Chapter 5

The reciprocity principle

The reciprocal property is capable of generalization so as to apply to all acoustical systems whatever capable of vibrating about a configuration of equilibrium, as I proved in the Proceedings of the Mathematical Society for June 1873 [Art. XXI], and is not lost even when the systems are subject to damping...

John William Strutt (Lord Rayleigh) (Rayleigh, 1899a)

The reciprocity principle relates two solutions in a medium where the sources and the field receivers are interchanged. The principle for static displacements is credited to Bétti (1872). Rayleigh (1873) extended the principle to vibrating bodies and included the action of dissipative forces[1] (see Rayleigh, 1945, vol. 1, p. 157f). Lamb (1888) showed how the reciprocal theorems of Helmholtz – in the theory of least action in acoustics and optics – and of Lord Rayleigh – in acoustics – can be derived from a formula established by Lagrange in the *Méchanique Analytique* (1809), thereby anticipating Lagrange's theory of the variation of arbitrary constants (Fung, 1965, p. 429).

In this century, the work of Graffi (1939, 1954, 1963) is notable. Graffi derived the first convolutional reciprocity theorem for an isotropic, homogeneous, elastic solid. Extension to inhomogeneous elastic anisotropic media was achieved by Knopoff and Gangi (1959). Gangi (1970) developed a volume integral, time-convolution formulation of the reciprocity principle for inhomogeneous anisotropic linearly elastic media. This formulation permits the use of distributed sources as well as multi-component sources (i.e., couples with and without moment). Gangi also derived a representation of particle displacement in terms of Green's theorem.

de Hoop (1966) generalized the principle to the anisotropic viscoelastic case. It is worth mentioning the work of Boharski (1983), who distinguished between convolution-type and correlation-type reciprocity relations. Recently, de Hoop and Stam (1988) derived a general reciprocity theorem valid for solids with relaxation, including reciprocity for stress, as well as for particle velocity (see also de Hoop, 1995). Laboratory experiments of the reciprocity principle were performed by Gangi (1980b), who used a granite block containing a cylindrical brass obstacle to act as a scatterer, and piezoelectric transducers to act as vertical source and vertical receiver. A direct numerical test of the principle in the inhomogeneous anisotropic elastic case was performed by Carcione and Gangi (1998).

[1] A reciprocity relation when the source is a dipole rather a monopole has been derived by Lord Rayleigh in 1876.

Useful applications of the reciprocity principle can be found in Fokkema and van den Berg (1993).

5.1 Sources, receivers and reciprocity

Reciprocity is usually applied to concentrated point forces and point receivers. However, reciprocity has a much wider application potential; in many cases, it is not used at its full potential, either because a variety of source and receiver types are not considered or their implementation is not well understood.

Reciprocity holds for the very general case of an inhomogeneous anisotropic viscoelastic solid, in the presence of boundary surfaces satisfying Dirichlet and/or Neumann boundary conditions (e.g., Lamb's problem, (Lamb, 1904)) (Fung, 1965, p. 214). However, it is not clear how the principle is applied when the sources are couples (Fenati and Rocca, 1984). For instance, Mittet and Hokstad (1995) use reciprocity to transform walk-away VSP data into reverse VSP data, for offshore acquisition. Nyitrai, Hron and Razavy (1996) claim that the analytical solution to Lamb's problem – expressed in terms of particle displacement – for a dilatational point source does not exhibit reciprocity when the source and receiver locations are interchanged. Hence, the following question arises: what, if any, source-receiver configuration is reciprocal in this particular situation? In order to answer this question, we apply the reciprocity principle to the case of sources of couples and demonstrate that for any particular source, there is a corresponding receiver-configuration that makes the source-receiver pair reciprocal.

We obtain reciprocity relations for inhomogeneous anisotropic viscoelastic solids, and for distributed sources and receivers. We show that, in addition to the usual relations involving directional forces, the following results exist: i) the diagonal components of the strain tensor are reciprocal for dipole sources (single couple without moment), ii) the off-diagonal components of the stress tensor are reciprocal for double couples with moments, iii) the dilatation due to a directional force is reciprocal to the particle velocity due to a dilatation source, and iv) some combinations of the off-diagonal strains are reciprocal for single couples with moments.

5.2 The reciprocity principle

Let us consider a volume Ω, enclosed by a surface S, in a viscoelastic solid of density $\rho(\mathbf{x})$ and relaxation tensor $\psi_{ijkl}(\mathbf{x}, t)$, where $\mathbf{x} = (x, y, z)$ denotes the position vector. In full explicit form, the equation of motion (1.23) and the stress-strain relation (2.9) can be written as

$$\rho(\mathbf{x})\partial^2_{tt}u_i(\mathbf{x}, t) = \partial_j\sigma_{ij}(\mathbf{x}, t) + f_i(\mathbf{x}, t), \tag{5.1}$$

$$\sigma_{ij}(\mathbf{x}, t) = \psi_{ijkl}(\mathbf{x}, t) * \partial_t\epsilon_{kl}(\mathbf{x}, t). \tag{5.2}$$

A reciprocity theorem valid for a general anisotropic viscoelastic medium can be derived from the equation of motion (5.1) and the stress-strain relation (5.2), and can be written in the form

$$\int_\Omega [u_i(\mathbf{x}, t) * f'_i(\mathbf{x}, t) - f_i(\mathbf{x}, t) * u'_i(\mathbf{x}, t)]d\Omega = 0 \tag{5.3}$$

(Knopoff and Gangi, 1959; de Hoop, 1995). Here u_i is the i-th component of the displacement due to the source $\mathbf{f}$, while u_i' is the i-th component of the displacement due to the source $\mathbf{f}'$. The derivation of equation (5.3) assumes that the displacements and stresses are zero on the boundary S. Zero initial conditions for the displacements are also assumed. Equation (5.3) is well known and can conveniently be used for deriving representations of the displacement in terms of Green's tensor (representation theorem, see Gangi, 1970).

Assuming that the time Fourier transform of displacements and sources exist, equation (5.3) can be transformed into the frequency domain and written as

$$\int_\Omega [\tilde{u}_i(\mathbf{x},\omega)\tilde{f}_i'(\mathbf{x},\omega) - \tilde{f}_i(\mathbf{x},\omega)\tilde{u}_i'(\mathbf{x},\omega)]d\Omega = 0. \tag{5.4}$$

Equation (5.4) can also be expressed in terms of the particle velocity $\tilde{v}_i(\mathbf{x},\omega) = \mathrm{i}\omega\tilde{u}_i(\mathbf{x},\omega)$ by multiplying both sides with $\mathrm{i}\omega$,

$$\int_\Omega [\tilde{v}_i(\mathbf{x},\omega)\tilde{f}_i'(\mathbf{x},\omega) - \tilde{f}_i(\mathbf{x},\omega)\tilde{v}_i'(\mathbf{x},\omega)]d\Omega = 0, \tag{5.5}$$

In the time domain, equation (5.5) reads

$$\int_\Omega [v_i(\mathbf{x},t) * f_i'(\mathbf{x},t) - f_i(\mathbf{x},t) * v_i'(\mathbf{x},t)]d\Omega = 0. \tag{5.6}$$

In the special case that the sources f_i and f_i' have the same time dependence and can be written as

$$f_i(\mathbf{x},t) = h(t)g_i(\mathbf{x}), \qquad f_i'(\mathbf{x},t) = h(t)g_i'(\mathbf{x}), \tag{5.7}$$

equation (5.5) reads

$$\int_\Omega [\tilde{v}_i(\mathbf{x},\omega)g_i'(\mathbf{x}) - g_i(\mathbf{x})\tilde{v}_i'(\mathbf{x},\omega)]d\Omega = 0. \tag{5.8}$$

In the time domain equation (5.8) reads

$$\int_\Omega [v_i(\mathbf{x},t)g_i'(\mathbf{x}) - g_i(\mathbf{x})v_i'(\mathbf{x},t)]d\Omega = 0. \tag{5.9}$$

5.3 Reciprocity of particle velocity. Monopoles

In the following discussion, the indices m and p indicate either x, y or z. The spatial part g_i of the source f_i is referred to as the body force. To indicate the direction of the body force, a superscript is used so that the i-th component g_i^m of a body force acting at $\mathbf{x} = \mathbf{x}_0$ in the m-direction is specified by

$$g_i^m(\mathbf{x};\mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0)\delta_{im}, \tag{5.10}$$

where $\delta(\mathbf{x})$ and δ_{im} are Dirac's and Kronecker's delta functions, respectively. The i-th component g_i^p of a body force acting at $\mathbf{x} = \mathbf{x}_0'$ in the p-direction is, similarly, given by

$$g_i^p(\mathbf{x};\mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0')\delta_{ip}. \tag{5.11}$$

We refer to body forces of the type given by equations (5.10) and (5.11) as monopoles. In the following formulation, we use a superscript on the particle velocity to indicate the direction of the corresponding body force. Then, v_i^m indicates the i-th component of the particle velocity due to a body force acting in the m-direction, while v_i^p indicates the i-th component of the particle velocity due to a body force acting in the p-direction. In addition, we indicate the position of the source in the argument of the particle velocity. A complete specification of the i-th component of the particle velocity due to a body force acting at $\mathbf{x}_0$ in the m-direction is written as $v_i^m(\mathbf{x}, t; \mathbf{x}_0)$. Similarly, we have $v_i^p(\mathbf{x}, t; \mathbf{x}_0')$ for the primed system.

Using the above notation, we can write the reciprocity relation (5.9) as

$$\int_\Omega [v_i^m(\mathbf{x}, t; \mathbf{x}_0) g_i^p(\mathbf{x}; \mathbf{x}_0') - g_i^m(\mathbf{x}; \mathbf{x}_0) v_i^p(\mathbf{x}, t; \mathbf{x}_0')] d\Omega = 0. \tag{5.12}$$

Substituting equations (5.10) and (5.11) into equation (5.12), we obtain

$$\int_\Omega [v_i^m(\mathbf{x}, t; \mathbf{x}_0) \delta(\mathbf{x} - \mathbf{x}_0') \delta_{ip} - \delta(\mathbf{x} - \mathbf{x}_0) \delta_{im} v_i^p(\mathbf{x}, t; \mathbf{x}_0')] d\Omega = 0. \tag{5.13}$$

Recalling the properties of Dirac's and Kronecker's functions, we note that equation (5.13) implies

$$v_p^m(\mathbf{x}_0', t; \mathbf{x}_0) = v_m^p(\mathbf{x}_0, t; \mathbf{x}_0'). \tag{5.14}$$

This equation reveals a fundamental symmetry of the wave field. In any given experiment, the source and receiver positions may be interchanged provided that the particle-velocity component indices and the force component indices are interchanged. Note that this equation only applies to the situation where the source consists of a simple body force. In order to illustrate the interpretation of equation (5.14), Figure 5.1 shows three possible 2-D reciprocal experiments.

5.4 Reciprocity of strain

For more complex sources than a body force oriented along one of the coordinate axes, the reciprocity relation will differ from equation (5.14). Equation (5.9) is, however, valid for an arbitrary spatially distributed source and can be used to derive reciprocity relations for couples of forces. A review of the use of couples for modeling earthquake sources can be found in Aki and Richards (1980, p. 50) and Pilant (1979, p. 356).

5.4.1 Single couples

We consider sources consisting of force couples where the i-th component of the body force takes the particular form

$$g_i^{mn}(\mathbf{x}; \mathbf{x}_0) = \partial_j \delta(\mathbf{x} - \mathbf{x}_0) \delta_{im} \delta_{jn}. \tag{5.15}$$

Here the double superscript mn indicates that the force couple depends on the m- and n-directions. Similarly, in the primed system, the source components are specified by

$$g_i^{pq}(\mathbf{x}; \mathbf{x}_0') = \partial_j \delta(\mathbf{x} - \mathbf{x}_0') \delta_{ip} \delta_{jk}. \tag{5.16}$$

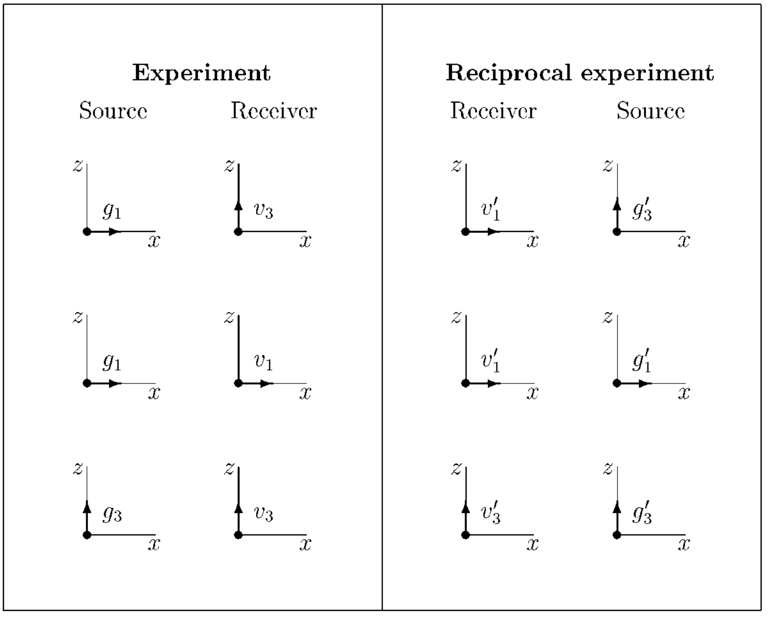

Figure 5.1: 2-D reciprocal experiments for single forces.

The corresponding particle velocities are expressed as $v_i^{mn}(\mathbf{x}, t; \mathbf{x}_0)$ and $v_i^{pq}(\mathbf{x}, t; \mathbf{x}_0')$, respectively. Following Aki and Richards (1980, p. 50), the forces in equations (5.15) and (5.16) may be thought of as composed of a simple (point) force in the positive m-direction and another force of equal magnitude in the negative m-direction. These two forces are separated by a small distance in the n-direction. The magnitude of the forces must be chosen such that the product of the distance between the forces and the magnitude is unity. This is illustrated by the examples in Figures 5.2 and 5.3. The source in the top left experiment of Figure 5.2 can be obtained from equation (5.15) by setting $\mathbf{x}_0 = 0$ and $m = n = 1$. Then, $g_2^{11} = g_3^{11} = 0$ and

$$g_1^{11}(\mathbf{x}; 0) = \partial_1 \delta(\mathbf{x}). \tag{5.17}$$

Consider now the source in the top left experiment of Figure 5.3. Using equation (5.15) and assuming $m = 1$ and $n = 3$, we have $g_2^{13} = g_3^{13} = 0$ and

$$g_1^{13}(\mathbf{x}; 0) = \partial_3 \delta(\mathbf{x}). \tag{5.18}$$

This body force possesses a moment around the y-axis, in contrast to the source considered in Figure 5.2, which has zero moment around the y-axis. Whenever $m = n$, the body force is referred to as a couple without moment, whereas when $m \neq n$ the corresponding body force is referred to as a couple with moment.

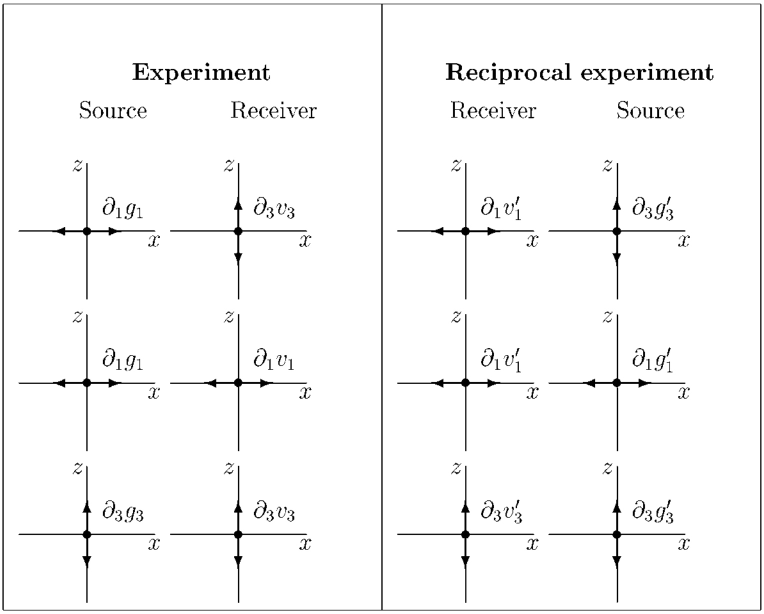

Figure 5.2: 2-D reciprocal experiments for couples without moment.

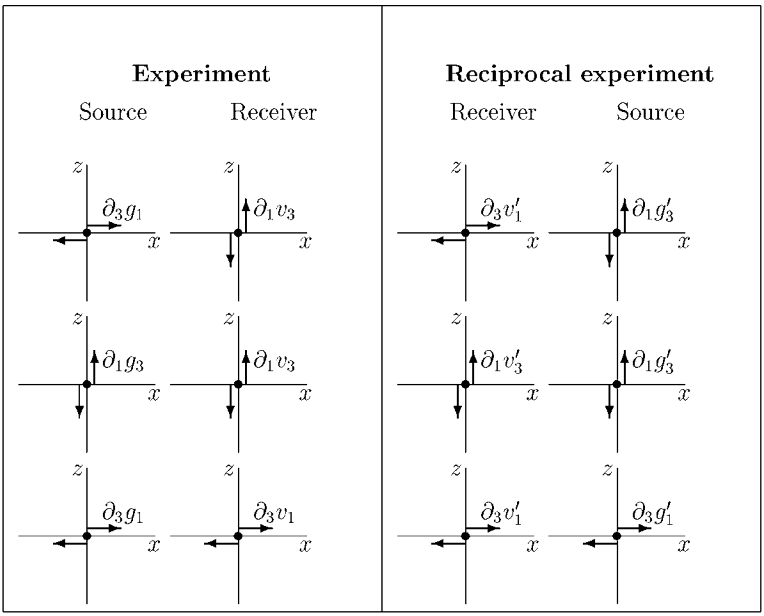

Figure 5.3: Some of the 2-D reciprocal experiments for single couples with moment.

Substituting equations (5.15) and (5.16) into equation (5.9), we obtain the reciprocity relation for couple forces,

$$\partial_q[v_p^{mn}(\mathbf{x}_0', t; \mathbf{x}_0)] = \partial_n[v_m^{pq}(\mathbf{x}_0, t; \mathbf{x}_0')]. \tag{5.19}$$

The interpretation of equation (5.19) is similar to that of equation (5.14), except that the spatial derivatives of the particle velocity are reciprocal rather the particle velocities themself. The following cases are most relevant.

Single couples without moment

When $m = n$ and $p = q$ in equation (5.19), the derivatives are calculated along the force directions. The resulting couples have orientations depending on those directions. This is illustrated in Figure 5.2 for three different experiments.

Single couples with moment

This situation corresponds to the case $m \neq n$ and $p \neq q$ in equation (5.19). The resulting couples have moments. Three cases are illustrated in Figure 5.3.

5.4.2 Double couples

Double couple without moment. Dilatation.

Two perpendicular couples without moments constitute a dilatational source. Such couples have the form

$$g_i(\mathbf{x}; \mathbf{x}_0) = \partial_i \delta(\mathbf{x} - \mathbf{x}_0), \tag{5.20}$$

and

$$g_i(\mathbf{x}; \mathbf{x}_0') = \partial_i \delta(\mathbf{x} - \mathbf{x}_0'). \tag{5.21}$$

The respective particle-velocity components are $v_i(\mathbf{x}, t; \mathbf{x}_0)$ and $v_i(\mathbf{x}, t; \mathbf{x}_0')$.

Substituting equation (5.20) and (5.21) into (5.9), we obtain

$$\int_\Omega [v_i(\mathbf{x}, t; \mathbf{x}_0)\partial_i \delta(\mathbf{x} - \mathbf{x}_0') - \partial_i \delta(\mathbf{x} - \mathbf{x}_0) v_i(\mathbf{x}, t; \mathbf{x}_0')] d\Omega = 0, \tag{5.22}$$

or

$$\dot{\vartheta}(\mathbf{x}_0', t; \mathbf{x}_0) = \dot{\vartheta}(\mathbf{x}_0, t; \mathbf{x}_0'). \tag{5.23}$$

where

$$\dot{\vartheta} = \partial_i v_i \tag{5.24}$$

(see equation (1.11)). Equation (5.23) indicates that for a dilatation point source (explosion), the time derivative of the dilatation fields are reciprocal when the source and receiver are interchanged.

Double couple without moment and monopole force

Let us consider a double couple without moment (dilatation source) at $\mathbf{x} = \mathbf{x}_0$,

$$g_i(\mathbf{x}; \mathbf{x}_0) = \partial_i \delta(\mathbf{x} - \mathbf{x}_0), \tag{5.25}$$

and a monopole force at $\mathbf{x} = \mathbf{x}'_0$

$$g_i^m(\mathbf{x}; \mathbf{x}'_0) = g_0 \delta(\mathbf{x} - \mathbf{x}'_0)\delta_{im}, \tag{5.26}$$

where g_0 is a constant with dimensions of 1/length. The respective particle-velocity components are $v_i(\mathbf{x}, t; \mathbf{x}_0)$ and $v_i^m(\mathbf{x}, t; \mathbf{x}'_0)$. Substituting equation (5.25) and (5.26) into (5.9), we have

$$\int_\Omega [v_i(\mathbf{x}, t; \mathbf{x}_0) g_0 \delta(\mathbf{x} - \mathbf{x}'_0)\delta_{im} - \partial_i \delta(\mathbf{x} - \mathbf{x}_0) v_i^m(\mathbf{x}, t; \mathbf{x}'_0)] d\Omega = 0. \tag{5.27}$$

Integration of (5.27) implies

$$g_0 v_i(\mathbf{x}'_0, t; \mathbf{x}_0)\delta_{im} - \partial_i v_i^m(\mathbf{x}_0, t; \mathbf{x}'_0) = 0, \tag{5.28}$$

which can be written as

$$g_0 v_m(\mathbf{x}'_0, t; \mathbf{x}_0) = \dot{\vartheta}_m(\mathbf{x}_0, t; \mathbf{x}'_0), \tag{5.29}$$

where

$$\dot{\vartheta}_m = \partial_i v_i^m. \tag{5.30}$$

Equation (5.29) indicates that the particle velocity and time derivative of the dilatation field must be substituted when the source and receiver are interchanged. The case $g_0 v_3 = \dot{\vartheta}_3$ is illustrated in Figure 5.4 (top).

The question posed by Nyitrai, Hron and Razavy (1996) regarding reciprocity in Lamb's problem (see Section 5.1) has then the following answer: the horizontal (vertical) particle velocity due to a dilatation source is reciprocal with the time derivative of the dilatation due to a horizontal (vertical) force.

Double couple without moment and single couple

Let us consider a double couple without moment at $\mathbf{x} = \mathbf{x}_0$,

$$g_i(\mathbf{x}; \mathbf{x}_0) = \partial_i \delta(\mathbf{x} - \mathbf{x}_0), \tag{5.31}$$

and a single couple at $\mathbf{x} = \mathbf{x}'_0$,

$$g_i^{mn}(\mathbf{x}; \mathbf{x}'_0) = \partial_j \delta(\mathbf{x} - \mathbf{x}'_0)\delta_{im}\delta_{jn}. \tag{5.32}$$

The particle-velocity components are $v_i(\mathbf{x}, t; \mathbf{x}_0)$ and $v_i^{mn}(\mathbf{x}, t; \mathbf{x}'_0)$, respectively.

Substituting equation (5.31) and (5.32) into (5.9), we obtain

$$\int_\Omega [v_i(\mathbf{x}, t; \mathbf{x}_0) \partial_j \delta(\mathbf{x} - \mathbf{x}'_0)\delta_{im}\delta_{jn} - \partial_i \delta(\mathbf{x} - \mathbf{x}_0) v_i^{mn}(\mathbf{x}, t; \mathbf{x}'_0)] d\Omega = 0, \tag{5.33}$$

Integration of (5.33) implies

$$\partial_j v_i(\mathbf{x}'_0, t; \mathbf{x}_0)\delta_{im}\delta_{jn} - \partial_i v_i^{mn}(\mathbf{x}_0, t; \mathbf{x}'_0) = 0, \tag{5.34}$$

which can be written as

$$\partial_n v_m(\mathbf{x}'_0, t; \mathbf{x}_0) = \dot{\vartheta}_{mn}(\mathbf{x}_0, t; \mathbf{x}'_0), \tag{5.35}$$

where

$$\dot{\vartheta}_{mn} = \partial_i v_i^{mn}. \tag{5.36}$$

In this case, the time derivative of the dilatation is reciprocal with the derivatives of the particle velocity. Two examples are illustrated in Figure 5.4 (middle and bottom pictures).

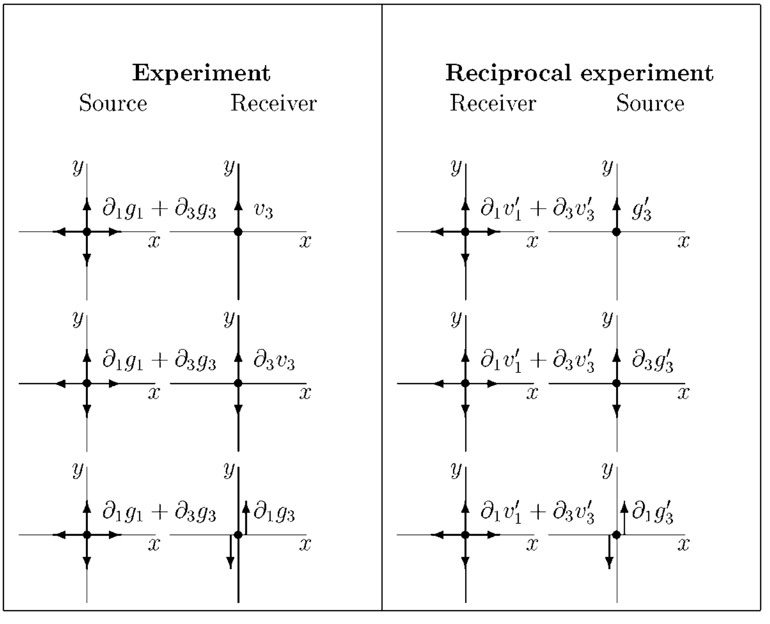

Figure 5.4: 2-D reciprocal experiments for double couples without moment and single couples.

5.5 Reciprocity of stress

A proper choice of the body forces f_i and f'_i leads to reciprocity relations for stress. This occurs for the following forces:

$$f_i^{mn}(\mathbf{x}, t; \mathbf{x}_0) = [\psi_{ijkl}(\mathbf{x}_0, t)\partial_j \delta(\mathbf{x} - \mathbf{x}_0)\delta_{km}\delta_{ln}] * h(t) \tag{5.37}$$

and

$$f_i^{pq}(\mathbf{x}, t; \mathbf{x}'_0) = [\psi_{ijkl}(\mathbf{x}'_0, t)\partial_j \delta(\mathbf{x} - \mathbf{x}'_0)\delta_{kp}\delta_{lq}] * h(t). \tag{5.38}$$

The associated particle-velocity components are $v_i(\mathbf{x}, t; \mathbf{x}_0)$ and $v_i(\mathbf{x}, t; \mathbf{x}'_0)$, respectively, where we have omitted the superscripts for simplicity. The corresponding components of the stress tensor are denoted by $\sigma_{ij}^{mn}(\mathbf{x}, t; \mathbf{x}_0)$ and $\sigma_{ij}^{pq}(\mathbf{x}, t; \mathbf{x}'_0)$. Substituting equation (5.37) and (5.38) into (5.6), we obtain

$$\int_\Omega \{v_i(\mathbf{x}, t; \mathbf{x}_0) * [\psi_{ijkl}(\mathbf{x}'_0, t)\partial_j \delta(\mathbf{x} - \mathbf{x}'_0)\delta_{kp}\delta_{lq}] * h(t)$$

$$-v_i(\mathbf{x}, t; \mathbf{x}'_0) * [\psi_{ijkl}(\mathbf{x}_0, t)\partial_j \delta(\mathbf{x} - \mathbf{x}_0)\delta_{mk}\delta_{ln}] * h(t)\} d\Omega = 0. \tag{5.39}$$

Integrating this equation, we obtain

$$\partial_j v_i(\mathbf{x}'_0, t; \mathbf{x}_0) * [\psi_{ijkl}(\mathbf{x}'_0, t)\delta_{kp}\delta_{lq}] * h(t)$$

$$-\partial_j v_i(\mathbf{x}_0, t; \mathbf{x}_0) * [\psi_{ijkl}(\mathbf{x}_0, t)\delta_{mk}\delta_{ln}] * h(t) = 0. \tag{5.40}$$

We now use the symmetry properties (2.24), to rewrite equation (5.40) as

$$[\psi_{ijkl}(\mathbf{x}'_0, t; \mathbf{x}_0) * \partial_l v_k(\mathbf{x}'_0, t; \mathbf{x}_0)\delta_{ip}\delta_{jq}] * h(t)$$

$$-[\psi_{ijkl}(\mathbf{x}_0, t; \mathbf{x}'_0) * \partial_l v_k(\mathbf{x}_0, 0, t; \mathbf{x}'_0)\delta_{im}\delta_{jn}] * h(t) = 0, \tag{5.41}$$

or

$$[\sigma_{ij}^{mn}(\mathbf{x}'_0, t; \mathbf{x}_0)\delta_{ip}\delta_{jq}] * h(t) - [\sigma_{ij}^{pq}(\mathbf{x}_0, t; \mathbf{x}'_0)\delta_{im}\delta_{jn}] * h(t) = 0, \tag{5.42}$$

where the stress-strain relation (5.2) and the relation $\psi_{ijkl} * (\partial_k v_l + \partial_l v_k) = 2\psi_{ijkl} * \partial_k v_l$ have been used. Contraction of indices implies

$$\sigma_{pq}^{mn}(\mathbf{x}'_0, t; \mathbf{x}_0) * h(t) - \sigma_{mn}^{pq}(\mathbf{x}_0, t; \mathbf{x}'_0) * h(t) = 0. \tag{5.43}$$

If h is such that $\bar{h}$ satisfying $h * \bar{h} = \delta$ exists, it is easy to show that equation (5.43) is equivalent to

$$\sigma_{pq}^{mn}(\mathbf{x}'_0, t; \mathbf{x}_0) = \sigma_{mn}^{pq}(\mathbf{x}_0, t; \mathbf{x}'_0). \tag{5.44}$$

The interpretation of equation (5.44) follows. The pq stress component at $\mathbf{x}'_0$ due to a body force with i-th component given by f_i^{mn} at $\mathbf{x}_0$ equals the mn stress component at $\mathbf{x}_0$ due to a body force with i-th component given by f_i^{pq} and applied at $\mathbf{x}'_0$.

Figure 5.5 illustrates the source and receiver configuration for an experiment corresponding to reciprocity of stress. The sources of the experiments are

$$f_1^{13}(\mathbf{x}, t; \mathbf{x}_0) = \psi_{55} * h(t)\partial_3 \delta(\mathbf{x} - \mathbf{x}_0), \quad f_3^{13}(\mathbf{x}, t; \mathbf{x}_0) = \psi_{55} * h(t)\partial_1 \delta(\mathbf{x} - \mathbf{x}_0),$$

and

$$f_1^{33}(\mathbf{x}, t; \mathbf{x}'_0) = \psi_{13} * h(t)\partial_1 \delta(\mathbf{x} - \mathbf{x}'_0), \quad f_3^{33}(\mathbf{x}, t; \mathbf{x}'_0) = \psi_{33} * h(t)\partial_3 \delta(\mathbf{x} - \mathbf{x}'_0),$$

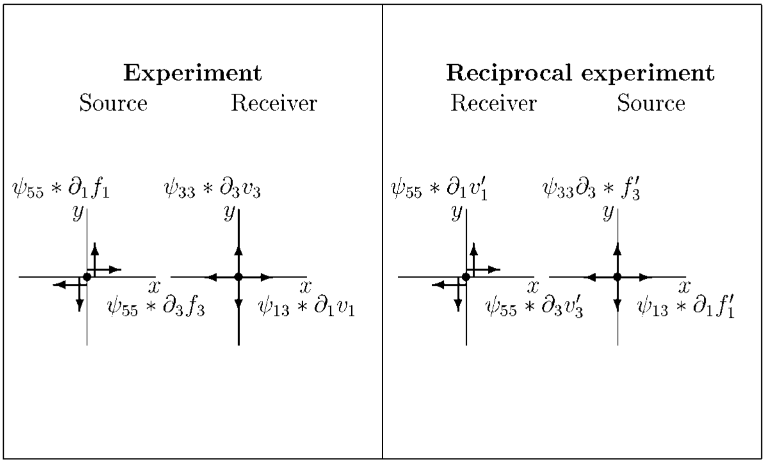

Figure 5.5: Source and receiver configuration for reciprocal stress experiments.

where ψ_{IJ} are the relaxation components in the Voigt's notation. In this case, σ_{33}^{13} is equal to σ_{13}^{33} when the source and receiver positions are interchanged.

We then conclude that for many types of sources, such as, for example, dipoles or explosions (dilatations), there is a field that satisfies the reciprocity principle. An example of the application of the reciprocity relations can be found, for instance, in offshore seismic experiments, since the sources are of dilatational type and the hydrophones record the pressure field, i.e., the dilatation multiplied by the water bulk modulus. In land seismic acquisition, an example is the determination of the radiation pattern for a point source on a homogeneous half-space (Lamb's problem). The radiation pattern can be obtained by using reciprocity and the displacements on the half-space surface due to incident plane waves (White, 1960). The reciprocity relations can be useful in borehole seismic experiments, where couples and pressure sources and receivers are employed. An example of how not to use the reciprocity principle is given by Gangi (1980a). It is the case of an explosive source in a bore that is capped so that the explosion is a pressure source, and displacements are measured on the surface using a vertical geophone. An explosion at the surface and a vertical geophone in the borehole will not necessarily provide the configuration that is reciprocal to the first experiment. The correct reciprocal configuration involves a hydrophone in the well and a directional vertical source at the surface. A set of numerical experiments confirming the reciprocity relations obtained in this chapter can be found in Arntsen and Carcione (2000).