim 2.5$ m derived by Brahic (1977) from numerical simulations, and it is much less than the values $h \simeq 1$ km deduced from observations made during the Earth's passage through the ring plane (Bobrov, 1972). The large observational values of h cannot be due to the bending of the ring plane since the most powerful perturbers (Titan and the Sun) warp the invariable plane by only a few tens of meters. The best explanation for the large apparent thickness is observational error since none of the observed values are more than three standard deviations from zero.

There are situations in which some of the approximations we have made fail. The kinetic theory we have used is only valid when the fractional volume F filled by disk particles is small. In terms of the particle number density n and the particle radius a, kinetic theory requires

$$F \simeq na^3 \ll 1. \tag{9}$$

We have assumed that the potential field of the central body dominates that of the disk. That is,

$$G\rho na^3/\Omega^2 \simeq F(G\rho/\Omega^2) \ll 1, \tag{10}$$

where ρ is the density of the particle material. Although we could include the contribution from the self-gravity of the disk in our analysis, disks which violate inequality (10) are gravitationally unstable to small-scale disturbances (Toomre, 1964). These instabilities produce clumping and tend to increase random motions; both of these effects are too difficult to model reliably.

We have neglected the gravitational forces between particles. This is valid if the surface escape velocity of a particle is much less than the velocity dispersion,

$$G\rho a^2 \ll \sigma^2. \tag{11}$$

This relation can be recast in terms of the optical depth $\tau \simeq na^2\sigma\Omega^{-1}$ as

$$(F/\tau)(G\rho/\Omega^2)^{1/2} \ll 1. \tag{12}$$

If the particles in Saturn's rings are made of ice ($\rho = 0.9$ g cm^{-3}) then for $\tau \simeq 1$ inequalities (9), (10), and (12) require $F \ll 1$. We discuss likely values for the filling factor F and the validity of our approximations in Section V. The same approximations are used in the more exact calculation of the $\epsilon(\tau)$ relation reported in the next section.

III. THE EQUILIBRIUM DISK

(a) *The Boltzmann Equation*

Let $f(\mathbf{x}, \mathbf{v})$ be the number density of disk particles in the phase–space volume centered at position $\mathbf{x}$ and velocity $\mathbf{v}$. The density f satisfies the Boltzmann equation

$$\frac{\partial f}{\partial t} + v_\alpha \frac{\partial f}{\partial x_\alpha} - \frac{\partial U}{\partial x_\alpha}\frac{\partial f}{\partial v_\alpha} = \left(\frac{\partial f}{\partial t}\right)_c, \tag{13}$$

where we have used Cartesian coordinates (x_α, v_α), $U(\mathbf{x})$ is the gravitational potential and $(\partial f/\partial t)_c$ is the rate of change of f due to collisions. The coordinates are the first

TABLE I

COORDINATE SYSTEMS

Label	Type	Orientation	Equation where used
x_α	Cartesian	Arbitrary	(15)–(18), (43)
$\tilde{\omega}$, θ, z	Cylindrical	Disk has azimuthal symmetry in θ and reflection symmetry in z	(19), (41), (44), (45)
x_i	Cartesian	Principal axis system	(20), (23)–(37), (45)
X, Y, Z	Cartesian	Z axis parallel to v_r, (see Fig. 1)	(23), (24), (26)–(28)
r, θ, φ	Spherical	$\theta = 0$ is Z axis, $\varphi = 0$ is XZ plane	(23), (24), (26)–(28)
$\|v_r\|$, θ_v, φ_v; $\mu = \cos\theta_v$	Spherical (velocity space)	$\theta_v = 0$ is x_i axis	(33)

of several listed in Table I which we will need in this section. The number density $n(\mathbf{x})$, the mean velocity vector $\mathbf{u}(\mathbf{x})$, and the pressure tensor $p_{\alpha\beta\gamma\ldots}(\mathbf{x})$ are defined by velocity moments over f:

$$n = \int f d\mathbf{v}, \tag{14}$$

$$nu_\alpha = \int f v_\alpha d\mathbf{v}, \tag{15}$$

$$p_{\alpha\beta\gamma\ldots} = \int f(v_\alpha - u_\alpha)(v_\beta - u_\beta) \times (v_\gamma - u_\gamma)\ldots d\mathbf{v}. \tag{16}$$

We obtain three sets of moment equations by multiplying (13) successively by 1, v_α, and $v_\alpha v_\beta$ and integrating over $d\mathbf{v}$:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_\alpha}(nu_\alpha) = \left(\frac{\partial n}{\partial t}\right)_c,$$

$$\frac{\partial}{\partial t}(nu_\alpha) + \frac{\partial}{\partial x_\beta}(p_{\alpha\beta} + nu_\alpha u_\beta) + n\frac{\partial U}{\partial x_\alpha} = \left(\frac{\partial}{\partial t} nu_\alpha\right)_c,$$

$$\frac{\partial}{\partial t}(p_{\alpha\beta} + nu_\alpha u_\beta) + \frac{\partial}{\partial x_\gamma}(p_{\alpha\beta\gamma} + u_\beta p_{\alpha\gamma} + u_\gamma p_{\alpha\beta} + u_\alpha p_{\beta\gamma} + nu_\alpha u_\beta u_\gamma) + n\left(u_\alpha\frac{\partial U}{\partial x_\beta} + u_\beta\frac{\partial U}{\partial x_\alpha}\right) = \left[\frac{\partial}{\partial t}(p_{\alpha\beta} + nu_\alpha u_\beta)\right]_c. \tag{17}$$

We model the particles as identical, indestructible, imperfectly elastic spheres, and neglect particle spins. This model ensures that $(\partial n/\partial t)_c$ and $(\partial nu_\alpha/\partial t)_c = 0$. We assume that the random motions are much smaller than the mean orbital motions so that the term $\partial p_{\alpha\beta\gamma}/\partial x_\gamma$ in (17) can be neglected. After some algebra (17) becomes

$$\begin{aligned} &\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_\alpha}(nu_\alpha) = 0, \\ &\frac{\partial u_\alpha}{\partial t} + u_\beta\frac{\partial u_\alpha}{\partial x_\beta} = -\frac{\partial U}{\partial x_\alpha} - \frac{1}{n}\frac{\partial p_{\alpha\beta}}{\partial x_\beta}, \\ &\frac{\partial p_{\alpha\beta}}{\partial t} + p_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma} + p_{\beta\gamma}\frac{\partial u_\alpha}{\partial x_\gamma} + \frac{\partial}{\partial x_\gamma}(p_{\alpha\beta}u_\gamma) = \left(\frac{\partial p_{\alpha\beta}}{\partial t}\right)_c. \end{aligned} \tag{18}$$

These are the continuity equation, Euler's equation, and the viscous-stress equation. We focus our attention on the viscous-stress equation.

Since we assume azimuthal symmetry, it is natural to transform Eqs. (18) into cylindrical coordinates ($\tilde{\omega}$, θ, z) where the disk is symmetric about $z = 0$ (see Table I). In the limit of small random motions, we can replace $\mathbf{u}(\mathbf{x})$ by $\tilde{\omega}\Omega(\tilde{\omega})\mathbf{e}_\theta = (\tilde{\omega}dU/d\tilde{\omega})^{1/2}\mathbf{e}_\theta$. The resulting equations are

$$\frac{\partial p_{\omega\tilde{\omega}}}{\partial t} - 4\Omega p_{\omega\theta} = \left(\frac{\partial p_{\omega\omega}}{\partial t}\right)_c,$$

$$\frac{\partial p_{\omega\theta}}{\partial t} - \frac{p_{\omega\tilde{\omega}}}{\tilde{\omega}}\frac{d}{d\tilde{\omega}}(\tilde{\omega}^2\Omega) - 2\Omega p_{\theta\theta} = \left(\frac{\partial p_{\omega\theta}}{\partial t}\right)_c, \quad (19)$$

$$\frac{\partial p_{\theta\theta}}{\partial t} + \frac{2p_{\omega\theta}}{\tilde{\omega}}\frac{d}{d\tilde{\omega}}(\tilde{\omega}^2\Omega) = \left(\frac{\partial p_{\theta\theta}}{\partial t}\right)_c,$$

$$\frac{\partial p_{zz}}{\partial t} = \left(\frac{\partial p_{zz}}{\partial t}\right)_c.$$

Note that $p_{\omega z} = p_{\theta z} = 0$ from the assumed symmetry about $z = 0$.

For any given position, we can find the principal axes $\mathbf{e}_i$, $i = 1, 2, 3$ of the pressure tensor (see Table I). We put $\mathbf{e}_3 = \mathbf{e}_z$ and define the angle δ by $\mathbf{e}_z \sin\delta = \mathbf{e}_{\tilde{\omega}} \times \mathbf{e}_1$. Without loss of generality we require $|\delta| \leq \pi/4$. We specialize to a Keplerian rotation law, $\Omega \propto \tilde{\omega}^{-3/2}$. Equations (19) then take the form

$$\frac{\partial p_{11}}{\partial t} - \frac{3\Omega}{2}(\sin 2\delta)p_{11} = \left(\frac{\partial p_{11}}{\partial t}\right)_c,$$

$$\frac{\partial p_{22}}{\partial t} + \frac{3\Omega}{2}(\sin 2\delta)p_{22} = \left(\frac{\partial p_{22}}{\partial t}\right)_c, \quad (20)$$

$$p_{11}(1 + 3\sin^2\delta) - p_{22}(1 + 3\cos^2\delta) = 0,$$

$$\frac{\partial p_{33}}{\partial t} = \left(\frac{\partial p_{33}}{\partial t}\right)_c.$$

To make any further progress, we have to evaluate the collision terms $(\partial p_{ii}/\partial t)_c$. Most of the traditional approximations used to evaluate the collision terms are inappropriate for this problem. In the limit of high collision frequency, $\omega_c \gg \Omega$ or $\tau \gg 1$, the viscous-stress equation becomes the Navier–Stokes equation of hydrodynamics, but this approximation fails when $\tau \lesssim 1$. The Fokker–Planck approximation requires small-angle scattering and physical impacts can scatter particles through large angles. Cook and Franklin (1964) use the Krook model for the collision term: $(\partial f/\partial t)_c = \omega_c(f_0 - f)$ where f_0 is a Maxwellian with the same number and energy densities as f. However, this model cannot account for the energy loss in inelastic collisions. Cook and Franklin have to introduce the energy loss artifically.

Our technique for calculating the collision terms is based on the assumption that f is a triaxial Gaussian distribution in velocity space. With this single assumption, the full Boltzmann collision integral for inelastic smooth sphere scattering can be evaluated analytically. Since the parameters of the Gaussian are completely specified by n, u_i, and p_{ij}, Eqs. (20) are closed.

(b) *The Collision Terms*

We now evaluate the terms $(\partial p_{ii}/\partial t)_c$ in (20). Consider a collision between two particles with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ which changes the velocities to $\mathbf{v}'_1$ and $\mathbf{v}'_2$. The relative velocities before and after the collision are $\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2$ and $\mathbf{v}_r' = \mathbf{v}_1' - \mathbf{v}'_2$. The center-of-mass velocity $\mathbf{v}_c$ is conserved and the relative motion of the two particles is found by assuming that one acts as a fixed center of force while the other has the reduced mass $\mu = m/2$.

As in Section II, we assume that the impact conserves the relative tangential velocity, but reduces the absolute value of the relative normal velocity by a factor ϵ. Let b be the impact parameter, and let $\boldsymbol{\lambda}$ be a unit vector pointing from the center of particle 1 to the center of particle 2 at the

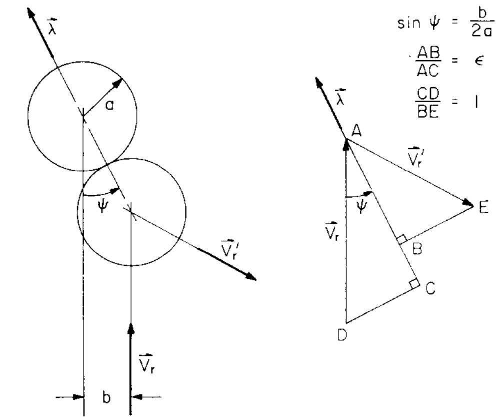

FIG. 1. The geometry of an inelastic collision of two smooth spheres.

time of impact. We have (see Fig. 1)

$$\mathbf{v}'_r = \mathbf{v}_r - \boldsymbol{\lambda}(1+\epsilon)\mathbf{v}_r\cdot\boldsymbol{\lambda},$$
$$\boldsymbol{\lambda}\cdot\mathbf{v}_r = |v_r|(1 - b^2/4a^2)^{1/2}. \quad (21)$$

The collision dynamics are conveniently described in a frame $(X, Y, Z) \equiv (r,\theta,\varphi)$ whose Z axis ($\theta = 0$) is in the direction of $\mathbf{v}_r$ (see Table I). A collision is completely specified by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\boldsymbol{\lambda} = (\theta_\lambda, \varphi_\lambda)$ or alternatively by $\mathbf{v}_1$, $\mathbf{v}_2$, b, and φ_λ since $\boldsymbol{\lambda}\cdot\mathbf{v}_r = |v_r|\cos\theta_\lambda$ is given in terms of b in (21). The collision rate per unit volume in the interval $\mathbf{v}_1 \to \mathbf{v}_1 + d\mathbf{v}_1$, $\mathbf{v}_2 \to \mathbf{v}_2 + d\mathbf{v}_2$, $b \to b + db$, $\varphi_\lambda \to \varphi_\lambda + d\varphi_\lambda$ is

$$f(\mathbf{v}_1)f(\mathbf{v}_2)d\mathbf{v}_1 d\mathbf{v}_2|\mathbf{v}_r|\,b\,db\,d\varphi_\lambda. \quad (22)$$

Thus the collision term is

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{1}{2}\int f(\mathbf{v}_1)f(\mathbf{v}_2)d\mathbf{v}_1 d\mathbf{v}_2|v_r| \times [(\mathbf{e}_i\cdot\mathbf{v}'_1)^2 + (\mathbf{e}_i\cdot\mathbf{v}'_2)^2 - (\mathbf{e}_i\cdot\mathbf{v}_1)^2 - (\mathbf{e}_i\cdot\mathbf{v}_2)^2]\,b\,db\,d\varphi_\lambda. \quad (23)$$

A factor $\frac{1}{2}$ has been inserted so that each collision is counted only once. We express $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}'_1$, $\mathbf{v}'_2$ in terms of $\mathbf{v}_c$, $\mathbf{v}_r$ and $\mathbf{v}_r'$, and obtain

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{1}{4}\int f(\mathbf{v}_1)f(\mathbf{v}_2)d\mathbf{v}_1 d\mathbf{v}_2|v_r| \times [(\mathbf{e}_i\cdot\mathbf{v}'_r)^2 - (\mathbf{e}_i\cdot\mathbf{v}_r)^2]\,b\,db\,d\varphi_\lambda. \quad (24)$$

From (21),

$$(\mathbf{e}_i\cdot\mathbf{v}'_r)^2 - (\mathbf{e}_i\cdot\mathbf{v}_r)^2 = (\mathbf{e}_i\cdot\boldsymbol{\lambda})^2(\mathbf{v}_r\cdot\boldsymbol{\lambda})^2(1+\epsilon)^2 - 2(\mathbf{e}_i\cdot\boldsymbol{\lambda})\cdot(\mathbf{e}_i\cdot\mathbf{v}_r)(\mathbf{v}_r\cdot\boldsymbol{\lambda})(1+\epsilon). \quad (25)$$

To evaluate the eight-dimensional integral in (24), we write

$$\boldsymbol{\lambda} = (\sin\theta_\lambda\cos\varphi_\lambda, \sin\theta_\lambda\sin\varphi_\lambda, \cos\theta_\lambda)$$

and $\mathbf{e}_i = (e_{iX}, e_{iY}, e_{iZ})$. Then

$$\int_0^{2\pi} d\varphi_\lambda(\mathbf{e}_i\cdot\boldsymbol{\lambda}) = 2\pi e_{iZ}\cos\theta_\lambda,$$

$$\int_0^{2\pi} d\varphi_\lambda(\mathbf{e}_i\cdot\boldsymbol{\lambda})^2 = \pi(e_{iX}^2 + e_{iY}^2)\sin^2\theta_\lambda + 2\pi e_{iZ}^2\cos^2\theta_\lambda = \pi(1 - e_{iZ}^2)\sin^2\theta_\lambda + 2\pi e_{iZ}^2\cos^2\theta_\lambda. \quad (26)$$

These are the only factors in (25) which depend on φ_λ. The integral of (25) over φ_λ yields

$$\int_0^{2\pi} d\varphi_\lambda[(\mathbf{e}_i\cdot\mathbf{v}'_r)^2 - (\mathbf{e}_i\cdot\mathbf{v}_r)^2] = -4\pi e_{iZ}^2 v_r^2\cos^2\theta_\lambda(1+\epsilon) + \pi[\sin^2\theta_\lambda(1 - e_{iZ}^2) + 2\cos^2\theta_\lambda e_{iZ}^2] \times v_r^2\cos^2\theta_\lambda(1+\epsilon)^2. \quad (27)$$

Next we use (21) to eliminate θ_λ from (27). Integrating over b, we obtain

$$\int_0^{2a} b\,db\int_0^{2\pi} d\varphi_\lambda[(\mathbf{e}_i\cdot\mathbf{v}'_r)^2 - (\mathbf{e}_i\cdot\mathbf{v}'_r)^2] = -4\pi a^2 v_r^2 e_{iZ}^2(1+\epsilon) + \tfrac{1}{3}\pi a^2 v_r^2(1 + 3e_{iZ}^2)(1+\epsilon)^2. \quad (28)$$

Replacing e_{iZ} by $v_{ri}/|v_r|$ and substituting back in (24), we find

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \pi a^2(1+\epsilon)\int f(\mathbf{v}_1)f(\mathbf{v}_2)d\mathbf{v}_1 d\mathbf{v}_2|v_r| \times [\tfrac{1}{4}(1+\epsilon)(v_{ri}^2 + \tfrac{1}{3}|v_r|^2) - v_{ri}^2]. \quad (29)$$

Note that the rate of loss of kinetic energy per unit volume

$$\tfrac{1}{2}m \sum_{i=1}^{3} \left(\frac{\partial p_{ii}}{\partial t}\right)_c = -\tfrac{1}{4}\pi m a^2(1-\epsilon^2) \times \int f(\mathbf{v}_1) f(\mathbf{v}_2) |v_r|^3 d\mathbf{v}_1 d\mathbf{v}_2 \quad (30)$$

is negative unless the collisions are perfectly elastic ($\epsilon = 1$).

Further progress depends on the explicit form of the distribution function f. As advertised in Section III(a), we assume that f is a triaxial Gaussian in velocity space. In the principal axis system (see Table I),

$$f(\mathbf{v}) = \frac{n}{(2\pi)^{3/2}\sigma_1\sigma_2\sigma_3} \exp\left(-\sum_{j=1}^{3} \frac{v_j^2}{2\sigma_j^2}\right), \quad (31)$$

where $\sigma_j^2 = p_{jj}/n$.

We substitute (31) into (29) and change velocity variables from $\mathbf{v}_1$, $\mathbf{v}_2$ to the center-of-mass and relative velocities $\mathbf{v}_c = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$ and $\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2$. After integration over $\mathbf{v}_c$,

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = \frac{n^2 a^2(1+\epsilon)}{8\pi^{1/2}\sigma_1\sigma_2\sigma_3} \int d\mathbf{v}_r |\mathbf{v}_r| \times \exp\left(-\sum_{j=1}^{3} \frac{v_{rj}^2}{4\sigma_j^2}\right) \times \left[\tfrac{1}{4}(1+\epsilon)(v_{ri}^2 + \tfrac{1}{3}|v_r|^2) - v_{ri}^2\right]. \quad (32)$$

To do the remaining integrals we change to polar coordinates in $\mathbf{v}_r$, $(|v_r|, \theta_v, \varphi_v)$, with $\theta_v = 0$ along the $\mathbf{e}_i$ axis. The integrals over $|v_r|$ and φ_v are easily done leaving only a single integral over $\mu = \cos\theta_v$. We denote the two principal axes normal to $\mathbf{e}_i$ by $\mathbf{e}_j$ and $\mathbf{e}_k$, and define

$$f_p(a, b) = (a^2 - a^2 b^2 + b^2)^{-p/2}.$$

Then

$$\left(\frac{\partial p_{ii}}{\partial t}\right)_c = 4\pi^{1/2} n^2 a^2 (1+\epsilon) \frac{\sigma_i^5}{\sigma_j\sigma_k} \int_0^1 d\mu \times \left[\tfrac{1}{4}(1+\epsilon)(\mu^2 + \tfrac{1}{3}) - \mu^2\right]$$
$$\times \left[3 f_5(\mu, \sigma_j/\sigma_i) f_1(\mu, \sigma_k/\sigma_i) + 2 f_3(\mu, \sigma_j/\sigma_i) f_3(\mu, \sigma_k/\sigma_i) + 3 f_1(\mu, \sigma_j/\sigma_i) f_5(\mu, \sigma_k/\sigma_i)\right]$$
$$= 4\pi^{1/2} n^2 a^2 (1+\epsilon) \frac{\sigma_i^5}{\sigma_j\sigma_k} \times \left[(1+\epsilon) J_P{}^i + J_Q{}^i\right]. \quad (33)$$

The collision integrals are now expressed in terms of two functions J_P and J_Q defined by (33). They are symmetrical functions of the variable pairs σ_j/σ_i, σ_k/σ_i and can be expressed in terms of elliptic integrals. For numerical work it is usually more convenient to do the integrals in (33) directly.

(c) *Equations of Equilibrium*

At this point, we make the additional assumption that the velocity ellipsoid is independent of z. In principle, we could solve the moment equations as a function of z and avoid this assumption. However, we feel that the errors it introduces are small. The density distribution in z is therefore isothermal and

$$n(z) = n(0)\exp[-U(r,z)/\sigma_3^2] = n(0)\exp[-\tfrac{1}{2}\Omega^2 z^2/\sigma_3^2], \quad (34)$$

where $U = \frac{1}{2}\Omega^2 z^2$ is the appropriate potential for a spherical central mass and $z \ll r$. Substituting the collision terms (33) into equations (20), setting $\partial/\partial t = 0$ and integrating over z using $p_{ii} = n(z)\sigma_i^2$, we obtain

$$\sin 2\delta = -\frac{4(2\pi)^{1/2} n(0) a^2}{3\Omega\sigma_2\sigma_3} \sigma_1^3 (1+\epsilon) \times \left[J_Q{}^1 + (1+\epsilon) J_P{}^1\right],$$
$$\sin 2\delta = +\frac{4(2\pi)^{1/2} n(0) a^2}{3\Omega\sigma_1\sigma_3} \sigma_2^3 (1+\epsilon) \times \left[J_Q{}^2 + (1+\epsilon) J_P{}^2\right], \quad (35)$$
$$\sigma_1^2(1 + 3\sin^2\delta) = \sigma_2^2(1 + 3\cos^2\delta),$$
$$J_Q{}^3 + (1+\epsilon) J_P{}^3 = 0.$$

In the limit $a \to 0$, we find $\delta = 0$ and $\sigma_1^2 = 4\sigma_2^2$, or equivalently, $p_{\tilde{\omega}\tilde{\omega}} = 4p_{\theta\theta}$. This is a well-known result for a collisionless disk in a Keplerian force field (e.g., Chandrasekhar, 1960, p. 159).

We can eliminate $n(0)$ using the equation for the optical depth[1]

$$\tau = \pi a^2 \int_{-\infty}^{\infty} n(z)dz$$
$$= (2\pi^3)^{1/2} n(0) a^2 \sigma_3 \Omega^{-1}. \quad (36)$$

Equations (35) now take their final form

$$\sin 2\delta = -\tfrac{4}{3}(\tau/\pi)\frac{\sigma_1^3}{\sigma_2\sigma_3^2}(1+\epsilon) \times [J_Q{}^1 + (1+\epsilon)J_P{}^1],$$
$$\sin 2\delta = +\tfrac{4}{3}(\tau/\pi)\frac{\sigma_2^3}{\sigma_1\sigma_3^2}(1+\epsilon) \times [J_Q{}^2 + (1+\epsilon)J_P{}^2], \quad (37)$$
$$\sigma_1^2(1 + 3\sin^2\delta) = \sigma_2^2(1 + 3\cos^2\delta),$$
$$J_Q{}^3 + (1+\epsilon)J_P{}^3 = 0.$$

For any assumed value of the optical depth τ, we can calculate the coefficient of restitution ϵ and both the orientation angle δ and axis ratios σ_2/σ_1, σ_3/σ_1 of the velocity ellipsoid (cf. Figs. 2 and 3). Note that only ratios of the axes of the velocity ellipsoid appear in Eqs. (37); the magnitude of the velocity dispersion enters indirectly through its effect on the coefficient of restitution.

Unfortunately the coefficient of restitution of ice (the most likely constituent of the ring particles) has not been measured.

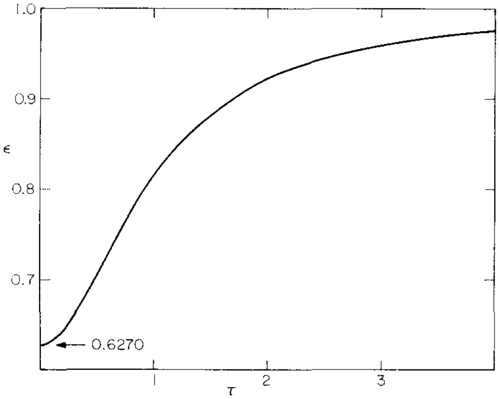

FIG. 2. The $\epsilon(\tau)$ relation.

The theoretical discussion in the next section suggests that even at velocities as low as 10^{-3} to 10^{-4} cm sec^{-1}, ϵ is appreciably less than 1 for ice. It would be very useful to have measurements of ϵ for ice at the temperature of the rings (90°K) over a range of impact velocities from 10^{-3} to 10 cm sec^{-1}.

(d) *Stability*

If the solutions of Section III(c) are realistic, they must be stable to arbitrary small changes from the equilibrium pressure tensor P°_{ij}. To check stability we return to Eqs. (20) and expand all of the terms to lowest order in the small quantity $p_{ij} - p^\circ_{ij}$. We assume $\partial(p_{ij} - p^\circ_{ij})/\partial t = \lambda(p_{ij} - p^\circ_{ij})$ and solve the resulting cubic characteristic equation for λ. The equilibrium is stable if and only if the real part of every root is less than or equal to zero.

Although the equilibria depend only on the local value of ϵ, the mean coefficient of restitution, their stability also depends on the local variation of ϵ with impact velocity. We model this variation crudely by assuming that ϵ depends only on the mean square velocity

$$\langle v^2 \rangle = \sum_{i=1}^{3} \sigma_i{}^2$$

[1] We use πa^2 rather than $2\pi a^2$ for the particle cross section. The smaller value does not include the contribution from diffracted light which is confined within a cone of opening angle $\Delta \simeq \lambda/a$, where a is the particle size and λ is the wavelength. If $a \gtrsim 10$ cm, $\Delta \lesssim 1''$, and under normal conditions the diffracted light lies within the seeing disk. For Saturn's rings, the large radar cross section of the rings implies $a \gtrsim 10$ cm (Goldstein and Morris, 1973).

according to

$$d\epsilon/d\langle v^2\rangle = -k(1-\epsilon)/\langle v^2\rangle. \quad (38)$$

Most materials have $k > 0$ corresponding to a coefficient of restitution which decreases with increasing impact velocity (Goldsmith, 1960).

We find by numerical experiments that the solutions are stable if and only if $k > 0$. This result is easy to understand. The rate of transfer of energy into random motions by viscosity is roughly proportional to $\langle v^2\rangle$, while the rate of loss is proportional to $(1-\epsilon^2)\langle v^2\rangle$. If the gain and loss rates are initially equal and $\langle v^2\rangle$ is increased, then for $k > 0$ the collisions become more inelastic and there is a net loss of random energy which returns the disk to equilibrium. Similarly, for $k < 0$, there is a net gain of energy and $\langle v^2\rangle$ continues to increase without bound.

(e) Viscosity

In any differentially rotating system, one of the important effects of collisions is the radial transport of angular momentum (e.g., Lynden-Bell and Pringle, 1974). So far we have worked entirely with the viscous-stress equation which is an energy equation. The momentum information is contained in Euler's equation

$$n\frac{D\mathbf{u}}{Dt} = -n\nabla U - \nabla\cdot\mathbf{p}, \quad (39)$$

where D/Dt is a Lagrangian derivative and $\mathbf{p}$ is the pressure tensor. The rate of gain of angular momentum per unit mass is

$$\frac{D\mathbf{J}}{Dt} = \frac{D}{Dt}(\mathbf{r}\times\mathbf{u}) = -\mathbf{r}\times\nabla U - \frac{1}{n}\mathbf{r}\times\nabla\cdot\mathbf{p}. \quad (40)$$

For a central potential ∇U is radial so $\mathbf{r}\times\nabla U = 0$. For an azimuthally symmetric disk with reflection symmetry about $z = 0$, $p_{\omega z} = p_{\theta z} = 0$ and $\partial/\partial\theta = 0$, so (40) becomes

$$\frac{DJ}{Dt} = -\frac{1}{n\tilde{\omega}}\frac{\partial}{\partial\tilde{\omega}}(\tilde{\omega}^2 p_{\theta\omega}). \quad (41)$$

Equation (39) is similar to the Euler equation for a viscous fluid disk (Lynden-Bell and Pringle, 1974):

$$\rho\frac{D\mathbf{u}}{Dt} = -\rho\nabla U + \nabla\cdot\mathbf{s}, \quad (42)$$

where ρ is the density and $\mathbf{s}$ is the stress

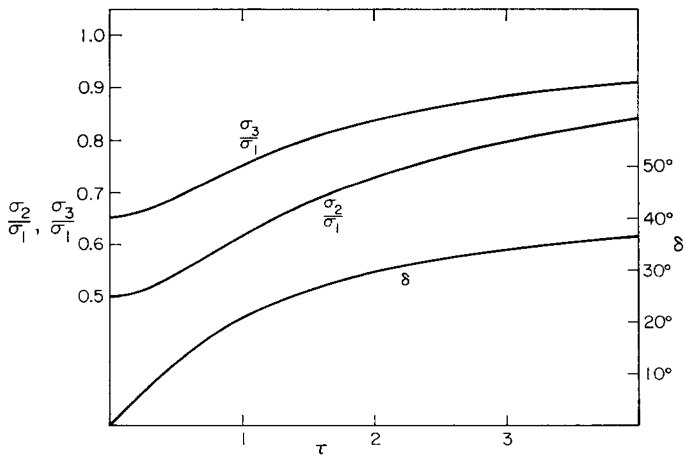

FIG. 3. The orientation angle δ and the axis ratios σ_2/σ_1 and σ_3/σ_1 as a function of optical depth. As $\tau \to \infty$, $\delta \to 45°$ and $\sigma_2/\sigma_1 \to 1$, $\sigma_3/\sigma_1 \to 1$.

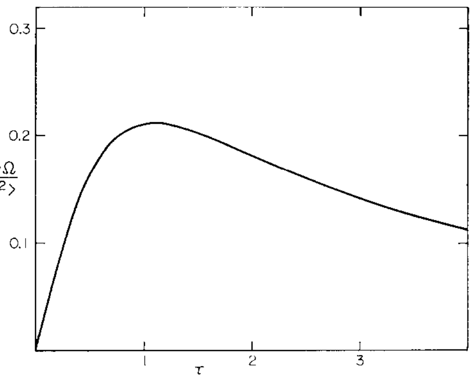

FIG. 4. The kinematic viscosity in units of $\frac{1}{3}\langle v^2\rangle/\Omega$ as a function of optical depth.

tensor

$$s_{\alpha\beta} = -p\delta_{\alpha\beta} + \eta\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} - \tfrac{2}{3}\delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial x_\gamma}\right) + \zeta\delta_{\alpha\beta}\frac{\partial u_\gamma}{\partial x_\gamma}, \quad (43)$$

where p is the pressure and η, ζ are the shear and bulk viscosity. From (42) and (43),

$$\frac{DJ}{Dt} = \frac{1}{\rho\tilde{\omega}}\frac{\partial}{\partial\tilde{\omega}}(\tilde{\omega}^2 s_{\theta\omega}) = \frac{1}{\rho\tilde{\omega}}\frac{\partial}{\partial\tilde{\omega}}\left(\eta\tilde{\omega}^3\frac{d\Omega}{d\tilde{\omega}}\right). \quad (44)$$

Comparison of (41) and (44) shows that the radial transport of angular momentum in the particle disk is exactly the same as in the fluid disk if the kinematic viscosity $\nu = \eta/\rho$ is assigned the value

$$\nu = \frac{1}{n}p_{\theta\omega}\left(-\tilde{\omega}\frac{d\Omega}{\alpha\tilde{\omega}}\right)^{-1} = \frac{\langle v^2\rangle}{3\Omega}\frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}\sin 2\delta \quad (45)$$

(cf. Fig. 4).

The limiting values of ν as a function of optical depth follow from solving Eqs. (37):

$$3\nu\Omega/\langle v^2\rangle = 0.4644\tau, \quad \tau \ll 1,$$
$$= 5\pi/32\tau, \quad \tau \gg 1. \quad (46)$$

The rough arguments of Section II predicted $\nu\Omega/\sigma^2 \rightarrow \tau$ as $\tau \rightarrow 0$ and $\nu\Omega/\sigma^2 \rightarrow \tau^{-1}$ as $\tau \rightarrow \infty$. These results are consistent with (46) since we can identify the one-dimensional velocity dispersion σ^2 with $\langle v^2\rangle/3$.

IV. PARTICLE EROSION

In this section we examine the effects of collisions on the ring particles themselves. The major question is whether the collisions erode the ring particles.

The stresses arising in a slow collision of two elastic bodies may be calculated from the Hertz law of contact (Goldsmith, 1960; Landau and Lifshitz, 1959). Hertz's law is valid if the particles behave elastically and the duration of contact is much greater than the travel time of elastic waves across the colliding bodies. For spheres, the latter condition is equivalent to the requirement that the impact velocity is much less than the elastic wave velocity. We calculate the maximum stress assuming Hertz theory is valid then test to see whether it exceeds the yield stress.

The maximum stress arising in the collision of two identical spheres is comparable to the maximum pressure exerted during the collision in the area of contact which is

$$p_{\rm m} = 0.63(\rho v^2)^{1/5}E^{4/5}(1 - \sigma^2)^{-4/5}. \quad (47)$$

Here v is the relative normal velocity, E is Young's modulus, σ is Poisson's ratio, and ρ is the density of the colliding bodies. Note that $p_{\rm m}$ is independent of the radius of the spheres.

For ice at the temperature of the rings, $E \simeq 10^{11}$ dyn cm^{-2}, $\sigma \simeq 0.5$, and $\rho \simeq 0.9$ g cm^{-3} (Hobbs, 1974). Thus

$$p_{\rm m} \simeq 5 \times 10^8\left(\frac{v}{1\ \text{cm sec}^{-1}}\right)^{2/5}\ \text{dyn cm}^{-2}. \quad (48)$$

This number should be compared with the critical stresses at which ice fractures. Compressive, tensile, and shear stresses all appear near the point of impact. Typical critical stresses are of order 0.5×10^7 to 2×10^7 dyn cm^{-2} (Hobbs, 1974; Weeks and Assur, 1972). These values apply at 273°K and ice at the ring temperature of 90°K is probably stronger. Weeks and Assur quote Charpy impact values which indicate that the strength of ice is about 30% stronger at the lower temperature. The critical stress values suggest that spherical ice particles are eroded at impact velocities exceeding 3×10^{-4} cm sec^{-1} although this number is rather uncertain because it depends on the $\frac{5}{2}$ power of the critical stress. Irregular particles erode more rapidly than spheres, since for fixed impact velocity and mass, the maximum stress is a decreasing function of the radii of curvature at the point of contact.

It is clearly impossible to theoretically determine the kinetic energy loss and the fracture pattern which result from the collision of ice particles. At best, theory might be used to rationalize experimental data. Unfortunately, the relevant experiments have not yet been performed. All that we can do is to outline the pertinent theoretical considerations. This is done below.

If fracture occurs, new surface is created and the minimum kinetic energy loss is given by the additional surface energy. The surface energy per unit area is approximately $E\iota$, where ι is a typical bond length of order 10^{-8} cm. The maximum fractional mass of a fragment is obtained by setting the center of mass kinetic energy before contact equal to the fragment's surface energy. This procedure yields

$$\left(\frac{\Delta m}{m}\right)_1 \simeq \left(\frac{\rho v^2}{E}\frac{a}{\iota}\right)^{3/2} = 3 \times 10^{-8} \times \left(\frac{v}{10^{-2}\text{ cm sec}^{-1}}\right)^3 \left(\frac{a}{10^2\text{ cm}}\right)^{3/2}. \quad (49)$$

A more conservative estimate of the fragment size is obtained as follows. We note that in the collision of two elastic spheres of radius a, the contact area attains a maximum radius

$$c_{\max} = a\left[\frac{5\pi}{16}\frac{(1-\sigma^2)}{E}\rho v^2\right]^{1/5} \quad (50)$$

when the particles have zero relative velocity. The stress is comparable to its maximum value in a volume of order $c_{\max}{}^3$. Outside this volume it falls off as the square of the distance. Therefore, a plausible size estimate for the fractured mass is $\Delta m \simeq \rho c_{\max}{}^3$. From (50),

$$\left(\frac{\Delta m}{m}\right)_2 \simeq \left(\frac{\rho v^2}{E}\right)^{3/5} \simeq 10^{-9}\left(\frac{v}{10^{-2}\text{ cm sec}^{-1}}\right)^{6/5}. \quad (51)$$

Of course, the validity of (51) requires $(\Delta m/m)_1 > (\Delta m/m)_2$.

A very crude estimate of the rate of erosion in Saturn's rings can be obtained by adopting $(\Delta m/m)_2$ as the fractional mass of the fragments broken off in each collision. Since the interval between collisions is roughly 10^4 sec, the particle lifetime is estimated to be $3 \times 10^5(v/10^{-2}$ cm sec$^{-1})^{-6/5}$ yr. This is less than the age of the solar system for reasonable velocities.

If the ring particles are eroded by collisions, the resulting debris will be recaptured by the particles. It is difficult to describe this process in detail, because there are few data on the adhesion coefficient of chips of ice. If small ice chips do not adhere efficiently, they will be ground down to molecular size where the adhesion coefficient is known to be near unity.

These considerations also provide some information on the coefficient of restitution of ice. It is likely that ϵ is significantly smaller than unity for velocities that are sufficient to chip ice particles, 10^{-3} to 10^{-4}

cm $\sec^{-1}$. The center of mass kinetic energy of the incident particles is dissipated in several ways. Some of the energy is used in the creation of new surface and some is dissipated as heat. Additional energy goes into the kinetic energy of the fragments that are broken off the colliding particles. The apportionment of energy among these different channels depends upon details such as the number and size of the fragments and the manner in which stress is relieved when they are formed. A detailed understanding of these processes can only be obtained experimentally.

V. DISCUSSION

The main result of this paper is the derivation of the relation between the coefficient of restitution and the optical depth for a differentially rotating disk of identical, inelastic, smooth spheres (see Fig. 2). We now examine the assumptions underlying this derivation to see whether the $\epsilon(\tau)$ relation is applicable to Saturn's rings.

In Section II, we discussed the requirements that kinetic theory should be valid, that the gravitational field of the ring could be neglected and that particle–particle gravitational scattering could be ignored [Eqs. (9)–(12)]. Because $G\rho/\Omega^2 \simeq 1$ (i.e., the ring lies near the Roche limit for ice) and $\tau \simeq 1$, all three requirements reduce to the requirement that the filling factor F is much less than unity.

Since the ring thickness h satisfies $h \simeq \sigma/\Omega$, we can relate the filling factor to the velocity dispersion σ and the particle radius a by

$$F \simeq na^3 \simeq \tau a\Omega/\sigma. \tag{52}$$

The upper limit to σ is 0.2 cm $\sec^{-1}$ [Eq. (8)] and the lower limit to a is 10 cm from radar measurements (Goldstein and Morris, 1973), giving a lower limit $F \gtrsim 10^{-2}$. This limit is barely consistent with the opposition effect being due to interparticle shadowing which requires $F \approx 10^{-2}$ (e.g., Pollack, 1975). However, the opposition effect could also be the result of intraparticle shadowing, in which case the filling factor could be of order unity. Although our numerical results are invalid if $F \simeq 1$ the qualitative behavior of the rings will probably be similar to what we described for $F \ll 1$.

Brahic (1977) has pointed out that results such as ours are also invalid if the rms particle height h is less than the particle size a, since we assume that the typical separation in mean radius of two colliding particles is the epicycle size h. However, if $h < a$, the typical separation is of order a. As a result, the kinematic viscosity $\nu \simeq a^2\omega_c$, and substituting in (3) and (4), we find that this viscosity can maintain a velocity dispersion $\sigma \simeq a\Omega$. For a minimum particle size of 10 cm, the lower limit to the velocity dispersion is $\sim 2 \times 10^{-3}$ cm $\sec^{-1}$.

In summary, we feel that if the rings' filling factor F is small, our calculations should give accurate quantitative results and even if $F \simeq 1$, our results should be qualitatively correct. We can therefore draw several conclusions.

The upper limit to the velocity dispersion of the ring particles is 0.2 cm $\sec^{-1}$ (from their present radial extent; cf. Section II), and the lower limit is 0.002 cm $\sec^{-1}$ (see above). The corresponding limits to the ring thickness are 10 m and 10 cm. For an optical depth $\tau \simeq 1$, the mean coefficient of restitution must be 0.8 at typical impact velocities (cf. Fig. 2). A measurement of the coefficient of restitution of ice as a function of impact velocity would yield an estimate of the velocity dispersion independent of the upper and lower limits given above.

The description of the collisions of ice spheres presented in Section IV indicates that the particles in the rings may be subject to a continual process of erosion and accretion. Unfortunately, we have not been able to determine the particle size distribution that this process would produce. Thus, we cannot offer any comment on the plausibility of the particle size distributions suggested by Greenberg *et al.* (1977).

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