Copositive Duality for Discrete Markets and Games

Cheng Guo*[1], Merve Bodur†[1], and Joshua A. Taylor‡[2]

[1]Department of Mechanical and Industrial Engineering, University of Toronto, Toronto, Ontario M5S 3G8, Canada

[2]The Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, ON M5S 3G4, Canada

Abstract

Optimization problems with binary decisions are nonconvex and thus lack strong duality, which limits the usefulness of tools such as shadow prices and the KKT conditions. We propose a copositive duality framework to provide a notion of duality for nonconvex problems. This framework convexify problems with binary variables via copositive programming to leverage its convexity and strong duality features. We use copositive duality in applications including discrete energy markets and discrete games. We obtain several novel theoretical and numerical results for those applications, including a revenue-sufficient pricing scheme for energy markets, and existence and uniqueness conditions for the Nash equilibrium in discrete games. We also propose a novel and easy-to-implement cutting-plane algorithm that solves copositive programs exactly.

1 Introduction

Convex problems in economics, such as convex markets and convex games, often have unique properties, e.g. strong duality and fixed point theorem, that nonconvex problems lack. Motivated by this perspective, many papers in the literature seeks to connect nonconvex problems to convex problems. For example, Danilov et al. (2001), Baldwin and Klemperer (2019), and Tran and Yu

*cguo@mie.utoronto.ca

†bodur@mie.utoronto.ca

‡josh.taylor@utoronto.ca

(2019) which deal with nonconvex markets with indivisible goods prove that competitive equilibrium exists in such markets when demands satisfy "discrete convexity", or equivalently the market problem can be formulated as a linear program that satisfies the unimodularity theorem. O'Neill et al. (2005) fix the binary decisions in nonconvex electricity markets in order to linearly relax the optimization problem and obtain the shadow prices. Inspired by Burer (2009) where the authors transform mixed-binary quadratic programs (MBQPs) to equivalent completely positive programs (CPPs), which are convex, we reformulate problems of nonconvex markets and games containing binary decisions with CPPs. This transformation enables us to work directly with the dual of a CPP, i.e. a copositive program (COP), and use properties such as strong duality and KKT conditions for those problems.

Copositive programming is a special type of conic optimization. It solves a linear optimization problem over a completely positive cone or a copositive cone. In literature copositive programming is used to formulate a variety of NP-hard problems, such as mixed-binary quadratic programs (MBQPs) and two-stage adjustable robust optimization (Hanasusanto and Kuhn, 2018; Xu and Burer, 2018). However, most previous work focus on the strong reformulation power of copositive programming, and rarely explore the benefits brought by copositive programming convexity and strong duality. In this work we investigate the usefulness of copositive duality in economics problems, where for certain problem structures convexity could bring more interesting implications.

We first propose a novel cutting plane algorithm which solves COPs exactly. COPs are NP-hard problems, and there is no solver to solve them directly. Existing solution methods for COPs are either inexact, or requires extensive efforts. Our proposed cutting plane algorithm is applicable to general COPs and is straightforward to implement. Our experiments show that the cutting plane method can produce more accurate solutions than commonly used approximation methods. We use this algorithm to solve two COP duality applications in this work.

Our first application of copositive duality focuses on the pricing in nonconvex markets. Due to the lack of strong duality, nonconvex markets generally do not have shadow prices which leads to market equilibrium in convex markets. Our interest in nonconvex markets stems in particular from the energy market, where nonconvexity is introduced by binary commitment decisions of electricity generators. There is a large literature on the pricing of nonconvex energy markets, as summarized in the review paper by Liberopoulos and Andrianesis (2016).

We also use copositive duality for games with discrete strategy sets that contain both binary and continuous decisions. It is well-known that convex (concave) games with convex strategy sets have nice results for Nash Equilibrium (NE) such as the existence of pure-strategy NE (PNE). Also, due to the convexity of strategy sets, the Karush-Kuhn-Tucker (KKT) condition is readily applicable for computing the NE. We extend those results for convex games to a class of Integer programming games (IPGs) with binary and continuous decisions in the strategy sets.

We believe that our contributions include the following:

(1) We use copositive programming to construct a notion of duality for MIPs and MBQPs. Such copositive duality is shown to be useful in application problems including discrete markets and discrete games. It also has the potential to be applied in other nonconvex problems where discrete duality is useful.

(2) We develop a novel cutting plane algorithm which exactly solves COPs when it terminates. The main advantage of our algorithm is that it is straightforward to implement as it only requires a MIP solver. With this cutting plane algorithm, we are able to solve COP formulations for nonconvex market pricing problems, as well as the COP models for obtaining the NE of bimatrix games.

(3) We propose a new pricing scheme for nonconvex energy markets using copositive duality, which is proved to be revenue neutral. We also provide sufficient conditions for our pricing scheme to be individually rational. In addition we provide a modified version of our copositive duality-based pricing scheme which ensures revenue adequacy, and present numerical results comparing our pricing scheme with traditional ones from literature. Our pricing scheme has the potential to be generalized for other nonconvex markets, and provides a theoretical framework for analyzing nonconvex markets, as it has direct access to the dual pricing problem.

(4) The literature on IPGs mainly focuses on algorithms of IPGs, with relatively limited theoretical developments, while we provide theoretical results for the existence of PNE in a special class of IPG called mixed-binary quadratic (MBQ) games, and formulate an optimization problem based on the KKT conditions for obtaining the PNE. In addition, for certain types of IPGs, we can reformulate its KKT conditions to a single COP problem, then solve it to obtain the NE.

The reminder of the paper is organized as follows. In section 2 we review the literature of copositive programming, pricing of nonconvex markets, and discrete games. In section 3 we prove some useful results for the CPP and its dual COP problem. In section 4 we present a new cutting plane algorithm for solving COP, and also include some conventional methods for solving COPs approximately. In section 5 we propose a COP-based pricing scheme for nonconvex electricity markets. In section 6 we introduce new results for discrete games, by connecting a discrete game to its completely positive counterpart. In section 7 we discuss the numerical results.

2 Literature Review

In this section, we review relevant papers, summarize our contributions, and include an introduction to the basics of copositive programming. Section 2.1 reviews the application of copositive programming in reformulating integer programs, and solution methods for COPs. Section 2.2 surveys the pricing schemes for nonconvex markets. Section 2.3 focuses on the literature of integer programming games.

2.1 Copositive Programming

Copositive programming has broad applications in representing NP-hard problems, such as quadratic optimization (Bomze and De Klerk, 2002), two-stage adjustable robust optimization (Hanasusanto and Kuhn, 2018; Xu and Burer, 2018), and MBQPs (Burer, 2009). In particular, Burer (2009) shows that an MBQP with continuous and binary variables is equivalent to a CPP. The copositive reformulation of MBQPs makes it possible to use copositive duality to address nonconvex market and game problems in this work.

To realize the potential of COP duality we need reliable ways to solve COP problems. However, currently there is no solver that can directly solve COPs. Parrilo (2000) provides a hierarchy of SDP approximations for the COP cones, which is widely used as an approximation algorithm for COPs. For example, De Klerk and Pasechnik (2002) show that the COP model for calculating the stability number of a graph can be approximately solved via SDPs; Hanasusanto and Kuhn (2018) reformulate a two-stage distributionally robust linear program with a COP, and solve the COP by SDP relaxations. An exact algorithm for COPs based on simplicial partitions is proposed by Bundfuss and Dür (2009). However, their algorithm is rather complex to implement. Bomze et al. (2008) and Bomze et al. (2010) use cutting planes to strengthen the SDP relaxation for COPs, but they are both problem specific and can only be used for standard quadratic programming and clique number problems, respectively. Recently, Anstreicher (2020) proposes a MIP model that checks whether a matrix is copositive, which provides a convenient way for separating non-copositive matrices. Based on this MIP model, we develop a novel cutting plane algorithm which exactly solves COPs when it terminates.

2.2 Pricing for Nonconvex Markets

Markets with nonconvexities (indivisibilities) have a large presence in the real world, with some of the examples include housing markets, and markets with investment or start-up/shut-down decisions. The pricing problem in such nonconvex markets has been studied for decades. Starr (1969) proves that when there are a large number of traders in the economy, the divergence from market equilibrium because of nonconvexity is negligible. By using a well-known result in combinatorial topology, Gale (1984) shows that in a market where traders have at most one indivisible object, a price equilibrium exists under mild assumptions. Bevia et al. (1999) consider the pricing problem in economies with several types of indivisible goods. Danilov et al. (2001) and Baldwin and Klemperer (2019) propose a general existence theorem for markets with indivisible goods via discrete convexity, or Unimodularity Theorem. Tran and Yu (2019) provide an integer programming (IP) perspective for the work of Baldwin and Klemperer (2019), and shows that their the Unimodularity Theorem follows from the unimodularity theorem in IP.

One important line of research in nonconvex markets focuses on the pricing problem in electricity markets, which becomes popular following the deregulation of electricity market in the late 1990s. The nonconvexity in electricity markets comes from the start-up and shut-down decisions in a *unit commitment (UC)* problem, which can be modelled with a MIP. The general idea for obtaining prices from such markets is to obtain approximated shadow prices from the MIP model. O'Neill et al. (2005) propose a pricing scheme that directly prices integral activities. They eliminate the nonconvexity from the problem by fixing the binary decisions at their optimal levels, and solving the fixed-binary problem as an LP. Then they get the duals of the demand constraints and the constraints for fixing the binary variables as electricity prices. We call this pricing scheme the *restricted pricing (RP)*. RP and its variantions is used by some independent system operators (ISOs) in the US, such as PJM (Pennsylvania-New Jersey-Maryland Interconnect). However, RP is often too low to cover the costs of generators, which leads to *uplift costs* for the utilities to pay off the uncovered portion of the costs. The uplift cost is a type of generator-dependent payment that is made on top of a uniform, generator-independent electricity price. It may cause discriminatory prices and thus is not desirable. The *convex hull pricing (CHP)* by Hogan and Ring (2003) and Gribik et al. (2007) is designed to minimize the uplift costs. This approach uses the Lagrangian multiplier of the demand constraints as prices, and is shown to reduce uplift costs compared with the RP. A modified version of CHP called the extended locational marginal pricing is used by MISO (Midcontinent ISO). Ruiz et al. (2012) propose a *primal-dual approach* for pricing, which combines the UC problem and the dual of its linear relaxation, as well as revenue adequacy constraints that ensure nonnegative profit for each generator. Thanks to the revenue adequacy constraints, this pricing scheme does not require any uplift costs. However, both CHP and primal-dual approach modifies the original UC problem and may result in suboptimal operational decisions. In our work, both of our proposed COP pricing and modified COP pricing provide prices that support the optimal UC solutions. In addition, utilizing the revenue adequacy constraints, the modified COP pricing ensures that each generator is revenue adequate.

The UC model assumes a central planner's viewpoint, as it solves the operational problem for all generators in the system at the same time. Therefore, it is desirable for a pricing scheme to also ensure individual rationality, or in other words, to ensure that individual generators have no incentive to deviate from the optimal UC solution. O'Neill et al. (2005) proves that the RP is individually rational. CHP satisfies individual rationality in some special cases, but in general it does not support individual generator's profit-maximizing solutions (Gribik et al., 2007). The primal-dual approach does not guarantee individual rationality either. Since our work uses a conic program for pricing, it is also interesting to see works that use similar tools. As a such example, Winnicki et al. (2019) study the pricing problem of alternating current optimal power flow (ACOPF) problem with semidefinite cone constraints. They find that their pricing scheme is individually rational when the ACOPF problem has a zero duality gap. For our COP pricing scheme, we provide a sufficient condition when individual rationality holds.

Table 1 compares different pricing schemes and some of the important features. Column "uplift free" shows whether the scheme is able to cover total costs without uplift payments. "Optimal UC" shows whether the obtained prices correspond to the optimal UC solution. For "Individual rationality", certain schemes are labeled as "depends", which means individual rationality does not hold for them in general, but there exist sufficient conditions that make them individually rational.

Table 1: Features of Pricing Schems

Scheme	Uplift free	Optimal UC	Individual rationality
RP	$\times$	✓	✓
CHP	$\times$	$\times$	depends
Primal-dual	✓	$\times$	$\times$
COP	$\times$	✓	depends
Modified COP	✓	$\times$	$\times$

2.3 Integer Programming Games

IPGs are a class of games where each player's action set contains integer decisions. Some applications of IPG include bimatrix games, Nash-Cournot energy production games (Gabriel et al., 2013), and Cournot game with indivisible goods (Kostreva, 1993). IPGs are also referred to as "games with at least one player solving a combinatorial optimization problem" (Kostreva, 1993), "games with discrete strategy sets" (Sagratella, 2016). Most papers in this field focus on algorithms that finds NE, and they are only applicable on IPGs with all integer decisions (Carvalho et al., 2017; Köppe et al., 2011; Sagratella, 2016). Theoretical works are relatively sparse, including Mallick (2011) and Sagratella (2016), which respectively provide NE existence conditions for two-person discrete games and 2-groups partitionable discrete games. In this work we provide conditions for the existence and uniqueness of PNE in IPGs with both continuous and discrete variables, as well as KKT conditions for obtaining the PNE.

Since we use convex optimization techniques for IPG, it is interesting to review the works that explore similar ideas. Parrilo (2006) reformulates zero-sum polynomial games to a single SDP problem. Ahmadi and Zhang (2020) use SDP on bimatrix games to find the additive ϵ-approximate NE. Closely related to our work, Sayin and Basar (2019) studies a Stackelberg game with binary decisions in the context of optimal hierarchical signaling. They reformulate each player's optimization problem to a CPP, and solve the CPP and its dual COP problems approximately with semi-definite programming. However, their scope is limited to one application problem, while we use COP duality to derive theoretical results for general types of IPGs. In addition, for certain types of IPG we can obtain its NE by solving a single COP problem.

3 Duality for Copositive Programming

In the application problems of this work, we need to reformulate the MBQP problem with CPP, and then obtain the dual COP problem to get useful results. Therefore, in this section we include some useful results for copositive programming. Section 3.1 contains a brief introduction to copositive programming. Section 3.2 presents some new results we derive for the CPP reformulation of MBQPs.

3.1 Preliminaries

In this section we review some basic results in copositive programming that are relevant to our work. For a more complete introduction to copositive programming, we refer the readers to the review papers of Dür (2010) and Bomze (2012).

Let $\mathcal{S}$ be the set of real symmetric matrices. The *copositive cone* $\mathcal{C}$ is defined as:

$$\mathcal{C} = \left\{ X \in \mathcal{S} \mid y^\top X y \geq 0 \text{ for all } y \in \mathbb{R}^n_+ \right\}. \tag{1}$$

If we have strict inequality in (1), then X is strictly copositive and $X \in \text{int}(\mathcal{C})$, where $\text{int}(\mathcal{C})$ represents the interior of cone $\mathcal{C}$.

The dual cone of the copositive cone is the *completely positive cone* $\mathcal{C}^*$:

$$\mathcal{C}^* = \left\{ XX^\top \mid X \in \mathbb{R}^{n \times r}, X \geq 0 \right\}. \tag{2}$$

If X is entrywise strictly positive, then $XX^\top \in \text{int}(\mathcal{C}^*)$(Groetzner and Dür, 2018).

A (linear) COP is an optimization problem defined over the copositive matrix $X \in \mathcal{C}$, with objective and constraints linear in X. Similarly, a (linear) CPP, which is the dual of COP, is an optimization problem defined over $X \in \mathcal{C}^*$ with objective and constraints linear in X. Since COP and CPP are convex programs, strong duality holds between them if certain regularity condition holds. One such condition is the Slater's condition, which requires either COP or CPP to have at least on interior point.

Let $\mathcal{S}^+$ be a positive semidefinite (PSD) cone and $\mathcal{N}$ be the cone for entrywise nonnegative matrices, then we have $\mathcal{C}^* \subseteq \mathcal{S}^+ \cap \mathcal{N}$ and $\mathcal{S}^+ + \mathcal{N} \subseteq \mathcal{C}$ (Dür, 2010). Taking advantage of these relationships, we can relax and approximately solve CPP as an optimization problem over the union of $\mathcal{S}^+$ and $\mathcal{N}$ cones, and restrict the COP problem as an optimization over the Minkowski sum of $\mathcal{S}^+$ and $\mathcal{N}$ cones.

In our work, we study MBQP models which can be reformulated to CPPs, to utilize the dual COP problems. We also solve CPP and COP problems approximately via SDPs. We summarize the relationship between all those mathematical programming problems in Figure 1.

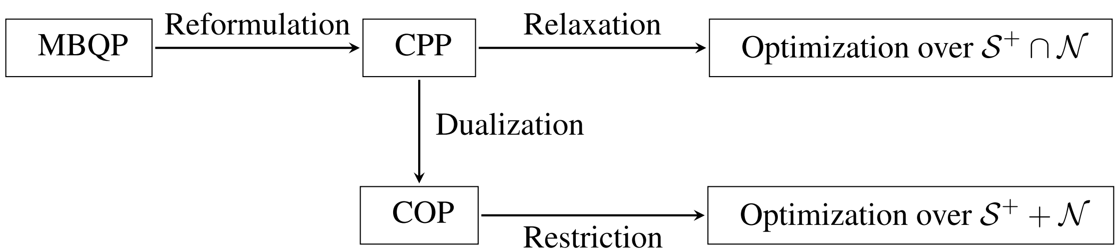

Figure 1: Relationships between MBQP, CPP, COP and their approximations

3.2 Optimality and Duality of CPP Reformulation

In this section we derive some important theorems and propositions for future use. We first present the dual COP problem for the CPP reformulation. Then we give a CPP reformulation that is slightly different than the CPP reformulations in Burer (2009). Moreover, we prove some useful results on the relationship of solutions from MBQP and its CPP reformulation.

We consider the following mixed-binary quadratic program (MBQP):

$$
\begin{aligned}
\mathcal{P}^{\text{MBQP}}: \min\ & \mathbf{x}^\top Q \mathbf{x} + 2\mathbf{c}^\top \mathbf{x} && && \text{(3a)}\\
\text{s.t.}\ & \mathbf{a}^{j\top} \mathbf{x} = b_j, && \forall j = 1, ..., m && \text{(3b)}\\
& x_k \in \{0, 1\}, && \forall k \in \mathcal{B} && \text{(3c)}\\
& \mathbf{x} \in \mathbb{R}^n_+ && && \text{(3d)}
\end{aligned}
$$

where $\mathcal{B} \subseteq \{1, ..., n\}$ is the set of indices for the binary elements of $\mathbf{x}$. In this paper, we use bold letters for vectors and capital letters for matrices.

$\mathcal{P}^{\text{MBQP}}$ can be reformulated to CPPs, as shown by models (C) and (C') in Burer (2009). We provide their model (C) as the problem $\mathcal{P}_o^{\text{CPP}}$ here, as we will use this reformulation later:

$$\mathcal{P}_o^{\text{CPP}} : \min \ \text{Tr}(QX) + 2\mathbf{c}^\top \mathbf{x} \tag{4a}$$
$$\text{s.t.} \ \mathbf{a}^{j\top}\mathbf{x} = b_j \qquad \forall j = 1, ..., m \tag{4b}$$
$$\mathbf{a}^{j\top} X \mathbf{a}^j = b_j^2 \qquad \forall j = 1, ..., m \tag{4c}$$
$$x_k = X_{kk} \qquad \forall k \in \mathcal{B} \tag{4d}$$
$$\begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & X \end{bmatrix} \in \mathcal{C}^* \tag{4e}$$

we call the reformulation $\mathcal{P}_o^{\text{CPP}}$ the original CPP reformulation in the following text.

For the convenience of taking the dual of CPP (4), we would like to rewrite the problem. Let us denote:

$$Y = \begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & X \end{bmatrix}$$

Also, define:

$$\tilde{Q} = \begin{bmatrix} 0 & \mathbf{0}^{1\times n\top} \\ \mathbf{0}^{n\times 1} & Q \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \mathbf{c}^\top \\ \mathbf{c} & \mathbf{0}^{n\times n} \end{bmatrix}$$

and

$$A_j = \begin{bmatrix} 0 & 1/2\mathbf{a}^{j\top} \\ 1/2\mathbf{a}^j & \mathbf{0}^{n\times n} \end{bmatrix}, \quad j = 1, ..., m$$

Let the numbers in bracket $[\cdot]([\cdot,\cdot])$ denote the index (indices) of a vector (matrix). Define $\tilde{A}_j$, $j = 1, ..., m$, such that

$$\tilde{A}_j[l_1, l_2] = \begin{cases} \mathbf{a}^j[l_1 - 1]\mathbf{a}^j[l_2 - 1] & \text{if } 2 \le l_1, l_2 \le n+1, \\ 0 & \text{otherwise} \end{cases}$$

and B_k, $k \in \mathcal{B}$, such that

$$B_k[l_1, l_2] = \begin{cases} 1/2 & \text{if } l_1 = j+1, \ l_2 = 1 \\ 1/2 & \text{if } l_1 = 1, \ l_2 = j+1 \\ -1 & \text{if } l_1 = j+1, \ l_2 = j+1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathcal{P}_o^{\text{CPP}}$ is equivalent to:

$$\min \quad \text{Tr}(\tilde{Q}Y) + \text{Tr}(CY) \tag{5a}$$

$$
\begin{aligned}
\text{s.t.}\quad & \text{Tr}(A_jY) = b_j, \quad j = 1, ..., m && \text{(5b)}\\
& \text{Tr}(\tilde{A}_jY) = b_j^2, \quad j = 1, ..., m && \text{(5c)}\\
& \text{Tr}(B_kY) = 0, \quad k \in \mathcal{B} && \text{(5d)}\\
& Y \in \mathcal{C}^* && \text{(5e)}
\end{aligned}
$$

Let $\boldsymbol{\gamma}^o$, $\boldsymbol{\beta}^o$, $\boldsymbol{\delta}^o$,and Ω^o be the respective dual variables of constraints (5b) - (5e), then the dual problem of $\mathcal{P}_o^{\text{CPP}}$ is the following COP:

$$
\begin{aligned}
\mathcal{P}_o^{\text{COP}} : \max_{\boldsymbol{\gamma}^o, \boldsymbol{\beta}^o, \boldsymbol{\delta}^o, \Omega^o} \quad & \sum_{j=1}^{m} (\gamma_j^o b_j + \beta_j^o b_j^2) && \text{(6a)}\\
\text{s.t.}\quad & \tilde{Q} + C - \sum_{j=1}^{m} \gamma_j^o A_j - \sum_{j=1}^{m} \beta_j^o \tilde{A}_j - \sum_{k\in\mathcal{B}} \delta_k^o B_k - \Omega^o = 0 && \text{(6b)}\\
& \Omega^o \in \mathcal{C} && \text{(6c)}
\end{aligned}
$$

where $\mathcal{C}$ is a cone of copositive matrices as defined in (1).

Next, we propose a CPP reformulation for $\mathcal{P}^{\text{MBQP}}$ that is slightly different from CPP models (C) (i.e. our model (4)) and (C') (i.e. the model (8) below) of Burer (2009). We provide this CPP reformulation in Proposition 5:

Proposition 1. *Assume problem $\mathcal{P}^{MBQP}$ satisfies the following assumption:*

$$
\exists \mathbf{y} \in \mathbb{R}^m \ \text{s.t.} \sum_{j=1}^{m} y_j \mathbf{a}^j \geq 0, \sum_{j=1}^{m} y_j b_j = 1 \qquad \text{(A)}
$$

and define $\boldsymbol{\alpha} : \sum_{j=1}^{m} y_j \mathbf{a}^j$, then the problem $\mathcal{P}^{MBQP}$ is equivalent to the following CPP:

$$
\begin{aligned}
\mathcal{P}^{\text{CPP}} : \min\ & \text{Tr}(QX) + 2\mathbf{c}^\top \mathbf{x} && \text{(7a)}\\
\text{s.t.}\ & \mathbf{a}^{j\top} \mathbf{x} = b_j && \forall j = 1, ..., m \quad \text{(7b)}\\
& \mathbf{a}^{j\top} X \mathbf{a}^j = b_j^2 && \forall j = 1, ..., m \quad \text{(7c)}\\
& x_k = X_{kk} && \forall k \in \mathcal{B} \quad \text{(7d)}\\
& \mathbf{x} = X\boldsymbol{\alpha} && \text{(7e)}\\
& X \in \mathcal{C}^* && \text{(7f)}
\end{aligned}
$$

where $\mathcal{C}^*$ is a cone for completely positive matrices as defined in (2).

Proof. Because of Theorem 3.1 in Burer (2009), if Assumption (A) holds, then problem $\mathcal{P}^{\text{MBQP}}$ is equivalent to the following CPP:

$$
\min\ \text{Tr}(QX) + 2\mathbf{c}^\top \mathbf{x} \qquad \text{(8a)}
$$

$$\text{s.t. } \mathbf{a}^{j\top} X \boldsymbol{\alpha} = b_j \quad \forall j = 1, ..., m \tag{8b}$$

$$\mathbf{a}^{j\top} X \mathbf{a}^j = b_j^2 \quad \forall j = 1, ..., m \tag{8c}$$

$$[X\boldsymbol{\alpha}]_k = X_{kk} \quad \forall k \in \mathcal{B} \tag{8d}$$

$$\boldsymbol{\alpha}^\top X \boldsymbol{\alpha} = 1 \tag{8e}$$

$$X \in \mathcal{C}^* \tag{8f}$$

Observe the transformation between problems (10) and (11) in Burer (2009), both of which are equivalent to our reformulation (8), from this transformation we know that the constraint $\mathbf{x} = X\boldsymbol{\alpha}$ is valid for problem (8). Add the constraint $\mathbf{x} = X\boldsymbol{\alpha}$ to (8), and replace all $X\boldsymbol{\alpha}$ in (8) with $\mathbf{x}$, we have:

$$\min \ \text{Tr}(QX) + 2\mathbf{c}^\top \mathbf{x} \tag{9a}$$

$$\text{s.t. } \mathbf{a}^{j\top} \mathbf{x} = b_j \quad \forall j = 1, ..., m \tag{9b}$$

$$\mathbf{a}^{j\top} X \mathbf{a}^j = b_j^2 \quad \forall j = 1, ..., m \tag{9c}$$

$$x_k = X_{kk} \quad \forall k \in \mathcal{B} \tag{9d}$$

$$\boldsymbol{\alpha}^\top \mathbf{x} = 1 \tag{9e}$$

$$\mathbf{x} = X\boldsymbol{\alpha} \tag{9f}$$

$$X \in \mathcal{C}^* \tag{9g}$$

But (9e) is already implied by the combination of constraints (9b), assumption (A), and the definition of $\boldsymbol{\alpha}$: by multiplying both sides of (9b) with y_j as defined in assumption (A), and sum up all $j = 1, ..., m$, we obtain $\sum_{j=1}^m (y_j \mathbf{a}^{j\top})\mathbf{x} = \sum_{j=1}^m y_j b_j$. This is equivalent to (9e) because $\boldsymbol{\alpha} := \sum_{j=1}^m y_j \mathbf{a}^j$ and $\sum_{j=1}^m y_j b_j = 1$. Therefore, we can eliminate (9e) from (9) and obtain $\mathcal{P}^{\text{CPP}}$, i.e. $\mathcal{P}^{\text{CPP}}$ is equivalent to $\mathcal{P}^{\text{MBQP}}$. □

Remark 1. It is proved in Burer (2009) that reformulation $\mathcal{P}_o^{\text{CPP}}$ does not have any interior points in the conic constraint (4e), which mean $\mathcal{P}_o^{\text{CPP}}$ does not satisfy Slater's condition for strong duality. On the other hand, $\mathcal{P}^{\text{CPP}}$ may have interior points. In addition, compared with $\mathcal{P}_o^{\text{CPP}}$ and reformulation (8), the reformulation $\mathcal{P}^{\text{CPP}}$ makes some of our subsequent proofs in Sections 6 more straightforward, because it contains constraints (3b) in their original linear form, and requires only X to be completely positive in the conic constraint.

Remark 2. Despite the lack of interior point, $\mathcal{P}_o^{\text{CPP}}$ and its dual $\mathcal{P}_o^{\text{COP}}$ can still be useful because:

(i) They do not require Assumption (A).

(ii) $\mathcal{P}_o^{\text{COP}}$ can have better computational performance than $\mathcal{P}^{\text{COP}}$. When solving the dual COP problem, both our proposed cutting plane algorithm and SDP solvers require the matrix in the conic constraint to be symmetric. $\mathcal{P}_o^{\text{COP}}$ naturally satisfies this requirement, as the symmetry of matrices

$\tilde{Q}$, C, A_j, $\tilde{A}_j$, and B_k ensures the symmetry of Ω in (6c). For $\mathcal{P}^{\text{COP}}$, however, symmetry of Ω is generally not guaranteed. This can be addressed by adding extra constraints in $\mathcal{P}^{\text{CPP}}$ to ensure the symmetry of X. Those extra constraints in $\mathcal{P}^{\text{CPP}}$ relaxes the dual, $\mathcal{P}^{\text{COP}}$, and can lead to a worse computational performance compared with $\mathcal{P}_o^{\text{COP}}$.

(iii) Slater's condition is not a necessary condition for the existence of strong duality. We observe that in our experiments, strong duality holds for $\mathcal{P}_o^{\text{CPP}}$ in some instances despite the lack of Slater's condition.

Notice that the equivalence between $\mathcal{P}^{\text{MBQP}}$ and $\mathcal{P}^{\text{CPP}}$ means (i) opt($\mathcal{P}^{\text{MBQP}}$) $=$ opt($\mathcal{P}^{\text{CPP}}$), where opt($\mathcal{P}$) represents the optimal objective value of the optimization problem $\mathcal{P}$; and (ii) if $(\mathbf{x}^*, X^*)$ is optimal for $\mathcal{P}^{\text{CPP}}$, then $\mathbf{x}^*$ is in the convex hull of optimal solutions for $\mathcal{P}^{\text{MBQP}}$. Item (ii) indicates that $\mathbf{x}^*$ is not necessarily feasible for $\mathcal{P}^{\text{MBQP}}$, and it is possible for some x^{k*} with $k \in \mathcal{B}$ to be fractional. However, if $\mathbf{x}^*$ is also feasible for $\mathcal{P}^{\text{MBQP}}$, and $Q \succeq 0$, then $\mathbf{x}^*$ is an optimal solution to $\mathcal{P}^{\text{MBQP}}$, as shown in the following proposition:

Proposition 2. *Let $(\mathbf{x}^*, X^*)$ be an optimal solution for $\mathcal{P}^{CPP}$, whose equivalent formulation is $\mathcal{P}^{MBQP}$. If additionally $Q \succeq 0$ and $\mathbf{x}^*$ is feasible for $\mathcal{P}^{MBQP}$, then $\mathbf{x}^*$ is an optimal solution of $\mathcal{P}^{MBQP}$.*

Proof. Let $\mathbf{x}^*(1), ..., \mathbf{x}^*(q)$ be all q optimal solutions of $\mathcal{P}^{\text{MBQP}}$. Then there exists $\lambda_j \geq 0, j = 1, ..., q$ and $\sum_{j=1}^{q} \lambda_j = 1$, such that $\mathbf{x}^* = \sum_{j=1}^{q} \lambda_j \mathbf{x}^*(j)$. Let us denote the objective function of $\mathcal{P}^{\text{MBQP}}$ as $f(\mathbf{x})$, i.e. $f(\mathbf{x}) = \mathbf{x}^\top Q \mathbf{x} + 2\mathbf{c}^\top \mathbf{x}$. Since $\mathbf{x}^*(j), j = 1, ..., q$ is optimal for $\mathcal{P}^{\text{MBQP}}$, we have opt($\mathcal{P}^{\text{MBQP}}$) $= f(\mathbf{x}^*(j)), j = 1, ..., q$. Because $Q \succeq 0$, $f(\mathbf{x})$ is a convex function on $\mathbf{x}$, we have the following:

$$f(\mathbf{x}^*) = f(\sum_{j=1}^{q} \lambda_j \mathbf{x}^*(j)) \leq \sum_{j=1}^{q} \lambda_j f(\mathbf{x}^*(j)) = \text{opt}(\mathcal{P}^{\text{MBQP}}) \sum_{j=1}^{q} \lambda_j = \text{opt}(\mathcal{P}^{\text{MBQP}}).$$

Since $\mathbf{x}^*$ is also feasible for the minimization problem $\mathcal{P}^{\text{MBQP}}$, which means $f(\mathbf{x}^*) \geq$ opt($\mathcal{P}^{\text{CPP}}$), we now have $f(\mathbf{x}^*) =$ opt($\mathcal{P}^{\text{MBQP}}$). Thus, $\mathbf{x}^*$ is an optimal solution for $\mathcal{P}^{\text{MBQP}}$. □

Conversely, for the optimal solution of $\mathcal{P}^{\text{MBQP}}$, $\mathbf{x}^*$, then $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ is optimal for $\mathcal{P}^{\text{CPP}}$, as shown in Proposition 3:

Proposition 3. *Let $\mathbf{x}^*$ be an optimal solution to $\mathcal{P}^{MBQP}$, then $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ is an optimal solution to $\mathcal{P}^{CPP}$.*

Proof. First, we prove that $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ is a feasible solution to $\mathcal{P}^{\text{CPP}}$. Since $\mathbf{x}^*$ satisfies constraints (3b), $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ should be feasible to constraints (7b) and (7c). It is feasible to (7d) because for all $x_k \in \{0, 1\}$ where $k \in \mathcal{B}$, $[\mathbf{x}^*\mathbf{x}^{*\top}]_{kk} = x_k^2 = x_k$. For (7e) the feasibility of $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ can be proved by definition of $\boldsymbol{\alpha}$, as $\mathbf{x}^*\mathbf{x}^{*\top}\boldsymbol{\alpha} = \sum_{j=1}^m y_j \mathbf{x}^*\mathbf{x}^{*\top}\mathbf{a}^j = \sum_{j=1}^m y_j \mathbf{x}^* b_j = (\sum_{j=1}^m y_j b_j)\mathbf{x}^* = \mathbf{x}^*$. Finally, $\mathbf{x}^*\mathbf{x}^{*\top} \in \mathcal{C}^*$ because $\mathbf{x}^* \geq 0$. Thus, $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ is a feasible solution.

It remains to prove that the feasible solution $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ is also optimal for $\mathcal{P}^{\text{CPP}}$. Notice that the corresponding objective value of the solution $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ is $\text{Tr}(Q\mathbf{x}^*\mathbf{x}^{*\top}) + c^\top \mathbf{x}^*$, which equals the optimal solution of $\mathcal{P}^{\text{MBQP}}$. Since the optimal values of $\mathcal{P}^{\text{MBQP}}$ and $\mathcal{P}^{\text{CPP}}$ are equivalent, $\text{Tr}(Q\mathbf{x}^*\mathbf{x}^{*\top}) + c^\top \mathbf{x}^*$ is also the optimal value of $\mathcal{P}^{\text{CPP}}$. Therefore, the solution $(\mathbf{x}^*, \mathbf{x}^*\mathbf{x}^{*\top})$ is both feasible and optimal to $\mathcal{P}^{\text{CPP}}$. □

The same result holds for the CPP reformulation $\mathcal{P}_o^{\text{CPP}}$, because of the equivalence between $\mathcal{P}_o^{\text{CPP}}$ and $\mathcal{P}^{\text{CPP}}$. As we will need this result in the following sections, we formally write it in Lemma 1:

Lemma 1. *Given an optimal solution to $\mathcal{P}^{\text{MBQP}}$, $\mathbf{x}^*$, there exists an optimal solution $(\mathbf{x}^*, X^*)$ to $\mathcal{P}_o^{\text{CPP}}$, where $X^* = \mathbf{x}^*\mathbf{x}^{*\top}$.*

Finally, we write down the dual of $\mathcal{P}^{\text{CPP}}$. Let $\boldsymbol{\gamma}$, $\boldsymbol{\beta}$, $\boldsymbol{\delta}$, $\boldsymbol{\xi}$, and Ω be the respective dual variables of constraints (7b) - (7f), then the dual problem of $\mathcal{P}^{\text{CPP}}$ is the following COP:

$$\mathcal{P}^{\text{COP}} : \max_{\boldsymbol{\gamma},\boldsymbol{\beta},\boldsymbol{\delta},\boldsymbol{\xi},\Omega} \quad \sum_{j=1}^m (\gamma_j b_j + \beta_j b_j^2) \tag{10a}$$

$$\text{s.t.} \quad 2\mathbf{c} - \sum_{j=1}^m \gamma_j \mathbf{a}^j - \sum_{k\in\mathcal{B}} \delta_k \mathbf{e}^k - \boldsymbol{\xi} = 0 \tag{10b}$$

$$Q - \sum_{j=1}^m \beta_j \mathbf{a}^j \mathbf{a}^{j\top} + \sum_{k\in\mathcal{B}} \delta_k \mathbf{e}^k \mathbf{e}^{k\top} + \boldsymbol{\alpha}\boldsymbol{\xi}^\top - \Omega = 0 \tag{10c}$$

$$\Omega \in \mathcal{C} \tag{10d}$$

where $\mathbf{e}^k \in \mathbb{R}^n$ is a vector with 1 for the kth element and 0 for other elements. $\mathcal{C}$ a cone of copositive matrices as defined in (1).

4 Solution Methods for COP

To use COP duality in application problems, we need to be able to solve COPs. Unfortunately, there is no solver for solving COPs (or its dual CPP problems) directly, and in the literature they are often solved approximately with SDPs. In Section 4.1 we first review the SDP approximations for COPs from literature. In Section 4.2, we propose a novel cutting plane algorithm for general COPs that guarantees to return an optimal solution when it terminates.

4.1 SDP Approximations of COPs

We can use the relationship between copositive cones and PSD cones to approximately solve a COP problem. As mentioned in our brief introduction for copositive programming of Section 3.1, for a copositive cone $\mathcal{C}$ we have $\mathcal{S}^+ + \mathcal{N} \subseteq \mathcal{C}$. This relationship gives us a restriction of the original COP. More specifically, we replace the conic constraint $\Omega \in \mathcal{C}$ with the following restriction:

$$\begin{aligned} \Omega - N &\in \mathcal{S}^+ \\ N &\geq 0 \end{aligned}$$

which can be solved with SDP solvers such as Mosek and SeDuMi.

Another method to obtain an optimal solution of COP is to solve its dual CPP problem using a commercial solver, then query the duals of CPP constraints via the solver. However, there is not any solver that directly solves CPPs, so we instead solve an SDP relaxation of the CPP problem, then query the duals of the SDP relaxation.

For a completely positive cone $\mathcal{C}^*$ we have $\mathcal{C}^* \subseteq \mathcal{S}^+ \cap \mathcal{N}$. Therefore, the conic constraint $X \in \mathcal{C}^*$ can be relaxed to the following constraints:

$$\begin{aligned} X &\in \mathcal{S}^+ \\ X &\geq 0 \end{aligned}$$

which can then be solved with SDP solvers.

4.2 A Novel Cutting Plane Algorithm for COP

IN this section we propose a novel cutting plane algorithm for the following general type of COPs with linear constraints over a copositive cone:

$$\begin{aligned} \min \;\; & \mathbf{c}^{l\top}\mathbf{x} + \mathrm{Tr}(C^{c\top} Y) && \text{(11a)} \\ \text{s.t.} \;\; & \mathbf{a}^{il\top}\mathbf{x} + \mathrm{Tr}(A_i^{c\top} Y) = b_i,\, i = 1, ..., m && \text{(11b)} \\ & \mathbf{x} \geq 0 && \text{(11c)} \\ & Y \in \mathcal{C}^{n_c} && \text{(11d)} \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^{n_l}$, $Y \in \mathbb{R}^{n_c \times n_c}$. $\mathcal{C}^{n_c}$ is an n_c-dimensional copositive cone. Notice that the COP problems $\mathcal{P}_o^{\text{COP}}$ (6) and $\mathcal{P}^{\text{COP}}$ (10) are special cases of the COP (11).

We relax problem (11) by removing the conic constraint (11d). Then we solve the relaxed problem to obtain an optimal solution and denote as $(\hat{x}, \hat{Y})$. We check whether $\hat{Y}$ is copositive by solving the following *separation problem* (Anstreicher, 2020):

$$\text{(SP)} \;: \max \;\; w \qquad \text{(12a)}$$

$$
\begin{aligned}
\text{s.t. } & \hat{Y}\mathbf{z} \le -w\mathbf{e} + \mathbf{m}(1-\mathbf{u}) && \text{(12b)}\\
& \mathbf{e}^\top \mathbf{u} \ge q && \text{(12c)}\\
& w \ge 0 && \text{(12d)}\\
& 0 \le \mathbf{z} \le \mathbf{u} && \text{(12e)}\\
& \mathbf{u} \in \{0,1\}^{n_c} && \text{(12f)}
\end{aligned}
$$

where $q = 1$, $\mathbf{e}$ is a vector of all ones, $\mathbf{m} \in \mathbb{R}^{n_c}_{++}$ is a vector of large numbers. By Theorem 2 of Anstreicher (2020), $\hat{Y}$ is copositive if and only if $\bar{w} = 0$, where $\bar{w}$ is the optimal solution for w and equals the optimal objective value of (12).

If the separation problem suggests that $\hat{Y}$ is copositive, then we can conclude that $(\hat{x}, \hat{Y})$ is optimal to the original COP problem (11). Otherwise, let $\bar{\mathbf{z}}$ be the optimal value of $\mathbf{z}$ in problem (12), we add the following *cut* to the relaxed problem to eliminate $\hat{Y}$ from its feasible region:

$$
\bar{\mathbf{z}}^\top Y \bar{\mathbf{z}} \ge 0 \tag{13}
$$

and re-solve the relaxed problem.

To prove that the cut (13) is valid, first notice that it does not cut off any feasible solution for problem (11). This is because for any $\bar{\mathbf{z}} \in \mathbb{R}^{n_c}_{+}$, a copositive matrix Y satisfies $\bar{\mathbf{z}}^\top Y \bar{\mathbf{z}} \ge 0$. Next, in Theorem 1 we prove that the cut (13) cuts off the infeasible solution $\hat{Y}$:

Theorem 1. *If the optimal value of separation problem* (12) *is nonzero when $Y = \hat{Y}$, then $\hat{Y}$ is infeasible for the cut* (13)*.*

Proof. Because of (12c) and the fact that $q = 1$, there is at least one element in u that is nonzero, i.e. $\beta = \{i | u_i = 1, i = 1, ..., n_c\} \ne \emptyset$. From the fact that optimal objective $\bar{w} > 0$, we know from constraint (12b) that $\hat{Y}_{\beta\beta}\bar{\mathbf{z}}_\beta < \mathbf{0}$ and that $\bar{\mathbf{z}}_\beta \ne \mathbf{0}$. Therefore, $\bar{\mathbf{z}}_\beta^\top \hat{Y}_{\beta\beta}\bar{\mathbf{z}}_\beta < 0$. Also, let $\alpha = \{1, ..., n_c\} \setminus \beta$, then $x_\alpha = \mathbf{0}$, which means $\bar{\mathbf{z}}^\top \hat{Y} \bar{\mathbf{z}} = \bar{\mathbf{z}}_\beta^\top \hat{Y}_{\beta\beta}\bar{\mathbf{z}}_\beta < 0$. Thus, $\hat{Y}$ violates the cut (13). $\square$

To make the algorithm more clear, we write out the relaxed problem (11) where the conic constraints are replaced by a set of cuts, which is conventionally named as the *master problem*:

$$
\begin{aligned}
\text{(MP) } : \min\ & \mathbf{c}^{l\top}\mathbf{x} + \mathrm{Tr}(C^{c\top} Y) && \text{(14a)}\\
\text{s.t. } & \mathbf{a}^{il\top}\mathbf{x} + \mathrm{Tr}(A_i^{c\top} Y) = b_i, i = 1, ..., m && \text{(14b)}\\
& \mathbf{x} \ge 0 && \text{(14c)}\\
& \{cuts\} && \text{(14d)}
\end{aligned}
$$

where $\{cuts\}$ is a set of cuts in the form of (13), which are added iteratively into the master problem.

The complete algorithm is as follows:

Algorithm 1: Cutting plane algorithm for linear copositive programs

Initialization: Set k = 0, $\{cuts\} = \emptyset$;
Solve (MP) to obtain an initial solution $\hat{Y}^0$;
Solve (SP) with $Y = \hat{Y}^0$ and obtain optimal solutions $\bar{w}$ and $\bar{\mathbf{z}}$;
while $\bar{w} > 0$ **do**
  $\{cuts\} = \{cuts\} \cup \{\bar{\mathbf{z}}^\top Y \bar{\mathbf{z}} \geq 0\}$;
  $k = k + 1$;
  Solve (MP) to obtain an initial solution $\hat{Y}^k$;
  Solve (SP) with $Y = \hat{Y}^k$ and update optimal solutions $\bar{w}$ and $\bar{\mathbf{z}}$;
end

Since (SP) is a MIP, we can strengthen its LP relaxation to improve its computational efficiency. One of the things we could do is to choose appropriate big-M constants $\mathbf{m}$. Anstreicher (2020) propose the following values for the ith element of $\mathbf{m}$:

$$m_i = 1 + \sum_{j=1,\ j\neq i}^{n_c} \{\hat{Y}_{ij} : \hat{Y}_{ij} > 0\}$$

Another way to strengthen the separation problem is to let $q = 2$. (SP) will still be a valid problem with $q = 2$ if diag$(\hat{Y}) \geq 0$ (Anstreicher, 2020). Luckily, diag$(\hat{Y}) \geq 0$ is always valid for the original COP (11) because $Y \in \mathcal{C}$. Therefore, we can add the constraint diag$(\hat{Y}) \geq 0$ to (MP) and increase the value of q to 2 in (SP) .

In some cases (MP) is unbounded at initialization. There are several methods to deal with this. One method is to impose a large constant as the bound for elements of Y in (MP) . When Algorithm 1 terminates, remove these bounds but keep all the added cuts, and continue the algorithm until it terminates again. In the context of our application, where the COP is obtained from the CPP reformulation of a MBQP, there is an easier way to judge if the first termination reaches optimality: compare the optimal result at the first termination with the optimal result of the original MBQP problem (3). If they are equivalent then because of strong duality this solution is optimal.

Using the optimal solution from the original MBQP problem (3) problem, x^*, another way to bound (MP) at initialization is to add the complementary slackness constraint $x^* x^{*\top} \Omega = 0$. This constraint is valid when strong duality holds for the problem. Therefore, if Algorithm 1 terminates at a solution where strong duality does not hold, then the solution obtained with the additional complementary slackness constraint may not be valid.

Notice that Algorithm 1 is not guaranteed to terminate in finite steps. If it does not terminate, we cannot evaluate the quality of master problem solution, because even a small nonzero value of $\bar{w}$ still means $\hat{Y}$ violates the conic constraint.

5 Application I: Pricing in Nonconvex Energy Markets

We now specialize the development of copositive duality to the nonconvex UC problem and develop an economic interpretation. A UC problem solves for optimal production and commitment (turn on/off) decisions for generators in a power system. This problem is usually solved for a *day-ahead* market, which decides the operation schedule of the following day. The nonconvexity of this a day-ahead energy market comes from the binary start-up/shut-down decisions. We obtain the electricity prices by first reformulate the MIP UC problem to a CPP (Section 5.1), then derive the optimal dual values for some constraints in the CPP reformulation and set them as prices (Section 5.2). We also prove that under some assumptions this set of prices are revenue neutral and individually rational. Additionally, in Section 5.3 we propose a modified version of our pricing scheme that ensures revenue adequacy for individual generators.

Let $g \in \mathcal{G}$ be the index for generators in set $|\mathcal{G}|$, $t \in \mathcal{T}$ be the index for time in time period $|\mathcal{T}|$, c_g^o and c_g^s are respectively the costs of production and start up for generator g, $\underline{P}_g$ and $\overline{P}_g$ are respectively the lower and upper bounds for production levels of generator g. Additionally, we have three decision variables: p_{gt} is the production level of generator g at hour t, u_{gt} is a binary variable that equals 1 if generator g is started up at hour t (0 otherwise), and z_{gt} is a binary variable that equals 1 if generator g is on at hour t (0 otherwise). We deal with the following UC problem with load constraints and production level constraints:

$$
\begin{aligned}
\min \quad & \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c_g^o p_{gt} + c_g^s u_{gt}\right) && && \text{(15a)}\\
\text{s.t.} \quad & \sum_{g\in\mathcal{G}} p_{gt} = d_t && \forall t\in\mathcal{T} \quad (\lambda) && \text{(15b)}\\
& u_{gt} \ge z_{gt} - z_{g,t-1} && \forall g\in\mathcal{G}, t\in\mathcal{T}\setminus\{1\} \; (\theta) && \text{(15c)}\\
& p_{gt} \ge z_{gt}\underline{P}_g && \forall g\in\mathcal{G}, t\in\mathcal{T} \quad (\mu) && \text{(15d)}\\
& p_{gt} \le z_{gt}\overline{P}_g && \forall g\in\mathcal{G}, t\in\mathcal{T} \quad (\gamma) && \text{(15e)}\\
& u_{gt}, z_{gt} \in \{0,1\} && \forall g\in\mathcal{G}, t\in\mathcal{T} && \text{(15f)}\\
& p_{gt} \ge 0 && \forall g\in\mathcal{G}, t\in\mathcal{T} && \text{(15g)}
\end{aligned}
$$

5.1 CPP Reformulation of UC

For the convenience of CPP reformulation, which only works with models containing equality constraints, we rewrite problem (15) to add slack variables for all inequality constraints:

$$
\mathcal{UC}: \quad \min \quad \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c_g^o p_{gt} + c_g^s u_{gt}\right) \tag{16a}
$$

$$
\begin{aligned}
\text{s.t.}\quad & \sum_{g\in\mathcal{G}} p_{gt} = d_t && \forall t\in\mathcal{T} && (\lambda_t) && \text{(16b)}\\
& u_{gt} - \ddot{\theta}_{gt} = z_{gt} - z_{g,t-1} && \forall g\in\mathcal{G}, t\in\mathcal{T}\setminus\{1\} && (\theta_{gt}) && \text{(16c)}\\
& p_{gt} - \ddot{\mu}_{gt} = z_{gt}\underline{P}_g && \forall g\in\mathcal{G}, t\in\mathcal{T} && (\mu_{gt}) && \text{(16d)}\\
& p_{gt} + \ddot{\gamma}_{gt} = z_{gt}\overline{P}_g && \forall g\in\mathcal{G}, t\in\mathcal{T} && (\gamma_{gt}) && \text{(16e)}\\
& u_{gt}, z_{gt} \in \{0,1\} && \forall g\in\mathcal{G}, t\in\mathcal{T} && && \text{(16f)}\\
& p_{gt}, u_{gt}, z_{gt}, \ddot{\theta}_{gt}, \ddot{\mu}_{gt}, \ddot{\gamma}_{gt} \geq 0 && \forall g\in\mathcal{G}, t\in\mathcal{T} && && \text{(16g)}
\end{aligned}
$$

We reformulate the problem (16) in the form of $\mathcal{P}_o^{\text{CPP}}$, i.e. the CPP reformulation (4). Let

$$
\mathbf{x} = (u_{[gt]}, z_{[gt]}, p_{[gt]}, \ddot{\theta}_{[gt]}, \ddot{\mu}_{[gt]}, \ddot{\gamma}_{[gt]}, \ddot{\eta}_{[gt]})
$$

where $[gt]$ represent all members of indices $g \in \mathcal{G}$ and $t \in \mathcal{T}$. Let X be the lifted matrix of $\mathbf{x}$. To make the correspondence between elements of X and variables in vector $\mathbf{x}$ more explicit, we denote X_{ii}, the diagonal terms of X, as V_k, where V is the upper case of ith variable and k is its index. For example, X_{22} is denoted as U_{12}, as the second variable in x is u_{12}. Similarly, denote off-diagonal terms in X as $X_{ij} = [V^1V^2]_{k_1,k_2} (\forall i \neq j, i, j = 1, ..., n)$, where V^1 is the upper case of the ith variable in x and k_1 is its index. So $X_{12} = [UU]_{11,12}$.

The CPP reformulation is as follows:

$$
\begin{aligned}
\min\quad & \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}} \Big(c_g^o p_{gt} + c_g^s u_{gt}\Big) && && && \text{(17a)}\\
\text{s.t.}\quad & \sum_{g\in\mathcal{G}} p_{gt} = d_t && \forall t\in\mathcal{T} && (\lambda_t) && \text{(17b)}\\
& u_{gt} - \ddot{\theta}_{gt} = z_{gt} - z_{g,t-1} && \forall g\in\mathcal{G}, t\in\mathcal{T}\setminus\{1\} && (\theta_{gt}) && \text{(17c)}\\
& p_{gt} - \ddot{\mu}_{gt} = z_{gt}\underline{P}_g && \forall g\in\mathcal{G}, t\in\mathcal{T} && (\mu_{gt}) && \text{(17d)}\\
& p_{gt} + \ddot{\gamma}_{gt} = z_{gt}\overline{P}_g && \forall g\in\mathcal{G}, t\in\mathcal{T} && (\gamma_{gt}) && \text{(17e)}\\
& z_{gt} + \ddot{\eta}_{gt} = 1 && \forall g\in\mathcal{G}, t\in\mathcal{T} && (\eta_{gt}) && \text{(17f)}\\
& \text{Tr}(\mathbf{a}^{\lambda_t}\mathbf{a}^{\lambda_t\top}X) = d_t^2 && \forall t\in\mathcal{T} && (\Lambda_t) && \text{(17g)}\\
& \text{Tr}(\mathbf{a}^{\theta_{gt}}\mathbf{a}^{\theta_t\top}X) = 0 && \forall g\in\mathcal{G}, t\in\mathcal{T}\setminus\{1\} && (\Theta_{gt}) && \text{(17h)}\\
& \text{Tr}(\mathbf{a}^{\mu_{gt}}\mathbf{a}^{\mu_{gt}\top}X) = 0 && \forall g\in\mathcal{G}, t\in\mathcal{T} && (M_{gt}) && \text{(17i)}\\
& \text{Tr}(\mathbf{a}^{\gamma_{gt}}\mathbf{a}^{\gamma_{gt}\top}X) = 0 && \forall g\in\mathcal{G}, t\in\mathcal{T} && (\Gamma_{gt}) && \text{(17j)}\\
& \text{Tr}(\mathbf{a}^{\eta_t}\mathbf{a}^{\eta_t\top}X) = 1 && \forall g\in\mathcal{G}, t\in\mathcal{T} && (H_{gt}) && \text{(17k)}\\
& z_{gt} = Z_{gt} && \forall g\in\mathcal{G}, t\in\mathcal{T} && (\delta_{gt}) && \text{(17l)}\\
& Y \in \mathcal{C}^* && && (\Omega) && \text{(17m)}
\end{aligned}
$$

Constraints (17f) and (17k) are necessary because the upper bound of binary variable z_{gt} is not implied by constraints in $\mathcal{UC}$. On the other hand, we can leave out the upper bound constraint for binary variable u_{gt} as it is implied by constraints (16c). Vectors $\mathbf{a}^{\lambda_t}$, $\mathbf{a}^{\theta_{gt}}$, $\mathbf{a}^{\mu_{gt}}$ $\mathbf{a}^{\gamma_{gt}}$, and $\mathbf{a}^{\eta_t}$ are respectively the parameter vectors of constraints (17b) - (17f). Also,

$$Y = \begin{bmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & X \end{bmatrix}$$

The the dual of (17) is as follows:

$$\max \sum_{t\in\mathcal{T}} \Big(\lambda_t d_t + \Lambda_t d_t^2 + \sum_{g\in\mathcal{G}} (\eta_{gt} + H_{gt})\Big) \tag{18a}$$

$$\text{s.t.} \ \ (\lambda_{[t]}, \theta_{[gt]}, \mu_{[gt]}, \gamma_{[gt]}, \eta_{[gt]}, \Lambda_{[t]}, \Theta_{[gt]}, M_{[gt]}, \Gamma_{[gt]}, H_{[gt]}, \delta_{[gt]}, \Omega) \in \mathcal{F}^{\text{COP}} \tag{18b}$$

where $\mathcal{F}^{\text{COP}}$ represents the feasible region of the dual problem characterized by constraints in the form of constraints (5b) - (5e).

Since CPP (17) is a convex program, if it satisfies certain sufficient conditions such as Slater's condition, then strong duality holds. This means at an optimum, the primal objective (17a) is equivalent to the dual objective (18a).

For the convenience of the discussion that follows, we assume whenever we mention an optimal solution of CPP (17), i.e. $[\mathbf{x}^*, X^*]$, we assume its componet $\mathbf{x}^*$ solves the original MIP formulation (16). This assumption is not restrictive, as such an $\mathbf{x}^*$ always exists according to Proposition 3.

5.2 COP Pricing Mechanism

We would like to find a market mechanism where the total payments collected by the utility are enough to cover the total costs. Strong duality of CPP enables us to derive such a market mechanism, where we use the dual variables that appear in the dual objective (18a) as prices. More specifically, let $\mathbf{x}^*$ be the optimal solution of the original UC MIP problem (15), and $X^* = \mathbf{x}^*\mathbf{x}^{*\top}$. Then according to Proposition 3, $(\mathbf{x}^*, X^*)$ is an optimal solution to CPP (17). Let $(\lambda^*_{[t]}, \eta^*_{[gt]}, \Lambda^*_{[t]}, H^*_{[gt]})$ be a set of optimal dual solutions corresponding to the solution $(\mathbf{x}^*, X^*)$, and multiply the left-hand side of constraints (17c), (17f), (17g), and (17k) with corresponding dual prices (substitute the primal variables with optimal values too). Denoting the total payments as π^{SO}, as this payment is obtained from solving the system operator's aggregated cost minimization problem, then π^{SO} is calculated as follows:

$$\sum_{t\in\mathcal{T}} \Bigg(\sum_{g\in\mathcal{G}} \Big(\lambda^*_t p^*_{gt} + \eta^*_{gt} z^*_{gt} + \eta^*_{gt} \ddot{\eta}^*_{gt} + \Lambda^*_t P^*_{gt} \Big) + 2 \sum_{g_1<g_2, g_1, g_2\in\mathcal{G}} \Lambda^*_t [PP]^*_{g_1t, g_2t}$$

$$+\sum_{g\in\mathcal{G}}\left(H^*_{gt}Z^*_{gt}+H^*_{gt}\ddot{H}_{gt}+2H^*_{gt}[Z\ddot{H}]^*_{gt,t}\right)\bigg) \tag{19}$$

where $\sum_{g\in\mathcal{G}}\Lambda^*_t P^*_{gt}+2\sum_{g_1<g_2,g_1,g_2\in\mathcal{G}}\Lambda^*_t[PP]^*_{g_1t,g_2t}$ represents the explicit form of $\Lambda^*_t\mathrm{Tr}(\mathbf{a}^{\lambda_t}\mathbf{a}^{\lambda_t\top}X)$, and $H^*_{gt}Z^*_{gt}+H^*_{gt}\ddot{H}_{gt}+2H^*_{gt}[Z\ddot{H}]^*_{gt,t}$ represents the explicit form of $H^*_{gt}\mathrm{Tr}(\mathbf{a}^{\eta_t}\mathbf{a}^{\eta_t\top}X)$. Because of (17f) and (17k), we can equivalently replace $\eta^*_{gt}z^*_{gt}+\eta^*_{gt}\ddot{\eta}^*_{gt}$ with η^*_{gt} and replace $H^*_{gt}Z^*_{gt}+H^*_{gt}\ddot{H}_{gt}+2H^*_{gt}[Z\ddot{H}]^*_{gt,t}$ with H^*_{gt}, and rewrite π^{SO} as follows:

$$\pi^{\mathrm{SO}}=\sum_{t\in\mathcal{T}}\left(\sum_{g\in\mathcal{G}}\left(\lambda^*_t p^*_{gt}+\eta^*_{gt}+\Lambda^*_t P^*_{gt}\right)+2\sum_{g_1<g_2,g_1,g_2\in\mathcal{G}}\Lambda^*_t[PP]^*_{g_1t,g_2t}+\sum_{g\in\mathcal{G}}H^*_{gt}\right) \tag{20}$$

The term $\lambda^*_t p^*_{gt}$ is the payment from the uniform production price λ^*_t, corresponding to the demand constraint (17b). Payment $\Lambda^*_t P^*_{gt}$ is also a payment from uniform price, Λ^*_t, which corresponds to the lifted demand constraint (17g). However, there is an extra payment from this lifted constraint, i.e. $\Lambda^*_t[PP]^*_{g_1t,g_2t}$, which depends on an off-diagonal term of X^*: $[PP]^*_{g_1t,g_2t}$. Since this payment involves two generators, it is an open question as to how this payment should be divided between those generators. Payments η^*_{gt} and H^*_{gt} are the generator-dependent payments corresponding to constraints (17f) and (17k), respectively. Since the prices η^*_{gt} and H^*_{gt} signals the availability of generators at each hour, we name them the *availability prices*. Finally, we use the name *COP prices* for the prices $(\lambda^*_{[t]},\eta^*_{[gt]},\Lambda^*_{[t]},H^*_{[gt]})$ that consist the total payment π^{SO}.

5.2.1 Revenue Neutrality

It can be proved via the strong duality of CPP that the market mechanism with the total payment π^{SO} covers the total costs:

Theorem 2. *Assume CPP* (17) *satisfies some sufficient condition for convex programming strong duality, then the COP pricing payment π^{SO} covers the total costs.*

Proof. Substitute the primal variables with optimal values in constraints (17c), (17f), (17g), and (17k), then multiply both sides of the constraints with corresponding optimal dual prices:

$$\sum_{g\in\mathcal{G}}\lambda^*_t p^*_{gt}=\lambda^*_t d_t \qquad \forall t\in\mathcal{T} \tag{21a}$$

$$\eta^*_{gt}z^*_{gt}+\eta^*_{gt}\ddot{\eta}^*_{gt}=\eta^*_{gt} \qquad \forall g\in\mathcal{G},t\in\mathcal{T} \tag{21b}$$

$$\sum_{g\in\mathcal{G}}\Lambda^*_t P^*_{gt}+2\sum_{g_1<g_2,g_1,g_2\in\mathcal{G}}\Lambda^*_t[PP]^*_{g_1t,g_2t}=\Lambda^*_t d^2_t \qquad \forall t\in\mathcal{T} \tag{21c}$$

$$H^*_{gt}Z^*_{gt}+H^*_{gt}\ddot{H}_{gt}+2H^*_{gt}[Z\ddot{H}]^*_{gt,t}=H^*_{gt} \qquad \forall g\in\mathcal{G},t\in\mathcal{T} \tag{21d}$$

Sum up the equations in (21), then the left-hand side represents the total payments π^{SO}, while the right-hand side is the optimal value of the dual objective (18a):

$$\pi^{\text{SO}} = \sum_{t \in \mathcal{T}} \Big(\lambda_t^* d_t + \Lambda_t^* d_t^2 + \sum_{g \in \mathcal{G}} (\eta_{gt}^* + H_{gt}^*) \Big) \tag{22}$$

Because CPP (17) satisfies strong duality, the right-hand side of (22) equals the optimal objective of the primal problem. Thus,

$$\pi^{\text{SO}} = \sum_{g \in \mathcal{G}} \sum_{t \in \mathcal{T}} \Big(c_g^o p_{gt}^* + c_g^s u_{gt}^* \Big)$$

Therefore, the total payments from our market mechanism is equivalent to the total costs. □

Equation (22) in the proof of Theorem 2 also indicates that the utility's revenue and costs are balance under COP pricing, as the left-hand side of (22) is utility's total payments to generators, and the right-hand side of (22) represents utility's total revenue from consumers.

5.2.2 Individual Rationality

Another useful property of market mechanism is called *individually rational*. A market mechanism is individually rational if given the prices, each participate does not want to deviate from the designated production level, despite the decisions of other players. More specifically, let $(\lambda_{[t]}^*, \eta_{[gt]}^*, \Lambda_{[t]}^*, H_{[gt]}^*)$ be the dual prices in the market, then a generator g faces the following profit-maximization problem:

$$\begin{aligned}
\pi_g^{\text{Gen}}(\mathbf{x}^g, \mathbf{x}_{-g}) := \quad \max_{\mathbf{x}^g} \quad & \sum_{t \in \mathcal{T}} \Big(\lambda_t^* p_{gt} + \Lambda_t^* p_{gt}^2 + \eta_{gt}^* + H_{gt}^* + \sum_{g' \in \mathcal{G} \setminus \{g\}} f(p_g, p_{g'}) \\
& - c_g^o p_{gt} - c_g^s u_{gt} \Big) && && (23a) \\
\text{s.t.} \quad & u_{gt} - \ddot{\theta}_{gt} = z_{gt} - z_{g,t-1} && \forall t \in \mathcal{T} \setminus \{1\} \quad (\theta_{gt}) && (23b) \\
& p_{gt} - \ddot{\mu}_{gt} = z_{gt} \underline{P}_g && \forall t \in \mathcal{T} \qquad (\mu_{gt}) && (23c) \\
& p_{gt} + \ddot{\gamma}_{gt} = z_{gt} \overline{P}_g && \forall t \in \mathcal{T} \qquad (\gamma_{gt}) && (23d) \\
& u_{gt}, z_{gt} \in \{0, 1\} && \forall t \in \mathcal{T} && (23e) \\
& p_{gt}, \ddot{\theta}_{gt}, \ddot{\mu}_{gt}, \ddot{\gamma}_{gt} \geq 0 && \forall t \in \mathcal{T} && (23f)
\end{aligned}$$

where $\mathbf{x}_{-g}$ represents the decision variables in $\mathbf{x}$ for generators other than g. The first three terms in the objective (23) represent the total revenue for generator g. $f(p_g, p_{g'})$ is the share of g's revenue from the cross-term payment $2\Lambda_t^*[PP]_{gt,g't}$, and $f(p_g, p_{g'}) + f(p_{g'}, p_g) = 2\Lambda_t^* p_g p_{g'}$. Also, if $p_g p_{g'} = 0$ then $f(p_g, p_{g'}) = 0$. Notice that in order to formulate the individual generator's profit-maximization problem, which does not have lifted decision variables X, we need to assume that for the CPP reformulation (17) at optimality $P^*_{gt} = p^{*2}_{gt}$ and $[PP]^*_{gt,g't} = p^*_{gt}p^*_{g't}$, so replacing lifted terms with polynomials of $p_{[gt]}$ for revenue is valid. Again, this is a reasonable assumption because of Lemma 1.

If the solution from the profit-maximization problem (23) for all generators $g \in \mathcal{G}$ matches the solution of problem $\mathcal{UC}$, then the market mechanism is individually rational.

Theorem 3 provides a sufficient condition for our market mechanism to be individually rational. But before proving Theorem 3 we need the following lemma:

Lemma 2. *Let $\mathbf{x}^g$ and X^g denote the portions of $\mathbf{x}$ and X that correspond to generator $g \in \mathcal{G}$, and let Y^g be defined as follows:*

$$Y^g = \begin{bmatrix} 1 & \mathbf{x_g}^\top \\ \mathbf{x_g} & X^g \end{bmatrix}.$$

If additionally all terms in X of the form $[V^1V^2]^*_{g_1t,g_2t}, \forall g_1 \neq g_2, g_1, g_2 \in \mathcal{G}, t \in \mathcal{T}$ are equal to zero, then the conic constraint $Y \in \mathcal{C}^*$ is equivalent to constraints (24):

$$Y^g \in \mathcal{C}^*, \;\; \forall g \in \mathcal{G} \tag{24}$$

which ensure that Y^g is copositive for all $g \in \mathcal{G}$.

Proof. First we define the matrix $Y^{'}$:

$$Y' = \begin{bmatrix} 1 & \mathbf{x}^{1\top} & \dots & \mathbf{x}^{|\mathcal{G}|\top} \\ \mathbf{x}^1 & X^1 & \dots & \mathbf{0}^{k\times k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}^{|\mathcal{G}|} & \mathbf{0}^{k\times k} & \dots & X^{|\mathcal{G}|} \end{bmatrix}$$

where k is the cardinality of vector $\mathbf{x}^g, g \in \mathcal{G}$. Notice that Y' is different from Y, as in Y' elements are grouped by g, while in Y elements are grouped in the same order of variables in $\mathbf{x}$, i.e. $(u_{[gt]}, z_{[gt]}, p_{[gt]}, \ddot{\theta}_{[gt]}, \ddot{\mu}_{[gt]}, \ddot{\gamma}_{[gt]}, \ddot{\eta}_{[gt]})$.

Next, we prove that the constraint $Y' \in \mathcal{C}^*$ is equivalent to constraints (24). By definition, $Y' \in \mathcal{C}^*$ means there exists a nonnegative matrix $M \in \mathbb{R}^{(1+k|\mathcal{G}|)\times l}$ such that $Y' = MM^\top$. Let us divide the elements of M into blocks:

$$M = \begin{bmatrix} \mathbf{a}^{0\top} \\ A^1 \\ \vdots \\ A^{|\mathcal{G}|} \end{bmatrix}$$

where $\mathbf{a}^0 \in \mathbb{R}_+^l$, and $A^g \in \mathbb{R}_+^{k\times l}, g \in \mathcal{G}$. Then $Y' = MM^\top$ can be rewritten as:

$$Y' = \begin{bmatrix} \mathbf{a}^{0\top} \\ A^1 \\ \vdots \\ A^{|\mathcal{G}|} \end{bmatrix} [\mathbf{a}^0 \; A^{1\top} \; ... A^{|\mathcal{G}|\top}\;] = \begin{bmatrix} \mathbf{a}^{0\top}\mathbf{a}^0 & \mathbf{a}^{0\top}A^{1\top} & ... & \mathbf{a}^{0\top}A^{|\mathcal{G}|\top} \\ A^1\mathbf{a}^0 & A^1A^{1\top} & ... & \mathbf{0}^{k\times k} \\ \vdots & \vdots & \ddots & \vdots \\ A^{|\mathcal{G}|}\mathbf{a}^0 & \mathbf{0}^{k\times k} & ... & A^{|\mathcal{G}|}A^{|\mathcal{G}|\top} \end{bmatrix}$$

Therefore, $1 = \mathbf{a}^{0\top}\mathbf{a}^0$, $\mathbf{x}^g = A^g\mathbf{a}^0$, and $X^g = A^gA^{g\top}, \forall g \in \mathcal{G}$. Using this relationship for matrix $Y^g, g \in \mathcal{G}$, we have:

$$Y^g = \begin{bmatrix} 1 & \mathbf{x}^g \\ \mathbf{x}^g & X^g \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{0\top}\mathbf{a}^0 & \mathbf{a}^{0\top}A^{g\top} \\ A^g\mathbf{a}^0 & A^gA^{g\top} \end{bmatrix} = \begin{bmatrix} \mathbf{a}^{0\top} \\ A^g \end{bmatrix} \left[\mathbf{a}^0 \; A^{g\top}\right] \Rightarrow Y^g \in \mathcal{C}^*$$

Finally, we need to prove that $Y' \in \mathcal{C}^*$ if and only if $Y \in \mathcal{C}^*$. To prove this relationship, notice that when $[V^1V^2]^*_{g_1t,g_2t} = 0, \forall g_1 \neq g_2, g_1, g_2 \in \mathcal{G}, t \in \mathcal{T}$, we can obtain Y' from Y by permutating rows and columns of Y. In other words, there exists a permutation matrix $P \in \{0,1\}^{(k|\mathcal{G}|+1)\times(k|\mathcal{G}|+1)}$ such that $P^\top YP = Y'$. Since for any permutation matrix $P^\top = P^{-1}$, when $Y' \in \mathcal{C}^*$, we have:

$$Y \; = P^{\top-1}\left(P^\top YP\right)P^{-1} = \; PY'P^\top = PMM^\top P^\top \; \Rightarrow Y \in \mathcal{C}^*$$

Therefore, $Y \in \mathcal{C}^*$ is equivalent to constraints (24) when $[V^1V^2]^*_{g_1t,g_2t} = 0, \forall g_1 \neq g_2, g_1, g_2 \in \mathcal{G}, t \in \mathcal{T}$. $\square$

Now let us prove Theorem 3:

Theorem 3. *Assume CPP (17) satisfies some sufficient condition for convex programming strong duality. Additionally, let* $[\mathbf{x}^*, X^*]$ *be an optimal solution for the problem (17), assume all items in* X^* *in the forms of* $[V^1V^2]^*_{g_1t,g_2t}, \forall g_1 \neq g_2, g_1, g_2 \in \mathcal{G}, t \in \mathcal{T}$ *are equal to zero. Then the market mechanism with COP pricing is also individually rational.*

Proof. In the CPP reformulation (17), we dualize demand constraints (17b) and lifted demand constraints (17g) with their respective optimal dual prices λ_t^* and Λ_t^* , we obtain the following Lagrangian relaxation problem:

$$\begin{aligned} \min \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\left(c_g^o p_{gt} + c_g^s u_{gt}\right) + \sum_{t\in\mathcal{T}}\lambda_t^*\Big(d_t - \sum_{g\in\mathcal{G}} p_{gt}\Big) \\ + \sum_{t\in\mathcal{T}}\Lambda_t^*\Big(d_t^2 - \sum_{g\in\mathcal{G}}\Big(P_{gt} + \sum_{g'\in\mathcal{G},g'\neq g}[PP]_{gt,g't}\Big)\Big) \end{aligned} \tag{25a}$$

$$\text{s.t.} \;\; (17c) - (17f), (17h) - (17m) \tag{25b}$$

Since we have strong duality, and because the Lagrangian multipliers in the Lagrangian relaxation problem (25) are fixed to their optimal values, an optimal solution $[\mathbf{x}^*, X^*]$ for CPP (17) is also optimal for its Lagrangian relaxation (25).

Because the terms in the form of $[V^1V^2]_{g_1t,g_2t}$ are zeros, the term $[PP]_{g_1t,g_2t}$ in the objective can be eliminated. Also, because of Lemma 2, the conic constraint (17m) can be equivalently replaced by the set of conic constraints for individual generators:

$$Y^g \in \mathcal{C}^*, \ \ \forall g \in \mathcal{G}$$

Now, the Lagrangian relaxation (25) can be separated into individual optimization problems for each generator. Converting the minimization problem into maximization, and ignoring the constant terms $\lambda_t^* d_t$ and $\Lambda_t^* d_t^2$ in the objective, solving problem (25) is equivalent to solving the following maximization problem for all $g \in \mathcal{G}$:

$$\max \ \sum_{t\in\mathcal{T}} \Big(\lambda_t^* p_{gt} + \Lambda_t^* P_{gt} - c_g^o p_{gt} - c_g^s u_{gt}\Big) \tag{26a}$$

$$\text{s.t.} \ \ (17c) - (17f), (17h) - (17l) \tag{26b}$$

$$Y^g \in \mathcal{C}^* \tag{26c}$$

Therefore, if a solution $[\mathbf{x}^*, X^*]$ is optimal for CPP (17), then its component corresponding to g, $[\mathbf{x}^{*g}, X^{g*}]$, also solves (26) optimally.

Notice that in the profit-maximization problem (23) we assumed that at optimality $P_{gt}^* = p_{gt}^{*2}$ and $[PP]_{gt,g't}^* = p_{gt}^* p_{g't}^*$, in addition to the assumption of $[V^1V^2]_{g_1t,g_2t}^* = 0$, we can eliminate the cross-term payment from the the objective of (23). Now if we reformulate (23) into a CPP, its objective is the same as (26a) (except for the constant terms η_{gt}^* and H_{gt}^*). Moreover, problem (26) is exactly the CPP equivalence of the profit-maximization problem (23), which means $\mathbf{x}^{*g}$ is optimal for (23). Therefore, if we charge electricity prices λ_t^* and Λ_t^*, individual generators will not have the incentive to deviate from the optimal dispatch levels set by the system operator, as those dispatch levels also maximize their own profits. □

It is straightforward to see that if there is only one generator in the system, our market mechanism is individually rational, because there will be no crossed terms in the form of $[V^1V^2]_{g_1t,g_2t}$ in the problem.

We can further extend the concept of individual rationality to the behaviour of a subset of generators that are owned by one company. Let generators in $\mathcal{G}$ be grouped into n disjoint subsets $\mathcal{G}_1, ..., \mathcal{G}_n$, and each subset of generators jointly solve their own profit-maximization problem. We call a market mechanism *subset rational* if given the prices, each subset of generators do not have the incentive to deviate from the designated production level. The profit-maximization problem solved by generators in the subset $\mathcal{G}_i, i \in \{1, ..., n\}$ is as follows:

$$\begin{aligned}
\max \quad & \sum_{t\in\mathcal{T}} \Bigg(\sum_{g\in\mathcal{G}_i} \Big(\lambda_t^* p_{gt} + \Lambda_t^* P_{gt} + \sum_{g'\notin\mathcal{G}_i} f(p_{gt}, p_{g't}) - c_g^o p_{gt} \\
& - c_g^s u_{gt} \Big) + \sum_{g<g', g, g'\in\mathcal{G}_i} 2\Lambda_t^* [PP]^*_{gt,g't} \Bigg) && \text{(27a)} \\
\text{s.t.} \quad & (23b) - (23f) \qquad\qquad \forall g \in \mathcal{G}_i && \text{(27b)}
\end{aligned}$$

Corollary 1 provides a sufficient condition for subset rationality, which is less restrictive than the sufficient condition for individual rationality in Theorem 3:

Corollary 1. *Assume CPP (17) satisfies some sufficient condition for convex programming strong duality. Additionally, let $[\mathbf{x}^*, X^*]$ be an optimal solution for the problem (17), assume all items in X^* in the forms of $[V^1V^2]^*_{g_1t,g_2t}, \forall g_1 \in \mathcal{G}_i, g_2 \in \mathcal{G}_j, i \neq j, i, j = 1, ..., n; t \in \mathcal{T}$ are equal to zero. Then the market mechanism with total payment π^{SO} is also subset rational.*

The proof of Corollary 1 follows similar steps as the proof of Theorem 3, so we do not repeat here.

5.3 Ensuring Individual Revenue Adequacy

Although revenue from our COP pricing scheme covers the total cost of all the generators, it does not necessarily ensure the revenue adequacy of each individual generator. Because we have full access to the dual pricing problem in our COP pricing scheme, we can directly add revenue adequacy constraints to ensure that each generator is profitable at optimal production level $p^*_{[gt]}$.

There are two options for the revenue adequacy constraints: generators either earn their income only from the uniform prices λ_t and Λ_t, or they get the income from both the uniform prices and availability prices. We start with the uniform price only version first:

Revenue Adequacy with Only Uniform Prices: As shown in the modified COP dual problem (28), the revenue adequacy constraints (28b) here ensure that for each generator, the revenue from uniform prices exceeds the cost. We name the prices obtained from solving this problem the *modified COP prices*:

$$\max \sum_{t\in\mathcal{T}} \Big(\lambda_t d_t + \Lambda_t d_t^2 + \sum_{g\in\mathcal{G}} (\eta_{gt} + H_{gt}) \Big) \tag{28a}$$

$$\text{s.t.} \quad \sum_{t\in\mathcal{T}} \Big(p^*_{gt}\lambda_t + p^{*2}_{gt}\Lambda_t + \sum_{g'\in\mathcal{G}\setminus\{g\}} p^*_{gt}p^*_{g't}\Lambda_t \Big) \geq \sum_{t\in\mathcal{T}} \Big(c^o_g p^*_{gt} + c^s_g u^*_{gt} \Big) \quad \forall g \in \mathcal{G} \tag{28b}$$

$$(\lambda_{[t]}, \theta_{[gt]}, \mu_{[gt]}, \gamma_{[gt]}, \eta_{[gt]}, \Lambda_{[t]}, \Theta_{[gt]}, M_{[gt]}, \Gamma_{[gt]}, H_{[gt]}, \delta_{[gt]}, \Omega) \in \mathcal{F}^{\text{COP}} \tag{28c}$$

where the objective (28a) and the constraint (28c) are from the original dual problem (18). The left-hand side of constraint (28b) is the total revenue of generator g via uniform prices (assuming generators dividing the cross-term payments evenly between them), while the right-hand side of (28b) is the total cost of g, so constraint (28b) ensures that each generator is profitable.

One benefit of using modified COP pricing is that there is no generator-dependent payment, or "uplift payment", from availability prices. All generators are paid with the same price at each hour, so generators with lower costs earn more profits. This encourages the investments of lower-cost generation technologies.

Note that since we use only the uniform prices and ignore the availability prices η_{gt} and H_{gt}, the total payments under modified COP pricing is no longer revenue neutral. However, the modified COP pricing still balances the utility's revenue and payments. More specifically, the utility's revenue from demand at hour t is $\lambda_t d_t + \Lambda_t d_t^2$, while its total payments to generators at hour t is $\sum_{g\in\mathcal{G}} \Big(p^*_{gt}\lambda_t + p^{*2}_{gt}\Lambda_t + \sum_{g'\in\mathcal{G}\setminus\{g\}} p^*_{gt}p^*_{g't}\Lambda_t \Big)$. According to equation (22) in the proof of Theorem 2, we have

$$\begin{aligned}
&\sum_{t\in\mathcal{T}} \Bigg(\sum_{g\in\mathcal{G}} \Big(\lambda^*_t p^*_{gt} + \eta^*_{gt} + \Lambda^*_t p^{*2}_{gt} \Big) + 2 \sum_{g_1<g_2, g_1,g_2\in\mathcal{G}} \Lambda^*_t p^*_{g_1 t} p^*_{g_2 t} + \sum_{g\in\mathcal{G}} H^*_{gt} \Bigg) \\
&= \sum_{t\in\mathcal{T}} \Big(\lambda^*_t d_t + \Lambda^*_t d_t^2 + \sum_{g\in\mathcal{G}} (\eta^*_{gt} + H^*_{gt}) \Big)
\end{aligned} \tag{29}$$

eliminating terms of availability prices from both sides, we obtain the result we want, i.e. utility's total payments equals the revenue. We formalize this result in the following theorem:

Theorem 4. *The modified COP pricing balances the utility's revenue and payments.*

Because the addition of constraints (28b) restrict the original dual problem (18), it is worth checking if the problem is still feasible. Since in conic programs, if one of the primal and dual conic programs is feasible, bounded and has an interior point, then the other is feasible too (Luenberger and Ye, 2015), we look at the dual problem of (28). Let the dual variable corresponding to (28b) be denoted as $q_g \geq 0$, then the dual of (28) is the following:

$$\min \quad \sum_{g\in\mathcal{G}} \sum_{t\in\mathcal{T}} \Big(c^o_g p_{gt} + c^s_g u_{gt} - (c^o_g p^*_{gt} + c^s_g u^*_{gt}) q_g \Big) \tag{30a}$$

$$\text{s.t.} \quad \sum_{g\in\mathcal{G}} \big(p_{gt} - p^*_{gt} q_g \big) = d_t \qquad \forall t \in \mathcal{T} \tag{30b}$$

$$\mathrm{Tr}(\mathbf{a}^{\lambda_t}\mathbf{a}^{\lambda_t\top}X) - \sum_{g\in\mathcal{G}}\Big(p_{gt}^{*2} + \sum_{g'\in\mathcal{G}\setminus\{g\}} p_{gt}^{*}p_{g't}^{*}\Big)q_g = d_t^2 \quad \forall t\in\mathcal{T} \tag{30c}$$

$$(17c) - (17f), (17h) - (17l) \tag{30d}$$

$$q_g \geq 0 \qquad g\in\mathcal{G} \tag{30e}$$

$$Y\in\mathcal{C}^* \tag{30f}$$

which basically is CPP problem (17) with extra terms of q_g in objective and constraints.

CPP problem (30) is feasible because we can always let all q_g be zero and get problem (17). It is bounded because in the objective q_g's cannot take very large values, as they are bounded by constraints (30b) and (30c). If additionally the problem has an interior, then its dual problem (28) should also be feasible. This reasoning leads us to the following proposition:

Proposition 4. *The modified COP pricing problem* (28) *is feasible if its dual problem* (30) *contains an interior.*

Revenue Adequacy with Both Uniform and Availability Prices: If we include both uniform and availability prices in the revenue adequacy constraints, we have the following pricing problem to solve:

$$\max \ \sum_{t\in\mathcal{T}}\Big(\lambda_t d_t + \Lambda_t d_t^2 + \sum_{g\in\mathcal{G}}(\eta_{gt} + H_{gt})\Big) \tag{31a}$$

$$\text{s.t.} \ \sum_{t\in\mathcal{T}}\Big(p_{gt}^{*}\lambda_t + p_{gt}^{*2}\Lambda_t + \sum_{g'\in\mathcal{G}\setminus\{g\}} p_{gt}^{*}p_{g't}^{*}\Lambda_t + \eta_{gt} + H_{gt}\Big)$$

$$\geq \sum_{t\in\mathcal{T}}\Big(c_g^o p_{gt}^* + c_g^s u_{gt}^*\Big) \qquad \forall g\in\mathcal{G} \tag{31b}$$

$$(\lambda_{[t]}, \theta_{[gt]}, \mu_{[gt]}, \gamma_{[gt]}, \eta_{[gt]}, \Lambda_{[t]}, \Theta_{[gt]}, M_{[gt]}, \Gamma_{[gt]}, H_{[gt]}, \delta_{[gt]}, \Omega) \in \mathcal{F}^{\text{COP}} \tag{31c}$$

If (31) is feasible, then prices from (31) should satisfy revenue neutrality. This is because if we sum up left-hand sides of (31b) over g, then the value equals the objective (31a) (proved by (22)), which represents the total revenue of generators. One the other hand, (31) can be viewed as imposing extra constraints on the original COP pricing problem (18), whose objective value, according to weak duality, is no more than the total costs $\sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{T}}\Big(c_g^o p_{gt}^* + c_g^s u_{gt}^*\Big)$. But (31) also restricts the total revenue of generators to be no less than the total costs. Therefore, a feasible solution of (31) should ensure revenue neutrality.

In addition, we actually have revenue neutrality for every generator, so each generator is paid exactly its cost. To understand why this result is true, assume towards contradiction if any generator has a strictly positive profit, then because of revenue neutrality of the whole system, some other generator must have a strictly negative profit, which violates (31b).

In comparison, we do not have revenue neutrality for individual generators in the modified COP pricing, i.e. in that case generators could have strictly positive profits.

Similar to the case of modified COP pricing, with this pricing scheme, it can be proved that (31) is guaranteed to be feasible if its dual problem has an interior.

Although our COP pricing schemes are derived for UC problems, it is easy to adapt them for any problems with both variable and fixed costs. For example, we use the COP pricing schemes to derive prices for the Scarf's example, a classical nonconvex market problem, in Section 7.2.

6 Application II: Mixed-Binary Quadratic (MBQ) Games

In this section we apply the COP duality concept to MBQ games, which is a special type of IPGs. An MBQ game is defined as a game where each player faces a MBQP in the form of $\mathcal{P}^{\text{MBQP}}$. Because of the binary variables in $\mathcal{P}^{\text{MBQP}}$, the PNE existence and uniqueness results for games with convex strategy set cannot be directly used for MBQ games. Also, we cannot apply KKT conditions for finding the NE as $\mathcal{P}^{\text{MBQP}}$ does not have strong duality. We overcome those difficulties by transform the discrete MBQ games to convex completely positive games (Section 6.1), then utilize this relationship to propose a PNE existence and uniqueness theorems (Section 6.2), as well as KKT conditions for computing the NE (Section 6.3), for MBQ games.

Formally, we study an n-person MBQ game $\mathcal{G}^{\text{MBQ}} = \langle \mathcal{I}, (\mathcal{X}_i)_{i\in\mathcal{I}}, (\mathbf{x}_i)_{i\in\mathcal{I}} \rangle$, where $\mathcal{I}$ is the set of indices for the players and $|\mathcal{I}|= n$; $\mathcal{X}_i$ is the *strategy set* for player i and $\mathbf{x}_i \in \mathcal{X}_i$ is the vector of player i's strategy that contains both continuous and binary variables. Moreover, let $\mathcal{X}_{-i}$ denote the strategy set of all players except i, and $\mathbf{x}_{-i}$ denotes the vector of strategy inside $\mathcal{X}_{-i}$. Player i faces the following MBQP:

$$
\begin{aligned}
\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}) : \min \quad & f_i(\mathbf{x}_i, \mathbf{x}_{-i}) && \text{(32a)}\\
\text{s.t.} \quad & \mathbf{a}^{j(i)\top}\mathbf{x}_i = b_j^{(i)}, \forall j = 1, ..., m_i && \text{(32b)}\\
& x_{ik} \in \{0, 1\}, \quad \forall k \in \mathcal{B}_i && \text{(32c)}\\
& \mathbf{x}_i \in \mathbb{R}_+^l && \text{(32d)}
\end{aligned}
$$

where the quadratic *payoff function* $f_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \mathbf{x}^\top Q^{(i)}\mathbf{x} + 2\mathbf{c}^{(i)\top}\mathbf{x}$, with $\mathbf{x}$ being the vector of all strategies and $Q^{(i)} \in \mathcal{S}^l$ being a symmetric matrix. Constraints (32b) are linear constraints for player i which are independent from decisions of other players. $\mathcal{B}_i$ is a subset of indices corresponding to binary variables for player i. Let $\text{Feas}(\mathcal{P})$ denote the feasible region of an optimization problem $\mathcal{P}$. Then in our setup $\text{Feas}(\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})) = X_i, \forall i \in \mathcal{I}$, because there are only private constraints in each player's problem.

We define the pure-strategy NE (PNE) of $\mathcal{G}^{\text{MBQ}}$ as the vector of actions $\mathbf{x}^* \in (\mathcal{X}_i)_{i\in\mathcal{I}}$ such that for each player i:

$$f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) \leq f_i(\mathbf{x}_i, \mathbf{x}_{-i}^*), \forall \mathbf{x}_i \in \mathcal{X}_i,$$

or in other words

$$f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) = \text{opt}(\mathcal{P}_i^{\text{MBQP}}(x_{-i}^*)), \tag{33}$$

i.e., at a PNE player i cannot achieve a better payoff by only changing her own behaviours.

6.1 Reformulation to Completely Positive (CP) Games

In this section we show that it is possible to transform an MBQ game in the form of $\mathcal{G}^{\text{MBQ}}$ to a CP game (which will be defined later in this section). Further, we show that from a PNE of a CP game we can obtain the PNE of $\mathcal{G}^{\text{MBQ}}$, and vice versa.

The following proposition reformulates $\mathcal{P}_i^{\text{MBQP}}$ to a CPP:

Proposition 5. *Assume problem $\mathcal{P}_i^{MBQP}(\mathbf{x}_{-i})$ satisfies Assumption A, and denote the $\boldsymbol{\alpha}$ in Assumption A for $\mathcal{P}_i^{MBQP}(\mathbf{x}_{-i})$ as $\boldsymbol{\alpha}^{(i)}$, then problem $\mathcal{P}_i^{MBQP}$ is equivalent to the following CPP:*

$$\begin{aligned}
\mathcal{P}_i^{CPP}(\mathbf{x}_{-i}) : \min \;& Tr(Q_{ii}^{(i)} X_i) + 2\mathbf{x}_{-i}^\top Q_{-i,i}^{(i)} \mathbf{x}_i + \mathbf{x}_{-i}^\top Q_{-i,-i}^{(i)} \mathbf{x}_{-i} + 2\mathbf{c}^{(i)\top}\mathbf{x} && && \text{(34a)}\\
\text{s.t. }\;& \mathbf{a}^{j(i)\top}\mathbf{x}_i = b_j^{(i)} && \forall j = 1, ..., m_i && \text{(34b)}\\
& \mathbf{a}^{j(i)\top} X_i \mathbf{a}^{j(i)} = b_j^{(i)2} && \forall j = 1, ..., m_i && \text{(34c)}\\
& x_{ik} = X_{i,kk} && \forall k \in \mathcal{B}_i && \text{(34d)}\\
& \mathbf{x}_i = X_i \boldsymbol{\alpha}^{(i)} && && \text{(34e)}\\
& X_i \in \mathcal{C}^* && && \text{(34f)}
\end{aligned}$$

Proof. The objective function of $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})$ can be decomposed in terms of $\mathbf{x}_i$ and $\mathbf{x}_{-i}$:

$$\begin{aligned}
f_i(\mathbf{x}_i, \mathbf{x}_{-i}) &= \mathbf{x}^\top Q^{(i)} \mathbf{x} + \mathbf{c}^{(i)\top}\mathbf{x}\\
&= \mathbf{x}_i^\top Q_{ii}^{(i)} \mathbf{x}_i + 2\mathbf{x}_{-i}^\top Q_{-i,i}^{(i)} \mathbf{x}_i + \mathbf{x}_{-i}^\top Q_{-i,-i}^{(i)} \mathbf{x}_{-i} + 2\mathbf{c}_i^{(i)\top}\mathbf{x}_i + 2\mathbf{c}_{-i}^{(i)\top}\mathbf{x}_{-i}\\
&= \mathbf{x}_i^\top Q_{ii}^{(i)} \mathbf{x}_i + 2(\mathbf{x}_{-i}^\top Q_{-i,i}^{(i)} + \mathbf{c}_i^{(i)\top})\mathbf{x}_i + \mathbf{x}_{-i}^\top Q_{-i,-i}^{(i)} \mathbf{x}_{-i} + 2\mathbf{c}_{-i}^{(i)\top}\mathbf{x}_{-i}.
\end{aligned}$$

$\mathbf{x}_{-i}$ is treated as a constant vector in $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})$. Therefore, via Proposition 5, $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})$ is equivalent to the following CPP:

$$\begin{aligned}
\min \;& \text{Tr}(Q_{ii}^{(i)} X_i) + 2(\mathbf{x}_{-i}^\top Q_{-i,i}^{(i)} + \mathbf{c}_i^{(i)\top})\mathbf{x}_i + \mathbf{x}_{-i}^\top Q_{-i,-i}^{(i)} \mathbf{x}_{-i} + 2\mathbf{c}_{-i}^{(i)\top}\mathbf{x}_{-i}\\
\text{s.t. }\;& \mathbf{a}^{j(i)\top}\mathbf{x}_i = b_j^{(i)} \qquad \forall j = 1, ..., m_i
\end{aligned}$$

$$
\begin{aligned}
&\mathbf{a}^{j(i)\top} X_i \mathbf{a}^{j(i)} = b_j^{(i)2} && \forall j = 1, ..., m_i \\
&x_{ik} = X_{i,kk} && \forall k \in \mathcal{B}_i \\
&\mathbf{x}_i = X_i \boldsymbol{\alpha}^{(i)} \\
&X_i \in \mathcal{C}^*
\end{aligned}
$$

Rearrange the terms in the objective, and we obtain $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i})$. □

We introduce a type of n-player *CP game* $\mathcal{G}^{\text{CP}} = \langle \mathcal{I}, (\mathcal{X}_i')_{i\in\mathcal{I}}, (\mathbf{x}_i, X_i)_{i\in\mathcal{I}} \rangle$. In this game, each player i solves a CPP $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i})$. Let $F_i(\mathbf{x}_i, X_i, \mathbf{x}_{-i})$ denote the objective of $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i})$. Then an PNE of the CP game $\mathcal{G}^{\text{CP}}$ is given by the strategies $(\mathbf{x}_i^*, X_i^*)_{i\in\mathcal{I}} \in (\mathcal{X}_i')_{i\in\mathcal{I}}$ such that for each player i:

$$
F_i(\mathbf{x}_i^*, X_i^*, \mathbf{x}_{-i}^*) = \text{opt}(\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i}^*)) \tag{35}
$$

i.e., with the decisions of other players fixed at $\mathbf{x}_{-i}^*$, $(\mathbf{x}_i^*, X_i^*)$ is the optimal solution for $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i})$.

With our definition of the CP game $\mathcal{G}^{\text{CP}}$, we can transform any MBQ game $\mathcal{G}^{\text{MBQ}}$ to a CP game $\mathcal{G}^{\text{CP}}$ (supposing Assumption A is satisfied). It remains to show the relationship between the Nash equilibria of those two types of games, which is the goal of Theorem 5:

Theorem 5. *Let $\mathbf{x}^*$ be a PNE of $\mathcal{G}^{MBQ}$, then $(\mathbf{x}_i^*, \mathbf{x}_i^*\mathbf{x}_i^{*\top})_{i\in\mathcal{I}}$ is a PNE of $\mathcal{G}^{CP}$.*

Conversely, if $(\mathbf{x}_i^*, X_i^*)_{i\in\mathcal{I}}$ is a PNE of $\mathcal{G}^{CP}$, and if $(Q_{ii}^{(i)} \succeq 0)_{i\in\mathcal{I}}$, $(x_{ik}^* \in \{0,1\}, \forall k \in \mathcal{B}_i)_{i\in\mathcal{I}}$, then $\mathbf{x}^*$ is a PNE of $\mathcal{G}^{MBQ}$.

Proof. ($\Rightarrow$) We first prove the conversion from $\mathcal{G}^{\text{MBQ}}$ to $\mathcal{G}^{\text{CP}}$. Since $\mathbf{x}^*$ is a PNE of $\mathcal{G}^{\text{MBQ}}$, it satisfies $f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*) = \text{opt}(\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}^*)), \forall i \in \mathcal{I}$. On the other hand, $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}^*)$ can be reformulated to $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i}^*)$. From Proposition 3, we know that $(\mathbf{x}_i^*, \mathbf{x}_i^*\mathbf{x}_i^{*\top})$ is an optimal solution of $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i}^*)$, which means $(\mathbf{x}_i^*, \mathbf{x}_i^*\mathbf{x}_i^{*\top})_{i\in\mathcal{I}}$ is a PNE for $\mathcal{G}^{\text{CPP}}$.

($\Leftarrow$) For the opposite direction, assume $(\mathbf{x}_i^*, X_i^*)_{i\in\mathcal{I}}$ is a PNE for $\mathcal{G}^{\text{CP}}$, which means $(\mathbf{x}_i^*, X_i^*)$ is an optimal solution for $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i}^*)$, and $\mathbf{x}_i^*$ is in the convex hull of optimal solutions for $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}^*)$. Now assume $(Q_{ii}^{(i)} \succeq 0)_{i\in\mathcal{I}}$, $(x_{ik}^* \in \{0,1\}, \forall k \in \mathcal{B}_i)_{i\in\mathcal{I}}$, we want to prove that $\mathbf{x}_i^*$ is a PNE. We start by proving $\mathbf{x}_i^*$ being feasible for $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}^*)$. Note that $\mathbf{x}_i^*$ satisfies (32b) because it satisfies (34b). $\mathbf{x}_i \in \mathbb{R}_+^l$ is guaranteed because $X_i^* \in \mathcal{C}^* \Rightarrow X_i^* \geq 0 \Rightarrow X_i^*\boldsymbol{\alpha}^{(i)} \geq 0 \Rightarrow \mathbf{x}_i^* \geq 0$, where the last step follows from the constraint (34e). Additionally, we assumed $x_{ik}^* \in \{0,1\}, \forall k \in \mathcal{B}_i$, which means (32c) is satisfied. Since $\mathbf{x}_i^*$ is feasible to $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})$ and $Q_{ii}^{(i)} \succeq 0$, by Proposition 2 we know that $\mathbf{x}_i^*$ is an optimal solution for $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}^*)$, where $\mathbf{x}_{-i}^*$ are also ensured to be pure strategies. Thus, $\mathbf{x}^*$ is an NE for $\mathcal{G}^{\text{MBQ}}$. □

Remark. The requirement $(x_{ik}^* \in \{0,1\}, \forall k \in \mathcal{B}_i)_{i\in\mathcal{I}}$ is important for ensuring that a PNE of $\mathcal{G}^{\text{CP}}$ can be projected to a PNE of $\mathcal{G}^{\text{MBQ}}$. In fact, we can replace this requirement with the (possibly more restrictive) condition that the matrix $\begin{bmatrix} 1 & \mathbf{x}_i^{*\top} \\ \mathbf{x}_i^* & X_i^* \end{bmatrix}$ being rank-1 feasible for $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i}), \forall i \in \mathcal{I}$, which is similar to the rank-1 feasible condition for recovering an NE from the SDP relaxation of an NE problem in Ahmadi and Zhang (2020). The following proposition shows the relationship between those two requirements:

Proposition 6. *Let $(\mathbf{x}_i, X_i)$ be a feasible solution for $\mathcal{P}_i^{CPP}(\mathbf{x}_{-i})$. If $Y_i = \begin{bmatrix} 1 & \mathbf{x}_i^\top \\ \mathbf{x}_i & X_i \end{bmatrix}$ is a rank-1 feasible solution for $\mathcal{P}_i^{CPP}(\mathbf{x}_{-i})$, then $x_i^k \in \{0,1\}, \forall k \in \mathcal{B}_i$.*

Proof. If Y_i is rank-1 feasible, there exist vectors $\mathbf{w}$ and $\mathbf{v}$ such that $Y_i = \mathbf{v}^\top \mathbf{w}$. Let $\mathbf{v} = \begin{bmatrix} v_1, & \mathbf{v}_2^\top \end{bmatrix}^\top$, $\mathbf{w} = \begin{bmatrix} w_1, & \mathbf{w}_2^\top \end{bmatrix}^\top$, and rewrite the rank-1 equation in blocks, we have:

$$\begin{aligned}
\begin{bmatrix} 1 & \mathbf{x}_i^\top \\ \mathbf{x}_i & X_i \end{bmatrix} &= \begin{bmatrix} v_1, & \mathbf{v}_2^\top \end{bmatrix} \begin{bmatrix} w_1 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} v_1 w_1 & w_1 \, \mathbf{v}_2^\top \\ v_1 \, \mathbf{w}_2 & \mathbf{v}_2^\top \mathbf{w}_2 \end{bmatrix} \\
\Rightarrow \mathbf{x}_i \mathbf{x}_i^\top &= w_1 \, \mathbf{v}_2^\top \cdot v_1 \, \mathbf{w}_2 = w_1 v_1 \cdot \mathbf{v}_2^\top \mathbf{w}_2 = \mathbf{v}_2^\top \mathbf{w}_2 = X_i \\
\Rightarrow x_{ik}^2 &= X_{i,kk} = x_{ik}, \forall k \in \mathcal{B}_i \\
\Rightarrow x_{ik} &\in \{0,1\}, \forall k \in \mathcal{B}_i
\end{aligned}$$

□

The second half of Theorem 5 can also be replaced with the following proposition:

Proposition 7. *If $(\mathbf{x}_i^*, X_i^*)_{i \in \mathcal{I}}$ is a PNE of $\mathcal{G}^{CP}$, and if $(Q_{ii}^{(i)} \succeq 0)_{i \in \mathcal{I}}$, and each one of the optimization problems $(\mathcal{P}_i^{MBQP}(\mathbf{x}_{-i}^*))_{i \in \mathcal{I}}$ has a unique optimal solution, then $\mathbf{x}^*$ is a PNE of $\mathcal{G}^{MBQ}$.*

This proposition is true because when $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}^*)$ has a unique optimal solution, the solution $\mathbf{x}_i^*$, which is in the convex hull of $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}^*)$'s optimal solutions, must be feasible for $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}^*)$. Then we can follow a similar proof as in Theorem 5 to show that $\mathbf{x}^*$ is a PNE of $\mathcal{G}^{\text{MBQ}}$.

Proposition 7 is more restrictive than the second half of Theorem 5, as when $(\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}^*))_{i \in \mathcal{I}}$ have unique optimal solutions, $(\mathbf{x}_i^*, X_i^*)_{i \in \mathcal{I}}$ must satisfy $(x_{ik}^* \in \{0,1\}, \forall k \in \mathcal{B}_i)_{i \in \mathcal{I}}$. The reverse, however, is not necessarily true.

6.2 Existence of NE in $\mathcal{G}^{\text{MBQ}}$

In this section we prove that PNE for $\mathcal{G}^{\text{MBQ}}$ exists with some conventional assumptions. To prove the existence of NE in $\mathcal{G}^{\text{MBQ}}$, we first prove the existence of PNE in a CP game $\mathcal{G}^{\text{CP}}$ by a well-known result for concave n-person games:

Proposition 8. *For each $\mathcal{P}_i^{CPP}(\mathbf{x}_{-i}), \forall i = 1, ..., n$, if $\mathcal{X}_i'$ is compact, then PNE exists for $\mathcal{G}^{CP}$.*

Proof. We use the following well-known existence theorem for PNE (see Ozdaglar (2015), for example): For every concave n-person game where the strategy set of a player i's maximization problem is convex and compact, and the payoff function is continuous on all players' strategies and concave (convex for minimization) for player i's strategies, then there exists at least one PNE point.

In our case, the feasible region $\mathcal{X}_i'$ of $\mathcal{P}_i^{\text{CPP}}$ is convex because there are only linear and conic constraints in $\mathcal{P}_i^{\text{CPP}}$. We also assumed $\mathcal{X}_i'$ to be compact. The objective of $\mathcal{P}_i^{\text{CPP}}$ is linear thus convex on player i's strategies $\mathbf{x}_i$ and X_i. It is also continuous for $\mathbf{x}$ and X_i. Thus, $\mathcal{G}^{\text{CP}}$ satisfy all the conditions for the existence of the PNE in a concave n-person game. $\square$

To prove the existence of NE in $\mathcal{G}^{\text{MBQ}}$, we also need the following lemma for relating the compactness of feasible regions $\mathcal{X}_i$ and $\mathcal{X}_i'$:

Lemma 3. *If $\mathcal{X}_i$ is compact, then $\mathcal{X}_i'$ is also compact.*

Proof. Define

$$\text{Feas}^+(\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})) := \text{conv}\left\{\begin{pmatrix}1\\ \mathbf{x}_i\end{pmatrix}\begin{pmatrix}1\\ \mathbf{x}_i\end{pmatrix}^\top : \mathbf{x}_i \in \mathcal{X}_i\right\}$$
$$\text{Feas}^+(\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i})) := \left\{\begin{pmatrix}1 & \mathbf{x}_i^\top\\ \mathbf{x}_i & X_i\end{pmatrix} : (\mathbf{x}_i, X_i) \in \mathcal{X}_i'\right\}.$$

By Corollary 2.5 in Burer (2009), if $\mathcal{X}_i$ is bounded, then $\text{Feas}^+(\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})) = \text{Feas}^+(\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i}))$. Since we assume $\mathcal{X}_i$ is compact, which means it is bounded and closed (assuming Euclidean space), we also have $\text{Feas}^+(\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})) = \text{Feas}^+(\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i}))$. The compactness of $\mathcal{X}_i$ also suggests that $\text{Feas}^+(\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i}))$ is closed and bounded, because it is defined as the convex hull of $\begin{pmatrix}1\\ \mathbf{x}_i\end{pmatrix}\begin{pmatrix}1\\ \mathbf{x}_i\end{pmatrix}^\top$, where $\mathbf{x}_i \in \mathcal{X}_i$. This implies that $\text{Feas}^+(\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i}))$ is also closed and bounded.

Because $\text{Feas}^+(\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i}))$ is defined as the set of $\begin{pmatrix}1 & x^\top\\ x & X\end{pmatrix}$ where $(x, X) \in \mathcal{X}_i'$, $\mathcal{X}_i'$ must also be closed and bounded, i.e. compact. $\square$

Now we provide a sufficient condition for the existence of PNE in $\mathcal{G}^{\text{MBQ}}$:

Theorem 6. (Existence of NE) *In an MBQ game $\mathcal{G}^{MBQ}$ for each player's optimization problem $\mathcal{P}_i^{MBQP}(\mathbf{x}_{-i})$, if*

(i) Assumption (A) is true,

(ii) $\mathcal{X}_i$ is compact,

then the corresponding CP game $\mathcal{G}^{CP}$ has at least one PNE. Additionally, if

(iii) $Q_{ii}^{(i)} \succeq 0$, and

(iv) there exists a PNE $(\mathbf{x}_i^*, X_i^*)_{i\in\mathcal{I}}$ for $\mathcal{G}^{CP}$ that satisfies $(x_{ik}^* \in \{0,1\}, \forall k \in \mathcal{B}_i)_{i\in\mathcal{I}}$,

then PNE exists for the $\mathcal{G}^{MBQ}$.

Proof. Conditions (i) and (ii) ensure that the optimization problems of $\mathcal{G}^{\text{MBQ}}$ can be reformulated as CPPs, and the obtained CP game $\mathcal{G}^{\text{CP}}$ has at least one NE: For each player i, because assumption (A) is true, we can reformulate $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})$ as the corresponding CPP problem $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i})$. Since $\mathcal{X}_i$ is compact, by Lemma 3, $\mathcal{X}_i'$ is also compact. By Proposition 8, there exists a PNE for $\mathcal{G}^{\text{CP}}$.

Let $(\mathbf{x}_i^*, X_i^*)_{i\in\mathcal{I}}$ be an PNE that satisfies condition (iv), along with the condition (iii), all requirements for the existence of PNE for $\mathcal{G}^{\text{MBQ}}$ in Theorem 5 are met and thus $\mathbf{x}^*$ is a PNE for $\mathcal{G}^{\text{MBQ}}$. Therefore, the PNE of $\mathcal{G}^{\text{MBQ}}$ exists with conditions (i)-(iv). $\square$

If condition (iv) is not satisfied, and $\mathcal{G}^{\text{CP}}$ does not have any PNE that satisfies $(x_{ik}^* \in \{0,1\}, \forall k \in \mathcal{B}_i)_{i\in\mathcal{I}}$, then $\mathcal{G}^{\text{MBQ}}$ does not have any PNE, as stated in the following corollary:

Corollary 2. *Assume the optimization problems of $\mathcal{G}^{MBQ}$ satisfy conditions (i) - (iii) in Theorem 6. If the PNE of the corresponding $\mathcal{G}^{CP}$ does not satisfy condition (iv), then $\mathcal{G}^{MBQ}$ does not have any PNE.*

Proof. This is a straightforward result of Theorems 5. Assume towards contradiction that $\mathcal{G}^{\text{MBQ}}$ has a PNE $\mathbf{x}^*$, then $(\mathbf{x}_i^*, \mathbf{x}_i^*\mathbf{x}_i^{*\top})_{i\in\mathcal{I}}$ is a PNE for $\mathcal{G}^{\text{CP}}$ that satisfies (iv), which is a contradiction. $\square$

Using Proposition 7, the condition (iv) in Theorem 6 can also be replaced with the requirement that the optimization problems in $\mathcal{G}^{\text{MBQ}}$ all have unique optimal solutions, since this requirement also ensures that a PNE of $\mathcal{G}^{\text{CP}}$ can be converted to a PNE of $\mathcal{G}^{\text{MBQ}}$. It is worth noting that a similar requirement is proposed for the existence of PNE in two-person discrete games, which is a special type of MBQ game, by Mallick (2011). In that paper the author proves for a two-person discrete game, if both players have unique best responses and if a condition called Minimal Acyclicity is satisfied, then the PNE must exists.

We next present conditions for the uniqueness of PNE in MBQ games. We use the concept of *diagonally strictly convex (concave)* from (Rosen, 1965): let $\nabla_i u(\mathbf{x})$ be the gradient of u with respect to $\mathbf{x}$'s subvector, $\mathbf{x}_i$. Additionally, let $u_1, ..., u_{|\mathcal{I}|}$ be a sequence of functions and define $\nabla u(\mathbf{x}) = [\nabla_1 u_1(\mathbf{x}), ..., \nabla_{|\mathcal{I}|} u_{|\mathcal{I}|}(\mathbf{x})]^\top$. Then the payoff functions $(f_1, ..., f_{|\mathcal{I}|})$ are diagonally strictly convex (or concave) for every pair of feasible strategies $\hat{\mathbf{x}}$ and $\bar{\mathbf{x}}$, we have $(\hat{\mathbf{x}} - \bar{\mathbf{x}})^\top \nabla f(\bar{\mathbf{x}}) + (\bar{\mathbf{x}} - \hat{\mathbf{x}})^\top \nabla f(\hat{\mathbf{x}}) < 0$ (or > 0). With this concept, we are ready to present the conditions for ensuring the uniqueness of PNE:

Theorem 7. (Uniqueness of PNE) *Suppose an MBQ game $\mathcal{G}^{MBQ}$ satisfies conditions (i) - (iv) of Theorem 6. Then $\mathcal{G}^{MBQ}$ has a corresponding CP game $\mathcal{G}^{CP}$ and has at least one PNE. Additionally, if*

(v) $\mathcal{P}_i^{CPP}(\mathbf{x}_{-i})$ satisfies certain regularity conditions (e.g., Slater's condition), $\forall i \in \mathcal{I}$ and

(vi) the payoff functions are diagonally strictly convex,

then $\mathcal{G}^{MBQ}$ has a unique PNE.

Proof. Let us first make some modifications to the problem $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i})$ in $\mathcal{G}^{\text{CP}}$, by replacing the term X_i in its objective function with $\mathbf{x}_i\mathbf{x}_i^\top$, which essentially changes the objective function back to $\mathcal{G}^{\text{MBQ}}$'s objective function, $f_i(\mathbf{x}_i^*, \mathbf{x}_{-i}^*)$. Let us denote the modified optimization problem as $\mathcal{P}_i^{\text{CPP}'}(\mathbf{x}_{-i})$. As $X_i = \mathbf{x}_i\mathbf{x}_i^\top$ is a valid constraint for the lifted problem $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i})$ (actually constraints (34b) - (34f) are derived as a convex relaxation for $X_i = \mathbf{x}_i\mathbf{x}_i^\top$), the problem $\mathcal{P}_i^{\text{CPP}'}(\mathbf{x}_{-i})$ is still equivalent to $\mathcal{P}_i(\mathbf{x}_{-i})$. Thus, it is easy to prove that all our previous results concerning $\mathcal{G}^{\text{CP}}$ in Sections 6.1 and 6.2 are applicable to a game $\mathcal{G}^{\text{CP}'}$ with optimization problems $\mathcal{P}_i^{\text{CPP}'}(\mathbf{x}_{-i}), \forall i \in \mathcal{I}$.

Suppose towards contradiction that distinct strategies, $\hat{\mathbf{x}}$ and $\bar{\mathbf{x}}$, are both pure-strategy Nash equilibria of $\mathcal{G}^{\text{MBQ}}$. Then $(\hat{\mathbf{x}}_i, \hat{\mathbf{x}}_i\hat{\mathbf{x}}_i^\top)_{i\in\mathcal{I}}$ and $(\bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i\bar{\mathbf{x}}_i^\top)_{i\in\mathcal{I}}$ are pure-strategy Nash equilibria of $\mathcal{G}^{\text{CP}'}$. Now we want to prove that it is impossible for $\mathcal{G}^{\text{CP}'}$ to have two distinct pure-strategy Nash equilibria. To prove this, we could extend the PNE uniqueness result for n-person concave games (Rosen, 1965) to $\mathcal{G}^{\text{CP}'}$. However, the proof in Rosen (1965) only considers inequality constraints in the feasible region, while we have either linear or conic constraints in $\mathcal{P}_i^{\text{CPP}'}(\mathbf{x}_{-i})$, which requires some modifications to the proof. Therefore, we provide a proof on the uniqueness of PNE for $\mathcal{G}^{\text{CP}'}$ below:

Because $\mathcal{P}_i^{\text{CPP}'}(\mathbf{x}_{-i})$ satisfies some regularity conditions, both $(\hat{\mathbf{x}}_i, \hat{\mathbf{x}}_i\hat{\mathbf{x}}_i^\top)_{i\in\mathcal{I}}$ and $(\bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i\bar{\mathbf{x}}_i^\top)_{i\in\mathcal{I}}$ should satisfy KKT conditions. To simplify the notation, we denote $\bar{X} = \bar{\mathbf{x}}_i\bar{\mathbf{x}}_i$ and $\hat{X} = \hat{\mathbf{x}}_i\hat{\mathbf{x}}_i^\top$. Let $\mathcal{L}_i$ denote the set of constraints in (34b) - (34e), $[h_{il}(\mathbf{x}_i, X_i) = 0, l \in \mathcal{L}_i]$ the individual constraints in (34b) - (34e), $[b_{il}, l \in \mathcal{L}_i]$ the constant terms in those constraints, and $[\lambda_{il}, l \in \mathcal{L}_i]$ their corresponding dual variables. We also use the existing notation Ω_i for the dual variable of the constraint (34f). Then there exist dual variables $[\bar{\lambda}_{il}, l \in \mathcal{L}_i]$, $[\hat{\lambda}_{il}, l \in \mathcal{L}_i]$, $\bar{\Omega}_i$, and $\hat{\Omega}_i$ that satisfy the following constraints in KKT conditions:

$$\nabla_i f_i(\bar{\mathbf{x}}) + \sum_{l\in\mathcal{L}_i} \bar{\lambda}_{il} \nabla_{\mathbf{x}_i} h_{il}(\bar{\mathbf{x}}_i, \bar{X}_i) = 0 \tag{36a}$$

$$\sum_{l\in\mathcal{L}_i} \bar{\lambda}_{il} \nabla_{X_i} h_{il}(\bar{\mathbf{x}}_i, \bar{X}_i) - \bar{\Omega}_i = 0 \tag{36b}$$

$$\text{Tr}(\bar{\Omega}_i \bar{X}_i) = 0 \tag{36c}$$

$$\nabla_i f_i(\hat{\mathbf{x}}) + \sum_{l\in\mathcal{L}_i} \hat{\lambda}_{il} \nabla_{\mathbf{x}_i} h_{il}(\hat{\mathbf{x}}_i, \hat{X}_i) = 0 \tag{36d}$$

$$\sum_{l\in\mathcal{L}_i}\hat{\lambda}_{il}\nabla_{X_i}h_{il}(\hat{\mathbf{x}}_i,\hat{X}_i)-\hat{\Omega}_i=0 \tag{36e}$$

$$\mathrm{Tr}(\hat{\Omega}_i\hat{X}_i)=0. \tag{36f}$$

We multiply both sides of (36a) with $(\hat{\mathbf{x}}_i-\bar{\mathbf{x}}_i)^\top$ and multiplying both sides of (36d) with $(\bar{\mathbf{x}}_i-\hat{\mathbf{x}}_i)^\top$, and multiply both sides of (36b) with $(\hat{X}_i-\bar{X}_i)^\top$ and multiplying both sides of (36e) with $(\bar{X}_i-\hat{X}_i)^\top$ and take the traces, and then sum them up over all $i\in\mathcal{I}$ to get:

$$\begin{aligned}0=&(\hat{\mathbf{x}}-\bar{\mathbf{x}})^\top\nabla f(\bar{\mathbf{x}})+(\bar{\mathbf{x}}-\hat{\mathbf{x}})^\top\nabla f(\hat{\mathbf{x}})\\&+\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\bar{\lambda}_{il}(\hat{\mathbf{x}}_i-\bar{\mathbf{x}}_i)^\top\nabla_{\mathbf{x}_i}h_{il}(\bar{\mathbf{x}}_i,\bar{X}_i)+\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\hat{\lambda}_{il}(\bar{\mathbf{x}}_i-\hat{\mathbf{x}}_i)^\top\nabla_{\mathbf{x}_i}h_{il}(\hat{\mathbf{x}}_i,\hat{X}_i)\\&+\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\bar{\lambda}_{il}\mathrm{Tr}\Big((\hat{X}_i-\bar{X}_i)^\top\nabla_{X_i}h_{il}(\bar{\mathbf{x}}_i,\bar{X}_i)\Big)+\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\hat{\lambda}_{il}\mathrm{Tr}\Big((\bar{X}_i-\hat{X}_i)^\top\nabla_{X_i}h_{il}(\hat{\mathbf{x}}_i,\hat{X}_i)\Big)\\&-\mathrm{Tr}(\bar{\Omega}_i\hat{X}_i)-\mathrm{Tr}(\hat{\Omega}_i\bar{X}_i).\end{aligned}$$

Note that in the above we have used complementary slackness constraints (36c) and (36f) to eliminate $\mathrm{Tr}(\bar{\Omega}_i\bar{X}_i)$ and $\mathrm{Tr}(\hat{\Omega}_i\hat{X}_i)$ terms.

Because the payoff functions are diagonally strictly convex, we have $(\hat{\mathbf{x}}-\bar{\mathbf{x}})^\top\nabla f(\bar{\mathbf{x}})+(\bar{\mathbf{x}}-\hat{\mathbf{x}})^\top\nabla f(\hat{\mathbf{x}})<0$. Thus,

$$\begin{aligned}&\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\bar{\lambda}_{il}(\hat{\mathbf{x}}_i-\bar{\mathbf{x}}_i)^\top\nabla_{\mathbf{x}_i}h_{il}(\bar{\mathbf{x}}_i,\bar{X}_i)+\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\hat{\lambda}_{il}(\bar{\mathbf{x}}_i-\hat{\mathbf{x}}_i)^\top\nabla_{\mathbf{x}_i}h_{il}(\hat{\mathbf{x}}_i,\hat{X}_i)\\&+\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\bar{\lambda}_{il}\mathrm{Tr}\Big((\hat{X}_i-\bar{X}_i)^\top\nabla_{X_i}h_{il}(\bar{\mathbf{x}}_i,\bar{X}_i)\Big)+\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\hat{\lambda}_{il}\mathrm{Tr}\Big((\bar{X}_i-\hat{X}_i)^\top\nabla_{X_i}h_{il}(\hat{\mathbf{x}}_i,\hat{X}_i)\Big)\\&-\mathrm{Tr}(\bar{\Omega}_i\hat{X}_i)-\mathrm{Tr}(\hat{\Omega}_i\bar{X}_i)>0.\end{aligned} \tag{37}$$

Note that $[h_{il}(\mathbf{x}_i,X_i),l\in\mathcal{L}_i]$ are all linear functions, which means $\nabla_{\mathbf{x}_i}h_{il}(\mathbf{x}_i,X_i)$ and $\nabla_{X_i}h_{il}(\mathbf{x}_i,X_i)$ are constant terms that do not contain $\mathbf{x}_i$ or X_i. In other words,

$$\begin{aligned}&\bar{\mathbf{x}}_i^\top\nabla_{\mathbf{x}_i}h_{il}(\bar{\mathbf{x}}_i,\bar{X}_i)+\mathrm{Tr}\Big(\bar{X}_i^\top\nabla_{X_i}h_{il}(\bar{\mathbf{x}},\bar{X}_i)\Big)+b_{il}=0\\&\hat{\mathbf{x}}_i^\top\nabla_{\mathbf{x}_i}h_{il}(\bar{\mathbf{x}}_i,\bar{X}_i)+\mathrm{Tr}\Big(\hat{X}_i^\top\nabla_{X_i}h_{il}(\bar{\mathbf{x}},\bar{X}_i)\Big)+b_{il}=0,\end{aligned}$$

and the same are true when we replace $\nabla_{\mathbf{x}_i}h_{il}(\bar{\mathbf{x}}_i,\bar{X}_i)$ and $\nabla_{X_i}h_{il}(\bar{\mathbf{x}},\bar{X}_i)$ with $\nabla_{\mathbf{x}_i}h_{il}(\hat{\mathbf{x}}_i,\hat{X}_i)$ and $\nabla_{X_i}h_{il}(\hat{\mathbf{x}},\hat{X}_i)$ above. Therefore, we can rewrite (37) as:

$$\begin{aligned}&\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\bar{\lambda}_{il}(-b_{il}+b_{il})+\sum_{i\in\mathcal{I},l\in\mathcal{L}_i}\hat{\lambda}_{il}(-b_{il}+b_{il})-\mathrm{Tr}(\bar{\Omega}_i\hat{X}_i)-\mathrm{Tr}(\hat{\Omega}_i\bar{X}_i)>0\\&\Rightarrow\mathrm{Tr}(\bar{\Omega}_i\hat{X}_i)+\mathrm{Tr}(\hat{\Omega}_i\bar{X}_i)<0.\end{aligned}$$

However, as $\bar{X}_i, \hat{X}_i \in \mathcal{C}^*_{n_i}$ and $\bar{\Omega}_i, \hat{\Omega}_i \in \mathcal{C}_{n_i}$, we have $\mathrm{Tr}(\hat{\Omega}_i \bar{X}_i) \geq 0$ and $\mathrm{Tr}(\bar{\Omega}_i \hat{X}_i) \geq 0$, which is a contradiction. Therefore, $\mathcal{G}^{\mathrm{CP}'}$ must have a unique PNE, which means $\mathcal{G}^{\mathrm{MBQ}}$ also has a unique PNE. □

6.3 Computing the NE

The NE of $\mathcal{G}^{\mathrm{CP}}$ is given by KKT conditions when certain constraint qualification, such as Slater's condition, is met. We write the KKT conditions for $\mathcal{G}^{\mathrm{CP}}$ explicitly. For all $i \in \mathcal{I}$, we have:

$$(34b) - (34f) \tag{38a}$$

$$2Q^{(i)\top}_{-i,i}\mathbf{x}_{-i} + 2\mathbf{c}_i - \sum_{j=1}^{m} \gamma_{ij}\mathbf{a}^{(i)}_j - \sum_{k\in\mathcal{B}} \delta_{ik}\mathbf{e}^k - \boldsymbol{\xi}_i = 0 \tag{38b}$$

$$Q^{(i)}_{ii} - \sum_{j=1}^{m} \beta_{ij}\mathbf{a}^{(i)}_j\mathbf{a}^{(i)\top}_j + \sum_{k\in\mathcal{B}} \delta_{ik}\mathbf{e}^k\mathbf{e}^{k\top} + \boldsymbol{\alpha}^{(i)}\boldsymbol{\xi}^\top_i - \Omega_i = 0 \tag{38c}$$

$$\Omega_i \in \mathcal{C} \tag{38d}$$

$$\mathrm{Tr}(\Omega_i X_i) = 0 \tag{38e}$$

where (38a) are constraints of $\mathcal{P}^{\mathrm{CPP}}_i(\mathbf{x}_{-i})$. $\boldsymbol{\gamma}_i$, $\boldsymbol{\beta}_i$, $\boldsymbol{\delta}_i$, $\boldsymbol{\xi}_i$, and Ω_i are respectively dual variables for (34b) - (34f). Let $\mathcal{L}(\mathbf{x}_i, X_i, \mathbf{x}_{-i})$ be the Lagrangian relaxation of $\mathcal{P}^{\mathrm{CPP}}_i(\mathbf{x}_{-i})$, i.e.,

$$\begin{aligned}\mathcal{L}(\mathbf{x}_i, X_i, \mathbf{x}_{-i}) =& F_i(\mathbf{x}_i, X_i, \mathbf{x}_{-i}) + \sum_{i=1}^{m}\Big(\gamma_{ij}(b^{(i)}_j - \mathbf{a}^{j(i)\top}\mathbf{x}_i) + \beta_{ij}(b^{(i)2}_j - \mathbf{a}^{j(i)\top}X_i\mathbf{a}^{j(i)})\Big)\\ &+ \sum_{k\in\mathcal{B}} \delta_{ik}\Big(X_{i,kk} - x_{ik}\Big) + \xi^\top_i(X_i\boldsymbol{\alpha}^{(i)} - \mathbf{x}_i) + \mathrm{Tr}(\Omega_i X_i),\end{aligned}$$

then $\partial\mathcal{L}(\mathbf{x}_i, X_i, \mathbf{x}_{-i})/\partial\mathbf{x}_i = 0$ and $\partial\mathcal{L}(\mathbf{x}_i, X_i, \mathbf{x}_{-i})/\partial X_i = 0$ respectively correspond to constraints (38b) and (38c). Constraint (38d) is necessary because of constraint (34f), i.e.$X_i \in \mathcal{C}^*$. Finally, constraint (38e) enforces complementary slackness. Note that the KKT conditions above are derived assuming $\mathcal{P}^{\mathrm{CPP}}_i(\mathbf{x}_{-i})$ is a minimization problem. If $\mathcal{P}^{\mathrm{CPP}}_i(\mathbf{x}_{-i})$ is a maximization problem, then the sign before Ω_i in (38c) is plus instead of minus.

Because of the existence condition for PNE in Theorem 6, we can add the following extra constraints to the KKT condition:

$$x_{ik} \in \{0, 1\}, \forall k \in \mathcal{B}_i, i \in \mathcal{I} \tag{39}$$

to make sure that the optimal solution is a PNE for $\mathcal{G}^{\mathrm{MBQ}}$ (assuming $Q^{(i)}_{ii} \succeq 0, \forall i \in \mathcal{I}$ is satisfied).

6.3.1 Special Case for KKT I: Dimension of $\mathcal{X}_i$ No More Than 4

The KKT conditions (38) contain conic constraints $X_i \in \mathcal{C}^*$ and $\Omega_i \in \mathcal{C}$. When the dimensions of X_i and Ω_i are no more than 4, we have $\mathcal{C}^* = \mathcal{S}^+ \cap \mathcal{N}$ and $\mathcal{C} = \mathcal{S}^+ + \mathcal{N}$, where $\mathcal{S}^+$ is the cone for PSD matrices and $\mathcal{N}$ is the cone of entrywise nonnegative matrices (Dür, 2010). Therefore, constraints $X_i \in \mathcal{C}^*$ and $\Omega_i \in \mathcal{C}$ can be equivalently replaced by the following constraints when $\dim X_i \leq 4$:

$$X_i \in \mathcal{S}^+ \tag{40a}$$

$$X_i \geq 0 \tag{40b}$$

$$\Omega_i - N_i \in \mathcal{S}^+ \tag{40c}$$

$$N_i \geq 0 \tag{40d}$$

Then the KKT conditions become a bilinear program over SDP conic constraints.

6.3.2 Special Case for KKT II: All Variables of $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})$ are Binary

Another special case for simplifying the KKT conditions is when $|B_i| = l$ in $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})$ (32), $\forall i \in \mathcal{I}$, i.e. all variables in $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})$ are binary. Then the KKT conditions (38) could be replaced with a MIP problem over copositive cone constraints as follows: for the primal problem constraints, we keep constraints (34b), and add constraints (39). Considering the fact that constraints (34c) - (34f) are actually a relaxation of the constraint $X_i = \mathbf{x}_i\mathbf{x}_i^\top$, or $X_{i,j_1,j_2} = x_{ij_1}x_{i,j_2}$, and all variables in $\mathcal{P}_i^{\text{MBQP}}(\mathbf{x}_{-i})$ are binary, we can replace (34c) - (34f) with the following mixed-integer constraints ($\forall i \in \mathcal{I}$):

$$X_{i,jj} = x_{ij} \qquad \forall j = 1, ..., l \tag{41a}$$

$$X_{i,j_1j_2} \leq x_{ij_1} \qquad \forall j_1 \neq j_2; j_1, j_2 = 1, ...l \tag{41b}$$

$$X_{i,j_1j_2} \leq x_{ij_2} \qquad \forall j_1 \neq j_2; j_1, j_2 = 1, ...l \tag{41c}$$

$$X_{i,j_1j_2} \geq x_{ij_1} + x_{ij_2} - 1 \quad \forall j_1 \neq j_2; j_1, j_2 = 1, ...l \tag{41d}$$

$$X_i \in \mathcal{S}_+^l \tag{41e}$$

On the dual side, we keep all constraints (38b) - (38d). For the bilinear complementary slackness constraint (38e), we linearize it by defining a new matrix $Z_i \in \mathcal{R}^{l\times l}$, and let $Z_{i,j_1j_2} = \Omega_{i,j_1j_2}X_{i,j_1j_2}$, then this equation can be linearized with big-M constraints. More specifically, constraint (38e) can be replaced with:

$$\text{Tr}(Z) = 0 \tag{42a}$$

$$-MX_{i,j_1j_2} \leq Z_{i,j_1j_2} \leq MX_{i,j_1j_2} \qquad \forall j_1, j_2 = 1, ..., l \tag{42b}$$

$$\Omega_{i,j_1j_2} - M(1 - X_{i,j_1j_2}) \leq Z_{i,j_1j_2} \leq \Omega_{i,j_1j_2} + M(1 - X_{i,j_1j_2}) \ \forall j_1, j_2 = 1, ..., l \tag{42c}$$

where M is a very large number. Here we use the fact that elements in X_i are binary, due to constraints (41).

Now we are ready to put together all the necessary constraints to replace KKT conditions (38), $\forall i \in \mathcal{I}$:

$$\mathbf{a}^{j(i)\top}\mathbf{x}_i = b_j^{(i)} \qquad \forall j = 1, ..., m_i \tag{43a}$$

$$(38b), (38c), (39), (41), (42) \tag{43b}$$

$$\Omega_i \in \mathcal{C} \tag{43c}$$

which is a MIP over copositive conic constraints, and can be solved exactly with the cutting plane algorithm in Section 4.2 with a MIP solver. If a solution for this system of constraints exists, then it is a PNE for the discrete game $\mathcal{G}^{\text{MBQ}}$.

As a final note to this section, we would like to mention that with some efforts the results for MBQ games in this section can be extended to mixed-*integer* quadratic games that contain general integer decisions, as any bounded general integer variable can be replaced by a set of binary variables.

7 Numerical Illustrations

In this section we present the results of numerical experiments for COP algorithms, pricing in nonconvex markets, and bimatrix games. In Section 7.1 we compare solution methods of COP for a maximum clique problem example. Section 7.2 presents a comparison of different pricing schemes for a classical nonconvex market example called Scarf's example. Section 7.3 includes the results of different pricing schemes for a nonconvex electricity market example. In Section 7.4 we compute the PNE of bimatrix games with the reformulated KKT conditions (43).

Unless otherwise specified, all experiments in this section are implemented in Julia v1.0.5 using the optimization package JuMP.jl v0.20.1. All experiments that solve COPs with our cutting plane algorithm uses CPLEX 12.8. For solving SDP problems in Sections 7.1 - 7.3, we conduct preliminary tests with both Mosek 9.1 and SeDuMi 1.3, then pick the solver which is faster or produces more consistent results. The computer we use for those experiments runs MacOS and has a 2.3 GHz Intel Core i5 processor and a 16GB RAM.

7.1 COP Algorithms Comparison for Max Clique Problem

To compare our COP cutting plane algorithm with the SDP approximation algorithm for COP in literature, we present the results from solving the max clique problem with those algorithms. The max clique problem tries to find the max clique number on a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$, which is equivalent to finding the stability number of $\mathcal{G}$'s complementary graph $\bar{\mathcal{G}} = (\mathcal{N}, \bar{\mathcal{E}})$. Let ω be the max clique number of graph $\mathcal{G}$, then we can formulate the max clique problem as the following MIP, which finds the stability number on graph $\bar{\mathcal{G}}$:

$$\omega = \max \sum_{i=1}^{n} x_i \tag{44a}$$

$$\text{s.t. } x_i + x_j \leq 1 \quad \forall (i,j) \in \bar{\mathcal{E}} \tag{44b}$$

$$x_i \in \{0,1\} \quad \forall i = 1, ..., n \tag{44c}$$

Let A be the adjacency matrix of $\mathcal{G}$, then we have $A = Q - \bar{A}$, where $Q = ee^\top - I$, $\bar{A}$ is the adjacency matrix of $\bar{\mathcal{G}}$. Apply this relationship to the COP problem in Corollary 2.4 of De Klerk and Pasechnik (2002), we obtain the following COP problem to calculate the max clique number of $\mathcal{G}$:

$$\omega = \min \ \lambda \tag{45a}$$

$$\text{s.t. } \lambda(ee^\top - A) - ee^\top = Y \tag{45b}$$

$$Y \in \mathcal{C} \tag{45c}$$

In our experiment we use 10 max-clique problem instances from the second DIMACS challenge. In terms of algorithm compare the following way of solving the COP (45):

(1) Approximately solve the COP with SDP, as shown in Section 4.1. This is the method suggested by De Klerk and Pasechnik (2002). We use Mosek 9.1.2 to solve those SDP approximations.

(2) Exactly solve the COP with the cutting plane algorithm of Section 4.2.

Notice that we can strengthen the master problem in our cutting plane algorithm by providing bounds for Y in the initialization stage. Since the max clique number λ cannot exceed the number of total nodes $|\mathcal{N}|$, and elements of $ee^\top$ are all 1's while elements of A are either 0 or 1, from constraint (45b) we have that the elements of Y should be in the range of $[-1, |\mathcal{N}|-1]$.

We present the results of the algorithmic comparison in Table 2, where we list the number of nodes $|\mathcal{N}|$, number of edges $|\mathcal{E}|$, and the max clique number of the graph ω for each instance. For the computational performance of solving the SDP approximation via Mosek, we list the objective ("Obj") , optimality gap ("Gap", compared with the true ω), and the computational time ("Time"). For the performance of our cutting plane algorithm we list the computational time and number of iterations needed for convergence. There is no need to list the objectives because the cutting plane method always converges to the exact solution.

When solving instances "hamming6-2", "hamming6-4" and "johnson16-2-4" with cutting planes, we encounter some very hard separation problems that takes very long time to solve. To speedup the process, we use instead a strengthened version of the separation problem with $q = \bar{\omega}$ in constraint (12c) (Anstreicher, 2020), where $\bar{\omega}$ is the current master problem solution for ω. In addition, even after our enhancement of $q = \bar{\omega}$, the instance "johnson16-2-4" still has a hard separation problem which achieves a nonzero lower bound early on (thus proves that the matrix is not copositive), but cannot converge after an extended period of time. In this case we set a time limit of 1 minute to help the separation problem stop early.

Table 2: Algorithm Comparison for Max Clique COP Model

				Mosek			Cutting plane	
Instance	$\lvert\mathcal{N}\rvert$	$\lvert\mathcal{E}\rvert$	ω	Obj	Gap(%)	Time(sec)	Time(sec)	#Iter
c-fat200-1	200	1534	12	12	0	566.81	13.87	2
c-fat200-2	200	3235	24	24	0	638.72	18.90	2
c-fat200-5	200	8473	58	60.35	3.89	606.33	12.19	2
hamming6-2[a]	64	1824	32	32	0	1.51	6.05	2
hamming6-4[a]	64	704	4	4	0	1.59	1.55	4
johnson8-2-4	28	210	4	4	0	0.20	9.53	2
johnson8-4-4	70	1855	14	14	0	2.47	11.82	2
johnson16-2-4[a,b]	120	5460	8	8	0	31.88	62.75	2
keller4	171	9435	11	13.47	18.34	426.16	-	-
MANN_a9	45	918	16	17.48	8.47	0.45	547.62	2

[a] Obtained by setting $q = \bar{\omega}$ in separation problem, see text for explanation.

[b] Obtained by early termination of separation problem.

From the results we can observe that for some instances, the SDP approximation fails to provide the correct max clique number. Also, in certain instances such as "c-fat200-1", "c-fat200-2" and "c-fat200-3", the cutting plane is faster than the SDP approximation.

It is also interesting to compare our algorithm with the simplicial partition method of Bundfuss and Dür (2009), which we believe is the only exact algorithm for general COPs in literature. Bundfuss and Dür (2009) also solve the max clique instances from the second DIMICS challenge. They report that their computation time for "johnson8-2-4" and "hamming6-4" are respectively 1 minutes 33 seconds and 57 minutes 52 seconds. For all other instances, their algorithm produces only poor bounds within two hours. Therefore, the performance of our algorithm is better than theirs in all test instances.

Notice that the cutting plane algorithm terminates in very few iterations for almost all test instances. It is not generally the case with the cutting plane algorithm when solving other COP problems. One reason for the small number of iterations could be the use of a strong formulation for the max clique problem. For example, if we use the weaker COP formulation (46) below, then the cutting plane algorithm takes longer to terminate: the simplest instance (in terms of number of nodes and edges) "johnson8-2-4" now costs 200.64 seconds and 690 iterations:

$$\omega = \min \ \lambda \tag{46a}$$

$$\text{s.t.} \ \lambda I + \sum_{(i,j)\in\bar{\mathcal{E}}} x_{ij}E_{ij} - ee^\top = Y \tag{46b}$$

$$Y \in \mathcal{C} \tag{46c}$$

where $E_{ij} \in \mathbb{R}^{n\times n}$ is a matrix with ones at ith row and jth column and jth row and ith column, and with zeros for all other positions.

7.2 Pricing in Scarf's Example

The Scarf's example is a classical nonconvex market example used in the literature for comparing different pricing schemes. We use the modified Scarf's example from Hogan and Ring (2003) to compare our COP pricing schemes with RP and CHP, which are the two pricing schemes used by utilities in the US. In the modified Scarf's example, there are three types of generators: smokestack, high technology and medium technology. Let $\mathcal{G}_i$ be the set of generators of type $i = 1, 2, 3$, we have $|\mathcal{G}_1|= 6, |\mathcal{G}_2|= 5, |\mathcal{G}_3|= 5$. Binary variables $u_{g_i}, g_i \in \mathcal{G}_i, i = 1, 2, 3$ represent startup decisions, continuous variables $p_{g_i}, g_i \in \mathcal{G}_i, , i = 1, 2, 3$ represent production decisions. Scarf's example solves the following cost minimization problem:

$$\min \sum_{g_1\in\mathcal{G}_1}(53u_{g_1} + 3p_{g_1}) + \sum_{g_2\in\mathcal{G}_2}(30u_{g_2} + 2p_{g_2}) + \sum_{g_3\in\mathcal{G}_3} 7p_{g_3} \tag{47a}$$

$$\text{s.t.} \sum_{g_1\in\mathcal{G}_1} p_{g_1} + \sum_{g_2\in\mathcal{G}_2} p_{g_2} + \sum_{g_3\in\mathcal{G}_3} p_{g_3} = d \tag{47b}$$

$$p_{g_3} \geq 2u_{g_3} \qquad \forall g_3 \in \mathcal{G}_3 \tag{47c}$$

$$p_{g_1} \leq 16u_{g_1} \qquad \forall g_1 \in \mathcal{G}_1 \tag{47d}$$

$$p_{g_2} \leq 7u_{g_2} \qquad \forall g_2 \in \mathcal{G}_2 \tag{47e}$$

$$p_{g_3} \leq 6u_{g_3} \qquad \forall g_3 \in \mathcal{G}_3 \tag{47f}$$

$$p_g \geq 0, u_g \in \{0, 1\} \qquad \forall g \in \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \tag{47g}$$

where the objective (47a) minimizes the total cost. Constraint (47b) ensure the total production equals the demand. Constraints (47c) sets the lower bound for the production of medium technology generators when they are on. Constraints (47d)-(47f) set the capacity of each generator.

We experiment with various demand levels from 5 to 160, with a step length of 5. In Figure 2 we compare RP, CHP, COP pricing and modified COP (MCOP) pricing schemes under different demand levels in the following aspects:

(a) The uniform prices. Notice that for COP and MCOP, both λ_t and Λ_t are uniform prices. However, in all our experiments the optimal Λ_t equals zero. Therefor, we report only λ_t for those schemes.

(b) The profits. This profit is calculated by deducting the costs from the total revenue, where the total revenue includes the revenue from uniform and generator-dependent prices, as well as the *make-whole* uplift payments (see (d) for details).

(c) The uplift payments from generator-dependent prices.

(d) The make-whole uplift payments. This payment is made when the revenue from electricity prices is not enough to cover the costs. Its value equals exactly the difference between the revenue of costs (“make whole”), so the generators do not operate under a loss.

The results for COP and MCOP are solved approximately with Mosek. For experiments in this section, Mosek solves all instances very fast (each of them under 10 seconds), while the cutting plane algorithm is much slower compared with Mosek.

Figure 2a shows that a small change in demand level can result in much volatility in RP prices. This observation is consistent with results in literature. Interestingly, CHP prices and COP prices are equivalent for all demand levels. MCOP prices ars higher than COP prices for lower demand, and equals COP prices when the demand is high.

For the comparison of profits in Figure 2b, we find that RP and COP have zero profit for all instances. CHP generates near-zero profits in lower demand levels, and higher profits for instances with higher demand levels. MCOP generates the highest profits among all pricing schemes, and its profits equals COP profits in higher demand levels.

In Figure 2c, both CHP and MCOP have zero generator-dependent prices, because those two pricing schemes only include uniform prices corresponding to the demand constraints. COP produces near-zero negative generator-dependent prices at low demand levels, and more negative prices at higher demand levels. RP produces volatile and large generator-dependent prices in many instances. As explained by O’Neill et al. (2005), the negative generator-dependent prices are used to discourage the entry of marginal plants when it is uneconomic to do so. In practice, utilities usually disregard such negative prices.

Figure 2d shows that RP requires zero make-whole payment, which is also consistent with the

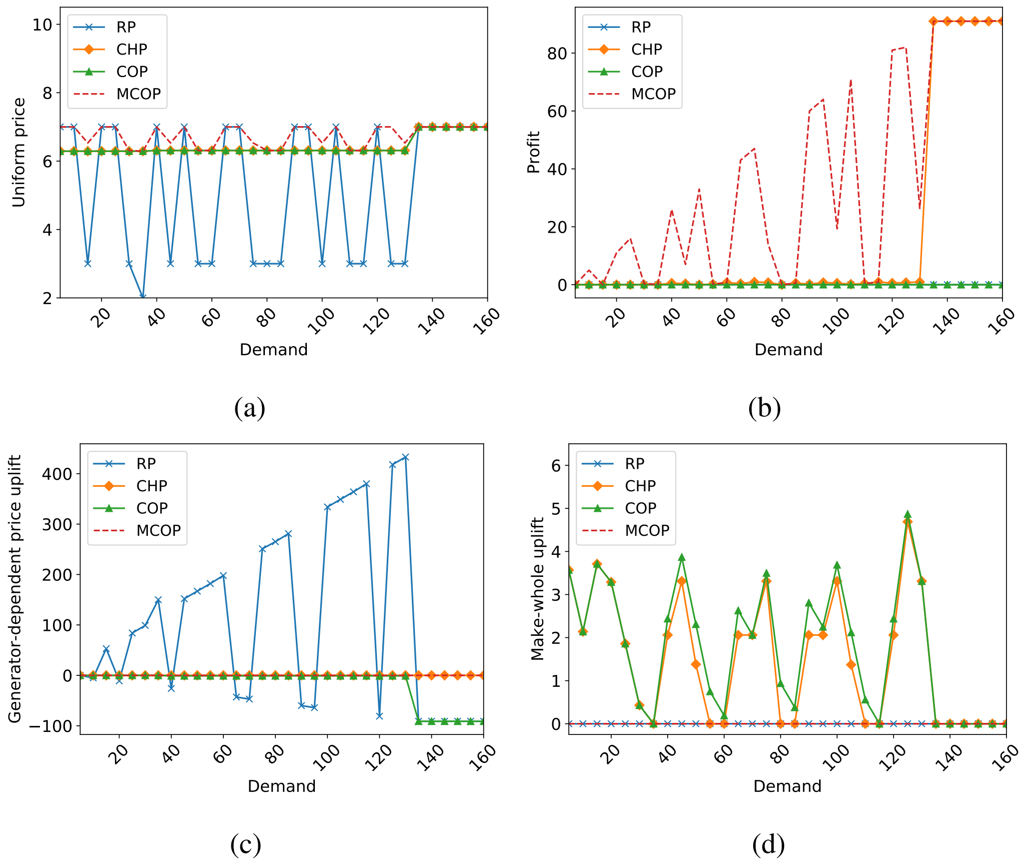

Figure 2: Comparison of different pricing schemes for (a) uniform prices, (b) profits, (c) uplift payments from generator-dependent prices, (d) make-whole uplift payments.

results in Azizan et al. (2020). MCOP ensures revenue adequacy thus also needs no make-whole payment. CHP requires make-whole payments, as expected, because Lagrangian dual problems of MIPs do not ensure strong duality. Interestingly, COP also requires make-whole payments in many instances, as in those instances strong duality does not hold.

Overall, we believe MCOP is a better choice compared with other three pricing schemes, as it only relies on uniform prices to ensure profitability. Plus, the change of MCOP uniform prices with demand levels is not too volatile.

7.3 Pricing in Electricity Markets

In this section we compare different pricing schemes for the energy market example in Section 5. We use the data from the adapted California ISO dataset of Guo et al. (2020). In our experiments we simplify the dataset so that the cutting plane algorithm can solve the problem to optimality or at least to a reasonable gap. The parameters for the generators we use in our experiments are listed in Table 3.

Table 3: Parameters for Generators

	Gen_1	Gen_2	Gen_3	Gen_4
c_g^o	25.0	25.5	44.68	44.68
c_g^s	140.94	140.94	86.31	86.31
$\underline{P}_g$	297	238	198	198
$\overline{P}_g$	620	496	620	620

We start with an instance with 2 generators and 4 hours in the planning horizon (we call it "the first UC instance"). We assume $d_t = [508, 644, 742, 776]$ and use the generators Gen_1 and Gen_2 in Table 3. We compare the profit and uplift payments of RP, CHP, COP and MCOP (all pricing schemes are solved to optimality). For COP and MCOP, we solve them with the cutting plane method, as it is more accurate. Also, Mosek fails when solves for MCOP (returns "UNKNOWN_RESULT_STATUS"), while cutting planes solves MCOP optimally within 282.85 seconds. When using the cutting plane method, we set the bounds of the absolute value for elements in copositive matrices as 1000, and add complementary slackness constraint when solving for COP (see Section 4.2). We present the results in Table 4, where we compare the profit for generators 1 and 2 (Profit1 and Profit 2, respectively), the total profit (ProfitTotal), the generator-dependent price uplift (GenDepUplift), and the make-whole uplift (MakewholeUplift). In the result, RP gives

Table 4: Comparison of Pricing Schemes for the First UC Instance

	RP	CHP	COP	MCOP
Profit1	0	839.45	292.02	1393.55
Profit2	0	0	0	72.13
ProfitTotal	0	839.45	292.02	1465.68
GenDepUplift	497.94	0	-201.45	0
MakewholeUplift	0	95.25	292.02	0

zero profit for both generators, but relies on the revenue from generator-dependent prices (for generator 2). CHP needs make-whole uplift payment (for generator 2) to fully recover the costs for all generators. Interestingly, COP generates a positive profit for generator 1, and needs a make-whole payment that equals generator 1's profit to cover the costs of generator 2. This demonstrates that COP is revenue neutral in the whole system, but is not revenue neutral/adequate for individual generators. MCOP is the only pricing scheme that is free from any uplift costs. It also makes sure the generators with a lower cost (generator 1) receives a higher profit, which is desirable and is expected from a market with only uniform prices.

Next, we scale up the instance to include all 4 generators in Table 3, and 4 hours in the planning horizon (we call it "the second UC instance"). Now assume $d_t = [1469, 1862, 2144, 2242]$. In Table 5 we again compare the four pricing schemes in ProfitTotal, GenDepUplift, and MakewholeUplift. For this instance, Mosek is unable to solve either COP nor MCOP, while cutting plane takes a long time to terminate so we terminate it after 3,000 iterations and report the results upon termination. Although the results for COP and MCOP are not optimal, we observe that the total revenue from uniform and availability prices, as well as the profit, are monotonically decreasing when more cuts are added. This indicates that by adding the cutting planes, the prices are getting closer to more reasonable levels.

When using the cutting plane method, we set the bounds of the absolute value for elements in copositive matrices as 5000. Since we do not solve the problems to optimality, we do not add complementary slackness constraints, as those constraints interfere with the monotonicity of revenue and profit trends.

Table 5: Comparison of Pricing Schemes for the Second UC Instance

	RP	CHP	COP	MCOP
ProfitTotal	86859.52	86528.41	56560.10	129918.33
GenDepUplift	86.31	0	-89354.68	0
MakewholeUplift	0	83.44	0	0

The observation from results of the second instance is similar to the first instance: MCOP pricing is the only pricing schemes that does not generate any uplift costs. It also generates a larger profit than other pricing scheme. Although not solved to optimality, COP pricing still has a smaller profit than RP and CHP. This is to be expected as strong duality of COP guarantees zero profit. If given more time to run, COP pricing should eventually have zero profit (not counting the make-whole uplift).

7.4 KKT Conditions for Bimatrix Games

We illustrate the copositive programming formulation (43) for KKT in Section 6.3.2 with bimatrix games. A bimatrix game has two players, players 1 and 2. Each player has n_i strategies to choose from, and we use the binary vector $\mathbf{x}_i \in \{0, 1\}^{n_i}$ to denote player i's decision. The elements of $\mathbf{x}_i$ are binary since we only look for a pure strategy. Player i faces the following maximization problem:

$$
\begin{aligned}
\max \ & \mathbf{x}_1^\top R_i \mathbf{x}_2 && \text{(48a)}\\
\text{s.t.} \ & \sum_{j=1}^{n_i} x_{ij} = 1 && \text{(48b)}\\
& x_{ij} \in \{0,1\} \ \ \forall j = 1, ..., n_i && \text{(48c)}
\end{aligned}
$$

where $R_i \in \mathcal{R}^{n_1 \times n_2}$ is the payoff matrix. If player 1 plays the k_1th strategy and player 2 plays the k_2th strategy, then the payoff for player i is R_{i,k_1k_2}. Thus, (48a) maximizes the payoff of player i. Constraint (48b) ensures that player i can only take one strategy.

The problem (48) has all binary variables and only equality constraints. The upper bound of binary variables are implied by the constraint (48b), so we don't need to add additional constraints for the upper bounds in the CPP reformulation. Moreover, there is no unary quadratic term in the objective, thus $Q_{ii}^{(i)}$ in (34a) of $\mathcal{P}_i^{\text{CPP}}(\mathbf{x}_{-i})$ is a zero matrix, i.e. $Q_{ii}^{(i)} \succeq 0$. Therefore, bimatrix game satisfies the requirements of Section 6.3.2, and we can use the reformed KKT conditions (43) to solve for the PNE of its CP game counterpart, and then recover the PNE of the bimatrix game from the PNE of the CP game.

We test the KKT conditions on bimatrix games with 2 to 5 strategies. For each number of strategy we test 5 instances. Because KKT conditions need to be solved exactly, we choose to use the cutting plane algorithm to solve equations (43). To ensure the boundedness of initial master problems, we set an upper bound of 100 for the absolute values of elements in copositive matrices. Our computational results show that those bounds are big enough for obtaining copositive matrices upon termination (i.e. separation problem objectives equal 0).

We compare the obtained results from our method with the results from the lrsnash bimatrix game solver (Avis et al., 2010), and observe that for all instances our method returns the correct results. In Table 6 we summarize the performance of our method. For each number of strategies (Num_strat), we report the average solving time (Num_time) in seconds, and the average number of iterations (Mean_num_iter). All tested instances are solved in 20 seconds and within 110 iterations.

Table 6: Performance of KKT Conditions for Bimatrix Games

Num_strat	Mean_time	Mean_num_iter
2	1.09	4
3	1.37	2
4	6.03	40
5	10.59	59

8 Conclusion

Our work proposes a general framework of COP duality, which provides a notion of duality for MIPs and MBQPs, as well as a new tool for theoretically and computationally analyzing discrete problems. We specialize this framework in two applications: nonconvex electricity markets and MBQP games. More specifically, for electricity markets, we propose a new pricing scheme for electricity markets using COP duality. This pricing scheme has the potential to be generalized for other nonconvex markets. For MBQP games, we transform them to equivalent CP games, to obtain the existence theorem and KKT conditions for the PNE. We also propose a novel cutting plane algorithm for COPs, and use it to obtain numerical results for the applications.

For future works, it will be interesting to find other applications of our COP duality framework, e.g. nonconvex economic problems other than markets and games. It is also interesting to explore the economic interpretation of COP duality. For example, in discrete markets, can we project COP dual prices to the space of original MBQP problems? How can the COP dual prices be interpreted with the terminologies in the literature of indivisibility goods? On the technical side, it will be helpful to find out classes of discrete markets and games whose CPP reformulations always have strong duality, such as in the example of Sayin and Basar (2019). Finally, to improve the applicability of COP duality, it is important to improve our cutting plane algorithm, or develop more efficient algorithms to solve COPs.

References

[1] Amir Ali Ahmadi and Jeffrey Zhang. Semidefinite programming and nash equilibria in bimatrix games. *INFORMS J. Comput.*, 2020.

[2] Kurt M Anstreicher. Testing copositivity via mixed–integer linear programming. 2020.

[3] David Avis, Gabriel D Rosenberg, Rahul Savani, and Bernhard Von Stengel. Enumeration of nash equilibria for two-player games. *Economic theory*, 42(1):9–37, 2010.

[4] Navid Azizan, Yu Su, Krishnamurthy Dvijotham, and Adam Wierman. Optimal pricing in markets with nonconvex costs. *Oper. Res.*, 68(2):480–496, 2020.

[5] Elizabeth Baldwin and Paul Klemperer. Understanding preferences: demand types, and the existence of equilibrium with indivisibilities. *Econometrica*, 87(3):867–932, 2019.

[6] Carmen Bevia, Martine Quinzii, and Jose A Silva. Buying several indivisible goods. *Math. Soc. Sci.*, 37(1):1–23, 1999.

[7] Immanuel M Bomze. Copositive optimization–recent developments and applications. *European Journal of Operational Research*, 216(3):509–520, 2012.

[8] Immanuel M Bomze and Etienne De Klerk. Solving standard quadratic optimization problems via linear, semidefinite and copositive programming. *J. Global. Opt.*, 24(2):163–185, 2002.

[9] Immanuel M Bomze, Marco Locatelli, and Fabio Tardella. New and old bounds for standard

quadratic optimization: dominance, equivalence and incomparability. *Math. Program.*, 115 (1):31, 2008.

[10] Immanuel M Bomze, Florian Frommlet, and Marco Locatelli. Copositivity cuts for improving sdp bounds on the clique number. *Math. Program.*, 124(1-2):13–32, 2010.

[11] Stefan Bundfuss and Mirjam Dür. An adaptive linear approximation algorithm for copositive programs. *SIAM J. Optim.*, 20(1):30–53, 2009.

[12] Samuel Burer. On the copositive representation of binary and continuous nonconvex quadratic programs. *Math. Program.*, 120(2):479–495, 2009.

[13] Margarida Carvalho, Andrea Lodi, and Joao Pedro Pedroso. Computing nash equilibria for in-teger programming games. 2017.

[14] Vladimir Danilov, Gleb Koshevoy, and Kazuo Murota. Discrete convexity and equilibria in economies with indivisible goods and money. *Math. Soc. Sci.*, 41(3):251–273, 2001.

[15] Etienne De Klerk and Dmitrii V Pasechnik. Approximation of the stability number of a graph via copositive programming. *SIAM J. Optim.*, 12(4):875–892, 2002.

[16] Mirjam Dür. Copositive programming–a survey. In *Recent Advances in Optimization and Its Applications in Engineering*, pages 3–20. Springer, 2010.

[17] Steven A Gabriel, Sauleh Ahmad Siddiqui, Antonio J Conejo, and Carlos Ruiz. Solving discretely-constrained nash–cournot games with an application to power markets. *Netw. Spat. Econ.*, 13(3):307–326, 2013.

[18] David Gale. Equilibrium in a discrete exchange economy with money. *Int. J. Game Theory*, 13(1):61–64, 1984.

[19] Paul R Gribik, William W Hogan, and Susan L Pope. Market-clearing electricity prices and energy uplift. *Electricity Policy Group*, 2007.

[20] Patrick Groetzner and Mirjam Dür. A factorization method for completely positive matrices. *Optimization Online*, 2018.

[21] Cheng Guo, Merve Bodura, and Dimitri J Papageorgioub. Generation expansion planning with revenue adequacy constraints. 2020.

[22] Grani A Hanasusanto and Daniel Kuhn. Conic programming reformulations of two-stage distributionally robust linear programs over wasserstein balls. *Oper. Res.*, 66(3):849–869, 2018.

[23] William W Hogan and Brendan J Ring. On minimum-uplift pricing for electricity markets. *Electr. Policy Gr.*, pages 1–30, 2003.

[24] Matthias Köppe, Christopher Thomas Ryan, and Maurice Queyranne. Rational generating functions and integer programming games. *Oper. Res.*, 59(6):1445–1460, 2011.

[25] Michael M Kostreva. Combinatorial optimization in nash games. *Comput. Math. Appl.*, 25 (10-11):27–34, 1993.

[26] George Liberopoulos and Panagiotis Andrianesis. Critical review of pricing schemes in markets with non-convex costs. *Oper. Res.*, 64(1):17–31, 2016.

[27] David G Luenberger and Yinyu Ye. Linear and nonlinear programming. 2015.

[28] Indrajit Mallick. On the existence of pure strategy nash equilibria in two person discrete games. *Econ. Lett.*, 111(2):144–146, 2011.

[29] Richard P O'Neill, Paul M Sotkiewicz, Benjamin F Hobbs, Michael H Rothkopf, and William R Stewart Jr. Efficient market-clearing prices in markets with nonconvexities. *Euro-*

pean J. Oper. Res., 164(1):269–285, 2005.
[30] Asuman E Ozdaglar. Strategic form games and nash equilibrium., 2015.
[31] Pablo A Parrilo. *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization*. PhD thesis, California Institute of Technology, 2000.
[32] Pablo A Parrilo. Polynomial games and sum of squares optimization. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 2855–2860. IEEE, 2006.
[33] J Ben Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, pages 520–534, 1965.
[34] Carlos Ruiz, Antonio J Conejo, and Steven A Gabriel. Pricing non-convexities in an electricity pool. *IEEE Trans. Power Syst.*, 27(3):1334–1342, 2012.
[35] Simone Sagratella. Computing all solutions of nash equilibrium problems with discrete strategy sets. *SIAM J. Optim.*, 26(4):2190–2218, 2016.
[36] Muhammed O Sayin and Tamer Basar. Optimal hierarchical signaling for quadratic cost measures and general distributions: A copositive program characterization. *arXiv preprint arXiv:1907.09070*, 2019.
[37] Ross M Starr. Quasi-equilibria in markets with non-convex preferences. *Econometrica*, pages 25–38, 1969.
[38] Ngoc Mai Tran and Josephine Yu. Product-mix auctions and tropical geometry. *Math. Oper. Res.*, 44(4):1396–1411, 2019.
[39] Anna Winnicki, Mariola Ndrio, and Subhonmesh Bose. Convex relaxation based locational marginal prices. *arXiv preprint arXiv:1910.10673*, 2019.
[40] Guanglin Xu and Samuel Burer. A copositive approach for two-stage adjustable robust optimization with uncertain right-hand sides. *Comput. Optim. Appl.*, 70(1):33–59, 2018.