The science of functional programming

With examples in Scala

by Sergei Winitzki, Ph.D.

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This book presents the theoretical knowledge that helps to write code in the functional programming paradigm. Detailed explanations and derivations are logically developed and accompanied by worked examples tested in the Scala interpreter. Exercises with solutions are provided. Readers need to have a working knowledge of basic Scala (e.g. be able to write code that reads a small text file and prints all word counts, sorted in descending order by count). Readers should also have a basic command of school-level mathematics; for example, be able to compute and simplify $\frac{d}{dx}\left((x+1)\,e^{-x}\right)$.

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TO DO:

Implement scripts for Scala syntax highlighting.

If LyX-Code has `scala>` prepended to its first line, the result of running the code will be shown in the book text.

If LyX-Code has `!scala>` prepended to its first line, the result of running the code will be shown in the book text, and we expect it to be an error.

All other LyX-Code must be run through the compiler, but results are not shown.

If LyX-Code has `///` prepended to it, the code should not be run at all but just shown (maybe with syntax highlighting?).

All LyX-Code fragments must have font "smaller"; also, all inline code fragments must have font "smaller".

Special symbols must be replaced in all LyX-Code fragments: curly quotes by straight quotes; mathematical $\Rightarrow$, $\rightarrow$, $\leftarrow$, as well as ASCII =>,->, <- by the corresponding Unicode symbols.

Syntax highlighting in color must be introduced in all code fragments, both inline and LyX-Code.

1 Values, types, expressions, functions

1.1 Translating mathematics into code

1.1.1 First examples

We begin by writing Scala code for some computational tasks.

Factorial of 10   Compute the product of integers from 1 to 10 (the **factorial** of 10).

First, we write a mathematical formula for the result:

$$\prod_{k=1}^{10} k \quad .$$

We can now write Scala code in a way that resembles the formula:

```
scala> (1 to 10).product
```

The Scala interpreter indicates that the result is the value 3628800 of type `Int`. If we need a name for this value (e.g. to use it later), we use the "`val`" syntax and write

```
scala> val fac10 = (1 to 10).product
scala> assert(fac10 == 3628800)
```

The code `(1 to 10).product` is an **expression**, which means that (1) it can be computed and yields a value, as we have just seen, and (2) the code can be inserted anywhere as part of a larger expression; for example, we could write

```
scala> 100 + (1 to 10).product
```

Factorial as a function   Define a function that takes an integer n and computes the factorial of n.

A mathematical formula for this function can be written as

$$f(n) = \prod_{k=1}^{n} k \quad .$$

The corresponding Scala code is

```scala
def f(n:Int) = (1 to n).product
```

In Scala's `def` syntax, we need to specify the type of a function's argument; in this case, we write `n:Int`. In the usual mathematical notation, types of arguments are either not written at all, or written separately from the formula:

$$f(n) = \prod_{k=1}^{n} k, \quad \forall n \in \mathbb{N} \quad .$$

This indicates that n must be from the set of non-negative integers ($\mathbb{N}$). This is quite similar to specifying the type `Int` in the Scala code. So, the argument's type in the code specifies the mathematical domain of a function.

Having defined the function `f`, we can now apply it to an integer argument:

```scala
scala> f(10)
```

It will be an error to apply `f` to a non-integer value, e.g. to a string:

```scala
!scala> f("abc")
```

1.1.2 Nameless functions

The formula and the code, shown above, both involve *naming* the function as "f". What if we do not need to name that function, for instance, if f will be used only once? Some mathematics books denote nameless functions like this:

$$x \mapsto \text{(some formula)} \quad .$$

So, the mathematical notation for the nameless factorial function could be $n \mapsto \prod_{k=1}^{n} k$. This reads as "a function that maps n to the product of all k where k goes from 1 to n". The Scala expression implementing this mathematical formula is

```
(n: Int) ⇒ (1 to n).product
```

This expression shows Scala's syntax for a nameless function. Scala uses the double arrow symbol ⇒ to separate function arguments from function bodies.[1] In this book, I will also use the symbol ⇒ instead of ↦ in mathematical expressions.

Functions in Scala (whether named or nameless) are treated as values, which means that we can also define a Scala value `fac` as

```
scala> val fac = (n: Int) ⇒ (1 to n).product
```

We see that the value `fac` has the type Int ⇒ Int, which means that the function takes an integer argument and returns an integer result value. The Scala interpreter will print something like

```
/// Lambda$1924/1256837057@363a52f
```

as the "value" of `fac`. This seems merely to indicate a memory location, which is not very illuminating. We can visualize a function value as representing a *block of compiled machine code*, — code that will actually run and evaluate the function body when the function is applied. There is no easy way of printing that block of machine code for us to look at.

Once defined, a function value can be applied to an argument:

```
scala> fac(10)
```

However, functions can be used even without naming them. We can directly apply a nameless factorial function to an integer argument 10:

```
scala> ((n: Int) ⇒ (1 to n).product)(10)
```

One would not usually write code like this, for (1) the syntax is harder to read, and (2) there is no advantage in creating a simple nameless function and applying it right away to an argument. We can replace the expression

```
((n: Int) ⇒ (1 to n).product)(10)
```

by an equivalent expression where $n = 10$ is substituted,

```
(1 to 10).product
```

---

[1] In Scala, the two characters => and the single Unicode symbol ⇒ have the same meaning. I will use the symbol ⇒ in this book.

If a nameless function uses the argument several times, for example

```
((n: Int) ⇒ n*n*n + n*n)(12345)
```

it is still better to substitute the argument and to eliminate the nameless function. We could have written

```
12345*12345*12345 + 12345*12345
```

but we would of course like to avoid repeating the value 12345. To achieve that, we may define n as a local value in an expression block:

```
{
  val n = 12345
  n*n*n + n*n
}
```

Nameless functions are most useful when they are themselves arguments of other functions, as we will see next.

Checking integers for being prime   Define a function that takes an integer n and determines whether n is a prime.

A simple mathematical formula for this function can be written as

$$\text{is\_prime}\,(n) = \forall k \in [2, n-1] \,:\, n \neq 0 \mod k \quad . \tag{1.1}$$

This formula has two clearly separated parts: first, a range of integers from 2 to $n-1$, and second, a requirement that all these integers should satisfy a given condition, $n \neq 0 \mod k$.

This mathematical expression is translated into the Scala code as

```
def is_prime(n: Int) = (2 to n-1).forall(k ⇒ n % k != 0)
```

In this code, the two parts of the mathematical formula are implemented in a way that is closely similar to the mathematical notation, except for the arrow after k.

We can now apply the function is_prime to some integer values:

```
scala> is_prime(12)
scala> is_prime(13)
```

As we can see, the function returns a value of type `Boolean`. Therefore, `is_prime` has type `Int ⇒ Boolean`.

A function that returns a Boolean value is called a **predicate**.

In Scala, it is optional—but strongly recommended—to specify the return type of named functions. The required syntax looks like this,

```
def is_prime(n: Int): Boolean = (2 to n-1).forall(k ⇒ n % k != 0)
```

However, we do not need to specify the type `Int` for the argument `k` of the nameless function `k ⇒ n % k != 0`. This is because the Scala compiler knows that `k` is going to iterate over the integer elements of the range `(2 to n-1)`, which effectively forces `k` to be of type `Int`.

1.1.3 Nameless functions and bound variables

The Scala code for `is_prime` differs from the mathematical formula (1.1) in two ways.

One difference is that the interval $[2, n-1]$ comes first in the Scala expression. To understand this, look at the ways Scala allows programmers to define syntax.

The Scala syntax such as `(2 to n-1).forall(k ⇒ ...)` means to apply a function called `forall` to *two* arguments: the range `(2 to n-1)` and the nameless function `(k ⇒ ...)`. In Scala, the infix syntax `x.f(z)`, or equivalently `x f z`, means that a function `f` is applied to its *two* arguments, `x` and `z`. In the non-infix mathematical notation, this would be $f(x, z)$. Infix notation is often easier to read and is also widely used in mathematics, for instance when we write $x + y$ rather than something like $plus(x, y)$.

A single-argument function could be also defined with infix notation, and then the syntax is `x.f`, as in the expression `(1 to n).product` we have seen before.

The infix methods `.product` and `.forall` are already provided in the Scala standard library, so it is natural to use them. If we did not want to use the infix syntax, we would instead have to define a function `for_all` with two arguments and write code like this,

```
/// for_all(2 to n-1, k ⇒ n % k != 0)
```

This brings the Scala syntax somewhat closer to the formula (1.1).

However, there still remains the second difference: The symbol k is used as an *argument* of a nameless function `k ⇒ n % k != 0` in the Scala code, — while the

mathematical notation, such as

$$\forall k \in [2, n-1] \ : \ n \neq 0 \mod k \quad ,$$

does not seem to involve any nameless functions. Instead, the mathematical formula defines the symbol k that "goes over a certain set," as we might say. The symbol k is then used for writing the predicate $n \neq 0 \mod k$. However, let us investigate the role of the symbol k more closely.

The symbol k is a mathematical variable that is actually defined *only inside* the expression "$\forall k : ...$" and makes no sense outside that expression. This becomes clear by looking at Eq. (1.1): The variable k is not present in the left-hand side and could not possibly be used there. The name "k" is defined only in the right-hand side, where it is first mentioned as the arbitrary element $k \in [2, n-1]$ and then used in the sub-expression "$... \mod k$".

So, the mathematical notation

$$\forall k \in [2, n-1] \ : \ n \neq 0 \mod k$$

gives two pieces of information: first, that we are examining all values from the given range; second, that we choose the name k for the values from the given range, and for each of those k we need to evaluate the expression $n \neq 0 \mod k$, which is a certain given *function* of k that returns a `Boolean` value. Translating the mathematical notation into code, it is therefore natural to write that function of k as a nameless function, *explicitly*:

$$k \Rightarrow n \neq 0 \mod k \quad ,$$

and to write Scala code that applies this nameless function to each element of the range $[2, n-1]$ and then requires that all result values be `true`. The Scala code is therefore

```
(2 to n-1).forall(k ⇒ n % k != 0)
```

Just as the mathematical notation defines the variable k only on the right-hand side of Eq. (1.1), the argument `k` of the nameless Scala function `k ⇒ n % k != 0` is defined only within that function's body and cannot be used in any code outside the expression `n % k != 0`.

Variables that are defined only inside an expression and are invisible outside are called **bound variables**, or "variables bound in an expression". Variables that are used in an expression but are defined outside it are called **free variables**, or

"variables occurring free in an expression". These concepts apply equally well to mathematical formulas and to Scala code. For example, in the mathematical expression $k \Rightarrow n \neq 0 \mod k$ (which is a nameless function), the variable k is bound (because it is named and defined as the argument of the nameless function, which happens in this expression) but the variable n is free (it is defined outside this expression).

The main difference between free and bound variables is that bound variables can be renamed at will, unlike free variables. To see this, consider that we could rename k to x and write instead of Eq. (1.1) an equivalent definition

$$\text{is\_prime}\,(n) = \forall x \in [2, n-1] \; : \; n \neq 0 \mod x \quad ,$$

or in Scala code,

```scala
(2 to n-1).forall(x ⇒ n % x != 0)
```

This renaming does not change the results. In the nameless function `k ⇒ n % k != 0`, the variable k is bound and thus may be renamed to x or to anything else, without changing the value of the expression. No code outside this expression needs to be changed after renaming k to x. But the variable n is free and cannot be renamed at will. If, for any reason, we wanted to rename n in the expression `k ⇒ n % k != 0`, we would also need to change some code *outside* this expression, or else the program would become incorrect.

Mathematical formulas have bound variables in various constructions such as $\forall k : p(k)$, $\exists k : p(k)$, $\sum_{k=a}^{b} f(k)$, $\int_0^1 k^2 dk$, $\lim_{n \to \infty} f(n)$, $\text{argmax}_k f(k)$, and so on. When we translate mathematical expressions into code, we need to recognize the presence of bound variables, which the mathematical notation does not make quite so explicit. For each bound variable, we need to create a nameless function whose argument is that variable, e.g. $k \Rightarrow p(k)$ or $k \Rightarrow f(k)$ for the examples just shown. Only then will our code correctly reproduce the behavior of bound variables in mathematical expressions.

As an example, the mathematical formula

$$\forall k \in [1, n] : p(k) \quad ,$$

(where p is a given predicate) has a bound variable k and is translated into Scala code as

```scala
(1 to n).forall(k ⇒ p(k))
```

At this point we can apply the following simplification trick to this code. The nameless function $k \Rightarrow p(k)$ does exactly the same thing as the (named) function p: It takes an argument, which we may call k, and returns $p(k)$. So, we can simplify the Scala code above to

```
(1 to n).forall(p)
```

The simplification of $x \Rightarrow f(x)$ to just f is always possible for functions f of a single argument.[2]

1.1.4 Aggregating data from arrays

Consider the task of finding how many even numbers there are in a given list L of integers. For example, the list $[1, 2, 3, 4, 5]$ contains *two* even numbers: 2 and 4.

A mathematical formula for this task can be written like this,

$$\text{count\_even}\,(L) = \sum_{k \in L} \text{is\_even}\,(k) \quad ,$$

$$\text{is\_even}\,(k) = \begin{cases} 1 & \text{if } k = 0 \mod 2 \\ 0 & \text{otherwise} \end{cases} .$$

Here we defined a helper function `is_even` in order to write more easily a formula for `count_even`. In mathematics, complicated formulas are often split into simpler parts by defining helper expressions.

We can write the Scala code similarly. We first define the helper function `is_even`; the Scala code can be written in the style quite similar to the mathematical formula:

```
def is_even(k: Int): Int = (k % 2) match {
  case 1 ⇒  1
  case _ ⇒  0
}
```

For such a simple computation, we could also write a shorter code and use a nameless function,

---

[2]The flexibility of Scala syntax makes it possible to write code that looks like `f(x)` but actually involves additional implicit or default arguments to the function `f`, or an implicit conversion for its argument `x`. In those cases, replacing `x⇒f(x)` by `f` may fail to compile in Scala. But these complications do not arise when reasoning about simple functions on integers.

```
val is_even = (k: Int) ⇒ if (k % 2 == 0) 1 else 0
```

Given this function, we now need to translate into Scala code the expression $\sum_{k \in L}$ is_even (k). We can represent the list L using the class `List[Int]` from the Scala standard library.

To compute $\sum_{k \in L}$ is_even (k), we must apply the function `is_even` to each element of the list L, which will produce a list of some (integer) results, and then we will need to add all those results together. It is convenient to perform these two steps separately. This can be done with the functions `.map` and `.sum`, defined in the Scala standard library as infix methods on the class `List`.

The method `.sum` is defined for any `List` of numerical types (`Int`, `Float`, `Double`, etc.) and computes the sum of all numbers in the list. For example,

```
scala> List(1, 2, 3).sum
```

We have seen a similar method `.product` before.

The method `.map` needs more explanation. This method takes a *function* as its second argument, applies that function to each element of the list, and puts all the results into a *new* list, which is then returned as the result value:

```
scala> List(1, 2, 3).map(x ⇒ x*x + 10*x)
```

In this example, we used the nameless function $x \Rightarrow x^2 + 10x$ as the argument of `.map()`. This function is repeatedly applied by `.map` to transform the values from a given list, creating a new list as a result.

It is equally possible to define the transforming function separately, give it a name, and then pass it as the argument to `.map`:

```
scala> def func1(x: Int): Int = x*x + 10*x
scala> List(1, 2, 3).map(func1)
```

An infix method, such as `.map`, can be also used in a shorter syntax:

```
scala> List(1, 2, 3) map func1
```

If the transforming function `func1` is used only once, and especially for a simple function such as $x \Rightarrow x^2 + 10x$, it is easier to write code with a nameless function.

It is now clear how to use the methods `.map` and `.sum` to define `count_even`. The code looks like this:

```
def count_even(s: List[Int]) = s.map(is_even).sum
```

This code can be also written using a nameless function instead of `is_even`:

```
def count_even(s: List[Int]): Int =
  s
    .map { k ⇒ if (k % 2 == 0) 1 else 0 }
    .sum
```

It is customary to use infix methods to chain several operations, for instance `s.map(...).sum` means first apply `s.map(...)`, which returns a new list, and then apply `.sum` to that list. To make the code more readable, I put each of the chained methods on a new line.

1.1.5 Filtering

The Scala standard library defines several useful methods for lists. In addition to the methods `.sum`, `.product`, `.map`, `.forall` that we have already seen, there are methods `.max`, `.min`, `.exists`, `.size`, `.filter`, `.takeWhile`, and many others.

The methods `.max`, `.min`, and `.size` are self-explanatory:

```
scala> List(10, 20, 30).max
scala> List(10, 20, 30).min
scala> List(10, 20, 30).size
```

The methods `.forall`, `.exists`, `.filter`, and `.takeWhile` require a predicate as an argument. The `.forall` method returns `true` iff the predicate is true on all values in the list; the `.exists` method returns `true` iff the predicate is true on at least one value in the list. These methods can be written as mathematical formulas like this:

$$\text{forall}\,(S, p) = \forall k \in S : p(k) = \text{true}$$
$$\text{exists}\,(S, p) = \exists k \in S : p(k) = \text{true}$$

However, mathematical notation does not exist for operations such as "finding elements in a list", so we will now focus on the Scala syntax for these operations.

The `.filter` method returns a *new list* that contains only the values for which the predicate returns `true`:

```
scala> List(1, 2, 3, 4, 5).filter(k ⇒ k % 3 != 0)
```

The `.takeWhile` method returns a *new list* that contains the initial subset of values from the original list, which are taken until the predicate first returns `false`:

```scala
scala> List(1, 2, 3, 4, 5).takeWhile(k ⇒ k % 3 != 0)
```

In all these cases, the predicate's argument must be of the same type as the elements in the list. In the examples shown above, lists have integer elements (i.e. the lists have type `List[Int]`), therefore the predicate's argument `k` must be of type `Int`.

The methods `.max`, `.min`, `.sum`, and `.product` are defined on lists of *numeric types*, such as `Int`, `Double`, and so on. The other methods are defined on lists of all types.

Using these methods, we can solve many problems that involve transforming and aggregating data stored in lists (as well as in arrays, sets, or other similar data structures). By **aggregation** I mean a function from a list of values to a single value; examples of aggregation functions are `.max` and `.sum`. By **transformation** I here mean a function from a list of values to another list of values; examples of transformation functions are `.filter` and `.map`.

Writing programs by chaining together various operations of transformation and aggregation is known as programming in the **map/reduce style**.

1.1.6 Curried functions

Consider a function with type signature `Int ⇒ (Int ⇒ Int)`. This is a function that takes an integer and returns a *function* that again takes an integer and then returns an integer. So, we obtain an integer result only after we apply the function to *two* integer values, one after another. This is, in a sense, equivalent to a function having two arguments, except the application of the function needs to be done in two steps, applying it to one argument at a time. Functions of this sort are called **curried** functions.

One way of defining such a function is

```scala
def f0(x: Int): Int ⇒ Int = { y ⇒ x - y }
```

The function takes an integer argument `x` and returns the expression `y ⇒ x - y`, which is a function of type `Int ⇒ Int`. Thus, the type of `f0` is `Int ⇒ (Int ⇒ Int)`.

To use `f0`, we apply it to an integer value. The result is a value of function type:

```
scala> val r = f0(20)
```

The value `r` can be now applied to another integer argument:

```
scala> r(4)
```

In Scala, `Int ⇒ Int ⇒ Int` means the same as `Int ⇒ (Int ⇒ Int)` and `x ⇒ y ⇒ x - y` means the same as `x ⇒ (y ⇒ x - y)`. In other words, the function symbol ⇒ associates to the right. Thus, the type signature of f0 may be equivalently written as `Int ⇒ Int ⇒ Int`.
  An equivalent way of defining a function with the same type signature is

```
val f1: Int ⇒ Int ⇒ Int = x ⇒ y ⇒ x - y
```

We can also define a function that takes its two arguments at once. Such a function will have a different type signature, for instance `(Int, Int) ⇒ Int` as in this example:

```
def f2(x: Int, y: Int): Int = x - y
```

The syntax for calling the functions `f1` and `f2` is different:

```
scala> f1(20)(4)
scala> f2(20, 4)
```

The main difference between the usage of `f1` and `f2` is that `f2` must be applied at once to both arguments, while `f1(20)` can be evaluated separately, with only the first argument, 20, the result being a *function* that can be later applied to another argument:

```
scala> val r = f1(20)
scala> r(4)
```

Applying a curried function to some but not all of possible arguments is called **partial application**.
  More generally, a curried function may have a type signature of the form $A \Rightarrow B \Rightarrow C \Rightarrow ... \Rightarrow P \Rightarrow Z$, where A, B, C, ..., Z are some types. Arguments of types A, B, ..., P are sometimes called the **curried arguments** of the function. This function is, in a sense, equivalent to an **uncurried** function with type signature $(A, B, C, ..., P) \Rightarrow Z$. The uncurried function takes all arguments at once.

The equivalence of curried and uncurried functions is not equality — these functions are *different*; but one of them can be easily reconstructed from the other if necessary. I call such functions **isomorphic**.

From the point of view of programming language theory, curried functions are simpler because they always have a *single* argument (and may return a function that will consume further arguments). From the point of view of programming practice, curried functions are not often used and may be harder to read.

In the short notation (used e.g. in OCaml and Haskell), a curried function such as `f2` is applied to its arguments as `f2 20 4`. This departs further from the mathematical notation and requires some getting used to. If the two arguments are more complicated than just 20 and 4, the resulting expression may become significantly harder to read, since no commas are used to separate the arguments. For this reason, programmers may prefer to first define short names for complicated expressions and then use these names as curried arguments.

In Scala, the choice of whether to use curried or uncurried function signatures is largely a matter of syntactic convenience. Most Scala code tends to be written with uncurried functions, while curried functions are used when they produce more easily readable code. One of the syntactic features for curried functions in Scala is the ability to supply a curried argument using the curly brace syntax. Compare the two definitions of the function `summation` described earlier:

```
def summation1(a: Int, b: Int, g: Int ⇒ Int): Int =
  (a to b).map(g).sum
def summation2(a: Int, b: Int)(g: Int ⇒ Int): int =
  (a to b).map(g).sum
summation1(1, 10, x ⇒ x*x*x + 2*x)
summation2(1, 10) { x ⇒ x*x*x + 2*x }
```

The code for `summation2` may be easier to read because the curried argument is syntactically separated from the rest of the code by curly braces. This is especially useful when the curried argument is itself a function, since the Scala curly braces syntax allows function bodies to contain their own local definitions (`val` or `def`).

1.1.7 Calculations with nameless functions

We now need to gain experience working with nameless functions.

In mathematics, functions are evaluated by substituting their argument values into their body. Nameless functions are evaluated in the same way. For example,

applying the nameless function $x \Rightarrow x + 10$ to an integer 2, we substitute 2 instead of x in "$x + 10$" and get "$2 + 10$", which we then evaluate to 12. The computation is written like this,

$$(x \Rightarrow x + 10)(2) = 2 + 10 = 12 \quad .$$

Nameless functions are *values* and can be used as part of larger expressions, just as any other values. For instance, nameless functions can be arguments of other functions (nameless or not). Here is an example of applying a nameless function $f \Rightarrow f(2)$ to a nameless function $x \Rightarrow x + 4$:

$$(f \Rightarrow f(2))\,(x \Rightarrow x + 4) = (x \Rightarrow x + 4)(2) = 6 \quad .$$

In the nameless function $f \Rightarrow f(2)$, the argument f has to be itself a function, otherwise the expression $f(2)$ would make no sense. In this example, f must have type Int $\Rightarrow$ Int.

There are some standard conventions for reducing the number of parentheses when writing expressions involving nameless functions, especially curried functions:

- Function expressions group everything to the right:
  $x \Rightarrow y \Rightarrow z \Rightarrow e$ means the same as $x \Rightarrow (y \Rightarrow (z \Rightarrow e))$.

- Function applications group everything to the left:
  $f(x)(y)(z)$ means $((f(x))(y))(z)$.

- Function applications group stronger than infix operations:
  $x + f(y)$ means $x + (f(y))$, just like in mathematics.

To specify the type of the argument, I will use the underlined superscript, for example: $x^{\underline{\text{Int}}} \Rightarrow x + 2$.

The convention of grouping functions to the right reduces the number of parentheses for curried function types used most often. It is rare to find use for function types such as $((a \Rightarrow b) \Rightarrow c) \Rightarrow d$ that require many parentheses with this convention.

Here are some more examples of performing function applications symbolically. I will omit types for brevity, since every non-function value is of type Int in these examples.

$$\left(x^{\underline{\text{Int}}} \Rightarrow x * 2\right)(10) = 10 * 2 = 20 \quad .$$
$$(p \Rightarrow z \Rightarrow z * p)\,(t) = (z \Rightarrow z * t) \quad .$$
$$(p \Rightarrow z \Rightarrow z * p)\,(t)(4) = (z \Rightarrow z * t)(4) = 4 * t \quad .$$

Some results of these computation are integer values such as 20; in other cases, results are *function values* such as $z \Rightarrow z * t$.

In the following examples, some function arguments are themselves functions:

$$(f \Rightarrow p \Rightarrow f(p))\,(g \Rightarrow g(2)) = (p \Rightarrow p(2)) \quad .$$
$$(f \Rightarrow p \Rightarrow f(p))\,(g \Rightarrow g(2))\,(x \Rightarrow x + 4) = (p \Rightarrow p(2))\,(x \Rightarrow x + 4)$$
$$= 2 + 4 = 6 \quad .$$

Here I have been performing calculations step by step, as usual in mathematics. A Scala program is evaluated in a similar way at run time.

1.1.8 Short syntax for function applications

In mathematics, function applications are sometimes written without parentheses, for instance $\cos x$ or $\arg z$. There are also cases where formulas such as $\sin 2x = 2 \sin x \cos x$ imply parentheses as $\sin(2x) = 2 \cdot \sin(x) \cdot \cos(x)$.

Many programming languages (such as ML, OCaml, F#, Haskell, Elm, Pure-Script) have adopted this "short syntax", in which parentheses are optional for function arguments. The result is a concise notation where `f x` means the same as `f(x)`. Parentheses are still used where necessary to avoid ambiguity or for readability.[3]

The conventions for nameless functions in the short syntax become:

- Function expressions group everything to the right:
  $x \Rightarrow y \Rightarrow z \Rightarrow e$ means $x \Rightarrow (y \Rightarrow (z \Rightarrow e))$.

- Function applications group everything to the left:
  $f\,x\,y\,z$ means $((f\,x)\,y)\,z$.

- Function applications group stronger than infix operations: $x + f\,y$ means $x + (f\,y)$, just like in mathematics $x + \cos y$ groups $\cos y$ stronger than the infix "+" operation.

So, $x \Rightarrow y \Rightarrow a\,b\,c + p\,q$ means $x \Rightarrow (y \Rightarrow ((a\,b)\,c) + (p\,q))$. When this notation becomes hard to read correctly, one needs to add parentheses, e.g. to write $f(x \Rightarrow g\,h)$ instead of $f\,x \Rightarrow g\,h$.

---

[3]The no-parentheses syntax is also used in Unix shell commands, for example `cp file1 file2`, as well as in the programming language Tcl. In LISP and Scheme, each function application is enclosed in parentheses but the arguments are separated by spaces, for example `(+ 1 2 3)`.

In this book, I will sometimes use this "short syntax" when reasoning about code. Scala does not support the short syntax; in Scala, parentheses need to be put around every curried argument. The infix method syntax such as `List(1,2,3) map func1` does not apply to curried functions in Scala.

1.1.9 Higher-order functions

The **order** of a function is the number of function arrows "⇒" contained in the type signature of that function. If a function's type signature contains more than one function arrow, the function is called a **higher-order** function. A higher-order function takes a function as argument and/or returns a function as its result value.

Examples:

```
def f1(x: Int): Int = x + 10
```

The function `f1` has type signature Int ⇒ Int and order 1, so it is *not* a higher-order function.

```
def f2(x: Int): Int ⇒ Int = z ⇒ z + x
```

The function `f2` has type signature Int ⇒ Int ⇒ Int and is a higher-order function, of order 2.

```
def f3(g: Int ⇒ Int): Int = g(123)
```

The function `f3` has type signature (Int ⇒ Int) ⇒ Int and is a higher-order function of order 2.

Although `f2` is a higher-order function, its higher-orderness comes from the fact that the return value is of function type. An equivalent computation can be performed by an uncurried function that is not higher-order:

```
scala> def f2u(x: Int, z: Int): Int = z + x
```

The Scala library defines methods to transform between curried and uncurried functions:

```
scala> def f2u(x: Int, z: Int): Int = z + x
scala> val f2c = (f2u _).curried
scala> val f2u1 = Function.uncurried(f2c)
```

The syntax (`f2u _`) is used in Scala to convert methods to function values. Recall that Scala has two ways of defining a function: one as a method (using `def`), another as a function value (using `val`).

The methods `.curried` and `.uncurried` can be easily implemented in Scala code, as we will see in the worked examples.

Unlike `f2`, the function `f3` cannot be converted to a non-higher-order function because `f3` has an *argument* of function type, rather than a return value of function type. Converting to an uncurried form cannot eliminate an argument of function type.

1.1.10 Worked examples: transformation and aggregation

1. Compute $\prod_{k \in [1,10]} |\sin (k + 2)|$.

   ```scala
   scala> (1 to 10)
     .map(k ⇒ math.abs(math.sin(k + 2)))
     .product
   ```

2. Compute $\sum_{k \in [1,10]; \cos k > 0} \sqrt{\cos k}$.

   ```scala
   scala> (1 to 10)
     .filter(k ⇒ math.cos(k) > 0)
     .map(k ⇒ math.sqrt(math.cos(k)))
     .sum
   ```

   It is safe to compute $\sqrt{\cos k}$, because we have first filtered the list by keeping only values k for which $\cos k > 0$:

   ```scala
   scala> (1 to 10).filter(k ⇒ math.cos(k) > 0)
   ```

3. Compute the average of a list of numbers of type `Double` (assuming that the list is not empty).

   ```scala
   def average(s: List[Double]): Double = s.sum / s.size
   scala> average(List(1.0, 2.0, 3.0))
   ```

4. Given n, compute the Wallis product truncated up to $\frac{2n}{2n+1}$:

$$\text{wallis} (n) = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \frac{6}{5} \frac{6}{7} ... \frac{2n}{2n + 1} \quad .$$

We create a list of `Double` numbers from integers, using the method `.toDouble`:

```
def wallis_frac(i: Int): Double =
    (2*i).toDouble / (2*i - 1)*(2*i)/(2*i + 1)
def wallis(n: Int) = (1 to n).map(wallis_frac).product
scala> math.cos(wallis(10000)) // Should be close to 0.
```

The limit of the Wallis product is $\frac{\pi}{2}$, so the cosine of wallis (n) is close to zero for large n.

5. Another well-known series related to π is

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad .$$

Define a function of n that computes the partial sum of this series from $k = 1$ to $k = n$. Compute the result for a large value of n and compare with the limit value.

```
def euler_series(n: Int): Double =
    (1 to n).map(k ⇒ 1.0/k/k).sum
scala> euler_series(10000)
scala> val pi = math.atan(1)
scala> pi*pi/6
```

6. Check numerically the infinite product formula

$$\prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) = \frac{\sin \pi x}{\pi x} \quad .$$

We will compute the product up to $k = n$ for a given n and compare with the right-hand side:

```
def sine_product(n: Int, x: Double): Double =
    (1 to n).map(k ⇒ 1.0 - x*x/k/k).product
scala> sine_product(1000, 0.1)
scala> math.sin(pi*0.1)/pi/0.1
```

7. Define a function p that takes a list of integers and a function $f : \text{Int} \Rightarrow \text{Int}$, and returns the largest value of $f(x)$ among all x in the list.

```
def p(s: List[Int], f: Int ⇒ Int): Int = s.map(f).max
```

8. Given a list of lists, `s: List[List[Int]]`, select the inner lists of size at least 3. The result must be again of type `List[List[Int]]`.

   To select the inner lists satisfying a condition means to compute a new list containing only the desired inner lists. We use `.filter` on the outer list `s`. The predicate for the filter is a function that takes an inner list and returns `true` if the size of that list is at least 3. Write the predicate as a nameless function, `t ⇒ t.size >= 3`. Then the solution is

   ```
   def f(s: List[List[Int]]): List[List[Int]] =
       s.filter(t ⇒ t.size >= 3)
   scala> f(List( List(1,2), List(1,2,3), List(1,2,3,4) ))
   ```

   The predicate in the argument of `.filter` is a function `t ⇒ t.size >= 3` whose argument `t` is of type `List[Int]`.

9. Find all integers $k \in [1, 10]$ such that there are at least two different integers j, where $1 \leq j \leq k$, each j satisfying the condition $j^2 > k$.

   ```
   scala> (1 to 10)
       .filter(k ⇒ (1 to k).filter(j ⇒ j*j > k).size >= 1)
   ```

   The argument of the outer `.filter` is a nameless function that itself uses a `.filter` to compute the list of j's that satisfy the condition, and then to compute the length of that list.

1.1.11 Worked examples: nameless functions

1. Using both `def` and `val`, define a function that...

   a) ...adds 20 to its integer argument.

      ```
      def fa(i: Int): Int = i + 20
      val fa_v: (Int ⇒ Int) = k ⇒ k + 20
      ```

      It is not necessary to specify the type of the argument `k` because we already fully specified the type (`Int ⇒ Int`) of `fa_v`. The parentheses around the type of `fa_v` are optional, I added them for clarity.

   b) ...takes an integer x, and returns a *function* that adds x to *its* argument.

      ```
      def fb(x: Int): (Int ⇒ Int) = k ⇒ k + x
      val fb_v: (Int ⇒ Int ⇒ Int) = x ⇒ k ⇒ k + x
      def fb_v2(x: Int)(k: Int): Int = k + x
      ```

Since functions are values, we can directly return new functions. When defining the right-hand sides as function expressions in `fb` and `fb_v`, it is not necessary to specify the type of the arguments x and k because we already fully specified the type signatures of `fb` and `fb_v`. The last version, `fb_v2`, may be easier to read and is equivalent to `fb_v`.

c) ...takes an integer x and returns `true` iff $x + 1$ is a prime. Use the function `is_prime` defined previously.

```
def fc(x: Int): Boolean = is_prime(x + 1)
val fc_v: (Int ⇒ Boolean) = x ⇒ is_prime(x + 1)
```

d) ...returns its integer argument unchanged. (This is called the **identity function** for integer type.)

```
def fd(i: Int): Int = i
val fd_v: (Int ⇒ Int) = k ⇒ k
```

e) ...takes x and always returns 123, ignoring its argument x. (This is called a **constant function**.)

```
def fe(x: Int): Int = 123
val fe_v: (Int ⇒ Int) = x ⇒ 123
```

To emphasize the fact that the argument x is ignored, use the special syntax where x is replaced by the underscore:

```
val fe_v1: (Int ⇒ Int) = _ ⇒ 123
```

f) ...takes x and returns a constant function that always returns the fixed value x. (This is called the **constant combinator**.)

```
def ff(x: Int): Int ⇒ Int = _ ⇒ x
val ff_v: (Int ⇒ Int ⇒ Int) = x ⇒ _ ⇒ x
def ff_v2(x: Int)(y: Int): Int = x
```

The syntax of `ff_v2` may be easier to read, but then we cannot omit the name y for the unused argument.

2. Define a function `comp` that takes two functions f :Int ⇒ Double and g :Double ⇒ String as arguments, and returns a new function that computes $g(f(x))$. What is the type of the function `comp`?

```
def comp(f: Int ⇒ Double, g: Double ⇒ String): (Int ⇒ String) =
  x ⇒ g(f(x))
scala> val f: Int ⇒ Double = x ⇒ 5.67 + x
```

```
scala> val g: Double ⇒ String = x ⇒ f"x=%3.2f"
scala> val h = comp(f, g)
scala> h(10)
```

The function comp has two arguments, of types Int ⇒ Double and Double ⇒ String. The result value of comp is of type Int ⇒ String, because comp returns a new function that takes an argument *x* of type Int and returns a String. So the full type signature of the function comp is written as

```
/// (Int ⇒ Double, Double ⇒ String) ⇒ (Int ⇒ String)
```

This is an example of a function that both takes other functions as arguments *and* returns a new function.

3. Define a function uncurry2 that takes a curried function of type Int ⇒ Int ⇒ Int and returns an uncurried equivalent function of type (Int, Int) ⇒ Int.

```
def uncurry2(f: Int ⇒ Int ⇒ Int): (Int, Int) ⇒ Int =
  (x, y) ⇒ f(x)(y)
```

1.2 Summary

This table translates mathematical formulas into code.

Mathematical notation	Scala code
$x \mapsto \sqrt{x^2 + 1}$	`x ⇒ math.sqrt(x * x + 1)`
list $[1, 2, ..., n]$	`(1 to n)`
list $[f(1), ..., f(n)]$	`(1 to n).map(k ⇒ f(k))`
$\sum_{k=1}^{n} k^2$	`(1 to n).map(k ⇒ k*k).sum`
$\prod_{k=1}^{n} f(k)$	`(1 to n).map(f).product`
$\forall k$ such that $1 \le k \le n : p(k)$ holds	`(1 to n).forall(k ⇒ p(k))`
$\exists k, 1 \le k \le n$ such that $p(k)$ holds	`(1 to n).exists(k ⇒ p(k))`
$\sum_{k \in S: \, p(k) \, \text{holds}} f(k)$	`s.filter(p).map(f).sum`

What problems can we solve with this knowledge?

- Compute mathematical expressions involving sums, products, and quantifiers, based on integer ranges, such as $\sum_{k=1}^{n} f(k)$ etc.

- Transform and aggregate data from lists using `.map`, `.filter`, `.sum`, and other methods from the Scala standard library.

- Define higher-order functions, that is, functions that take other functions as arguments and/or return new functions.

What are examples of problems that are not solvable with these tools?

- Example 1: Compute the smallest $n \geq 1$ such that

$$f(f(f(...f(0)...)) > 1000 \quad,$$

where the given function f is applied n times.

- Example 2: Given a list s of numbers, compute the list r of running averages: $r_n = \sum_{k=0}^{n} s_k$.

- Example 3: Perform binary search over a sorted array.

These problems are not yet solvable because the corresponding formulas involve *mathematical induction*, which we cannot yet translate into code in the general case.

Library functions we have seen so far, such as `.map` and `.filter`, implement a restricted class of iterative operations on lists: namely, operations that process each element of a given list independently. For instance, when computing `s.map(f)`, the number of function applications is given by the size of the initial list. However, Example 1 requires applying a function f repeatedly to previous values until a given condition holds—that is, an *initially unknown* number of times. So it is impossible to write an expression containing `.map`, `.filter`, `.takeWhile`, etc., that solves Example 1. We need mathematical induction to write the solution of Example 1 as a formula.

Similarly, Example 2 defines a new list r from s by induction,

$$r_0 = s_0; \quad r_i = s_i + r_{i-1}, \forall i > 0 \quad.$$

However, operations such as `.map` and `.filter` cannot compute r_i depending on r_{i-1}.

Example 3 defines the search result by induction and again requires an initially unknown number of steps. Chapter 2 explains how to solve these problems by translating mathematical induction into code using recursion.

1.3 Exercises

1. Define a function of type `List[Double]` ⇒ `List[Double]` that "normalizes" the list: finds the element having the largest absolute value and, if that value is nonzero, divides all elements by that factor and returns a new list; otherwise returns the original list.

2. Machin's formula gives a way of computing π more efficiently than the Wallis fraction:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239} \quad,$$

$$\arctan\frac{1}{n} = \frac{1}{n} - \frac{1}{3}\frac{1}{n^3} + \frac{1}{5}\frac{1}{n^5} - ...$$

Implement a function that computes the series for $\arctan\frac{1}{n}$ up to a given number of terms, and compute an approximation of π using this formula. Show that about 12 terms of the series are already sufficient for a full-precision `Double` approximation of π.

3. Check numerically the Euler product formula for the Riemann zeta function $\zeta(s)$ at $s = 4$, where it is known that $\zeta(4) = \frac{\pi^4}{90}$:

$$\prod_{k\in\mathbb{N};k\text{ is prime}} \frac{1}{1-p^{-4}} = \frac{\pi^4}{90} \quad.$$

Use the function `is_prime` defined earlier.

4. Define a function `add_20` of type `List[List[Int]]` ⇒ `List[List[Int]]` that adds 20 to every element of every inner list. A test:

```
/// add_20( List( List(1), List(2, 3) ) )
///    == List( List(21), List(22, 23) )
```

5. An integer n is called a "3-factor" if it is divisible by only three different integers j such that $2 \le j < n$. Compute the set of all "3-factor" integers n among $n \in [1, ..., 1000]$.

6. Given a function f : Int ⇒ Boolean, an integer n is called a "3-f" if there are only three different integers $j \in [1, ..., n]$ such that $f(j)$ returns true.

Define a function that takes f as an argument and returns a sequence of all "3-f" integers among $n \in [1, ..., 1000]$. What is the type of that function? Implement Exercise 5 using that function.

7. Define a function `sel_100` of type `List[List[Int]]` $\Rightarrow$ `List[List[Int]]` that selects only those inner lists whose largest value is at least 100. A test:

   ```
   /// sel_100( List( List(0, 100), List(60, 80), List(1000) ) )
   ///    == List( List(0, 100), List(1000) )
   ```

8. Define a function of type `Int` $\Rightarrow$ `List[List[Int]]` $\Rightarrow$ `List[List[Int]]` similar to Exercise 7 except that the hard-coded number 100 must be a *curried* first argument. Implement Exercise 7 using this function.

9. Define a function q that takes a function f :`Int` $\Rightarrow$ `Int` as its argument, and returns a new function that computes $f(f(f(x)))$. What is the required type of the function q?

10. Define a function `curry2` that takes an uncurried function of type `(Int, Int)` $\Rightarrow$ `Int` and returns a curried equivalent function of type `Int` $\Rightarrow$ `Int` $\Rightarrow$ `Int`.

1.4 Discussion

1.4.1 Functional programming as a paradigm

Functional programming (FP) is a **paradigm** of programming,—that is, an approach that guides programmers to write code in specific ways, for a wide range of programming tasks.

The main principle of FP is *to write code as a mathematical expression or formula*. This allows programmers to derive code through logical reasoning rather than through guessing, similarly to how books on mathematics reason about mathematical formulas and derive results systematically, without guessing or "debugging." Similarly to mathematicians and scientists who reason about formulas, functional programmers can *reason about code* systematically and logically, based on rigorous principles. This is possible only because code is written as a mathematical formula.

Mathematical intuition is backed by the vast experience accumulated while working with data over thousands of years of human history. It took centuries to invent flexible and powerful notation such as $\forall k \in S : p(k)$ and to develop the corresponding rules of reasoning. Functional programmers are fortunate to have at their disposal such a superior reasoning tool.

As we have seen, the Scala code for certain computational tasks corresponds quite closely to mathematical formulas. (Scala conventions and syntax, of course, require programmers to spell out certain things that the mathematical notation leaves out.) Just as in mathematics, large code expressions may be split into parts in a suitable way, so that the parts can be easily reused, flexibly composed together, and written independently from each other. The FP community has developed a toolkit of functions (such as `.map`, `.filter`, etc.) that proved especially useful in real-life programming, although many of them are not standard in mathematical literature.

Mastering FP involves practicing to reason about programs as formulas, building up the specific kind of applied mathematical intuition, familiarizing oneself with concepts adapted to programming needs, and learning how to translate the mathematics into code in various cases. The FP community has discovered a number of specific design patterns, founded on mathematical principles but driven by practical necessities of programming rather than by the needs of academic mathematics. This book explains the required mathematical principles in detail, developing them through intuition and practical coding tasks.

1.4.2 Functional programming languages

It is possible to apply the FP paradigm while writing code in any programming language. However, some languages lack certain features that make FP techniques much easier to use in practice. For example, in a language such as Python or Ruby, one can productively apply only a small number of the idioms of FP, such as the map/reduce operations. More advanced FP constructions are impractical in these languages because the corresponding code becomes too complicated to read, and mistakes are too easy to make, which negates the advantage of easier reasoning about the FP code.

Some programming languages, such as Haskell and OCaml, were designed specifically for advanced use in the FP paradigm. Other languages, such as ML, F#, Scala, Swift, Elm, PureScript, and Rust, had different design goals but still support enough FP features to be considered FP languages. I will be using Scala in this book, but exactly the same constructions could be implemented in other

FP languages in a similar way. At the level of detail needed in this book, the differences between languages such as ML, OCaml, Haskell, F#, Scala, Swift, Elm, PureScript, or Rust will not play a significant role.

1.4.3 The mathematical meaning of variables

I will now illustrate how the usage of variables in functional programming closely corresponds to the usage of variables in mathematical literature.

In mathematics, **variables** are used first of all as arguments of functions. For example, the mathematical formula

$$f(x) = x^2 + x$$

contains the variable x and defines a function f that takes a number x as its argument (to be definite, let us assume that x is an integer) and computes the value $x^2 + x$. The body of the function is the expression $x^2 + x$.

Mathematics has the convention that a variable, such as x, does not change its value within a formula. Indeed, there is no mathematical notation even to talk about "modifying" the value of x inside the formula $x^2 + x$. It would be quite confusing if a mathematics textbook said "before adding the last x in the formula $x^2 + x$, we modify x by adding 4 to it". If the "last x" in $x^2 + x$ needs to have a 4 added to it, a mathematics textbook will just write the formula $x^2 + x + 4$.

An example involving nameless functions is

$$f(n) = \sum_{k=0}^{n} k^2 + k \quad .$$

Here, n is the argument of the function f, while k is the argument of the nameless function $k \Rightarrow k^2 + k$. Neither n nor k can be "modified" in any sense within the expressions where they are used. The symbols k and n stand for some integer values, and these values are immutable. Indeed, it is meaningless to say that we want to "modify the value 4".

So, a variable in mathematics does not actually vary *within* the expression where it is used. We could say that a variable is a "named constant value of a fixed type," as seen within the expression where it is used. One could apply the function f to different values x, and each time one would compute a different result $f(x)$. However, each time a given value of x will remain unmodified within f while the body of the function f is being computed.

Functional programming adopts the same convention: variables are immutable named constants.

In Scala, function arguments are immutable within the function body:

```
def f(x: Int) = x*x + x // Can't modify x here.
```

The type of each mathematical variable (say, integer, string, etc.) is also fixed in advance. In mathematics, each variable is a value from a specific set, known in advance (the set of all integers, the set of all strings, etc.). Mathematical formulas such as $x^2 + x$ do not express any "checking" that x is indeed an integer and not, say, a string, before starting to evaluate $x^2 + x$.

Functional programming adopts the same view: Each argument of each function must be labeled by a **type**, which represents *the set of possible allowed values* for that function argument. The programming language's compiler will automatically check all types of all arguments, and will print an error message if a function is called on arguments of incorrect types. So, the programmer does not need to write any code that checks the types of arguments before starting to evaluate the function.

The second usage of **variables** in mathematics is to denote expressions that will be reused. For example, one writes: let $z = ...$ and now compute $\cos z + \cos 2z + \cos 3z$. Again, the variable z remains immutable, and its type remains fixed.

In Scala, this construction (defining an expression to be reused later) is written with the "val" syntax. Each variable defined using "val" is a named constant, and its type and value are fixed at the time of definition. Types for "val"s are optional in Scala, for instance we could write

```
val x: Int = 123
```

or more concisely

```
val x = 123
```

because it is obvious that this x is of type Int. However, when types are complicated, it helps to write them out. The compiler will check that the types match correctly everywhere and give an error message if we write

```
val x: Int = "123" // A String instead of an Int.
```

1.4.4 Iteration without loops

Another distinctive feature of functional programming is the absence of explicit loops or repetition.

Iterative computations are ubiquitous in mathematics; one can see formulas such as

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j - \left(\frac{1}{n} \sum_{i=1}^{n} s_i \right)^2 \quad .$$

To compute these expressions, we need to iterate over various values of i, j, etc. And yet, no mathematics textbook ever mentions "loops" or says "now repeat this formula ten times". Indeed, it would be pointless to evaluate a formula such as $x^2 + x$ ten times, or to "repeat" an equation such as

$$(x - 1)(x^2 + x + 1) = x^3 - 1 \quad .$$

Instead of loops, mathematicians write *expressions* such as $\sum_{i=1}^{n} s_i$, which denote repetitive computations. Mathematically, such expressions are defined using mathematical induction. The FP paradigm has developed rich tools for translating mathematical induction into code. In this chapter, we have seen methods such as `.map`, `.filter`, and `.sum`, which implement certain kinds of iterative computations. These and other operations can be combined in very flexible ways, which allows programmers to write iterative code *without loops*.

The programmer can avoid writing loops because the iteration is delegated to the library functions `.map`, `.filter`, `.sum`, and so on. It is the job of the library and the compiler to translate these functions into machine code that, perhaps, *will* contain loops; but the functional programmer does not need to see that code or to reason about it.

1.4.5 Nameless functions in mathematical notation

Functions in mathematics are mappings from one set to another. A function does not necessarily *need* a name; the mapping just needs to be defined. However, nameless functions have not been widely used in the conventional mathematical notation. It turns out that nameless functions are quite important in functional programming because, in particular, they allow programmers to write code more concisely and use a straightforward, consistent syntax.

Nameless functions have the property that their bound variables are invisible outside their scope. This property is directly reflected by the prevailing mathematical conventions. Compare the formulas

$$f(x) = \int_0^x \frac{dx}{1+x} \quad ; \quad f(x) = \int_0^x \frac{dz}{1+z} \quad .$$

The mathematical convention is that these formulas define the same function f, and that one may rename the integration variable at will.

In programming, the only situation when a variable "may be renamed at will" is when the variable represents an argument of a function. It follows that the notations $\frac{dx}{1+x}$ and $\frac{dz}{1+z}$ correspond to a nameless function whose argument was renamed from x to z. In FP notation, this nameless function would be denoted as $z \Rightarrow \frac{1}{1+z}$, and the integral rewritten as code such as

$$\text{integration } (0, x, g) \text{ where } g = \left(z \Rightarrow \frac{1}{1+z} \right) \quad .$$

To compare the traditional mathematical notations for sums and for integrals, consider the formula

$$\sum_{k=0}^x \frac{1}{1+k} \quad .$$

In sums, the bound variable k is introduced under the $\sum$ symbol; but in integrals, the bound variable follows the special symbol "d". This notational inconsistency could be removed if we were to use nameless functions explicitly, for example like this:

$$\sum_0^x k \Rightarrow \frac{1}{1+k} \text{ instead of } \sum_{k=0}^x \frac{1}{1+k} \quad ,$$

$$\int_0^x z \Rightarrow \frac{1}{1+z} \text{ instead of } \int_0^x \frac{dz}{1+z} \quad .$$

In this notation, the new summation symbol $\sum_0^x$ does not mention the name "k" but takes a function as an argument. Similarly, the new integration symbol $\int_0^x$ does not mention "z" and does not use the special symbol "d" but now takes a function as an argument. Written in this way, the operations of summation and integration become *functions* that take a function as argument (i.e. these op-

erations are higher-order functions). The summation shown above may be expressed as

$$\text{summation } (0, x, g) \text{ where } g = \left(y \Rightarrow \frac{1}{1+y} \right) \quad .$$

In this way, nameless functions remove inconsistency from the traditional mathematical notation and express summation and integration in a uniform and straightforward manner as higher-order functions. Expressed as functions, these operations are easy to implement and to use in programming.

The Scala standard library does not define functions `integration` or `summation`. We could implement `summation(a,b,g)` as

```
def summation(a: Double, b: Double, g: Double ⇒ Double) =
  (a to b).map(g).sum
scala> summation(1, 10, x ⇒ x*x)
```

Numerical integration requires more code, since the formula is more complicated. For instance, Simpson's rule can be written as

$$\text{int } (a, b, g) = \frac{\delta}{3} \left(g(a) + g(b) + 4s_1 + 2s_2 \right) \quad ,$$

$$n = 2 \left\lfloor \frac{b-a}{\varepsilon} \right\rfloor , \quad \delta = \frac{b-a}{n} \quad ,$$

$$s_1 = \sum_{i=1,3,\dots,n-1} g(a + i\delta) \quad ,$$

$$s_2 = \sum_{i=2,4,\dots,n-2} g(a + i\delta) \quad .$$

A straightforward translation of this formula into Scala is

```
def integration(a: Double, b: Double,
  g: Double ⇒ Double, eps: Double) = {
  // Define some helper values.
  val n: Int = (math.round((b-a)/eps/2)*2).toInt
  val delta_x = (b - a) / n
  val g_i = (i: Int) ⇒ g(a + i*delta_x)
  val s1 = (1 to (n-1) by 2).map(g_i).sum
  val s2 = (2 to (n-2) by 2).map(g_i).sum
  // Compute the final result.
```

```
      delta_x / 3 * (g(a) + g(b) + 4*s1 + 2*s2)
   }
   scala> integration(0, 2, x ⇒ x*x*x, 0.001)
```

The entire code is a large expression whose sub-expressions are defined for convenience as a few helper values and helper functions. In other words, this code is written in the FP paradigm.

1.4.6 Scope of bound variables

A bound variable is invisible outside the scope of the expression (often called **local scope** whenever it is clear which expression is being considered). This is why bound variables may be renamed at will: no outside code could possibly use them and depend on their values. However, outside code may define variables that (by chance or by mistake) have the same name as a bound variable inside the scope.

Consider this example from calculus: In the integral

$$f(x) = \int_0^x \frac{dx}{1+x} \quad ,$$

a bound variable named x is defined in *two* local scopes: in the scope of f and in the scope of the nameless function $x \Rightarrow \frac{1}{1+x}$. The convention in mathematics is to treat these two x's as two *completely different* variables that just happen to have the same name. In sub-expressions where both of these bound variables are visible, priority is given to the bound variable defined in the closest inner scope. The outer definition of x is **shadowed**, i.e. hidden, by the definition of the inner x. For this reason, mathematicians expect that evaluating $f(10)$ will give

$$f(10) = \int_0^{10} \frac{dx}{1+x} \quad ,$$

rather than $\int_0^{10} \frac{dx}{1+10}$, because the outer definition $x = 10$ is shadowed, within the expression $\frac{1}{1+x}$, by the closer definition of x in the local scope of $x \Rightarrow \frac{1}{1+x}$.

Since this is the prevailing mathematical convention, the same convention is adopted in FP. A variable defined in a local scope (i.e. a bound variable) is invisible outside that scope but will shadow any outside definitions of a variable with the same name.

It is better to avoid name shadowing, because it usually decreases the clarity of code and thus invites errors. Consider this function,

$$x \Rightarrow x \Rightarrow x \quad .$$

This code may look confusing; let us decipher this syntax. The symbol $\Rightarrow$ associates to the right, so $x \Rightarrow x \Rightarrow x$ is the same as $x \Rightarrow (x \Rightarrow x)$. So, it is a function that takes x and returns $x \Rightarrow x$. Since the returned nameless function, $(x \Rightarrow x)$, may be renamed to $(y \Rightarrow y)$ without changing its value, we can rewrite the code to

$$x \Rightarrow (y \Rightarrow y) \quad .$$

It is now easier to understand this code and reason about it. For instance, it becomes immediately clear that this function actually ignores its argument x.

1.4.7 Named and nameless expressions and their uses

It is an advantage if a programming language supports unnamed (or "nameless") expressions. To see this, consider a familiar situation where we take the absence of names for granted.

In most programming languages today, we can directly write arithmetical expressions such as (x+123)*y/(2+x). Here, x and y are variables with names. Note, however, that the entire expression does not need to have a name. Parts of that expression (such as x+123 or 2+x) also do not need to have separate names. Indeed, it would be quite cumbersome if we had to assign a name separately to each sub-expression. The code would then look like this:

```
/// a = 123
/// b = x + a
/// c = b * y
/// d = 2
/// e = d + x
/// f = c / e
/// return f
```

This style of programming resembles assembly language, where *every* sub-expression, every step of any calculation must be computed separately and labeled with a name (or a memory address). Clearly, this is a less productive style of programming than writing

```
/// val f = (x+123)*y/(2+x)
```

Programmers gain productivity when their programming language supports expressions without names (which I call "nameless expressions").

This is also common practice in mathematics; names are assigned when needed, but many expressions remain nameless.

It is similarly quite useful if data structures can be declared without a name. For instance, a dictionary (or a "hashmap") is declared in Scala as

```
scala> Map("a" → 1, "b" → 2, "c" → 3)
```

This is a nameless expression representing a dictionary. Without this construction, programmers have to write cumbersome, repetitive code that fills an initially empty dictionary step by step (this is the case with Java).

Nameless functions are useful for the same reason: they allow the programmer to build a larger program as one expression from simpler parts, with more concise code.

1.4.8 Nameless functions: historical perspective

Nameless functions were first used in 1936 within a theoretical programming language called "λ-calculus" (see Appendix D for more details). In that language, all functions are nameless and curried. The letter λ is a syntax separator denoting curried function arguments in nameless functions. For example, the nameless function $x \Rightarrow x + 1$ could be written as $\lambda x.add\ x\ 1$ in λ-calculus, if it had a built-in function *add* for adding integers (which it does not).

Many programming languages were invented since that time. In most programming languages that were in use until around 1990, all functions required names. But by 2015, most languages added support for nameless functions, possibly because programming in the map/reduce style (which invites frequent use of nameless functions) proved to be quite useful. Table 1.1 shows when nameless functions were introduced in each language and shows the syntax for the function $k^{\underline{Int}} \Rightarrow k + k$.

What I call in this book a "nameless function" is also elsewhere called anonymous function, function expression, function literal, closure, lambda function, lambda expression, or just a "lambda". I use the term "nameless function" in this book because it is the most descriptive and unambiguous both in speech and in writing.

Language	Year	Code for $k^{\underline{\text{Int}}} \Rightarrow k + k$
λ-calculus	1936	$\lambda k.\ add\ k\ k$
typed λ-calculus	1940	$\lambda k : int.\ add\ k\ k$
LISP	1958	`(lambda (k) (+ k k))`
Standard ML	1973	`fn (k:  int) => k+k`
OCaml, later F#	1985	`fun (k:  int) -> k+k`
Haskell	1990	`\ (k:  Int) -> k+k`
Ruby	1993	`lambda {|k| k+k }`
R	1993	`function(k) k+k`
Python	1994	`lambda k:  k+k`
JavaScript	1995	`function(k) { return k+k; }`
Scala	2003	`(k:  Int) => k+k`
C++ 11	2011	`[] (int k) { return k+k; }`
Go	2012	`func(k int) { return k+k }`
Kotlin	2012	`{ k:  Int -> k+k }`
Swift	2014	`{(k:  int) -> int in return k+k}`
Java 8	2014	`(int k) -> k+k`
Rust	2015	`|k:  i32| k+k`

Table 1.1: The nameless function $k^{\underline{\text{Int}}} \Rightarrow k + k$ in various languages.

2 Mathematical induction

2.1 Discussion

3 The formal logic of types

3.1 Higher-order functions

3.1.1 Discussion

3.2 Disjunction types

3.2.1 Discussion

3.3 The Curry-Howard correspondence

3.3.1 Discussion

4 Functors

4.1 Discussion

4.2 Practical use

4.3 Laws and structure

5 Type-level functions and type classes

5.1 Combining typeclasses

5.2 Inheritance

5.3 Functional dependencies

5.4 Discussion

6 Filterable functors

6.1 Practical use

6.1.1 Discussion

6.2 Laws and structure

6.2.1 Discussion

7 Semimonads and monads

7.1 Practical use

7.1.1 Discussion

7.2 Laws and structure

7.2.1 Discussion

8 Applicative functors, contrafunctors, and profunctors

8.1 Practical use

8.1.1 Discussion

8.2 Laws and structure

9 Traversable functors and profunctors

9.1 Discussion

10 Free constructions

10.1 Discussion

11 Monad transformers

11.1 Practical use

11.2 Laws and structure

11.3 Composed-outside transformers: Rigid monads

A monad R is **rigid** if it has the following properties:

1. R is a **rigid functor** with the natural transformation

$$\text{fi} : \left(A \Rightarrow R^B \right) \Rightarrow R^{A \Rightarrow B}$$

satisfying the nondegeneracy law $\text{fi}\,\mathring{,}\,\text{fo} = \text{id}$, where

$$\text{fo} : R^{A \Rightarrow B} \Rightarrow A \Rightarrow R^B$$

is a natural transformation defined by

$$\text{fo}\,(r) = a \Rightarrow (ab \Rightarrow ab\,(a))^{\uparrow R}\,r \quad .$$

2. $T^A \equiv R^{M^A}$ is a monad for any monad M.

3. The **compatibility law** for the `flatMap` methods of R and for T,

$$\text{flm}_R\,f^{:A \Rightarrow R^{M^B}} = \text{pure}_M^{\uparrow R}\,\mathring{,}\,\text{flm}_T\,f^{:A \Rightarrow R^{M^B}} \quad .$$

(Both sides must be equal as functions $R^A \Rightarrow R^{M^B}$.) An equivalent formulation of this law in terms of Kleisli products $\diamond_R$ and $\diamond_T$ is

$$f^{:A \Rightarrow R^B} \diamond_R g^{:B \Rightarrow R^{M^C}} = \left(f\,\mathring{,}\,\text{pure}_M^{\uparrow R} \right)^{:A \Rightarrow R^{M^B}} \diamond_T g^{:B \Rightarrow R^{M^C}} \quad .$$

The second property suggests that $R^{M^\bullet}$ is a monad transformer for R. The third property appears to be natural, and is also necessary for the validity of the construction of the general selector monad.

At this point, I do not know how to derive any of the properties 1-3 from each other. Instead, I will show that all three properties hold for the four known rigid monad constructions.

11.3.1 Rigid monad construction 1: contrafunctor selection

The construction of **contrafunctor selection**, $R^A \equiv H^A \Rightarrow A$, defines a rigid monad R for *any* contrafunctor H.

This monad's effect is to return a value of type A given a contrafunctor H that consumes values of type A (and presumably checks some conditions on those values). The contrafunctor H could be a constant contrafunctor $H^A \equiv Q$, a function such as $H^A \equiv A \Rightarrow Q$, or a more complicated contrafunctor.

Different choices of the contrafunctor H will yield simple rigid monad examples such as $R^A \equiv 1$ (unit monad), $R^A \equiv A$ (identity monad), $R^A \equiv Z \Rightarrow A$ (reader monad), as well as the **search monad** $R^A \equiv (A \Rightarrow Q) \Rightarrow A$.

The search monad represents the effect of searching for a value of type A that satisfies a condition expressed through a function of type $A \Rightarrow Q$. The simplest example of a search monad is found by setting $Q \equiv \text{Bool}$. One may implement a function of type $(A \Rightarrow \text{Bool}) \Rightarrow A$ that somehow finds a value of type A that satisfies the given predicate of type $A \Rightarrow \text{Bool}$.

Assume that H is a contrafunctor and M is a monad, and denote for brevity

$$T^A \equiv R^{M^A} \equiv H^{M^A} \Rightarrow M^A \quad .$$

We need to prove that (1) a non-degenerate fi method is defined for R^A; (2) T^A is a monad, and (3) T and R satisfy the compatibility law.

Statement 1. The nondegeneracy law fi $\mathbin{\circ}$ fo = id holds for the rigid functor R.

Proof. The fi method for R is

$$\text{fi} : (A \Rightarrow H^B \Rightarrow B) \Rightarrow H^{A \Rightarrow B} \Rightarrow A \Rightarrow B$$

$$\text{fi} = f^{:A \Rightarrow H^B \Rightarrow B} \Rightarrow h^{:H^{A \Rightarrow B}} \Rightarrow a \Rightarrow f(a)\big((b \Rightarrow \_ \Rightarrow b)^{\downarrow H}\, h\big)$$

It is easier to manipulate the methods fi and fo if we flip the curried arguments of the function type $A \Rightarrow R^B \equiv A \Rightarrow H^B \Rightarrow B$ and instead consider the equivalent methods $\tilde{\text{fi}}$ and $\tilde{\text{fo}}$ defined by

$$\tilde{\text{fi}} : (H^B \Rightarrow A \Rightarrow B) \Rightarrow H^{A \Rightarrow B} \Rightarrow A \Rightarrow B$$

$$\tilde{\text{fi}} = f^{:H^B \Rightarrow A \Rightarrow B} \Rightarrow h^{:H^{A \Rightarrow B}} \Rightarrow f((b \Rightarrow \_ \Rightarrow b)^{\downarrow H} h)$$

$$\tilde{\text{fo}} : \left(H^{A \Rightarrow B} \Rightarrow A \Rightarrow B\right) \Rightarrow H^B \Rightarrow A \Rightarrow B$$

$$\tilde{\text{fo}} = g^{:H^{A \Rightarrow B} \Rightarrow A \Rightarrow B} \Rightarrow h^{:H^B} \Rightarrow a^{:A} \Rightarrow g\left(\left(p^{:A \Rightarrow B} \Rightarrow p\, a\right)^{\downarrow H} h\right) a$$

To show the non-degeneracy law for R, compute

$$\tilde{\text{fo}} \left(\tilde{\text{fi}}\, f\right) h^{:H^B}\, a^{:A}$$

$$
\begin{aligned}
\text{definition of } \tilde{\text{fo}}: \quad &= \left(\tilde{\text{fi}}\, f\right) ((p \Rightarrow p\, a)^{\downarrow H} h)a \\
\text{definition of } \tilde{\text{fi}}: \quad &= f((b \Rightarrow \_ \Rightarrow b)^{\downarrow H} (p \Rightarrow p\, a)^{\downarrow H} h)a \\
\text{composition law for } H: \quad &= f((b \Rightarrow \_ \Rightarrow b\, \mathring{,}\, p \Rightarrow p\, a)^{\downarrow H} h)a \\
\text{simplify } b \Rightarrow \_ \Rightarrow b\, \mathring{,}\, p \Rightarrow p\, a \text{ to}: \quad &= f\left((b \Rightarrow b)^{\downarrow H} h\right) a \\
\text{identity law for } H: \quad &= f\, h\, a \quad .
\end{aligned}
$$

We obtained $\tilde{\text{fo}} \left(\tilde{\text{fi}}\, f\right) h\, a = f\, h\, a$, so the non-degeneracy law $\tilde{\text{fi}}\, \mathring{,}\, \tilde{\text{fo}} = \text{id}$ holds.

Statement 2.   The non-degeneracy law $\text{fi}_T\, \mathring{,}\, \text{fo}_T = \text{id}$ holds for the rigid functor $T^\bullet \equiv R^{M^\bullet}$ as long as M is rigid.

Proof.   Since it is given that M is rigid, we may use its method fi_M satisfying the non-degeneracy law $\text{fo}_M\, (\text{fi}_M\, f) = f$.

Flip the curried arguments of the function type $A \Rightarrow T^B \equiv A \Rightarrow H^{M^B} \Rightarrow M^B$, to obtain $H^{M^B} \Rightarrow A \Rightarrow M^B$, and note that $A \Rightarrow M^B$ can be mapped to $M^{A \Rightarrow B}$ using fi_M. So we can implement $\tilde{\text{fi}}_T$ using fi_M:

$$\tilde{\text{fi}}_T : (H^{M^B} \Rightarrow A \Rightarrow M^B) \Rightarrow H^{M^{A \Rightarrow B}} \Rightarrow M^{A \Rightarrow B}$$

$$\tilde{\text{fi}}_T = f \Rightarrow h \Rightarrow \text{fi}_M \left(f((b \Rightarrow \_ \Rightarrow b)^{\uparrow M \downarrow H} h)\right)$$

$$\tilde{\text{fo}}_T : \left(H^{M^{A \Rightarrow B}} \Rightarrow M^{A \Rightarrow B}\right) \Rightarrow H^{M^B} \Rightarrow A \Rightarrow M^B$$

$$\tilde{\text{fo}}_T = g \Rightarrow h \Rightarrow a \Rightarrow \text{fo}_M \left(g\left(\left(p^{:A \Rightarrow B} \Rightarrow p\, a\right)^{\uparrow M \downarrow H} h\right)\right) a$$

To show the non-degeneracy law for T, compute

$$\tilde{\text{fo}}_T \left(\tilde{\text{fi}}_T f \right) h^{:H^{M^B}} a^{:A}$$

insert the definition of $\tilde{\text{fo}}_T$: $= \text{fo}_M \left(\left(\tilde{\text{fi}}_T f \right) \left((p \Rightarrow p\, a)^{\uparrow M \downarrow H} h \right) \right) a$

insert the definition of $\tilde{\text{fi}}_T$: $= \text{fo}_M \left(\text{fi}_M \left(f\left((b \Rightarrow \_ \Rightarrow b)^{\uparrow M \downarrow H} (p \Rightarrow p\, a)^{\uparrow M \downarrow H} h \right) \right) \right) a$

nondegeneracy law for fi_M: $= f\left((b \Rightarrow \_ \Rightarrow b)^{\uparrow M \downarrow H} (p \Rightarrow p\, a)^{\uparrow M \downarrow H} h \right) a$

composition laws for M, H: $= f\left((b \Rightarrow \_ \Rightarrow b \,\mathring{\circ}\, p \Rightarrow p\, a)^{\uparrow M \downarrow H} h \right) a$

simplify: $= f \left((b \Rightarrow b)^{\uparrow M \downarrow H} h \right) a$

identity laws for M, H: $= f\, h\, a$.

We obtained $\tilde{\text{fo}}_T \left(\tilde{\text{fi}}_T f \right) h\, a = f\, h\, a$. Therefore the non-degeneracy law $\tilde{\text{fi}}_T \,\mathring{\circ}\, \tilde{\text{fo}}_T = \text{id}$ holds.

Statement 3. $\quad T^\bullet \equiv R^{M^\bullet} \equiv H^{M^\bullet} \Rightarrow M^\bullet$ is a monad if M is any monad and H is any contrafunctor.

    Proof.   We need to define the monad instance for T and prove the identity and the associativity laws for T, assuming that the monad M satisfies these laws.

    To define the monad instance for T, it is convenient to use the Kleisli category formulation of the monad. In this formulation, we consider Kleisli morphisms of type $A \Rightarrow T^B$ and then define the Kleisli identity morphism, $\text{pure}_T : A \Rightarrow T^A$, and the Kleisli product operation $\diamond_T$,

$$f^{:A \Rightarrow T^B} \diamond_T g^{:B \Rightarrow T^C} : A \Rightarrow T^C \quad .$$

We are then required to define the operation $\diamond_T$ and to prove identity and associativity laws for it.

    Since the type constructor R is itself a function type $H^A \Rightarrow A$, calculations become much quicker if we flip the curried arguments of the Kleisli morphism type $A \Rightarrow T^B \equiv A \Rightarrow H^{M^B} \Rightarrow M^B$ and instead consider the "**flipped Kleisli**" morphisms of type $H^{M^B} \Rightarrow A \Rightarrow M^B$. This type is equivalent to $A \Rightarrow T^B$, so any laws we prove for the flipped Kleisli morphisms will correspond to laws for the standard Kleisli morphisms. The use of flipped Kleisli morphisms makes all computations significantly shorter.

We temporarily denote by $\text{p\u016bre}_T$ and $\tilde{\diamond}_T$ the flipped Kleisli operations:

$$\text{p\u016bre}_T : \ H^{M^A} \Rightarrow A \Rightarrow M^A$$

$$f^{:H^{M^B} \Rightarrow A \Rightarrow M^B} \tilde{\diamond}_T g^{:H^{M^C} \Rightarrow B \Rightarrow M^C} : \ H^{M^C} \Rightarrow A \Rightarrow M^C \quad .$$

To define the operations $\text{p\u016bre}_T$ and $\tilde{\diamond}_T$, we may use the methods pure_M and flm_M as well as the Kleisli product $\diamond_M$ for the given monad M. The definitions are

$$\text{p\u016bre}_T = q \Rightarrow \text{pure}_M \quad \text{(the argument } q \text{ is unused)},$$

$$f \tilde{\diamond}_T g = q \Rightarrow (f\, p) \diamond_M (g\, q) \quad \text{where}$$

$$p^{:H^{M^B}} = (\text{flm}_M\, (g\, q))^{\downarrow H}\, q \quad .$$

This definition works by using the Kleisli product $\diamond_M$ on values $f\, p : A \Rightarrow M^B$ and $g\, q : B \Rightarrow M^C$. To obtain a value $p : H^{M^B}$, we use the function $\text{flm}_M\, (g\, q) : M^B \Rightarrow M^C$ to H-contramap $q : H^{M^C}$ into $p : H^{M^B}$.

Written as a single expression, the definition of $\tilde{\diamond}_T$ is

$$f \tilde{\diamond}_T g = q \Rightarrow f\left((\text{flm}_M\, (g\, q))^{\downarrow H}\, q\right) \diamond_M (g\, q) \quad .$$

Checking the left identity law:

$$\text{p\u016bre}_T \tilde{\diamond}_T g$$

$$\begin{aligned}
\text{definition of } \tilde{\diamond}_T : \quad &= q \Rightarrow \text{p\u016bre}_T \left((\text{flm}_M\, (g\, q))^{\downarrow H}\, q\right) \diamond_M (g\, q) \\
\text{definition of } \text{p\u016bre}_T : \quad &= q \Rightarrow \text{pure}_M \diamond_M g\, q \\
\text{left identity law for } M : \quad &= q \Rightarrow g\, q \\
\text{function expansion} : \quad &= g
\end{aligned}$$

Checking the right identity law:

$$f \tilde{\diamond}_T \text{p\u016bre}_T$$

$$\begin{aligned}
\text{definition of } \tilde{\diamond}_T : \quad &= q \Rightarrow f\left(\left(\text{flm}_M\left(\text{p\u016bre}_T q\right)\right)^{\downarrow H}\, q\right) \diamond_M \left(\text{p\u016bre}_T q\right) \\
\text{definition of } \text{p\u016bre}_T : \quad &= q \Rightarrow f\left(\left(\text{flm}_M\left(\text{pure}_M\right)\right)^{\downarrow H}\, q\right) \diamond_M \text{pure}_M \\
\text{right identity law for } M : \quad &= q \Rightarrow f\left((\text{id})^{\downarrow H}\, q\right) \\
\text{identity law for } H : \quad &= q \Rightarrow f\, (q) \\
\text{function expansion} : \quad &= f
\end{aligned}$$

Checking the associativity law: $(f \mathbin{\tilde{\diamond}}_T g) \mathbin{\tilde{\diamond}}_T h$ must equal $f \mathbin{\tilde{\diamond}}_T (g \mathbin{\tilde{\diamond}}_T h)$. We have

$$(f \mathbin{\tilde{\diamond}}_T g) \mathbin{\tilde{\diamond}}_T h$$
$$= \left(r \Rightarrow f \left((\mathrm{flm}_M \, (g \, r))^{\downarrow H} \, r \right) \diamond_M (g \, r) \right) \mathbin{\tilde{\diamond}}_T h$$
$$= q \Rightarrow f \left((\mathrm{flm}_M \, (g \, r))^{\downarrow H} \, r \right) \diamond_M (g \, r) \diamond_M (h \, q) \quad \text{where}$$
$$r \equiv (\mathrm{flm}_M \, (h \, q))^{\downarrow H} \, q \quad ;$$

while

$$f \mathbin{\tilde{\diamond}}_T (g \mathbin{\tilde{\diamond}}_T h)$$
$$= f \mathbin{\tilde{\diamond}}_T \left(q \Rightarrow g \left((\mathrm{flm}_M \, (h \, q))^{\downarrow H} \, q \right) \diamond_M (h \, q) \right)$$
$$= q \Rightarrow f \left((\mathrm{flm}_M \, k)^{\downarrow H} \, q \right) \diamond_M k \quad \text{where}$$
$$r \equiv (\mathrm{flm}_M \, (h \, q))^{\downarrow H} \, q \quad \text{and}$$
$$k \equiv (g \, r) \diamond_M (h \, q) \quad .$$

It remains to show that the following two expressions are equal,

$$f \left((\mathrm{flm}_M \, (g \, r))^{\downarrow H} \, r \right) \diamond_M (g \, r) \diamond_M (h \, q) \quad \text{and}$$
$$f \left((\mathrm{flm}_M \, ((g \, r) \diamond_M (h \, q)))^{\downarrow H} \, q \right) \diamond_M (g \, r) \diamond_M (h \, q) , \quad \text{where}$$
$$r \equiv (\mathrm{flm}_M \, (h \, q))^{\downarrow H} \, q \quad .$$

These long expressions differ only by the following two sub-expressions, namely

$$(\mathrm{flm}_M \, (g \, r))^{\downarrow H} \, r$$

and

$$(\mathrm{flm}_M \, ((g \, r) \diamond_M (h \, q)))^{\downarrow H} \, q \quad ,$$

where $r \equiv (\mathrm{flm}_M \, (h \, q))^{\downarrow H} \, q$. Writing out the value r in the last argument of $(\mathrm{flm}_M \, (g \, r))^{\downarrow H} \, r$ but leaving r unexpanded everywhere else, we now rewrite the differing sub-expressions as

$$(\mathrm{flm}_M \, (g \, r))^{\downarrow H} \, (\mathrm{flm}_M \, (h \, q))^{\downarrow H} \, q \quad \text{and}$$
$$(\mathrm{flm}_M \, ((g \, r) \diamond_M (h \, q)))^{\downarrow H} \, q \quad .$$

Now it becomes apparent that we need to put the two "flm$_M$"s closer together and to combine them by using the associativity law of the monad M. Then we can rewrite the first sub-expression and transform it into the second one:

$$(\text{flm}_M\,(g\,r))^{\downarrow H}\,(\text{flm}_M\,(h\,q))^{\downarrow H}\,q$$

$$\text{composition law for } H: \quad = (\text{flm}_M\,(g\,r)\,\mathring{;}\,\text{flm}_M\,(h\,q))^{\downarrow H}\,q$$

$$\text{associativity law for } M: \quad = (\text{flm}_M\,((g\,r)\,\mathring{;}\,\text{flm}_M\,(h\,q)))^{\downarrow H}\,q$$

$$\text{definition of } \diamond_M \text{ via flm}_M: \quad = (\text{flm}_M\,((g\,r)\,\diamond_M\,(h\,q)))^{\downarrow H}\,q \quad .$$

This proves the associativity law for $\bar{\diamond}_T$.

Statement 4. The monad instance for T defined in Statement 3 can be defined equivalently as

$$\text{pure}_T\left(a^{:A}\right): \; H^{M^A} \Rightarrow M^A$$

$$\text{pure}_T\,a = \_ \Rightarrow \text{pure}_M\,a$$

$$\text{flm}_T\,(f^{:A \Rightarrow H^{M^B} \Rightarrow M^B}): \; \left(H^{M^A} \Rightarrow M^A\right) \Rightarrow H^{M^B} \Rightarrow M^B$$

$$\text{flm}_T\,f = a^{:R^A} \Rightarrow q^{:H^{M^B}} \Rightarrow p^{\uparrow R}a \quad \text{where}$$

$$p^{:M^A \Rightarrow M^B} \equiv \text{flm}_M\left(x^{:A} \Rightarrow f\,x\,q\right) \quad .$$

Proof. The definition of $\bar{\diamond}_T$ in Statement 3 used the flipped types of Kleisli morphisms, which is not the standard way of defining the methods of a monad. To restore the standard types, we need to flip the arguments again:

$$f^{:A \Rightarrow H^{M^B} \Rightarrow M^B} \diamond_T g^{:B \Rightarrow H^{M^C} \Rightarrow M^C}: \; A \Rightarrow H^{M^C} \Rightarrow M^C \quad ;$$

$$f \diamond_T g = a \Rightarrow q \Rightarrow \left(\tilde{f}\left((\text{flm}_M\,(b \Rightarrow g\,b\,q))^{\downarrow H}\,q\right) \diamond_M (b \Rightarrow g\,b\,q)\right)a \quad ,$$

where $\tilde{f} \equiv h \Rightarrow k \Rightarrow f\,k\,h$ is the flipped version of f. To replace $\diamond_M$ by flm$_M$, express $x \diamond_M y = x\,\mathring{;}\,\text{flm}_M\,y$ to find

$$f \diamond_T g = a \Rightarrow q \Rightarrow \left(\tilde{f}\left(p^{\downarrow H}\,q\right)\,\mathring{;}\,p\right)a \quad \text{where } p = \text{flm}_M\,(x \Rightarrow g\,x\,q) \quad .$$

Since we would like to obtain the implementation of flm$_T$, we express flm$_T$ through $\diamond_T$ as

$$\text{flm}_T\,g^{:A \Rightarrow T^B} = \text{id}^{:T^A \Rightarrow T^A} \diamond_T g \quad .$$

Now we need to substitute $f^{:T^A \Rightarrow T^A}$ = id into $f \diamond_T g$. Noting that $\tilde{f}$ will then become

$$\tilde{f} = (h \Rightarrow k \Rightarrow \text{id } k\, h) = (h \Rightarrow k \Rightarrow k\, h) \quad,$$

we get

$$\text{flm}_T\, g^{:A \Rightarrow T^B} = \text{id}\, \mathring{,}\, \text{flm}_T\, g$$

$$\text{definition of } \diamond_T : \quad = a^{:T^A} \Rightarrow q^{H^{M^B}} \Rightarrow \left(\tilde{f}\left(p^{\downarrow H}q\right) \mathring{,} p\right) a$$

$$\text{where } p = \text{flm}_M\,(x \Rightarrow g\, x\, q)$$

$$\text{substitute } f = \text{id} : \quad = a \Rightarrow q \Rightarrow \left((h \Rightarrow k \Rightarrow k\, h)\left(p^{\downarrow H}q\right) \mathring{,} p\right) a$$

$$\text{apply } k \text{ to } p^{\downarrow H}q : \quad = a \Rightarrow q \Rightarrow \left(\left(k \Rightarrow k\left(p^{\downarrow H}q\right)\right) \mathring{,} p\right) a$$

$$\text{definition of } \mathring{,} : \quad = a \Rightarrow q \Rightarrow p\left(a\left(p^{\downarrow H}q\right)\right) \quad.$$

By definition of the functor $R^A \equiv H^A \Rightarrow A$, we lift any function $p^{:A \Rightarrow B}$ as

$$p^{\uparrow R} = a^{H^A \Rightarrow A} \Rightarrow q^{H^B} \Rightarrow p\left(a\left(p^{\downarrow H}q\right)\right) \quad.$$

Finally, renaming g to f, we obtain the desired code,

$$\text{flm}_T\, f = a \Rightarrow q \Rightarrow p^{\uparrow R}a \quad \text{where } p \equiv \text{flm}_M\,(x \Rightarrow f\, x\, q) \quad.$$

Statement 5. The rigid monad $R^A \equiv H^A \Rightarrow A$ satisfies the compatibility law with respect to any monad M.

Proof. We need to show that

$$\text{flm}_R\, f^{:A \Rightarrow R^{M^B}} = \text{pure}_M^{\uparrow R}\, \mathring{,}\, \text{flm}_T\, f^{:A \Rightarrow R^{M^B}} \quad.$$

The definition of flm_R is obtained from flm_T by choosing the identity monad $M^A \equiv A$ instead of an arbitrary monad M. This replaces flm_M by id:

$$\text{flm}_T\, f^{:A \Rightarrow R^{M^B}} = a^{:R^A} \Rightarrow q^{H^{M^B}} \Rightarrow \left(\text{flm}_M\left(x^{:A} \Rightarrow f\, x\, q\right)\right)^{\uparrow R} a \quad;$$

$$\text{flm}_R\, f^{:A \Rightarrow R^{M^B}} = a^{:R^A} \Rightarrow q^{H^{M^B}} \Rightarrow \left(x^{:A} \Rightarrow f\, x\, q\right)^{\uparrow R} a \quad.$$

Then we can rewrite

$$\text{pure}_M^{\uparrow R} \,\mathring{\,}\, \text{flm}_T \, f^{:A \Rightarrow R^{M^B}}$$

$$\text{definition of flm}_T : \quad = \left(a^{:R^A} \Rightarrow \text{pure}_M^{\uparrow R} a\right) \mathring{\,}\, \left(a \Rightarrow q \Rightarrow \left(\text{flm}_M \, (x \Rightarrow f \, x \, q)\right)^{\uparrow R} a\right)$$

$$\text{expand } \mathring{\,} \text{ and simplify}: \quad = a \Rightarrow q \Rightarrow \left(\text{flm}_M \, (x \Rightarrow f \, x \, q)\right)^{\uparrow R} \text{pure}_M^{\uparrow R} a$$

$$\text{composition law for } R: \quad = a \Rightarrow q \Rightarrow \left(\text{pure}_M \,\mathring{\,}\, \text{flm}_M \, (x \Rightarrow f \, x \, q)\right)^{\uparrow R} a$$

$$\text{left identity law for } M: \quad = a \Rightarrow q \Rightarrow (x \Rightarrow f \, x \, q)^{\uparrow R} a$$

$$\text{definition of flm}_R: \quad = \text{flm}_R \, f \quad .$$

11.3.2 Rigid monad construction 2: composition

Statement 1.   The composition $R_1^{R_2^\bullet}$ is a rigid monad if both R_1 and R_2 are rigid monads.

Proof.   Property 1 was proved in Statement 2 in the previous section.

To prove property 2 for $R_1^{R_2^\bullet}$, first consider that $R_1^{R_2^\bullet}$ is itself a monad, which is property 2 of the rigid monad R_1. Now consider an arbitrary monad M and the type constructor $Q^\bullet \equiv R_1^{R_2^{M^\bullet}}$. Property 2 of rigid monads requires that Q be a monad. Since R_2 is rigid, $R_2^{M^\bullet}$ is a monad, and since R_1 is rigid, Q is a monad.

To prove property 3 for $R_1^{R_2^\bullet}$, assume that this property already holds for R_1 and R_2 separately, and consider an arbitrary monad M and the type constructor $Q^\bullet \equiv R_1^{R_2^{M^\bullet}}$. ???

11.3.3 Rigid monad construction 3: product

The product of rigid monads, $R_1^A \times R_2^A$, is a rigid monad.

11.3.4 Rigid monad construction 4: selector monad

The **selector monad** $S^A \equiv F^{A \Rightarrow R^Q} \Rightarrow R^A$ is rigid if $R^\bullet$ is a rigid monad, $F^\bullet$ is any functor, and Q is any fixed type.

11.3.5 The codensity monad

Codensity monad over a functor F is defined as

$$\mathrm{Cod}^{F,A} \equiv \forall B. \left(A \Rightarrow F^B \right) \Rightarrow F^B$$

Properties:

- $\mathrm{Cod}^{F,\bullet}$ is a monad for any functor F

- If $F^\bullet$ is itself a monad then we have monadic morphisms $\mathrm{inC} : F^\bullet \rightsquigarrow \mathrm{Cod}^{F,\bullet}$ and $\mathrm{outC} : \mathrm{Cod}^{F,\bullet} \rightsquigarrow F^\bullet$ such that $\mathrm{inC} \mathbin{\fatsemi} \mathrm{outC} = \mathrm{id}$

- A monad transformer for the codensity monad is

$$T_{\mathrm{Cod}}^{M,A} = \forall B. \left(A \Rightarrow M^{F^B} \right) \Rightarrow M^{F^B}$$

11.4 Discussion

12 Recursive types

12.1 Fixpoints and type recursion schemes

12.2 Row polymorphism and OO programming

12.3 Column polymorphism

12.4 Discussion

13 Co-inductive typeclasses. Comonads

13.1 Practical use

13.2 Laws and structure

13.3 Co-free constructions

13.4 Co-free comonads

13.5 Discussion

14 Irregular type classes*

14.1 Distributive functors

14.2 Rigid functors

14.3 Lenses and prisms

14.4 Discussion

15 Type quantifiers*

16 Value-dependent types*

17 Conclusions and discussion

18 Essay: Software engineers and software artisans

Let us look at the differences between the kind of activities we ordinarily call engineering, as opposed to artisanship or craftsmanship. It will then become apparent that today's computer programmers are better understood as "software artisans" rather than software engineers.

18.1 Engineering disciplines

Consider what kinds of process a mechanical engineer, a chemical engineer, or an electrical engineer follows in their work, and what kind of studies they require for proficiency in their work.

A mechanical engineer studies calculus, linear algebra, differential geometry, and several areas of physics such as theoretical mechanics, thermodynamics, and elasticity theory, and then uses calculations to guide the design of a bridge, say. A chemical engineer studies chemistry, thermodynamics, calculus, linear algebra, differential equations, some areas of physics such as thermodynamics and kinetic theory, and uses calculations to guide the design of a chemical process, say. An electrical engineer studies advanced calculus, linear algebra, as well as several areas of physics such as electrodynamics and quantum physics, and uses calculations to guide the design of an antenna or a microchip.

The pattern here is that an engineer uses mathematics and natural sciences in order to design new devices. Mathematical calculations and scientific reasoning are required *before* drawing a design, let alone building a real device or machine.

Some of the studies required for engineers include arcane abstract concepts such as a "rank-4 elasticity tensor" (used in calculations of elasticity of materials), "Lagrangian with non-holonomic constraints" (used in robotics), the "Gibbs free energy" (for chemical reactor design), or the "Fourier transform of the delta function" and the "inverse Z-transform" (for digital signal processing).

To be sure, a significant part of what engineers do is not covered by any theory: the *know-how*, the informal reasoning, the traditional knowledge passed on from expert to novice, — all those skills that are hard to formalize. Nevertheless, engineering is crucially based on natural science and mathematics for some of its decision-making about new designs.

18.2 Artisanship: Trades and crafts

Now consider what kinds of things shoemakers, plumbers, or home painters do, and what they have to learn in order to become proficient in their profession.

A novice shoemaker, for example, would begin by copying some drawings and then cutting leather in a home workshop. Apprenticeships proceed via learning by doing while listening to comments and instructions from an expert. After a few years of apprenticeship (for example, a painter apprenticeship in California can be as short as 2 years), a new specialist is ready to start productive work.

All these trades operate entirely from tradition and practical experience. The trades do not require any academic study because there is no formal theory from which to proceed. To be sure, there is *a lot* to learn in the crafts, and it takes a large amount of effort to become a good artisan in any profession. But there are no rank-4 tensors to calculate, nor any differential equations to solve; no Fourier transforms to apply to delta functions, and no Lagrangians to check for non-holonomic constraints.

Artisans do not study any formal science or mathematics because their professions do not make use of any *formal computation* for guiding their designs or processes.

18.3 Programmers today are artisans, not engineers

Now I will argue that programmers are *not engineers* in the sense we normally see the engineering professions.

18.3.1 No requirement of formal study

According to this recent Stack Overflow survey, about half of the programmers do not have a degree in Computer Science. I am one myself; my degrees are in

physics, and I have never formally studied computer science. I took no academic courses in algorithms, data structures, computer networks, compilers, programming languages, or any other topics ordinarily included in the academic study of "computer science". None of the courses I took at university or at graduate school were geared towards programming. I am a completely self-taught software developer.

There is a large number of successful programmers who *never* studied at a college, or perhaps never studied formally in any sense. They acquired all their knowledge and skills through self-study and practical work. Robert C. Martin is one such prominent example; an outspoken guru in the arts of programming who has seen it all, he routinely refers to programmers as artisans and uses the appropriate imagery: novices, trade and craft, the "honor of the guild", etc. He compares programmers to plumbers, electricians, lawyers, and surgeons, but not to mathematicians, physicists, or engineers of any kind. According to one of his blog posts, he started working at age 17 as a self-taught programmer, and then went on to more jobs in the software industry; he never mentions going to college. It is clear that R. C. Martin *is* an expert craftsman, and that he did not need academic study to master his craft.

Here is another opinion (emphasis is theirs):

> Software Engineering is unique among the STEM careers in that it absolutely does *not* require a college degree to be successful. It most certainly does not require licensing or certification. *It requires experience*.

This is a description that fits a career in crafts—but certainly not a career, say, in electrical engineering.

The high demand for software developers gave rise to "developer boot camps"—vocational schools that prepare new programmers very quickly, with no formal theory or mathematics involved, through purely practical training. These vocational schools are successful in job placement. But it is unimaginable that a 6-month crash course or even a 2-year vocational school could prepare an engineer to work successfully on designing, say, quantum analog computers, without ever having studied quantum physics or calculus.

18.3.2 No mathematical formalism to guide the *development* of software

Most books on software engineering contain no formulas or equations, no mathematical derivations of any results, and no precise definitions of the various technical terms they are using (such as "object-oriented" or "software architecture"). Some books on software engineering even have no program code in them—just words and illustrative diagrams. These books talk about how programmers should approach their job, how to organize the work flow and the code architecture, in vague and general terms: "code is about detail", "you must never abandon the big picture", "you should avoid tight coupling in your modules", "a class must serve a single responsibility", and so on. Practitioners such as R. C. Martin never studied any formalisms and do not think in terms of formalisms; instead they think in vaguely formulated, heuristic "principles".

In contrast, every textbook on mechanical engineering or electrical engineering has a significant amount of advanced mathematics in it. The design of a microwave antenna is guided first and foremost not by "avoiding tight coupling in modules" but by calculations of wave propagation, based on electrodynamics.

Donald Knuth's classic textbook is called "*The art of programming*". It is full of tips and tricks about how to program; but it does not provide any formal theory that could guide programmers while actually *writing* programs. There is nothing in that book that would be similar to the way mathematical formalism guides designs in electrical or mechanical engineering. If Knuth's books were based on such formalism, they would have looked quite differently: some theory would be first explained and then applied to help us write code.

Knuth's books provide many algorithms, including mathematical ones. But algorithms are similar to patented inventions: They can be used immediately without further study. Algorithms are not similar to theory. Knowing one algorithm does not make it easier to develop another, unrelated algorithm. In comparison, knowing how to solve differential equations will be applicable to thousands of different areas of science and engineering.

A book exists with the title "Science of programming", but the title is misleading. The author does not propose a science, similar to physics, at the foundation of the process of designing programs, similarly to how calculations in quantum physics predict the properties of a quantum device. The book claims to give precise methods that guide programmers in writing code, but the scope of proposed methods is narrow: the design of simple algorithms for iterative manipulation of data. The procedure suggested in that book is far from a formal mathematical

derivation of programs from specification. (A book with that title also exists, and similarly disappoints.) Programmers today are mostly oblivious to these books and do not use methods explained there.

Standard computer science courses today do not teach a true *engineering* aproach to software construction. They do teach analysis of programs using formal mathematical methods; the main such methods are complexity analysis (the so-called "big O notation"), and formal verification. But programs are analyzed only *after* they are complete. Theory does not guide the actual *process* of writing code, does not suggest good ways of organizing the code (e.g. which classes or functions or modules should be defined), and does not tell programmers which data structures or APIs would be best to implement. Programmers make these design decisions purely on the basis of experience and intuition, trial-and-error, copy-paste, and guesswork.

The theory of program analysis and verification is analogous to writing a mathematical equation for the shape of the surface of a shoe made by a fashion designer. True, the "shoe surface equations" are mathematically unambiguous and can be "analyzed" or "verified"; but the equations are written after the fact and do not guide the fashion designers in actually making shoes. It is understandable that fashion designers do not study the mathematical theory of surfaces.

18.3.3 Programmers avoid academic terminology

Programmers appear to be taken aback by such terminology as "*functor*", "*monad*", or "*lambda-functions*".

> Those fancy words used by functional programmers purists really annoy me. Monads, Functors... Nonsense!!!

In my experience, only a tiny minority of programmers actually complain about this. The vast majority has never heard these words and are unaware of functors or monads.

However, chemical engineers do not wince at "phase diagram" or "Gibbs free energy", and apparently accept the need for studying differential equations. Electrical engineers do not complain that the word "Fourier" is foreign and difficult to spell, or that "delta-function" is such a weird thing to say. Mechanical engineers take it for granted that they need to calculate with "tensors" and "Lagrangians" and "non-holonomic constraints". Actually, it seems that the arcane terminology is the least of their difficulties! Their textbooks are full of complicated equations and long, difficult derivations.

Similarly, software engineers would not complain about the word "functor", or about having to study the derivation of the algebraic laws for "monads,"—if they were actually *engineers*. True software engineers' textbooks would be full of equations and derivations, which would be used to perform calculations required *before* starting to write code.

18.4 Towards software engineering

It is now clear that we do not presently have true software engineering. The people employed under that job title are actually artisans. They employ artisanal methods for their work, and their culture and work process are that of a crafts guild.

One could point out that certain programming tasks, for example numerical simulations in physics or machine learning, are mathematical and require formal theory. However, mathematical subject matter (aerospace control, physics or astronomy experiments, statistics, data science) does not automatically make the *process of programming* into engineering. Data scientists, aerospace engineers, and natural scientists are all working as artisans when they write code.

True software engineering would be achieved if we had theory that guides and informs our process of creating programs,—not theory that describes or analyzes programs *after* they are somehow written.

We expect that software engineers' textbooks should be full of equations. What theory should those equations represent?

I believe this theory already exists, and I propose to call it **functional type theory**. It is the algebraic foundation of modern functional programming, as implemented in languages such as OCaml, Haskell, and Scala. This theory is a blend of type theory, category theory, and logical proof theory. It has been in development since late 1990s and is still being actively worked on by a community of academic computer scientists and advanced software practitioners.

To appreciate that functional programming, unlike any other programming paradigm, *has a theory that guides coding*, we can look at some recent software engineering conferences such as Scala By the Bay or BayHac, or at the numerous FP-related online tutorials and blogs. We cannot fail to notice that much time is devoted not to writing code but to a peculiar kind of mathematical reasoning. Rather than focusing on one or another API or algorithm, as it is often the case with other software engineering blogs or presentations, an FP speaker describes a *mathematical structure*—such as the "applicative functor" or the "free

Figure 18.1: Example calculation in functional type theory.

monad"—and illustrates its use for practical coding.

These people are not some graduate students showing off their theoretical research; they are practitioners, software engineers who use FP on their jobs. It is just the nature of FP that certain mathematical tools—coming from formal logic and category theory—are now directly applicable to practical programming tasks.

These mathematical tools are not mere tricks for a specific programming language; they apply equally to all FP languages. Before starting to write code, the programmer can jot down something that resembles calculations in abstract algebra (see Fig. 18.1), which will help design the code fragment they are about to write. This activity is quite similar to that of an engineer who first performs some mathematical calculations and only then embarks on a real-life design project.

A concrete example of this hand-in-hand development of the functional type theory and its applications is the so called "free applicative functor" construction. It was first described in a 2014 paper; a couple of years later, a combined free applicative + free monad data type was designed and its implementation proposed in Scala as well as in Haskell. This technique allows programmers to work with declarative side-effect computations where some parts can be computed in parallel, and to achieve the parallelism *automatically* while maintaining the composability of the resulting programs. The new technique has distinct advantages over using monad transformers, which was the previous method of composing declarative side-effects.

The "free applicative + free monad" combination was designed and implemented by true software engineers. They first wrote down the types and derived the necessary algebraic properties; the obtained results directly guided them about how to proceed writing the library API.

Another example of a development in functional type theory is the so called "tagless final" encoding of data types, first described in 2009. This technique, developed from category theory and type theory motivations, has several advantages over the free monad technique and can improve upon it in a number of cases—just as the free monad itself was designed to cure certain problems with monad transformers. The new technique is also not a trick in a specific programming language; rather, it is a theoretical development that is available to programmers in any language (even in Java).

This example shows that we may need several more years of work before the practical aspects of using "functional type theory" are sufficiently well understood by the FP community. The theory is in active development, and its design patterns—as well as the exact scope of the requisite theoretical material—are still being figured out. If the 40-year gap hypothesis holds, we should expect functional type theory (perhaps under a different name) to become mainstream by 2030. This book is a step towards a clear designation of the scope of that theory.

18.5 Do we need software *engineering*, or are the artisans good enough?

The demand for programmers is growing. Software developer was #1 best job in the US in 2018. But is there a demand for engineers, or just for artisans?

We do not seem to be able to train enough software artisans. Therefore, it is probably impossible to train as many software engineers in the true sense of the word. Modern courses in Computer Science do not actually train engineers in that sense; at best, they train academics who act as software artisans when writing code. The few existing true software *engineers* are all self-taught. Recalling the situation in construction business, with a few architects and hundreds of construction workers, we might also conclude that, perhaps, only a few software engineers are required per hundred software artisans.

What is the price of *not* having engineers, of replacing them with artisans?

Software practitioners have long bemoaned the mysterious difficulty of software development. Code "becomes rotten with time", programs grow in size "out of control", and Microsoft's operating systems have been notorious for ever-appearing security flaws despite many thousands of programmers and testers Microsoft employed. I think this shows we are overestimating the artisanal creative capacity of the human brain.

It is precisely in designing very large and robust software systems that we would benefit from true engineering. Consider that humanity has been using chemical reactions and building bridges by trial, error, and adherence to tradition, long before mechanical or chemical engineering disciplines were developed and founded upon rigorous theory. Once the theory became available, humanity proceeded to create unimaginably more complicated and powerful structures and devices than ever before.

For building large and reliable software, such as new mobile or embedded

operating systems or distributed peer-to-peer trust architectures, we will most likely need the qualitative increase in productivity and reliability that can only come from transforming artisanal programming into a proper engineering discipline. Functional type theory and functional programming are first steps in that direction.

19 Essay: Towards functional data engineering with Scala

Data engineering is among the most in-demand novel occupations in the IT world today. Data engineers create software pipelines that process large volumes of data efficiently. Why did the Scala programming language emerge as a premier tool for crafting the foundational data engineering technologies such as Spark or Akka? Why is Scala in such demand within the world of big data software?

There are reasons to believe that the choice of Scala was quite far from pure happenstance.

19.1 Data is math

Humanity has been working with data at least since Babylonian tax tables and the ancient Chinese number books. Mathematics summarizes several millennia's worth of data processing experience into a few tenets:

- Data is *immutable*, because facts are immutable.

- Each *type* of values—population count, land area, distances, prices, dates, times,—needs to be handled separately; e.g. it is meaningless to add a distance to a population count.

- Data processing is to be codified by *mathematical formulas*.

Violating these tenets produces nonsense (see Fig. 19.1).

The power of the basic principles of mathematics extends over all epochs and all cultures; they are the same in Rio de Janeiro, in Kuala-Lumpur, and even in Pyongyang (see Fig. 19.2).

Figure 19.1: A nonsensical calculation arises when mixing incompatible data types.

19.2 Functional programming is math

The functional programming paradigm is based on similar principles: values are immutable, data processing is coded through formula-like expressions, and each type of data is required to match correctly during the computations. A flexible system of data types helps programmers automatically prevent many kind of coding errors. In addition, modern programming languages such as Scala and Haskell have a set of features adapted to building powerful abstractions and domain-specific languages. This power of abstraction is not accidental. Since mathematics is the ultimate art of building abstractions, math-based functional programming languages capitalize on the advantage of several millennia of mathematical experience.

A prominent example of how mathematics informs the design of programming languages is the connection between constructive logic and the programming language's type system, called the Curry-Howard (CH) correspondence. The main idea of the CH correspondence is to think of programs as mathematical formulas that compute a value of a certain type *A*. The CH correspondence is between programs and logical propositions. To any program that computes a

value of type *A*, there corresponds a proposition stating that "a value of type *A* can be computed".

This may sound rather theoretical so far. To see the real value of the CH correspondence, recall that formal logic has operations "*and*", "*or*", and "*implies*". For any two propositions *A*, *B*, we can construct the propositions "*A and B*", "*A or B*", "*A implies B*". These three logical operations are foundational; without one of them, the logic is *incomplete* (you cannot derive some theorems).

A programming language **obeys the CH correspondence** to the logic if for any two types *A*, *B*, the language also contains composite types corresponding to the logical formulas "*A or B*", "*A and B*", "*A implies B*". In Scala, these composite types are Either[A,B], the tuple (A,B), and the function type, A⇒B. All modern functional languages such as OCaml, Haskell, Scala, F#, Swift, Rust, Elm, PureScript support these three type constructions and thus are faithful to the CH correspondence. Having a *complete* logic in a language's type system enables declarative domain-driven code design.

It is interesting to note that most older programming languages (C/C++, Java, JavaScript, Python) do not support some of these composite types. In other words, these programming languages have type systems based on an incomplete logic. As a result, users of these languages have to implement burdensome workarounds that make for error-prone code. Failure to follow mathematical principles has real costs.

19.3 The power of abstraction

Data engineering at scale poses problems of such complexity that many software companies adopt functional programming languages as their main implementation tool. Netflix, LinkedIn, Twitter started using Scala early on and were able to reap the benefits of the powerful abstractions Scala affords, such as asynchronous streams and parallelized functional collections. In this way, Scala enabled these businesses to engineer and scale up their massively concurrent computations. What exactly makes Scala suitable for big data processing?

The only way to manage massively concurrent code is to use sufficiently high-level abstractions that make application code declarative. The two most important such abstractions are the "resilient distributed dataset" (RDD) of Apache Spark and the "reactive stream" used in systems such as Kafka, Apache Storm, Akka Streams, and Apache Flink. While these abstractions are certainly implementable in Java or Python, true declarative and type-safe usage is possible only

Figure 19.2: The Pyongyang method of error-free programming.

in a programming language with a sufficiently sophisticated functional type system. Among the currently available mature functional languages, only Scala and Haskell would be technically adequate for that task, due to their support for type classes and higher-order generic collections.

It remains to see why Scala became the *lingua franca* of big data and not, say, Haskell.

19.4 Scala is Java on math

The recently invented general-purpose functional programming languages can be grouped into "industrial" (F#, Scala, Swift) and "academic" (OCaml, Haskell).

The "academic" languages are clean-room implementations of well-researched mathematical principles of programming language design (the CH correspondence being one such principle). These languages are unencumbered by requirements of compatibility with any existing platform or libraries. Because of this, the "academic" languages are perfect playgrounds for taking various mathematical ideas to their logical conclusion. At the same time, software practitioners strug-

gle to adopt these languages due to a steep learning curve, a lack of enterprise-grade libraries and tool support, and immature package management.

The languages from the "industrial" group are based on existing and mature software ecosystems: F# on .NET, Scala on JVM, and Swift on Apple's Cocoa/iOS platform. One of the important design requirements for these languages is 100% binary compatibility with their "parent" platforms and languages (F# with C#, Scala with Java, and Swift with Objective-C). Because of this, developers can immediately take advantage of the existing tooling, package management, and industry-strength libraries, while slowly ramping up the idiomatic usage of new language features. However, the same compatibility requirement necessitated certain limitations in the languages, making their design less than fully satisfactory from the functional programming viewpoint.

It is now easy to see why the adoption rate of the "industrial" group of languages is much higher than that of the "academic" languages. The transition to the functional paradigm is also made smoother for software developers because F#, Scala, and Swift seamlessly support the familiar object-oriented programming paradigm. At the same time, these new languages still have logically complete type systems, which gives developers an important benefit of type-safe domain modeling.

Nevertheless, the type systems of these languages are not equally powerful. For instance, F# and Swift are similar to OCaml in many ways but omit OCaml's parameterized modules and some other features. Of all mentioned languages, Scala and Haskell are the only ones that directly support type classes and higher-order generics, which are necessary for expressing abstractions such as an automatically parallelized data set or an asynchronous data stream.

To see the impact of these advanced features of Scala and Haskell, consider LINQ, a domain-specific language for database queries on .NET, implemented in C# and F# through a special syntax supported by Microsoft's compilers. Analogous functionality is provided in Scala as a *library*, without need to modify the Scala compiler, by several open-source projects such as Slick, Squeryl, or Quill. Similar libraries exist for Haskell—but are impossible to implement in languages with less powerful type systems.

19.5 Conclusion

The decisive advantages of Scala over other contenders (such as OCaml, Haskell, F#, Swift, or Rust) are

1. functional collections in the standard library;

2. a highly sophisticated type system, with support for type classes and higher-order generics;

3. seamless compatibility with a mature software ecosystem (JVM).

Based on this assessment, we may be confident in Scala's great future as a main implementation language for big data engineering.

A Notations

F^A means a type constructor F with a type argument A. In Scala, this is `F[A]`.

$x^{:A}$ means a value x that has type A. The colon, :, in the superscript is to show that it is not a type argument. In Scala, this is `x :   A`. The notation $x : A$ is used as well, but $x^{:A}$ is easier to read when x is inside a code expression.

$A + B$ means the disjunction type made of types A and B. In Scala, this is the type expression `Either[A, B]`.

$A \times B$ means the product type made of types A and B. In Scala, this is the tuple `(A,B)`.

$A \Rightarrow B$ means a function type from A to B. In Scala, this is the function type `A ⇒ B`.

$\equiv$ means "equal by definition". Examples:

- $f \equiv x^{:Int} \Rightarrow x + 1$ is a definition of the function f. In Scala, this is `val f = (x:   Int) ⇒ x + 1`.

- $F^A \equiv 1 + A$ is a definition of a type constructor F. In Scala, this is `type F[A] = Option[A]`.

$\cong$ means "equivalent" according to an equivalence relation that needs to be established in the text. For example, if we have established the equivalence that allows nested tuples to be reordered whenever needed, we can write $(a \times b) \times c \cong a \times (b \times c)$.

$(A)^{:F^B}$ in type expressions means that the type constructor $F^\bullet$ assigns the type F^B to the type expression A. This is used in defining unfunctors (GADTs) such as $F^A \equiv 1^{:F^{Int}} + A^{:F^{A \times String}}$. In Scala, this is

```scala
sealed trait F[A]
case class F1() extends F[Int]
case class F2(a: A) extends F[(A, String)]
```

fmap_F means the standard method fmap pertaining to the functor F. In Scala, this is typically implemented as `Functor[F].fmap()`. Since each functor F will

have its own specific definition of fmap, the subscript "F" is not a type parameter of fmap. The method fmap actually has *two* type parameters, and its type signature is $\mathrm{fmap}_F^{A,B} : (A \Rightarrow B) \Rightarrow F^A \Rightarrow F^B$. For clarity, one may write explicitly the type parameters A, B in the expression $\mathrm{fmap}_F^{A,B}$, but in most cases these type parameters may be omitted.

Similarly, a monad's standard method pure_F has the subscript denoting its monad F. This function has type signature $A \Rightarrow F^A$ that contains a type parameter A. If we are using this method with a complicated type, e.g. $1 + P^A$, instead of A, we might want to indicate this type for clarity as a type parameter and write $\mathrm{pure}_F^{1+P^A}$. The type signature of that function is

$$\mathrm{pure}_F^{1+P^A} : 1 + P^A \Rightarrow F^{1+P^A} \quad .$$

$F^\bullet$ means the type constructor F understood as a type-level function, – that is, with its type argument unspecified. In Scala, this is F[_]. The bullet symbol, $\bullet$, is used as a placeholder for the missing type parameter. I also simply write F when no type argument is needed, and it means the same as $F^\bullet$. (For example, I say "a functor F" and "a functor $F^\bullet$" to mean the same thing.) However, it is useful for clarity to be able to indicate the place where the type argument would appear. For instance, $T_L^{M,\bullet}$ denotes a monad transformer for the base monad L and the foreign monad M. The foreign monad M is a type parameter in $T_L^{M,\bullet}$, and so is the missing type parameter denoted by the placeholder symbol $\bullet$. (However, the base monad L is not a type parameter in $T_L^{M,\bullet}$ because the construction of the monad transformer depends sensitively on the internal details of L.)

$F^\bullet \rightsquigarrow G^\bullet$ or $F^A \rightsquigarrow G^A$ means a natural transformation between two functors F and G.

$\forall A.P^A$ is a universally quantified type expression, in which A is a bound type parameter.

$\exists A.P^A$ is an existentially quantified type expression, in which A is a bound type parameter.

$\,\mathbin{\raise.2ex{\hbox{$\scriptstyle\fatsemi$}}}$ means the forward composition of functions: $f \mathbin{\raise.2ex{\hbox{$\scriptstyle\fatsemi$}}} g$ (reads "f before g") is the function defined as $x \Rightarrow g(f(x))$.

$\circ$ means the backward composition of functions: $f \circ g$ (reads "f after g") is the function defined as $x \Rightarrow f(g(x))$.

$f^{\uparrow G}$ means a function f lifted to a functor G. For a function $f^{:A \Rightarrow B}$, the application of $f^{\uparrow G}$ to a value $g : G^A$ is written as $f^{\uparrow G} g$. In Scala, this is g.map(f). Nested liftings can be written as $f^{\uparrow G \uparrow H}$, which means $\left(f^{\uparrow G} \right)^{\uparrow H}$ and produces a

function of type $H^{G^A} \Rightarrow H^{G^B}$. Applying a nested lifting to a value h of type H^{G^A} is written as $f^{\uparrow G \uparrow H} h$. In Scala, this is `h.map(_.map(f))`.

$f^{\downarrow H}$ means a function f lifted to a contrafunctor H. For a function $f^{:A \Rightarrow B}$, the application of $f^{\downarrow H}$ to a value $h : H^B$ is written as $f^{\uparrow G} g$. In Scala, this is `g.map(f)`.

$\diamond_M$ means the Kleisli product operation for the monad M.

B Glossary of terms

I chose certain terms in this book to be different from the terms currently used in the functional programming community. My proposed terminology is designed to help readers understand and remember the concepts behind the terms.

Nameless function An expression of function type, representing a function. For example, x ⇒ x * 2. Also known as function expression, function literal, anonymous function, closure, lambda-function, lambda-expression, or simply a "lambda".

Product type A type representing several values given at once. In Scala, product types are the tuple types, for example (Int, String), and case classes. Also known as **tuple** type, **struct** (in C and C++), and **record**.

Disjunction type A type representing one of several distinct possibilities. In Scala, this is usually implemented as a sealed trait extended by several case classes. The standard Scala disjunction types are Option[A] and Either[A, B]. Also known as **sum** type, **co-product** type, and variant type (in OCaml and in Object Pascal). It is shorter to say "sum type," but the English word "sum" is more ambiguous to the ear than "disjunction".

Polynomial functor A type built out of disjunctions (sums) and products of type parameters and constant types. For example, in Scala, type F[A] = Either[(Int, A), A]. Also known as algebraic data type (ADT). A polynomial type constructor is always a functor in any of its type parameters, hence the shorter name "polynomial functor" instead of "polynomial type constructor".

Unfunctor A type constructor that cannot possibly be a functor, a contrafunctor, or a profunctor. An example is a type constructor with explicitly indexed type parameters, such as $F^A \equiv (A \times A)^{:F^{\text{Int}}} + (\text{Int} \times A)^{:F^1}$. Also called a **GADT** (generalized algebraic data type).

Method 1) A function defined as a member of a typeclass. For example, `flatMap` is a method defined in the `Monad` typeclass. 2) In Scala, a function defined as a member of a data type declared as a Java-compatible `class` or `trait`. Trait methods are necessary in Scala when implementing functions whose arguments have type parameters (because ordinary Scala function values cannot have type parameters). So, many type classes such as `Functor` or `Monad`, whose methods$_1$ require type parameters, will use Scala `trait`s with methods$_2$ for their implementation. The same applies to type constructions with quantified types, such as Church encoding.

Exponential-polynomial type A type built out of disjunctions (sums), products, and function types. For brevity, I also say "exp-poly". For example, in Scala, `type F[A] = Either[(A, A), Int ⇒ A]` is an exp-poly type constructor. Such type constructors can be functors, contrafunctors, or profunctors.

Short type notation A mathematical notation for type expressions that are adapted to practical use in functional programming. Disjunction types are denoted by +, product types by ×, and function types by ⇒. The function arrow ⇒ has weaker precedence than +, which is in turn weaker than ×. Type parameters are denoted by superscripts. As an example of using these conventions, the Scala definition

```
type F[A] = Either[(A, A ⇒ Option[Int]), String ⇒ A]
```

is denoted in the short type notation as

$$F^A \equiv A \times (A \Rightarrow 1 + \text{Int}) + \left(\text{String} \Rightarrow A\right) \quad .$$

B.1 On the current misuse of the term "algebra"

In this book, I do not use the terms "algebra" or "algebraic" because these terms are too ambiguous. In the current practice, the functional programming community is using the word "algebra" in at least *four* incompatible ways.

Definition 0. In mathematics, an "algebra" is a vector space with multiplication and certain standard properties. For example, you need $1 * x = x$, the addition must be commutative, the multiplication must be distributive over addition, and so on. As an example, the set of all 10×10 matrices with real coefficients is an

"algebra" in this sense. These matrices form a 100-dimensional vector space, and can be multiplied and added. This standard definition of "algebra" is not actually used in functional programming.

Definition 1. An "algebra" is a function with type signature $F^A \Rightarrow A$, where F^A is some fixed functor. This definition comes from category theory, where such types are called F**-algebras**. There is no direct connection between this "algebra" and Definition 0, except when the functor F is defined by $F^A \equiv A \times A$, and then a function of type $A \times A \Rightarrow A$ may be interpreted as a "multiplication" operation (but, in any case, A is a type and not a vector space, and there are no distributivity or commutativity laws). I prefer to call such functions "F-algebras", emphasizing that they characterize and depend on a chosen functor F.

Definition 2. Polynomial functors are often called "algebraic data types". However, they are not "algebraic" in the sense of Definition 0 or 1. For example, consider the "algebraic data type" `Either[Option[A], String]`, or $F^A \equiv 1 + A +$ String in the short type notation. The set of all values of the type F^A does not admit the addition and multiplication operations required by the mathematical definition of "algebra". The type F^A may admit some binary or unary operations (e.g. that of a monoid), but these operations will not be commutative or distributive. Also, there cannot be a function with type $F^A \Rightarrow A$, as required for Definition 1. It seems that the usage of the word "algebra" here is to refer to "school-level algebra" with polynomials; these data types are built from sums and products of types. In this book, I call such types "polynomial". However, if the data type contains a function type, e.g. `Option[Int ⇒ A]`, the type is no longer polynomial. So I use the more precise terms "polynomial type" and "exponential-polynomial type".

Definition 3. People talk about the "algebra" of properties of functions such as `map` or `flatMap`, referring to the fact that these functions must satisfy certain equational laws (e.g. the identity, composition, or associativity laws). But these laws do not form an "algebra" in the sense of Definition 0, nor do the functions such as `map` or `flatMap` themselves (there are no binary operations on them). Neither do they form an algebra in the sense of Definition 1. The laws for `map` or `flatMap` are in no way related to "algebraic data types" of Definition 2. So here the word "algebra" is used in a way that is unrelated to the three previous definitions. To me, it does not seem helpful to say the word "algebra" or "algebraic"

when talking about equational laws.

Definition 4.   In the Church encoding of a free monad (currently called the "final tagless" encoding), the term "algebra" refers to the *type constructor* parameter F. This definition has nothing to do with any of the previous definitions. Clearly, Definition 0 cannot apply to a type constructor. Definition 1 does not apply since F is not itself a function type of the form $G^A \Rightarrow A$. (A function of type $F^A \Rightarrow A$, which I call a "runner" for the type constructor F, is not usually called an "algebra" in discussions of the "final tagless" encoding.) Definition 2 seems to be the most closely related meaning, since F is *sometimes* a polynomial type in practical usage (although in most cases F will be an unfunctor). However, it is not helpful to call the polynomial functor F an "algebraic data type" and, at the same time, an "algebra". Definition 3 does not apply since the free monad construction does not assume that any laws hold about F, and .

I propose to reserve the word "algebra" to denote the branch of mathematics, as in "school-level algebra" or "graduate-level algebra". Instead of "algebra" as in Definitions 1 to 4, I propose to talk about "F-algebras" with a specific functor F; "polynomial types" or "polynomial functors" or "exponential-polynomial functors" etc.; "equational laws"; and an "effect constructor" F.

C Scala syntax and features

C.0.1 Function syntax

Functions have arguments, body, and type. The function type lists the type of all arguments and the type of the result value.

```
def f(x: Int, y: Int): Int ⇒ Int = { z ⇒ x + y + z }
```

Functions may be used with infix syntax as well. For this syntax to work, the function must be defined **as a Scala method**, that is, using `def` within the declaration of x's class, or as an extension method. The infix syntax cannot work with functions defined using `val`. For clarity, I call Scala functions **infix methods** when defined and used in this way.

The type syntax `List[Int]` means "a list of integer values." In the type expression `List[Int]`, the "Int" is called the **type parameter** and `List` is called the **type constructor**. A list can contain values of any type; for example, `List[List[List[Int]]]` means a list of lists of lists of integers. So, the type constructor can be seen as a function from types to types: A type constructor takes a type parameter as an argument, and produces a new type as a result.

C.0.2 Scala collections

The Scala standard library defines collections of several kinds, the main ones being sequences, sets, and dictionaries. These collections have many map/reduce-style methods defined on them.

Sequences are "subclasses" of the class `Seq`. The standard library will sometimes choose automatically a suitable subclass of `Seq`, such as `List`, `IndexedSeq`, `Vector`, `Range`, etc.; for example:

```
scala> 1 to 5
scala> (1 to 5).map(x ⇒ x*x)
scala> (1 to 5).toList
scala> 1 until 5
```

```
scala> (1 until 5).toList
```

For our purposes, all these "sequence-like" types are equivalent.

Sets are values of class `Set`, and dictionaries are values of class `Map`.

```
scala> Set(1, 2, 3).filter(x ⇒ x % 2 == 0)
```

D Typed λ-calculus

Typed λ-calculus is a simple programming language[1] that is intended to help understand how to design functional programming languages. Indeed, languages such as OCaml, Haskell, Scala, F#, Swift, Rust, Elm, and PureScript include typed λ-calculus as a sub-language.

---

[1] Although called a "calculus," it is really just a programming language. It has nothing to do with differential or integral calculus.

E Intuitionistic (constructive) propositional logic

The intuitionistic propositional logic describes how programs in functional programming languages may be able to compute values of different types.

The main difference between IPL and the classical (Boolean) logic is that IPL does not include the axiom of excluded middle ("*tertium non datur*"), which is

$$\forall A : (A \text{ or } (\text{not}A)) \text{ is true} \quad .$$

The reason this axiom is not included in IPL is that IPL propositions such as $\mathcal{CH}(A)$ correspond to the *possibility* of values of type A *to be computed* by a program. For the proposition $\mathcal{CH}(A)$ to be true in IPL, a program needs to actually compute a value of type A. It is not sufficient merely to show that the non-existence of such a value would be somehow contradictory. (In classical logic, that would be sufficient.)

Another significant difference between IPL and the Boolean logic is that propositions in IPL cannot be assigned a fixed set of "truth values". This was proved by Gödel in 1935. It means that a proposition in IPL cannot be decided by writing out a truth table, even if we allow many truth values.

F Category theory

Examples of categories

1. Objects: types Int, String, ...; morphisms (arrows) are functions Int $\rightarrow$ String etc. – this is the "standard" category corresponding to a given programming language

2. Objects: types A, B, ...; morphisms are pairs of functions $(A \rightarrow B)$, $(B \rightarrow A)$

3. * Objects: types List^A, List^B, ...; morphisms are functions of type $\text{List}^A \rightarrow \text{List}^B$

4. Objects: types A, B, ...; morphisms are functions of type $\text{List}^A \rightarrow \text{List}^B$

5. Objects: types A, B, ...; morphisms are functions of type $A \rightarrow \text{List}^B$

6. * Objects: types List^A, List^B, ...; morphisms are functions $A \rightarrow B$

7. Objects: types A, B, ...; morphisms are $\text{List}^{A \rightarrow B}$

8. Objects: types A, B, ...; morphisms are functions $B \rightarrow A$

9. * Objects: things A, B, ...; morphisms are pairs (A, B) of things – this is the same as a preorder

Examples marked with * are for illustration only, they are probably not very useful

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