# Stochastic Geometry and Wireless Networks

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# What is stochastic geometry?

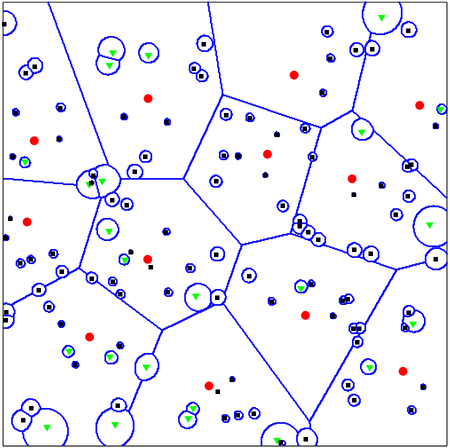
Stochastic geometry is the study of random spatial patterns

- Point processes
- Random tessellations
- Stereology

Applications

- Astronomy
- Communications
- Material science
- Image analysis and stereology
- Forestry
- Random matrix theory

# Application to wireless networks



- Interference is a major limitation
- Networks are getting heterogeneous and decentralized

# Outline

* Primer on Point Processes
* Ad hoc Networks
* Cellular Networks
* Heterogeneous Networks

# Primer on Point Processes

* Primer on Point Processes
  - Poisson point process
  - Transformations of PPP
  - Reduced Palm probability
* Ad hoc Networks
* Cellular Networks
* Heterogeneous Networks

# What is a spatial point process?

Let $\mathbb{N}$ be the set of all sequences $\phi \subset \mathbb{R}^2$ satisfying

1. (Finite) Any bounded set $A \subset \mathbb{R}^2$ contains finite number of points.
2. (Simple) $x_i \neq x_j$ if $i \neq j$.

## Definition

A point process[1] in $\mathbb{R}^2$ is a **random variable** taking values in the space $\mathbb{N}$.

A simple representation: $\Phi = \sum_i \delta_{X_i}$

*Notation:*

1. Point process is denoted by $\Phi$; An instance of the point process is denoted by $\phi$
2. Number of points of the point process in a set $A \subset \mathbb{R}^2$: $\Phi(A)$

[1] D.J. Daley, D. Vere-Jones , *An introduction to the theory of point processes, Vol 1 and 2*, Springer

# Example 1: An interesting but trivial point process

1. Contains only only one point
2. The random point $x$ is uniformly distributed in a bounded set $A$.

## Uniform distribution

Let $B \subset A$

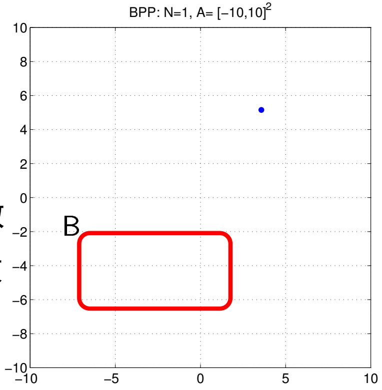
$$\mathbb{P}(x \in B) = \frac{|B|}{|A|},$$

B中点的个数

A中点的个数

where $|A|$ denotes the area of the set $A$.

*Caveat:* Defined only on bounded set, *i.e.*, $|A| < \infty$.



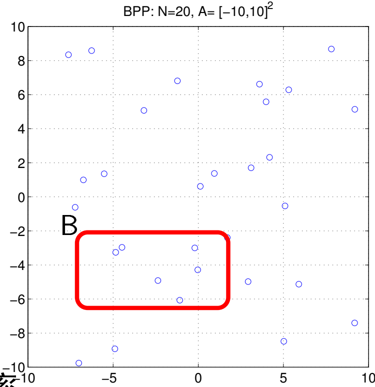
# Example 2: Binomial point process (BPP)

A BPP on a set $A$ is the superposition of $N$ independent uniformly distributed points on the set $A$.

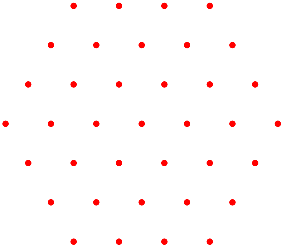
Let $B \subset A$, then $\mathbb{P}(\Phi(B) = k) =$

$$\binom{N}{k}\left(\frac{|B|}{|A|}\right)^{k}\left(1-\frac{|B|}{|A|}\right)^{N-k}$$

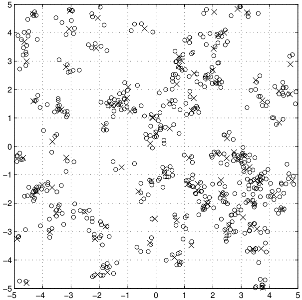
N个不同的点中有K个落在B中的概率



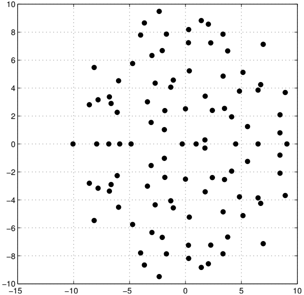
# Other interesting examples



Hexagonal lattice




Thomas cluster process




Determinantal point process (eigenvalues of a Gaussian matrix)

# Characterization of a point process

Given two point processes, is there a simple way to see if both of them are equivalent?

A simple point process $\Phi$ is determined by its void probabilities over all compact sets, *i.e.*, $\mathbb{P}(\Phi(K) = 0)$ for $K \subset \mathbb{R}^2$ and compact.

- This means that two point processes are equivalent if they have the same void probability distribution (for all sets).

# Stationary point processes

## Definition (Stationary point process)

A point process is stationary if its distribution is invariant with respect to translations.

- The point process looks statistically similar from any point in space.
- BPP is not a stationary point process.
- A stationary point process cannot be defined on a subset of $\mathbb{R}^2$

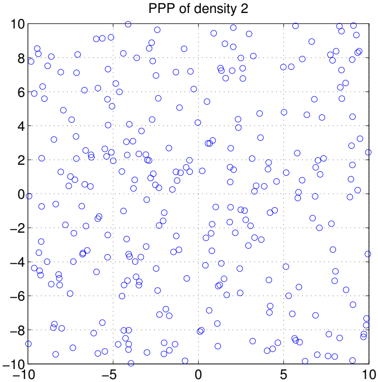
The density of a stationary point process $\Phi$ is defined as

$$\frac{\mathbb{E}[\Phi(B)]}{|B|}, \quad B \subset \mathbb{R}^2.$$

The RHS does not depend on the particular choice of the set $B$.

# Stationary Poisson point process (PPP)

1. The most widely used model for spatial locations of nodes
    - Most amicable to analysis
    - "Gaussian of point processes"
2. No dependence between node locations
3. Random number of nodes
4. Can be defined on the entire plane
    - Limiting distribution of a BPP



# PPP: Formal definition

A stationary Poisson point process $\Phi$ of density $\lambda$ is characterized by

1. The number of points in a bounded set $A \subset \mathbb{R}^2$ has a **Poisson distribution** with mean $\lambda|A|$, *i.e.*,

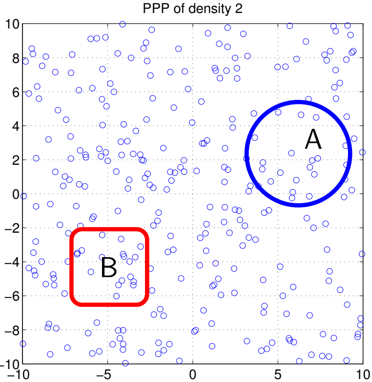
$$\mathbb{P}(\Phi(A) = n) = \exp(-\lambda|A|)\frac{(\lambda|A|)^n}{n!}$$

A中有n个点

2. The number of points in disjoint sets are independent, *i.e.*, for $A \subset \mathbb{R}^2$, $B \subset \mathbb{R}^2$ and $A \cap B = \emptyset$,

两个不同区域中点的个数是独立的

$$\Phi(A) \perp \Phi(B)$$




A stationary PPP is completely characterized by a single number $\lambda$.

# Properties of PPP

**Lemma**

*The density of the PPP (as defined in previous slide) is* $\lambda$.

Proof: Let $A \subset \mathbb{R}^2$. Then

$$\mathbb{E}[\Phi(A)] = \sum_{n=0}^{\infty} n \boxed{\exp(-\lambda|A|)\frac{(\lambda|A|)^n}{n!}}$$

A中点的个数的均值

那个点的概率

$$= \lambda|A|,$$

which follows from the mean of a Poisson random variable. Hence

$$\frac{\mathbb{E}[\Phi(A)]}{|A|} = \lambda.$$

A的面积

Observe that the above expression does not depend on the set $A$. $\square$

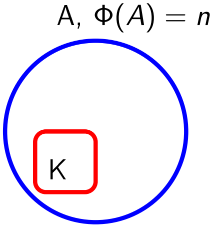
# Properties...

## Lemma

*Let $A \subset \mathbb{R}^2$. Conditioned on the number of points $\Phi(A)$, the points are independently and uniformly distributed in the set $A$,* i.e., *the points form a BPP.*

Proof: We consider the void probability of a set $K \subset A$.

$$\begin{aligned}
\mathbb{P}(\Phi(K)=0|\Phi(A)=n) &= \frac{\mathbb{P}(\Phi(K)=0 \cap \Phi(A)=n)}{\mathbb{P}(\Phi(A)=n)} \\
&= \frac{\mathbb{P}(\Phi(K)=0)\mathbb{P}(\Phi(A\setminus K)=n)}{\mathbb{P}(\Phi(A)=n)} \\
&= \frac{e^{-\lambda|K|}e^{\lambda|A\setminus K|}(\lambda|A\setminus K|)^n/n!}{e^{\lambda|A|}(\lambda|A|)^n/n!} \\
&= \frac{|W\setminus K|^n}{|A|^n} = \left(1-\frac{|K|}{|A|}\right)^n . \quad \square
\end{aligned}$$



# Simulation of a PPP

How to simulate a PPP of density $\lambda$ on $A = [-L, L]^2$?

1. The number of points in the set $A$ is a Poisson random variable with mean $\lambda|A|$.
2. Conditioned on the number of points, the points are uniformly distributed as a BPP.

## Matlab code

$N = \text{poissrnd}(\lambda|A|)$
Points $= \text{unifrnd}(-L, L, N, 2)$

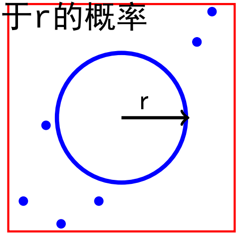
# Distance properties of a PPP

## First contact distribution

The CCDF of the distance of the nearest point of the process from the origin denoted by $D$ is $\boxed{\mathbb{P}(D \geq r) = \exp(-\lambda\pi r^2)}$.

Proof: 到远点最近的点的距离大于等于r的概率

$$\begin{aligned}
\mathbb{P}(D \geq r) &= \mathbb{P}(B(o,r) \text{ is empty }) \\
&= \exp(-\lambda|B(o,r)|) \\
&= \exp(-\lambda\pi r^2) \quad \square
\end{aligned}$$




Hence the PDF equals $f_N(r) = 2\lambda\pi r \exp(-\lambda\pi r^2)$. The average distance is

$$\mathbb{E}[D] = \int_0^\infty r2\lambda\pi r f_N(r)\mathrm{d}r = \frac{1}{2\sqrt{\lambda}}.$$

# $N$-th closest point

## 第n近的点到原点的距离的概率分布

The CDF[2] of the $N$-th closest point to the origin equals

$$\mathbb{P}(D_n \geq r) = \sum_{k=0}^{n-1} e^{-\lambda\pi r^2} \frac{(\lambda\pi r^2)^k}{k!} = \frac{\Gamma(n, \lambda\pi r^2)}{(n-1)!}$$

The average distance to the $N$-th closest point equals

$$\mathbb{E}[D_n] = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi\lambda}\Gamma(n)} \sim \sqrt{\frac{n}{\pi\lambda}}$$

[2] M. Haenggi, "On Distances in Uniformly Random Networks," IEEE Transactions on Information Theory, vol. 51, pp. 3584-3586, Oct. 2005

# Sums over PPP

## Lemma (Campbells theorem)

*Let $\Phi$ be a PPP of density $\lambda$ and $f(x) : \mathbb{R}^2 \to \mathbb{R}^+$.*

$$\boxed{\mathbb{E}[\sum_{x\in\Phi} f(x)] = \lambda \int_{\mathbb{R}^2} f(x)\mathrm{d}x}$$

x是一个泊松点过程

Proof: We have

$$\mathbb{E}[\sum_{x\in\Phi} f(x)] = \lim_{R\to\infty} \mathbb{E}[\sum_{x\in\Phi\cap B(o,R)} f(x)].$$

Let $n = \Phi(B(o,R))$. Conditioning on the number of points $n$,

$$\mathbb{E}\left[\sum_{x\in\Phi\cap B(o,R)} f(x)\right] = \mathbb{E}_n\left[\mathbb{E}\left[\sum_{x\in\Phi\cap B(o,R)} f(x)\Big| n\right]\right]$$

Since conditioned on the number of points, the points are i.i.d uniform

个数为n

$$\mathbb{E}[\sum_{x\in\Phi\cap B(o,R)} f(x)|n] = n\int_{B(o,R)} \frac{f(x)}{|B(o,R)|}\mathrm{d}x.$$

限定在半径为R的圆内

Averaging over $n$

$$\mathbb{E}[\sum_{x\in\Phi\cap B(o,R)} f(x)] = \mathbb{E}[n]\int_{B(o,R)} \frac{f(x)}{|B(o,R)|}\mathrm{d}x$$

As $\mathbb{E}[n] = \lambda|B(o,R)|$, and tending $R\to\infty$ we obtain the result. □

Let $g(x,y) : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}^+$. Then[3]

$$\mathbb{E}\sum_{x,y\in\Phi}^{\neq} g(x,y) = \lambda^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} g(x,y)\mathrm{d}x\mathrm{d}y.$$

[3] D. Stoyan, W. Kendall, and J. Mecke, "Stochastic Geometry and Its Applications", 2nd ed. John Wiley and Sons, 1996

# Example: Mean and variance of interfernece

Let the transmitters be distributed as a PPP $\Phi$ of density $\lambda$.

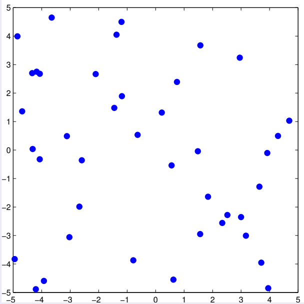
## Definition (Interfernce)

The interfernce (sum power) at location $y \in \mathbb{R}^2$ is

$$I(y) = \sum_{x\in\Phi} \ell(x-y),$$

干扰叠加

where $\ell(x)$ is the path-loss function.




*Mean of interference*: By Campbell theorem,

$$\mathbb{E}[I(y)] = \mathbb{E}[\sum_{x\in\Phi} \ell(x-y)] = \lambda \int_{\mathbb{R}^2} \ell(x-y)\mathrm{d}x = \lambda \int_{\mathbb{R}^2} \ell(x)\mathrm{d}x$$

*Variance of interference:*

$$\mathbb{E}[I(y)^2] = \mathbb{E}\left[\left(\sum_{x\in\Phi}\ell(x-y)\right)^2\right] = \mathbb{E}\left[\left(\sum_{x\in\Phi}\ell(x-y)\right)\left(\sum_{z\in\Phi}\ell(z-y)\right)\right]$$

$$= \mathbb{E}[\sum_{x\in\Phi}\ell(x-y)^2] + \mathbb{E}[\sum_{x,z\in\Phi}^{\neq}\ell(x-y)\ell(z-y)]$$

$$= \lambda\int_{\mathbb{R}^2}\ell(x-y)^2\mathrm{d}x + \lambda^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\ell(x-y)\ell(z-y)\mathrm{d}x\mathrm{d}z$$

$$= \lambda\int_{\mathbb{R}^2}\ell(x)^2\mathrm{d}x + (\lambda\int_{\mathbb{R}^2}\ell(x)\mathrm{d}x)^2$$

Hence variance equals,

$$\mathrm{var}(I(y)) = \lambda\int_{\mathbb{R}^2}\ell(x)^2\mathrm{d}x.$$

# Products over PPP

## Lemma (Probability generating functional (PGFL))

*Let $\Phi$ be a PPP of density $\lambda$ and $f(x) : \mathbb{R}^2 \to [0,1]$ be a real valued function. Then*

$$\mathbb{E}\left[\prod_{x\in\Phi} f(x)\right] = \exp\left(-\lambda \int_{\mathbb{R}^2} (1 - f(x))\mathrm{d}x\right).$$

Proof: We prove the result for $\Psi_r = \Phi \cap B(o,r)$. Observe that $\Psi_r$ is a PPP with number of points $n$ distributed as a Poisson random variable with mean $\lambda\pi r^2$.

$$\boxed{\begin{aligned}\mathbb{E}\left[\prod_{x\in\Psi_r} f(x)\right] &= \mathbb{E}_n\mathbb{E}\left[\prod_{x\in\Psi_r} f(x)\middle| n\right] \\ &= \mathbb{E}_n\mathbb{E}[f(x)]^n\end{aligned}}$$

But $\mathbb{E}[f(x)] = \frac{1}{\pi r^2}\int_{B(o,r)} f(x)\mathrm{d}x$. Hence

$$\mathbb{E}\left[\prod_{x\in\Psi_r} f(x)\right] = \mathbb{E}_n\left[\left(\frac{1}{\pi r^2}\int_{B(o,r)} f(x)\mathrm{d}x\right)^n\right].$$

Let $z > 0$. Let $n$ be a Poisson random variable with mean $a$. Then

$$\mathbb{E}[z^n] = \exp(-a(1-z)).$$

$$\begin{aligned}\mathbb{E}\left[\prod_{x\in\Psi_r} f(x)\right] &= \exp\left(-\lambda\pi r^2\left(1-\frac{1}{\pi r^2}\int_{B(o,r)} f(x)\mathrm{d}x\right)\right)\\ &= \exp\left(-\lambda\int_{B(o,r)}(1-f(x))\mathrm{d}x\right). \quad \square\end{aligned}$$

# Application of PGFL: Laplace transform of interference

$$\mathbb{E}[\exp(-sI(y))] = \mathbb{E}\left[\exp\left(-s\sum_{x\in\Phi}\ell(x-y)\right)\right].$$

This can be rewritten as,

$$\mathbb{E}[\exp(-sI(y))] = \mathbb{E}\left[\prod_{x\in\Phi}\exp\left(-s\ell(x-y)\right)\right].$$

Using the PGFL,

$$\mathbb{E}[\exp(-sI(y))] = \exp\left(-\lambda\int_{\mathbb{R}^2} 1 - e^{-s\ell(x-y)}\mathrm{d}x\right).$$

Substituting $x-y \to x$, we have

$$\mathcal{L}_I(s) = \exp\left(-\lambda\int_{\mathbb{R}^2} 1 - e^{-s\ell(x)}\mathrm{d}x\right).$$

# Transformations of PPP: Independent Thinning

Let $\Phi$ be a PPP of density $\lambda$

1. A node $x \in \Phi$ is coloured red with probability $p$ and blue with probability $1 - p$.
2. Let $\Phi_r$ denote the red point process and $\Phi_b$ denote the blue point process. So we have $\Phi = \Phi_r \cup \Phi_b$.

Can be used to model ALOHA MAC protocol.

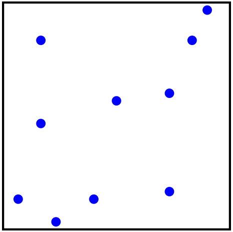
**Lemma (Thinning)**

1. *$\Phi_r$ is a PPP of density $\lambda p$*
2. *$\Phi_b$ is a PPP of density $\lambda(1 - p)$*
3. *$\Phi_r$ is independent of $\Phi_b$.*

Proof: Look at void probabilities of $\Phi_r$ and $\Phi_b$. $\square$

# Matern hard-core process: Dependent thinning

1. Begin with a PPP $\Phi$ of density $\lambda$.
2. To each $x \in \Phi$, associate a mark $m_x \sim U[0,1]$ independent of every other point.
3. A node $x \in \Phi$ selected if it has the lowest mark among all the points in the ball $B(x, R)$.



$$\Psi = \{y : y \in \Phi, m_y \leq m_x, \forall x \in B(y, R) \cap \Phi\}$$

A minimum distance process for modelling CSMA MAC.

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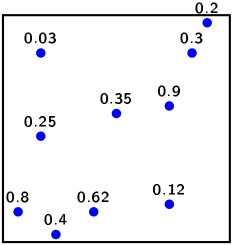


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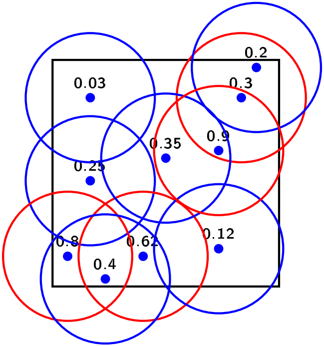


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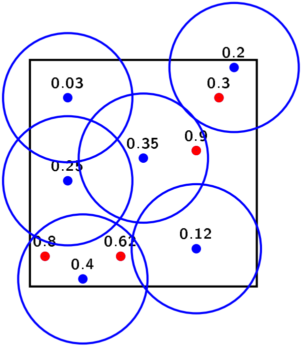



A minimum distance process for modelling CSMA MAC.

# Density of Matern hard-core process

A node $x \in \phi$ is retained with probability

$$p = \mathbb{P}(m_x \leq m_y, \forall y \in B(x,R) \cap \Phi).$$

- Let the mark of $x$ be equal to $t \in [0,1]$.
- Let $n$ represent the number of points of $\Phi$ in $B(x,R)$.
  - $n \sim Poi(\lambda \pi R^2)$.
- Conditioned on the mark $t$, the probability that $x$ is selected equals $\mathbb{E}[(1-t)^n] = \exp(-\lambda \pi R^2 t)$.
- Averaging over $t$,

$$p = \int_0^1 \exp(-\lambda \pi R^2 t) \mathrm{d}t = \frac{1 - \exp(-\lambda \pi R^2)}{\lambda \pi R^2}.$$

So the final density of the process equals $\lambda_m = p\lambda$.

$$\lambda_m = \frac{1 - \exp(-\lambda \pi R^2)}{\pi R^2}$$

# Conditioning: Reduced Palm probability

- Notion of a typical point, *i.e.*, conditioning on the existence of a node at a particular location

## Nearest-neighbour distribution function

The distance of the nearest neighbour from a "typical" point.

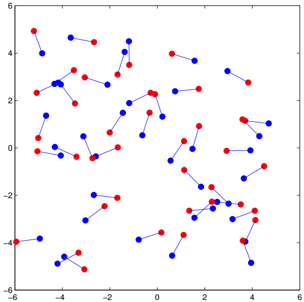
$$\begin{aligned} D(r) &= \mathbb{P}(\Phi(B(o,r) = 1 | o \in \Phi) \\ &= \mathbb{P}^o(\Phi(B(o,r) = 1) \\ &= \mathbb{P}^{!o}(\Phi(B(o,r) = 0) \end{aligned}$$

Probability conditioned on there being a point at the origin (but not counting it).

# A spatial average interpretation

The reduced Palm probability can also be interpreted as a spatial average

$$\mathbb{P}^{!o}(Y) = \lim_{R\to\infty} \mathbb{E} \sum_{x\in\Phi\cap B(o,R)} \frac{\mathbb{P}(\Phi_{-x} \setminus \{x\} \in Y)}{\lambda \pi R^2}.$$




## Palm distribution of PPP: Slivnyak theorem

$$\mathbb{P}^{!o} = \mathbb{P},$$

*i.e.*, reduced Palm distribution of a PPP equals the original distribution.

Hence for a PPP, a new point can be added to the process without disturbing other points of the process.

# Campbell Mecke Theorem

Let $f(x, \phi) : \mathbb{R}^2 \times \mathbb{N} \to [0, \infty]$ be a real valued function,

$$\mathbb{E}[\sum_{x \in \Phi} f(x, \Phi \setminus \{x\})] = \lambda \int_{\mathbb{R}^2} \mathbb{E}^{!o}[f(x, \Phi)] \mathrm{d}x.$$

Hence for a PPP,

$$\mathbb{E}[\sum_{x \in \Phi} f(x, \Phi \setminus \{x\})] = \lambda \int_{\mathbb{R}^2} \mathbb{E}[f(x, \Phi)] \mathrm{d}x.$$

# Analysis of Ad Hoc Networks

* Primer on Point Processes

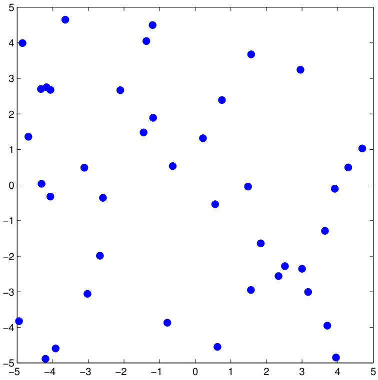
* Ad hoc Networks
– *SINR* analysis
– Interfernece correlation

* Cellular Networks

* Heterogeneous Networks

# Dipole model

- The transmitters are distributed as a PPP $\Phi$ of density $\lambda$
- Each transmitter has a receiver at a distance $d$ in a random direction
  - Not part of the process $\Phi$.
- Path loss function is denoted by $\ell(x)$
  1. Examples: $\ell(x) = \|x\|^{-\alpha}$, $\ell(x) = \min\{1, \|x\|^{-\alpha}\}$.
  2. $\alpha > 2$
  3. Path loss between nodes $x$ and $y$ is $\ell(x - y)$.



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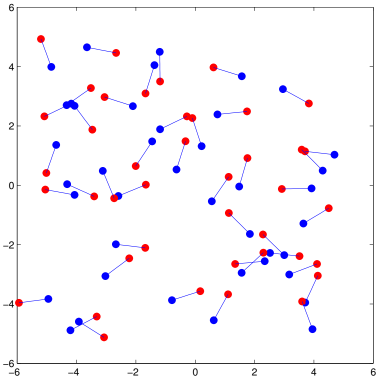
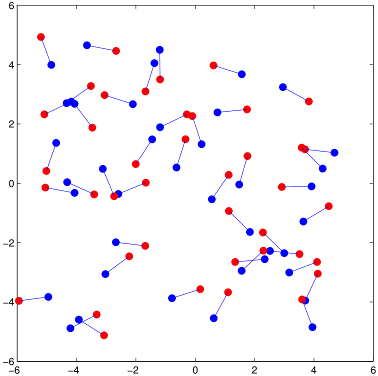


# Dipole model

- The transmitters are distributed as a PPP $\Phi$ of density $\lambda$
- Each transmitter has a receiver at a distance $d$ in a random direction
  - Not part of the process $\Phi$.
- Path loss function is denoted by $\ell(x)$
  1. Examples: $\ell(x) = \|x\|^{-\alpha}$, $\ell(x) = \min\{1, \|x\|^{-\alpha}\}$.
  2. $\alpha > 2$
  3. Path loss between nodes $x$ and $y$ is $\ell(x-y)$.



# System model

- All nodes transmit at the same power
- TX has $N_t$ transmit antenna
- RX has $N_r$ receive antenna
- Fading between any two nodes is i.i.d Rayleigh
  1. The fading power is exponentially distributed with unit mean.
  2. The fading power between nodes is denoted by $\mathrm{h}_{xy}$
     - $\mathbb{P}(\mathrm{h}_{xy} \geq z) = \exp(-z)$

What is the performance of a "typical" link?

[3] F. Baccelli, B. Blaszczyszyn, P. Muhlethaler, "An ALOHA protocol for multihop mobile wireless networks," Information Theory, IEEE Transactions on , vol.52, no.2, pp. 421- 436, Feb. 2006

# SISO ad hoc network: $N_t = N_r = 1$

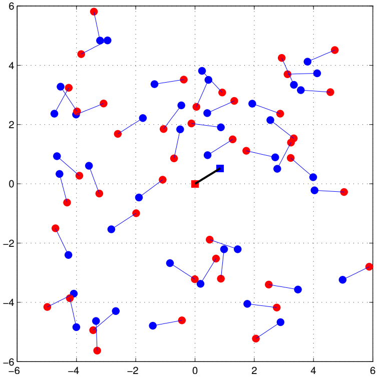
Typical link: By Slivnyaks theorem, we can add a new reference transmitter with its receiver at the origin

The Signal-to-interference-noise ratio (after processing) between the receiver at the origin and its corresponding transmitter is

$$\mathrm{SINR} = \frac{\mathrm{h}\, d^{-\alpha}}{\sigma^2 + I(o)},$$

where $I(o)$ is the interference at receiver at origin

$$I(o) = \sum_{y \in \Phi \setminus \{x\}} \mathrm{h}_{yo} \|y\|^{-\alpha}.$$




How to compute $\mathbb{P}(\mathrm{SINR} \geq \theta)$ for the reference link?

# SINR distribution

$$
\begin{aligned}
\mathrm{p}_s(\theta,\lambda) = \mathbb{P}(\mathrm{SINR}(o) > \theta) &= \mathbb{P}\left(\frac{\mathrm{h}\, d^{-\alpha}}{\sigma^2 + I(o)} \geq \theta\right) \\
&= \mathbb{P}\left(\mathrm{h} \geq d^{\alpha}\theta(\sigma^2 + I(o))\right) \\
&= \mathbb{E}\exp(-d^{\alpha}\theta(\sigma^2 + I(o))) \\
&= \exp(-d^{\alpha}\theta\sigma^2)\underbrace{\mathbb{E}\exp(-\mathrm{d}^{\alpha}\theta I(o))}_{T_1 = \mathcal{L}_{I(o)}(d^{\alpha}\theta)}
\end{aligned}
$$

Observe that $T_1$ is the Laplace transform of $I(o)$ evaluated at $s = d^{\alpha}\theta$.
We now evaluate the Laplace transform of interference

$$
\begin{aligned}
\mathcal{L}_{I(o)}(s) &= \mathbb{E}\exp(-s\sum_{y\in\Phi}\mathrm{h}_{yo}\|y\|^{-\alpha}) \\
&= \mathbb{E}\prod_{y\in\Phi}\exp(-s\mathrm{h}_{yo}\|y\|^{-\alpha})
\end{aligned}
$$

# Laplace transform of interference

$$
\begin{aligned}
\mathcal{L}_{I(o)}(s) &= \mathbb{E} \prod_{y \in \Phi} \exp(-s h_{yo} \|y\|^{-\alpha}) \\
&\stackrel{(a)}{=} \mathbb{E} \prod_{y \in \Phi} \mathbb{E}_{h_{yo}} \exp(-s h_{yo} \|y\|^{-\alpha}) \\
&\stackrel{(b)}{=} \mathbb{E} \prod_{y \in \Phi} \frac{1}{1 + s\|y\|^{-\alpha}}
\end{aligned}
$$

$(a)$ follows from the independence of the fading random variables and $(b)$ follows from the Laplace transform of an exponential random variable.

## Recall PGFL of a PPP

$$
\mathbb{E} \prod_{y \in \Phi} f(x) = \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - f(x) \mathrm{d}x\right).
$$

Using the PGFL of a PPP,

$$\begin{aligned}\mathcal{L}_{I(o)}(s) &= \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \frac{1}{1+s\|y\|^{-\alpha}} \mathrm{d}x\right)\\ &= \exp\left(-\lambda \int_{\mathbb{R}^2} \frac{1}{1+s^{-1}\|y\|^{\alpha}} \mathrm{d}x\right)\\ &= \exp(-\lambda s^{2/\alpha} C(\alpha)),\end{aligned}$$

where $C(\alpha) = \frac{2\pi^2}{\alpha \sin(2\pi/\alpha)}$.

**The CCDF of SINR is**

$$\begin{aligned}\mathrm{p}_s(\theta,\lambda) = \mathbb{P}(\mathrm{SINR}(o) > \theta) &= \exp(-d^{\alpha}\theta\sigma^2)\mathcal{L}_{I(o)}(d^{\alpha}\theta)\\ &= \underbrace{\exp(-d^{\alpha}\theta\sigma^2)}_{\textit{Noise}}\underbrace{\exp(-\lambda d^2\theta^{2/\alpha} C(\alpha))}_{\textit{Interference}}\end{aligned}$$

# Multiple antenna systems:

Post-processing SIR depends on the processing at the receiver

- *Maximal-ratio combining (MRC)*, $N_t = 1, N_r = n$ : Let $h_x$ denote $1 \times n$ channel vector from a node $x$ to the receiver at the origin. The received signal is

$$Y = \frac{h_o d^{-\alpha/2}}{\sqrt{n}} a_o + \sum_{x \in \Phi} \frac{h_x \|x\|^{-\alpha/2}}{\sqrt{n}} a_x,$$

where $a_x$ are the transmitted symbols. Hence the received SIR after multiplying with $h_o^H$ is

$$\mathrm{SIR} = \frac{\frac{1}{n} |h_o^H h_o|^2 d^{-\alpha}}{\frac{1}{n} \sum_{x \in \Phi} |h_o^H h_x|^2 \|x\|^{-\alpha}} = \frac{\|h_o\|^2 d^{-\alpha}}{\sum_{x \in \Phi} |\frac{h_o^H}{\|h_o\|} h_x|^2 \|x\|^{-\alpha}}$$

  - $\|h_o\|^2$ is $\chi^2$ distributed with $2n$ degrees of freedom.
  - $|\frac{h_o^H}{\|h_o\|} h_x|^2$ is exponentially distributed with unit mean.

- *Zero forcing receiver (ZF), $N_t = n, N_t = n$, # streams =n.* Looking at the $k$-th stream the received vector is

$$\sqrt{n}Y_k = h_o(k)d^{-\alpha/2}a_{ok} + \sum_{i=1, i\neq k}^{n} h_o(i)d^{-\alpha/2}a_{oi} + \sum_{x\in\Phi}\sum_{i=1}^{n} h_x(n)\|x\|^{-\alpha/2}a_x$$

$h_x(i)$ is the $i$-th column of the channel matrix between the node $x$ and the tagged receiver. Multiplying with the $v$ that is orthogonal to the vectors $h_o(i), i \neq k$ the received SINR is

$$\text{SIR} = \frac{Sd^{-\alpha}}{\sum_{x\in\Phi} S_x\|x\|^{-\alpha}}$$

1. $S$ is exponentially distributed.
2. $S_x$ is also exponentially distributed.

The post process signal-to-interference ratio (after processing) is generally of the form

$$\mathrm{SIR} = \frac{Sd^{-\alpha}}{\sigma^2 + I(o)},$$

where $I(o) = \sum_{y \in \Phi \setminus \{x\}} \hat{\mathrm{h}_{yo}} \|y\|^{-\alpha}$ is the interference at receiver at origin. Let the CCDF of $S$ be

$$F(x) = \sum_{k=0}^{n} a_k x^k \exp(-b_k x)$$

For example

1. When $S$ is exponentially distributed, $n = 1$, $a_1 = 1$ and $b_1 = 1$
2. When $S$ is $\chi^2$ with $2m$ degrees of freedom, $n = m - 1$, $a_k = \frac{1}{k!}$ and $b_k = 1$.

# Laplace trick

$$\begin{aligned}
p_s(\theta,\lambda) &= \mathbb{P}\left(\frac{Sd^{-\alpha}}{I(o)} \geq \theta\right) = \mathbb{P}\left(S \geq \theta d^{\alpha} I(o)\right) \\
&= \mathbb{E}F(S \geq \theta d^{\alpha} I(o)) = \sum_{k=0}^{n} a_k \mathbb{E}\left[(\theta d^{\alpha} I(o))^k \exp(-b_k \theta d^{\alpha} I(o))\right] \\
&= \sum_{k=0}^{n} a_k (\theta d^{\alpha})^k \mathbb{E}\left[I(o)^k \exp(-b_k \theta d^{\alpha} I(o))\right]
\end{aligned}$$

$$\mathbb{E}[x^k e^{-x}] = \mathbb{E}\left[(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} e^{-xs}\Big|_{s=1}\right] = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \mathcal{L}_X(s)\Big|_{s=1}$$

Hence,

$$p_s(\theta,\lambda) = \sum_{k=0}^{n} a_k \theta^k d^{k\alpha} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}s^k} \mathcal{L}_{I(o)}(s)\Big|_{s=b_k\theta d^{\alpha}}$$

# Summary: Key steps in the SINR evaluation

$$
\begin{aligned}
\mathrm{p}_s(\theta,\lambda) &= \mathbb{P}\left(\mathrm{h} \geq d^\alpha\theta(\sigma^2 + I(o))\right) \overset{(a)}{=} \mathbb{E}\exp(-d^\alpha\theta(\sigma^2 + I(o))) \\
&= \exp(-d^\alpha\theta\sigma^2)\mathbb{E}\exp(-\mathrm{d}^\alpha\theta I(o)) \\
&= \exp(-d^\alpha\theta\sigma^2)\mathbb{E}\prod_{y\in\Phi}\mathcal{L}_h(d^\alpha\theta\|y\|^{-\alpha}) \\
&= \exp(-d^\alpha\theta\sigma^2)\mathbb{E}\prod_{y\in\Phi}\frac{1}{1+d^\alpha\theta\|y\|^{-\alpha}} \\
&\overset{(b)}{=} \exp(-d^\alpha\theta\sigma^2)\exp(-\lambda d^2\theta^{2/\alpha}C(\alpha))
\end{aligned}
$$

1. The distribution of h being exponential in $(a)$.
2. The distribution of the fading between the interferers and the tagged receiver is not crucial.
3. Using PGFL in $(b)$.

# Interference distribution

The Laplace transform of interference $I = \sum_{y \in \Phi} \mathrm{h}_{yo} \|y\|^{-\alpha}$ is given by

$$\mathcal{L}_I(s) = \exp(-\lambda s^{2/\alpha} C(\alpha)), \quad \alpha > 2.$$

- The Laplace transform of an alpha stable distribution with parameter $2/\alpha$.
  1. Heavy tailed distribution. Not Gaussian[4].
  2. No closed form expression for CDF
  3. Integer moments don't exist.
     - $\mathbb{E}[I] = \lambda \int_{\mathbb{R}^2} \|x\|^{-\alpha} \mathrm{d}x = \infty$
     - An artefact of the singularity of path loss model $\ell(x) = \|x\|^{-\alpha}$ at $x = o$.
- $I = \sum_{y \in \Phi} \mathrm{h}_{yo} \|y\|^{-\alpha}$ is also referred as shot noise (SN) process.

[4] R.K. Ganti and M. Haenggi. "Interference in ad hoc networks with general motion-invariant node distributions", ISIT, 2008

# Tail bounds on interference[5]

Evaluate the CCDF of $\mathbb{P}(I \geq y)$, where $I = \sum_{y \in \Phi} \mathrm{h}_{yo} \|y\|^{-\alpha}$.

- Divide the point process into two sets

$$\Phi_y = \{x \in \Phi, \mathrm{h}_{xo} \|x\|^{-\alpha} > y\}$$

$$\Phi_y^c = \{x \in \Phi, \mathrm{h}_{xo} \|x\|^{-\alpha} \leq y\}$$

- Lower bound:

$$\begin{aligned}\mathbb{P}(I \geq y) &= \mathbb{P}(I_{\Phi_y} + I_{\Phi_y^c} \geq y) \\ &\geq \mathbb{P}(I_{\Phi_y} \geq y) = 1 - \mathbb{P}(I_{\Phi_y} \leq y) \\ &= 1 - \mathbb{P}(\Phi_y = \emptyset)\end{aligned}$$

[4] S. Weber, X. Yang, J. G. Andrews and G. de Veciana, "Transmission Capacity of Wireless Ad Hoc Networks with Outage Constraints", IEEE Transactions on Information Theory, Vol. 51, No. 12, Dec. 2005

- $$\mathbb{P}(\Phi_y = \emptyset) = \mathbb{E}\prod_{x\in\Phi} \mathbf{1}(\mathsf{h}_{xo}\|x\|^{-\alpha} < y)$$
$$= \mathbb{E}\prod_{x\in\Phi} \mathbb{E}_{\mathsf{h}_{xo}}\mathbf{1}(\mathsf{h}_{xo}\|x\|^{-\alpha} < y) = \mathbb{E}\prod_{x\in\Phi} 1 - e^{-y\|x\|^{\alpha}}$$
$$= \exp\left(-\lambda\int_{\mathbb{R}^2} e^{-y\|x\|^{\alpha}}\mathrm{d}x\right) = \exp\left(-\lambda y^{-2/\alpha}\pi\Gamma(1+2/\alpha)\right)$$

- Upper bound

$$\mathbb{P}(I \geq y) = \mathbb{P}(I \geq y | I_{\Phi_y} > y)\mathbb{P}(I_{\Phi_y} > y) + \mathbb{P}(I \geq y | I_{\Phi_y} \leq y)\mathbb{P}(I_{\Phi_y} \leq y)$$
$$= \mathbb{P}(I_{\Phi_y} > y) + \mathbb{P}(I \geq y | I_{\Phi_y} \leq y)\mathbb{P}(I_{\Phi_y} \leq y)$$
$$= 1 - \mathbb{P}(\Phi_y = \emptyset) + \mathbb{P}(I \geq y | I_{\Phi_y} \leq y)\mathbb{P}(\Phi_y = \emptyset)$$
$$= 1 - (1 - \mathbb{P}(I \geq y | I_{\Phi_y} \leq y))\mathbb{P}(\Phi_y = \emptyset)$$

- Using Markov inequality

$$
\begin{aligned}
\mathbb{P}(I \geq y | I_{\Phi_y} \leq y) &= \mathbb{P}(I \geq y | \Phi_y = \emptyset) \\
&\leq \frac{\mathbb{E}[I \geq y | \Phi_y = \emptyset]}{y} \\
&= \frac{1}{y} \mathbb{E} \sum_{x \in \Phi} \mathsf{h}_{xo} \|x\|^{-\alpha} \mathbf{1}(\mathsf{h}_{xo} \|x\|^{-\alpha} \leq y) \\
&= \frac{\lambda}{y} \int_{\mathbb{R}^2} \|x\|^{-\alpha} \mathbb{E}[\mathsf{h}_{xo} \mathbf{1}(\mathsf{h}_{xo} \|x\|^{-\alpha} \leq y)] \mathrm{d}x \\
&= \frac{\lambda}{y} \int_{\mathbb{R}^2} \|x\|^{-\alpha} \int_0^{y\|x\|^{\alpha}} h e^{-h} \mathrm{d}h \mathrm{d}x \\
&= \frac{2\pi \Gamma\left(1 + 2/\alpha\right)}{2 - \alpha} y^{-2/\alpha}
\end{aligned}
$$

$$
1 - e^{-\lambda y^{-2/\alpha} \mathbb{E}[\mathsf{h}^{2/\alpha}]} \leq \mathbb{P}(I \geq y) \leq 1 - \left(1 - \frac{2\pi \mathbb{E}[\mathsf{h}^{2/\alpha}]}{2 - \alpha} y^{-2/\alpha}\right) e^{-\lambda y^{-2/\alpha} \mathbb{E}[\mathsf{h}^{2/\alpha}]}
$$

# Interference is heavy tailed

**Lemma**

*When path loss is given by $\ell(x) = \|x\|^{-\alpha}$, interference is heavy tailed*

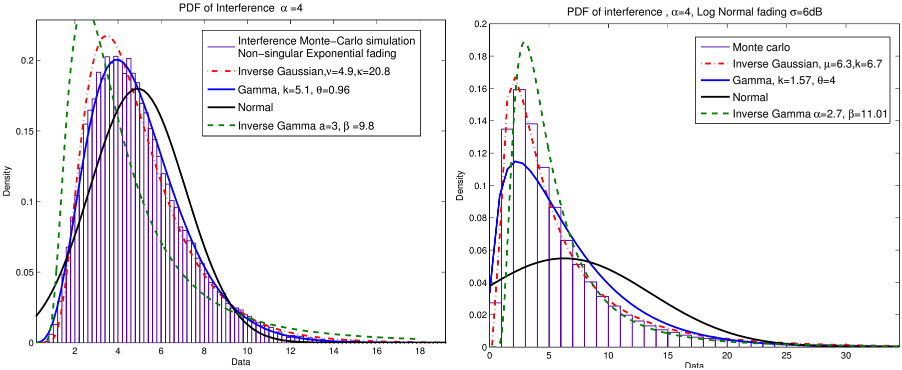
$$\mathbb{P}(I \geq y) \sim \lambda y^{-2/\alpha}\mathbb{E}[\mathsf{h}^{2/\alpha}], \; y \to \infty$$

Proof: We have

$$1 - e^{-\lambda y^{-2/\alpha}\mathbb{E}[\mathsf{h}^{2/\alpha}]} \leq \mathbb{P}(I \geq y) \leq 1 - \left(1 - \frac{2\pi\mathbb{E}[\mathsf{h}^{2/\alpha}]}{2-\alpha} y^{-2/\alpha}\right) e^{-\lambda y^{-2/\alpha}\mathbb{E}[\mathsf{h}^{2/\alpha}]}.$$

Use $\exp(-x) \sim 1 - x$, $x \to 0$. $\square$

# Is Gaussian modelling of interference appropriate?




PDF of interference for Rayleigh and log normal fading with path loss $\ell(x) = (1 + \|x\|^{\alpha})^{-1}$.

# Transmission capacity(TC)

Let $\epsilon \in (0,1)$. TC is defined as

$$TC(\epsilon) = (1-\epsilon)\arg\max_{\lambda>0}\{p_s(\theta,\lambda) > 1-\epsilon\}$$

1. $\arg\max_{\lambda>0}\{p_s(\theta,\lambda) > 1-\epsilon\}$ is the maximum density of transmitting nodes that can be supported for an outage constraint $\epsilon$.
2. $1-\epsilon$ fraction of these nodes are successful.
3. Hence TC measures the maximum spatial intensity of successful transmissions per unit area for a given outage capacity.
4. Can be related to area spectral efficiency (ASE) by multiplying with $\log_2(1+\theta)$.

# Transmission capacity of the PPP dipole network

## Lemma

*When $\sigma^2 \approx 0$, the TC of a PPP dipole network with $N_t = N_r = 1$ is*

$$TC(\epsilon) = \frac{(1-\epsilon)}{d^2 C(\alpha)\theta^{2/\alpha}} \ln\left(\frac{1}{1-\epsilon}\right).$$

Proof: We have,

$$\mathrm{p}(\theta,\lambda) = \exp(-\lambda d^2 \theta^{2/\alpha} C(\alpha))$$

Observe that $\mathrm{p}(\theta,\lambda)$ increases with $\lambda$. Hence solving for

$$\mathrm{p}(\theta,\lambda) > 1-\epsilon,$$

we obtain the result. $\square$

# Sphere packing interpretation of TC

When $\epsilon \approx 0$, by Taylor series expansion, $\ln\left(\frac{1}{1-\epsilon}\right) = \epsilon + o(\epsilon)$. Hence

$$TC(\epsilon) = \frac{(1-\epsilon)}{d^2 C(\alpha)\theta^{2/\alpha}} \ln\left(\frac{1}{1-\epsilon}\right) \sim \frac{\epsilon}{d^2 C(\alpha)\theta^{2/\alpha}} = \frac{1}{\pi\left(d\sqrt{\frac{2\pi}{\epsilon\alpha\sin(2\pi/\alpha)}}\right)^2}.$$

## Interpretation (heuristic)

Hence each transmission approximately requires an interference free disc of radius

$$R = d\left(\frac{2\pi}{\epsilon\alpha\sin(2\pi/\alpha)}\right)^{1/2}.$$

- The disc radius increases as $\frac{1}{\sqrt{\epsilon}}$.
- The disc radius decreases with increasing $\alpha$
  - Higher path loss exponent $\rightarrow$ better packing.

# Transmission capacity for other schemes when $\epsilon \approx 0$

- SISO: $TC(\epsilon) = \frac{\epsilon}{d^2 C(\alpha)\theta^{2/\alpha}}$
- MIMO: MRC with $n$ receive antenna

$$\frac{n^{2/\alpha}\epsilon}{d^2 C(\alpha)\theta^{2/\alpha}} \leq TC(\epsilon) \leq \frac{n^{2/\alpha}\Gamma(1-2/\alpha)\epsilon}{d^2 C(\alpha)\theta^{2/\alpha}}$$

- MIMO eigen-beamforming: $m$ transmit and $n$ receive

$$\frac{\max\{n,m\}^{2/\alpha}\epsilon}{d^2 C(\alpha)\theta^{2/\alpha}} \leq TC(\epsilon) \leq \frac{(nm)^{2/\alpha}\Gamma(1-2/\alpha)\epsilon}{d^2 C(\alpha)\theta^{2/\alpha}}$$

Can be used to analyse a multitude of systems with interference.

A. M. Hunter, J. G. Andrews and S. P. Weber, "Transmission Capacity of Ad Hoc Networks with Spatial Diversity", IEEE Transactions on Wireless Communications, Vol. 7, No. 12, pp. 5058-71, Dec. 2008

# References


- MIMO

  R.H.Y. Louie, M. R. McKay, I. B. Collings , "Open-Loop Spatial Multiplexing and Diversity Communications in Ad Hoc Networks",/ IEEE Transactions on Information Theory, Vol. 57, No.1, pp. 317-344, Jan. 2011

- Power control

  N. Jindal, S. Weber, and J. Andrews, "Fractional Power Control for Decentralized Wireless Networks, IEEE Trans. Wireless Communications, Vol. 7, No. 12, pp. 5482-5492, Dec. 2008

  X. Zhang and M. Haenggi, "Random Power Control in Poisson Networks," IEEE Transactions on Communications, 2012.

- Multihop

  R. Vaze,"Throughput-Delay-Reliability Tradeoff with ARQ in Wireless Ad Hoc Networks", Wireless Communications, IEEE Transactions on , vol.10, no.7, pp.2142-2149, July 2011

## Monographs

F. Baccelli and B. Blaszczyszyn "Stochastic Geometry and Wireless Networks" NOW Publishers

S. Weber and J. G. Andrews, "Transmission Capacity of Wireless Networks", NOW Publishers

M. Haenggi and R. K. Ganti, "Interference in Large Wireless Networks", NOW Publishers

# Spatial and temporal correlation of interference in PPPs with ALOHA

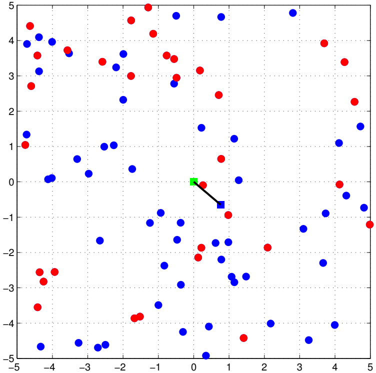
- At each time instant each node transmits with probability $p$
- Let $\Phi_k$ denote the set of active transmitters at time $k$. *i.e.*,

$$\Phi_k = \{x; x \in \Phi, x \text{ is on at time } k\}$$

- The interference is

$$I(\Phi_k, z) = \sum_{\mathsf{x}\in\Phi_k} \mathsf{h}_{\mathsf{x}z}[k]\ell(\mathsf{x} - z)$$

  - We assume that the fading is indpendent across space and time.




Time instant 1.

# Spatial and temporal correlation of interference in PPPs with ALOHA

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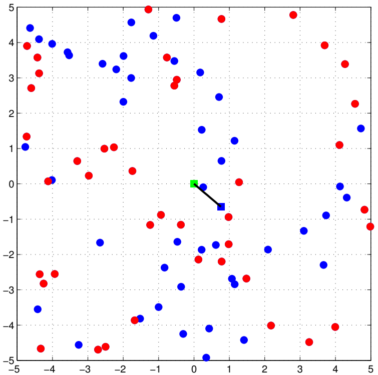



Time instant 2.

# Spatial and temporal correlation of interference in PPPs with ALOHA

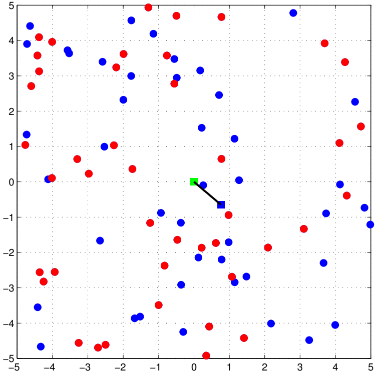
- At each time instant each node transmits with probability $p$
- Let $\Phi_k$ denote the set of active transmitters at time $k$. *i.e.*,

$$\Phi_k = \{x; x \in \Phi, x \text{ is on at time } k\}$$

- The interference is

$$I(\Phi_k, z) = \sum_{\mathrm{x} \in \Phi_k} \mathrm{h}_{\mathrm{x}z}[k] \ell(\mathrm{x} - z)$$

  - We assume that the fading is indpendent across space and time.




Time instant 3.

# Spatio-temporal correlation coefficient

$$\text{Correlation coefficient, } \rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y}.$$

## Lemma

*The spatio-temporal correlation coefficient of the interferences* $I(\Phi_m, u)$ *and* $I(\Phi_n, v)$, $m \neq n$ *for ALOHA and path loss functions* $\ell(x)$ *satisfying* $\int_{\mathbb{R}^2} \ell(x)\mathrm{d}x < \infty$, *is*

$$\zeta(u,v) = \frac{p\int_{\mathbb{R}^2} \ell(x)\ell(x - \|u-v\|)\mathrm{d}x}{\mathbb{E}[h^2]\int_{\mathbb{R}^2} \ell^2(x)\mathrm{d}x}.$$

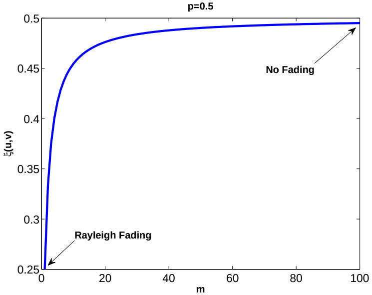
Proof: Follows[6] from Campbell's theorem. □

[6] R. K. Ganti and M. Haenggi. "Spatial and temporal correlation of the interference in ALOHA ad hoc networks", IEEE Communications Letters, 13(9):631 -633, September 2009

When $u = v$, the temporal correlation is equal to

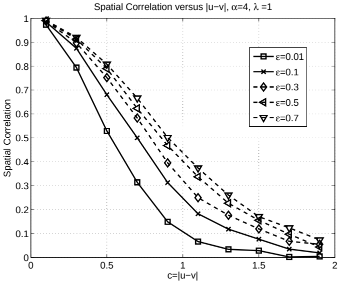
$$\frac{p}{\mathbb{E}[h^2]}.$$

Hence for Nakagami-m fading, it is equal to $\frac{pm}{1+m}$.




When the path loss is given by $\ell(x) = 1/(\epsilon + \|x\|^\alpha)$ and $u \neq v$ the correlation is equal to

$$\lim_{\epsilon \to 0} \xi(u, v) = 0.$$



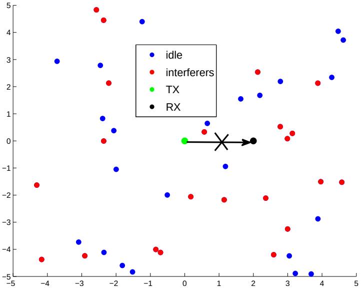

Interference is spatio-temporally correlated.

# Link formation delay

- A TX at $x$ can connect to a RX at $y$ if

$$\mathrm{SIR}(x,y) = \frac{\mathrm{h}_{xy}[k]\ell(d)}{I(\Phi_k, y)} \geq \theta$$

- ALOHA MAC with access probability $p$
- $D$: No of attempts required for a connection to form.




**First attempt**

No connection

# Link formation delay

- A TX at $x$ can connect to a RX at $y$ if

$$\mathrm{SIR}(x,y) = \frac{\mathrm{h}_{xy}[k]\ell(d)}{I(\Phi_k, y)} \geq \theta$$

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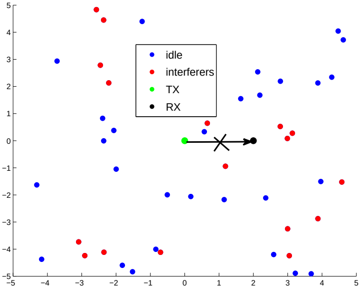



**Second attempt**

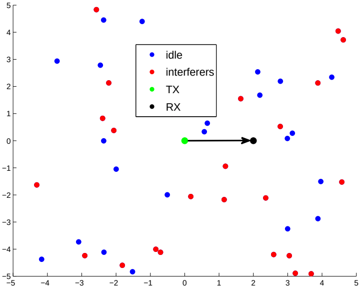
No connection

# Link formation delay

- A TX at $x$ can connect to a RX at $y$ if

$$\mathrm{SIR}(x,y) = \frac{\mathrm{h}_{xy}[k]\ell(d)}{I(\Phi_k, y)} \geq \theta$$

- ALOHA MAC with access probability $p$
- $D$: No of attempts required for a connection to form.




**Third attempt**

Connected

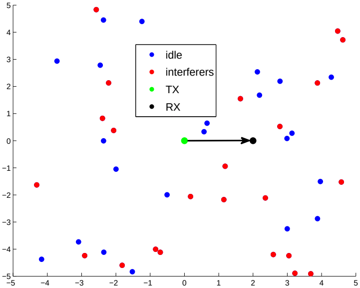
$$D = 3$$

# Link formation delay

- A TX at $x$ can connect to a RX at $y$ if

$$\mathrm{SIR}(x,y) = \frac{\mathrm{h}_{xy}[k]\ell(d)}{I(\Phi_k, y)} \geq \theta$$

- ALOHA MAC with access probability $p$
- $D$: No of attempts required for a connection to form.




What is the average delay $\mathbb{E}[D]$ ?

# Average delay $\mathbb{E}[D]$: Neglecting interference correlation

## Recall

- ALOHA corresponds to independent thinning
  - $\Phi_m$ (the set of transmitters at time $m$) is a PPP of density $\lambda p$.
- The probability of link formation in a PPP network with density $p\lambda$ is

$$\mathrm{p}_s(\theta, p\lambda) = \exp(-p\lambda d^2 \theta^{2/\alpha} C(\alpha))$$

1. At each time instant a link is formed with probability $\mathrm{p}_s(\theta, \lambda)$ independent of every other time
2. So the delay $D$ is a geometric random variable with mean $\frac{1}{\mathrm{p}_s(\theta,\lambda)}$.

$$\mathbb{E}[D] = \exp(p\lambda d^2 \theta^{2/\alpha} C(\alpha))$$

3. Observe that the delay increases with $p$.

# Average delay $\mathbb{E}[D]$: With interference correlation

- Let $E_k$ denote the event $\frac{\mathrm{h}[k]\ell(d)}{I(\Phi_k,o)} \geq \theta$
- Then $\mathbb{P}(D > k) = \mathbb{P}(E_1^c \cap E_2^c \cap \ldots \cap E_k^c)$ Fail for $k$ times
- $\mathbb{E}[D] = \sum_{k=0}^{\infty} \mathbb{P}(D > k)$ Average of a positive random variable
  - $\mathbb{E}[D] = \mathbb{E}_\Phi \mathbb{E}[D|\Phi]] = \mathbb{E}_\Phi[\sum_{k=0}^{\infty} \mathbb{P}(D > k|\Phi)]$ Conditioning on $\Phi$
- $\mathbb{P}(D > k|\Phi) = \mathbb{P}(E_1^c|\Phi)\mathbb{P}(E_2^c|\Phi) \ldots \mathbb{P}(E_k^c|\Phi)$ Conditional Independence
- Probability that a link is not formed at time $m$ is

$$\begin{aligned}\mathbb{P}(E_m^c|\Phi) &= \mathbb{P}\left(\frac{\mathrm{h}[m]d^{-\alpha}}{I(\Phi_m,y)} \leq \theta \;\middle|\; \Phi\right)\\ &= 1 - \underbrace{\mathbb{E}[\exp\left(-d^\alpha \theta I(\Phi_m,o)\right)|\Phi]}_{T_1}\end{aligned}$$

- Two sources on randomness: Fading and ALOHA MAC

$$
\begin{aligned}
T_1 &= \mathbb{E} e^{-d^\alpha \theta \sum_{x\in\Phi} \mathrm{h}_{xo}[k]\|x\|^{-\alpha} \mathbf{1}(\text{x is Tx at time m})} \\
&= \mathbb{E} \prod_{x\in\Phi} e^{-d^\alpha \theta \mathrm{h}_{xo}[k]\|x\|^{-\alpha} \mathbf{1}(\text{x is Tx})} \\
&= \mathbb{E} \prod_{x\in\Phi} \left[ e^{-d^\alpha \theta \mathrm{h}_{xo}[k]\|x\|^{-\alpha}} \mathbf{1}(\text{x is Tx}) + 1 - \mathbf{1}(\text{x is Tx}) \right]
\end{aligned}
$$

- First averaging over ALOHA,

$$
T_1 = \mathbb{E} \prod_{x\in\Phi} \left[ e^{-d^\alpha \theta \mathrm{h}_{xo}[k]\|x\|^{-\alpha}} p + 1 - p \right]
$$

- Averaging over fading,

$$
\mathbb{P}(E_m^c|\Phi) = 1 - T_1 = 1 - \prod_{x\in\Phi} \left[ 1 - \frac{p}{1 + d^\alpha \theta \|x\|^\alpha} \right]
$$

Observe that $\mathbb{P}(E_m^c|\Phi)$ does not depend on the time index $m$.

- $$\mathbb{P}(D > k|\Phi) = \mathbb{P}(E_1^c|\Phi)\mathbb{P}(E_2^c|\Phi)\ldots\mathbb{P}(E_k^c|\Phi)$$
$$= \left(1 - \prod_{x\in\Phi}\left[1 - \frac{p}{1 + d^\alpha\theta\|x\|^\alpha}\right]\right)^k$$

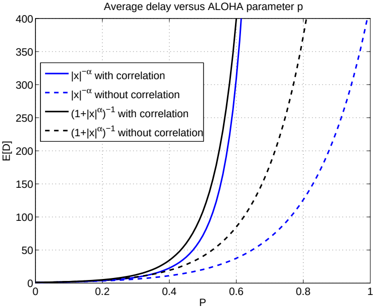
- Hence the conditional average of delay is

$$\mathbb{E}[D|\Phi] = \sum_{k=0}^{\infty}\mathbb{P}(D > k|\Phi) = \sum_{k=0}^{\infty}\left(1 - \prod_{x\in\Phi}\left[1 - \frac{p}{1 + d^\alpha\theta\|x\|^\alpha}\right]\right)^k$$
$$= \frac{1}{\prod_{x\in\Phi}\left[1 - \frac{p}{1+d^\alpha\theta\|x\|^\alpha}\right]}$$
$$= \prod_{x\in\Phi}\left[1 - \frac{p}{1 + d^\alpha\theta\|x\|^\alpha}\right]^{-1}$$

Using PGFL of a PPP,

$$\mathbb{E}\prod_{x\in\Phi}\left[1-\frac{p}{1+d^\alpha\theta\|x\|^\alpha}\right]^{-1}$$

$$=\exp\left(-\lambda\int_{\mathbb{R}^2}1-\left[1-\frac{p}{1+d^\alpha\theta\|x\|^\alpha}\right]^{-1}\mathrm{d}x\right)$$




**With correlation**

$$\mathbb{E}[D]=\exp\left(\frac{p\lambda d^2\theta^{2/\alpha}C(\alpha)}{(1-p)^{1-2/\alpha}}\right)$$

Observe $\mathbb{E}[D]=\infty$ for $p=1$.

**Recall: Without considering correlation**

$$\mathbb{E}[D]=\exp\left(p\lambda d^2\theta^{2/\alpha}C(\alpha)\right)$$

- Relying on fading is not sufficient for ARQ to succeed.
- Correlation of interference cannot be neglected.

F. Baccelli, B. Baszczyszyn, "A New Phase Transitions for Local Delays in MANETs," INFOCOM, 2010

Proceedings IEEE , vol., no., pp.1-9, 14-19 March 2010

# Analysis of Cellular Networks

* Primer on Point Processes

* Ad hoc Networks

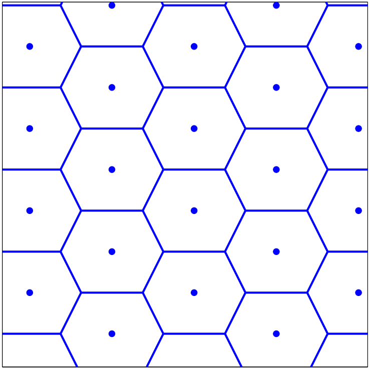
* Cellular Networks
– SINR distribution
– Frequency reuse

* Heterogeneous Networks

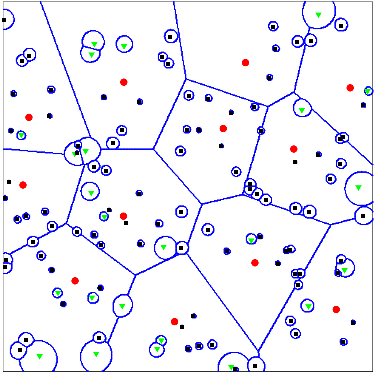
# Cellular Trends

- Interference is a main challenge in cellular systems
  1. SINR more important than SNR
     - Universal frequency reuse
     - Denser and denser deployments
  2. BS cooperation and other interference-suppression techniques require good models for other-cell interference
- Networks are becoming unplanned, decentralized and heterogeneous
  - Picocells placed strategically in high-traffic areas
  - Femtocells/relays being placed randomly
  - BS deployments increasingly driven by capacity needs rather than coverage needs

# Emerging cellular networks



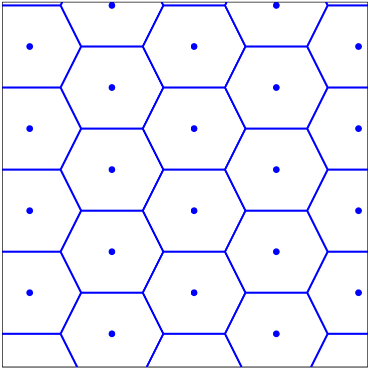
What we think of



4G+femto/pico

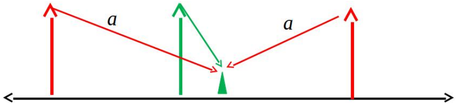
Cells getting smaller, more random and chaotic

# Some current models



Hexagonal model

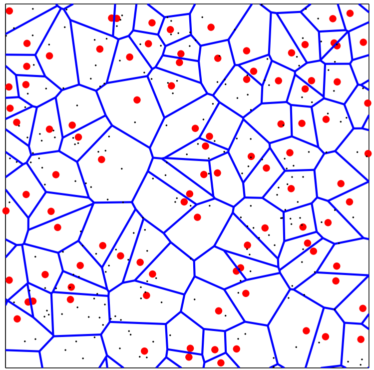
1. Is it accurate?
2. Not very tractable.




Wyner model

1. Fixed background interference
2. Highly inaccurate (averaging)

# Proposed Model: PPP Base Stations




- BS locations are drawn from a PPP of density $\lambda$
- Each mobile associates with the closest BS
  - Cells are Voronoi tessellations

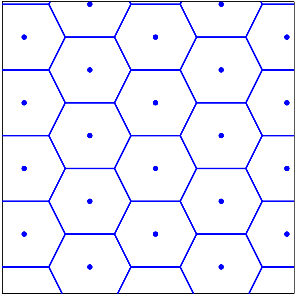
## Advantages

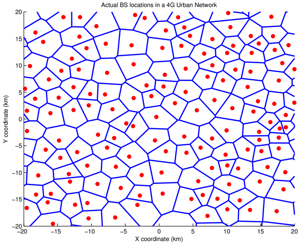
- Non uniform cell sizes
- Tractable?

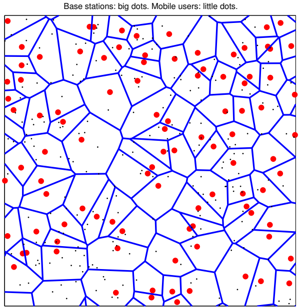
## Disadvantages

- BSs might get very close

# Comparison with real deployment





# System model: Downlink

- The BSs are spatially distributed as a PPP of density $\lambda$
- Mobile users connect to the nearest (geographical) BS
- The path loss is given by $\ell(x) = \|x\|^{-\alpha}$, $\alpha > 2$.
- All BSs transmit at the same power $P$
- The fading between a BS $x$ and a mobile $y$ is denoted by $\mathrm{h}_{xy}$

What is the SINR distribution of a typical mobile user?

Without loss of generality, we can assume the typical mobile user to be at the origin $o$.

# Analysis of SINR

Let $x \in \Phi$ be the BS that is closest to the reference MS at the origin.

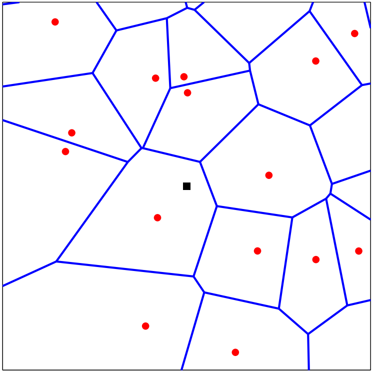
The downlink SINR of the MS at the origin is

$$\mathrm{SINR} = \frac{\mathsf{h}_{xo} r^{-\alpha}}{\sum_{y \in \Phi \setminus \{x\}} \mathsf{h}_{yo} \|y\|^{-\alpha}}$$

where $r = \|x\|$.

Compute $\mathbb{P}(\mathrm{SINR} > \theta)$

Base stations: big dots. Mobile users: little dots.



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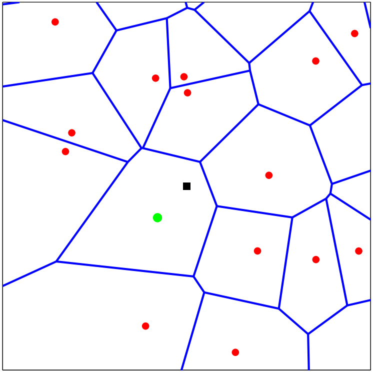

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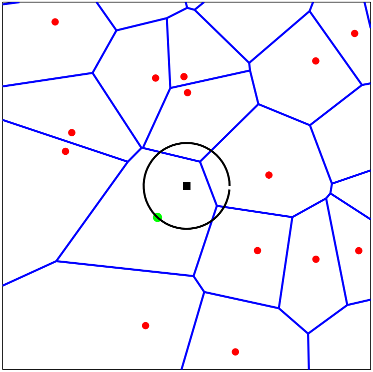


# Analysis of SINR

Recall that the distribution of the nearest BS equals the first contact distribution

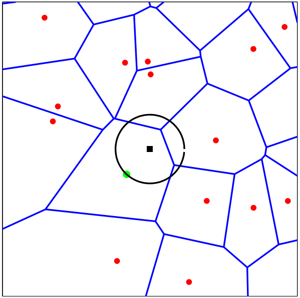
$$f(r) = \lambda 2\pi r \exp(-\lambda\pi r^2)$$

- We first condition on the distance to the nearest BS $r$.

$$\begin{aligned}\mathbb{P}(\mathrm{SINR} > \theta) &= \mathbb{E}_r \mathbb{P}(\mathrm{SINR} > \theta \mid r) \\ &= \int_0^\infty \mathbb{P}(\mathrm{SINR} > \theta \mid r) \lambda 2\pi r \exp(-\lambda\pi r^2) \mathrm{d}r\end{aligned}$$

- Focusing on $\mathbb{P}(\mathrm{SINR} > \theta \mid r)$ which we denote by $\mathrm{p}_r$

$$\begin{aligned}\mathrm{p}_r &= \mathbb{P}\left( \frac{\mathrm{h}_{xo} r^{-\alpha}}{\sum_{y\in\Phi\setminus\{x\}} \mathrm{h}_{yo}\|y\|^{-\alpha}} \geq \theta \middle| r \right) \\ &= \mathbb{E}\left[ \exp\left( -\theta r^{\alpha} \sum_{y\in\Phi\setminus\{x\}} \mathrm{h}_{yo}\|y\|^{-\alpha} \right) \middle| r \right]\end{aligned}$$




- $$p_r = \mathbb{E}\left[\prod_{y\in\Phi\setminus\{x\}} \exp\left(-\theta r^{\alpha} h_{yo}\|y\|^{-\alpha}\right) \Big| r\right]$$
- Since the fades are independent and exponentially distributed with unit mean,

$$p_r = \mathbb{E}\left[\prod_{y\in\Phi\setminus\{x\}} \frac{1}{1+\theta r^{\alpha}\|y\|^{-\alpha}} \Big| r\right]$$

- Using PGFL on $\Phi \cap B(o,r)^c$

$$p_r = e^{-\lambda \int_{B(o,r)^c} \frac{1}{1+\theta^{-1} r^{-\alpha}\|y\|^{\alpha}} \mathrm{d}y}.$$

- Un-conditioning on $r$,

$$\mathbb{P}(\mathrm{SINR} > \theta) = \int_0^{\infty} e^{-\lambda \int_{B(o,r)^c} \frac{1}{1+\theta^{-1} r^{-\alpha}\|y\|^{\alpha}} \mathrm{d}y} f(r)\mathrm{d}r.$$

# SINR distribution

The SIR distribution in a PPP BS network where a MS connects to the nearest BS is given by[7]

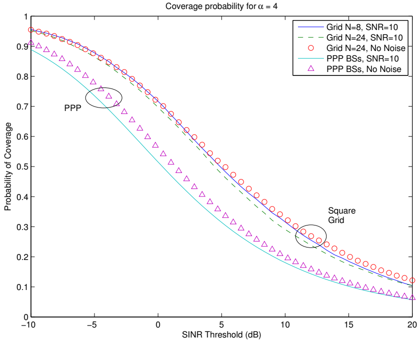
$$\mathbb{P}(\mathsf{SIR} \geq \theta) = \frac{1}{1+\rho(\theta,\alpha)}$$

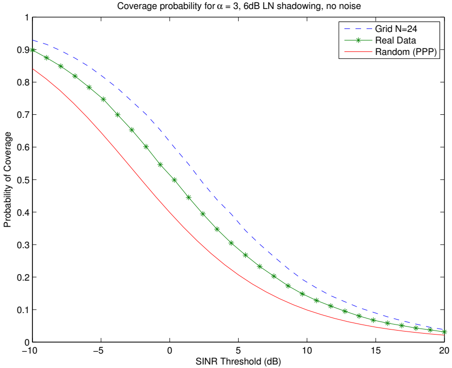
where $\rho(\theta,\alpha) = \theta^{2/\alpha} \int_{\theta^{-2/\alpha}}^{\infty} \frac{1}{1+u^{\alpha/2}} \mathrm{d}u$.

- Does not depend on the density of BSs $\lambda$.
  - Increasing the density of BSs does not increase the coverage probability.
- Simple expression
  - For $\alpha = 4$ reduces to $(1+\sqrt{\theta}(\pi/2 - \arctan(1/\sqrt{\theta})))^{-1}$.
- Extensions to general fading distributions and noise possible.
- Since the PDF of SIR is known the average ergodic can be computed
  - For $\alpha = 4$, the computed ergodic rate is $1.49 \mathrm{nats}/\mathrm{sec}/\mathrm{Hz}$

[7] J. G. Andrews, F. Baccelli, and R. K. Ganti. A tractable approach to coverage and rate in cellular networks. IEEE Trans. on Communications, Nov. 2011

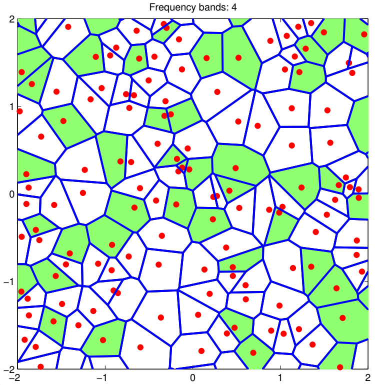
# Numerical results



# Frequency reuse

- Frequency planning necessary for increasing the coverage.
- Assume that there are $\delta$ frequency bands
  - PPP: random allocation of bands
  - Corresponds to thinning of a PPP $\Phi$
    - Density of $\Phi_m$ is $\lambda/\delta$.
- We first consider the BS $x \in \Phi$ to which the MS at the origin connects.
  - $\|x\| = r \sim \lambda 2\pi r \exp(-\lambda\pi r^2)$
  - The interferers density is now $\lambda/\delta$.




The coverage probability is

$$\mathbb{P}(\mathrm{SIR} \geq \theta) = \frac{1}{1 + \frac{1}{\delta}\rho(\theta, \alpha)}$$

# Coverage versus rate

1. The coverage probability is

$$\mathbb{P}(\mathsf{SIR} \geq \theta) = \frac{1}{1 + \frac{1}{\delta}\rho(\theta, \alpha)} \quad \Uparrow \delta$$

2. The ergodic rate equals $\frac{1}{\delta}\mathbb{E}[\ln(1 + \mathsf{SIR})]$, which equals

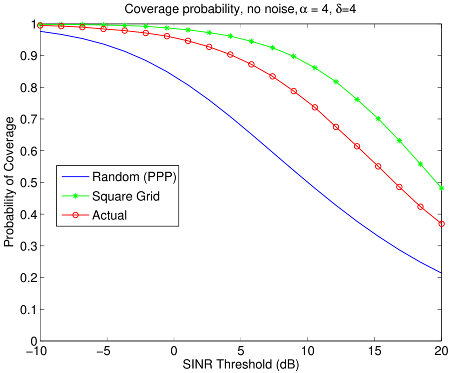
$$\frac{1}{\delta}\int_0^\infty \mathbb{P}(\ln(1 + \mathsf{SIR}) > \theta)\mathrm{d}\theta = \frac{1}{\delta}\int_0^\infty \frac{1}{1 + \frac{1}{\delta}\rho(e^\theta - 1, \alpha)}\mathrm{d}\theta,$$

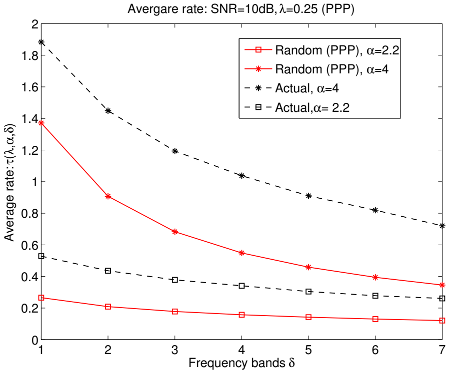
which equals

$$R = \int_0^\infty \frac{1}{\delta + \rho(e^\theta - 1, \alpha)}\mathrm{d}\theta \quad \Downarrow \delta$$

Coverage increases with $\delta$ while the average rate decreases with $\delta$.

# Numerical results



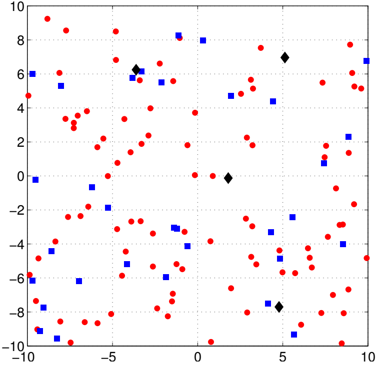



# Analysis of Heterogeneous Networks

* Primer on Point Processes
* Ad hoc Networks
* Cellular Networks
* Heterogeneous Networks

# System Model

- $K$ tier network
- The BS locations in tier $i$ are modelled by a PPP $\Phi_i$
  - Density of $\Phi_i$ is $\lambda_i$
  - BSs in tier $i$ transmit with power $P_i$
  - A mobile can connect to a BS in tier $i$ if the received SIR is greater than $\theta_i$
- All tiers transmit in the same frequency band and hence contribute to interference
- Fading is i.i.d Rayleigh
- Path loss is given by $\|x\|^{-\alpha}$, $\alpha > 2$




An example of $k = 3$ network. Red: FemtoBS, blue: PicoBS, black: MacroBS

## Assumption

The SIR thresholds $\theta_i > 1$.

# Max SIR model

## Connectivity model

A mobile user can connect to a BS of any tier provided that the SIR constraint is satisfied, *i.e.*, to connect to a BS of tier $i$, the SIR should be greater than $\theta_i$.

## Lemma

*If $\theta_i > 1$, a mobile user can connect to at most one BS*

## Proof.

Let $a_i$ denote the received power from a BS $i$, then only one of the following terms can be greater than 1

$$\frac{a_i}{\sum_{j \neq i} a_j}, \quad i = 1, 2, \ldots$$

$\square$

# Coverage probability

The receive SIR of a mobile at origin and a BS $x \in \Phi_m$ is

$$\mathrm{SIR}(x) = \frac{\mathrm{h}_{xo}\|x\|^{-\alpha}}{\sum_{i=1}^{K} \sum_{y \in \Phi_i} \mathrm{h}_{yo}\|y\|^{-\alpha} - \mathrm{h}_{xo}\|x\|^{-\alpha}}$$

Let $\mathrm{p}_c$ denote the coverage probability.

$$\begin{aligned} 1 - \mathrm{p}_c &= \mathbb{E}\left( \prod_{m=1}^{K} \prod_{x \in \Phi_m} \mathbf{1}(\mathrm{SIR}(x) < \theta_m) \right) \\ &= \mathbb{E}\left( \prod_{m=1}^{K} \prod_{x \in \Phi_m} 1 - \mathbf{1}(\mathrm{SIR}(x) > \theta_m) \right) \end{aligned}$$

# Contd...

Expanding the inner product,

$$1 - \mathrm{p}_c = 1 - \mathbb{E} \sum_{m=1}^{K} \sum_{x \in \Phi_m} \mathbf{1}(\mathrm{SIR}(x) > \theta_m)$$
$$+ \mathbb{E} \sum \underbrace{\mathbf{1}(\mathrm{SIR}(x) > \theta_m) \mathbf{1}(\mathrm{SIR}(y) > \theta_n)}_{T_2} - (\text{ three terms }) \ldots$$

The term $T_2$ and higher order terms are zero since the MS can connect to at most 1 BS. Hence

$$\mathrm{p}_c = \sum_{m=1}^{K} \mathbb{E} \sum_{x \in \Phi_m} \mathbf{1}(\mathrm{SIR}(x) > \theta_m)$$

# Contd...

How to evaluate $\mathbb{E}\sum_{x\in\Phi_m}\mathbf{1}(\mathrm{SIR}(x)>\theta_m)$.

## Recall Campbell Mecke theorem

For a PPP of density $\lambda$

$$\mathbb{E}\sum_{x\in\Phi} f(x,\Phi\setminus\{x\}) = \lambda\int_{\mathbb{R}^2}\mathbb{E}[f(x,\Phi)]\mathrm{d}x$$

$$\mathbb{E}\sum_{x\in\Phi_m}\mathbf{1}(\mathrm{SIR}(x)>\theta_m) = \lambda_m\int_{\mathbb{R}^2}\mathbb{P}(\mathrm{SIR}(x)>\theta_m)\mathrm{d}x$$

As before,

$$\mathbb{P}\left(\frac{P_m \mathrm{h}_{xo}\|x\|^{-\alpha}}{I}>\theta_m\right) = \mathcal{L}_I(\|x\|^{\alpha}\theta_m P_m^{-1}).$$

# Contd...

Observe that the total interference is the sum of the interference from each tier which are independent. Hence

$$\mathcal{L}_I(\|x\|^\alpha \theta_m P_m^{-1}) = \prod_{j=1}^{K} \mathcal{L}_{I_j}(\|x\|^\alpha \theta_m P_m^{-1})$$

Recall that the Laplace transform of interference $I_j = \sum_{x \in \Phi_j} P_j \mathrm{h}_{xo} \|x\|^{-\alpha}$ is

$$\mathcal{L}_{I_j}(s) = \exp(-\lambda_j P_j^{2/\alpha} s^{2/\alpha} C(\alpha)),$$

where $C(\alpha) = \frac{2\pi^2}{\alpha \sin(2\pi/\alpha)}$

Hence

$$\mathcal{L}_I(\|x\|^\alpha \theta_m P_m^{-1}) = \exp(-\|x\|^2 \theta_m^{2/\alpha} P_m^{-2/\alpha} \sum_{i=1}^{K} \lambda_i P_i^{2/\alpha} C(\alpha))$$

# Coverage results[8]

Combining everything,

$$p_c = \sum_{m=1}^{k} \lambda_m \int_{\mathbb{R}^2} \exp(-\|x\|^2 \theta_m^{2/\alpha} P_m^{-2/\alpha} \sum_{i=1}^{K} \lambda_i P_i^{2/\alpha} C(\alpha)) \mathrm{d}x$$

## Lemma

*The coverage probability in a $K$ tier heterogeneous network is*

$$p_c = \frac{\pi}{C(\alpha)} \frac{\sum_{m=1}^{K} \lambda_m P_m^{2/\alpha} \theta_m^{-2/\alpha}}{\sum_{m=1}^{K} \lambda_m P_m^{2/\alpha}}, \quad \theta_i > 1$$

1. A simple expression. Convex combination of $\theta_m^{-2/\alpha}$
2. $p_c$ does not depend on the densities if all the thresholds $\theta_m$ are equal
   - Can add more tiers without changing the coverage

[7] H. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews. "Modelling and analysis of k-tier downlink heterogeneous cellular networks ". IEEE JSAC, April 2012

Fraction of users connected to $j$-th tier is

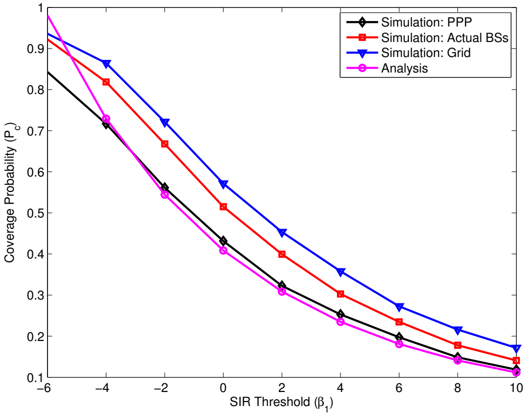
$$\beta_j = \frac{\lambda_j P_j^{2/\alpha} \theta_j^{-2/\alpha}}{\sum_{m=1}^{K} \lambda_m P_m^{2/\alpha} \theta_m^{-2/\alpha}}$$

### Lemma (Closed access)

*When the user is allowed to connect to only a subset of tiers $B \subset \{1, 2, 3, .., K\}$, the coverage probability is*

$$\mathrm{p}_c = \frac{\pi}{C(\alpha)} \frac{\sum_{m \in B} \lambda_m P_m^{2/\alpha} \theta_m^{-2/\alpha}}{\sum_{m=1}^{K} \lambda_m P_m^{2/\alpha}}, \quad \theta_i > 1$$

# Numerical results




A two-tier HCN, $K = 2$, $\alpha = 3$, $P_1 = 100P_2$, $\lambda_2 = 2\lambda_1$, $\beta_2 = 1$dB

# Conclusions

- Networks are getting more random and chaotic
- Random spatial models are necessary for modelling current networks.
- A rich set of mathematical tools are provided by stochastic geometry