Identification and Estimation of Dynamic Discrete Demand Using Aggregate Data

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… dynamic discrete demand

- Discrete: consumer i considers whether or not to purchase a product from $\mathcal{J}_t \subset \mathcal{J} \equiv \{0, 1, \ldots, J\}$.
- 0 is outside good.
- for durable goods, "dynamic" reflects the trade-off between buying now and waiting so he can decide whether to purchase in the next period
- Some applications: Gordon (2009, CPU market), Schiraldi (2011, car), Gowrisankaran and Rysman (2012, digital camera)

… dynamic discrete demand *using aggregate data*

- do not observe individual choices.
- instead we observe market shares s_{jt} and observable product characteristics (including prices) of products (x'_{jt}, p_{jt}),
- unobservable product characteristics ξ_{jt} affects price p_{jt} — price is endogenous in econometrics sense
- multiple markets (50 states of the US) and time periods (18 months since January of 2007)
- Same data structure as BLP model (Berry, 1994; Berry, Levinsohn and Pakes, 1995)
- *Panel* data structure is necessary to us

Identification and estimation ...

- Consumer's decisions will be dynamic programming problem
- Given the aggregate data, identify and estimate
  - consumer's per period utility function,
  - state transition distribution function $F(\Omega_{i,t+1} \mid \Omega_{it})$
  - and initial distribution function $F(\Omega_{it})$.
- necessary for counterfactual and welfare analysis
- e.g. Blackberry's market share if iPhone was not invented
- difference from BLP: explicitly model consumer's dynamic decisions. Hence, richer analysis, e.g. the price elasticity on future market share.

Dynamic demand literature

- Value function is headache in estimation.
- literature now focuses on simplifying estimation
- relies on approximation approach (inclusive value approach by Gowrisankaran and Rysman, 2012)
- Curse of dimensionality in both approximating value function and state transition law
- identification is unclear and estimation is complicated
- mostly estimate only consumer's per period utility function
- not discount factor, joint distribution of observable and unobservable product characteristics

Identification literature

- identification does not follow from Berry and Haile (2014) which requires index form (the distribution of utility of product j depends on ξ_{jt} only by $\delta_{jt} = x^1_{1j}\beta_j + \xi_{jt}$)
- index form may fail because ξ_{jt} enters the value function
- Berry and Haile (2014) cannot separately identify current and expected future payoff of product
- mixture literature does not work well here, Kasahara and Shimotsu (2009), Hu and Shum (2012), Allman, Matias and Rhodes (2009), Bonhomme, Jochmans and Robin (2016)

What we got

- show identification of all structural parameters explicitly
- a simple two step estimation method that estimates per period utility function, discount factor, joint distribution of price and unobservable characteristics, and joint distribution of unobservable characteristics in different periods
  - no approximation of value function
  - no solving dynamic programming
  - no curse of dimensionality in state transition
- LIMITATION: NO RANDOM COEFFICIENT (nested logit is allowed)
- hard to include random coefficient

Outline

- Model
- Identification
- Estimation details
- Monte Carlo

Model setup

- model follows from the literature; we did not invent it
- Conclusion is that the market share s_{jt} of product j at time t is
$$s_{jt} = \frac{\exp(v_{jt}(m_t))}{\sum_{k \in \mathcal{J}_t} \exp(v_{kt}(m_t))}.$$
- v_{jt} is the expected remaining lifetime payoff from purchasing product j.
- Structural parameters (per period utility, discount factor et al) and value functions enter into v_{jt}.
- $m_t = (x_t, p_t, \xi_t)$ is the vector of market level state variables, which is partially observable.
- similar to BLP; different payoff

Consumer i's dynamic programming problem

- Choose from $\mathcal{J}_t \subset \mathcal{J} \equiv \{0, 1, \ldots, J\}$. "0" is outside product.
- Per period utility u_{ijt}: if don't buy any product,

$$u_{i0t} = 0 + \varepsilon_{i0t}.$$

If buy j in period t,

$$u_{ijt} = x'_{jt}\gamma - \alpha p_{jt} + \xi_{jt} + \varepsilon_{ijt};$$

after purchasing j,

$$u_{ij\tau} = x'_{jt}\gamma + \xi_{jt} + \varepsilon_{ij\tau}, \qquad \tau > t.$$

- Once a consumer purchased a product, he exits the market completely
- $\Omega_{it} = (p'_t, x'_t, \xi'_t, \varepsilon'_{it})'$, $m_t = (x'_t, p'_t, \xi'_t)$
- consumers know Ω_{it}

Restrictions

- Standard restrictions:
  - Ω_{it} is stationary Markov process, $F(\Omega_{i,t+1} \mid \Omega_{it})$ is time invariant (not essential to our arguments)
  - $\varepsilon_{ijt} \perp\!\!\!\perp m_t = (x_t, p_t, \xi_t)$
  - $\Omega_{i,t+1} \equiv (m_{t+1}, \varepsilon_{i,t+1}) \perp\!\!\!\perp \varepsilon_{it} \mid m_t$ (ε_{ijt} is serially uncorrelated; no feedback effect)
  - $\varepsilon_{i,t+1} \perp\!\!\!\perp \Omega_{it} \equiv (m_t, \varepsilon_{it}) \mid m_{t+1}$ (no lagged effect)
- Issues:
  - taste (γ, α) are NOT random. Limitation, hard to relax in dynamic demand
  - Nonparametric extension $u_{ijt} = f(x_{jt}, p_{jt}) + \xi_{jt}$ is easy.
  - Difficult: "nonadditive" ξ_{jt}; higher (or unknown) dimension of ξ_{jt}

Expected lifetime payoff for purchasing product j at time t: v_{jt}

- β is discount factor
- outside option 0:

$$v_{0t}(\Omega_{it}) = 0 + \beta \, \mathrm{E}(\bar{V}_{t+1}(m_{t+1}) \mid m_t),$$

- product $j \neq 0$:

$$\begin{aligned} v_{jt}(m_t) &= x_{jt}'\gamma + \xi_{jt} - \alpha p_{jt} + \beta(x_{jt}'\gamma + \xi_{jt}) + \beta^2(x_{jt}'\gamma + \xi_{jt}) + \cdots \\ &= \frac{x_{jt}'\gamma + \xi_{jt}}{1-\beta} - \alpha p_{jt}. \end{aligned}$$

- $V_{it}(\Omega_{it})$ is the value function, and

$$\bar{V}_t(m_t) \equiv \int V_{it}(\Omega_{it}) F(\mathrm{d}\,\varepsilon_{it}).$$

Our starting point: market shares formula

- $\varepsilon_{it} = (\varepsilon_{i0t}, \ldots, \varepsilon_{iJt})$ are independent type 1 evd with mean zero.
- Consumer's choice is logit. The choice probability for product j at time t is the market share s_{jt}

$$s_{jt} = \frac{\exp(v_{jt}(m_t))}{\sum_{k \in \mathcal{J}_t} \exp(v_{kt}(m_t))},$$

where

$$v_{jt} = \begin{cases} \beta \, \mathrm{E}(\bar{V}_{t+1}(m_{t+1}) \mid m_t), & j = 0 \\ \frac{x'_{jt}\gamma + \xi_{jt}}{1-\beta} - \alpha p_{jt}, & j \neq 0. \end{cases}$$

- We also did nested logit case

Identification

- We need $\theta = (\alpha, \beta, \gamma)$, $F(\Omega_{i,t+1} \mid \Omega_{it})$ and $F(\Omega_{it})$ to simulate consumer's dynamic decisions from period t.
- By the conditional independence assumptions,

$$F(\Omega_{it}) = F(m_t)F(\varepsilon_{it}) \qquad \text{and} \qquad F(\Omega_{i,t+1} \,|\, \Omega_{it}) = F(m_{t+1} \,|\, m_t)F(\varepsilon_{i,t+1}).$$

- The target is θ, $F(m_t \equiv (x_t, p_t, \xi_t))$, $F(m_{t+1} \mid m_t)$.
- The difficulty is that the vector ξ_t is unobservable, serially correlated and at least correlated with p_t
- It also enters into the market share in unknown nonlinear form through value function.

Recipe (summary of results)

Different objects (meals) require different ingredients ...

To identify

- preference $\theta = (\alpha, \beta, \gamma)$, we only need some instrumental variables
- $\mathrm{E}(\xi_{jt})$, $\mathrm{E}(\xi_{jt} \mid x_t, p_t)$, we further assume $F(\xi_t)$ and $F(\xi_t \mid x_t, p_t)$ are time invariant.
- $\mathrm{Var}(\xi_{jt})$, $\mathrm{Var}(\xi_{jt} \mid x_t, p_t)$, we need $\xi_{1t}, \ldots, \xi_{Jt}$ are independent and homoscedastic conditional on x_t, p_t (e.g. $\mathrm{Var}(\xi_{1t} \mid x_{1t}, x_{2t}, p_{1t}, p_{2t}) = \mathrm{Var}(\xi_{2t} \mid x_{1t}, x_{2t}, p_{1t}, p_{2t})$)
- $F(\xi_t \mid x_t, p_t)$ nonparametrically, we assume qualities (e.g. ξ_{1t} and ξ_{2t}) have identical conditional distribution excepting for conditional mean
- $F(m_{t+1} \mid m_t)$, we would require ξ_{t+1} follows autoregressive process and $x_{t+1} \perp\!\!\!\perp (\xi_t, \xi_{t+1}) \mid (x_t, p_t)$

Main cooking method: log shares ratio

- Market share:

$$s_{jt} = \frac{\exp(v_{jt}(m_t))}{\sum_{k \in \mathcal{J}_t} \exp(v_{kt}(m_t))}.$$

- Given $0, 1, \ldots, J$, we have at most J independent log shares ratio,

$$\ln(s_{2t}/s_{1t}), \ln(s_{3t}/s_{2t}), \ldots, \ln(s_{Jt}/s_{J-1,t}), \ln(s_{Jt}/s_{0t}).$$

- demo with $\mathcal{J} = \{0, 1, 2\}$

Preference

- log shares ratio of product 2 to 1:

$$\ln(s_{2t}/s_{1t}) = (x_{2t} - x_{1t})' \frac{\gamma}{1-\beta} - \alpha(p_{2t} - p_{1t}) + \frac{\xi_{2t} - \xi_{1t}}{1-\beta}.$$

- We can identify $\gamma/(1-\beta)$ and α with mean zero IV z_t,
  - $E(z_t) = 0$;
  - z_t is uncorrelated with ξ_{1t} and ξ_{2t}
  - similar to BLP-type IV (the third product characteristics)

Identify discount factor

- log shares ratio of product 2 to 0:

$$\ln(s_{2t}/s_{0t}) = x_{2t}' \frac{\gamma}{1-\beta} - \alpha p_{2t} + \frac{\xi_{2t}}{1-\beta} - \beta\, \mathrm{E}\big(\bar{V}_{t+1}(m_{t+1}) \bigm| m_t\big). \qquad (1)$$

- Define the identified object y_t

$$y_t = \ln(s_{2t}/s_{0t}) - x_{2t}' \frac{\gamma}{1-\beta} + \alpha p_{2t}.$$

  Note that y_t is a function of m_t only.

- rewrite eq. (1) with y_t

$$y_t = \frac{\xi_{2t}}{1-\beta} - \beta\, \mathrm{E}\big(\bar{V}_{t+1}(m_{t+1}) \bigm| m_t\big). \qquad (2)$$

- headache is the last term

- It follows from multinomial logit property that

$$\bar{V}_t(m_t) = v_2(m_t) - \ln s_{2t}(m_t) = \left(x'_{2t} \frac{\gamma}{1-\beta} - \alpha p_{2t} + \frac{\xi_{2t}}{1-\beta} \right) - \ln s_{2t}(m_t).$$

- Define another identified term w_t:

$$w_t = x'_{2t} \frac{\gamma}{1-\beta} - \alpha p_{2t} - \ln s_{2t}(m_t).$$

- Clearly, w_t is a function of m_t only. Then we have

$$\bar{V}_t(m_t) = w_t(m_t) + \frac{\xi_{2t}}{1-\beta}, \qquad \text{for all } t.$$

- Replacing $\bar{V}_{t+1}$ in

$$y_t = \frac{\xi_{2t}}{1-\beta} - \beta \, \mathrm{E}\big(\bar{V}_{t+1}(m_{t+1}) \mid m_t\big).$$

with the above expression,

Useful conditional moment condition

- we have

$$y_t(m_t) = \xi_{2t}/(1-\beta) - \beta\,\mathrm{E}(w_{t+1}(m_{t+1}) + \xi_{2,t+1}/(1-\beta) \mid m_t).$$

- $\mathrm{E}(y_t \mid m_t) = y_t$ and $\mathrm{E}(\xi_{2t} \mid m_t) = \xi_{2t}$
- As a result, the above display implies

$$\mathrm{E}\left(y_t + \beta w_{t+1} - \frac{1}{1-\beta}\xi_{2t} + \frac{\beta}{1-\beta}\xi_{2,t+1} \,\middle|\, m_t \right) = 0. \tag{3}$$

- Hence for any integrable $\eta(m_t)$,

$$\mathrm{E}\left[\left(y_t + \beta w_{t+1} - \frac{1}{1-\beta}\xi_{2t} + \frac{\beta}{1-\beta}\xi_{2,t+1} \right) \eta(m_t) \right] = 0. \tag{4}$$

Identify discount factor

- If we have at least one exogenous observed characteristic with mean zero, denoted by $x_{t,IV} \in m_t$, we can identify β.
- Given $\text{Cov}(x_{t,IV}, \xi_{2,t}) = \text{Cov}(x_{t,IV}, \xi_{2,t+1}) = 0$ (exogenous),

$$\beta = -\,\text{E}(y_t x_{t,IV}) / \,\text{E}(w_{t+1} x_{t,IV}).$$

- γ is identified from $\gamma/(1-\beta) \times (1-\beta)$
- We need at least two periods
- IV z used earlier may not work here:
  - z is required to be uncorrelated with ξ_{1t} and ξ_{2t}
  - z may not be a function of m_t
  - BLP-type IV may work with additional restriction that it is uncorrelated with next period $\xi_{2,t+1}$

- a bit of intuition:

$$\mathsf{E}\left[\left(\underbrace{y_t}_{\text{current market share}} + \beta \underbrace{w_{t+1}}_{\text{value of waiting}} - \frac{1}{1-\beta}\xi_{2t} + \frac{\beta}{1-\beta}\xi_{2,t+1}\right) x_{t,IV}\right] = 0.$$

  β measures the "causality effect" of future value on current market share.

- Conclusion: $\theta = (\alpha, \beta, \gamma)$ is identified
- The conditions are similar Berry (1993) or Berry and Haile (2014), though we need two periods panel data and BLP IV is uncorrelated with future ξ

Preference is not enough

- For example, we cannot even identify price elasticity
- It can be shown that the price elasticity

$$e_{jk,t} = \frac{\partial s_{jt}}{\partial p_{kt}} \frac{p_{kt}}{s_{jt}}$$

  depends on

$$\begin{aligned}
a &= \frac{\partial}{\partial p_{kt}} \mathrm{E}(x_{j,t+1} \mid x_t, p_t, \xi_t), \\
b &= \frac{\partial}{\partial p_{kt}} \mathrm{E}(p_{j,t+1} \mid x_t, p_t, \xi_t), \\
c &= \frac{\partial}{\partial p_{kt}} \mathrm{E}(\ln s_{j,t+1} \mid x_t, p_t, \xi_t), \\
d &= \frac{\partial}{\partial p_{kt}} \mathrm{E}(\xi_{j,t+1} \mid x_t, p_t, \xi_t).
\end{aligned}$$

- To determine these terms, we need $F(m_{t+1} \mid m_t)$.

Identify $F(x_t, p_t, \xi_t) = F(x_t, p_t) F(\xi_t \mid x_t, p_t)$

- The focus is $F(\xi_t \mid x_t, p_t)$.
- Show first $\mathrm{E}(\xi_{jt})$ and $\mathrm{E}(\xi_{jt} \mid x_t, p_t)$
- We need
  - (i) $F(\xi_t)$ and $F(\xi_t \mid x_t, p_t)$ are time invariant (*stationary*)
  - (ii) $\xi_{t+1} \perp\!\!\!\perp (x_t, p_t) \mid (x_{t+1}, p_{t+1})$ (*no direct feedback; x_t, p_t could only affect ξ_{t+1} via x_{t+1}, p_{t+1}*)

$E(\xi_{jt})$

- $E(\xi_{2t}) = E(y_t + \beta w_{t+1})$ follows from

$$E\left(y_t + \beta w_{t+1} - \frac{1}{1-\beta}\xi_{2t} + \frac{\beta}{1-\beta}\xi_{2,t+1}\right) = 0.$$

- From log shares ratio, we have identified

$$\xi_{2t} - \xi_{1t} = d_t, \tag{5}$$

$$d_t \equiv (1-\beta)\ln(s_{2t}/s_{1t}) - (x_{2t} - x_{1t})'\gamma + (1-\beta)\alpha(p_{2t} - p_{1t}).$$

  The variable d_t is an identified object.

- Obviously, we identify $E(\xi_{1t})$.
- a bit of intuition: letting $\beta = 0$ (extremely impatient consumers)

$$E(\xi_{2t}) = E(y_t).$$

  Recall

$$y_t = \ln(s_{2t}/s_{0t}) - (x'_{2t}\gamma - \alpha p_{2t}) - 0$$

$\mathrm{E}(\xi_{jt} \mid x_t, p_t)$

- Applying law of iterated expectation, we have

$$\mathrm{E}\left[(1-\beta)y_t + \beta(1-\beta)w_{t+1} - \xi_{2t} + \beta\xi_{2,t+1} \,\middle|\, x_t, p_t\right] = 0. \tag{6}$$

- We know

$$h(x,p) = \mathrm{E}[(1-\beta)y_t + \beta(1-\beta)w_{t+1} \mid x_t = x, p_t = p],$$

- We are interested in

$$\pi(x,p) = \mathrm{E}(\xi_{2t} \mid x_t = x, p_t = p).$$

- We can derive integral equation from eq. (6)

$$\pi(x,p) - \beta \int \pi(x',p') F(\mathrm{d}\, x', \mathrm{d}\, p' \mid x, p) = h(x,p).$$

- We have unique solution from this integral equation.
- $\mathrm{E}(\xi_{1t} \mid x_t, p_t) = \mathrm{E}(\xi_{2t} \mid x_t, p_t) - \mathrm{E}(d_t \mid x_t, p_t)$.

$$
\begin{aligned}
h(x_t, p_t) &= \pi(x_t, p_t) - \beta \, \mathrm{E}(\xi_{2,t+1} \mid x_t, p_t) \\
&= \pi(x_t, p_t) - \beta \, \mathrm{E}[\mathrm{E}(\xi_{2,t+1} \mid x_{t+1}, p_{t+1}, x_t, p_t) \mid x_t, p_t] \\
&= \pi(x_t, p_t) - \beta \, \mathrm{E}[\mathrm{E}(\xi_{2,t+1} \mid x_{t+1}, p_{t+1}) \mid x_t, p_t] \\
&= \pi(x_t, p_t) - \beta \, \mathrm{E}(\pi(x_{t+1}, p_{t+1}) \mid x_t, p_t) \\
&= \pi(x_t, p_t) - \beta \int \pi(x, p) F(\,\mathrm{d}\, x, \,\mathrm{d}\, p \mid x_t, p_t).
\end{aligned}
$$

$\mathrm{Var}(\xi_t \mid x_t, p_t)$

- use $d_t = \xi_{2t} - \xi_{1t}$ to get $\sigma^2(x_t, p_t)$.
- We need additional assumptions:
  (i) $\xi_{1t}, \ldots, \xi_{Jt}$ are independent conditional on x_t, p_t
  (ii) Assume that $\mathrm{Var}(\xi_{1t} \mid x_t, p_t) = \cdots = \mathrm{Var}(\xi_{Jt} \mid x_t, p_t) = \sigma^2(x_t, p_t)$.
- So we only need $F(\xi_{jt} \mid x_t, p_t)$ for each j,
- and $\sigma^2(x_t, p_t)$.
- Since we have $d_t = \xi_{2t} - \xi_{1t}$, we get $\sigma^2(x_t, p_t)$.
- For location scale family, mean and variance is enough.

$F(\xi_{jt} \mid x_t, p_t)$

- One more assumption: conditional on (x_t, p_t), ξ_{jt} and ξ_{kt} have identical distribution excepting for their conditional mean.
- We then can nonparametrically identify $F(\xi_{jt} \mid x_t, p_t)$.
- Let $\tilde{\xi}_{jt} = \xi_{jt} - \mathrm{E}(\xi_{jt} \mid x_t, p_t)$. We have

$$\underbrace{\tilde{\xi}_{2t} - \tilde{\xi}_{1t}}_{\text{iid}} = \underbrace{d_t + \mathrm{E}(\xi_{1t} \mid x_t, p_t) - \mathrm{E}(\xi_{2t} \mid x_t, p_t)}_{\text{known}}$$

- constrained deconvolution problem in statistics (Belomestinyi, 2002)
- statisticians know both existence of the solution and how to solve it

$F(m_{t+1} \mid m_t)$

- We assume the following decomposition holds

$$F(m_{t+1} \mid m_t) = F(\xi_{t+1} \mid \xi_t) F(x_{t+1} \mid x_t, p_t) F(p_{t+1} \mid x_{t+1}, \xi_{t+1}).$$

- One interpretation: at the beginning of period $t+1$, manufacturers receive quality ξ_{t+1}, which depends on ξ_t. Meanwhile, x_{t+1} is generated based only on x_t and p_t. Given the quality ξ_{t+1} and x_{t+1}, manufactures determine prices for period $t+1$.
- The focus is then $F(\xi_{t+1} \mid \xi_t)$.
- The $F(p_{t+1} \mid x_{t+1}, \xi_{t+1})$ is identified from $F(m_{t+1})$. It is reduced form of supply side. More structural model can be attached to the supply side.

- more restrictions:
  (i) $F(\xi_{t+1} \mid \xi_t) = F(\xi_{1,t+1} \mid \xi_{1t}) \cdots F(\xi_{J,t+1} \mid \xi_{Jt})$
  (ii) $\xi_{j,t+1}$ and ξ_{kt} are uncorrelated for $j \neq k$
  (iii) Assume that

$$\xi_{j,t+1} = \phi_{j0} + \phi_{j1}\xi_{jt} + \nu_{j,t+1},$$

  where $\nu_{j,t+1}$ has mean zero and is independent of ξ_{jt}.
- For stationary ξ_{jt}, $\mathrm{E}(\xi_{jt}) = \phi_{j0}/(1 - \phi_{j1})$.
- only need ϕ_{j1} which corresponds to autocovariance $\mathrm{E}(\xi_{j,t+1}\xi_{jt})$.
- By

$$\mathrm{E}[(\xi_{2t} - \xi_{1t})(\xi_{2,t+1} - \xi_{1,t+1})] = \mathrm{E}(d_t d_{t+1}),$$

  $\mathrm{E}(\xi_{2t}\xi_{2,t+1})$ is enough

- To get $\mathrm{E}(\xi_{2t}\xi_{2,t+1})$,
- consider

$$\mathrm{E}\left[(y_t + \beta w_{t+1})(\xi_{2t} - \xi_{1t}) - \frac{1}{1-\beta}\xi_{2t}(\xi_{2t} - \xi_{1t}) + \frac{\beta}{1-\beta}\xi_{2,t+1}(\xi_{2t} - \xi_{1t})\right] = 0.$$

- We have formula,

$$\mathrm{E}(\xi_{2,t+1}\xi_{2t}) = \mathrm{E}(\xi_{2,t+1})\,\mathrm{E}(\xi_{1t}) + \frac{\mathrm{E}(\xi_{2t}^2) - \mathrm{E}(\xi_{2t}\xi_{1t})}{\beta} - \frac{1-\beta}{\beta}\,\mathrm{E}[(y_t + \beta w_{t+1})d_t].$$

$$\mathrm{E}(\xi_{2t}\xi_{1t}) = \mathrm{E}\big[\mathrm{E}(\xi_{2t} \mid x_t, p_t)\,\mathrm{E}(\xi_{1t} \mid x_t, p_t)\big].$$

- We have identified $F(\xi_{jt} \mid x_t, p_t)$, hence $F(\xi_{jt})$.
- Under the assumption $\nu_{j,t+1} \perp\!\!\!\perp \xi_{jt}$, we identify $F(\nu)$ by deconvolution.

Estimation

- Data: panel data $(s_{j,r,t}, x_{j,r,t}, p_{j,r,t})$ for market $r = 1, \ldots, n$ and $t = 1, 2, \ldots, T$
- I explicitly show $T = 2$ case.
- Asymptotic results can be derived either using big n or big T or both.
- Estimand: $\theta = (\alpha, \beta, \gamma)$, $F(m_t)$ and $F(m_{t+1} \mid m_t)$.
- not going to pursue full nonparametric estimator of $F(m_t)$ and $F(m_{t+1} \mid m_t)$ (cross-sectionally, we have only 50 states)

- two step procedure:
  (i) estimate θ (only standard GMM involved)
  (ii) estimate $F(m_t)$ and $F(m_{t+1} \mid m_t)$ using generated $\hat{y}_{rt}(\hat{\theta}), \hat{w}_{rt}(\hat{\theta}), \hat{d}_{rt}(\hat{\theta})$ (only sample average formulas involved)
- I calculated the influence functions of the second step estimators.
- Conclusion from the calculation: the first step estimation error does not affect the asymptotic variance of the second step estimators
- hardly surprising because the sample average formulas in the second step are linear in the generated regressors (Newey, 1994, Hahn and Ridder, 2013)

GMM first step

- The moments are from

$$E(g_{2,1,t}(\theta_o)) = 0 \qquad \text{and} \qquad E(g_{2,0,t}(\theta_o)) = 0,$$

  where

$$g_{2,1,t}(\theta) = [\ln(s_{2t}/s_{1t}) - (x_{2t} - x_{1t})'\gamma/(1-\beta) + \alpha(p_{2t} - p_{1t})]z_t,$$

$$g_{2,0,t}(\theta) = (y_t + \beta w_{t+1})x_{t,IV}$$

- Consistent initial estimate of θ is readily available from IV regression and closed-form expression of β.
- free from dirty work about value function approximation
- free from specification about state transition

Closed form second step

- We made bivariate normal specification:
  (i) $x_t \perp\!\!\!\perp \xi_t \mid p_t$ and $\xi_{t+1} \perp\!\!\!\perp p_t \mid p_{t+1}$.
  (ii) Assume

$$F(\xi_{1t}, \ldots, \xi_{Jt} \mid p_{1t}, \ldots, p_{Jt}) = F(\xi_{1t} \mid p_{1t}) \cdots F(\xi_{Jt} \mid p_{Jt}).$$

  (iii) For each product j,

$$\begin{pmatrix} p_{jt} \\ \xi_{jt} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_{pjt} \\ \mu_j \end{pmatrix}, \begin{pmatrix} \sigma^2_{pjt} & \rho_j \sigma_{pjt} \sigma \\ & \sigma^2 \end{pmatrix} \right).$$

- By bivariate normal,

$$\mathrm{E}(\xi_{jt} \mid p_{jt}) = \mu_j + \rho_j \sigma \tilde{p}_{jt},$$

  where $\tilde{p}_{jt} = (p_{jt} - \mu_{pjt})/\sigma_{pjt}$.

- Moreover, $F(\xi_{j,t+1} \mid \xi_{jt})$ is normal

- In the paper, we derived formulas for the parameters related to $F(\xi_t \mid x_t, p_t)$ and $F(\xi_{t+1} \mid \xi_t)$.
- They all have sample average form, e.g. for $\rho_2 = \text{Corr}(p_{2t}, \xi_{2t})$,

$$\hat{\rho}_2 = n^{-1} \sum_{r=1}^{n} \frac{(1-\beta)(\hat{y}_{r1} + \beta \hat{w}_{r2}) \tilde{p}_{2,r1}}{\hat{\sigma}(1 - \beta\, \text{Corr}(p_{2,t=1}, p_{2,t=2}))}.$$

- $\hat{y}$ and $\hat{w}$ are "generated regressors",

$$\hat{y}_t = \ln(s_{2t}/s_{0t}) - x_{2t}' \frac{\hat{\gamma}}{1-\hat{\beta}} + \hat{\alpha} p_{2t},$$

$$\hat{w}_t = x_{2t}' \frac{\hat{\gamma}}{1-\hat{\beta}} - \hat{\alpha} p_{2t} - \ln s_{2t}.$$

A few numerical exercises

- At most 10 products
- Per period utility during the purchasing period t is

$$\begin{aligned} u_{ijt} &= \gamma x_{jt} - \alpha p_{jt} + \xi_{jt} + \varepsilon_{ijt} \\ &= 0.25 x_{jt} - 0.5 p_{jt} + \xi_{jt} + \varepsilon_{ijt}. \end{aligned}$$

  $\mathrm{E}(\xi_{jt}) = 0.5$ for all j.

- Transition:

$$\begin{aligned} p_{jt} &= \theta_p + \rho_j^p p_{j,t-1} + \xi_{jt} + \omega_{jt}^p \\ x_{jt} &= \theta_x + \rho_j^x x_{j,t-1} + \omega_{jt}^x. \end{aligned}$$

  p_{jt} is correlated with ξ_{jt}; x_{jt} is uncorrelated with ξ_{jt}.

- Instrument z is $p_{j,t-1}$; $x_{t,IV}$ is just x_{jt}.
- $\xi_{j,t+1} = 0.25 + 0.5\xi_{jt} + \eta_{j,t+1}$
- $\beta = 0.9$

Table: Estimation with Known Discount Factor

	5 markets per period		
	$\mathrm{E}(\xi_{jt}) = 0.5$	$\gamma = 0.25$	$\alpha = 0.5$
$T = 50, J = 2$	0.47(0.037)	0.25(0.022)	0.49(0.024)
$T = 50, J = 3$	0.47(0.035)	0.25(0.018)	0.49(0.021)
$T = 50, J = 4$	0.46(0.025)	0.26(0.015)	0.49(0.017)
$T = 50, J = 5$	0.46(0.023)	0.25(0.010)	0.49(0.014)
$T = 50, J = 10$	0.47(0.018)	0.25(0.010)	0.49(0.009)

Table: Estimation with Unknown Discount Factor

	5 markets per period			
	$\beta = 0.9$	$\mathrm{E}(\xi_{jt}) = 0.5$	$\gamma = 0.25$	$\alpha = 0.5$
$T = 50, J = 2$	0.92(0.047)	0.43(0.104)	0.20(0.109)	0.49(0.024)
$T = 50, J = 3$	0.92(0.036)	0.45(0.068)	0.20(0.091)	0.49(0.021)
$T = 50, J = 4$	0.92(0.035)	0.45(0.042)	0.22(0.092)	0.49(0.017)
$T = 50, J = 5$	0.91(0.025)	0.46(0.025)	0.22(0.065)	0.49(0.014)
$T = 50, J = 10$	0.91(0.026)	0.48(0.019)	0.23(0.016)	0.49(0.009)

Cheers!

takeaway