Tightening Quadratic Convex Relaxations for the AC Optimal Transmission Switching Problem

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Joint work with Harsha Nagarajan and Merve Bodur

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Alternating Current Optimal Transmission Switching (ACOTS)

- ACOTS: switch off lines to reduce line congestion, subject to AC power flow constraints

ACOTS Background

- Reduce congestion without building new lines

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- Reduce congestion without building new lines
- Similar to Braess's paradox in transportation networks
- Incorporating switching decisions to the AC optimal power flow (ACOPF) problem

A Simplified ACOTS Formulation

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\min \quad & f(p_i) && && \rightarrow \text{Convex quadratic cost function} \\
\text{s.t.} \quad & S_{ij} = V_i(V_i^* - V_j^*)Y_{ij}^* z_{ij} && \forall (i,j) \in \mathcal{A} \\
& S_i = \sum_{(i,j)\in\mathcal{A}\cup\mathcal{A}^R} S_{ij} && \forall i \in \mathcal{N} \\
& \underline{\boldsymbol{v}}_i \le |V_i| \le \overline{\boldsymbol{v}}_i && \forall i \in \mathcal{N} \\
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& \underline{\boldsymbol{S}}_i \le S_i \le \overline{\boldsymbol{S}}_i && \forall i \in \mathcal{N} \\
& z_{ij} \in \{0,1\} && \forall (i,j) \in \mathcal{A}
\end{aligned}
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V_i:	AC voltage
$S_{ij} = p_{ij} + \mathbf{j}q_{ij}$:	AC power flow
$S_i = p_i + \mathbf{j}q_i$:	Difference between generation and demand
z_{ij}:	Line on/off

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- A mixed-integer nonlinear program (MINLP)
- Need to linearize $S_{ij} = V_i(V_i^* - V_j^*)Y_{ij}^* z_{ij}$

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Challenge

$$S_{ij} = V_i(V_i^* - V_j^*)Y_{ij}^* z_{ij}$$

- Reformulation: Lift $W_{ii} = V_i V_i^*$ and $W_{ij} = V_i V_j^*$

$$\begin{aligned}
&S_{ij} = (W_{ii} - W_{ij})Y_{ij}^* && \forall (i,j) \in \mathcal{A} \\
&\underline{\boldsymbol{v}}_i^2 \leq W_{ii} \leq \overline{\boldsymbol{v}}_i^2 && \forall i \in \mathcal{N} \\
&W \succeq 0 \\
&\text{rank}(W) = 1
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- Semidefinite programming (SDP) relaxation: Remove the rank constraint $\text{rank}(W) = 1$
- Second-order cone (SOC) relaxation: 1×1 and 2×2 minors of W are nonnegative

Quadratic Convex (QC) Relaxation

- Proposed by Hijazi, Coffrin, and Van Hentenryck (2017)
  - More efficient compared with SDP relaxation
  - Stronger compared with SOC relaxation

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w_i &= v_i^2 \\
w_{ij}^R &= z_{ij}(v_i v_j \cos(\theta_i - \theta_j)) \\
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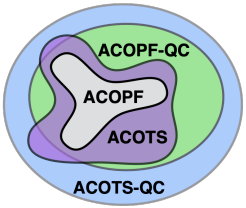
  - Solution: lifting + disjunctive programming

Literature Review: Convex Relaxations for ACOTS

- Became more popular recently
  - Most previous works solve ACOTS approximately: DCOTS, heuristics, etc.



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- ACOTS SOC relaxation [Kocuk et al., 2017]
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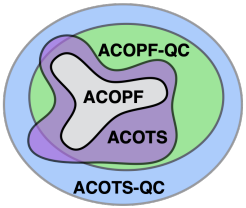

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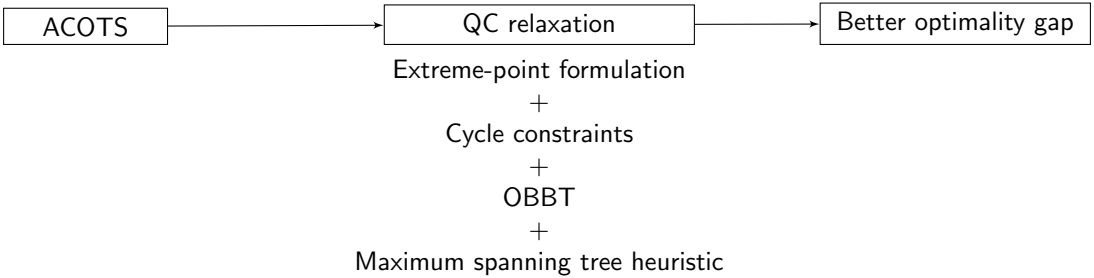

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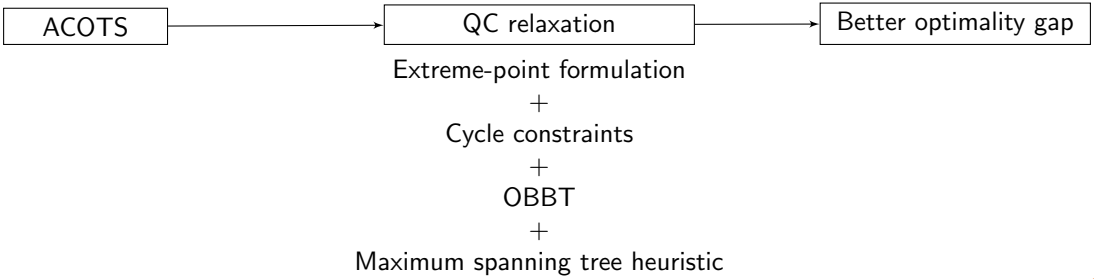

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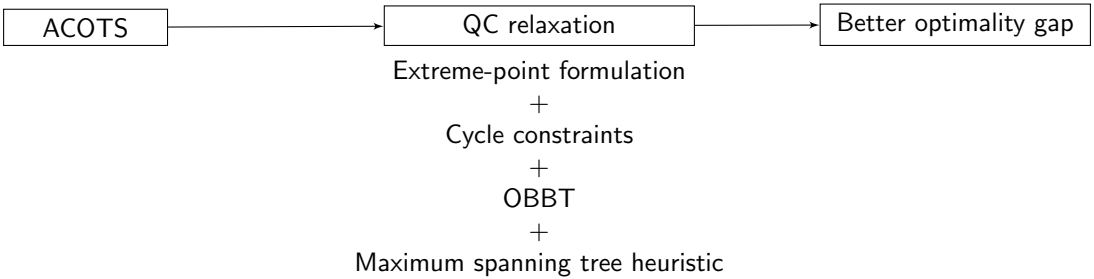

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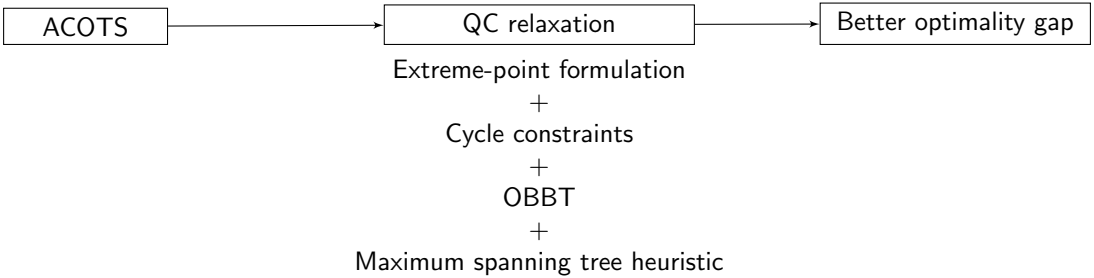

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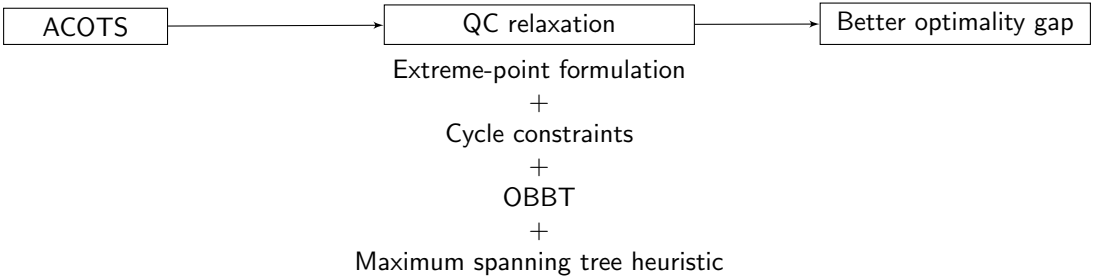

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- Let $c_{ij} = cos(\theta_{ij})$ and relax:

$$c_{ij} \leqslant 1 - \frac{1 - \cos(\boldsymbol{\theta}_{ij}^{\mathbf{u}})}{(\boldsymbol{\theta}_{ij}^{\mathbf{u}})^2}(\theta_{ij}^2)$$

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  - $z_{ij} = 0 \Rightarrow c_{ij} = 0$: use disjunctive programming, enforce inequalities only when $z_{ij} = 1$
  - Similar for $s_{ij} = \sin(\theta_{ij})$

Extreme-point Formulation for Sum of Two Trilinear Terms

$$f = \boldsymbol{a}_1 w_{ij}^R + \boldsymbol{a}_2 w_{ij}^I = \boldsymbol{a}_1 v_i v_j c_{ij} + \boldsymbol{a}_2 v_i v_j s_{ij}$$

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  - Tighter than recursive McCormick

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- H^0 ($z_{ij} = 0$): $c_{ij} = s_{ij} = w_{ij}^R = w_{ij}^I = 0$

Extreme-point Formulation for Sum of Two Trilinear Terms (Continued)

Theorem

$H = \text{conv}(H^0 \cup H^1)$

Extreme-point Formulation for Sum of Two Trilinear Terms (Continued)

Theorem

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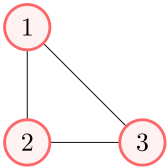
- An extreme-point formulation for ACOTS-QC
- Previous works: use recursive McCormick-based relaxations, not as tight

Lifted Cycle Constraints



- Strengthening constraints over cycles of power networks
  - $\theta_{ik} = \theta_{ij} + \theta_{jk} \Rightarrow cos(\theta_{ik}) = cos(\theta_{ij})cos(\theta_{jk}) - sin(\theta_{ij})sin(\theta_{jk})$

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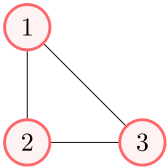

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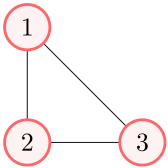

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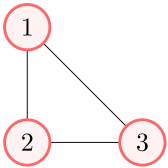

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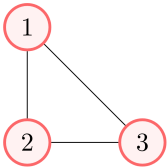

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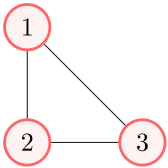

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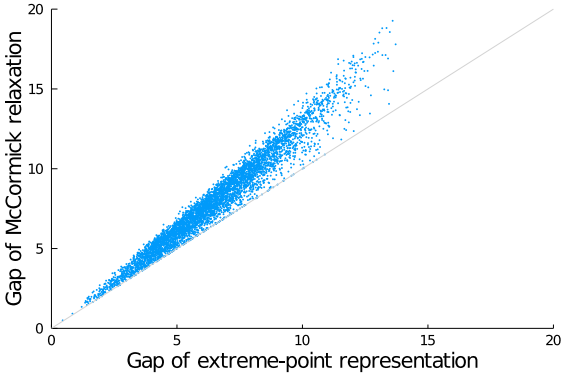
- Add line-switching variables via big-M constraints
- Also implement for 4-cycles

Extreme-point Formulation vs. McCormick Relaxation

- Relaxation for summation of bilinear terms $\sum_{(j_1,j_2)\in\mathcal{P}} x^c_{j1}x^c_{j2}$, with $x^c_i \in [-1,1]$
- Gap between upper and lower bounds of the relaxation



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- Lifted cycle constraints add extra constraints and variables
- Add only violated constraints:
  - Benders cuts from extreme point formulation (without binary variables)
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☹ Separation problem building time in Julia is a bottleneck

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- Tighten the bounds of v_i, θ_{ij}, z_{ij} and y_C
- Bound tightening optimization problem for variable x:

$$\begin{aligned} \min / \max \quad & x \\ \text{s.t.} \quad & f(p_i) \leq f^* \\ & \text{ACOTS-QC constraints with relaxed binary variables} \end{aligned}$$

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- Solves large instances with 500 to 2,312 buses

Experiment: Optimality Gap Comparison

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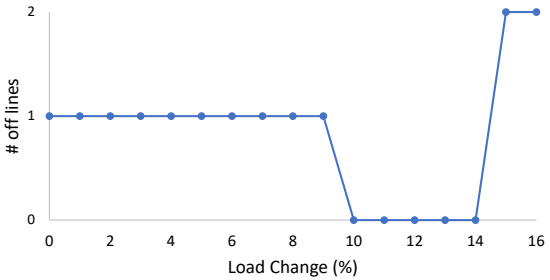
- Main findings:
  - OBBT is most helpful for SAD (small voltage angle) and API (congested) instances
  - Extreme-point formulation and lifted cycle constraints are most helpful for SAD instances
  - The heuristic is more accurate for larger instances: more flexibility for switching

Experiment: Number of Lines Off

- Various load profiles for "TYP" case30_ieee



Experiment: Lifted Cycle Constraints for ACOPF-QC

- ACOPF-QC + lifted cycle constraints is also novel
- Most helpful for SAD instances

instances	improvement
pglib_opf_case14_ieee__sad	6.06%
pglib_opf_case24_ieee_rts__sad	0.53%
pglib_opf_case73_ieee_rts__sad	0.57%
pglib_opf_case3_lmbd__api	0.68%
pglib_opf_case24_ieee_rts__api	0.14%
pglib_opf_case73_ieee_rts__api	0.22%
pglib_opf_case179_goc__api	0.11%
nesta_case6_c__sad	0.21%
nesta_case24_ieee_rts__sad	0.53%
nesta_case29_edin__sad	4.85%
nesta_case73_ieee_rts__sad	0.57%
nesta_case118_ieee__sad	0.24%
nesta_case240_wecc__sad	0.12%
nesta_case3_lmbd__api	0.33%

Summary

- ACOTS: a MINLP problem
- Tight relaxations for ACOTS
  - ACOTS-QC relaxation
  - Extreme-point formulation
  - Lifted cycle constraints
  - OBBT
  - Maximum spanning tree heuristic
- Solve large-scale instances instances with 500 to 2,312 buses
- Minor revision at *INFORMS Journal on Computing*