Academic Notes Series: No. 1

Financial Decisions and Markets: A Course for Asset Pricing

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Date: September 16, 2022

Version: 0.1

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Work Hard, Study Hard.

Contents

Introduction

This is academic note for Campbell 2017 and this project is beginning from September 10, 2022.

Chapter 1 Choice under Uncertainty

Introduction

- *Expected Utility*

This chapter review the basic theory of choice under Uncertainty, ignoring time by assuming that all Uncertainty is resolved at a single future data.

Related Literature:

- Gollier 2001
- Ingersoll 1987

1.1 Expected Utility

Proposition 1.1

- *An ordinal utility function $\Upsilon(.)$ tells you that an agent is indifferent between x and y if $\Upsilon(x) = \Upsilon(y)$ and prefers x to y if $\Upsilon(x) > \Upsilon(y)$.*
- *For any strictly increasing function Θ, the preferences expressed by $\Theta(\Upsilon(.))$ are the same as those expressed by Υ.*

♠

Chapter 2 Static Portfolio Choice

Chapter 3 Static Equilibrium Asset Pricing

Introduction

- *CAPM*

3.1 The Capital Asset Pricing Model (CAPM)

Some basic assumption:

- All investors are price-takers
- Evaluate portfolios using the means and variances of single-period returns
- Investors have common beliefs about the means, variances, and covariances of returns.
- There are no nontraded assets, taxes, or transactions costs.
- Investors can borrow or lend at a given riskfree interest rate.

Note *The market portfolio is mean-variance efficient.*

3.1.1 Asset Pricing Implication of the Sharpe-Lintner CAPM

An increase in the weight of asset i in portfolio p, ω_i, financed by a decrease in the weight on the riskless asset, affect the mean and variance of the return on portfolio p as follows:

$$\begin{aligned} \overline{R}_p &= \sum_i w_i \left(\overline{R_i} - R_f\right) \\ \frac{dR_p}{dw_i} &= \overline{R_i} - R_f \end{aligned} \tag{3.1}$$

$$\frac{d\,\mathrm{Var}\left(R_p\right)}{dw_i} = 2\,\mathrm{Cov}\left(R_i, R_p\right) \tag{3.2}$$

Proof The individual-asset variance and covariances in $Var(R_p)$ are

$$Var(R_p) = 2w_iw_1\,\mathrm{Cov}\left(R_i, R_1\right) + \cdots + w_i^2\,\mathrm{Var}\left(R_i\right) + \cdots + 2w_iw_N\,\mathrm{Cov}\left(R_i, R_N\right)$$

$$\frac{d\,\mathrm{Var}\left(R_p\right)}{dw_i} = 2w_1\,\mathrm{Cov}\left(R_i, R_1\right) + \cdots + 2w_i\,\mathrm{Var}\left(R_i\right) + \cdots + 2w_N\,\mathrm{Cov}\left(R_i, R_N\right) = 2\,\mathrm{Cov}\left(R_i, R_p\right) \tag{3.3}$$

The ratio of the effect on mean, (3.1), to the effect on variance, (3.2) in

$$\frac{d\bar{R}_p/dw_i}{d\,\mathrm{Var}\left(R_p\right)/dw_i} = \frac{\bar{R}_i - R_f}{2\,\mathrm{Cov}\left(R_i, R_p\right)} \tag{3.4}$$

Proposition 3.1

If portfolio p is mean-variance efficient, this ratio should be the same for all individual assets i.

Proof

$$d\bar{R}_p = \left(\bar{R}_i - R_f\right) dw_i + \left(\bar{R}_j - R_f\right) dw_j \tag{3.5}$$

and

$$d\,\mathrm{Var}\left(R_p\right) = 2\,\mathrm{Cov}\left(R_i, R_p\right) dw_i + 2\,\mathrm{Cov}\left(R_j, R_p\right) dw_j \tag{3.6}$$

Now consider setting dw_j so that the mean portfolio return is unchanged, $d\overline{R_p} = 0$:

$$dw_j = -\frac{\left(\bar{R}_i - R_f\right)}{\left(\bar{R}_j - R_f\right)} dw_i \tag{3.7}$$

The portfolio variance must also be unchanged. We have

$$d\,\mathrm{Var}\left(R_p\right) = \left[2\,\mathrm{Cov}\left(R_i, R_p\right) - 2\,\mathrm{Cov}\left(R_j, R_p\right)\frac{\left(\bar{R}_i - R_f\right)}{\left(\bar{R}_j - R_f\right)}\right] dw_i = 0 \tag{3.8}$$

This requires

$$\frac{\bar{R}_i - R_f}{2\,\mathrm{Cov}\left(R_i, R_p\right)} = \frac{\bar{R}_j - R_f}{2\,\mathrm{Cov}\left(R_j, R_p\right)} \tag{3.9}$$

This equation must hold for all assets j, including the original portfolio itself. Setting $j = p$, we get

$$\frac{\bar{R}_i - R_f}{2\,\mathrm{Cov}\left(R_i, R_p\right)} = \frac{\bar{R}_p - R_f}{2\,\mathrm{Var}\left(R_p\right)} \tag{3.10}$$

or

$$\bar{R}_i - R_f = \frac{\mathrm{Cov}\left(R_i, R_p\right)}{\mathrm{Var}\left(R_p\right)}\left(\bar{R}_p - R_f\right) = \beta_{ip}\left(\bar{R}_p - R_f\right) \tag{3.11}$$

where $\beta_{ip} \equiv \mathrm{Cov}\left(R_i, R_p\right) / \mathrm{Var}\left(R_p\right)$ is the regression coefficient of asset i's return on portfolio p's return.

The market portfolio m is mean-variance efficient. Under the restriction (3.11) describes the market portfolio:

$$\bar{R}_i - R_f = \beta_{im}\left(\bar{R}_m - R_f\right) \tag{3.12}$$

The regression of excess return on the market excess return,

$$R_{it} - R_{ft} = \alpha_i + \beta_{im}\left(R_{mt} - R_{ft}\right) + \epsilon_{it} \tag{3.13}$$

the intercept α_i should be 0 for all assets. α_i is called *Jensen's alpha*.

3.1.2 The Black CAPM

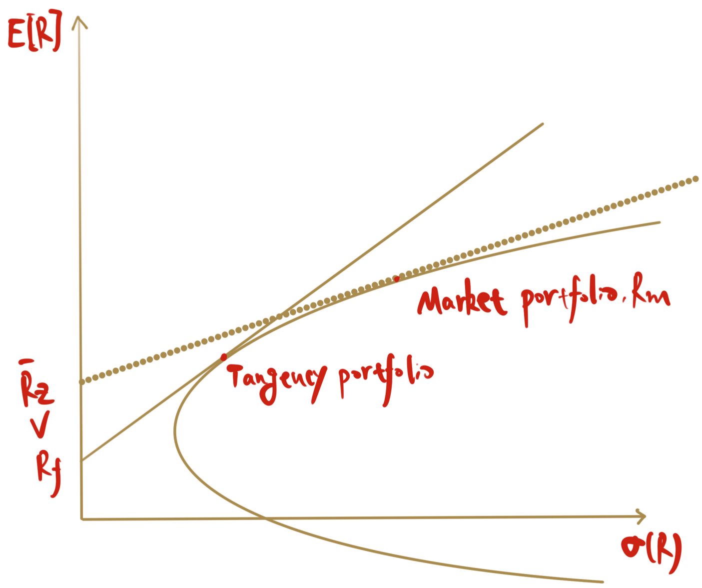

Figure 3.1: The Black (1972) CAPM

3.1.3 Beta Pricing and Portfolio Choice

Problem 3.1 Assume that the Sharpe-Lintner CAPM holds, so the mean-variance efficient frontier consists of combinations of Treasury bills and the market portfolio. Nonetheless, some households make the mistake of holding undiversified portfolios that contain only one stock or a few stocks. (Empirical evidence on such behavior is discussed in Chapter 10.)

1. Show that the Sharpe ratio of any portfolio divided by the Sharpe ratio of the market portfolio equals the correlation of that portfolio with the market portfolio.
2. Suppose the market is made up of identical stocks, each of which has the same market capitalization, the same mean and variance of return, and the same correlation $\rho > 0$, with every other individual stock. Consider the limit as the number of stocks in the market increases. What is the Sharpe ratio of an equally-weighted portfolio that contains N stocks divided by the Sharpe ratio of the market portfolio? Interpret.

Solution

1. *The Sharpe ratio of portfolio p divided by the Sharpe ratio of the market portfolio equals to*

$$\begin{aligned}
&\frac{E\left(R_p-R_f\right)}{\sigma\left(R_p\right)} / \frac{E\left(R_m-R_f\right)}{\sigma\left(R_m\right)} \\
&=\frac{\beta_{pm} E\left(R_m-R_f\right)}{\sigma\left(R_p\right)} / \frac{E\left(R_m-R_f\right)}{\sigma\left(R_m\right)} \\
&=\frac{\beta_{pm} \sigma\left(R_m\right)}{\sigma\left(R_p\right)} \\
&=\frac{\operatorname{cov}\left(R_p \cdot R_m\right) \cdot \sigma\left(R_m\right)}{\operatorname{Var}\left(R_m\right) \cdot \sigma\left(R_p\right)} \\
&=\frac{\rho \sigma\left(R_p\right) \sigma\left(R_m\right) \cdot \sigma\left(R_m\right)}{\operatorname{Var}\left(R_m\right) \cdot \sigma\left(R_p\right)} \\
&=\rho
\end{aligned}$$

where ρ is the correlation coefficient between portfolio r_p and r_m.

2. *The equal-weighted portfolio p contains N stocks. So the return and variance of this portfolio is*

$$r_p=\frac{1}{N} r_1+\frac{1}{N} r_2+\ldots+\frac{1}{N} r_N=r$$

$$\begin{aligned}
\sigma_p^2= & \frac{1}{N^2} \sigma^2\left(r_1\right)+\frac{1}{N^2} \sigma^2\left(r_1\right)+\cdots \\
& +2 \frac{1}{N^2} \operatorname{cov}\left(r_1, r_2\right)+\cdots+2 \frac{1}{N^2} \operatorname{cov}\left(r_{N-1}, r_N\right) \\
= & \frac{1}{N} \sigma^2+\frac{2}{N^2} \cdot \frac{N-1}{2} \cdot N \cdot \rho \sigma^2 \\
= & \frac{1}{N} \sigma^2+\frac{N-1}{N} \rho \sigma^2 \\
= & \rho \sigma^2+\frac{1}{N} \sigma^2(1-\rho)
\end{aligned}$$

when $N \rightarrow \infty$, the variance of market portfolio is

$$\sigma_m^2=\rho \sigma^2$$

The Sharpe ratio's ratio is

$$\frac{\sqrt{\rho\sigma^2}}{\sqrt{\rho\sigma^2 + \frac{1}{N}\sigma^2(1-\rho)}} = \sqrt{\frac{N\rho}{N\rho + 1 - \rho}}$$

3.2 Arbitrage Pricing and Multifactor Models

3.2.1 Arbitrage Pricing in a Single-Factor Model

$$\begin{aligned} R_{it}^e &= \alpha_i + \beta_{im} R_{mt}^e + \epsilon_{it} \\ \mathrm{E}\left[\epsilon_{it}\epsilon_{jt}\right] &= 0 \\ \mathrm{Cov}\left(R_{it}^e, R_{jt}^e\right) &= \beta_{im}\beta_{jm}\sigma_m^2 \end{aligned}$$

The implication of above assumption is that:

Remark If many assets are available, we should expect α_i typically to be very small in absolute value. This is the *arbitrage pricing theory* of Ross 1976.

Why?

Portfolio of N assets i, the excess return on the portfolio will be

$$R_{pt}^e = \alpha_p + \beta_{pm} R_{mt}^e + \epsilon_{pt}$$

where $\alpha_p = \sum_{j=1}^N w_j \alpha_j, \beta_{pm} = \sum_{j=1}^N w_j \beta_{jm}$, and $\epsilon_{pt} = \sum_{j=1}^N w_j \epsilon_{jt}$.

The variance of ϵ_{pt} will be

$$\mathrm{Var}\left(\epsilon_{pt}\right) = \sum_{j=1}^N w_j^2 \,\mathrm{Var}\left(\epsilon_{jt}\right)$$

which will shrink rapidly with N provided that no single weight w_j is too large.

Suppose that the portfolio has enough stocks, with a small enough weight in each one, that the residual risk $\mathrm{Var}\left(\epsilon_{pt}\right)$ is negligible. We say that the portfolio is **well diversified**. For such a portfolio, we can neglect ϵ_{pt} and write the excess return as

$$R_{pt}^e = \alpha_p + \beta_{pm} R_{mt}^e$$

But we must have $\alpha_p = 0$. If not, there is an arbitrage opportunity: go short β_{pm} units of the market and go long on unit of portfolio p, while funding all positions with riskless borrowing and lending. This delivers a riskless excess return of α_p.

Ross 1976 sda

Chapter 4   The Stochastic Discount Factor

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Appendix A Nothing