Chapter 1

Anisotropic elastic media

About two years since I printed this Theory in an Anagram at the end of my Book of the Descriptions of Helioscopes, viz.ceiiinosssttuu,ideft, Ut tensio sic vis; That is The Power of any Spring is in the same proportion with the Tension thereof: That is, if one power stretch[es] *or bend*[s] *it one space, two will bend it two, and three will bend it three, and so forward...*

Heterogeneous motions from without are propagated within the solid in a direct line if they hit perpendicular to the superficies or bounds, but if obliquely in ways not direct, but different and deflected, according to the particular inclination of the body striking, and according to the proportion of the Particles striking and being struck.

Robert Hooke (Hooke, 1678)

The stress-strain law and/or wave propagation in anisotropic elastic (lossless) media are discussed in several books, notably, Love (1944), Musgrave (1970), Fedorov (1968), Beltzer (1988), Payton (1983), Nye (1985), Hanyga (1985), Aboudi (1991), Auld (1990a,b), Helbig (1994) and Ting (1996). Crampin (1981), Winterstein (1990), Mavko, Mukerji and Dvorkin (1998), Tsvankin (2001) and Červený (2001) provide a comprehensive review of the subject with respect to seismic applications. In this chapter, we review the main features of anisotropy in order to understand the physics of wave propagation in anisotropic elastic media, and to provide the basis for the theoretical developments regarding more complex rheologies, discussed in the next chapters.

1.1 Strain-energy density and stress-strain relation

Defining strain energy is the first step in determining the constitutive equations or stress-strain relations, which provide the basis for the description of static and dynamic deformations of physical media. Invoking the symmetry of the stress and strain tensors[1], the most general form of the strain-energy volume density is

$$2V = \sum_{I}^{6} \sum_{J \geq I}^{6} a_{IJ} e_I e_J. \tag{1.1}$$

[1]See Auld (1990a) and Klausner (1991), and Nowacki (1986) for a theory of non-symmetric stress and strain tensors.

According to Voigt's notation,

$$e_I = e_{i(i)} = \partial_i u_{(i)}, \quad I = 1, 2, 3, \qquad e_I = e_{ij} = \partial_j u_i + \partial_i u_j, \quad i \neq j \ (I = 4, 5, 6), \tag{1.2}$$

where u_i are the displacement components, and a_{IJ} are 21 coefficients related to the elasticity constants c_{IJ} as $a_{I(I)} = c_{I(I)}$ and $a_{IJ} = 2c_{IJ}$ for $I \neq J$ (Love, 1994, p. 100, 159). Note that the strains in standard use are

$$\epsilon_{i(i)} = e_I, \quad I = 1, 2, 3, \qquad \epsilon_{ij} = \frac{1}{2} e_{ij} = \frac{1}{2} e_I, \quad i \neq j \ (I = 4, 5, 6). \tag{1.3}$$

Alternatively, using the Cartesian components, the strain-energy density can be expressed in terms of a fourth-order elasticity tensor c_{ijkl}, as

$$2V = c_{ijkl} \epsilon_{ij} \epsilon_{kl}, \tag{1.4}$$

where the symmetries

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij} \tag{1.5}$$

reduce the number of independent elasticity constants from 81 to 21. The first and second equalities arise from the symmetry of the strain and stress tensors. The last equality is obtained by noting that the second partial derivatives of V are independent of the order of differentiation with respect to the strain components (Auld, 1990a, p. 138, 144; Ting, 1996, p. 32).

The strain tensor can be expressed as $\boldsymbol{\epsilon} = \sum \epsilon_{ij}\ \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$. Let us consider a medium that possesses at each point the (x, z)-plane as its plane of symmetry. This medium has monoclinic symmetry. A reflection with respect to this plane ($y \to -y$) should leave the strain energy unaltered. Such a transformation implies $\epsilon_{12} \to -\epsilon_{12}$ and $\epsilon_{23} \to -\epsilon_{23}$, which implies $c_{14} = c_{16} = c_{24} = c_{26} = c_{34} = c_{36} = c_{45} = c_{56} = 0$ (see Love, 1944, p. 154). The result is

$$2V = c_{11} e_{11}^2 + c_{22} e_{22}^2 + c_{33} e_{33}^2 + 2c_{12} e_{11} e_{22} + 2c_{13} e_{11} e_{33} + 2c_{23} e_{22} e_{33}$$
$$+ c_{44} e_{23}^2 + c_{55} e_{13}^2 + c_{66} e_{12}^2 + 2(c_{15} e_{11} + c_{25} e_{22} + c_{35} e_{33}) e_{13} + 2c_{46} e_{23} e_{12}. \tag{1.6}$$

Similar reflections with respect to the other Cartesian planes of symmetry imply that other coefficients become equal to each other. Thus, the number of coefficients required to describe a medium possessing orthorhombic symmetry – three mutually orthogonal planes of symmetry – is reduced. The result is

$$2V = c_{11} e_{11}^2 + c_{22} e_{22}^2 + c_{33} e_{33}^2 + 2c_{12} e_{11} e_{22} + 2c_{13} e_{11} e_{33} + 2c_{23} e_{22} e_{33}$$
$$+ c_{44} e_{23}^2 + c_{55} e_{13}^2 + c_{66} e_{12}^2. \tag{1.7}$$

If the material possesses an axis of rotational symmetry – as in a transversely isotropic medium – the strain energy should be invariant to rotations about that axis. Then,

$$2V = c_{11}(e_{11}^2 + e_{22}^2) + c_{33} e_{33}^2 + 2(c_{11} - 2c_{66}) e_{11} e_{22} + 2c_{13}(e_{11} + e_{22}) e_{33}$$
$$+ c_{44}(e_{23}^2 + e_{13}^2) + c_{66} e_{12}^2 \tag{1.8}$$

(Love, 1944, p. 152-160; Helbig, 1994, p. 87).

If the medium is isotropic, every plane is a plane of symmetry, and every axis is an axis of symmetry. Consequently, some of the coefficients vanish, and we obtain

$$2V = c_{11}(e_{11}^2+e_{22}^2+e_{33}^2)+2(c_{11}-2c_{66})(e_{11}e_{22}+e_{11}e_{33}+e_{22}e_{33})+c_{66}(e_{12}^2+e_{13}^2+e_{23}^2), \quad (1.9)$$

where $c_{11} = \lambda + 2\mu$, and $c_{66} = \mu$, with λ and μ being the Lamé constants.

Alternatively, the strain energy for isotropic media can be expressed in terms of invariants of strain – up to the second-order. These invariants can be identified in equation (1.9). In fact, this equation can be rewritten as

$$2V = c_{11}\vartheta^2 - 4c_{66}\varpi, \quad (1.10)$$

where

$$\vartheta = e_{11} + e_{22} + e_{33} \quad (1.11)$$

and

$$\varpi = e_{11}e_{22} + e_{11}e_{33} + e_{22}e_{33} - \frac{1}{4}(e_{12}^2 + e_{13}^2 + e_{23}^2) \quad (1.12)$$

or

$$\varpi = \epsilon_{11}\epsilon_{22} + \epsilon_{11}\epsilon_{33} + \epsilon_{22}\epsilon_{33} - (\epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2) \quad (1.13)$$

are invariants of strain (Love, 1944, p. 43). These invariants are the coefficients of the second and first powers of the polynomial in r, $\det(\boldsymbol{\epsilon} - r\mathbf{I}_3)$, where $\boldsymbol{\epsilon} = \sum \epsilon_{ij}\ \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j$. The roots of this polynomial are the principal strains that define the strain quadric – an ellipsoid (Love, 1944, p. 41).

We know, a priori, (for instance, from experiments) that a homogeneous isotropic medium "supports" two pure deformation modes, i.e., a dilatational one and a shear one. These correspond to a change of volume, without a change in shape, and a change in shape without a change of volume, respectively. It is, therefore, reasonable to follow the physics of the problem and write the strain energy in terms of the dilatation ϑ and the deviator

$$d^2 = d_{ij}d_{ij}, \quad (1.14)$$

where

$$d_{ij} = \epsilon_{ij} - \frac{1}{3}\vartheta\delta_{ij} \quad (1.15)$$

are the components of the deviatoric strain tensor, with δ_{ij} being the components of the Kronecker matrix. Since,

$$d^2 = e_{11}^2 + e_{22}^2 + e_{33}^2 + \frac{1}{2}(e_{12}^2 + e_{13}^2 + e_{23}^2) - \frac{\vartheta^2}{3} \quad (1.16)$$

and $\varpi = (\vartheta^2/3) - d^2/2$, we have the following expression:

$$2V = \left(c_{11} - \frac{4}{3}c_{66}\right)\vartheta^2 + 2c_{66}d^2. \quad (1.17)$$

This form is used in Chapter 7 to derive the dynamical equations of poroelasticity.

Having obtained the strain-energy expression, we now consider stress. The stresses are given by

$$\sigma_{ij} = \frac{\partial V}{\partial e_{ij}} \quad (1.18)$$

(Love, 1944, p. 95) or, using the shortened matrix notation,

$$\sigma_I = \frac{\partial V}{\partial e_I}, \tag{1.19}$$

where

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)^\top = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12})^\top. \tag{1.20}$$

Having made use of the standard strain components from the outset, and having calculated the stresses as $\sigma_{ij} = \partial V / \partial \epsilon_{ij}$ from equation (1.4), we are required to distinguish between (ij) and (ji) components $(i \neq j)$. However, the use of Love's notation, to express both the strain components e_{ij} and the form (1.1), avoids the necessity of this distinction.

Using the Cartesian and shortened notations, we can write Hooke's law for the anisotropic elastic case as

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \tag{1.21}$$

and

$$\sigma_I = c_{IJ} e_J, \tag{1.22}$$

respectively.

1.2 Dynamical equations

In this section, we derive the differential equations describing wave propagation in terms of the displacements of the material. The conservation of linear momentum implies

$$\partial_j \sigma_{ij} + f_i = \rho \partial^2_{tt} u_i \tag{1.23}$$

(Auld, 1990a, p. 43), where u_i are the components of the displacement vector, ρ is the mass density and f_i are the components of the body forces per unit volume. Assuming a volume Ω bounded by a surface S, the volume integral of equation (1.23) is the balance between the surface tractions on S – obtained by applying the divergence theorem to $\partial_j \sigma_{ij}$ – and the body forces with the inertia term $\rho \partial^2_{tt} u_i$. Equations (1.23) are known as Euler's equations for elasticity, corresponding to Newton's law of motion for particles.

The substitution of Hooke's law (1.21) into equation (1.23) yields

$$\partial_j (c_{ijkl}\epsilon_{kl}) + f_i = \rho \partial^2_{tt} u_i. \tag{1.24}$$

In order to use the shortened matrix notation, we introduce Auld's notation (Auld, 1990a,b) for the differential operators. The symmetric gradient operator has the following matrix representation

$$\nabla = \begin{pmatrix} \partial_1 & 0 & 0 & 0 & \partial_3 & \partial_2 \\ 0 & \partial_2 & 0 & \partial_3 & 0 & \partial_1 \\ 0 & 0 & \partial_3 & \partial_2 & \partial_1 & 0 \end{pmatrix}. \tag{1.25}$$

The strain-displacement relation (1.2) can then be written as

$$\mathbf{e} = \nabla^\top \cdot \mathbf{u}, \quad (e_I = \nabla_{Ij} u_j), \tag{1.26}$$

with

$$\mathbf{e} = (e_1, e_2, e_3, e_4, e_5, e_6)^\top = (e_{11}, e_{22}, e_{33}, e_{23}, e_{13}, e_{12})^\top = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{13}, 2\epsilon_{12})^\top. \tag{1.27}$$

The divergence of the stress tensor $\partial_i \sigma_{ij}$ can be expressed as $\nabla \cdot \boldsymbol{\sigma}$, and equation (1.23) becomes

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \partial^2_{tt} \mathbf{u}, \tag{1.28}$$

where $\boldsymbol{\sigma}$ is defined in equation (1.20), and where

$$\mathbf{u} = (u_1, u_2, u_3) \tag{1.29}$$

and

$$\mathbf{f} = (f_1, f_2, f_3). \tag{1.30}$$

Similarly, using the matrix notation, the stress-strain relation (1.22) reads

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \mathbf{e}, \tag{1.31}$$

with the elasticity matrix given by

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix}. \tag{1.32}$$

The zero strain state corresponds to static equilibrium with minimum strain energy ($V = 0$). Because this energy must always increase when the medium is deformed, we have $c_{IJ} e_I e_J > 0$. Mathematically, this expression involving non-zero components e_I defines a positive definite quadratic function, which, by definition, imposes some constraints on the elasticity constants (stability condition, see Auld, 1990a, p. 147, and Ting, 1996, p. 56); namely, all principal determinants should be greater than zero,

$$c_{I(I)} > 0, \quad \det \begin{pmatrix} c_{I(I)} & c_{IJ} \\ c_{IJ} & c_{J(J)} \end{pmatrix} > 0, \quad \ldots \quad \det(c_{IJ}) > 0. \tag{1.33}$$

Alternatively, the strain-energy density can be expressed in terms of the eigenvalues of matrix $\mathbf{C}$, namely, $\Lambda_I, I = 1, \ldots 6$, called eigenstiffnesses (Kelvin, 1856) (see Section 4.1 in Chapter 4); that is, $2V = \Lambda_I \mathbf{e}_I^\top \cdot \mathbf{e}_I$, wherein $\mathbf{e}_I$ are the eigenvectors or eigenstrains. It is clear that a positive strain energy implies the condition $\Lambda_I > 0$ (see Pipkin, 1976).

Equations (1.26), (1.28) and (1.31) combine to give

$$\nabla \cdot [\mathbf{C} \cdot (\nabla^\top \cdot \mathbf{u})] + \mathbf{f} = \rho \partial^2_{tt} \mathbf{u}, \tag{1.34}$$

or

$$\boldsymbol{\Gamma}_\nabla \cdot \mathbf{u} + \mathbf{f} = \rho \partial^2_{tt} \mathbf{u}, \quad (\Gamma_{\nabla ij} u_j + f_i = \rho \partial^2_{tt} u_i), \tag{1.35}$$

where

$$\boldsymbol{\Gamma}_\nabla = \nabla \cdot \mathbf{C} \cdot \nabla^\top, \quad (\Gamma_{\nabla ij} = \nabla_{iI} c_{IJ} \nabla_{Jj}) \tag{1.36}$$

is the 3×3 symmetric Kelvin-Christoffel differential-operator matrix.

1.2.1 Symmetries and transformation properties

Differentiation of the strain energies (1.6), (1.7) and (1.8), in accordance with equation (1.18) yields the elasticity matrices for the monoclinic, orthorhombic and transversely isotropic media. Hence, we obtain

$$\mathbf{C}(\text{monoclinic}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & c_{15} & 0 \\ c_{12} & c_{22} & c_{23} & 0 & c_{25} & 0 \\ c_{13} & c_{23} & c_{33} & 0 & c_{35} & 0 \\ 0 & 0 & 0 & c_{44} & 0 & c_{46} \\ c_{15} & c_{25} & c_{35} & 0 & c_{55} & 0 \\ 0 & 0 & 0 & c_{46} & 0 & c_{66} \end{pmatrix}, \tag{1.37}$$

$$\mathbf{C}(\text{orthorhombic}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} \tag{1.38}$$

and

$$\mathbf{C}(\text{transversely isotropic}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}, \quad 2c_{66} = c_{11} - c_{12}, \tag{1.39}$$

which imply 13, 9 and 5 independent elasticity constants, respectively. In the monoclinic case, the symmetry plane is the (x, z)-plane. A rotation by an angle θ – with $\tan(2\theta) = 2c_{46}/(c_{66} - c_{44})$ – about the y-axis removes c_{46}, so that the medium can actually be described by 12 elasticity constants. The isotropic case is obtained from the transversely isotropic case, where $c_{11} = c_{33} = \lambda + 2\mu$, $c_{55} = c_{66} = \mu$ and $c_{13} = \lambda$, in terms of the Lamé constants. The aforementioned material symmetries are enough to describe most of the geological systems at different scales. For example, matrix (1.39) may represent a finely layered medium (see Section 1.5), matrix (1.38) may represent two sets of cracks with crack normals at 90°, or a vertical set of cracks in a finely layered medium, and matrix (1.37) may represent two sets of cracks with crack normals other than 0° or 90° (Winterstein, 1990).

Let us consider the conditions of existence for a transversely isotropic medium according to equations (1.33). The first condition implies $c_{11} > 0$, $c_{33} > 0$, $c_{55} > 0$ and $c_{66} > 0$; the second-order determinants imply $c_{11}^2 - c_{12}^2 > 0$ and $c_{11}c_{33} - c_{13}^2 > 0$; and the relevant third-order determinant implies $(c_{11}^2 - c_{12}^2)c_{33} - 2c_{13}^2(c_{11} - c_{12}) > 0$. All these conditions can be combined into

$$c_{11} > |c_{12}|, \quad (c_{11} + c_{12})c_{33} > 2c_{13}^2, \quad c_{55} > 0. \tag{1.40}$$

In isotropic media, expressions (1.40) reduce to

$$3\lambda + 2\mu > 0, \qquad 2\mu > 0, \tag{1.41}$$

where these stiffnesses are the eigenvalues of matrix $\mathbf{C}$, the second eigenvalue having a multiplicity of five.

It is useful to express explicitly the equations of motion for a particular symmetry that are suitable for numerical simulation of wave propagation in inhomogeneous media. The particle-velocity/stress formulation is widely used for this purpose. Consider, for instance, the case of a medium exhibiting monoclinic symmetry. From equations (1.34) and (1.37), we obtain the following expressions.

Particle velocity:

$$\begin{aligned} \partial_t v_1 &= \rho^{-1}\left(\partial_1\sigma_{11} + \partial_2\sigma_{12} + \partial_3\sigma_{13} + f_1\right) \\ \partial_t v_2 &= \rho^{-1}\left(\partial_1\sigma_{12} + \partial_2\sigma_{22} + \partial_3\sigma_{23} + f_2\right) \\ \partial_t v_3 &= \rho^{-1}\left(\partial_1\sigma_{13} + \partial_2\sigma_{23} + \partial_3\sigma_{33} + f_3\right). \end{aligned} \tag{1.42}$$

Stress:

$$\begin{aligned} \partial_t\sigma_{11} &= c_{11}\partial_1 v_1 + c_{12}\partial_2 v_2 + c_{13}\partial_3 v_3 + c_{15}(\partial_1 v_3 + \partial_3 v_1) \\ \partial_t\sigma_{22} &= c_{12}\partial_1 v_1 + c_{22}\partial_2 v_2 + c_{23}\partial_3 v_3 + c_{25}(\partial_1 v_3 + \partial_3 v_1) \\ \partial_t\sigma_{33} &= c_{13}\partial_1 v_1 + c_{23}\partial_2 v_2 + c_{33}\partial_3 v_3 + c_{35}(\partial_1 v_3 + \partial_3 v_1) \\ \partial_t\sigma_{23} &= c_{44}(\partial_2 v_3 + \partial_3 v_2) + c_{46}(\partial_1 v_2 + \partial_2 v_1) \\ \partial_t\sigma_{13} &= c_{15}\partial_1 v_1 + c_{25}\partial_2 v_2 + c_{35}\partial_3 v_3 + c_{55}(\partial_1 v_3 + \partial_3 v_1) \\ \partial_t\sigma_{12} &= c_{46}(\partial_2 v_3 + \partial_3 v_2) + c_{66}(\partial_1 v_2 + \partial_2 v_1), \end{aligned} \tag{1.43}$$

where the particle-velocity vector is

$$\mathbf{v} = (v_1, v_2, v_3) = \partial_t\mathbf{u} = (\partial_t u_1, \partial_t u_2, \partial_t u_3). \tag{1.44}$$

Symmetry plane of a monoclinic medium

In the (x, z)-plane ($\partial_2 = 0$), we identify two sets of uncoupled differential equations

$$\begin{aligned} \partial_t v_1 &= \rho^{-1}\left(\partial_1\sigma_{11} + \partial_3\sigma_{13} + f_1\right) \\ \partial_t v_3 &= \rho^{-1}\left(\partial_1\sigma_{13} + \partial_3\sigma_{33} + f_3\right) \\ \partial_t\sigma_{11} &= c_{11}\partial_1 v_1 + c_{13}\partial_3 v_3 + c_{15}(\partial_1 v_3 + \partial_3 v_1) \\ \partial_t\sigma_{33} &= c_{13}\partial_1 v_1 + c_{33}\partial_3 v_3 + c_{35}(\partial_1 v_3 + \partial_3 v_1) \\ \partial_t\sigma_{13} &= c_{15}\partial_1 v_1 + c_{35}\partial_3 v_3 + c_{55}(\partial_1 v_3 + \partial_3 v_1) \end{aligned} \tag{1.45}$$

and

$$\begin{aligned} \partial_t v_2 &= \rho^{-1}\left(\partial_1\sigma_{12} + \partial_3\sigma_{23} + f_2\right) \\ \partial_t\sigma_{23} &= c_{44}\partial_3 v_2 + c_{46}\partial_1 v_2 \\ \partial_t\sigma_{12} &= c_{46}\partial_3 v_2 + c_{66}\partial_1 v_2. \end{aligned} \tag{1.46}$$

The first set describes in-plane particle motion while the second set describes cross-plane particle motion, that is, the propagation of a pure shear wave. Using the appropriate elasticity constants, equations (1.45) and (1.46) hold in the three symmetry planes of an orthorhombic medium, and at every point of a transversely isotropic medium, by virtue of the azimuthal symmetry around the z-axis. The uncoupling implies that a cross-plane shear wave exists at a plane of mirror symmetry (Helbig, 1994, p. 142).

Equations (1.42) and (1.43) can be restated as a matrix equation

$$\partial_t\underline{\mathbf{v}} = \mathbf{H}\cdot\underline{\mathbf{v}} + \underline{\mathbf{f}}, \tag{1.47}$$

where

$$\underline{\mathbf{v}} = (v_1, v_2, v_3, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12})^\top \tag{1.48}$$

is the 9×1 column matrix of the unknown field,

$$\rho\underline{\mathbf{f}} = (f_1, f_2, f_3, 0, 0, 0, 0, 0, 0)^\top \tag{1.49}$$

and $\mathbf{H}$ is the 9×9 differential-operator matrix. The formal solution of equation (1.47) is

$$\underline{\mathbf{v}}(t) = \exp(\mathbf{H}t) \cdot \underline{\mathbf{v}}_0 + \exp(\mathbf{H}t) * \underline{\mathbf{f}}(t), \tag{1.50}$$

where $\underline{\mathbf{v}}_0$ is the initial condition. A numerical solution of equation (1.50) requires a polynomial expansion of the so-called evolution operator $\exp(\mathbf{H}t)$ in powers of $\mathbf{H}t$. This is shown in Chapter 9, where the numerical methods are presented.

It is important to distinguish between the principal axes of the material and the Cartesian axes. The principal axes – called crystal axes in crystallography – are intrinsic axes, that define the symmetry of the medium. For instance, to obtain the strain energy (1.7), we have chosen the Cartesian axes in such a way that they coincide with the three principal axes defined by the three mutually orthogonal planes of symmetry of the orthorhombic medium. The Cartesian axes may be arbitrarily oriented with respect to the principal axes. It is, therefore, necessary to analyze how the form of the elasticity matrix may be transformed for use in other coordinate systems.

The displacement vector and the strain and stress tensors transform from a system (x, y, z) to a system (x', y', z') as

$$u'_i = a_{ij} u_j, \quad \epsilon'_{ij} = a_{ik} a_{jl} \epsilon_{kl}, \quad \sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}, \tag{1.51}$$

where

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{1.52}$$

is the orthogonal transformation matrix. Orthogonality implies $\mathbf{a}^{-1} = \mathbf{a}^\top$ and $\mathbf{a} \cdot \mathbf{a}^\top = \mathbf{I}_3$; $\det(\mathbf{a}) = 1$ for rotations and $\det(\mathbf{a}) = -1$ for reflections. For instance, a clockwise rotation through an angle θ about the z-axis requires $a_{11} = a_{22} = \cos\theta$, $a_{12} = -a_{21} = \sin\theta$, $a_{33} = 1$, and $a_{13} = a_{23} = a_{31} = a_{32} = 0$. The transformations (1.51) provide the tensorial character for the respective physical quantities – first rank in the case of the displacement vector, and second rank in the case of the strain and stress tensors.

After converting the stress components to the shortened notation, each component of equation (1.51) must be analyzed individually. Using the symmetry of the stress tensor, we have

$$\boldsymbol{\sigma}' = \mathbf{M} \cdot \boldsymbol{\sigma}, \qquad (\sigma'_I = M_{IJ} \sigma_J), \tag{1.53}$$

where

$$\mathbf{M} = \begin{pmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{12}a_{13} & 2a_{13}a_{11} & 2a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{22}a_{23} & 2a_{23}a_{21} & 2a_{21}a_{22} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{32}a_{33} & 2a_{33}a_{31} & 2a_{31}a_{32} \\ a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{22}a_{33} + a_{23}a_{32} & a_{21}a_{33} + a_{23}a_{31} & a_{22}a_{31} + a_{21}a_{32} \\ a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & a_{12}a_{33} + a_{13}a_{32} & a_{13}a_{31} + a_{11}a_{33} & a_{11}a_{32} + a_{12}a_{31} \\ a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{12}a_{23} + a_{13}a_{22} & a_{13}a_{21} + a_{11}a_{23} & a_{11}a_{22} + a_{12}a_{21} \end{pmatrix} \tag{1.54}$$

(Auld, 1990a, p. 74). Due to the 1/2 factor in ϵ_{ij} (see equation (1.3)), the transformation matrix for the strain component is different from $\mathbf{M}$. We have

$$\mathbf{e}' = \mathbf{N} \cdot \mathbf{e}, \qquad (e'_I = N_{IJ} e_J), \tag{1.55}$$

where

$$\mathbf{N} = \begin{pmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & a_{12}a_{13} & a_{13}a_{11} & a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & a_{22}a_{23} & a_{23}a_{21} & a_{21}a_{22} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & a_{32}a_{33} & a_{33}a_{31} & a_{31}a_{32} \\ 2a_{21}a_{31} & 2a_{22}a_{32} & 2a_{23}a_{33} & a_{22}a_{33} + a_{23}a_{32} & a_{21}a_{33} + a_{23}a_{31} & a_{22}a_{31} + a_{21}a_{32} \\ 2a_{31}a_{11} & 2a_{32}a_{12} & 2a_{33}a_{13} & a_{12}a_{33} + a_{13}a_{32} & a_{13}a_{31} + a_{11}a_{33} & a_{11}a_{32} + a_{12}a_{31} \\ 2a_{11}a_{21} & 2a_{12}a_{22} & 2a_{13}a_{23} & a_{12}a_{23} + a_{13}a_{22} & a_{13}a_{21} + a_{11}a_{23} & a_{11}a_{22} + a_{12}a_{21} \end{pmatrix} \tag{1.56}$$

(Auld, 1990a, p. 75). Matrices $\mathbf{M}$ and $\mathbf{N}$ are called Bond matrices after W. L. Bond who developed the approach from which they are obtained.

Let us now find the transformation law for the elasticity tensor from one system to the other. From equations (1.31), (1.53) and (1.55), we have

$$\boldsymbol{\sigma}' = \mathbf{C}' \cdot \mathbf{e}', \qquad \mathbf{C}' = \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{N}^{-1}. \tag{1.57}$$

Because matrix $\mathbf{a}$ in (1.52) is orthogonal, the matrix $\mathbf{N}^{-1}$ can be found by transposing all subscripts in equation (1.56). The result is simply $\mathbf{M}^\top$, and (1.57) becomes

$$\mathbf{C}' = \mathbf{M} \cdot \mathbf{C} \cdot \mathbf{M}^\top. \tag{1.58}$$

Transformation of the stiffness matrix

In the current seismic terminology, a transversely isotropic medium refers to a medium represented by the elasticity matrix (1.39), with the symmetry axis along the vertical direction, i.e., the z-axis. By performing appropriate rotations of the coordinate system, the medium may become azimuthally anisotropic (e.g., Thomsen, 1988). An example is a transversely isotropic medium whose symmetry axis is horizontal and makes an angle θ with the x-axis. To obtain this medium, we perform a clockwise rotation by $\pi/2$ about the y-axis followed by a counterclockwise rotation by θ about the new z-axis. The corresponding rotation matrix is given by

$$\mathbf{a} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.59}$$

The corresponding Bond transformation matrix is

$$\mathbf{M} = \begin{pmatrix} 0 & \sin^2\theta & \cos^2\theta & \sin(2\theta) & 0 & 0 \\ 0 & \cos^2\theta & \sin^2\theta & -\sin(2\theta) & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin\theta & \cos\theta \\ 0 & 0 & 0 & 0 & -\cos\theta & -\sin\theta \\ 0 & -\frac{1}{2}\sin(2\theta) & \frac{1}{2}\sin(2\theta) & -\cos(2\theta) & 0 & 0 \end{pmatrix}. \tag{1.60}$$

Using (1.58), we note that the elasticity constants in the new system are

$$
\begin{aligned}
c'_{11} &= c_{33}\cos^4\theta + \tfrac{1}{2}(c_{13}+2c_{55})\sin^2(2\theta) + c_{11}\sin^4\theta \\
c'_{12} &= \tfrac{1}{8}[c_{11}+6c_{13}+c_{33}-4c_{55}-(c_{11}-2c_{13}+c_{33}-4c_{55})\cos(4\theta)] \\
c'_{13} &= c_{13}\cos^2\theta + c_{12}\sin^2\theta \\
c'_{16} &= \tfrac{1}{4}[-c_{11}+c_{33}+(c_{11}-2c_{13}+c_{33}-4c_{55})\cos(2\theta)]\sin(2\theta) \\
c'_{22} &= c_{11}\cos^4\theta + \tfrac{1}{2}(c_{13}+2c_{55})\sin^2(2\theta) + c_{33}\sin^4\theta \\
c'_{23} &= c_{12}\cos^2\theta + c_{13}\sin^2\theta \\
c'_{26} &= \tfrac{1}{4}[-c_{11}+c_{33}-(c_{11}-2c_{13}+c_{33}-4c_{55})\cos(2\theta)]\sin(2\theta) \\
c'_{33} &= c_{11} \\
c'_{36} &= \tfrac{1}{2}(c_{13}-c_{12})\sin(2\theta) \\
c'_{44} &= \tfrac{1}{2}(c_{11}-c_{12})\cos^2\theta + c_{55}\sin^2\theta \\
c'_{45} &= \tfrac{1}{4}(-c_{11}+c_{12}+2c_{55})\sin(2\theta) \\
c'_{55} &= \tfrac{1}{2}(c_{11}-c_{12})\sin^2\theta + c_{55}\cos^2\theta \\
c'_{66} &= \tfrac{1}{8}[c_{11}-2c_{13}+c_{33}+4c_{55}-(c_{11}-2c_{13}+c_{33}-4c_{55})\cos(4\theta)]
\end{aligned}
\tag{1.61}
$$

and the other components are equal to zero.

1.3 Kelvin-Christoffel equation, phase velocity and slowness

A plane-wave analysis yields the Kelvin-Christoffel equations and the expressions for the phase velocity and slowness of the different wave modes. A general plane-wave solution for the displacement vector of body waves is

$$\mathbf{u} = \mathbf{u}_0 \exp[\mathrm{i}(\omega t - \boldsymbol{\kappa}\cdot\mathbf{x})], \tag{1.62}$$

where $\mathbf{u}_0$ represents a constant complex vector, ω is the angular frequency and $\boldsymbol{\kappa}$ is the wavenumber vector or wavevector. We recall that when using complex notation for plane waves, the field variables are obtained as the real part of the corresponding wave fields. The particle velocity is given by

$$\mathbf{v} = \partial_t \mathbf{u} = \mathrm{i}\omega\mathbf{u}. \tag{1.63}$$

In the absence of body forces ($\mathbf{f} = 0$), we consider plane waves propagating along the direction

$$\hat{\boldsymbol{\kappa}} = l_1\hat{\mathbf{e}}_1 + l_2\hat{\mathbf{e}}_2 + l_3\hat{\mathbf{e}}_3, \tag{1.64}$$

(or (l_1, l_2, l_3)), where l_1, l_2 and l_3 are the direction cosines. We have

$$\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3) = \kappa(l_1, l_2, l_3) = \kappa\hat{\boldsymbol{\kappa}}, \tag{1.65}$$

where κ is the magnitude of the wavevector. In this case, the time derivative and the spatial differential operator (1.25) can be replaced by

$$\partial_t \to \mathrm{i}\omega \tag{1.66}$$

and

$$\nabla \to -\mathrm{i}\kappa \begin{pmatrix} l_1 & 0 & 0 & 0 & l_3 & l_2 \\ 0 & l_2 & 0 & l_3 & 0 & l_1 \\ 0 & 0 & l_3 & l_2 & l_1 & 0 \end{pmatrix} \equiv -\mathrm{i}\kappa\mathbf{L}, \tag{1.67}$$

respectively.

Substitution of these operators into the equation of motion (1.35) yields

$$\kappa^2 \mathbf{\Gamma} \cdot \mathbf{u} = \rho\omega^2 \mathbf{u}, \qquad (\kappa^2 \Gamma_{ij} u_j = \rho\omega^2 u_i), \tag{1.68}$$

where

$$\mathbf{\Gamma} = \mathbf{L} \cdot \mathbf{C} \cdot \mathbf{L}^\top, \qquad (\Gamma_{ij} = l_{iI} c_{IJ} l_{Jj}) \tag{1.69}$$

is the symmetric Kelvin-Christoffel matrix . Defining the phase-velocity vector as

$$\mathbf{v}_p = v_p \hat{\boldsymbol{\kappa}}, \qquad v_p = \frac{\omega}{\kappa}, \tag{1.70}$$

we find that equation (1.68) becomes an "eigenequation" (the Kelvin-Christoffel equation),

$$(\mathbf{\Gamma} - \rho v_p^2 \mathbf{I}_3) \cdot \mathbf{u} = 0 \tag{1.71}$$

for the eigenvalues $(\rho v_p^2)_m$ and eigenvectors $(\mathbf{u})_m$, $m = 1, 2, 3$. The dispersion relation is given by

$$\det(\mathbf{\Gamma} - \rho v_p^2 \mathbf{I}_3) = 0. \tag{1.72}$$

In explicit form, the components of the Kelvin-Christoffel matrix are

$$\begin{aligned}
\Gamma_{11} &= c_{11} l_1^2 + c_{66} l_2^2 + c_{55} l_3^2 + 2c_{56} l_2 l_3 + 2c_{15} l_3 l_1 + 2c_{16} l_1 l_2 \\
\Gamma_{22} &= c_{66} l_1^2 + c_{22} l_2^2 + c_{44} l_3^2 + 2c_{24} l_2 l_3 + 2c_{46} l_3 l_1 + 2c_{26} l_1 l_2 \\
\Gamma_{33} &= c_{55} l_1^2 + c_{44} l_2^2 + c_{33} l_3^2 + 2c_{34} l_2 l_3 + 2c_{35} l_3 l_1 + 2c_{45} l_1 l_2 \\
\Gamma_{12} &= c_{16} l_1^2 + c_{26} l_2^2 + c_{45} l_3^2 + (c_{46} + c_{25}) l_2 l_3 + (c_{14} + c_{56}) l_3 l_1 + (c_{12} + c_{66}) l_1 l_2 \\
\Gamma_{13} &= c_{15} l_1^2 + c_{46} l_2^2 + c_{35} l_3^2 + (c_{45} + c_{36}) l_2 l_3 + (c_{13} + c_{55}) l_3 l_1 + (c_{14} + c_{56}) l_1 l_2 \\
\Gamma_{23} &= c_{56} l_1^2 + c_{24} l_2^2 + c_{34} l_3^2 + (c_{44} + c_{23}) l_2 l_3 + (c_{36} + c_{45}) l_3 l_1 + (c_{25} + c_{46}) l_1 l_2 .
\end{aligned} \tag{1.73}$$

The three solutions obtained from considering $m = 1, 2, 3$ correspond to the three body waves propagating in an unbounded homogeneous medium. At a given frequency ω, $v_p(l_1, l_2, l_3)$ defines a surface in the wavenumber space as a function of the direction cosines. The slowness is defined as the inverse of the phase velocity, namely as

$$s = \frac{\kappa}{\omega} = \frac{1}{v_p}. \tag{1.74}$$

Similarly, we can define the slowness surface $s(l_1, l_2, l_3)$. The slowness vector is closely related to the wavevector by the expression

$$\mathbf{s} = \frac{\boldsymbol{\kappa}}{\omega} = s\hat{\boldsymbol{\kappa}}. \tag{1.75}$$

1.3.1 Transversely isotropic media

Let us consider wave propagation in a plane containing the symmetry axis (z-axis) of a transversely isotropic medium. This problem illustrates the effects of anisotropy on the velocity and polarization of the body waves. For propagation in the (x, z)-plane, $l_2 = 0$ and equation (1.71) reduces to

$$\begin{pmatrix} \Gamma_{11} - \rho v_p^2 & 0 & \Gamma_{13} \\ 0 & \Gamma_{22} - \rho v_p^2 & 0 \\ \Gamma_{13} & 0 & \Gamma_{33} - \rho v_p^2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0, \tag{1.76}$$

or

$$\begin{pmatrix} c_{11}l_1^2 + c_{55}l_3^2 - \rho v_p^2 & 0 & (c_{13}+c_{55})l_1l_3 \\ 0 & c_{66}l_1^2 + c_{55}l_3^2 - \rho v_p^2 & 0 \\ (c_{13}+c_{55})l_1l_3 & 0 & c_{33}l_3^2 + c_{55}l_1^2 - \rho v_p^2 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0. \quad (1.77)$$

We obtain two uncoupled dispersion relations,

$$\begin{array}{l} c_{66}l_1^2 + c_{55}l_3^2 - \rho v_p^2 = 0 \\ (c_{11}l_1^2 + c_{55}l_3^2 - \rho v_p^2)(c_{33}l_3^2 + c_{55}l_1^2 - \rho v_p^2) - (c_{13}+c_{55})^2 l_1^2 l_3^2 = 0, \end{array} \quad (1.78)$$

giving the phase velocities

$$\begin{array}{l} v_{p1} = \sqrt{(\rho)^{-1}(c_{66}l_1^2 + c_{55}l_3^2)} \\ v_{p2} = (2\rho)^{-1/2}\sqrt{c_{11}l_1^2 + c_{33}l_3^2 + c_{55} - C} \\ v_{p3} = (2\rho)^{-1/2}\sqrt{c_{11}l_1^2 + c_{33}l_3^2 + c_{55} + C} \\ C = \sqrt{[(c_{11}-c_{55})l_1^2 + (c_{55}-c_{33})l_3^2]^2 + 4[(c_{13}+c_{55})l_1l_3]^2}. \end{array} \quad (1.79)$$

From equation (1.77), we see that the first solution has a displacement (or polarization) given by $(0, u_2, 0)$, which is normal to the (x, z)-plane of propagation. Therefore, this solution describes a pure shear wave – termed SH wave in the geophysical literature – with H denoting horizontal polarization if the z-axis is oriented in the vertical direction. Note that the dispersion relation $(1.78)_1$ can be written as

$$\frac{s_1^2}{\rho/c_{66}} + \frac{s_3^2}{\rho/c_{55}} = 1, \quad (1.80)$$

where $s_1 = s^{(1)}l_1$ and $s_3 = s^{(1)}l_3$, with $s^{(1)} = 1/v_{p1}$. Hence, the slowness surface is an ellipse, with semiaxes ρ/c_{66} and ρ/c_{55} along the x- and z-directions, respectively.

For the coupled waves, the normalized polarizations are obtained from equation (1.76) by using the dispersion relation $(1.78)_2$. Hence, we obtain

$$\begin{pmatrix} u_1 \\ u_3 \end{pmatrix} = \frac{1}{\sqrt{\Gamma_{11} + \Gamma_{33} - 2\rho v_p^2}} \begin{pmatrix} \sqrt{\Gamma_{33} - \rho v_p^2} \\ \sqrt{\Gamma_{11} - \rho v_p^2} \end{pmatrix}. \quad (1.81)$$

Using the fact that $l_1^2 + l_3^2 = 1$, as well as equations (1.79) and (1.81), we can identify the wave modes along the x and z axes, which may be written as

$$\begin{array}{llll} x-\text{axis } (l_1 = 1), & \rho v_{p2}^2 = c_{55}, & u_1 = 0, & \rightarrow \text{S wave} \\ x-\text{axis } (l_1 = 1), & \rho v_{p3}^2 = c_{11}, & u_3 = 0, & \rightarrow \text{P wave} \\ z-\text{axis } (l_1 = 0), & \rho v_{p2}^2 = c_{55}, & u_3 = 0, & \rightarrow \text{S wave} \\ z-\text{axis } (l_1 = 0), & \rho v_{p3}^2 = c_{33}, & u_1 = 0, & \rightarrow \text{P wave.} \end{array} \quad (1.82)$$

These expressions denote pure mode directions for which the polarization of the P wave coincides with the wavevector direction and the S-wave polarization is normal to this direction. There exists another pure mode direction defined by

$$\tan^2\theta = \frac{c_{33} - 2c_{55} - c_{13}}{c_{11} - 2c_{55} - c_{13}}, \qquad \theta = \arcsin(l_1) \quad (1.83)$$

(Brugger, 1965), which extends azimuthally about the z-axis. The polarizations along the other directions are not parallel or perpendicular to the propagation directions, and, therefore, the waves are termed quasi P and quasi S . The latter is usually called the qSV wave, with V denoting the vertical plane if the z-axis is oriented in the vertical direction. The (x, y)-plane of a transversely isotropic medium is a plane of isotropy, where the velocity of the SV wave is $\sqrt{c_{55}/\rho}$ and the velocity of the SH wave is $\sqrt{c_{66}/\rho}$. The velocity of the compressional wave is $\sqrt{c_{11}/\rho}$.

1.3.2 Symmetry planes of an orthorhombic medium

In the symmetry planes of an orthorhombic medium, the physics of wave propagation is similar to the previous case, i.e., there is a pure shear wave (labeled 1 below) and two coupled waves.

The respective slowness surfaces are:

(x, y)-plane $\quad (l_1 = \sin\theta,\ l_2 = \cos\theta)$:

$$\begin{aligned} & c_{55}l_1^2 + c_{44}l_2^2 - \rho v_p^2 = 0 \\ & (c_{11}l_1^2 + c_{66}l_2^2 - \rho v_p^2)(c_{22}l_2^2 + c_{66}l_1^2 - \rho v_p^2) - (c_{12} + c_{66})^2 l_1^2 l_2^2 = 0; \end{aligned} \tag{1.84}$$

(x, z)-plane $\quad (l_1 = \sin\theta,\ l_3 = \cos\theta)$:

$$\begin{aligned} & c_{66}l_1^2 + c_{44}l_3^2 - \rho v_p^2 = 0 \\ & (c_{11}l_1^2 + c_{55}l_3^2 - \rho v_p^2)(c_{33}l_3^2 + c_{55}l_1^2 - \rho v_p^2) - (c_{13} + c_{55})^2 l_1^2 l_3^2 = 0; \end{aligned} \tag{1.85}$$

(y, z)-plane $\quad (l_2 = \sin\theta,\ l_3 = \cos\theta)$:

$$\begin{aligned} & c_{66}l_2^2 + c_{55}l_3^2 - \rho v_p^2 = 0 \\ & (c_{22}l_2^2 + c_{44}l_3^2 - \rho v_p^2)(c_{33}l_3^2 + c_{44}l_2^2 - \rho v_p^2) - (c_{23} + c_{44})^2 l_2^2 l_3^2 = 0. \end{aligned} \tag{1.86}$$

The corresponding phase velocities are:

(x, y)-plane:

$$\begin{aligned} v_{p1} &= \sqrt{(\rho)^{-1}(c_{55}l_1^2 + c_{44}l_2^2)} \\ v_{p2} &= (2\rho)^{-1/2}\sqrt{c_{11}l_1^2 + c_{22}l_2^2 + c_{66} - C} \\ v_{p3} &= (2\rho)^{-1/2}\sqrt{c_{11}l_1^2 + c_{22}l_2^2 + c_{66} + C} \\ C &= \sqrt{[(c_{22} - c_{66})l_2^2 - (c_{11} - c_{66})l_1^2]^2 + 4[(c_{12} + c_{66})l_1 l_2]^2}; \end{aligned} \tag{1.87}$$

(x, z)-plane:

$$\begin{aligned} v_{p1} &= \sqrt{(\rho)^{-1}(c_{66}l_1^2 + c_{44}l_3^2)} \\ v_{p2} &= (2\rho)^{-1/2}\sqrt{c_{11}l_1^2 + c_{33}l_3^2 + c_{55} - C} \\ v_{p3} &= (2\rho)^{-1/2}\sqrt{c_{11}l_1^2 + c_{33}l_3^2 + c_{55} + C} \\ C &= \sqrt{[(c_{33} - c_{55})l_3^2 - (c_{11} - c_{55})l_1^2]^2 + 4[(c_{13} + c_{55})l_1 l_3]^2}; \end{aligned} \tag{1.88}$$

(y, z)-plane:

$$\begin{aligned} v_{p1} &= \sqrt{(\rho)^{-1}(c_{66}l_2^2 + c_{55}l_3^2)} \\ v_{p2} &= (2\rho)^{-1/2}\sqrt{c_{22}l_2^2 + c_{33}l_3^2 + c_{44} - C} \\ v_{p3} &= (2\rho)^{-1/2}\sqrt{c_{22}l_2^2 + c_{33}l_3^2 + c_{44} + C} \\ C &= \sqrt{[(c_{33} - c_{44})l_3^2 - (c_{22} - c_{44})l_2^2]^2 + 4[(c_{23} + c_{44})l_2 l_3]^2}. \end{aligned} \tag{1.89}$$

Angle θ is measured from the y-axis in the (x, y)-plane, and from the z-axis in the (x, z)- and (y, z)-planes.

The velocities along the principal axes are:

(x, y)-plane:

$$\begin{aligned}
v_{p1}(0^o) &= v_{p_S}(0^o) = \sqrt{c_{44}/\rho} \\
v_{p1}(90^o) &= v_{p_S}(90^o) = \sqrt{c_{55}/\rho} \\
v_{p2}(0^o) &= v_{p_{qS}}(0^o) = \sqrt{c_{66}/\rho} \\
v_{p2}(90^o) &= v_{p_{qS}}(90^o) = \sqrt{c_{66}/\rho} \\
v_{p3}(0^o) &= v_{p_{qP}}(0^o) = \sqrt{c_{22}/\rho} \\
v_{p3}(90^o) &= v_{p_{qP}}(90^o) = \sqrt{c_{11}/\rho};
\end{aligned} \tag{1.90}$$

(x, z)-plane:

$$\begin{aligned}
v_{p1}(0^o) &= v_{p_S}(0^o) = \sqrt{c_{44}/\rho} \\
v_{p1}(90^o) &= v_{p_S}(90^o) = \sqrt{c_{66}/\rho} \\
v_{p2}(0^o) &= v_{p_{qS}}(0^o) = \sqrt{c_{55}/\rho} \\
v_{p2}(90^o) &= v_{p_{qS}}(90^o) = \sqrt{c_{55}/\rho} \\
v_{p3}(0^o) &= v_{p_{qP}}(0^o) = \sqrt{c_{33}/\rho} \\
v_{p3}(90^o) &= v_{p_{qP}}(90^o) = \sqrt{c_{11}/\rho};
\end{aligned} \tag{1.91}$$

(y, z)-plane:

$$\begin{aligned}
v_{p1}(0^o) &= v_{p_S}(0^o) = \sqrt{c_{66}/\rho} \\
v_{p1}(90^o) &= v_{p_S}(90^o) = \sqrt{c_{55}/\rho} \\
v_{p2}(0^o) &= v_{p_{qS}}(0^o) = \sqrt{c_{44}/\rho} \\
v_{p2}(90^o) &= v_{p_{qS}}(90^o) = \sqrt{c_{44}/\rho} \\
v_{p3}(0^o) &= v_{p_{qP}}(0^o) = \sqrt{c_{33}/\rho} \\
v_{p3}(90^o) &= v_{p_{qP}}(90^o) = \sqrt{c_{22}/\rho}.
\end{aligned} \tag{1.92}$$

1.3.3 Orthogonality of polarizations

In order to determine if the polarizations of the waves are orthogonal, we consider two solutions "a" and "b" of the eigensystem (1.71)

$$\mathbf{\Gamma} \cdot \mathbf{u}_a = \rho v_{pa}^2 \mathbf{u}_a, \qquad \mathbf{\Gamma} \cdot \mathbf{u}_b = \rho v_{pb}^2 \mathbf{u}_b, \tag{1.93}$$

and take the scalar product from the left-hand side with the displacements $\mathbf{u}_b$ and $\mathbf{u}_a$, respectively,

$$\mathbf{u}_b \cdot \mathbf{\Gamma} \cdot \mathbf{u}_a = \rho v_{pa}^2 \mathbf{u}_b \cdot \mathbf{u}_a, \qquad \mathbf{u}_a \cdot \mathbf{\Gamma} \cdot \mathbf{u}_b = \rho v_{pb}^2 \mathbf{u}_a \cdot \mathbf{u}_b. \tag{1.94}$$

Since $\mathbf{\Gamma}$ is symmetric, we have $\mathbf{u}_b \cdot \mathbf{\Gamma} \cdot \mathbf{u}_a = \mathbf{u}_a \cdot \mathbf{\Gamma} \cdot \mathbf{u}_b$. Subtracting one equation from the other, we get

$$\rho(v_{pa}^2 - v_{pb}^2)\mathbf{u}_b \cdot \mathbf{u}_a = 0. \tag{1.95}$$

If the phase velocities are different, we have $\mathbf{u}_b \cdot \mathbf{u}_a = 0$ and the polarizations are orthogonal. Note that this property is a consequence of the symmetry of the Kelvin-Christoffel matrix.

1.4 Energy balance and energy velocity

Energy-balance equations are important for characterizing the energy stored and the transport properties in a field. In particular, the concept of energy velocity is useful in determining how the energy transferred by the wave field is related to the strength of the field, i.e., the location of the wave front. Although, in lossless media, this velocity can be obtained from "kinematic" considerations – we shall see that the group and the energy velocities are the same – an analysis of this media provides a basis to study more complex situations, such as wave propagation in anelastic and porous media.

The equation of motion (1.28) corresponding to the plane wave (1.62) is

$$-\mathrm{i}\kappa \mathbf{L} \cdot \boldsymbol{\sigma} = \mathrm{i}\omega\rho\mathbf{v}, \tag{1.96}$$

where we assumed no body forces and used equations (1.66) and (1.67). The scalar product of $-\mathbf{v}^*$ and equation (1.96) is

$$-\kappa \mathbf{v}^* \cdot \mathbf{L} \cdot \boldsymbol{\sigma} = \omega\rho\mathbf{v}^* \cdot \mathbf{v}. \tag{1.97}$$

Moreover, the strain-displacement relation (1.26) is replaced by

$$\omega \mathbf{e} = -\kappa \mathbf{L}^\top \cdot \mathbf{v}. \tag{1.98}$$

The scalar product of the complex conjugate of equation (1.98) and $\boldsymbol{\sigma}^\top$ gives

$$-\kappa \boldsymbol{\sigma}^\top \cdot \mathbf{L}^\top \cdot \mathbf{v}^* = \omega \boldsymbol{\sigma}^\top \cdot \mathbf{e}^*. \tag{1.99}$$

The left-hand sides of equations (1.97) and (1.99) coincide and can be written in terms of the Umov-Poynting vector[2] (or power-flow vector)

$$\mathbf{p} = -\frac{1}{2}\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \cdot \mathbf{v}^* \tag{1.100}$$

as

$$2\boldsymbol{\kappa} \cdot \mathbf{p} = \omega\rho\mathbf{v}^* \cdot \mathbf{v} \tag{1.101}$$

and

$$2\boldsymbol{\kappa} \cdot \mathbf{p} = \omega \boldsymbol{\sigma}^\top \cdot \mathbf{e}^*. \tag{1.102}$$

Adding equations (1.101) and (1.102), we get

$$4\boldsymbol{\kappa} \cdot \mathbf{p} = \omega(\rho\mathbf{v}^* \cdot \mathbf{v} + \boldsymbol{\sigma}^\top \cdot \mathbf{e}^*), \tag{1.103}$$

or, using the stress-strain relation (1.31) and the symmetry of $\mathbf{C}$, we obtain

$$4\boldsymbol{\kappa} \cdot \mathbf{p} = \omega(\rho\mathbf{v}^* \cdot \mathbf{v} + \mathbf{e}^\top \cdot \mathbf{C} \cdot \mathbf{e}^*). \tag{1.104}$$

For generic field variables $\mathbf{a}$ and $\mathbf{b}$, and a symmetric matrix $\mathbf{D}$, the time average over a cycle of period $2\pi/\omega$ has the following properties:

$$\langle \mathrm{Re}(\mathbf{a}^\top) \cdot \mathrm{Re}(\mathbf{b}) \rangle = \frac{1}{2}\mathrm{Re}(\mathbf{a}^\top \cdot \mathbf{b}^*) \tag{1.105}$$

[2]Vector of the density of energy flux introduced independently by N. Umov in 1874 and J. Poynting in 1884 (Alekseev, 1986).

(Booker, 1992), and

$$\begin{aligned} \langle \mathrm{Re}(\mathbf{a}^\top) \cdot \mathrm{Re}(\mathbf{D}) \cdot \mathrm{Re}(\mathbf{a}) \rangle &= \tfrac{1}{2}\mathrm{Re}(\mathbf{a}^\top \cdot \mathbf{D} \cdot \mathbf{a}^*), \\ \langle \mathrm{Re}(\mathbf{a}^\top) \cdot \mathrm{Im}(\mathbf{D}) \cdot \mathrm{Re}(\mathbf{a}) \rangle &= \tfrac{1}{2}\mathrm{Im}(\mathbf{a}^\top \cdot \mathbf{D} \cdot \mathbf{a}^*) \end{aligned} \tag{1.106}$$

(Carcione and Cavallini, 1993). Using equation (1.105), we obtain the time average of the real Umov-Poynting vector (1.100), namely

$$-\mathrm{Re}(\mathbf{\Sigma}) \cdot \mathrm{Re}(\mathbf{v}), \tag{1.107}$$

where

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}, \tag{1.108}$$

is

$$\langle \mathbf{p} \rangle = \mathrm{Re}(\mathbf{p}), \tag{1.109}$$

which represents the magnitude and direction of the time-averaged power flow.

We identify, in equation (1.104), the time averages of the kinetic- and strain-energy densities, namely,

$$\langle T \rangle = \frac{1}{2}\langle \mathrm{Re}(\mathbf{v}) \cdot \mathrm{Re}(\mathbf{v}) \rangle = \frac{1}{4}\mathrm{Re}(\mathbf{v}^* \cdot \mathbf{v}) \tag{1.110}$$

and

$$\langle V \rangle = \frac{1}{2}\langle \mathrm{Re}(\mathbf{e}^\top) \cdot \mathbf{C} \cdot \mathrm{Re}(\mathbf{e}) \rangle = \frac{1}{4}\mathrm{Re}(\mathbf{e}^\top \cdot \mathbf{C} \cdot \mathbf{e}^*). \tag{1.111}$$

The substitution of equations (1.110) and (1.111) into the real part of equation (1.104) yields the energy-balance equation

$$\boldsymbol{\kappa} \cdot \langle \mathbf{p} \rangle = \omega(\langle T \rangle + \langle V \rangle) = \omega(\langle T + V \rangle) = \omega\langle E \rangle, \tag{1.112}$$

where $\langle E \rangle$ is the time-averaged energy density.

The wave surface is the locus of the end of the energy-velocity vector multiplied by one unit of propagation time, with the energy-velocity vector defined as the ratio of the time-averaged power-flow vector $\langle \mathbf{p} \rangle$ to the total energy density $\langle E \rangle$. Because this is equal to the sum of the time-averaged kinetic- and strain-energy densities $\langle T \rangle$ and $\langle V \rangle$, the energy-velocity vector is

$$\mathbf{v}_e = \frac{\langle \mathbf{p} \rangle}{\langle E \rangle} = \frac{\langle \mathbf{p} \rangle}{\langle T + V \rangle}. \tag{1.113}$$

Using this definition, we note that equation (1.112) gives

$$\hat{\boldsymbol{\kappa}} \cdot \mathbf{v}_e = v_p, \qquad (\mathbf{s} \cdot \mathbf{v}_e = 1), \tag{1.114}$$

where v_p and $\mathbf{s}$ are the phase velocity and slowness vector defined in equations (1.70) and (1.75), respectively. Relation (1.114) means that the phase velocity is equal to the projection of the energy velocity onto the propagation direction. The wave front is associated with the higher energy velocity. Since, in the elastic case, all the wave surfaces have the same velocity – there is no velocity dispersion – the concepts of wave front and wave surface are the same. In anelastic media, the wave front is the wave surface associated with the unrelaxed energy velocity.

Equation (1.113) allows further simplifications. Let us calculate the time averages of the kinetic and strain energies explicitly. The substitution of equation (1.62) into equation (1.110) yields

$$\langle T \rangle = \frac{1}{4}\rho\omega^2 |\mathbf{u}_0|^2. \tag{1.115}$$

From equations (1.26) and (1.67), we have

$$\mathbf{e} = -\mathrm{i}\kappa \mathbf{L}^\top \cdot \mathbf{u}, \tag{1.116}$$

which implies

$$\mathbf{e}^\top \cdot \mathbf{C} \cdot \mathbf{e}^* = \kappa^2 \mathbf{u} \cdot \mathbf{L} \cdot \mathbf{C} \cdot \mathbf{L}^\top \cdot \mathbf{u}^* = \kappa^2 \mathbf{u} \cdot \mathbf{\Gamma} \cdot \mathbf{u}^*, \tag{1.117}$$

where we have used equation (1.69). In view of the complex conjugate of equation (1.68), equation (1.117) can be written as

$$\mathbf{e}^\top \cdot \mathbf{C} \cdot \mathbf{e}^* = \rho\omega^2 \mathbf{u} \cdot \mathbf{u}^* = \rho\omega^2 |\mathbf{u}_0|^2. \tag{1.118}$$

Using this relation, we find that the time-averaged strain-energy density (1.111) becomes

$$\langle V \rangle = \frac{1}{4}\rho\omega^2 |\mathbf{u}_0|^2 = \langle T \rangle. \tag{1.119}$$

Hence, in elastic media, the time averages of the strain- and kinetic-energy densities are equal and the energy-velocity vector (1.113) can be simplified to

$$\mathbf{v}_e = \frac{\langle \mathbf{p} \rangle}{2\langle T \rangle}. \tag{1.120}$$

It can be shown that for a traveling wave, whose argument is $t - \mathbf{s} \cdot \mathbf{x}$ – the plane wave (1.62) is a particular case – the instantaneous kinetic- and strain-energy densities are the same. On the other hand, an exchange of kinetic and potential energies occurs in forced oscillators (exercise left to the reader).

1.4.1 Group velocity

A wave packet can be seen as a superposition of harmonic components. In general, each component may travel with a different phase velocity. This is not the case in homogeneous elastic media, since the phase velocity is frequency independent (see, for instance, the transversely isotropic case, equation (1.79)). Following the superposition principle, the wave packet propagates with the same velocity as each harmonic component. However, the relation between the group and the energy velocities, as well as the velocity of propagation of the pulse as a function of the propagation direction, merits careful consideration.

Let us consider two harmonic components "a" and "b" given by

$$u = u_0[\cos(\omega_a t - \kappa_a x) + \cos(\omega_b t - \kappa_b x)], \tag{1.121}$$

and assume that the frequencies are slightly different

$$\omega_b = \omega_a + \delta\omega, \qquad \kappa_b = \kappa_a + \delta\kappa. \tag{1.122}$$

Equation (1.121) can then be written as

$$u = 2u_0 \cos\left[\frac{1}{2}(\delta\omega\ t - \delta k\ x)\right]\cos(\bar{\omega}t - \bar{\kappa}x), \tag{1.123}$$

where

$$\bar{\kappa} = \frac{1}{2}(\kappa_a + \kappa_b), \qquad \bar{\omega} = \frac{1}{2}(\omega_a + \omega_b). \tag{1.124}$$

The first term in equation (1.123) is the modulation envelope and the second term is the carrier wave, which has a phase velocity equal to $\bar{\omega}/\bar{\kappa}$. The velocity of the modulation wave is equal to $\delta\omega/\delta\kappa$, which, by taking the limit $\bar{\kappa} \to 0$, gives the group velocity

$$v_g = \frac{\partial\omega}{\partial\kappa}. \tag{1.125}$$

Generalizing this equation to the 3-D case, we obtain the group-velocity vector

$$\mathbf{v}_g = \frac{\partial\omega}{\partial\kappa_1}\hat{\mathbf{e}}_1 + \frac{\partial\omega}{\partial\kappa_2}\hat{\mathbf{e}}_2 + \frac{\partial\omega}{\partial\kappa_3}\hat{\mathbf{e}}_3, \qquad \left(v_{gi} = \frac{\partial\omega}{\partial\kappa_i}\right). \tag{1.126}$$

(Lighthill, 1964; 1978, p. 312)

In general, the dispersion relation $\omega = \omega(\kappa_i)$ is not available in explicit form. For instance, using equations (1.65) and (1.70), we note that equation $(1.78)_2$ has the form

$$(c_{11}\kappa_1^2 + c_{55}\kappa_3^2 - \rho\omega^2)(c_{33}\kappa_3^2 + c_{55}\kappa_1^2 - \rho\omega^2) - (c_{13} + c_{55})^2\kappa_1^2\kappa_3^2 = 0. \tag{1.127}$$

In general, we have, from (1.71),

$$\det(\kappa^2\mathbf{\Gamma} - \rho\omega^2\mathbf{I}_3) \equiv F(\omega, \kappa_i) = 0. \tag{1.128}$$

Using implicit differentiation, we have for each component

$$\frac{\partial F}{\partial\omega}\delta\omega + \frac{\partial F}{\partial\kappa_i}\delta\kappa_i = 0, \tag{1.129}$$

which is obtained by keeping the other components constant. Thus, the final expression of the group velocity is

$$\mathbf{v}_g = -\left(\frac{\partial F}{\partial\omega}\right)^{-1}\left(\frac{\partial F}{\partial\kappa_1}\hat{\mathbf{e}}_1 + \frac{\partial F}{\partial\kappa_2}\hat{\mathbf{e}}_2 + \frac{\partial F}{\partial\kappa_3}\hat{\mathbf{e}}_3\right) \equiv -\left(\frac{\partial F}{\partial\omega}\right)^{-1}\nabla_{\boldsymbol{\kappa}}F,$$

$$\left[\mathbf{v}_g = -\left(\frac{\partial F}{\partial\omega}\right)^{-1}\left(\frac{\partial F}{\partial\kappa_i}\hat{\mathbf{e}}_i\right)\right]. \tag{1.130}$$

1.4.2 Equivalence between the group and energy velocities

In order to find the relation between the group and energy velocities, we use Cartesian notation. Rewriting the Kelvin-Christoffel matrix (1.69) in terms of this notation, we get

$$\Gamma_{ij} = c_{ijkl}l_kl_l. \tag{1.131}$$

We have, from equation (1.68), after using (1.62) and (1.65),

$$\rho\omega^2 u_{0i} = c_{ijkl}\kappa_j\kappa_k u_{0l}. \tag{1.132}$$

Differentiating this equation with respect to κ_j, we obtain

$$2\rho\omega\frac{\partial\omega}{\partial\kappa_j}u_{0i} = 2c_{ijkl}\kappa_k u_{0l}, \tag{1.133}$$

since $\partial(\kappa_j\kappa_k)/\partial\kappa_j = \kappa_k + \kappa_j\delta_{jk} = 2\kappa_k$. Taking the scalar product of equation (1.133) and u_{0i}^*, and using the definition of group velocity (1.126), we obtain

$$v_{gj} = \frac{\partial\omega}{\partial\kappa_j} = \frac{c_{ijkl}\kappa_k u_{0l}u_{0i}^*}{\rho\omega|\mathbf{u}_0|^2}. \tag{1.134}$$

On the other hand, the Cartesian components of the complex power-flow vector (1.100) can be expressed as

$$p_j = -\frac{1}{2}\sigma_{ji}v_i^*. \tag{1.135}$$

Using the stress-strain relation (1.21) and $v_i^* = -\mathrm{i}\omega u_i^*$, we have

$$\sigma_{ji}v_i^* = -\mathrm{i}\omega c_{jikl}\epsilon_{kl}u_i^*. \tag{1.136}$$

The strain-displacement relations (1.2) and (1.3) imply

$$\sigma_{ji}v_i^* = -\frac{1}{2}\mathrm{i}\omega c_{jikl}(\partial_l u_k + \partial_k u_l)u_i^* = -\frac{1}{2}\omega c_{jikl}(\kappa_l u_k + \kappa_k u_l)u_i^*, \tag{1.137}$$

where we have used the property $\partial_l u_k = -i\kappa_l u_k$ (see equation (1.67)). Using the symmetry properties (1.5) of c_{ijkl}, we note that equation (1.137) becomes

$$\sigma_{ji}v_i^* = -\omega c_{ijkl}\kappa_l u_k u_i^* = -\omega c_{ijkl}\kappa_k u_l u_i^*. \tag{1.138}$$

The Cartesian components of the energy velocity (1.120) can be obtained by using equations (1.62), (1.109), (1.119), (1.135) and (1.138). Thus, we obtain

$$v_{ej} = \frac{\omega c_{ijkl}\kappa_k u_l u_i^*}{\rho\omega^2|\mathbf{u}_0|^2} = \frac{c_{ijkl}\kappa_k u_{0l}u_{0i}^*}{\rho\omega|\mathbf{u}_0|^2}, \tag{1.139}$$

which, when compared to equation (1.134), shows that, in elastic media, the energy velocity is equal to the group velocity, namely,

$$\mathbf{v}_e = \mathbf{v}_g. \tag{1.140}$$

This fact simplifies the calculations since the group velocity is easier to compute than the energy velocity.

1.4.3 Envelope velocity

The spatial part of the phase of the plane wave (1.62) can be written as $\boldsymbol{\kappa} \cdot \mathbf{x} = \kappa(l_1 x + l_2 y + l_3 z)$. An equivalent definition of wave surface in anisotropic elastic media is given by the envelope of the plane

$$l_1 x + l_2 y + l_3 z = v_p, \qquad (l_i x_i = v_p) \tag{1.141}$$

(Love, 1944, p. 299), because the velocity of the envelope of plane waves at unit propagation time, which we call $\mathbf{v}_{env}$, has the components

$$v_{env})_i = x_i = \frac{\partial v_p}{\partial l_i}, \tag{1.142}$$

and

$$v_{env} = \sqrt{x^2 + y^2 + z^2}. \tag{1.143}$$

To compute the components of the envelope velocity, we need the function $v_p = v_p(l_i)$, which is available only in simple cases, such as those describing the symmetry planes (see equations (1.87)-(1.89)). However, note that $v_p = \omega/\kappa$ and l_i are related by the function F defined in equation (1.128), since using (1.131) and dividing by κ^6, we obtain

$$F(\omega, \kappa_i) = \kappa^{-6} \det(c_{ijkl}\kappa_k\kappa_l - \rho\omega^2\delta_{ij}) = \det(c_{ijkl}l_k l_l - \rho v_p^2 \delta_{ij}), \tag{1.144}$$

and ω and κ_i, and v_p and l_i are related by the same function. Hence,

$$v_{env})_i = \frac{\partial v_p}{\partial l_i} = \frac{\partial \omega}{\partial \kappa_i} = v_{gi} = v_{ei} \tag{1.145}$$

from (1.140), and, in anisotropic elastic media, the envelope velocity is equal to the group and energy velocities.

If we restrict our analysis to a given plane, say the (x, z)-plane ($l_2 = 0$), we obtain another well-known expression of the envelope velocity (Postma, 1955; Berryman, 1979). In this case, the wavevector directions can be defined by $l_1 = \sin\theta$ and $l_3 = \cos\theta$, where θ is the angle between the wavevector and the z-axis. Differentiating equation (1.141) with respect to θ, squaring it and adding the results to the square of equation (1.141), we get

$$v_{env} = \sqrt{v_p^2 + \left(\frac{dv_p}{d\theta}\right)^2}. \tag{1.146}$$

Postma (1955) obtained this equation for a transversely isotropic medium. Although the group velocity is commonly called the envelope velocity in literature, we show in Chapter 4 that they are not the same in attenuating media. Rather, the envelope velocity is equal to the energy velocity in isotropic anelastic media. In anisotropic anelastic media, the three velocities are different.

1.4.4 Example: Transversely isotropic media

The phase velocity of SH waves is given in equation $(1.79)_1$. The calculation of the group velocity makes use of the dispersion relation $(1.78)_1$ in the form

$$F(\kappa_1, \kappa_3, \omega) = c_{66}\kappa_1^2 + c_{55}\kappa_3^2 - \rho\omega^2 = 0. \tag{1.147}$$

From equation (1.130), and using (1.70), we obtain

$$\mathbf{v}_g = \frac{1}{\rho\omega}(c_{66}\kappa_1\hat{\mathbf{e}}_1 + c_{55}\kappa_3\hat{\mathbf{e}}_3) = \frac{1}{\rho v_p}(c_{66}l_1\hat{\mathbf{e}}_1 + c_{55}l_3\hat{\mathbf{e}}_3), \tag{1.148}$$

and

$$v_g = \frac{1}{\rho v_p}\sqrt{c_{66}^2 l_1^2 + c_{55}^2 l_3^2}. \tag{1.149}$$

It is rather easy to show that, using the dispersion relation $(1.78)_1$, we obtain the same result from equations (1.142) and (1.146).

To compute the energy velocity, we use equation (1.120). Thus, we need to calculate the complex Umov-Poynting vector (1.100), which for SH-wave propagation in the (x, z)-plane can be expressed as

$$\mathbf{p} = -\frac{1}{2}(\sigma_{12}\hat{\mathbf{e}}_1 + \sigma_{23}\hat{\mathbf{e}}_3)v_2^*. \tag{1.150}$$

From equations (1.26) and (1.31), we note that

$$\sigma_{12} = c_{66}\partial_1 u_2, \qquad \sigma_{23} = c_{55}\partial_3 u_2, \tag{1.151}$$

and using equations (1.62) and (1.67), we have

$$\sigma_{12} = -\mathrm{i}\kappa_1 c_{66} u_2, \qquad \sigma_{23} = -\mathrm{i}\kappa_3 c_{55} u_2. \tag{1.152}$$

Since $v_2^* = -\mathrm{i}\omega u_2^*$ and $u_2 u_2^* = |\mathbf{u}_0|^2$, we use equation (1.152) to obtain

$$\mathbf{p} = \frac{1}{2}\omega(c_{66}\kappa_1\hat{\mathbf{e}}_1 + c_{55}\kappa_3\hat{\mathbf{e}}_3)u_2 u_2^* = \frac{1}{2}\omega\kappa|\mathbf{u}_0|^2(c_{66}l_1\hat{\mathbf{e}}_1 + c_{55}l_3\hat{\mathbf{e}}_3). \tag{1.153}$$

Substituting equation (1.115) into equation (1.120), and using expressions (1.109) and (1.153), we get

$$\mathbf{v}_e = \frac{1}{\rho v_p}(c_{66}l_1\hat{\mathbf{e}}_1 + c_{55}l_3\hat{\mathbf{e}}_3) = \mathbf{v}_g. \tag{1.154}$$

Note that because $v_{e1} = c_{66}l_1/(\rho v_p)$ and $v_{e3} = c_{55}l_3/(\rho v_p)$, we have

$$\frac{v_{e1}^2}{c_{66}/\rho} + \frac{v_{e3}^2}{c_{55}/\rho} = 1, \tag{1.155}$$

where we have used equation $(1.79)_1$. Hence, the energy-velocity curve – and the wave front – is an ellipse, with semiaxes c_{66}/ρ and c_{55}/ρ along the x- and z-directions, respectively. We have already demonstrated that the slowness surface for SH waves is an ellipse (see equation (1.80)).

To obtain the energy velocity for the coupled qP and qS waves, we compute, for simplicity, the group velocity using equation (1.130) by rewriting the dispersion relation $(1.78)_2$ as

$$F(\kappa_1, \kappa_3, \omega) = (c_{11}\kappa_1^2 + c_{55}\kappa_3^2 - \rho\omega^2)(c_{33}\kappa_3^2 + c_{55}\kappa_1^2 - \rho\omega^2) - (c_{13} + c_{55})^2\kappa_1^2\kappa_3^2 = 0. \tag{1.156}$$

Then, after some calculations,

$$v_{e1} = \left(\frac{l_1}{v_p}\right)\frac{(\Gamma_{33} - \rho v_p^2)c_{11} + (\Gamma_{11} - \rho v_p^2)c_{55} - (c_{13} + c_{55})^2 l_3^2}{\rho(\Gamma_{11} + \Gamma_{33} - 2\rho v_p^2)} \tag{1.157}$$

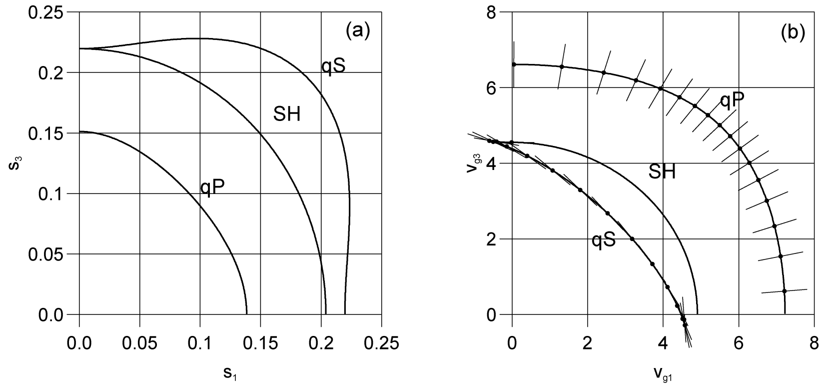

Figure 1.1: Slowness (a) and group-velocity curves (b) for apatite (c_{11} = 167 GPa, c_{12} = 13.1 GPa, c_{13} = 66 GPa, c_{33} = 140 GPa, c_{55} = 66.3 GPa, and ρ = 3200 kg/m^3). This mineral is transversely isotropic. The curves represent sections of the respective slowness and group-velocity surfaces across a plane containing the symmetry axis. The polarization directions are indicated in the curves (the SH polarization is perpendicular to the plane of the page).

$$v_{e3} = \left(\frac{l_3}{v_p}\right) \frac{(\Gamma_{33} - \rho v_p^2)c_{55} + (\Gamma_{11} - \rho v_p^2)c_{33} - (c_{13} + c_{55})^2 l_1^2}{\rho(\Gamma_{11} + \Gamma_{33} - 2\rho v_p^2)}, \tag{1.158}$$

where Γ_{11} and Γ_{33} are defined in equations (1.76) and (1.77). The phase and energy velocities of each mode coincide at the principal axes – the Cartesian axes in these examples.

Figure 1.1 shows the slowness (a) and group-velocity curves (b) for apatite (Payton, 1983, p. 3; Carcione, Kosloff and Kosloff, 1988a). Only one quarter of the curves are displayed because of symmetry considerations. The cusps, folds or lacunas, on the qS wave are due to the presence of inflection points in the slowness surface. This phenomenon implies three qS waves around the cusps. One of the remarkable effects of anisotropy on acoustic waves is the possible appearance of these folds (triplications) in wave fronts. Frequency slices taken through anisotropic field data exhibit rings of interference patterns (Ohanian, Snyder and Carcione, 1997). The phenomenon by which a single anisotropic wave front interferes with itself was reported by Maris (1983) in his study of the effect of finite phonon wavelength on phonon focusing. The phenomenon by which shear waves have different velocities along a given direction is termed shear-wave splitting in seismic wave propagation.

1.4.5 Elasticity constants from phase and group velocities

Elasticity constants can be obtained from five phase velocity measurements. For typical transducer widths ($\approx$ 10 mm), for which the measured signal in ultrasonic experiments is a plane wave, the travel times correspond to the phase velocity (Dellinger and Vernik, 1992). Let us consider the (x, z)-plane of an orthorhombic medium. The corresponding phase velocities are given in equations (1.88) and (1.91). Moreover, using the dispersion relations (1.85) with $\theta = 45^\circ$ ($l_1 = l_3 = 1/\sqrt{2}$), we obtain

$$\begin{aligned}
c_{11} &= \rho v^2_{p_{\rm qP}}(90^\circ) \\
c_{33} &= \rho v^2_{p_{\rm qP}}(0^\circ) \\
c_{44} &= \rho v^2_{p_{\rm SH}}(0^\circ) = c_{55} = \rho v^2_{p_{\rm qS}}(0^\circ) = \rho v^2_{p_{\rm qS}}(90^\circ) \\
c_{66} &= \rho v^2_{p_{\rm SH}}(90^\circ) \\
c_{13} &= -c_{55} + \sqrt{4\rho^2 v^4_{p_{\rm qP}}(45^\circ) - 2\rho v^2_{p_{\rm qP}}(45^\circ)(c_{11} + c_{33} + 2c_{55}) + (c_{11} + c_{55})(c_{33} + c_{55})}.
\end{aligned} \tag{1.159}$$

When the signal is not a plane wave but a localized wave packet – transducers less than 2 mm wide – the measured travel time is related to the energy velocity not to the phase velocity. In this case, c_{13} can be obtained as follows. If the receiver is located at 45°, equation (1.114) implies

$$v_e \cos\psi = v_e \cos(\theta - 45^\circ) = v_p, \tag{1.160}$$

or

$$v_p = \frac{v_e}{\sqrt{2}}(l_1 + l_3), \tag{1.161}$$

where ψ is the angle between the ray and propagation directions (see Figure 1.2).

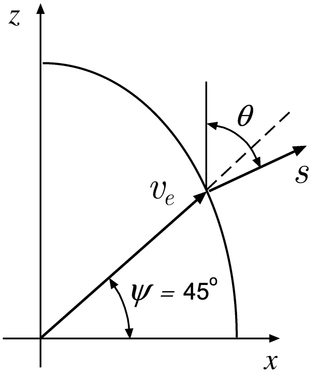

Figure 1.2: Relation between the phase-velocity angle θ and the group-velocity angle ψ.

Now, noting that equations (1.157) and (1.158) for transversely isotropic media are also the energy velocity components for our case, we perform the scalar product between (v_{e1}, v_{e3}) and $(l_1, -l_3)$ and use $v_{e1} = v_{e3} = v_e \cos 45^\circ = v_e/\sqrt{2}$. Hence, we obtain

$$\begin{aligned}
\rho v_p v_e (l_1 - l_3)(\Gamma_{11} + \Gamma_{33} - 2\rho v_p^2) = \sqrt{2}[(\Gamma_{33} - \rho v_p^2)(c_{11} l_1^2 - c_{55} l_3^2) \\
+ (\Gamma_{11} - \rho v_p^2)(c_{55} l_1^2 - c_{33} l_3^2)],
\end{aligned} \tag{1.162}$$

where

$$\Gamma_{11} = c_{11} l_1^2 + c_{55} l_3^2, \qquad \Gamma_{33} = c_{55} l_1^2 + c_{33} l_3^2. \tag{1.163}$$

By substituting equation (1.161), we can solve equation (1.162) for $\theta = \arcsin(l_1)$ (note that $l_1^2 + l_3^2 = 1$) as a function of the elasticity constants c_{11}, c_{33} and c_{55}, and v_e. Then, the elasticity constant c_{13} can be obtained from the dispersion relation (1.85) as

$$c_{13} = -c_{55} + \frac{1}{l_1 l_3}\sqrt{(\Gamma_{11} - \rho v_p^2)(\Gamma_{33} - \rho v_p^2)}. \tag{1.164}$$

1.4.6 Relationship between the slowness and wave surfaces

The normal to the slowness surface $F(\kappa_i, \omega) = F(s_i)$ – use $s_i = \kappa_i/\omega$ in equation (1.128) – is $\nabla_{\mathbf{s}} F$, where $\nabla_{\mathbf{s}} = (\partial/\partial s_1, \partial/\partial s_2, \partial/\partial s_3)$. Because $\boldsymbol{\kappa} = \omega \mathbf{s}$, this implies that the group-velocity vector (1.130) and, therefore, the energy-velocity vector, are both normal to the slowness surface.

On the other hand, since the energy velocity, which defines the wave front, is equal to the envelope velocity in lossless media (see equation (1.145)), the wave surface can be defined by the function

$$W(x_i) = \kappa_i x_i - \omega, \tag{1.165}$$

in accordance with equation (1.141), and using (1.70) and $\kappa_i = \kappa l_i$. The normal vector to the wave surface is grad W. But grad $W = (\kappa_1, \kappa_2, \kappa_3) = \boldsymbol{\kappa}$. Therefore, the wavevector is normal to the wave surface, a somehow obvious fact, because the wave surface is the envelope of the plane waves. A geometrical illustration of these perpendicularity properties is shown in Figure 1.3.

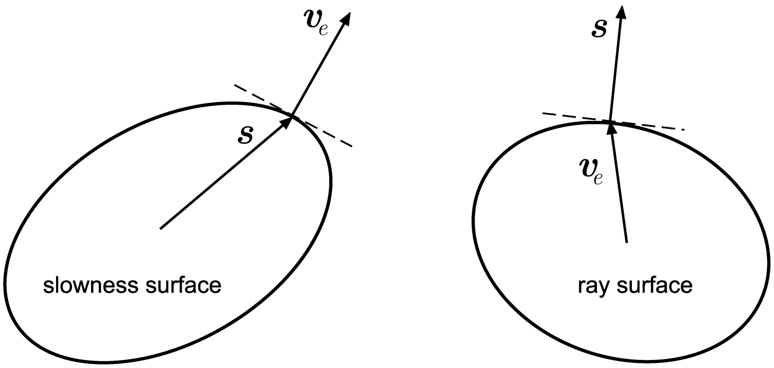

Figure 1.3: Relationships between the slowness and the ray (or wave) surfaces (perpendicularity properties).

SH-wave propagation

We obtain the slowness and wave surfaces from equations (1.80) and (1.155), namely

$$F(s_1, s_3) = \frac{s_1^2}{\rho/c_{66}} + \frac{s_3^2}{\rho/c_{55}} - 1 \tag{1.166}$$

and

$$W(v_{e1}, v_{e3}) = \frac{v_{e1}^2}{c_{66}/\rho} + \frac{v_{e3}^2}{c_{55}/\rho} - 1. \tag{1.167}$$

Taking the respective gradients, and using equation (1.154), we have

$$\nabla_{\mathbf{s}} F = \frac{2}{\rho}(c_{66}s_1, c_{44}s_3) = \frac{2}{\rho v_p}(c_{66}l_1, c_{44}l_3) = 2\mathbf{v}_e, \tag{1.168}$$

and

$$\nabla_{\mathbf{v}_e} W = 2\left(\frac{v_{e1}}{c_{66}}, \frac{v_{e3}}{c_{44}}\right) = \frac{2}{v_p}(l_1, l_3) = \frac{2}{\omega}\boldsymbol{\kappa} = 2\mathbf{s}, \tag{1.169}$$

which agree with the statements demonstrated earlier in this section.

1.5 Finely layered media

Most geological systems can be modeled as fine layering, which refers to the case where the dominant wavelength of the pulse is much larger than the thicknesses of the individual layers. When this occurs, the medium is effectively transversely isotropic. The first to obtain a solution for this problem was Bruggeman (1937). Later, other investigators studied the problem using different approaches, e.g., Riznichenko (1949) and Postma (1955). To illustrate the averaging process and obtain the equivalent transversely isotropic medium, we consider a two-constituent periodically layered medium, as illustrated in Figure 1.4, and follow Postma's reasoning (Postma, 1955). We assume that all the stress and strain components in planes parallel to the layering are the same in all layers. The other components may differ from layer to layer and are represented by average values.

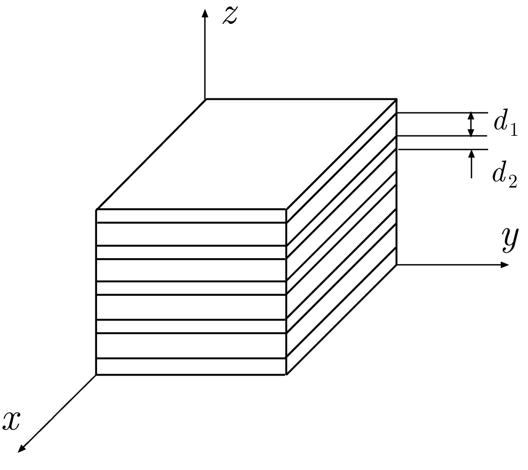

Figure 1.4: Representative volume of stratified medium.

Assume that a stress σ_{33} is applied to the faces perpendicular to the z-axis, and that there are no tangential components, namely σ_{13} and σ_{23}. This stress does not generate shear strains. On the faces perpendicular to the x-axis, we impose

$$\begin{array}{ll} \sigma_{11}^{(1)} & \text{on medium 1,} \\ \sigma_{11}^{(2)} & \text{on medium 2,} \\ \text{and} & e_{11}^{(1)} = e_{11}^{(2)} = e_{11}. \end{array} \tag{1.170}$$

Similarly, on the faces perpendicular to the y-axis, we require

$$\begin{aligned} &\sigma_{22}^{(1)} \quad \text{on medium 1,} \\ &\sigma_{22}^{(2)} \quad \text{on medium 2,} \\ &\text{and} \quad e_{22}^{(1)} = e_{22}^{(2)} = e_{22}. \end{aligned} \tag{1.171}$$

We also must have

$$\sigma_{33}^{(1)} = \sigma_{33}^{(2)} = \sigma_{33}. \tag{1.172}$$

The preceding equations guarantee the continuity of displacements and normal stress across the interfaces. The changes in thickness in media 1 and 2 are $e_{33}^{(1)} d_1$ and $e_{33}^{(2)} d_2$, respectively.

The stress-strain relations of each isotropic medium can be obtained from equations (1.17) and (1.18). We obtain for medium 1 (l =1) and medium 2 (l =2),

$$\begin{aligned} \sigma_{11}^{(l)} &= E_l e_{11} + \lambda_l (e_{22} + e_{33}^{(l)}) \\ \sigma_{22}^{(l)} &= E_l e_{22} + \lambda_l (e_{11} + e_{33}^{(l)}) \\ \sigma_{33} &= E_l e_{33}^{(l)} + \lambda_l (e_{11} + e_{22}), \end{aligned} \tag{1.173}$$

where $E_l = \lambda_l + 2\mu_l$. The average stresses on the faces perpendicular to the x- and y-axes are

$$\sigma_{11} = \frac{d_1 \sigma_{11}^{(1)} + d_2 \sigma_{11}^{(2)}}{d} \qquad \sigma_{22} = \frac{d_1 \sigma_{22}^{(1)} + d_2 \sigma_{22}^{(2)}}{d}, \tag{1.174}$$

where $d = d_1 + d_2$. Eliminating the stresses $\sigma_{11}^{(l)}$ and $\sigma_{22}^{(l)}$ from equations (1.173) and (1.174), we obtain

$$\begin{aligned} d\,\sigma_{11} &= (E_1 d_1 + E_2 d_2) e_{11} + (\lambda_1 d_1 + \lambda_2 d_2) e_{22} + \lambda_1 d_1 e_{33}^{(1)} + \lambda_2 d_2 e_{33}^{(2)} \\ d\,\sigma_{22} &= (E_1 d_{+} E_2 d_2) e_{22} + (\lambda_1 d_1 + \lambda_2 d_2) e_{11} + \lambda_1 d_1 e_{33}^{(1)} + \lambda_2 d_2 e_{33}^{(2)} \\ d\,\sigma_{33} &= (\lambda_1 d_1 + \lambda_2 d_2)(e_{11} + e_{22}) + E_1 d_1 e_{33}^{(1)} + E_2 d_2 e_{33}^{(2)}. \end{aligned} \tag{1.175}$$

The strain along the z-axis is the average, given by

$$e_{33} = \frac{d_1 e_{33}^{(1)} + d_2 e_{33}^{(2)}}{d}. \tag{1.176}$$

Then, we can compute the normal strains along the z-axis by using $(1.173)_3$ and (1.176). Hence, we obtain

$$e_{33}^{(1)} = \frac{d\,E_2 e_{33} - (\lambda_1 - \lambda_2)(e_{11} + e_{22}) d_2}{d_1 E_2 + d_2 E_1}, \qquad e_{33}^{(2)} = \frac{d\,E_1 e_{33} + (\lambda_1 - \lambda_2)(e_{11} + e_{22}) d_1}{d_1 E_2 + d_2 E_1}. \tag{1.177}$$

Substituting these results into equations (1.175), we obtain a stress-strain relation for an effective transversely isotropic medium, for which

$$\begin{aligned} \sigma_{11} &= c_{11} e_{11} + c_{12} e_{22} + c_{13} e_{33} \\ \sigma_{22} &= c_{12} e_{11} + c_{22} e_{22} + c_{13} e_{33} \\ \sigma_{33} &= c_{13} e_{11} + c_{13} e_{22} + c_{33} e_{33}, \end{aligned} \tag{1.178}$$

where

$$
\begin{aligned}
c_{11} &= [d^2 E_1 E_2 + 4 d_1 d_2 (\mu_1 - \mu_2)(\lambda_1 + \mu_1 - \lambda_2 - \mu_2)]\mathrm{D}^{-1} \\
c_{12} &= [d^2 \lambda_1 \lambda_2 + 2(\lambda_1 d_1 + \lambda_2 d_2)(\mu_2 d_1 + \mu_1 d_2)]\mathrm{D}^{-1} \\
c_{13} &= d(\lambda_1 d_1 E_2 + \lambda_2 d_2 E_1)\mathrm{D}^{-1} \\
c_{33} &= d^2 E_1 E_2 \mathrm{D}^{-1},
\end{aligned}
\tag{1.179}
$$

and

$$\mathrm{D} = d(d_1 E_2 + d_2 E_1). \tag{1.180}$$

Next, we apply a stress σ_{23} to the faces perpendicular to the z-axis. Continuity of tangential stresses implies $\sigma_{23}^{(1)} = \sigma_{23}^{(2)} = \sigma_{23}$. The resulting strain is shown in Figure 1.5.

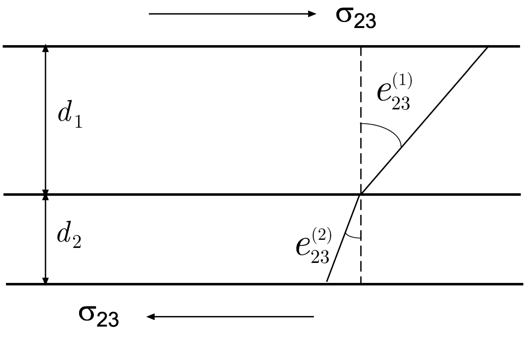

Figure 1.5: Tangential stress and strain.

We have, in this case,

$$e_{23} = (d_1 e_{23}^{(1)} + d_2 e_{23}^{(2)}) d^{-1}, \quad \sigma_{23} = \mu_l e_{23}^{(l)}. \tag{1.181}$$

Hence, eliminating $e_{23}^{(l)}$, we obtain a relation between σ_{23} and e_{23}. Similarly, we find the relation between σ_{13} and e_{13}. Thus, we obtain

$$\sigma_{23} = c_{44} e_{23}, \quad \sigma_{13} = c_{44} e_{13}, \tag{1.182}$$

with

$$c_{44} = \frac{d \mu_1 \mu_2}{d_1 \mu_2 + d_2 \mu_1}. \tag{1.183}$$

To obtain c_{66}, we apply a stress $\sigma_{12}^{(l)}$ to the faces perpendicular to the y-axis and note that $e_{12}^{(1)} = e_{12}^{(2)} = e_{12}$, because for thin layers the displacement inside a layer cannot differ greatly from the displacement at its boundaries. Then

$$\sigma_{12}^{(l)} = \mu_l e_{12}, \tag{1.184}$$

and, since the average stress satisfies

$$d\ \sigma_{12} = d_1 \sigma_{11}^{(1)} + d_2 \sigma_{11}^{(2)} = e_{12}(d_1 \mu_1 + d_2 \mu_2), \tag{1.185}$$

we have

$$\sigma_{12} = c_{66} e_{12}, \tag{1.186}$$

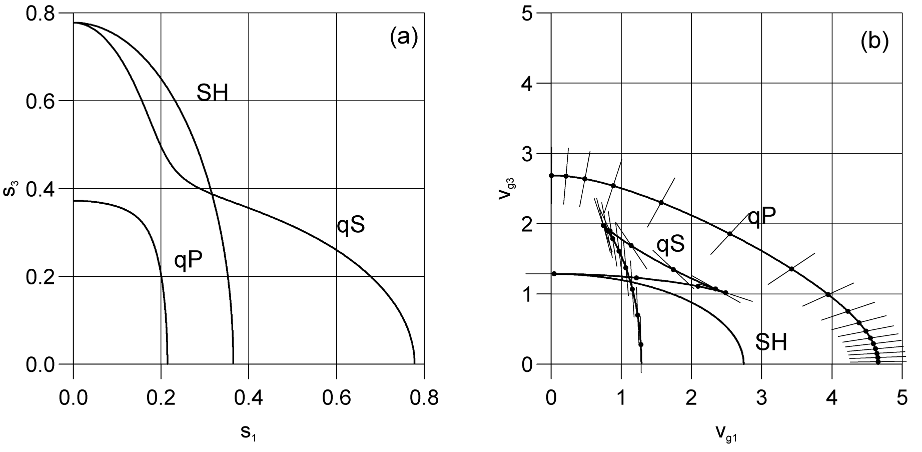

Figure 1.6: Slowness section (a) and group-velocity section (b) corresponding to the medium long-wavelength equivalent to an epoxy-glass sequence of layers with equal composition (c_{11} = 39.4 GPa, c_{12} = 12.1 GPa, c_{13} = 5.8 GPa, c_{33} = 13.1 GPa, c_{55} = 3 GPa, and ρ = 1815 kg/m^3). Only one quarter of the curves are displayed because of symmetry considerations. The polarization directions are indicated in the curves (the SH polarization is perpendicular to the plane of the page).

where

$$c_{66} = (d_1\mu_1 + d_2\mu_2)d^{-1}. \tag{1.187}$$

Note the relation $c_{66} = (c_{11} - c_{12})/2$. The equivalent anisotropic media possess four cuspidal triangles at 45° from the principal axes.

Figure 1.6 shows an example where the slowness and group-velocity sections can be appreciated (Carcione, Kosloff and Behle, 1991). The medium is an epoxy-glass sequence of layers with equal composition. We may infer from equations (1.183) and (1.187) that $c_{44} \le c_{66}$, and Postma (1955) shows that $c_{11} \ge c_{33}/2$.

Backus (1962) obtained the average elasticity constants in the case where the single layers are transversely isotropic with the symmetry axis perpendicular to the layering plane. Moreover, he assumed stationarity; that is, in a given length of composite medium much smaller than the wavelength, the proportion of each material is constant (periodicity is not required). The equations were further generalized by Schoenberg and Muir (1989) for anisotropic single constituents. The transversely isotropic equivalent medium is described by the following constants:

$$\begin{aligned}
c_{11} &= \langle c_{11} - c_{13}^2 c_{33}^{-1} \rangle + \langle c_{33}^{-1} \rangle^{-1} \langle c_{33}^{-1} c_{13} \rangle^2 \\
c_{33} &= \langle c_{33}^{-1} \rangle^{-1} \\
c_{13} &= \langle c_{33}^{-1} \rangle^{-1} \langle c_{33}^{-1} c_{13} \rangle \\
c_{55} &= \langle c_{55}^{-1} \rangle^{-1} \\
c_{66} &= \langle c_{66} \rangle,
\end{aligned} \tag{1.188}$$

where the weighted average of a quantity a is defined as

$$\langle a \rangle = \sum_{l=1}^{L} p_l a_l, \tag{1.189}$$

where p_l is the proportion of material l. More details about these media (for instance, constraints in the values of the different elasticity constants) are given by Helbig (1994, p. 315).

The dispersive effects are investigated by Norris (1992). Carcione, Kosloff and Behle (1991) evaluate the long-wavelength approximation using numerical modeling experiments. An acceptable rule of thumb is that the wavelength must be larger than eight times the layer thickness. A complete theory, for all frequencies, is given in Burridge, de Hoop, Le and Norris (1993) and Shapiro and Hubral (1999). This theory, which includes Backus averaging in the low-frequency limit and ray theory in the high-frequency limit, can be used to study velocity dispersion and frequency-dependent anisotropy for plane waves propagating at any angle in a layered medium. The extension of the low-frequency theory to poroelastic media can be found in Norris (1993), Bakulin and Moloktov (1997) and Gelinsky and Shapiro (1997).

1.6 Anomalous polarizations

In this section[3], we show that there are media with the same phase velocity or slowness surface that exhibit drastically different polarization behaviors. Such media are kinematically identical but dynamically different. Therefore, classification of the media according to wave velocity alone is not sufficient, and the identification of the wave type should be based on both velocity and polarization.

"Anomalous Polarization" refers to the situation where the slowness and wave surfaces of two elastic media are identical, but the polarization fields are different. Examples of anomalous polarization have been discussed for transverse isotropy by Helbig and Schoenberg (1987), and for orthorhombic symmetry by Carcione and Helbig (2000). In this note we determine, without prior restriction of the symmetry class, under what conditions the phenomenon can occur. Since the three slownesses in a given direction are the square roots of the eigenvalues of the Kelvin-Christoffel matrix, while the polarizations are the corresponding eigenvectors, the condition for the existence of anomalous polarization can be formulated as: Two media with different stiffness matrices are "anomalous companions" if the characteristic equations (1.72) of their respective Kelvin-Christoffel matrices $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ are identical, i.e., if $\det(\mathbf{\Gamma} - \lambda \mathbf{I}_3) = \det(\mathbf{\Gamma}' - \lambda \mathbf{I}_3)$, where $\lambda = \rho v_p^2$.

1.6.1 Conditions for the existence of anomalous polarization

Without loss of generality we assume that the elastic fourth-rank stiffness tensors (and the corresponding 6×6 stiffness matrices) are referred to a natural coordinate system of the media. Inspection of the Kelvin-Christoffel dispersion relation

$\det(\mathbf{\Gamma} - \lambda \mathbf{I}_3) = -\lambda^2 + (\Gamma_{11} + \Gamma_{22} + \Gamma_{33})\lambda^2 - (\Gamma_{22}\Gamma_{33} - \Gamma_{23}^2 + \Gamma_{11}\Gamma_{33} - \Gamma_{13}^2 + \Gamma_{11}\Gamma_{22} - \Gamma_{12}^2)\lambda +$

[3]This section has been written in collaboration with Klaus Helbig.

$$\Gamma_{11}\Gamma_{22}\Gamma_{33} + 2\Gamma_{23}\Gamma_{13}\Gamma_{12} - \Gamma_{11}\Gamma_{23}^2 - \Gamma_{22}\Gamma_{13}^2 - \Gamma_{33}\Gamma_{12}^2 \tag{1.190}$$

indicates that two stiffness tensors have identical characteristic equations if for all propagation directions the following three conditions hold:

1. The diagonal terms of the Kelvin-Christoffel matrices $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ are identical;
2. The squares of their off-diagonal terms are identical; and
3. The products of their three off-diagonal terms are identical.

The second and third conditions can be satisfied simultaneously if all corresponding off-diagonal terms have the same magnitude, and precisely two corresponding terms have opposite sign.

Let us consider the three conditions:

1. The diagonal terms of the Kelvin-Christoffel matrices of an anomalous companion pair are equal for all propagation directions if they share the 15 stiffnesses occurring in equations $(1.73)_1$, $(1.73)_2$ and $(1.73)_3$, i.e.,

$$c_{11},\ c_{22},\ c_{33},\ c_{44},\ c_{55},\ c_{66},\ c_{15},\ c_{16},\ c_{56},\ c_{24},\ c_{26},\ c_{46},\ c_{34},\ c_{35} \text{ and } c_{45}. \tag{1.191}$$

Two anomalous companion matrices can thus differ only in

$$c_{23},\ c_{13},\ c_{12},\ c_{14},\ c_{25} \text{ and } c_{36}. \tag{1.192}$$

The position of these elasticity constants in the stiffness matrix are

	c_{12}	c_{13}	c_{14}		
c_{12}		c_{23}		c_{25}	
c_{13}	c_{23}				c_{36}
c_{14}					
	c_{25}				
		c_{36}			

2. Two of the three off-diagonal terms of the Kelvin-Christoffel matrices for an anomalous companion pair must be of equal magnitude but opposite sign for all propagation directions, thus for these terms all coefficients of the product of direction cosines must change sign. The off-diagonal terms of the Kelvin-Christoffel matrix are given by equations $(1.73)_4$, $(1.73)_5$ and $(1.73)_6$. The nine stiffnesses c_{15}, c_{16}, c_{24}, c_{26}, c_{34}, c_{35}, c_{45}, c_{46} and c_{56} are listed in equation (1.191), as being equal in both terms, thus they can change sign only if they vanish. Thus, the off-diagonal terms of the Kelvin-Christoffel matrix in a pair of companion matrices must have the form

$$\begin{aligned}
\Gamma_{23} &= (c_{23} + c_{44})l_2l_3 + c_{36}l_1l_3 + c_{25}l_1l_2 \\
\Gamma_{13} &= c_{36}l_2l_3 + (c_{13} + c_{55})l_1l_3 + c_{14}l_1l_2 \\
\Gamma_{12} &= c_{25}l_2l_3 + c_{14}l_1l_3 + (c_{12} + c_{66})l_1l_2,
\end{aligned} \tag{1.193}$$

and

$$\begin{aligned}
\Gamma'_{23} &= (c'_{23} + c_{44})l_2l_3 + c'_{36}l_1l_3 + c'_{25}l_1l_2 \\
\Gamma'_{13} &= c'_{36}l_2l_3 + (c'_{13} + c_{55})l_1l_3 + c'_{14}l_1l_2 \\
\Gamma'_{12} &= c'_{25}l_2l_3 + c'_{14}l_1l_3 + (c'_{12} + c_{66})l_1l_2,
\end{aligned} \tag{1.194}$$

with

$$\Gamma'_{23} = \pm\Gamma_{23} \quad \Gamma'_{13} = \pm\Gamma_{13} \quad \Gamma'_{12} = \pm\Gamma_{12}. \tag{1.195}$$

There are eight sign combinations of off-diagonal terms of the Kelvin-Christoffel matrix, each corresponding to a characteristic equation (1.190) with identical coefficients for the terms with λ_m, $m = 1, \ldots, 3$. The condition $\Gamma'_{23}\Gamma'_{13}\Gamma'_{12} = \Gamma_{23}\Gamma_{13}\Gamma_{12}$ divides the corresponding eight slowness surfaces into two classes containing each four elements with the same product $\Gamma_{23}\Gamma_{13}\Gamma_{12}$. The two classes share the intersections with the coordinate planes, but differ outside these planes. The following table shows the sign combinations for the two sets of four slowness each:

Γ_{23}	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
Γ_{13}	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$+$
Γ_{12}	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
$\Gamma_{23}\ \Gamma_{13}\ \Gamma_{12}$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$

(1.196)

This table shows that any two anomalous companion media differ in the algebraic signs of precisely two off-diagonal terms of the Kelvin-Christoffel matrix. Inspection of equations (1.193)-(1.195) shows that this is possible only if either all three or precisely two of the three stiffnesses $\{c_{14}, c_{25}, c_{36}\}$ vanish: if two of these stiffnesses would not vanish, all three off-diagonal terms would be affected and would have to change sign. The two slowness surfaces would share the intersections with the coordinate planes, but would not be identical outside these planes. It follows that anomalous polarization is possible for any stiffness matrix that can be brought – through rotation of the coordinate system and/or exchange of subscripts – into the following forms:

i. Medium with an (x, y)-symmetry plane:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & c_{36} & 0 & 0 & c_{66} \end{pmatrix}; \tag{1.197}$$

ii. Medium with an (x, z)-symmetry plane:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & c_{25} & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & c_{25} & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}; \tag{1.198}$$

iii. Medium with a (y, z)-symmetry plane:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ c_{14} & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}. \tag{1.199}$$

The media defined by these matrices have normal polarization in the symmetry plane and anomalous polarization in the other orthogonal planes.

1.6.2 Stability constraints

In the previous section the formal conditions for the existence of anomalous companion pairs were derived without regard to the stability of the corresponding media. Only stable media can exist under the laws of physics. An elastic medium is stable if and only if every deformation requires energy. This means that all principal minors of the stiffness matrix must be positive (in this terminology, a "minor" is the determinant of the corresponding sub-matrix; the main diagonal of the sub-matrix corresponding to a "principal minor" is a non-empty subset of the main diagonal of the matrix). This is equivalent with the requirement that the stiffness matrix must be positive definite. The condition for positive definiteness can be relaxed to "all leading principal minors must be positive" (see equation (1.33)). The sub-matrix corresponding to a leading principal minor is contiguous and contains the leading element of the matrix.

Let us consider the matrix defined in (1.199). The first-order principal minors are positive if

$$c_{11} > 0,\ c_{22} > 0,\ c_{33} > 0,\ c_{44} > 0,\ c_{55} > 0,\ c_{66} > 0. \tag{1.200}$$

The second-order principal minors are positive if

$$\begin{aligned} -\sqrt{c_{22}c_{33}} < c_{23} < \sqrt{c_{22}c_{33}}, \quad -\sqrt{c_{11}c_{33}} < c_{13} < \sqrt{c_{11}c_{33}}, \\ -\sqrt{c_{11}c_{22}} < c_{12} < \sqrt{c_{11}c_{22}}, \quad -\sqrt{c_{11}c_{44}} < c_{14} < \sqrt{c_{11}c_{44}}. \end{aligned} \tag{1.201}$$

The last inequality (constraint on c_{14}) is easily changed to the constraints on c_{25} and c_{36}. The leading principal third-order minor

$$D_3 = c_{11}c_{22}c_{33} + 2c_{12}c_{23}c_{13} - c_{11}c_{23}^2 - c_{22}c_{13}^2 - c_{33}c_{12}^2 \tag{1.202}$$

is positive if c_{23}, c_{13} and c_{12} satisfy

$$1 + 2\frac{c_{23}c_{13}c_{12}}{c_{11}c_{22}c_{33}} - \frac{c_{23}^2}{c_{22}c_{33}} - \frac{c_{13}^2}{c_{11}c_{33}} - \frac{c_{12}^2}{c_{11}c_{22}} > 0. \tag{1.203}$$

The leading principal fourth-order minor is obtained by development about the fourth column:

$$D_4 = c_{44}D_3 - c_{14}^2(c_{22}c_{33} - c_{23}). \tag{1.204}$$

If inequalities (1.200), (1.201) and (1.203) are satisfied, D_4 is positive if and only if

$$c_{14}^2 < \frac{c_{44}D_3}{c_{22}c_{33} - c_{23}^2} \rightarrow -\sqrt{\frac{c_{44}D_3}{c_{22}c_{33} - c_{23}^2}} < c_{14} < \sqrt{\frac{c_{44}D_3}{c_{22}c_{33} - c_{23}^2}}, \tag{1.205}$$

with obvious generalizations to the constraints on c_{25} and c_{36}:

$$c_{25}^2 < \frac{c_{55}D_3}{c_{11}c_{33} - c_{13}^2} \rightarrow -\sqrt{\frac{c_{55}D_3}{c_{11}c_{33} - c_{13}^2}} < c_{25} < \sqrt{\frac{c_{55}D_3}{c_{11}c_{33} - c_{13}^2}} \tag{1.206}$$

and

$$c_{36}^2 < \frac{c_{66}D_3}{c_{11}c_{22} - c_{12}^2} \rightarrow -\sqrt{\frac{c_{66}D_3}{c_{11}c_{22} - c_{12}^2}} < c_{36} < \sqrt{\frac{c_{66}D_3}{c_{11}c_{22} - c_{12}^2}}. \tag{1.207}$$

1.6.3 Anomalous polarization in orthorhombic media

It follows from equations (1.193)-(1.195) that for orthorhombic media the off-diagonal terms of the Kelvin-Christoffel matrices of a pair of companion matrices are

$$\begin{aligned}\Gamma_{23} &= (c_{23}+c_{44})l_2l_3\\ \Gamma_{13} &= (c_{13}+c_{55})l_1l_3\\ \Gamma_{12} &= (c_{12}+c_{66})l_1l_2,\end{aligned} \tag{1.208}$$

and

$$\begin{aligned}\Gamma'_{23} &= (c'_{23}+c_{44})l_2l_3\\ \Gamma'_{13} &= (c'_{13}+c_{55})l_1l_3\\ \Gamma'_{12} &= (c'_{12}+c_{66})l_1l_2,\end{aligned} \tag{1.209}$$

with

$$\Gamma'_{23} = \pm\Gamma_{23}, \quad \Gamma'_{13} = \pm\Gamma_{13}, \quad \Gamma'_{12} = \pm\Gamma_{12}, \tag{1.210}$$

where in the last line precisely two of the minus signs must be taken. We obtain the elasticity constants of the anomalous companions as:

i. Medium with an (x, y)-symmetry plane

$$\begin{aligned}c'_{13}+c_{55} = -(c_{13}+c_{55}) \to c'_{13} = -(c_{13}+2c_{55}),\\ c'_{23}+c_{44} = -(c_{23}+c_{44}) \to c'_{23} = -(c_{23}+2c_{44});\end{aligned} \tag{1.211}$$

ii. Medium with an (x, z)-symmetry plane

$$\begin{aligned}c'_{12}+c_{66} = -(c_{12}+c_{66}) \to c'_{12} = -(c_{12}+2c_{66}),\\ c'_{23}+c_{44} = -(c_{23}+c_{44}) \to c'_{23} = -(c_{23}+2c_{44});\end{aligned} \tag{1.212}$$

iii. Medium with a (y, z)-symmetry plane

$$\begin{aligned}c'_{12}+c_{66} = -(c_{12}+c_{66}) \to c'_{12} = -(c_{12}+2c_{66}),\\ c'_{13}+c_{55} = -(c_{13}+c_{55}) \to c'_{13} = -(c_{13}+2c_{55}),\end{aligned} \tag{1.213}$$

where the polarization is normal in the symmetry planes.

Only companion pairs where $\{c_{23}, c_{13}, c_{12}\}$ and $\{c'_{23}, c'_{13}, c'_{12}\}$ satisfy the stability conditions are meaningful.

1.6.4 Anomalous polarization in monoclinic media

It follows from equations (1.193)-(1.195) and (1.197) that for monoclinic media with the (x, y)-plane as symmetry plane, the off-diagonal terms of the Kelvin-Christoffel matrices of a pair of companion matrices are

$$\begin{aligned}\Gamma_{23} &= (c_{23}+c_{44})l_2l_3 + c_{36}l_1l_3\\ \Gamma_{13} &= c_{36}l_2l_3 + (c_{13}+c_{55})l_1l_3\\ \Gamma_{12} &= (c_{12}+c_{66})l_1l_2,\end{aligned} \tag{1.214}$$

and

$$\begin{aligned}\Gamma'_{23} &= -\Gamma_{23} = (c'_{23}+c_{44})l_2l_3 - c_{36}l_1l_3\\ \Gamma'_{13} &= -\Gamma_{13} = -c_{36}l_2l_3 + (c'_{13}+c_{55})l_1l_3\\ \Gamma'_{12} &= \Gamma_{12} = (c_{12}+c_{66})l_1l_2.\end{aligned} \tag{1.215}$$

This is easily satisfied if the orthorhombic "root" medium that is obtained by setting $c_{36} = 0$ has an anomalous companion. Then, $(c'_{23} + c_{44}) = -(c_{23} + c_{44})$ and $(c'_{13} + c_{55}) = -(c_{13} + c_{55})$, and because the leading third-order minor $D_3 > 0$, the interval (1.207) for the addition of $\pm c_{36}$ is not empty.

Therefore, the anomalous companions of monoclinic media are

i. Medium with an (x, y)-symmetry plane:

$$\begin{aligned} c'_{36} &= -c_{36} \\ c'_{13} &= -(c_{13} + 2c_{55}) \\ c'_{23} &= -(c_{23} + 2c_{44}); \end{aligned} \tag{1.216}$$

ii. Medium with an (x, z)-symmetry plane:

$$\begin{aligned} c'_{25} &= -c_{25} \\ c'_{12} &= -(c_{12} + 2c_{66}) \\ c'_{23} &= -(c_{23} + 2c_{44}); \end{aligned} \tag{1.217}$$

iii. Medium with a (y, z)-symmetry plane:

$$\begin{aligned} c'_{14} &= -c_{14} \\ c'_{12} &= -(c_{12} + 2c_{66}) \\ c'_{13} &= -(c_{13} + 2c_{55}). \end{aligned} \tag{1.218}$$

1.6.5 The polarization

The components of the polarization vector, u_m, corresponding to propagation direction l_i and one of the three eigenvalues λ_i, stand in the same ratio as the corresponding cofactors of $(\mathbf{\Gamma} - \lambda \mathbf{I}_3)_{ji}$ in the development of $\det(\mathbf{\Gamma} - \lambda \mathbf{I}_3)$ for an arbitrary j, i.e.,

$$\begin{aligned} & u_1 : u_2 : u_3 = \\ & [(\Gamma_{22} - \lambda)(\Gamma_{22} - \lambda) - \Gamma_{23}^2] : [\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda)] : [\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda)], \quad j = 1, \\ & [\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda)] : [(\Gamma_{11} - \lambda)(\Gamma_{33} - \lambda) - \Gamma_{13}^2] : [\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda)], \quad j = 2, \\ & [\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda)] : [\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda)] : [(\Gamma_{11} - \lambda)(\Gamma_{22} - \lambda) - \Gamma_{12}^2], \quad j = 3. \end{aligned} \tag{1.219}$$

It follows from the expressions for j =1 and $j = 3$ that

$$[(\Gamma_{11} - \lambda)(\Gamma_{22} - \lambda) - \Gamma_{12}^2] = \frac{[\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda)][\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda)]}{[\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda)]}. \tag{1.220}$$

After substitution of this equation into (1.219) and division of all three terms by $[\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda)][\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda)]$, one obtains the symmetric expression

$$\begin{aligned} & u_1 : u_2 : u_3 = \\ & \frac{1}{[\Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - \lambda)]} : \frac{1}{[\Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - \lambda)]} : \frac{1}{[\Gamma_{23}\Gamma_{13} - \Gamma_{12}(\Gamma_{33} - \lambda)]}. \end{aligned} \tag{1.221}$$

For a pair of companion media with $\Gamma'_{23} = -\Gamma_{23}$ and $\Gamma'_{13} = -\Gamma_{13}$ one has

$$\begin{aligned} & u'_1 : u'_2 : u'_3 = \\ & \frac{1}{[\Gamma_{12}\Gamma'_{13} - \Gamma'_{23}(\Gamma_{11} - \lambda)]} : \frac{1}{[\Gamma_{12}\Gamma'_{23} - \Gamma'_{13}(\Gamma_{22} - \lambda)]} : \frac{1}{[\Gamma'_{23}\Gamma'_{13} - \Gamma_{12}(\Gamma_{33} - \lambda)]}, \end{aligned} \tag{1.222}$$

i.e.,

$$u_1' : u_2' : u_3' = -u_1 : -u_2 : u_3, \tag{1.223}$$

or, since $|\mathbf{u}'| = |\mathbf{u}|$,

$$\mathbf{u}' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{u}. \tag{1.224}$$

1.6.6 Example

We consider an orthorhombic medium. The four polarization distributions corresponding to the "normal" slowness surface – with the sign combinations (1.196) – are shown in Figure 1.7. This figure shows the intersections of the slowness surface with the three planes of symmetry, and the polarization vector for the fastest (innermost) sheet wherever it makes an angle greater than $\pi/4$ with the propagation vector. The "zones" of anomalous polarization are clearly visible in Figure 1.7.

In the following example, we assume a simultaneous change of sign of $c_{12} + c_{66}$ and $c_{13} + c_{55}$. The stiffness matrix of the orthorhombic medium with normal polarization is

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} = \begin{pmatrix} 10 & 2 & 1.5 & 0 & 0 & 0 \\ 2 & 9 & 1 & 0 & 0 & 0 \\ 1.5 & 1 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.225}$$

(normalized by $\rho\times$ MPa, where ρ is the density in kg/m^3). Then, according to equation (1.213), $c_{12}' = -4$ GPa and $c_{13}' = -5.5$ GPa.

Figure 1.8 shows the group velocities and corresponding snapshots of the wave field in the three symmetry planes of the normal and anomalous media. The polarization is indicated on the curves; when it is not plotted, the particle motion is perpendicular to the respective plane (cross-plane shear waves). Only one octant is shown due to symmetry considerations.

As can be seen, a sign change in $c_{12} + c_{66}$ and $c_{13} + c_{55}$ only affects the (x, y)- and (x, z)- planes, leaving the polarizations in the (y, z)-plane unaltered. The anomaly is more pronounced about 45° where the polarization of the fastest wave is quasi-transverse and the cusp lid is essentially longitudinal. Moreover, the cross-plane shear wave with polarization perpendicular to the respective symmetry plane can be clearly seen in Figures 1.8c and d. More details about this example and anomalous polarization can be found in Carcione and Helbig (2001).

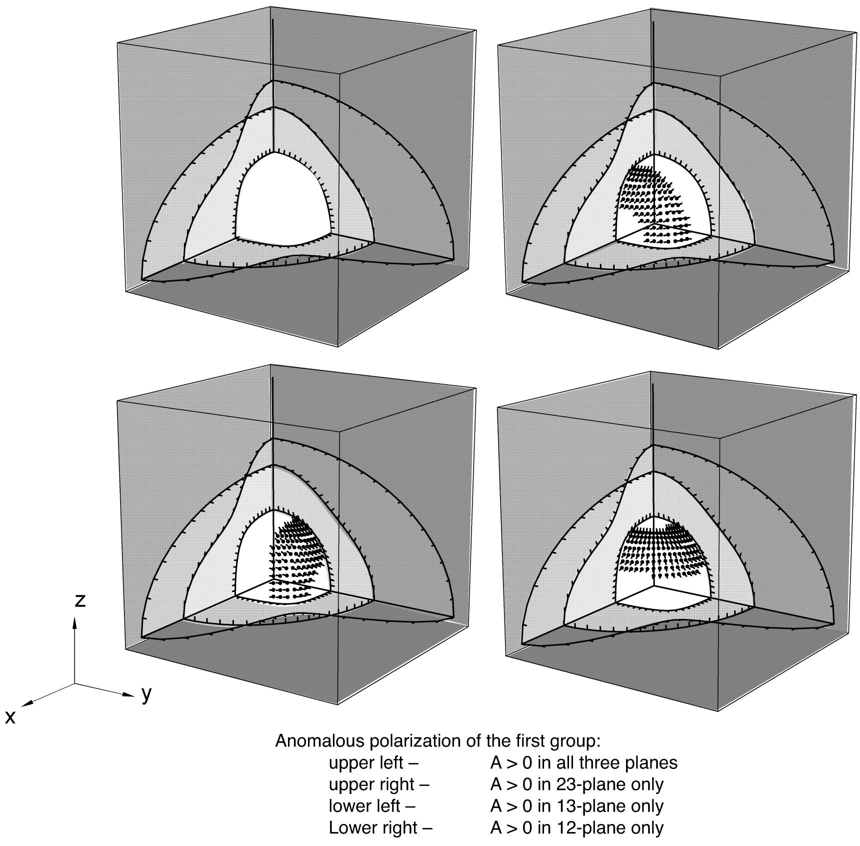

Figure 1.7: Distribution of anomalous polarization of the fastest sheet of the slowness surface for $A_{12}A_{23}A_{13} > 0$, where $A_{12} = c_{12} + c_{66}$, $A_{23} = c_{23} + c_{44}$ and $A_{13} = c_{13} + c_{55}$. Polarization vectors are plotted if they make an angle greater than $\pi/4$ with the propagation direction. Top left: all three $A_{IJ} > 0$; top right: only $A_{23} > 0$, bottom left: only $A_{13} > 0$; bottom right: only $A_{12} > 0$.

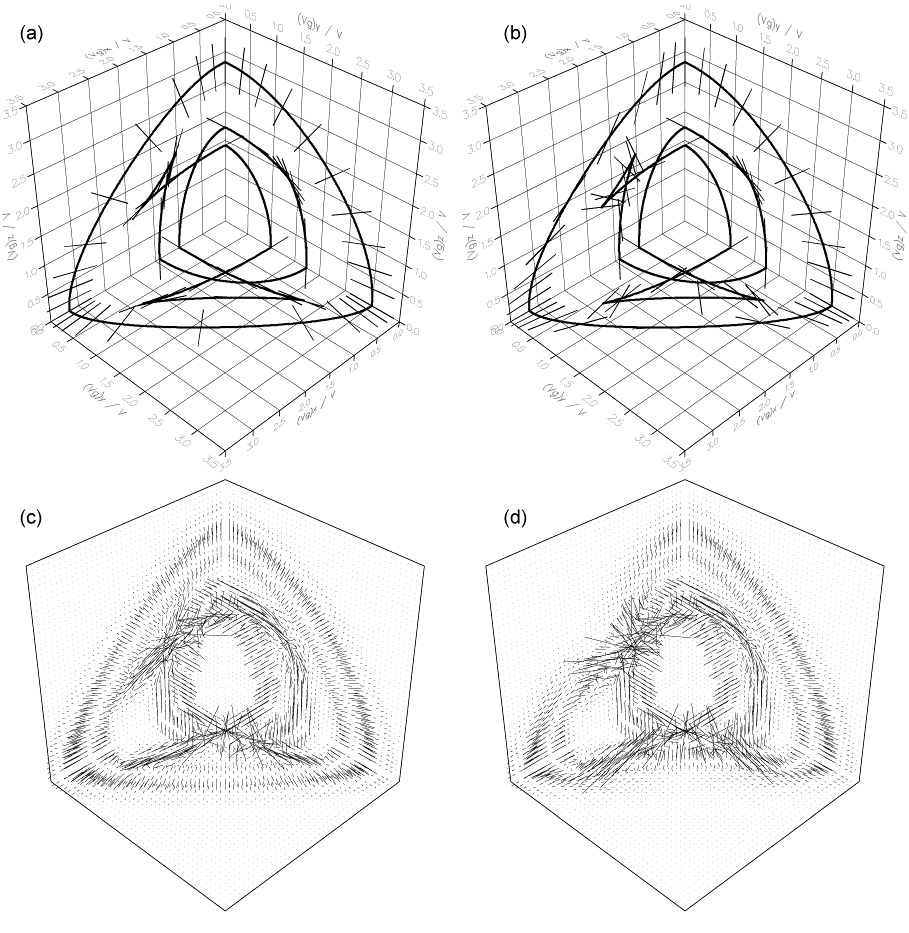

Figure 1.8: Ray-velocity sections and snapshots of the displacement vector at the symmetry planes of an orthorhombic medium. Figures (b) and (d) correspond to the anomalous medium. Only one octant of the model space is displayed due to symmetry considerations.

1.7 The best isotropic approximation

We address in this section the problem of finding the best isotropic approximation of the anisotropic stress-strain relation and quantifying anisotropy with a single numerical index. Fedorov (1968) and Backus (1970) obtained the bulk and shear moduli of the best isotropic medium using component notation. Here, we follow the approach of Cavallini (1999), who used a shorter and coordinate-free derivation of equivalent results. The reader may refer to Gurtin (1981) for background material on the corresponding mathematical methods.

Let $\mathbb{X}$ be any real finite-dimensional vector space, with a scalar product $\mathbf{a} \cdot \mathbf{b}$ for $\mathbf{a}$, $\mathbf{b}$ in $\mathbb{X}$. The tensor (dyadic) product $\mathbf{a} \otimes \mathbf{b}$ is the linear operator such that

$$(\mathbf{a} \otimes \mathbf{b})\,\mathbf{x} = (\mathbf{b} \cdot \mathbf{x})\,\mathbf{a} \tag{1.226}$$

for all $\mathbf{x}$ in $\mathbb{X}$. The space $L(\mathbb{X})$ of linear operators over $\mathbb{X}$ inherits from $\mathbb{X}$ a scalar product, which is defined by

$$\mathbf{a} \cdot \mathbf{b} = \mathrm{tr}(\mathbf{a}^\top \circ \mathbf{b}) \qquad \text{for}\, \mathbf{a},\, \mathbf{b}\, \text{in}\, L(\mathbb{X}), \tag{1.227}$$

where tr denotes the trace (the sum of all eigenvalues, each counted with its multiplicity), and symbol $\circ$ denotes the composition of maps. We denote by $\mathbb{R}^n$ the $n-$dimensional Euclidean space. Moreover, Lin is the space of linear operators over $\mathbb{R}^3$, Sym is the subspace of Lin formed by symmetric operators, S is the subspace of Sym formed by all the operators proportional to the identity operator $\mathbf{I}_3$, D is the subspace of Sym formed by all the operators with zero trace. The operators $\mathbf{S}$ (spherical) and $\mathbf{D}$ (deviatoric), defined by

$$\mathbf{S} = \frac{1}{3}\mathbf{I}_3 \otimes \mathbf{I}_3 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.228}$$

and

$$\mathbf{D} = \mathbf{I}_6 - \mathbf{S} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} \tag{1.229}$$

are orthogonal projections from Sym into S and D, respectively.

We consider the stress-strain relation (1.31), written in tensorial notation instead of the Voigt matrix notation. Using tensor notation, also termed "Kelvin's notation", the stress-strain relation reads

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{23} \\ \sqrt{2}\sigma_{13} \\ \sqrt{2}\sigma_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \sqrt{2}c_{14} & \sqrt{2}c_{15} & \sqrt{2}c_{16} \\ c_{12} & c_{22} & c_{23} & \sqrt{2}c_{24} & \sqrt{2}c_{25} & \sqrt{2}c_{26} \\ c_{13} & c_{23} & c_{33} & \sqrt{2}c_{34} & \sqrt{2}c_{35} & \sqrt{2}c_{36} \\ \sqrt{2}c_{14} & \sqrt{2}c_{24} & \sqrt{2}c_{34} & 2c_{44} & 2c_{45} & 2c_{46} \\ \sqrt{2}c_{15} & \sqrt{2}c_{25} & \sqrt{2}c_{35} & 2c_{45} & 2c_{55} & 2c_{56} \\ \sqrt{2}c_{16} & \sqrt{2}c_{26} & \sqrt{2}c_{36} & 2c_{46} & 2c_{56} & 2c_{66} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \sqrt{2}\epsilon_{23} \\ \sqrt{2}\epsilon_{13} \\ \sqrt{2}\epsilon_{12} \end{pmatrix} \tag{1.230}$$

(Mehrabadi and Cowin, 1990; Helbig, 1994, p. 406). The three arrays in equation (1.230) are true tensors in 6-D space. Using the same symbols for simplicity, equation (1.230) is similar to (1.31) ($\boldsymbol{\sigma} = \mathbf{C} \cdot \mathbf{e}$) where $\mathbf{C} : \mathrm{Sym} \to \mathrm{Sym}$ is a linear operator.

Accordingly, isotropy is a special case of anisotropy, and the isotropic stiffness operator has the form

$$\mathbf{C}_{\mathrm{iso}} = 3\mathcal{K}\,\mathbf{S} + 2\mu\,\mathbf{D} \tag{1.231}$$

where $\mathcal{K}$ and μ are the bulk and shear moduli, respectively.

The norms of $\mathbf{S}$ and $\mathbf{D}$ are 1 and $\sqrt{5}$, respectively, where the norm has the usual definition $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. Therefore, $\mathbf{S}$ and $(1/\sqrt{5})\,\mathbf{D}$ constitute an orthonormal pair, and the projector onto the space of isotropic elasticity tensors is[4]

$$P_{\mathrm{iso}} = \mathbf{S} \otimes \mathbf{S} + \frac{1}{5}\,\mathbf{D} \otimes \mathbf{D}. \tag{1.232}$$

Thus, given an anisotropic stiffness tensor $\mathbf{C}$, its best isotropic approximation is

$$P_{\mathrm{iso}}\,\mathbf{C} = (\mathbf{S} \cdot \mathbf{C})\mathbf{S} + \frac{1}{5}(\mathbf{D} \cdot \mathbf{C})\mathbf{D}, \tag{1.233}$$

where we have used equation (1.226). Now, comparing (1.231) and (1.233), the dilatational term is $3\mathcal{K} = \mathbf{S} \cdot \mathbf{C} = \mathrm{tr}(\mathbf{S}^\top \circ \mathbf{C}) = \mathrm{tr}(\mathbf{C} \cdot \mathbf{I}_3)/3$, according to equations (1.227) and (1.228). The shear term is obtained in the same way by using equation (1.229). Hence, the corresponding bulk and shear moduli are

$$\mathcal{K} = \frac{1}{9}\mathrm{tr}(\mathbf{C} \cdot \mathbf{I}_3) \quad \text{and} \quad \mu = \frac{1}{10}\mathrm{tr}\,\mathbf{C} - \frac{1}{30}\mathrm{tr}(\mathbf{C} \cdot \mathbf{I}_3). \tag{1.234}$$

In Voigt's notation, we have

$$\mathrm{tr}(\mathbf{C} \cdot \mathbf{I}_3) = \sum_{I,J=1}^{3} c_{IJ} \quad \text{and} \quad \mathrm{tr}\,\mathbf{C} = \sum_{I=1}^{3} c_{II} + 2\sum_{I=4}^{6} c_{II}, \tag{1.235}$$

to obtain

$$\mathcal{K} = \frac{1}{9}[c_{11} + c_{22} + c_{33} + 2(c_{12} + c_{13} + c_{23})] \tag{1.236}$$

and

$$\mu = \frac{1}{15}[c_{11} + c_{22} + c_{33} + 3(c_{44} + c_{55} + c_{66}) - (c_{12} + c_{13} + c_{23})]. \tag{1.237}$$

Note that the bulk modulus (1.236) can be obtained by assuming an isotropic strain state, i.e, $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon_0$, and $\epsilon_{23} = \epsilon_{13} = \epsilon_{12} = 0$. Then, the mean stress $\bar{\sigma} = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$ can be expressed as $\bar{\sigma} = 3\mathcal{K}\epsilon_0$.

The eigenvalues of the isotropic stiffness operator (1.231) are $3\mathcal{K}$ and 2μ, with corresponding eigenspaces S (of dimension 1) and D (of dimension 5), respectively (see Chapter 4, Section 4.1.2). Then, from the symmetry of the stiffness operator, we immediately get the orthogonality between the corresponding projectors $\mathbf{S}$ and $\mathbf{D}$, as mentioned before.

[4]In order to get a geometrical picture of the projector P_{iso}, imagine that $\mathbf{S}$ and $\mathbf{D}$ represent two orthonormal unit vectors along the Cartesian axes x and y, respectively. To project a general vector $\mathbf{x}$ onto the x-axis, we perform the scalar product $\mathbf{S} \cdot \mathbf{x}$ and obtain the projected vector as $(\mathbf{S} \cdot \mathbf{x})\mathbf{S}$, which is equal to $(\mathbf{S} \otimes \mathbf{S})\mathbf{x}$ according to equation (1.226).

In order to quantify with a single number the level of anisotropy present in a material, we introduce the anisotropy index

$$I_A = \frac{\|\mathbf{C} - P_{\rm iso}\,\mathbf{C}\|}{\|\mathbf{C}\|} = \sqrt{1 - \frac{\|P_{\rm iso}\,\mathbf{C}\|^2}{\|\mathbf{C}\|^2}}, \tag{1.238}$$

where the second identity follows from the n-dimensional Pythagoras' theorem. We obviously have $0 \leq I_A \leq 1$, with $I_A = 0$ corresponding to isotropic materials.

The quantity $\|P_{\rm iso}\,\mathbf{C}\|^2$ that appears in equation (1.238) is easily computed using equation (1.231) and the orthonormality of the pair $\{\mathbf{S},\ (1/\sqrt{5})\mathbf{D}\}$:

$$\|P_{\rm iso}\,\mathbf{C}\|^2 = 9\,\mathcal{K}^2 + 20\,\mu^2 \tag{1.239}$$

To compute $\|\mathbf{C}\|^2$, we need to resort to component notation; for example, in Voigt's notation we have

$$\|\mathbf{C}\|^2 = \sum_{I,J=1}^{3} c_{IJ}^2 + 4\sum_{I,J=4}^{6} c_{IJ}^2 + 4\sum_{I=1}^{3}\sum_{J=4}^{6} c_{IJ}^2 \tag{1.240}$$

As an example, let us consider the orthorhombic elastic matrix (1.225) and its corresponding anomalously-polarized medium whose matrix is obtained by using the relations (1.213). Both media have the same slowness surfaces, but their anisotropy indices are 0.28 (normal polarization) and 0.57 (anomalous polarization). Thus, polarization alone has a significant influence on the degree of anisotropy. Regarding geological media, Cavallini (1999) computed the anisotropic index for 44 shales, with the result that the index ranges from a minimum value of 0.048 to a maximum value of 0.323. The median is 0.17.

A general anisotropic stress-strain relation can also be approximated by symmetries lower than isotropy, such as transversely isotropic and orthorhombic media. This has been done by Arts (1993) using Federov's approach.

1.8 Analytical solutions for transversely isotropic media

2-D and 3-D analytical solutions are available for the Green function – the response to $\delta(t)\delta(\mathbf{x})$ – in the symmetry axis of a transversely isotropic medium. This section shows how these exact solutions can be obtained. The complete Green's tensor for ellipsoidal slowness surfaces has been obtained by Burridge, Chadwick and Norris (1993).

1.8.1 2-D Green's function

Payton (1983, p. 38) provides a classification of the wave-front curves on the basis of the location of the cusps. We consider here class IV materials, for which there are four cusps, two of them centered on the symmetry axis. Let us consider the (x, z)-plane and define

$$\check{\alpha} = c_{33}/c_{55}, \qquad \check{\beta} = c_{11}/c_{55}, \qquad \check{\gamma} = 1 + \check{\alpha}\check{\beta} - (c_{13}/c_{55} + 1)^2, \tag{1.241}$$

the dimensionless variable

$$\zeta = \left(\frac{z}{t}\right)^2 \left(\frac{\rho}{c_{55}}\right), \tag{1.242}$$

and

$$t_P = z\sqrt{\frac{\rho}{c_{33}}}, \qquad t_S = z\sqrt{\frac{\rho}{c_{55}}}. \tag{1.243}$$

The following Green's function is given in Payton (1983, p. 78) and is valid for materials satisfying the conditions

$$\check{\gamma} < (\check{\beta} + 1), \qquad (\check{\gamma}^2 - 4\check{\alpha}\check{\beta}) < 0. \tag{1.244}$$

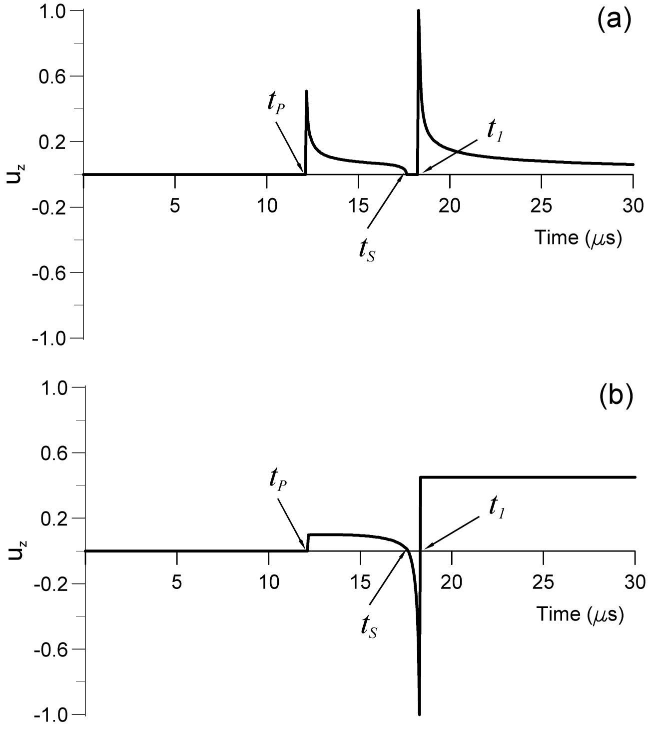

Figure 1.9: Two-dimensional Green's function (a) and three-dimensional response to Heaviside's function (b) as a function of time. The source is a z-directional point force. The medium is apatite and the source-receiver distance is 8 cm.

Due to a force directed in the z-direction, the Green function is

$$u_1(z, t) = 0, \tag{1.245}$$

$$u_3(z,t) = \begin{cases} 0, & 0 \le t \le t_P, \\ G_1(\zeta), & t_P < t < t_S, \\ 0, & t_S \le t \le t_1, \\ G_3(\zeta), & t > t_1, \end{cases} \tag{1.246}$$

with

$$G_1(\zeta) = \frac{\sqrt{\zeta}}{4\pi z}\left[1 - \frac{2(1-\zeta) - \check{\gamma} + (\check{\beta}+1)\zeta}{\sqrt{\varsigma}}\right]\sqrt{\frac{\check{\gamma} - (\check{\beta}+1)\zeta - \sqrt{\varsigma}}{2(\check{\alpha} - \zeta)(1-\zeta)}}, \tag{1.247}$$

$$G_3(\zeta) = \frac{\sqrt{\zeta}}{2\pi z}\left[\sqrt{\breve{\beta}} + \sqrt{\frac{1-\zeta}{\breve{\alpha}-\zeta}}\right]\left\{\breve{\gamma} - (\breve{\beta}+1)\zeta + 2\sqrt{\breve{\beta}(\breve{\alpha}-\zeta)(1-\zeta)}\right\}^{-1/2}, \quad (1.248)$$

where

$$t_1 = \frac{t_S}{\breve{\eta}}, \quad (1.249)$$

$$\breve{\eta} = \sqrt{\breve{\gamma}(\breve{\beta}+1) - 2\breve{\beta}(\breve{\alpha}+1) + 2\sqrt{\breve{\beta}(1+\breve{\alpha}\breve{\beta}-\breve{\gamma})(\breve{\alpha}+\breve{\beta}-\breve{\gamma})}/(\breve{\beta}-1)}, \quad (1.250)$$

and

$$\varsigma = [\breve{\gamma} - (\breve{\beta}+1)\zeta]^2 - 4\breve{\beta}(\breve{\alpha}-\zeta)(1-\zeta). \quad (1.251)$$

Figure 1.9a shows the Green function for apatite at 8 cm from the source location (see Figure 1.1 for an illustration of the slowness and group-velocity sections). The singularities are located at times t_P and t_1, and the lacuna – due to the cusps – can be seen between times t_S and t_1. The last singularity is not present in an isotropic medium because $t_1 \to \infty$. For more details see Carcione, Kosloff and Kosloff (1988a).

1.8.2 3-D Green's function

In the 3-D case, the response to Heaviside's function $H(t)$ is available (Payton, 1983, p. 108). (Condition (1.244) must be satisfied in the following solution.) Let us consider a force along the z-direction, that is

$$\mathbf{f} = (0,0,1)\delta(x)\delta(y)\delta(z)H(t). \quad (1.252)$$

The solution is

$$u_1(z,t) = u_2(z,t) = 0, \quad (1.253)$$

$$u_3(z,t) = \frac{1}{4\pi z}\left(\frac{\rho}{c_{55}}\right)\begin{cases} 0, & 0 \le t \le t_P, \\ h(\zeta), & t_P < t < t_S, \\ 2h(\zeta), & t_S \le t \le t_1, \\ 1, & t > t_1, \end{cases} \quad (1.254)$$

with

$$h(\zeta) = \frac{1}{2} - \frac{2(1-\zeta) - \breve{\gamma} + (\breve{\beta}+1)\zeta}{2\sqrt{\varsigma}}, \quad (1.255)$$

where the involved quantities have been introduced in the previous section. The Green function is the time derivative of (1.254).

Figure 1.9b shows the response to Heaviside's function for apatite at 8 cm from the source location. A seismogram can be obtained by time convolution of (1.254) with the time derivative of the corresponding source wavelet. For more details see Carcione, Kosloff, Behle and Seriani (1992).

1.9 Reflection and transmission of plane waves

An analysis of the reflection-transmission problem in anisotropic elastic media can be found in Musgrave (1960), Henneke II (1971), Daley and Hron (1977), Keith and Crampin (1977), Rokhlin, Bolland and Adler (1986), Graebner (1992), Schoenberg and Protazio

(1992), Chapman (1994), Pšenčík and Vavryčuk (1998) and Ursin and Haugen (1996). In the anisotropic case, we study the problem in terms of energy flow rather than amplitude, since the energy-flow direction, in general, does not coincide with the propagation (wavevector) direction. Critical angles occur when the ray (energy-flow) direction is parallel to the interface.

In this section, we formally introduce the problem for the general 3-D case and discuss in detail the reflection-transmission problem of cross-plane waves in the symmetry plane of a monoclinic medium (Schoenberg and Costa, 1991, Carcione, 1997a). This problem, considered in the context of a single wave mode, illustrates most of the phenomena related to the presence of anisotropy.

Let us consider a plane wave of the form (1.62), incident from the upper medium on a plane boundary between two anisotropic media. The incident wave generates three reflected waves and three transmitted waves. For a welded contact, the boundary conditions are continuity of displacement (or particle velocity) and stresses on the interface:

$$\mathbf{u}^I + \mathbf{u}^R_{\rm qP} + \mathbf{u}^R_{\rm qS1} + \mathbf{u}^R_{\rm qS2} = \mathbf{u}^T_{\rm qP} + \mathbf{u}^T_{\rm qS1} + \mathbf{u}^T_{\rm qS2}, \tag{1.256}$$

$$(\boldsymbol{\sigma}^I + \boldsymbol{\sigma}^R_{\rm qP} + \boldsymbol{\sigma}^R_{\rm qS1} + \boldsymbol{\sigma}^R_{\rm qS2}) \cdot \hat{\mathbf{n}} = (\boldsymbol{\sigma}^T_{\rm qP} + \boldsymbol{\sigma}^T_{\rm qS1} + \boldsymbol{\sigma}^T_{\rm qS2}) \cdot \hat{\mathbf{n}}, \tag{1.257}$$

where I, R and T denote the incident, reflected and transmitted waves, and $\hat{\mathbf{n}}$ is a unit vector normal to the interface. These are six boundary conditions, constituting a system of six algebraic equations in terms of the six unknown amplitudes of the reflected and transmitted waves.

Snell's law implies the following:

- All slowness vectors should lie in the plane formed by the slowness vector of the incident wave and the normal to the interface.
- The projections of the slowness vectors on the interface coincide.

Since the slowness vectors lie in the same plane, it is convenient to choose this plane as one of the Cartesian planes, say, the (x, z)-plane. Once the elasticity constants are transformed into this system, the slowness vectors have two components. Let the interface be in the x-direction. The y-component of the slowness vectors is zero, and the x-components are equal to that of the incident wave,

$$s_1^I = s^R_{1\rm qP} = s^R_{1\rm qS1} = s^R_{1\rm qS2} = s^T_{1\rm qP} = s^T_{1\rm qS1} = s^T_{1\rm qS2} \equiv s_1. \tag{1.258}$$

The unknown s_3 components are found by solving the dispersion relation (1.72), which can be rewritten as

$$\det(c_{ijkl}s_k s_l - \rho\delta_{ij}) = 0, \tag{1.259}$$

since $s_k = l_k/v_p$. This six-order equation is solved for the upper and lower media. Of the 12 solutions – 6 for the lower media and 6 for the upper media – we should select three physical solutions for each half-space. The criterion is that the Umov-Poynting vector (the energy-flow vector) must point into the incidence medium for the reflected waves and into the transmission medium for the transmitted waves. At critical angles and for evanescent waves, the energy-flow vector is parallel to the interface. Calculation of the energy-flow vector requires the calculations of the eigenvectors for each reflected and transmitted wave

– special treatment is required along the symmetry axes. A root of the dispersion relation can be real or complex. In the latter case, the chosen sign of its imaginary part must be such that it has an exponential decay away from the interface. However, this criterion is not always valid in lossy media (see Chapter 6). It is then convenient to check the solutions by computing the energy balance normal to the interface.

Our analysis is simplified when the incidence plane coincides with a plane of symmetry of both media, because the incident wave does not generate all the reflected-transmitted modes at the interface. For example, at a symmetry plane, there always exists a cross-plane shear wave (Helbig, 1994, p. 111). An incident cross-plane shear wave generates a reflected and a transmitted wave of the same nature. Incidence of qP and qS waves generates qP and qS waves. Most of the examples found in the literature correspond to these cases. Figure 1.10 shows an example of analysis for normally polarized media. A qP wave reaches the interface, generating two reflected waves and two transmitted waves. The projection of the wavevector onto the interface is the same for all the waves, and the group-velocity vector is perpendicular to the slowness curve.

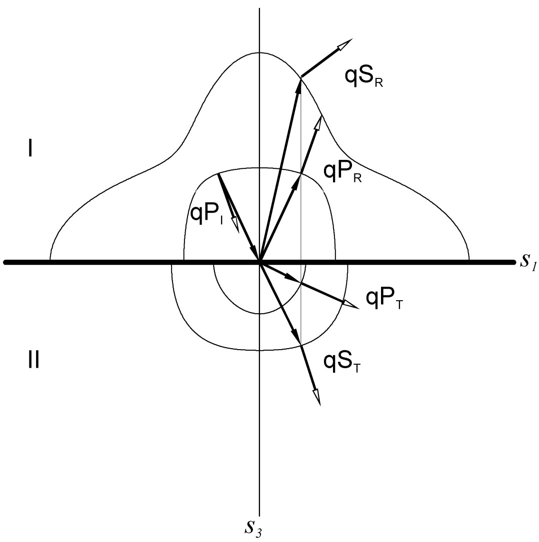

Figure 1.10: Example of analysis using the slowness surfaces of the reflection-transmission problem between two anisotropic media (I and II). The media are transversely isotropic with the symmetry axes along the directions perpendicular to the interface. The inner and outer curves are the qP and qS slowness sections, respectively. Full arrows correspond to the wavevector and empty arrows to the group-velocity vector.

Strange effects are caused by the deviation of the energy-flow vector from the wavevector direction. In the phenomenon of external conical refraction, the Umov-Poynting vector may be normal to the slowness surface at an infinite set of points. If the symmetry axis of the incidence medium is normal to the interface and the transmission medium is isotropic, Snell's law implies the existence of a divergent circular cone of transmitted rays (Musgrave, 1970, p. 144).

1.9.1 Cross-plane shear waves

Equations (1.46) describe cross-plane shear motion in the plane of symmetry of a monoclinic medium. Let us introduce the plane wave

$$v_2 = v = \mathrm{i}\omega u_0 \exp[\mathrm{i}\omega(t - s_1 x - s_3 z)], \tag{1.260}$$

where u_0 is a constant complex displacement and $s_i = \kappa_i/\omega$ are the slowness components. Substitution of this plane wave into equations (1.46) gives the slowness relation

$$F(s_1, s_3) = c_{66}s_1^2 + 2c_{46}s_1s_3 + c_{44}s_3^2 - \rho = 0, \tag{1.261}$$

which, in real (s_1, s_3) space, is an ellipse due to the positive definite conditions

$$c_{44} > 0, \quad c_{66} > 0, \quad c^2 \equiv c_{44}c_{66} - c_{46}^2 > 0, \tag{1.262}$$

which can be deduced from equations (1.33).

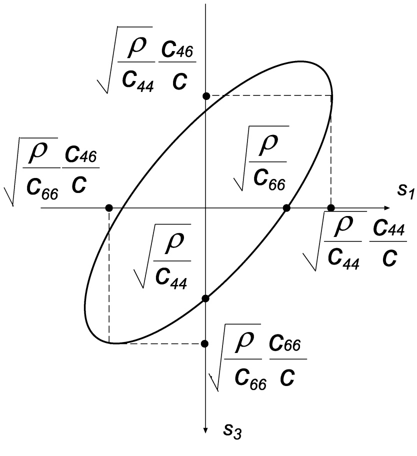

Figure 1.11: Characteristics of the slowness surface corresponding to an SH wave in the plane of symmetry of a monoclinic medium.

Figure 1.11 illustrates the characteristics of the slowness curve. The group or energy velocity can be calculated by using equation (1.130),

$$v_{e1} = (c_{66}s_1 + c_{46}s_3)/\rho, \quad v_{e3} = (c_{46}s_1 + c_{44}s_3)/\rho. \tag{1.263}$$

Solving for s_1 and s_3 in terms of v_{e1} and v_{e3}, and substituting the result into equation (1.261), we obtain the energy-velocity surface

$$c_{44}v_{e1}^2 - 2c_{46}v_{e1}v_{e3} + c_{66}v_{e3}^2 - c^2/\rho = 0, \tag{1.264}$$

which is also an ellipse. In order to distinguish between down and up propagating waves, the slowness relation (1.261) is solved for s_3, given the horizontal slowness s_1. It yields

$$s_{3\pm} = \frac{1}{c_{44}}\left(-c_{46}s_1 \pm \sqrt{\rho c_{44} - c^2 s_1^2}\right). \tag{1.265}$$

In principle, the + sign corresponds to downward or $+z$ propagating waves, while the $-$ sign corresponds to upward or $-z$ propagating waves.

Substituting the plane wave (1.260) into equations $(1.46)_2$ and $(1.46)_3$, we get

$$\sigma_{12} = -(c_{46}s_3 + c_{66}s_1), \quad \text{and} \quad \sigma_{23} = -(c_{44}s_3 + c_{46}s_1). \tag{1.266}$$

The Umov-Poynting vector (1.100) is given by

$$\mathbf{p} = -\frac{1}{2}(\sigma_{12}\hat{\mathbf{e}}_1 + \sigma_{23}\hat{\mathbf{e}}_3)v^*. \tag{1.267}$$

Substituting the plane wave (1.260) and the stress-strain relations (1.266) into equation (1.267), we obtain

$$\mathbf{p} = \frac{1}{2}\omega^2|u_0|^2(X\hat{\mathbf{e}}_1 + Z\hat{\mathbf{e}}_3), \tag{1.268}$$

where

$$X = c_{66}s_1 + c_{46}s_3, \quad \text{and} \quad Z = c_{46}s_1 + c_{44}s_3. \tag{1.269}$$

Using equation (1.265), we have

$$Z = \pm\sqrt{\rho c_{44} - c^2 s_1^2}. \tag{1.270}$$

The particle velocity of the incident wave can be written as

$$v^I = \mathrm{i}\omega \exp[\mathrm{i}\omega(t - s_1 x - s_3^I z)], \tag{1.271}$$

where

$$s_1 = \sin\theta^I / v_p(\theta^I), \qquad s_3^I = \cos\theta^I / v_p(\theta^I), \tag{1.272}$$

where θ^I is the incidence propagation angle (see Figure 1.11), and

$$v_p(\theta) = \sqrt{(c_{44}\cos^2\theta + c_{66}\sin^2\theta + c_{46}\sin 2\theta)/\rho} \tag{1.273}$$

is the phase velocity.

Snell's law, i.e., the continuity of the horizontal slowness,

$$s_1^R = s_1^T = s_1, \tag{1.274}$$

is a necessary condition to satisfy the boundary conditions.

Denoting the reflection and transmission coefficients by R_{SS} and T_{SS}, the particle velocities of the reflected and transmitted waves are given by

$$v^R = \mathrm{i}\omega R_{\mathrm{SS}} \exp[\mathrm{i}\omega(t - s_1 x - s_3^R z)] \tag{1.275}$$

and

$$v^T = \mathrm{i}\omega T_{\mathrm{SS}} \exp[\mathrm{i}\omega(t - s_1 x - s_3^T z)], \tag{1.276}$$

respectively.

Then, continuity of v and σ_{23} at $z = 0$ gives

$$T_{\mathrm{SS}} = 1 + R_{\mathrm{SS}} \tag{1.277}$$

and

$$Z^I + R_{\rm SS} Z^T = T_{\rm SS} Z^T, \tag{1.278}$$

which have the following solution:

$$R_{\rm SS} = \frac{Z^I - Z^T}{Z^T - Z^R}, \qquad T_{\rm SS} = \frac{Z^I - Z^R}{Z^T - Z^R}, \tag{1.279}$$

where Z is defined in equation $(6.10)_2$. Since both the incident and reflected waves satisfy the slowness relation (1.261), the vertical slowness s_3^R can be obtained by subtracting $F(s_1, s_3^I)$ from $F(s_1, s_3^R)$ and assuming $s_3^R \neq s_3^I$. This yields

$$s_3^R = -\left(s_3^I + \frac{2c_{46}}{c_{44}} s_1\right). \tag{1.280}$$

Then, using equation (1.278) we obtain

$$Z^R = -Z^I \tag{1.281}$$

and the reflection and transmission coefficients (1.279) become

$$R_{\rm SS} = \frac{Z^I - Z^T}{Z^I + Z^T}, \qquad T_{\rm SS} = \frac{2Z^I}{Z^I + Z^T}. \tag{1.282}$$

Denoting the material properties of the lower medium by primed quantities, we see that the slowness relation (1.261) of the transmission medium gives s_3^T in terms of s_1:

$$s_3^T = \frac{1}{c'_{44}}\left(-c'_{46} s_1 + \sqrt{\rho' c'_{44} - c'^2 s_1^2}\right), \tag{1.283}$$

with

$$c'^2 = c'_{44} c'_{66} - {c'_{46}}^2. \tag{1.284}$$

Alternatively, from equation (1.270),

$$s_3^T = \frac{1}{c'_{44}}\left(Z^T - c'_{46} s_1\right). \tag{1.285}$$

Let us consider an isotropic medium above a monoclinic medium, with the wavevector of the incidence wave lying in the (x, z)-plane, which is assumed to be the monoclinic-medium symmetry plane. Then, an incident cross-plane shear wave generates only reflected and transmitted cross-plane shear waves. This case is discussed by Schoenberg and Costa (1991).

The isotropic medium has elasticity constants $c_{46} = 0$ and $c_{44} = c = \mu$. The vertical slowness is

$$s_3^I = \sqrt{\rho/\mu - s_1^2}, \tag{1.286}$$

and

$$Z^I = \mu s_3^I. \tag{1.287}$$

From equation (1.270), we have

$$Z^T = \pm\sqrt{\rho' c'_{44} - c'^2 s_1^2}, \tag{1.288}$$

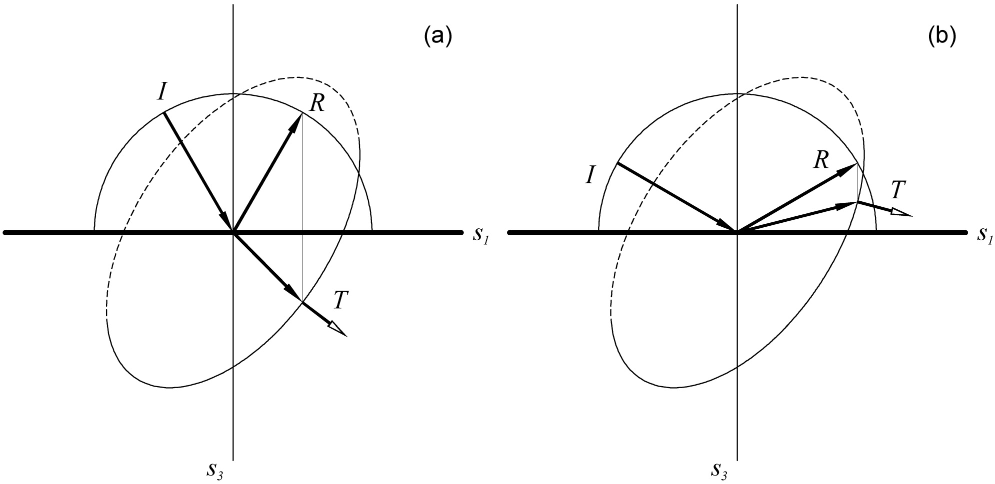

Figure 1.12: The reflection-transmission problem for an SH wave incident on an interface between an isotropic medium and a monoclinic medium: (a) illustrates the slowness vectors when the transmitted wave has a downward-pointing slowness vector, and (b) shows the slowness vectors when the transmitted wave has an upward-pointing slowness vector. The corresponding group-velocity vector (empty arrow) is normal to the slowness surface and points downwards.

Two different situations are shown in Figure 1.12a-b. The slowness sections are shown, together with the respective wavevectors (full arrows) and energy-velocity vectors (empty arrows). In Figure 1.12a, the transmitted slowness vector has a positive value of s_3 and points downwards. In Figure 1.12b, it has a negative value of s_3 and points upwards. However, the energy-velocity vector points downwards and the solution is a valid transmitted wave. The transmitted slowness vector must be a point on the lower section (solid line) of the slowness surface since there the energy-velocity vector points downwards.

Example: Let us consider the following properties

$$\mu = 10 \text{ GPa}, \quad \rho = 2500 \text{ kg/m}^3,$$

and

$$c'_{44} = 15 \text{ GPa}, \quad c'_{46} = -7 \text{ GPa}, \quad c'_{66} = 22 \text{ GPa}, \quad \rho' = 2700 \text{ kg/m}^3.$$

The absolute values of the reflection and transmission coefficients versus the incidence angle are shown in Figure 1.13.

According to equation (1.268), the condition $Z^T = 0$ yields the critical angle θ_C. From equation (1.288), we obtain

$$\theta_C = \arcsin\left(\frac{1}{c'}\sqrt{\frac{\rho'}{\rho}c'_{44}\mu}\right). \tag{1.289}$$

In this case $\theta_C = 49^\circ$. Beyond the critical angle, the time-averaged power-flow vector of the transmitted wave is parallel to the interface because $\mathrm{Re}(Z^T) = 0$ (see equation

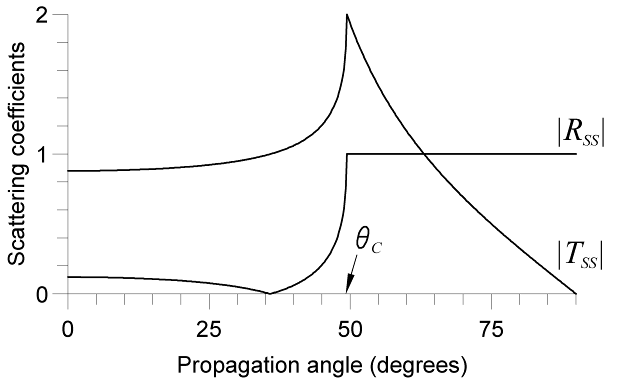

Figure 1.13: Absolute values of the reflection and transmission coefficients versus the incidence propagation angle for an SH wave incident on an interface separating an isotropic medium and a monoclinic medium.

(1.268)) and the wave becomes evanescent. This problem is discussed in more detail in Chapter 6, where dissipation is considered.

Note that any lower medium with constants $\bar{c}_{44}$, $\bar{c}_{46}$ and density $\bar{\rho}$ satisfying

$$\bar{\rho}\bar{c}_{44} = \rho' c'_{44}, \qquad \bar{c}^2 = c'^2, \tag{1.290}$$

will have the same $R_{\rm SS}$ and $T_{\rm SS}$ for all s_1. If we choose the material properties of the isotropic medium to satisfy

$$\rho\mu = \rho' c'_{44}, \qquad \mu = c'^2, \tag{1.291}$$

then $Z^I = Z^T$, $R_{\rm SS} = 0$ and $T_{\rm SS}$=1 for all s_1. In such a case, there is no reflected wave, and, thus, the interface would be impossible to detect using a reflection method based on cross-plane shear waves.