Solution for Equation 15 in Mineshige1993

1. Simplification the equation

Equation 15 read as:

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left\{\sqrt{r}\frac{\partial}{\partial r}\left[\nu_0\left(\frac{\Sigma}{\Sigma_0}\right)^m\left(\frac{r}{r_0}\right)^n\Sigma\sqrt{r}\right]\right\}, \tag{1}$$

where $\Sigma = \Sigma(r,t)$. Multiplying Equation 1 by $r^{3/2}$

$$\frac{\partial \Sigma r^{3/2}}{\partial t} = \frac{3r^{3/2}}{r}\frac{\partial}{\partial r}\left\{\sqrt{r}\frac{\partial}{\partial r}\left[\nu_0\left(\frac{\Sigma}{\Sigma_0}\right)^m\left(\frac{r}{r_0}\right)^n\Sigma\sqrt{r}\right]\right\}, \tag{2}$$

$$= 3\sqrt{r}\frac{\partial}{2\sqrt{r}\partial\sqrt{r}}\left\{\sqrt{r}\frac{\partial}{2\sqrt{r}\partial\sqrt{r}}\left[\nu_0\left(\frac{\Sigma}{\Sigma_0}\right)^m\left(\frac{r}{r_0}\right)^n\Sigma\sqrt{r}\right]\right\}, \tag{3}$$

$$= 3\frac{\partial}{2\partial\sqrt{r}}\left\{\frac{\partial}{2\partial\sqrt{r}}\left[\nu_0\left(\frac{\Sigma}{\Sigma_0}\right)^m\left(\frac{r}{r_0}\right)^n\Sigma\sqrt{r}\right]\right\}, \tag{4}$$

Define

$$Y = \left(\frac{\Sigma}{\Sigma_0}\right)\left(\frac{r}{r_0}\right)^{3/2}$$

$$x = \left(\frac{r}{r_0}\right)^{1/2}$$

thus

$$\frac{\partial Y}{\partial t} = \frac{3}{4r_0\sqrt{r_0}\sqrt{r_0}}\frac{\partial^2}{\partial x^2}\left(\nu_0 Y^m (x)^{-3m} x^{2n} Y x^{-2}\right) \tag{5}$$

$$\frac{\partial Y}{\partial t} = \frac{3\nu_0}{4r_0^2}\frac{\partial^2}{\partial x^2}\left(Y^{m+1}x^{2n-3m-2}\right) \tag{6}$$

then we define: $\tau = 3\nu_0 t/4r_0^2$, and now we have the final form of Equation 1:

$$\frac{\partial Y}{\partial \tau} = \frac{\partial^2}{\partial x^2}\left(Y^{m+1}x^{2n-3m-2}\right). \tag{7}$$

2. Self-similar solution

$$Y = p(\xi)\tau^{-\mu} \tag{8}$$

$$\xi = x\tau^{-\lambda} \tag{9}$$

$$p(\xi) = \xi^c(1 - k\xi^b)^a \tag{10}$$

$$\begin{aligned}
a &= \frac{1}{m} && (11)\\
b &= c + 2 && (12)\\
c &= \frac{3m + 2 - 2n}{m + 1} && (13)\\
\lambda &= \frac{1}{4m + 4 - 2n} && (14)\\
\mu &= \lambda && (15)\\
k &= \frac{m}{(4m + 4 - 2n)(5m + 4 - 2n)} && (16)
\end{aligned}$$

3. Check for the solution

In this section, we check the self-similar solution.

$$Y = (x\tau^{-\lambda})^c(1 - k(x\tau^{-\lambda})^b)^a\tau^{-\mu} \tag{17}$$

$$Y = x^c\tau^{-\lambda c - \lambda}(1 - kx^b\tau^{-\lambda b})^{1/m} \tag{18}$$

$$Y = x^c\tau^{-\lambda(c+1)}(1 - kx^b\tau^{-\lambda b})^{1/m} \tag{19}$$

$$Y^{m+1} * x^{2n-3m-2} = x^{\frac{3m+2-2n}{m+1}(m+1)}\tau^{-\lambda(c+1)(m+1)}(1 - kx^b\tau^{-\lambda b})^{\frac{1+m}{m}}x^{2n-3m-2} \tag{20}$$

$$= \tau^{-\frac{1}{4m+4-2n}(\frac{3m+2-2n}{m+1}+1)(m+1)}(1 - kx^b\tau^{-\lambda b})^{\frac{1+m}{m}} \tag{21}$$

$$= \tau^{-\frac{1}{4m+4-2n}(\frac{3m+2-2n+m+1}{m+1})(m+1)}(1 - kx^b\tau^{-\lambda b})^{\frac{1+m}{m}} \tag{22}$$

$$= \tau^{-\frac{4m+3-2n}{4m+4-2n}}(1 - kx^b\tau^{-\lambda b})^{\frac{1+m}{m}} \tag{23}$$

$$\frac{\partial^2}{\partial x^2}\left(\tau^{-\frac{4m+3-2n}{4m+4-2n}}(1-kx^b\tau^{-\lambda b})^{\frac{1+m}{m}}\right) \tag{24}$$

$$\tau^{-\frac{4m+3-2n}{4m+4-2n}}\frac{\partial^2}{\partial x^2}(1-kx^b\tau^{-\lambda b})^{\frac{1+m}{m}} \tag{25}$$

$$\tau^{-\frac{4m+3-2n}{4m+4-2n}}\frac{\partial}{\partial x}\frac{1+m}{m}(1-kx^b\tau^{-\lambda b})^{\frac{1+m}{m}-1}(-k)\tau^{-\lambda b}bx^{b-1} \tag{26}$$

$$\frac{1+m}{m}(-k)b\tau^{-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1})}\tau^{-\frac{4m+3-2n}{4m+4-2n}}\frac{\partial}{\partial x}(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}}x^{b-1} \tag{27}$$

$$-\frac{1}{4m+4-2n}\tau^{-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1})}\tau^{-\frac{4m+3-2n}{4m+4-2n}}\frac{\partial}{\partial x}(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}}x^{b-1} \tag{28}$$

$$\frac{\partial}{\partial x}(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}}x^{b-1} \tag{29}$$

$$=\frac{\partial}{\partial x}(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}}\times x^{b-1}+(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}}\frac{\partial}{\partial x}x^{b-1} \tag{30}$$

$$=\frac{1}{m}(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}-1}(-kb\tau^{-\lambda b}x^{b-1})\times x^{b-1}+(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}}(b-1)x^{b-2} \tag{31}$$

$$\lambda b=\frac{1}{4m+4-2n}\frac{5m+4-2n}{m+1} \tag{32}$$

$$\lambda(c+1)=\frac{1}{4m+4-2n}\frac{4m+3-2n}{m+1} \tag{33}$$

$$\frac{1+m}{m}kb=\frac{1+m}{m}\frac{m}{(4m+4-2n)(5m+4-2n)}\frac{5m+4-2n}{m+1}=\frac{1}{4m+4-2n} \tag{34}$$

$$a = \frac{1}{m} \tag{35}$$
$$b = c+2 \tag{36}$$
$$c = \frac{3m+2-2n}{m+1} \tag{37}$$
$$\lambda = \frac{1}{4m+4-2n} \tag{38}$$
$$\mu = \lambda \tag{39}$$
$$k = \frac{m}{(4m+4-2n)(5m+4-2n)} \tag{40}$$

$$\frac{\partial}{\partial\tau}x^c\tau^{-\lambda(c+1)}(1-kx^b\tau^{-\lambda b})^{1/m} \tag{41}$$

$$x^c \frac{\partial}{\partial \tau} \tau^{-\lambda(c+1)} (1 - kx^b \tau^{-\lambda b})^{1/m} \tag{42}$$

$$\frac{\partial}{\partial \tau} \tau^{-\lambda(c+1)} (1 - kx^b \tau^{-\lambda b})^{1/m} \tag{43}$$

$$= \frac{\partial}{\partial \tau} \tau^{-\lambda(c+1)} \times (1 - kx^b \tau^{-\lambda b})^{1/m} + \tau^{-\lambda(c+1)} \frac{\partial}{\partial \tau} (1 - kx^b \tau^{-\lambda b})^{1/m} \tag{44}$$

$$= -\lambda(c+1) \tau^{-\lambda(c+1)-1} \times (1 - kx^b \tau^{-\lambda b})^{1/m} + \tau^{-\lambda(c+1)} \frac{1}{m} (1 - kx^b \tau^{-\lambda b})^{\frac{1}{m}-1} (-kx^b(-\lambda b)\tau^{-\lambda b-1}) \tag{45}$$

Coefficient of $(1 - kx^b \tau^{-\lambda b})^{1/m}$:

$$-\frac{1}{4m+4-2n} \tau^{-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1})} \tau^{-\frac{4m+3-2n}{4m+4-2n}} (1 - kx^b \tau^{-\lambda b})^{\frac{1}{m}} (b-1) x^{b-2} \tag{46}$$

and

$$-x^c \lambda(c+1) \tau^{-\lambda(c+1)-1} \times (1 - kx^b \tau^{-\lambda b})^{1/m} \tag{47}$$

thus

$$-\frac{1}{4m+4-2n} \tau^{-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1})} \tau^{-\frac{4m+3-2n}{4m+4-2n}} (b-1) \tag{48}$$

and

$$-\lambda(c+1) \tau^{-\lambda(c+1)-1} \tag{49}$$

$$\tau^{-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1})} \tau^{-\frac{4m+3-2n}{4m+4-2n}} (b-1) \tag{50}$$

and

$$(c+1) \tau^{-\lambda(c+1)-1} \tag{51}$$

$$\tau^{-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1})} \tau^{-\frac{4m+3-2n}{4m+4-2n}} \tag{52}$$

and

$$\tau^{-\lambda(c+1)-1} \tag{53}$$

$$-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1}) - \frac{4m+3-2n}{4m+4-2n} - (-\lambda(c+1)-1) \tag{54}$$

and

$$-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1}) - \frac{4m+3-2n}{4m+4-2n} + \frac{1}{4m+4-2n} \frac{4m+3-2n}{m+1} + 1 == 0 \tag{55}$$

So the coefficients of $(1 - kx^b \tau^{-\lambda b})^{1/m}$ are equal.

Then the coefficients of $(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}-1}$

$$x^c\tau^{-\lambda(c+1)}\frac{1}{m}(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}-1}(kx^b\lambda b\tau^{-\lambda b-1}) \tag{56}$$

and

$$-\frac{1}{4m+4-2n}\tau^{-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1})}\tau^{-\frac{4m+3-2n}{4m+4-2n}}\frac{1}{m}(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}-1}(-kb\tau^{-\lambda b}x^{b-1})\times x^{b-1} \tag{57}$$

$$x^c\tau^{-\lambda(c+1)}(kx^b\lambda b\tau^{-\lambda b-1}) \tag{58}$$

and

$$\frac{1}{4m+4-2n}\tau^{-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1})}\tau^{-\frac{4m+3-2n}{4m+4-2n}}(kb\tau^{-\lambda b}x^{b-1})\times x^{b-1} \tag{59}$$

$$\tau^{-\lambda(c+1)}(\tau^{-1}) \tag{60}$$

and

$$\tau^{-\frac{1}{4m+4-2n}(\frac{5m+4-2n}{m+1})}\tau^{-\frac{4m+3-2n}{4m+4-2n}} \tag{61}$$

, and thus the coefficients of $(1-kx^b\tau^{-\lambda b})^{\frac{1}{m}-1}$ is the same.

So we can see:

$$\frac{\partial Y}{\partial \tau}=\frac{\partial}{\partial x}\left(\frac{\partial\left(Y^{m+1}x^{2n-3m-2}\right)}{\partial x}\right) \tag{62}$$

i.e.

$$\frac{\partial Y}{\partial \tau}=\frac{\partial^2}{\partial x^2}\left(Y^{m+1}x^{2n-3m-2}\right) \tag{63}$$