Chapter 11: Computations in a functor context III

Monad transformers

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Computations within a functor context: Combining monads

Programs often need to combine monadic effects

- "Effect" $\equiv$ what else happens in $A \Rightarrow M^B$ besides computing B from A
- Examples of effects for some standard monads:
  - `Option` – computation will have no result or a single result
  - `List` – computation will have zero, one, or multiple results
  - `Either` – computation may fail to obtain its result, reports error
  - `Reader` – computation needs to read an external context value
  - `Writer` – some value will be appended to a (monoidal) accumulator
  - `Future` – computation will be scheduled to run later
- How to combine several effects in the same functor block (`for`/`yield`)?

```
// This is not valid Scala!
val result = for { i ← 1 to n
   j ← Future { q(i) }
   k ← maybeError(j) : Try[Int]
} yield f(k)
// What should be the type of result??
```

```
// This is not valid Scala!
(1 to n).flatMap { i ⇒
   Future(q(i)).flatMap { j ⇒
     maybeError(j).map { k ⇒
       f(k)
     }}}
```

- The code will work if we "unify" all effects in a new, larger monad
- Need to compute the type of new monad that contains all given effects

Combining monadic effects I. Trial and error

There are several ways of combining two monads into a new monad:

- If M_1^A and M_2^A are monads then $M_1^A \times M_2^A$ is also a monad
  - But $M_1^A \times M_2^A$ describes two separate values with two separate effects
- If M_1^A and M_2^A are monads then $M_1^A + M_2^A$ is usually not a monad
  - If it worked, it would be a choice between two different values / effects
- If M_1^A and M_2^A are monads then one of $M_1^{M_2^A}$ or $M_2^{M_1^A}$ is often a monad
- Examples and counterexamples for functor composition:
  - Combine $Z \Rightarrow A$ and List^A as $Z \Rightarrow \text{List}^A$
  - Combine `Future[A]` and `Option[A]` as `Future[Option[A]]`
  - But `Either[Z, Future[A]]` and `Option[Z ⇒ A]` are not monads
  - Neither `Future[State[A]]` nor `State[Future[A]]` are monads
- The order of effects matters when composition works both ways:
  - Combine `Either` ($M_1^A = Z + A$) and `Writer` ($M_2^A = W \times A$)
    - ★ as $Z + W \times A$ – either compute result and write a message, or all fails
    - ★ as $(Z + A) \times W$ – message is always written, but computation may fail
- Find a general way of defining a new monad with combined effects
- Derive properties required for the new monad

Combining monadic effects II. Lifting into a larger monad

If a "big monad" `BigM[A]` *somehow* combines all the needed effects:

```
// This could be valid Scala...
val result: BigM[Int] = for {
   i ← lift₁(1 to n)
   j ← lift₂(Future{ q(i) })
   k ← lift₃(maybeError(j))
} yield f(k)
```

```
// If we define the various
// required "lifting" functions:
def lift₁[A]: Seq[A] ⇒ BigM[A] = ???
def lift₂[A]: Future[A] ⇒ BigM[A] = ???
def lift₃[A]: Try[A] ⇒ BigM[A] = ???
```

- Example 1: combining as `BigM[A] = Future[Option[A]]` with liftings:

  ```
  def lift₁[A]: Option[A] ⇒ Future[Option[A]] = Future.successful(_)
  def lift₂[A]: Future[A] ⇒ Future[Option[A]] = _.map(x ⇒ Some(x))
  ```

- Example 2: combining as `BigM[A] = List[Try[A]]` with liftings:

  ```
  def lift₁[A]: Try[A] ⇒ List[Try[A]] = x ⇒ List(x)
  def lift₂[A]: List[A] ⇒ List[Try[A]] = _.map(x ⇒ Success(x))
  ```

Remains to be understood:

- Finding suitable laws for the liftings; checking that the laws hold
- Building a "big monad" out of "smaller" ones, with lawful liftings
  - Is this always possible? Unique? Are there alternative solutions?
- Ways of reducing the complexity of code; make liftings automatic

Laws for monad liftings I. Identity laws

Whatever identities we expect to hold for monadic programs must continue to hold after lifting `M₁` or `M₂` values into the "big monad" `BigM`

- We assume that `M₁`, `M₂`, and `BigM` already satisfy all the monad laws

Consider the various functor block constructions containing the liftings:

- Left identity law after `lift₁`

```
// Anywhere inside a for/yield:        // Must be equivalent to...
i ← lift₁(M₁.pure(x))                  i = x
j ← bigM(i) // Any BigM value.         j ← bigM(x)
```

`lift₁(M₁.pure(x)).flatMap(b) = b(x)` — in terms of Kleisli composition ($\diamond$):
$\left(\text{pure}_{M_1} ; \text{lift}_1\right)^{:X\Rightarrow\text{BigM}^X} \diamond b^{:X\Rightarrow\text{BigM}^Y} = b$ with $f^{:X\Rightarrow M^Y} \diamond g^{:Y\Rightarrow M^Z} \equiv x \Rightarrow f(x).\text{flatMap}(g)$

- Right identity law after `lift₁`

```
// Anywhere inside a for/yield:        // Must be equivalent to...
x ← bigM // Any BigM value.            x ← bigM
i ← lift₁(M₁.pure(x))                  i = x
```

`b.flatMap(M₁.pure andThen lift₁) = b` — in terms of Kleisli composition:

$$b^{:X\Rightarrow\text{BigM}^Y} \diamond \left(\text{pure}_{M_1} ; \text{lift}_1\right)^{:Y\Rightarrow\text{BigM}^Y} = b$$

- The same identity laws must hold for `M₂` and `lift₂` as well

Laws for monad liftings II. Simplifying the laws

$(\mathrm{pure}_{M_1} ⨾ \mathrm{lift}_1)$ is a unit for the Kleisli composition $\diamond$ in the monad `BigM`

- But the monad `BigM` already has a unit element, namely $\mathrm{pure}_{\mathrm{BigM}}$
- The two-sided unit element is always unique: $\mathrm{u} = \mathrm{u} \diamond \mathrm{u}' = \mathrm{u}'$
- So the two identity laws for $(\mathrm{pure}_{M_1} ⨾ \mathrm{lift}_1)$ can be reduced to one law:

$$\mathrm{pure}_{M_1} ⨾ \mathrm{lift}_1 = \mathrm{pure}_{\mathrm{BigM}}$$

Refactoring a portion of a monadic program under `lift₁` gives another law:

```
// Anywhere inside a for/yield, this...
i ← lift₁(p) // Any M₁ value.
j ← lift₁(q(i)) // Any M₁ value.
```

```
// must be equivalent to...
pq = p.flatMap(q) // In M₁.
j ← lift₁(pq) // Now lift it.
```

```
lift₁(p).flatMap(q andThen lift₁) = lift₁(p flatMap q)
```

- Rewritten equivalently through $\mathrm{flm}_M : (A \Rightarrow M^B) \Rightarrow M^A \Rightarrow M^B$ as $\mathrm{lift}_1 ⨾ \mathrm{flm}_{\mathrm{BigM}} (q ⨾ \mathrm{lift}_1) = \mathrm{flm}_{M_1} q ⨾ \mathrm{lift}_1$ – both sides are functions $M_1^A \Rightarrow \mathrm{BigM}^B$
- Rewritten equivalently through $\mathrm{ftn}_M : M^{M^A} \Rightarrow M^A$, the law is $\mathrm{lift}_1 ⨾ \mathrm{fmap}_{\mathrm{BigM}} \mathrm{lift}_1 ⨾ \mathrm{ftn}_{\mathrm{BigM}} = \mathrm{ftn}_{M_1} ⨾ \mathrm{lift}_1$ – both sides are functions $M_1^{M_1^A} \Rightarrow \mathrm{BigM}^A$
- In terms of Kleisli composition $\diamond_M$ it becomes the **composition law**:

$$\left(b^{:X \Rightarrow M_1^Y} ⨾ \mathrm{lift}_1\right) \diamond_{\mathrm{BigM}} \left(c^{:Y \Rightarrow M_1^Z} ⨾ \mathrm{lift}_1\right) = (b \diamond_{M_1} c) ⨾ \mathrm{lift}_1$$

- Liftings lift_1 and lift_2 must obey an identity law and a composition law
  - The laws say that the liftings **commute with** the monads' operations

Laws for monad liftings III. The naturality law

Show that $\text{lift}_1 : M_1^A \Rightarrow \text{BigM}^A$ is a natural transformation

- It maps pure_{M_1} to $\text{pure}_{\text{BigM}}$ and flm_{M_1} to flm_{BigM}
  - lift_1 is a **monadic morphism** between monads $M_1^\bullet$ and $\text{BigM}^\bullet$
  - example: monad "interpreters" $M^A \Rightarrow N^A$ are monadic morphisms

The (functor) naturality law: for any $f : X \Rightarrow Y$,

$$\text{lift}_1 \fatsemi \text{fmap}_{\text{BigM}} f = \text{fmap}_{M_1} f \fatsemi \text{lift}_1$$

$$\begin{tikzcd}
M_1^X \arrow[r, "\mathbf{lift_1}"] \arrow[d, "\mathbf{fmap}_{M_1} f^{:X\Rightarrow Y}"'] & \text{BigM}^X \arrow[d, "\mathbf{fmap_{BigM}} f^{:X\Rightarrow Y}"] \\
M_1^Y \arrow[r, "\mathbf{lift_1}"] & \text{BigM}^Y
\end{tikzcd}$$

Derivation of the functor naturality law for lift_1:

- Express fmap as $\text{fmap}_M f = \text{flm}_M \left(f \fatsemi \text{pure}_M\right)$ for both monads
- Given $f^{:X\Rightarrow Y}$, use the law $\text{flm}_{M_1} q \fatsemi \text{lift}_1 = \text{lift}_1 \fatsemi \text{flm}_{\text{BigM}} \left(q \fatsemi \text{lift}_1\right)$ to compute $\text{flm}_{M_1} \left(f \fatsemi \text{pure}_{M_1}\right) \fatsemi \text{lift}_1 = \text{lift}_1 \fatsemi \text{flm} \left(f \fatsemi \text{pure}_{M_1} \fatsemi \text{lift}_1\right) = \text{lift}_1 \fatsemi \text{flm} \left(f \fatsemi \text{pure}_{\text{BigM}}\right) = \text{lift}_1 \fatsemi \text{fmap}_{\text{BigM}} f$

A monadic morphism is always also a natural transformation of the functors

Monad transformers I: Motivation

- Combine $Z \Rightarrow A$ and $1 + A$: only $Z \Rightarrow 1 + A$ works, not $1 + (Z \Rightarrow A)$
  - It is not possible to combine monads via a natural bifunctor B^{M_1, M_2}
  - It is not possible to combine arbitrary monads as $M_1^{M_2^\bullet}$ or $M_2^{M_1^\bullet}$
    - Example: state monad $\text{St}_S^A \equiv S \Rightarrow A \times S$ does not compose
- The trick: for a fixed **base** monad $L^\bullet$, let $M^\bullet$ (**foreign** monad) vary
- Call the desired result the "L's monad transformer", $T_L^{M,\bullet}$
  - In Scala: `LT[M[_]: Monad, A]` – e.g. `ReaderT`, `StateT`, etc.
- $T_L^{M,\bullet}$ is generic in M but not in L
  - No general formula for monad transformers seems to exist
  - For each base monad L, a different construction is needed
  - Some monads L do not seem to have a transformer (?)
- To combine 3 or more monads, compose the transformers: $T_{L_1}^{T_{L_2}^{M,\bullet}}$
  - Example in Scala: `StateT[S, ListT[Reader[R, ?], ?], A]`
- This is called a **monad stack** – but may not be *functor composition*
  - because e.g. `State[S, List[Reader[R, A]]]` is not a monad

Monad transformers II: The requirements

A **monad transformer** for a **base** monad $L^{\bullet}$ is a type constructor $T_L^{M,\bullet}$ parameterized by a monad $M^{\bullet}$, such that for all monads $M^{\bullet}$

- $T_L^{M,\bullet}$ is a monad (the monad M **transformed with** T_L)
- "Lifting" – a monadic morphism $\text{lift}_L^M : M^A \leadsto T_L^{M,A}$
- "Base lifting" – a monadic morphism $\text{blift} : L^A \leadsto T_L^{M,A}$
  - The "base lifting" could not possibly be natural in $L^{\bullet}$
- Transformed identity monad (Id) must become L, i.e. $T_L^{\text{Id},\bullet} \cong L^{\bullet}$
- $T_L^{M,\bullet}$ is **monadically natural** in $M^{\bullet}$ (but not in $L^{\bullet}$)
  - $T_L^{M,\bullet}$ is natural w.r.t. a monadic functor $M^{\bullet}$ as a type parameter
  - For any monad $N^{\bullet}$ and a monadic morphism $f : M^{\bullet} \leadsto N^{\bullet}$ we need to have a monadic morphism $T_L^{M,\bullet} \leadsto T_L^{N,\bullet}$ for the transformed monads: $\text{mrun}_L^M : (M^{\bullet} \leadsto N^{\bullet}) \Rightarrow T_L^{M,\bullet} \leadsto T_L^{N,\bullet}$
    - If we implement $T_L^{M,\bullet}$ only via M's monad methods, naturality will hold
  - Cf. `traverse`: $L^A \Rightarrow \left(A \Rightarrow F^B\right) \Rightarrow F^{L^B}$ – natural w.r.t. applicative $F^{\bullet}$
  - This can be used for lifting a "runner" $M^A \leadsto A$ to $T_L^{M,\bullet} \leadsto T_L^{\text{Id},\bullet} = L^{\bullet}$
- "Base runner": lifts $L^A \leadsto A$ into a monadic morphism $T_L^{M,\bullet} \leadsto M^{\bullet}$; $\text{brun}_L^M : (L^{\bullet} \leadsto \bullet) \Rightarrow T_L^{M,\bullet} \leadsto M^{\bullet}$

Monad transformers III: First examples

Recall these monad constructions:

- If M^A is a monad then $R \Rightarrow M^A$ is also a monad (for a fixed type R)
- If M^A is a monad then $M^{Z+A\times W}$ is also a monad (for fixed W, Z)

This gives the monad transformers for base monads `Reader`, `Writer`, `Either`:

```
type ReaderT[R, M[_], A] = R ⇒ M[A]
type EitherT[Z, M[_], A] = M[Either[Z, A]]
type WriterT[W, M[_], A] = M[(W, A)]
```

- `ReaderT` wraps the foreign monad from the outside
- `EitherT` and `WriterT` require the foreign monad to wrap *them* outside

Remaining questions:

- What are transformers for other standard monads (`List`, `State`, `Cont`)?
  - These monads do not compose (neither "inside" nor "outside" works)
- How to derive a monad transformer for an arbitrary given monad?
  - For monads obtained via known monad constructions?
  - For monads constructed via other monad transformers?
  - Is it always possible? (unknown; may be impossible for some monads)
- For a given monad, is the corresponding monad transformer unique?
- How to avoid the boilerplate around `lift`? (`mtl`-style transformers)

Monad transformers IV: The zoology of monads

Need to select the correct monad transformer construction, per monad:

- "Composed-inside", base monad is inside foreign monad: $T_L^{M,A} = M^{L^A}$
  - Examples: the "single-value monads" `OptionT`, `WriterT`, `EitherT`
- "Composed-outside" – the base monad is outside: $T_L^{M,A} = L^{M^A}$
  - Examples: `ReaderT`; `SearchT` for search monad `S[A] = (A ⇒ Z) ⇒ A`
  - More generally: all rigid monads have "outside" transformers
    - Definition: a **rigid monad** has the method `fuseIn`: $\left(A \Rightarrow R^B\right) \Rightarrow R^{A \Rightarrow B}$
- "Recursive": interleaves the base monad and the foreign monad
  - Examples: `ListT`, `NonEmptyListT`, `FreeMonadT`
- "Irregular": none of the above constructions work
  - Examples: `StateT`, `ContT`, "codensity monads"
- Monad constructions: defining a transformer for new monads
  - Product monads $L_1^A \times L_2^A$ – "product transformer" $T_{L_1}^{M,A} \times T_{L_2}^{M,A}$
  - Consumer choice monads $H^A \Rightarrow A$ – "composed-outside" transformer
  - Free pointed monads $A + L^A$ – transformer $M^{A + T_L^{M,A}}$
  - "Selectors" $\left(A \Rightarrow P^Q\right) \Rightarrow P^A$ – transformer $\left(M^A \Rightarrow P^Q\right) \Rightarrow T_P^{M,A}$
- Examples of monads for which no transformers exist? (unknown)

Composed-inside transformers I

Base monad $L^{\bullet}$, foreign monad $M^{\bullet}$, transformer $T_L^{M,\bullet} \equiv T^{\bullet} \equiv M^{L^{\bullet}}$

- Monad instance: use the natural transformation $\text{seq}_L^{M,A} : L^{M^A} \rightsquigarrow M^{L^A}$
  - $\text{pure}_T : A \Rightarrow M^{L^A}$ is defined as $\text{pure}_T = \text{pure}_M \fatsemi \text{pure}_L^{\uparrow M}$
  - $\text{ftn}_T : T^{T^A} \Rightarrow T^A$ is defined as $\text{ftn}_T = \text{seq}^{\uparrow M} \fatsemi \text{ftn}_L^{\uparrow M \uparrow M} \fatsemi \text{ftn}_M$

$$T^{T^A} \equiv M^{L^{M^{L^A}}} \xrightarrow[\text{fmap}_M\,\text{seq}_L^{M,L^A}]{} M^{M^{L^{L^A}}} \xrightarrow[\text{fmap}_M(\text{fmap}_M \text{ftn}_L)]{} M^{M^{L^A}} \xrightarrow[\text{ftn}_M]{} M^{L^A} \equiv T^A$$

- Monad laws must hold for T^A (must check this separately)
  - This depends on special properties of $\text{seq}_L^{M,A}$ (denoted seq for brevity), e.g. $\text{pure}_L \fatsemi \text{seq} = \text{pure}_L^{\uparrow M}$ (L-identity); $\text{pure}_M^{\uparrow L} \fatsemi \text{seq} = \text{pure}_M$ (M-identity)
    - ★ See example code that verifies these properties for $L^A \equiv E + W \times A$
    - ★ It is not enough to have *any* traversable functor $L^{\bullet}$ here!
- Monad transformer methods for $T_L^{M,\bullet} \equiv M^{L^{\bullet}}$:
  - Lifting, $\text{lift} : M^A \Rightarrow M^{L^A}$ is defined as $\text{lift} = \text{pure}_L^{\uparrow M}$
  - Base lifting, $\text{blift} : L^A \Rightarrow M^{L^A}$ is equal to pure_M
  - Runner, $\text{mrun} : \left(\forall B. M^B \Rightarrow N^B\right) \Rightarrow M^{L^A} \Rightarrow N^{L^A}$ is simply id
  - Base runner, $\text{brun} : \left(\forall B. L^B \Rightarrow B\right) \Rightarrow M^{L^A} \Rightarrow M^A$ is simply fmap_M

Composed-inside transformers II

Base monad $L^\bullet$, foreign monad $M^\bullet$, transformer $T_L^{M,\bullet} \equiv T^\bullet \equiv M^{L^\bullet}$

- Identity laws for the monad $T^\bullet$ hold:
- $\text{pure}_T \fatsemi \text{ftn}_T = \text{id}$. Proof:
- $\text{pure}_T^{\uparrow T} \fatsemi \text{ftn}_T = \text{id}$
- if $\text{pure}_L \fatsemi \text{seq} = \text{pure}_L^{\uparrow M}$ and $\text{pure}_M^{\uparrow L} \fatsemi \text{seq} = \text{pure}_M$

Rigid monads, their laws and structure I

- A **rigid monad** $R^\bullet$ has the method `fuseIn`: $(A \Rightarrow R^B) \Rightarrow R^{A \Rightarrow B}$
  - Examples: $R^A \equiv A \times A$ and $R^A \equiv Z \Rightarrow A$ are rigid; $R^A \equiv 1 + A$ is not
  - Compare with `fuseOut`: $R^{A \Rightarrow B} \Rightarrow A \Rightarrow R^B$, which exists for any functor
  - Implementation: $\text{fo}\, h^{:R^{A \Rightarrow B}} = x^{:A} \Rightarrow \left(f^{:A \Rightarrow B} \Rightarrow f\, x\right)^{\uparrow R} h$

Laws: the `fuseIn` method (`fi`) must be "compatible with the monad R"

- `fi` must be a lawful lifting from $A \Rightarrow R^B$ to $R^{A \Rightarrow B}$
- A (generalized) functor from Kleisli category to Applicative category
  - identity law: $\text{fi}\,(\text{pure}_R) = \text{pure}_R\,(\text{id})$
  - composition law: $\text{fi}\,(f \diamond_R g) = (p \times q \Rightarrow p ; q)^{\uparrow R}\,(\text{fi}\, f \bowtie \text{fi}\, g)$

$$\begin{array}{ccccccc}
A \Rightarrow R^B & \times & B \Rightarrow R^C & & \xrightarrow[\text{use } \diamond_R]{\hspace{8em}} & & A \Rightarrow R^C \\
\downarrow{\scriptstyle \text{fi}} & & \downarrow{\scriptstyle \text{fi}} & & & & \downarrow{\scriptstyle \text{fi}} \\
R^{A \Rightarrow B} & \times & R^{B \Rightarrow C} & \xrightarrow[\text{use } \bowtie]{} & R^{(A \Rightarrow B) \times (B \Rightarrow C)} & \xrightarrow[\text{fmap}(;)]{} & R^{A \Rightarrow C}
\end{array}$$

  - Alternative formulation: $\text{flm} = \text{fi} ; \text{pa}$ where $\text{pa} : R^{A \Rightarrow B} \Rightarrow R^A \Rightarrow R^B$
  - Then $\text{fi} ; \text{fo} = \text{id}$. Proof: $\text{fo}\, x\, a = \text{pa}\, x\ (\text{pure}\, a)$; set $x^{:R^{A \Rightarrow B}} = \text{fi}\, h^{:A \Rightarrow R^B}$ and get $\text{fo}\, x\, a = \text{pa}\ (\text{fi}\, h)\,(\text{pure}\, a) = \text{flm}\, h\,(\text{pure}\, a) = h\, a$, so $\text{fo}\,(\text{fi}\, h) = h$
- Rigid monads $R^\bullet$ have "composed-outside" transformers, $T_R^{M,A} \equiv R^{M^A}$

Rigid monads, their laws and structure II

Examples and constructions of rigid and non-rigid monads:

- Rigid: `Id`, `Reader`, and $R^A \equiv H^A \Rightarrow A$ (where $H^\bullet$ is a contrafunctor)
  - The construction $R^A \equiv H^A \Rightarrow A$ covers $R^A \equiv 1$, $R^A = A$, $R^A = Z \Rightarrow A$
- Not rigid: $R^A \equiv W \times A$, $R^A \equiv E + A$, List^A, Cont^A, State^A
- The composition of rigid monads is rigid: $R_1^{R_2^A}$
- The product of rigid monads is rigid: $R_1^A \times R_2^A$
- The selector monad $S^A \equiv \left(A \Rightarrow R^Q\right) \Rightarrow R^A$ is rigid if R^A is rigid

Use cases for rigid monads:

- Any rigid functor is pointed: a method $A \Rightarrow R^A$ can be defined
- Handle multiple $M^\bullet$ effects at once: For a rigid monad $R^\bullet$ and any monad $M^\bullet$, have "R-valued `flatMap`": $M^A \times \left(A \Rightarrow R^{M^B}\right) \Rightarrow R^{M^B}$
- Uptake monadic API: For a rigid monad $R^\bullet$, can implement a general refactoring function, `monadify`: $((A \Rightarrow B) \Rightarrow C) \Rightarrow \left(A \Rightarrow R^B\right) \Rightarrow R^C$

Codensity monads

Codensity monad over a functor $F^{\bullet}$ is $\text{Cod}^{F,A} \equiv \forall B.\left(A \Rightarrow F^{B}\right) \Rightarrow F^{B}$
Properties:

- $\text{Cod}^{F,\bullet}$ is a monad for any functor $F^{\bullet}$
- If $F^{\bullet}$ is itself a monad, we have a monadic morphism $F^{\bullet} \rightsquigarrow \text{Cod}^{F,\bullet}$

Invalid attempts to create a general monad transformer

General recipes for combining two functors $L^\bullet$ and $M^\bullet$ all fail:

- "Fake" transformers: $T_L^{M,A} \equiv L^A$; or $T_L^{M,A} \equiv M^A$; or just $T_L^{M,A} \equiv 1$
  - no `lift` and/or no base runner and/or $T_L^{\mathrm{Id},A} \not\equiv L^A$
- Functor composition, disjunction, or product: $L^{M^\bullet}$, $M^{L^\bullet}$, $L^\bullet + M^\bullet$ – not a monad in general; $L^\bullet \times M^\bullet$ – no lifting $M^\bullet \leadsto L^\bullet \times M^\bullet$
- Making a monad out of functor composition:
  - free monad over $L^{M^\bullet}$, Free^{L^M} – `lift` violates lifting laws
  - free monad over $L^\bullet + M^\bullet$, $\mathrm{Free}^{L^\bullet + M^\bullet}$ – `lift` violates lifting laws
    - ★ Laws will hold after interpreting the free monad into a concrete monad
  - codensity monad over $L^{M^\bullet}$: $F^A \equiv \forall B.\left(A \Rightarrow L^{M^B}\right) \Rightarrow L^{M^B}$ – no `lift`
- Codensity-L transformer: $\mathrm{Cod}_L^{M,A} \equiv \forall B.\left(A \Rightarrow L^B\right) \Rightarrow L^{M^B}$ – no `lift`
  - applies the continuation transformer to $M^A \cong \forall B.(A \Rightarrow B) \Rightarrow M^B$
- Codensity composition: $F^A \equiv \forall B.\left(M^A \Rightarrow L^B\right) \Rightarrow L^B$ – not a monad
  - Counterexample: $M^A \equiv R \Rightarrow A$ and $L^A \equiv S \Rightarrow A$
- "Monoidal" convolution: $(L \star M)^A \equiv \exists P \exists Q.(P \times Q \Rightarrow A) \times L^P \times M^Q$
  - combines $L^A \cong \exists P.L^P \times (P \Rightarrow A)$ with $M^A \cong \exists Q.M^Q \times (Q \Rightarrow A)$
  - $L \star M$ is not a monad for e.g. $L^A \equiv 1 + A$ and $M^A \equiv R \Rightarrow A$

Exercises

1. Show that the method `pure`: $A \Rightarrow M^A$ is a monadic morphism between monads $\text{Id}^A \equiv A$ and M^A. Show that $1 \Rightarrow 1 + A$ is not a monadic morphism.
2. Show that $M_1^A + M_2^A$ is *not* a monad when $M_1^A \equiv 1 + A$ and $M_2^A \equiv Z \Rightarrow A$.
3. Derive the composition law for `lift` written using ftn as $\text{lift}_1 \,\mathring{,}\, \text{fmap}_{\text{BigM}} \text{lift}_1 \,\mathring{,}\, \text{ftn}_{\text{BigM}} = \text{ftn}_{M_\mathbf{1}} \,\mathring{,}\, \text{lift}_\mathbf{1}$ from the flm-based law $\text{lift}_1 \,\mathring{,}\, \text{flm}_{\text{BigM}} \left(q \,\mathring{,}\, \text{lift}_1\right) = \text{flm}_{M_\mathbf{1}} q \,\mathring{,}\, \text{lift}_1$. Draw type diagrams for both laws.
4. Show that the continuation monad is not rigid and does not compose with arbitrary other monads.