Team No. 366

Problem A

Peak Mission: Ion Thrusters to Saturn Saving the Maximum Fuel Consumption

Abstract

An interplanetary mission "Peak" from 90-minute-period orbit around Earth to the 40-hour-period orbit around Saturn is designed to achieve the minimum fuel consumption. The spacecraft used to carry out this mission is 5,000 kg weighed and is equipped with an engine which is capable of providing a thrust of 400 millinewtons, with a specific impluse of 4,000 seconds. Using the graphic-analytic method, a numerical piecewise hohmann method and a simplified gravity assist maneuver model, the optimal trajectory is found and the whole mission is divided into five parts. The mission consumes 6,542 days of time and 42.00 % amount of the total mass to store the fuel. The thrusters are only turned on during hohmann transfers at the initial and final stage of the mission to speed up the spacecraft. In comparison with the Cassisni Mission carried out by NASA, the fuel storage proportion is smaller, while the mission duration is significantly streched.

1 Introduction

In various interplanetary missions, the gravity assist maneuver can be used to help the spacecraft to reach distinct planets and return to Earth. This is a technique that considers certain gravitational force field's acceleration or deacceleration impact on the spacecraft. It can reduce the required energy per transfer at the cost of time of arrangement and complexity of optimization.

This method first showed up as the simplest model in works by K. E. Tsiolkovskii and F. Zander, founders of astronautics [1][2]. Now, wide studies and implementations have highlighted the iconic details of the magnificent method. For example, the Cassini Mission Carried out by NASA let the spacecraft, Cassini-Huygens, took a long trip from Earth to Saturn, which is the target planet in this paper as well. It had undergone two Venus flybys, an Earth-Moon flyby and a Jupiter flyby by the time it arrived at Saturn in July 2004 [3].

In this paper, a 5,000 kg weighed spacecraft is required to start its journey beginning at the 90-minute-period orbit around Earth to the 40-hour-period orbit around Saturn. It is equiped with ion thrusters. They are capable of providing a constant thurst of 400 millinewtons, with a specific impluse of 4,000 seconds. This mission, called as Peak, aims to send the spacecraft to the target with a minimum amount of fuel taken so that a maximum proportion of weight can be remained for other scientific researches. Note that here the goal is not to cut down the time needed but to save more mass for other important equipments. The primary aim is to carry less proportion of fuel than the Cassini mission, which is more than half of its 5,712 kg weight.

The following methods are going to be used to find an optimal path to Saturn. Initially, scheme of the 2-D forming orbits of the planets are drawn out. After the steps of orbit modifications are specified using the graphic-analytical method for various paths, the energy and kinematic characteristics of the trajectories are assessed with a first attempt. Some adequate paths are picked out for analysis. Here, the table of flybies provided in the chapter 4 of the book [2] by Labunsky is very useful. Then, the simplified Hohmann transfer model, numerical piecewise Hohmann transfer method and the simplified gravity assist maneuver model are used to study the process of entering the orbit of Saturn, leaving Earth and planets' flybies.

This paper begins with modelling. with the help of graphic-analytical method, a model is set up to divide the Peak mission into five parts and the important methods are gone through. The results of the model are presented in the form of direct calculations, concept mappings and programming outputs. At last, limitations and advantages of the modelling of the mission are concluded.

2 Model

2.1 Problem overview

Our goal is to design a trajectory of the spacecraft so that the spacecraft can enter the orbit of Saturn and reduce fuel consumption as much as possible. This article will divide the spacecraft's journey to Saturn into five steps, and show detailed modeling and calculation of these five parts, so as to obtain the final fuel required and the flight time of the spacecraft. Among them, we will use the gravity assist to accelerate the spacecraft on Venus and Earth, and use Hohmann transfer orbit to transfer to the target orbit when landing on Saturn, so as to minimize fuel consumption.

2.2 Definition

2.2.1 Effective Radius

When an object moves in the universe, its gravitational force will come from many different celestial bodies, which leads to the complexity of modeling and in some cases, we can not get the analytical solution of a motion, only through computer simulation to get the numerical solution. So, we will define the concept of effective radius to simplify the model.

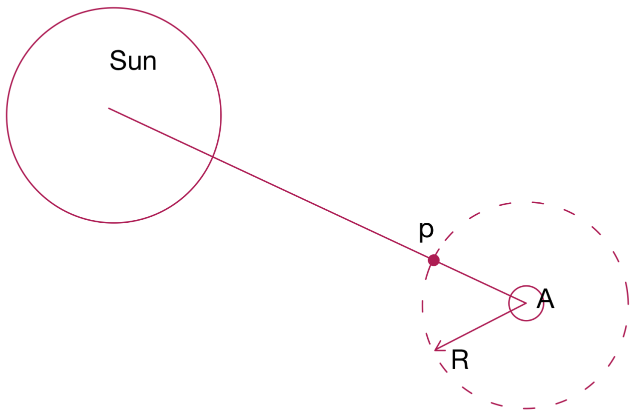

Figure 1: Effective Radius

Suppose that we have a spacecraft on the point p, the gravitation due to the Sun is F_1 and the gravitation due to the planet A is F_2, when

$$F_2 = 10F_1$$

We call the radius R from the center of the planet to the point p the effective radius. And we can further derive the equation of the effective radius as

$$R = \frac{R_{AS}}{\sqrt[3]{\frac{10M_s}{M_A}} + 1}$$

Here, R_{AS} is the distance between the planet A and the sun, M_s is the mass of the sun and M_A is the mass of the planet.

When the spacecraft is within the effective radius, we only calculate the gravitational effect brought by the planet, while when the spacecraft is outside the effective radius, we only calculate the gravitational effect brought by the sun. In this way, the whole model and calculation will be simplified.

2.3 Assumptions and Laws

2.3.1 Initial and final condition

Because the orbital period of the spacecraft around the earth and the aimed orbit period around the Saturn are given. We can calculate the corresponding speed and distance from the planet based on the following three equations.

$$\frac{2\pi}{\omega} = T$$
$$\omega = \frac{v}{R}$$
$$\frac{mv^2}{R} = \frac{GMm}{R^2}$$

Here ω is the angular velocity, T is the orbit period, v is the velosity of the spacecraft, R is the distance from the center of the planet, M is the mass of the planet and G is the gravitational constant.

For m vanishes in these equations, the value of T is given and the value of G can be found on ...... which equals to $6.6726 \times 10^{-11}[N \cdot m^2/kg^2]$ we can solve v and R.

$$v = \sqrt[3]{\frac{2\pi GM}{T}} \tag{1}$$
$$R = \frac{Tv}{2\pi} = \sqrt[3]{\frac{GMT^2}{4\pi^2}} \tag{2}$$

Finally, we can get (v_i, R_i) for initial condition around the earth and (v_f, R_f) for the final state around the Saturn based on the two equations.

2.3.2 Escape Velocity

When we accelerate an spacecraft to a certain velocity, it will be able to escape from the influence of the planet's gravity. We call this velocity the escape velocity.

For the total enegy can be calculated by

$$E_{total} = U + E_k = \frac{1}{2}mv^2 - \frac{GMm}{R}$$

Where, m is the mass of the spacecraf, v is its velocity, M is the mass of the planet and r is the distance from the spacecraft to the center of the planet. When $E_{total} = 0$, the spacecraft can escape from the planet so that the velosity can be caculated by

$$v_{esp} = \sqrt{\frac{2GM}{R}}$$

2.3.3 Hohmann transfer orbit

When an object is small enough comparing with the planet, we can use a method called Hohmann transfer orbit to transfer orbit in the same plane. In our model, we will use Hohmann transfer orbit twice and it can help us save fuel.

The following graph shows the basic approch of the Hohmann transfer orbit, their are two states of the transfer, first, the orbit becomes an ellipse and then the object successfully transfer to the aimed orbit.

Suppose that the initial radius of the orbit is r and the aimed orbit radius is R, we can get the following two equations [2]

$$\Delta v_1 = \sqrt{\frac{GM}{r}}\left(\sqrt{\frac{2R}{r+R}} - 1\right)$$
$$\Delta v_2 = \sqrt{\frac{GM}{R}}\left(1 - \sqrt{\frac{2r}{r+R}}\right)$$

By Kepler's third law, the time for transfering can be calculated by

$$t_{tran} = \pi\sqrt{\frac{(r+R)^3}{8GM}}$$

2.3.4 The Tsiolkovsky Rocket Equation

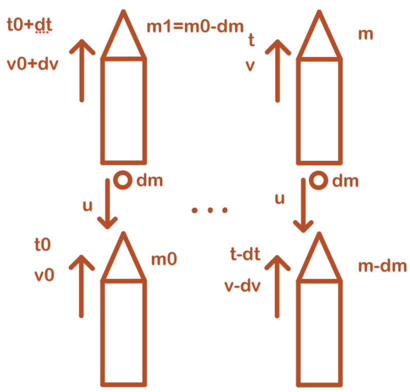

Figure 2: Tsiolkovsky Rocket Equation

Thinking about a continuous process, in which a rocket of initial mass m_0 and velocity v_0 keep throwing parts of its mass dm in the different direction of movement with velocity u. The momentum change of the thrown out fuel will be like the equations in [2]

$$(v_0 + dv - u)dm - vdm = -udm$$

which is equal to the change of the momentum for the rocket and yields

$$mdv = -udm$$

After integration, the desired representation of velocity by u, initial mass m_0 and final mass m is shown at the beginning of the section.

$$\Delta v = uIn(\frac{m_0}{m})$$

2.3.5 Specific Impulse

Theoretically, specific impulse is used to describe the total impulse that can be delivered by consuming unit mass of propellant. It is defined in [2] to be as

$$I_{sp} = \frac{dI}{dmg_0}$$

It is also called as specific thrust. Its specific value is related to the chemical energy per unit propellant, the efficiency of burning and the design of the orifice. For example, the rectangular orifice should have better performance than a circular orifice. Then, recall that the rocket thrust equation is

$$F = \frac{dm}{dt}V_{eq}$$

and that the equivalent velocity is equal to $\frac{I}{m}$, the following useful equation can be finally derived for the calculation when the engine is turned on.

$$F_{thrust} = I_{sp}g_0\frac{dm}{dt}$$

2.4 Determine the Possible Paths

Before everything sets off, this mission needs care analysis for the choice of paths. There are five planets, including Jupiter, Mars, Venus, Mercury and Earth itself, between the Sun and Saturn. They can all be chosen as possible sources of the gravity assists.

There're, certainly, endless possibilities. Even for just twice gravity assist maneuver, there could be theoretically $5^2 = 25$ paths that could be chosen. So, before drawing out the possible paths, a criteria is given as follows:

1. Less than two times (including two) of gravity assist maneuver will be considered.

2. The spacecraft won't seek for the gravity assist maneuver that must cross the orbit of the planet that is closer to the sun than itself.
   ex. Since the spacecraft is launched from the orbit around Earth. It won't go for Mercury for gravity assist maneuver because it crosses the orbit of Venus. But, gravity assist maneuver offered by Venus could be considered.

3. The spacecraft won't seek for the gravity assist maneuver provided by the planet it has visited just a time before.
   ex. The spacecraft is launched from the orbit around Earth and it won't go back for Earth's help again. But, it could seek for Earth's help after Venus's gravity assist maneuver.

These ”rules” are decided quite subjectively, aiming at cutting down the number of paths. However, after careful analysis, we find that they make sense in eliminating unwanted paths. This will be revisited in the discussion section.

Now that the basic characteristics of the paths have been decided, the graphic-analytic method, learned from chapter 2 in the book [2] by Labunsky, can be applied to draw out the paths:

The figure above shows 8 possible paths that starts from Earth and ends at Saturn. The black circles are the orbits of the planets, while the dots and curves show the positions of the planets and the paths. The outer-most black circle is the orbit of Saturn and the inner-most black circle is the orbit of Mercury. At last, we checked the table of chapter 4, ”flights to Saturn”, of the book [2] by Labunsky. Regarding the starting and ending velocities of the spacecraft, we finally decided to choose the path Earth-Venus Flyby-Earth Flyby-(Jupiter Flyby)-Saturn.

2.5 Model

The overall model consists of five different stages

1. Escape from the earth through Piecewise Hohmann Transfer Method.
2. Use the sun's gravity to move towards Venus.
3. Use gravity assist on Venus, and then use it on earth.
4. Flying from earth to Saturn.
5. Land in the Saturn through Hohmann transfer orbit.

2.5.1 Escape from the earth

Based on Hohmann Transfer Method, we develop a new method which is **Piecewise Hohmann Transfer Method** to help the spacecraft to escape from the earth and here is a brief introduction of this method.

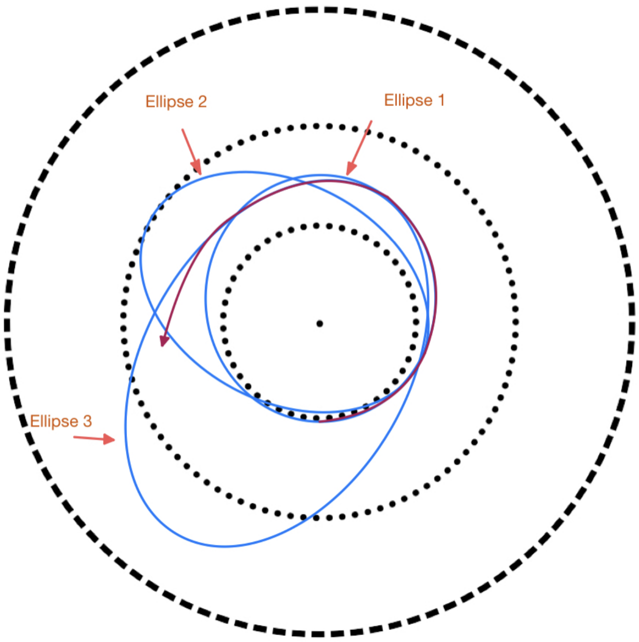

Figure 3: Piecewise Hohmann Transfer Method

Real World Limit As stated in the theory of Hohmann transfer orbit, an important assumption is that the velocity increment is done in infinitesimal time without bending the orbit in the process.

However, this can never achieved as long as the thruster of our spacecraft has a finite specific impulse and thrust force. Practically, we would like to consider the maximal velocity increment can be achieved in a relatively short period of time, and only apply Hohmann transfer whose required velocity increment is than the maximum.

In order to adopt the Hohmann transfer's instantaneous velocity increment in a more real-life case, we would like to set up the following numerical method, which is called by us as Piecewise Hohmann Transfer Method.

For example, as the spacecraft starts from the orbit around earth, it undergoes an increment of velocity dv in the infinitesimal time interval dt. Its orbit should be changed a little bit according to the equations of Hohmann Transfer.

Definitions In application, we can define what is relatively a short period of time with respect to the period of the orbit before Hohmann transfer is applied.

A time interval t is said to be short if it is less than or equal to a tenth of the orbit period T.

$$t \leq \frac{1}{10}T$$

Using the rocket projection formula and the definition of the specific impulse mentioned before, we can find the maximal velocity increment that can be achieved.

First, we write out the 2 equations.

$$m(\Delta v) = m\exp(-\frac{\Delta v}{I_{sp}g_0}), \Delta m = \frac{F_{\text{thrust}}}{I_{sp}g_0}\Delta t$$

Letting $\Delta t = 0.1T$, we can obtain the maximal velocity increment Δv.

$$\Delta v_{max} = -I_{sp}g_0 \ln(1 - \frac{F_{\text{thrust}}T}{10I_{sp}g_0 m})$$

Here, m refers to the mass of the spacecraft right before the application of Hohmann transfer orbit.

Applying a Hohmann transfer orbit whose required velocity increment is less than or equal to the maximal velocity increment, the assumptions of Hohmann transfer orbit are considered satisfied. This means the formulas of standard Hohmann transfer orbits can be used as a good approximation.

In cases where the required velocity increment is larger than the maximal velocity increment, the target orbit radius can be achieved by performing multiple Hohmann transfers with corresponding maximal velocity increment.

Futher explanation and calculation can be found on result part.

2.5.2 Move Towards Venus

After escape from the earth, the space craft can move towards the Venus without active its ion thrusters. Before entering the effective radius of the Venus, our spacecraft is only gravitated by the sun because of our assumption of effective radius. Also, the speed of the spacecraft will be increase beacuse the angle between the direction of the gravitational force and the velosity of the spacecraft is less than 90 degree.

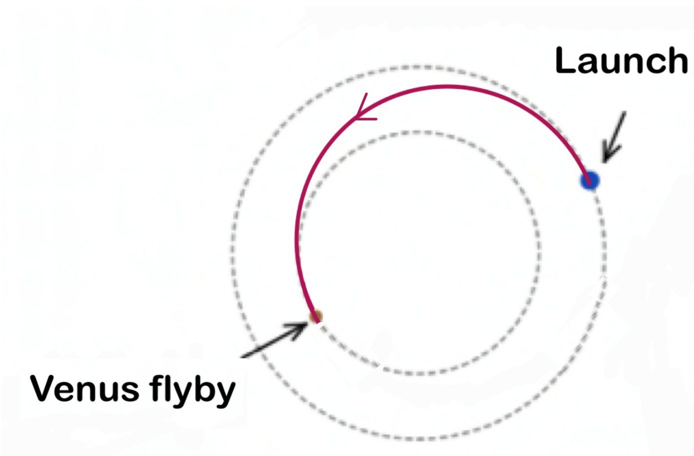

Figure 4: Move Towards Venus

2.5.3 Gavity Assist

The following graph shows the model for gravity assist, when the spacecraft enter into the effective radius of the planet, we can use the relative motion and gravition of a planet to speed up our spacecraft. It can help us save fuel and time.

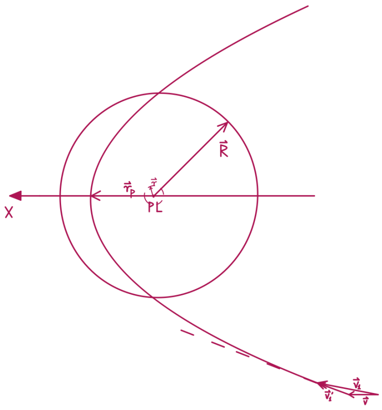

Figure 5: Gavity Assist

Here we just show the brief model of the gravity assist and further calculation will be shown on the next part. The following graph shows the whole process including two gravity assists. After the spacecraft escape from the earth and enter into the effective radius of the Venus, we will get use of the gravitation of the Venus to speed up to earth again and then we can use the gravition of the earth to speed up our spacecraft again.

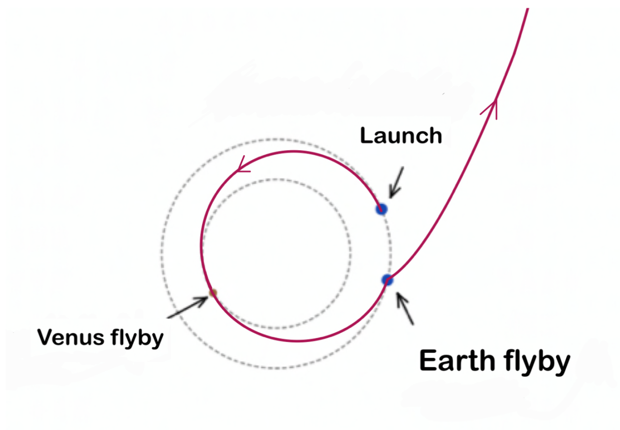

Figure 6: Whole Process

2.5.4 Gravity Assist Model

Assumptions In order to simplify the analysis of gravity assist, we make the following assumptions.

1. The duration of Gravity Assist is short with respect to the period of the planet's orbit.

2. In this short period, the motion of the planet simplifies to a mass particle moving uniformly in a constant direction.

3. The frame of reference of the planet can be considered inertial, as the planet moves uniformly in a constant direction.
4. The Gravity Assist happens inside Effective Radius of the planet, where the gravitational force due to the Sun can be ignored.
5. No relativistic effect will appear.

As the assumptions above are satisfied, the nature of Gravity Assist is in fact a kind of non-head-to-head completely elastic collision. However, the duration of Gravity Assist can still be calculated instead of an infinitesimal time element.

Collision Black Box Collision black box allows us to find the velocity output. When given a velocity input, without the neccesity to know the actual trajectory of the spacecraft near the planet, we can derive the output velocity directly.

To establish the theoretical background of this claim, let us focus on the following figure:

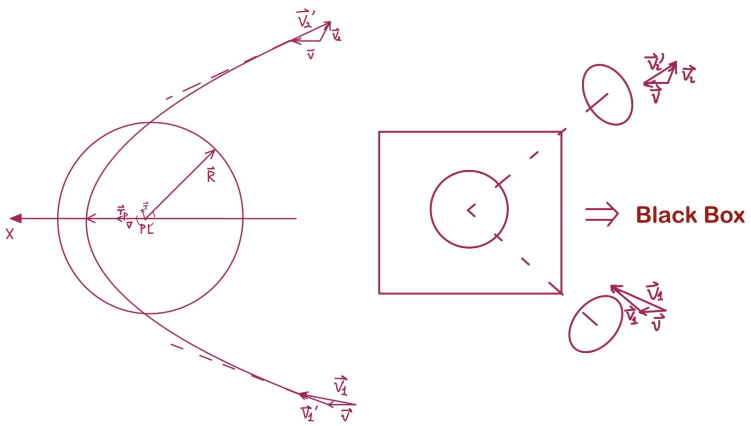

Figure 7: Gravity Assist

As is shown, for the gravity assist, assuming that the planet is moving at a velocity of V, the intersecting angle is θ, velocity is V_1' and the final magnitude of the leaving velocity V_2' becomes

$$V_2' = (V_1 + 2V)\sqrt{1 - \frac{4VV_1'(1 - cos(\theta))}{(V_1 + 2V)^2}}$$

On the other hand, the center collision yields the following result:

$$V_{2x} = V_1 cos(\theta) + 2V$$
$$V_{2y} = V_1 sin(\theta)$$

And thus, the leaving velocity's magnitude will be

$$V_2' = (V_1 + 2V)\sqrt{1 - \frac{4VV_1'(1 - cos(\theta))}{(V_1 + 2V)^2}}$$

which is exactly the same as the gravity assist maneuver. These calculations lay the foundation of the previous statement that the gravity assist maneuver can be viewed as a collision black box to derive the velocity only.

Change of Frame of Reference In order to find the duration of Gravity Assist, we can take the center of the planet as the frame of reference. Then, the trajectory of the spacecraft in this frame of reference is a conic section, since the only force exerted on it is the gravitational force of the planet, which is a central force. The velocity input under this frame of reference is transformed using $\vec{v}'_{in} = \vec{v}_{in} - \vec{u}$. Thus, the transformed velocity is

$$v'_{in} = \sqrt{v_{in}^2 \sin^2\theta + (v_{in}\cos\theta - u)^2}, \theta' = \arccos(\frac{u - 2v_{in}\cos\theta}{2v'_{in}})$$

Here, v_{in} refers to the magnitude of input velocity, u refers to the velocity of the planet, θ refers to the angle of the input velocity. The boundary condition would be that the spacecraft need to be able to leave the planet, that is the total mechanical energy must be greater than or equal to 0. Then we can derive the expression for minimal entering speed v_0 in the Sun frame of reference.

$$E = \frac{1}{2}mv'^2_{in} - G\frac{Mm}{R_0} \geq 0$$

$$v'_{in} \geq \sqrt{\frac{2GM}{R_0}}$$

Here, v'_{in} refers to the entering velocity, R_0 refers to the effective radius and GM refers to the standard gravitational parameter. Take the direction of planet motion in the Sun frame of reference as the polar axis of this frame of reference, the conic sections under polar coordinates follows the following equation (Consider the case in figure).

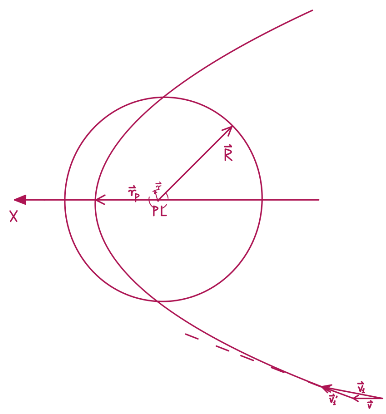

Figure 8: Gravity Assist

$$r(\varphi) = \frac{k}{1 - \mathrm{sgn}(u - 2v_{in}\cos\theta)\epsilon\cos\varphi}, k = \frac{L^2}{GMm^2}, \epsilon = \sqrt{1 + \frac{2EL^2}{(GM)^2m^3}}$$

Here, ϵ refers to the eccentricity of the trajectory, L refers to the angular momentum of the system. Considering our input as the magnitude of entering velocity and the angle it forms with the polar axis, the angular momentum is a unknown quantity that needs to be found. It can be solved from the following equations.

$$\begin{cases} L = mR_0 v'_{in} \sin(\varphi_0 - \theta) \\ R_0 = \dfrac{k}{1 - \operatorname{sgn}(u - 2v_{in}\cos\theta)\epsilon\cos\varphi_0} \end{cases}$$

Analysis in Changed Frame After changing the frame of reference, the assumptions ensures this is an inertial frame of reference. In this reference, the conservation of angular momentum and energy is observed. So, we can write out 2 equations with respect to L and E.

$$\begin{cases} mr^2\dot{\varphi} = L \\ \dfrac{1}{2}mv^2 - G\dfrac{Mm}{r} = E \\ \dot{r}^2 + r^2\dot{\varphi}^2 = v^2 \end{cases}$$

A differential equation about $r(t)$ can be derived.

$$\dot{r}^2 + \frac{L^2}{m^2r^2} - \frac{2GM}{r} = \frac{2E}{m}$$

Based on the symmetry of the conic section, the time to elapse lower half equals to half of the total time. When the spacecraft moves on the lower half, $\dot{r} \geq 0$, this allows us to convert the above equation in to an separable one.

$$\frac{rdr}{\sqrt{\frac{2E}{m}r^2 + 2GMr - \frac{L^2}{m^2}}} = dt$$

Thus, we can integrate both sides to find relationship between t and r.

$$t = \int_{r_{min}}^{r} \frac{rdr}{\sqrt{\frac{2E}{m}r^2 + 2GMr - \frac{L^2}{m^2}}}$$

So the total time duration of the Gravity Assist can be calculated.

$$t_{\text{total}} = 2\int_{r_{min}}^{R_0} \frac{rdr}{\sqrt{\frac{2E}{m}r^2 + 2GMr - \frac{L^2}{m^2}}}$$

2.5.5 Move Towards Saturn

Then, we can travel towards the Saturn after the two gravity assists. In this stage, Our spacecraft is not in any planet's effective radius so that our spacecraft is subject to only one force which is the gravitation of the sun. What should be mentioned is that as we pass through Jupiter, we can further use Jupiter's gravity for an additional acceleration.

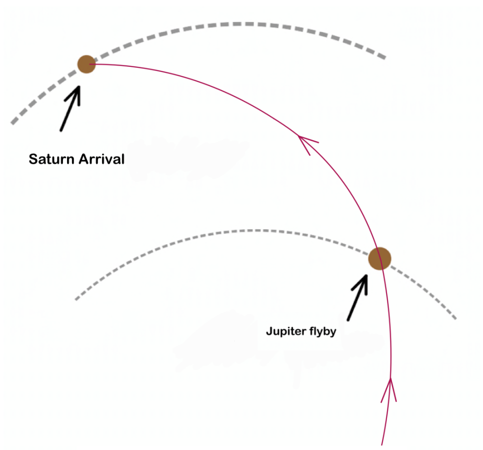

Figure 9: Move Towards Saturn

2.5.6 Arrive the Saturn

After we enter into the effective radius of the Saturn, the ship's thruster will be activated to speed up and use the Hohmann transfer orbit to move to our target orbit to complete our journey.

In the following graph, the spacecraft first enter into the blue orbit which is the effective radius of the Saturn, then, it transfer to our aimed red orbit through Hohmann transfer orbit.

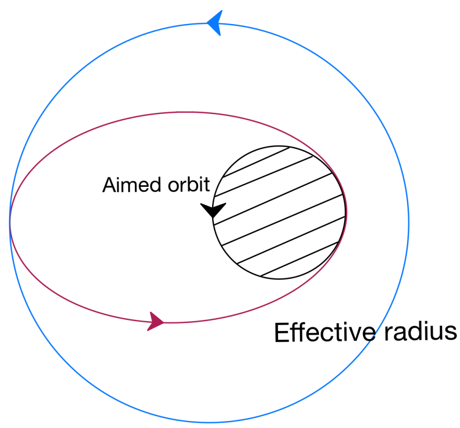

Figure 10: Arrive the Saturn

3 Results

3.1 The Whole Model

This paper is concerned with a mission, Peak, all the way from Earth to Mars. Peak consumes 42.00 % of its 5,000 kg to store fuel. With the aid of a Venus flyby, a Earth flyby and a possible Jupiter flyby, only turning on the thrusters to speed up during the first stage and the last stage is necessary. However, it lasts for a long time for it to complete the trip. It takes approximately 13 years, 19 days, 6 days, 2.3 years and 0.6 year to leave Earth, leave Earth for Venus, leave Venus for Earth, leave Earth for Saturn and finally port in the targeted orbit around Saturn respectively.

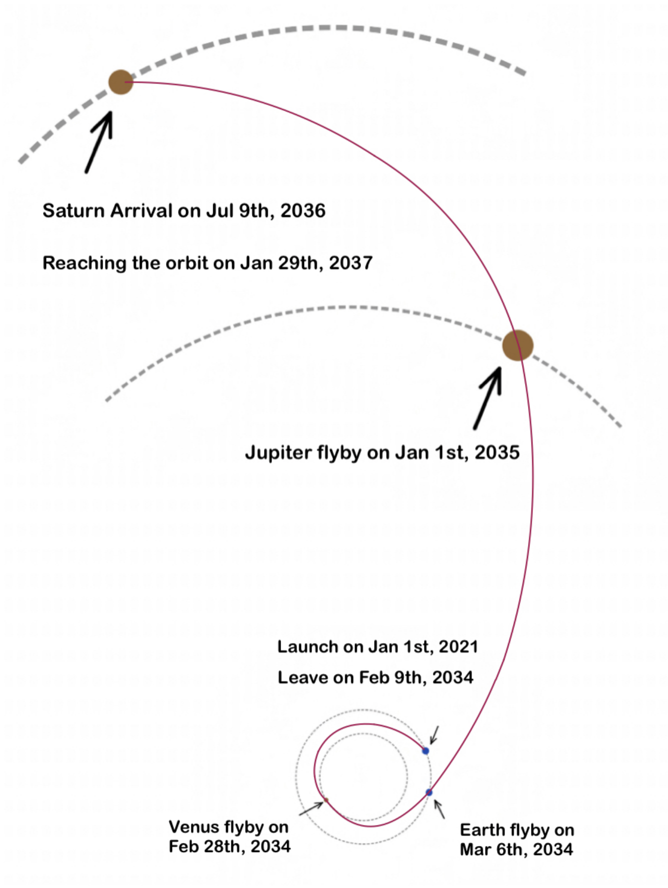

Figure 11: Whole Model

3.2 Analysis and Calculation

Detailed analysis and calculation will be shown on the following pages and here are some constant that will be used in our calculation. All of them are from [1], a useful data source book for astronomy.

- Mass of the earth $M_{earth} = 5.965 \times 10^{24} [kg]$
- Mass of the Venus $M_{venus} = 4.869 \times 10^{24} [kg]$
- Mass of the Saturn $M_{saturn} = 2.6846 \times 10^{26} [kg]$
- Mass of the Earth $M_{sun} = 1.9891 \times 10^{30} [kg]$
- Initial mass of the spacecraft $m_0 = 5000 [kg]$

- Gravitational constant $G = 6.67259 \times 10^{-11}[Nm/kg]$
- Specific impulse of the aircraft $I_{sp} = 4000[s]$
- Constant thrust of the aircraft $F_{thrust} = 0.4[N]$
- Rocket jet velocity $v_e = I_{sp}g_0 = 39226.6[m/s]$
- Standard gravity $g_0 = 9.80665[m/s^2]$

3.2.1 Calculation of the Initial and Final Condition

Recall the following two equations that we have already obtained from 3.2.1.

$$
\begin{aligned}
v &= \sqrt[3]{\frac{2\pi GM}{T}} \\
R &= \frac{Tv}{2\pi} = \sqrt[3]{\frac{GMT^2}{4\pi^2}}
\end{aligned}
$$

We can then use them to calculate (v_i, R_i) for initial condition around the earth and (v_f, R_f) for the final state around the Saturn.

$$
\begin{aligned}
v_i &= 7736.84[m/s] \\
R_i &= 6.64933 \times 10^6[m] \\
v_f &= 11828.7[m/s] \\
R_f &= 2.71094 * 10^8[m]
\end{aligned}
$$

3.2.2 Time required for fuel release

For we have already introduce the equation for the specific impulse which is

$$F_{thrust} = I_{sp}g_0\frac{dm}{dt}$$

Then we can get the function for the time required for fuel release which is

$$
\begin{aligned}
dt &= 98066.5dm \\
\Delta t &= 98066.5\Delta m
\end{aligned}
$$

Together woth the Tsiolkovsky Rocket Equation, we can get

$$
\begin{aligned}
\Delta t &= 98066.5 \times (m_0 - m_1) \\
&= 98066.5 \times m_0(1 - e^{-\frac{\Delta v}{v_e}})
\end{aligned}
$$

3.2.3 Effective radius

We should get the effective radius for the earth, the Venus and the Saturn first. From the following equation, we can calculate them.

$$R = \frac{R_{AS}}{\sqrt[3]{\frac{10M_s}{M_A}} + 1}$$

Finally

$$R_{earth} = 9.93 \times 10^8 [m]$$
$$R_{venus} = 6.67 \times 10^8 [m]$$
$$R_{saturn} = 3.15 \times 10^{10} [m]$$

3.2.4 Using Piecewise Hohmann Transfer Orbit to Leave the Earth

Since the thrust force is relatively small in our problem setup, the required velocity increment of shifting the spacecraft to an orbit of another radius is often larger than the maximal velocity increment.

This means multiple Hohmann transfers need to be performed in many cases. Applying the spirit of **Greedy Algorithms**, we would like to achieve the target orbit radius by utilizing maximal velocity increment in each step of a Piecewise Hohmann Transfer Orbit.

In this case, we can solve for the orbit radius after each step and apply an iterative algorithm to find the quantities needed at the final state. The total velocity increment of a step of Hohmann transfer can be found directly using the standard formula.

$$\Delta v_{\text{total}} = \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) + \sqrt{\frac{GM}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right)$$

Here, r_1 refers to the orbit radius before the apply of a Hohmann transfer, r_2 refers to the radius after a Hohmann transfer, and GM refers to the standard gravitational parameter.

Letting $\Delta v_{\text{total}} = \Delta v_{max}$, we can solve for $r_2 = r_{2,\text{optimal}}$.

$$\sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1\right) + \sqrt{\frac{GM}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1 + r_2}}\right) = -I_{sp} g_0 \ln\left(1 - \frac{F_{\text{thrust}} T}{10 I_{sp} g_0 m}\right)$$

Thus, the optimal step radius $r_{2,\text{optimal}}$ can be defined as the solution of r_2 in the equation above.

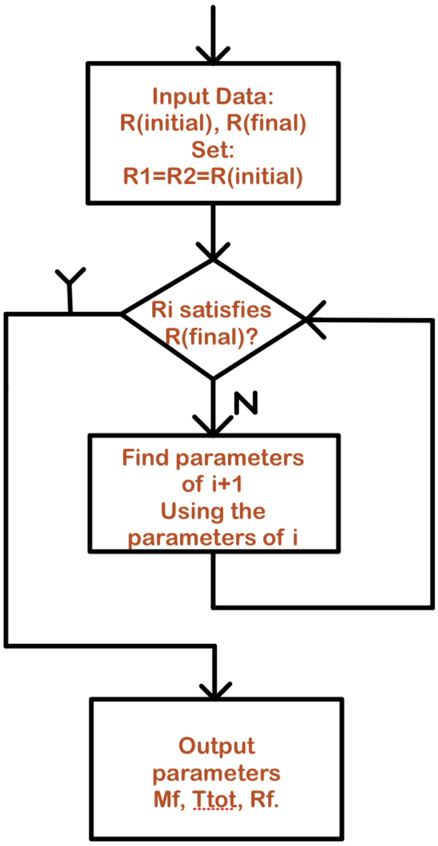

Figure 12: algorithm

Numerical Solution In order to find quantities that we are interested in after the application of a Piecewise Hohmann Transfer Orbit, we can use the **Method of Iteration**. For example, the quantities we are interested in are the time consumption and fuel consumption (change in mass) for our space craft to reach the final state. In a step i, these quantities can be find by the following formulas.

$$T_i = 2\pi\sqrt{\frac{r_i^3}{GM}}, r_i = r_{i,\text{optimal}}$$

$$t_{H,i+1} = t_{H,i} + \pi\sqrt{\frac{(r_i + r_{i+1})^3}{8GM}}$$

$$m_{i+1} = m_i - \frac{F_{\text{thrust}}}{10 I_{sp} g_0} T_i$$

Then, keep iterating and updating the variable of corresponding quantities until the final state is reached, we will be able to obtain the total time usage t_{total} and final mass m_f. The fuel consumption is $m_1 - m_f$, where m_1 denotes the mass of the initial state.

Final Calculation Use the previous methods, we can finally get the following results.

$$T = 4.134 \times 10^8 [s]$$
$$R_f = 1.34 \times 10^9 [m]$$
$$v_f = 770.6 [m/s]$$
$$m_1 = 4161.6 [kg]$$

Further Approximations Qualitatively speaking, one may notice that when transferring from smaller orbit to a larger orbit, the increment at first would be small and than become larger and larger as radius increases, or reversely, the decrement will become smaller and smaller when the radius gets smaller as transferring from a larger orbit to a smaller one.

Based on this property, we can ignore some later steps in iterations while only calculating the very first steps when transferring from a larger orbit to a smaller one.

To validate this, we take a transfer from $9.93 \times 10^8[m]$ to $7 \times 10^8[m]$, the first optimal step would result in $7.85 \times 10^8[m]$, which covers nearly 71.1% of total radius change.

This result allows us to ignore later steps in a transfer from larger orbits to smaller ones and simplifies calculations.

3.2.5 Move towards Venus

Before calculation, we should clarify something about the initial condition.

In 4.2.4, the final speed v_f is $770.6[m/s]$. However, when we transfer spacecraft-earth system to spacecraft-solar system, we should add the speed of the earth around the sun which is $29783[m/s]$.

So, we can get

$$v_0 = 30552.6[m/s]$$

Similarly, we can conclude that the initial radius of our motion is $R_i + R_{SE}$ where R_{SE} is the distance between the sun and the earth which is $1.496 \times 10^{11}[m]$.

$$R_0 = 1.482 \times 10^{11}[m]$$

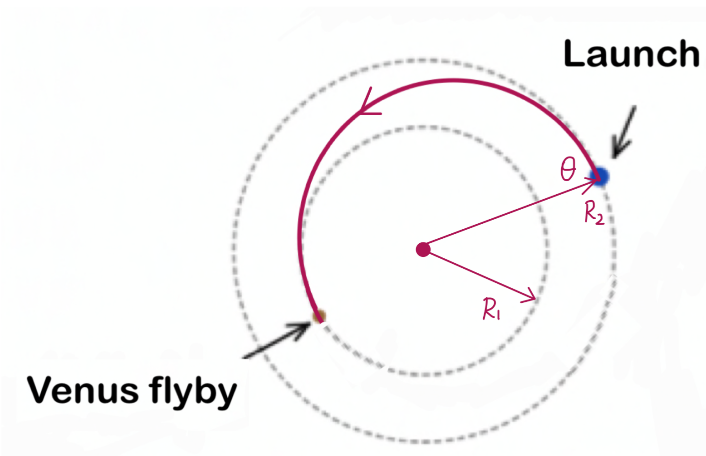

Figure 13: Move Towards Venus

Then, From the Conservation of energy, we can get

$$\frac{1}{2}mv_0^2 - G\frac{M_s m}{R_2} = \frac{1}{2}mv_1^2 - G\frac{M_s m}{R_1}$$

For in this stage, there are just the gravational force add on the spacecraft, we have these equations.

$$v_\theta = v_0 sin(\theta)$$
$$v_r = \sqrt{v_1^2 - v_\theta^2}$$

So that we can finally get

$$v_1 = 30562[m/s]$$
$$\alpha = arctan(\frac{v_r}{v_\theta}) = \frac{2\pi}{3}[rad]$$

During this stage, we won't open the ion thrusters of our spacecraft so that the fuel consumption is 0, and we can futher calculate the time needed.

First, we have the following two equations

$$\frac{d\theta}{dt} = v_0 sin(\theta)$$
$$\frac{d^2r}{dt^2} = G\frac{M_s m}{r^2}$$

So that by solving the difficiential equation, we can get

$$\Delta t = 1.6536 \times 10^6[s]$$

3.2.6 Gravity Assist through Venus

For on 2.5.4 we have detailed discussed Gravity Assist Model and related calcultion, here we just show the initial condition and results.

The initial speed and angle of the spacecraft is

$$v_0 = 30562[m/s]$$
$$\theta_0 = \frac{2\pi}{3}[rad]$$

And from our formular we can finally get

$$v_1 = 41557[m/s]$$
$$\theta_1 = 2.1930[rad]$$

3.2.7 From Venus to Earth

After the first gravity assist, we should go back to earth to execute the second gravity assist.

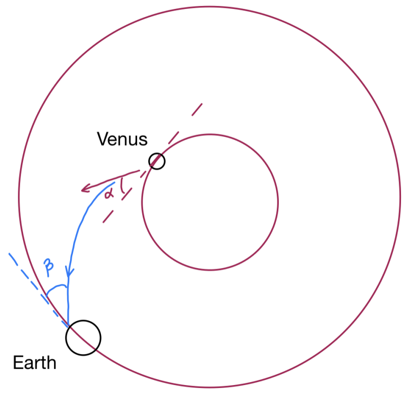

Figure 14: From Venus to Earth

The calculation is similar to 3.2.5 so that we just give equations and initial conditions and directly get final redults. However, this time the angle between the gravitional force of the sun and our moving direction is larger than 90 degree so that we should get new formulars.

For the initial velosity on the solar system is $41557[m/s]$ and initial angle is $2.1930[rad]$.

$$\frac{1}{2}mv_0^2 - G\frac{M_sm}{R_0} = \frac{1}{2}mv_1^2 - G\frac{M_sm}{R_1}$$
$$v_\beta = v_0cos(\alpha)$$
$$v_r = \sqrt{v_1^2 - v_\theta^2}$$

The final value will be

$$v_1 = 40057[m/s]$$
$$\beta = arctan(\frac{v_r}{v_\theta}) = 2.3539[rad]$$
$$\Delta t = 1.9534 \times 10^6[s]$$

3.2.8 Gravity Assist through Earth

Similarly, the initial speed and angle of the spacecraft is

$$v_0 = 40057[m/s]$$
$$\theta_0 = 2.3539[rad]$$

And from our formular we can finally get

$$v_1 = 50389[m/s]$$
$$\theta_1 = 2.3955[rad]$$

3.2.9 Move towards Saturn

After we using the gravity assist to speed up our spacecraft, we can move towards the Saturn. The model and calculation is very similar to 3.2.7.

Suppose that our initial condition is $(R_0, 0)$ with polar coordinates and our final condition is (R_1, α). Moreover, from the previous section, the initial speed is $v_0 = 50.389[m/s]$.

$$\frac{1}{2}mv_0^2 - G\frac{M_s m}{R_0} = \frac{1}{2}mv_1^2 - G\frac{M_s m}{R_1}$$
$$v_\theta = v_0 cos(\alpha)$$
$$v_r = \sqrt{v_1^2 - v_0^2}$$

The final value will be

$$v_1 = 35.679[m/s]$$
$$\alpha = arctan(\frac{v_r}{v_\theta}) = 2.6702[rad]$$
$$\Delta t = 21.31 \times 10^7[s]$$

3.2.10 Enter Saturn's aimed orbit

Finally, after a long journey, we arrive the Saturn, recall that when the spacecraft arrive the effective radius which is $3.15 \times 10^{10}[m]$, we just consider the influence of the Saturn towards the spacecraft.

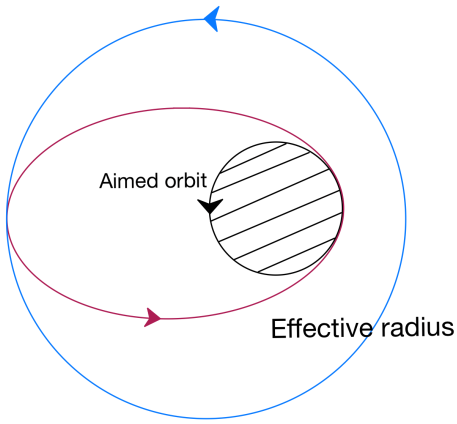

Figure 15: Arrive the Saturn

So that we can use Hohmann Transfer to reach our aimed orbit. Here we should open our ion thrusters again to reach the aimed orbit.

$$\Delta v = \sqrt{\frac{GM}{r_1}}(\sqrt{\frac{2r_2}{r_1 + r_2}} - 1) + \sqrt{\frac{GM}{r_2}}(1 - \sqrt{\frac{2r_1}{r_1 + r_2}})$$

Because the speed $v_{01} = 35.679[m/s]$ is calculated on the solar system, when we change to Saturn system, the speed becomes

$$v_0 = 25989[m/s]$$
$$\Delta v = 2832.5[m/s]$$

Finally, in order to reach our final speed $v_f = 11828.7[m/s]$, we should change our speed again so that the total speed chang can be calculated by

$$\Delta v_{total} = \Delta v + [(v_0 - \Delta v) - v_f] = 14160.3[m/s]$$

Based on the Tsiolkovsky Rocket Equation, we can calculate the final mass of our spaceship and based on Kepler's third law, we can calculate the time that we need to change orbit to our aimed orbit.

$$t_{tran} = \pi\sqrt{\frac{(r+R)^3}{8GM}} = 1.76 \times 10^7[s]$$
$$m_1 = m_0' e^{-\frac{\Delta v}{v_e}} = 2899.875[kg]$$

4 Discussions

4.1 Limitations of Paths' Choosing Criteria

Recall that the criteria includes the three points. First, only less than two times of gravity assist should be considered. This leads directly to an elimination of infinite possibilities. But, in real missions, catching up with the planets need lots of efforts, calculations and, the most importantly, fuel. So, apart from multi-purpose missions, the spacecraft will not undergo a large number of gravity assists.

Second, the spacecraft does not cross the inner orbit of planet. This indicates that the planet with less maximum velocity gain mentioned in chapter 1 of the book [2] by Labunsky during a single gravity assist maneuver won't be considered. As is shown already, the planets of a bigger orbit could contribute more during a single gravity maneuver to a spacecraft. However, in real missions, these planets could be considered for a change of orbit to catch up with the targeted gravity assist maneuver.

Third, the spacecraft will not revisit the planet that has been visited one time before. This makes sense because the mission's goal is to save as much fuel as possible. In real missions, the revisit is possible, but it only becomes possible with the aid of extra fuel comsumption. For example, the revisit by Cassini-Huegens to Venus was done to reajust the orbit's parameters to catch up with Earth and led it to Jupiter and Saturn.

4.2 Piecewise Hohmann Transfer

Advantages The Piecewise Hohmann Transfer Orbit expands the domain where Hohmann Transfer Orbit can be utilized. Multiple Hohmann Transfer is performed in order to achieve a certain target radius. This gives a good approximation and approach to take thruster limit into account but yet trying to minimize fuel cost. The error or refinement of the piecewise condition, which is the definition of short time can be changed according to demands. For example, a finer Piecewise Hohmann Transfer Orbit can be constructed if $0.01T$ is considered as short instead of $0.1T$. Moreover, such piecewise construction makes it very easy to implement with iteration algorithms and greedy algorithms.

Disadvantage Applying a Piecewise Hohmann Transfer Orbit takes longer than simply turns on the thruster for a long time.

Also, time needed to solve such a problem would be quite long due to the large iteration number. And whether the fuel consumption is a global minimum or not is not quite sure since the greedy algorithm doesn't ensure a global minimum.

4.3 Gravity Assist Model

Advantages The Gravity Assist model simplifies the calculation a lot, the time duration can be calculated directly using numerical integration.

Disadvantages The limitations of the gravity assist model mainly lie in the assumptions.

First, for the calculation of velocity, we assume that this time interval is small enough and could be neglected. This assumption is biased because it could be large with respect to a very big velocity travelling passively interpanetarily. In this report, though the Venus flyby only consumes half of a day, it is not so small regarding the 18 days and 6 days consumed commuting between Venus and Earth.

Second, based on the previous discussion, the claim that gravity assist only happens within the so-called effective range is under doubt.

References

[1] P. Moore and R. Rees, *Patric Moore's Data Book of Astronomy*, 2nd edition, 2011, Cambridge University Press, ISBN = 978-0-521-89935-2.

[2] A. V. Labunsky, O. V. Papkov and K. G. Sukhanov, *Muptiple Gravity Assist Interplanetary Trajectories*, Editor, P. Kleber, 1st edition, 1998, Gordon and Breach Science Publishers, ISBN = 90-5699-090-X.

[3] R. C. Johnson, "The Slingshot Effect", Department of Mathematical Sciences, University of Durhan, Jan. 2003.

A Calculation Codes

```
% piecewise Hohmann transfer
clearvars;clc;
EarthEffectiveRadius=9.93e8;
massRate=10000*9.8;
```

```
dvConst=4000*9.8;
r1=6.64933e6;r2=0;
mass=5000;
syms r2sym positive;
mu=6.67259*5.972e13;
time=0;
while r2<EarthEffectiveRadius
dvsym=sqrt(mu/r1)*(sqrt(2*r2sym/(r1+r2sym))-1)+...
        sqrt(mu/r2sym)*(1-sqrt(2*r1/(r1+r2sym)));
period=2*pi/sqrt(mu/r1^3);
dv=-dvConst*log(1-period/(10*mass*massRate));
equation=(dvsym==dv);
assume(r2sym>r1);
r2=double(vpasolve(equation,r2sym,r1));
mass=mass*exp(-dv/dvConst);
time=time+pi*sqrt((r1+r2)^3/(8*mu));
r1=r2;
end

% gravity assist
clearvars;clc;
%Parameters
EffectiveRadius=9.93e8;
%EffectiveRadius=6.67e8; %for venus
v=50.3e3;u=29.8e3;
%v=40359;u=35e3; %for venus
m=4161.6;M=5.972e24;
%m=4161.6;M=4.8967e24; %for venus
G=6.674e-11;
theta=69.41*pi/180;
%theta=pi/4; %for venus
%Solve for L,E
syms L p r positive;
vTransferred=norm([v*cos(theta)-u,v*sin(theta)]);
thetaTransferred=acos((u-v*cos(theta))/(vTransferred));
E=0.5*m*vTransferred^2-G*M*m/EffectiveRadius;
phi=double(vpasolve(sqrt(1+2*E*(EffectiveRadius...
*vTransferred*sin(p-thetaTransferred))^2/(G^2*M^2*m))...
*cos(p)==1-sign(u-v*cos(theta))*vTransferred^2...
*EffectiveRadius/(G*M)*(sin(p-thetaTransferred))^2,...
p,pi/2.9));
L=m*EffectiveRadius*vTransferred*sin(phi-...
        thetaTransferred);
%Solve for others
ecc=sqrt(1+2*E*L^2/(G^2*M^2*m^3));
```

```
rmin=L^2/(G*M*m^2)/(1+ecc);
t=r/sqrt(2*E/m*r^2+2*G*M*r-L^2/m^2);
T=2*double(vpaintegral(t,r,rmin,EffectiveRadius));
vout=(v+2*u)*sqrt(1-4*u*v*(1-cos(pi-theta))/((v+2*u)^2));
thetaOut=atan(v*sin(theta)/(v*cos(theta)+2*u));
```