

Chapter 2

Viscoelasticity and wave propagation

The quantity ET , by which the rate of displacement must be multiplied to get the force, may be called the coefficient of viscosity. It is the product of a coefficient of elasticity, E , and a time T , which may be called the “time of relaxation” of the elastic force. In the case of a collection of moving molecules such as we suppose a gas to be, there is also a resistance to change of form, constituting what may be called the linear elasticity, or “rigidity” of the gas, but this resistance gives way and diminishes at a rate depending on the amount of the force and on the nature of the gas.

James Clerk Maxwell (Maxwell, 1867)

The basic formulation of linear (infinitesimal) viscoelasticity has been developed by several scientists, including Maxwell (1867), Voigt (1892), Lord Kelvin (William Thomson (Kelvin, 1875)), Boltzmann (1874), Volterra (1909, 1940) and Graffi (1928). Boltzmann (1874), in particular, introduced the concept of memory, in the sense that at a fixed point of the medium, the stress at any time depends upon the strain at all preceding times. Viscoelastic behavior is a time-dependent, mechanical non-instantaneous response of a material body to variations of applied stress. Unlike a lossless elastic medium, a viscoelastic solid once set into vibration would not continue to vibrate indefinitely. Because the response is not instantaneous, there is a time-dependent function that characterizes the behavior of the material. The function embodies the stress or strain history of the viscoelastic body. The strength of the dependence is greater for events in the most recent past and diminishes as they become more remote in time: it is said that the material has memory. In a linear viscoelastic material, the stress is linearly related to the strain history until a given time. The strain arising from any increment of the stress will add to the strain resulting from stresses previously created in the body. This is expressed in mathematical form by Boltzmann's superposition principle or Boltzmann's law.

Notation: Let f and g be scalar time-dependent functions. The time convolution of f and g is defined by

$$f * g = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau. \quad (2.1)$$

Hooke's law can be expressed in 3-D space or 6-D space depending on whether the stress and the strain are tensors or column matrices. In the shortened matrix notation, the definition of convolution may be extended easily to include 6×1 column matrices (a)

and 6×6 tensors (or matrices) (\mathbf{A}):

$$f * \mathbf{a} = \int_{-\infty}^{\infty} f(\tau) \mathbf{a}(t - \tau) d\tau, \quad (2.2)$$

$$\mathbf{A} * \mathbf{a} = \int_{-\infty}^{\infty} \mathbf{A}(\tau) \cdot \mathbf{a}(t - \tau) d\tau. \quad (2.3)$$

As a convention, any function $f(t)$ is said to be of the Heaviside type if the past history of f up to time $t = 0$ vanishes. That is,

$$f(t) = \check{f}(t)H(t), \quad (2.4)$$

where $H(t)$ is Heaviside's or step function, and there is no restriction on \check{f} . If f and g are of the Heaviside type, we can write

$$f * g = \int_0^t f(\tau)g(t - \tau) d\tau, \quad (2.5)$$

If f is of the Heaviside type, we define the Boltzmann operation as

$$f \odot g = f(0)g + (\partial_t \check{f} H) * g, \quad (2.6)$$

corresponding to the time derivative of the convolution between f and g , that is $f * (\partial_t g)$.

2.1 Energy densities and stress-strain relations

In order to obtain the stress-strain relation for anisotropic elastic media, we defined the strain-energy function (1.1) and used equation (1.21) (or equation (1.22)) to calculate the stress components in terms of the strain components. In materials with dissipation, a unique free-energy density function (the strain energy here) cannot be defined (e.g., Morro and Vianello, 1990). There are cases where the strain energy is unique, such as that of viscoelastic materials with internal variables based on exponential relaxation functions (Fabrizio and Morro, 1992, p. 61). The uniqueness holds when the number of internal variables is less than the number of physical (observable) variables (Graffi and Fabrizio, 1982).

We assume that the properties of the medium do not vary with time (non-aging material), and, as in the lossless case, the energy density is quadratic in the strain field. We introduce the constitutive equation as a convolutional relation between stress and strain, with the assumption of isothermal conditions. However, as stated above, it is important to note that the form of the strain-energy density is not unique (see Rabotnov, 1980, p. 72). Analogy with mechanical models provides a quite general description of anelastic phenomena. The building blocks are the spring and the dashpot. In these elements, it is assumed that energy is “stored” in the springs and “dissipated” in the dashpots. An arbitrary – series and parallel – connection of these elements provides a good phenomenological model to describe the behavior of many materials, from polymers to rocks. Christensen (1982, p. 86), Hunter (1983, p. 542), and Golden and Graham (1988, p. 12) define appropriate forms of the strain energy in the linear viscoelastic case (see also Carcione, 1999a).

A form of the strain-energy density, which can be made consistent with the mechanical model description, is

$$V(t) = \frac{1}{2} \int_{-\infty}^t \int_{-\infty}^t G_{ijkl}(t - \tau_1, t - \tau_2) \partial_{\tau_1} \epsilon_{ij}(\tau_1) \partial_{\tau_2} \epsilon_{kl}(\tau_2) d\tau_1 d\tau_2 \quad (2.7)$$

(Christensen, 1982, p. 79; Golden and Graham, 1988, p. 12). As we shall see below, the general expression of the strain-energy density is not uniquely determined by the relaxation function.

Differentiation of V yields

$$\begin{aligned} \partial_t V &= \partial_t \epsilon_{ij} \int_{-\infty}^t G_{ijkl}(t - \tau_2, 0) \partial_{\tau_2} \epsilon_{kl}(\tau_2) d\tau_2 \\ &+ \frac{1}{2} \int_{-\infty}^t \int_{-\infty}^t \partial_t G_{ijkl}(t - \tau_1, t - \tau_2) \partial_{\tau_1} \epsilon_{ij}(\tau_1) \partial_{\tau_2} \epsilon_{kl}(\tau_2) d\tau_1 d\tau_2. \end{aligned} \quad (2.8)$$

We define the stress-strain relation

$$\sigma_{ij} = \psi_{ijkl} * \partial_t \epsilon_{kl}, \quad (2.9)$$

where ψ_{ijkl} are the components of the relaxation tensor, such that

$$\psi_{ijkl}(t) = G_{ijkl}(t, 0) H(t), \quad (2.10)$$

where $H(t)$ is Heaviside's function. Then,

$$\int_{-\infty}^t G_{ijkl}(t - \tau_2, 0) \partial_{\tau_2} \epsilon_{kl}(\tau_2) d\tau_2 = \sigma_{ij} \quad (2.11)$$

and (2.8) becomes

$$\sigma_{ij} \partial_t \epsilon_{ij} = \partial_t V + \dot{D}, \quad (2.12)$$

where

$$\dot{D}(t) = -\frac{1}{2} \int_{-\infty}^t \int_{-\infty}^t \partial_t G_{ijkl}(t - \tau_1, t - \tau_2) \partial_{\tau_1} \epsilon_{ij}(\tau_1) \partial_{\tau_2} \epsilon_{kl}(\tau_2) d\tau_1 d\tau_2 \quad (2.13)$$

is the rate of dissipated-energy density. Note that the relation (2.10) does not determine the stored energy, i.e., this cannot be obtained from the stress-strain relation. However, if we assume that

$$G_{ijkl}(t, \tau_1) = \check{\psi}_{ijkl}(t + \tau_1), \quad (2.14)$$

such that

$$\psi_{ijkl}(t) = \check{\psi}_{ijkl}(t) H(t), \quad (2.15)$$

this choice will suffice to determine G_{ijkl} , and

$$V(t) = \frac{1}{2} \int_{-\infty}^t \int_{-\infty}^t \check{\psi}_{ijkl}(2t - \tau_1 - \tau_2) \partial_{\tau_1} \epsilon_{ij}(\tau_1) \partial_{\tau_2} \epsilon_{kl}(\tau_2) d\tau_1 d\tau_2, \quad (2.16)$$

$$\dot{D}(t) = - \int_{-\infty}^t \int_{-\infty}^t \partial \check{\psi}_{ijkl}(2t - \tau_1 - \tau_2) \partial_{\tau_1} \epsilon_{ij}(\tau_1) \partial_{\tau_2} \epsilon_{kl}(\tau_2) d\tau_1 d\tau_2, \quad (2.17)$$

where ∂ denotes differentiation with respect to the argument of the corresponding function. Equation (2.14) is consistent with the corresponding theory implied by mechanical models (Christensen, 1982, p. 120; Hunter, 1983, p. 542), i.e., these expressions describe the energy stored in the springs and the energy dissipated in the dashpots (Cavallini and Carcione, 1994).

The strain-energy density must be positive; therefore $V \geq 0$. Substituting the strain function $\epsilon_{ij}(t) = \check{\epsilon}_{ij}H(t)$ into equation (2.7), we obtain the condition $G_{ijkl}(t,t)\check{\epsilon}_{ij}\check{\epsilon}_{kl} \geq 0$, which from (2.10) and (2.14) implies

$$\psi_{ijkl}(t)\check{\epsilon}_{ij}\check{\epsilon}_{kl} \geq 0. \quad (2.18)$$

Similarly, since $\dot{D}(t) \geq 0$, the same test implies

$$\partial_t \psi_{ijkl}(t)\check{\epsilon}_{ij}\check{\epsilon}_{kl} \leq 0. \quad (2.19)$$

The definitions of stored-(free-)energy and energy-dissipation rate are controversial, both in electromagnetism (Oughstun and Sherman, 1994, p. 31) and viscoelasticity (Caviglia and Morro, 1992, p. 53-57). The problem is particularly intriguing in the time domain, since different definitions may give the same time-average value for harmonic fields. Although the forms (2.16) and (2.17) may lead to ambiguous partitions of the rate of work (equation (2.12) is one of these possibilities), this ambiguity is not present when the stress-strain relation can be described in terms of springs and dashpots (Hunter, 1983, p. 542; Cavallini and Carcione, 1994).

2.1.1 Fading memory and symmetries of the relaxation tensor

On the basis of observations and experiments, we may postulate the fading memory hypothesis, which states that the value of the stress depends more strongly upon the recent history than upon the remote history of the strain (Christensen, 1982, p. 9). It is then sufficient that the magnitude of each component of the relaxation tensor be a decreasing function of time,

$$|\partial_t \check{\psi}_{ijkl}|_{t=t_1} \leq |\partial_t \check{\psi}_{ijkl}|_{t=t_2}, \quad t_1 > t_2 > 0. \quad (2.20)$$

As in the lossless case, the symmetry of the stress and strain tensors gives

$$\psi_{ijkl} = \psi_{jikl} = \psi_{ijlk}, \quad (2.21)$$

implying 36 independent components. In the shortened matrix notation, the stress-strain relation (2.9) has the form

$$\boldsymbol{\sigma} = \boldsymbol{\Psi} * \partial_t \mathbf{e}, \quad (\sigma_I = \psi_{IJ} \partial_t e_J), \quad (2.22)$$

where $\boldsymbol{\sigma}$ and \mathbf{e} are defined in equations (1.20) and (1.27), respectively. In general, under the assumption that the stress-strain relation is given by Boltzmann's law, and without a precise definition of a strain-energy function, it can be shown that $\boldsymbol{\Psi}$ is a symmetric matrix in the low- and high-frequency limits only, that is

$$\psi_{ijkl}(t = \infty) = \psi_{klij}(t = \infty), \quad \psi_{ijkl}(t = 0) = \psi_{klij}(t = 0), \quad (2.23)$$

(Christensen, 1982, p. 86; Fabrizio and Morro, 1992, p. 46). The number of components of the relaxation matrix can be reduced to 21 if we consider that the matrix is symmetric, i.e.,

$$\psi_{IJ}(t) = \psi_{JI}(t), \quad (\psi_{ijkl}(t) = \psi_{klji}(t)). \quad (2.24)$$

There is no rigorous demonstration of this property¹, and equation (2.24) is generally assumed to be valid (e.g., Golden and Graham, 1988, p. 37).

2.2 Stress-strain relation for 1-D viscoelastic media

The complex modulus is the key quantity in the following analysis. We determine its properties – closely related to those of the relaxation function – and its significance in terms of stored and dissipated energies. To introduce the basic concepts, it is simplest to start in one dimension.

2.2.1 Complex modulus and storage and loss moduli

Hooke's law in the lossless case is

$$\sigma = M_e \epsilon, \quad (2.25)$$

where M_e is the elastic modulus. ($M_e = \lambda$ is the Lamé constant if we assume $\mu = 0$). According to equation (2.9), the relaxation function in this case is

$$\psi(t) = M_e H(t), \quad (2.26)$$

because

$$\sigma = \psi * \partial_t \epsilon = \partial_t \psi * \epsilon = M_e \delta(t) * \epsilon = M_e \epsilon. \quad (2.27)$$

In the lossy case,

$$\sigma = \psi * \partial_t \epsilon, \quad (2.28)$$

where

$$\psi = \check{\psi} H(t). \quad (2.29)$$

The Fourier transform of equation (2.28) gives

$$\mathcal{F}[\sigma(\omega)] = M(\omega) \mathcal{F}[\epsilon(\omega)] \quad (\tilde{\sigma} = M\tilde{\epsilon}), \quad (2.30)$$

where \mathcal{F} is the Fourier-transform operator, and

$$M(\omega) = \mathcal{F}[\partial_t \psi(\omega)] = \int_{-\infty}^{\infty} \partial_t \psi(t) \exp(-i\omega t) dt \quad (2.31)$$

¹The symmetry can be proved if one can show that the Hermitian (H) and antihermitian (A) parts of the relaxation matrix are even and odd functions, respectively. Any complex matrix can be written as $\psi_{IJ}(\omega) = \psi_{IJ}^H(\omega) + \psi_{IJ}^A(\omega)$, where $\psi_{IJ}^H = \frac{1}{2}[\psi_{IJ}(\omega) + \psi_{JI}^*(\omega)]$ and $\psi_{IJ}^A = \frac{1}{2}[\psi_{IJ}(\omega) - \psi_{JI}^*(\omega)]$. Moreover, since $\psi_{IJ}(\omega)$ is the Fourier transform of a real quantity, it must satisfy the reality condition $\psi_{IJ}^*(\omega) = \psi_{IJ}^*(-\omega)$. The first statement implies $\psi_{IJ}^H(-\omega) = \psi_{IJ}^H(\omega)$ and $\psi_{IJ}^A(-\omega) = -\psi_{IJ}^A(\omega)$. Combining these relations into one by using the reality condition implies $\psi_{IJ}(\omega) = \psi_{JI}(\omega)$. Melrose and McPhedran (1991, p. 83) justify the first statement for the dielectric-permittivity tensor by invoking the time-reversal invariance of the equation of motion, under certain transformations of the field variables (Onsager's relations).

is the complex modulus. Since $\partial_t \psi = \delta(t)\check{\psi} + \partial_t \check{\psi} H(t)$,

$$M(\omega) = \psi(0^+) + \int_0^\infty \partial_t \check{\psi}(t) \exp(-i\omega t) dt, \quad (2.32)$$

because $\check{\psi}(0) = \psi(0^+)$. Equation (2.32) becomes

$$M(\omega) = \psi(\infty) + i\omega \int_0^\infty [\psi(t) - \psi(\infty)] \exp(-i\omega t) dt, \quad (2.33)$$

since $\check{\psi}(t) = \psi(t)$ for $t > 0$. (To demonstrate (2.33) it is convenient to derive (2.32) from (2.33) using integration by parts²).

We decompose the complex modulus into real and imaginary parts

$$M(\omega) = M_1(\omega) + iM_2(\omega), \quad (2.34)$$

where

$$M_1(\omega) = \psi(0^+) + \int_0^\infty \partial_t \check{\psi}(t) \cos(\omega t) dt = \psi(\infty) + \omega \int_0^\infty [\psi(t) - \psi(\infty)] \sin(\omega t) dt, \quad (2.35)$$

or,

$$M_1(\omega) = \omega \int_0^\infty \psi(t) \sin(\omega t) dt \quad (2.36)$$

is the storage modulus, and

$$M_2(\omega) = - \int_0^\infty \partial_t \check{\psi}(t) \sin(\omega t) dt = \omega \int_0^\infty [\psi(t) - \psi(\infty)] \cos(\omega t) dt \quad (2.37)$$

is the loss modulus. To obtain equation (2.36), we have used the property

$$\omega \int_0^\infty \sin(\omega t) dt = 1 \quad (2.38)$$

(Golden and Graham, 1988, p. 243).

In the strain-stress relation

$$\epsilon = \chi * \partial_t \sigma, \quad (2.39)$$

the function χ is referred to as the creep function. Since

$$\sigma = \partial_t \psi * \epsilon = \partial_t \psi * (\partial_t \chi * \sigma) = (\partial_t \psi * \partial_t \chi) * \sigma, \quad (2.40)$$

we have

$$\partial_t \psi(t) * \partial_t \chi(t) = \delta(t), \quad (2.41)$$

and

$$M(\omega) J(\omega) = 1, \quad (2.42)$$

where

$$J(\omega) = \mathcal{F}[\partial_t \chi] \quad (2.43)$$

is the complex creep compliance.

²Proof: $i\omega \int_0^\infty [\psi(t) - \psi(\infty)] \exp(-i\omega t) dt = - \int_0^\infty [\psi(t) - \psi(\infty)] d \exp(-i\omega t) = - [\psi(t) - \psi(\infty)] \exp(-i\omega t)|_{t=0}^{t=\infty} + \int_0^\infty \exp(-i\omega t) d\psi = -[\psi(\infty) - \psi(\infty)] \exp(-i\omega \infty) + [\psi(0) - \psi(\infty)] \exp(-i\omega 0) + \int_0^\infty \partial_t \psi(t) \exp(-i\omega t) dt = \psi(0) - \psi(\infty) + \int_0^\infty \partial_t \psi(t) \exp(-i\omega t) dt.$

2.2.2 Energy and significance of the storage and loss moduli

Let us calculate the time-averaged strain-energy density (2.16) for harmonic fields of the form $[\cdot] \exp(i\omega t)$. The change of variables $\tau_1 \rightarrow t - \tau_1$ and $\tau_2 \rightarrow t - \tau_2$ yields

$$V(t) = \frac{1}{2} \int_0^\infty \int_0^\infty \check{\psi}(\tau_1 + \tau_2) \partial\epsilon(t - \tau_1) \partial\epsilon(t - \tau_2) d\tau_1 d\tau_2. \quad (2.44)$$

We now average this equation over a period $2\pi/\omega$ using the property (1.105) and obtain

$$\langle \partial\epsilon(t - \tau_1) \partial\epsilon(t - \tau_2) \rangle = \frac{1}{2} \text{Re}\{\partial\epsilon(t - \tau_1)[\partial\epsilon(t - \tau_2)]^*\} = \frac{1}{2} \omega^2 |\epsilon|^2 \cos[\omega(\tau_2 - \tau_1)]. \quad (2.45)$$

Then, the time average of equation (2.44) is

$$\langle V \rangle = \frac{1}{4} \omega^2 |\epsilon|^2 \int_0^\infty \int_0^\infty \check{\psi}(\tau_1 + \tau_2) \cos[\omega(\tau_1 - \tau_2)] d\tau_1 d\tau_2. \quad (2.46)$$

A new change of variables $\zeta = \tau_1 + \tau_2$ and $\varsigma = \tau_1 - \tau_2$ gives

$$\langle V \rangle = \frac{1}{8} \omega^2 |\epsilon|^2 \int_0^\infty \int_{-\zeta}^\zeta \check{\psi}(\zeta) \cos(\omega\varsigma) d\zeta d\varsigma = \frac{1}{4} \omega |\epsilon|^2 \int_0^\infty \check{\psi}(\zeta) \sin(\omega\zeta) d\zeta. \quad (2.47)$$

Using equation (2.36), we finally get

$$\langle V \rangle = \frac{1}{4} |\epsilon|^2 M_1. \quad (2.48)$$

A similar calculation shows that

$$\langle \dot{D} \rangle = \frac{1}{2} \omega |\epsilon|^2 M_2. \quad (2.49)$$

These equations justify the terminology used for the storage and loss moduli M_1 and M_2 . Moreover, since the time-averaged strain and dissipated energies should be non-negative, it follows that

$$M_1(\omega) \geq 0, \quad M_2(\omega) \geq 0. \quad (2.50)$$

2.2.3 Non-negative work requirements and other conditions

The work done to deform the material from the initial state must be non-negative

$$\frac{1}{t} \int_0^t \sigma(\tau) \partial_\tau \epsilon(\tau) d\tau \geq 0 \quad (2.51)$$

(Christensen, 1982, p. 86). Let us consider oscillations in the form of sinusoidally time variations

$$\epsilon(\tau) = \epsilon_0 \sin(\omega\tau), \quad (2.52)$$

and let $t = 2\pi/\omega$ be one period, corresponding to a cycle. Using equation (2.31) (see equations (2.36) and (2.37)), we note that the stress-strain relation (2.28) becomes

$$\sigma(t) = \epsilon_0 \int_0^\infty \partial_\tau \psi \sin[\omega(t - \tau)] d\tau = \epsilon_0 [M_1 \sin(\omega t) + M_2 \cos(\omega t)]. \quad (2.53)$$

Substitution of (2.53) into the inequality (2.51) gives

$$\frac{\omega^2}{2\pi} \epsilon_0^2 \left[M_1 \int_0^{2\pi/\omega} \sin(\omega\tau) \cos(\omega\tau) d\tau + M_2 \int_0^{2\pi/\omega} \cos^2(\omega\tau) d\tau \right] d\tau \geq 0. \quad (2.54)$$

We now make use of the primitive integral $\int \cos^2(ax) dx = (x/2) + [\sin(2ax)/(4a)]$. The first integral vanishes, and the second integral is equal to π/ω . The condition is then

$$\frac{\omega}{2} M_2 \epsilon_0^2 \geq 0, \quad \text{or} \quad M_2 \geq 0, \quad (2.55)$$

as found earlier (equation (2.50)₂). This result can also be obtained by using complex notation and the time-average formula (1.105).

We have shown, in addition, that $\langle \sigma \partial_t \epsilon \rangle = \langle \dot{D} \rangle$, if we compare our results to equation (2.49). From equation (2.12), this means that the time average of the strain-energy rate, $\langle \partial_t V \rangle$, is equal to zero.

Equation (2.37) and condition (2.55) imply that

$$\psi(t) - \psi(\infty) \geq 0. \quad (2.56)$$

Then,

$$\psi(t=0) \geq \psi(t=\infty). \quad (2.57)$$

Note that from (2.33), we have

$$M(\omega=0) = \psi(t=\infty), \quad (2.58)$$

i.e., a real quantity. Moreover, from (2.33) and using $i\omega \mathcal{F}[f(t)] = f(t=0)$, for $\omega \rightarrow \infty$ (Golden and Graham, 1988, p. 244), we have

$$M(\omega=\infty) = \psi(t=0), \quad (2.59)$$

also a real quantity. We then conclude that $M_2 = 0$ at the low- and high-frequency limits, and

$$M(\omega=\infty) \geq M(\omega=0). \quad (2.60)$$

(As shown in the next section, the validity of some of these properties requires $|M(\omega)|$ to be a bounded function).

Additional conditions on the relaxation function, based on the requirements of positive work and positive rate of dissipation, can be obtained from the general conditions (2.18) and (2.19),

$$\psi(t) \geq 0, \quad (2.61)$$

$$\partial_t \psi(t) \leq 0. \quad (2.62)$$

2.2.4 Consequences of reality and causality

Equation (2.28) can also be written as

$$\sigma = \dot{\psi} * \epsilon, \quad (2.63)$$

where

$$\dot{\psi} \equiv \partial_t \psi. \quad (2.64)$$

Note that $M = \mathcal{F}[\dot{\psi}]$ (equation (2.31)). Since $\dot{\psi}(t)$ is real, $M(\omega)$ is Hermitian (Bracewell, 1965, p. 16); that is

$$M(\omega) = M^*(-\omega), \quad (2.65)$$

or

$$M_1(\omega) = M_1(-\omega), \quad M_2(\omega) = -M_2(-\omega). \quad (2.66)$$

Furthermore, $\dot{\psi}$ can split into even and odd functions of time, $\dot{\psi}_e$ and $\dot{\psi}_o$, respectively, as

$$\dot{\psi}(t) = \frac{1}{2}[\dot{\psi}(t) + \dot{\psi}(-t)] + \frac{1}{2}[\dot{\psi}(t) - \dot{\psi}(-t)] \equiv \dot{\psi}_e + \dot{\psi}_o. \quad (2.67)$$

Since $\dot{\psi}$ is causal, $\dot{\psi}_o(t) = \text{sgn}(t)\dot{\psi}_e(t)$, and

$$\dot{\psi}(t) = [1 + \text{sgn}(t)]\dot{\psi}_e(t), \quad (2.68)$$

whose Fourier transform is

$$\mathcal{F}[\dot{\psi}(t)] = M_1(\omega) - \left(\frac{i}{\pi\omega} \right) * M_1(\omega), \quad (2.69)$$

because $\mathcal{F}[\dot{\psi}_e] = M_1$ and $\mathcal{F}[\text{sgn}(t)] = -i/(\pi\omega)$ (Bracewell, 1965, p. 272). Equation (2.69) implies

$$M_2 = - \left(\frac{1}{\pi\omega} \right) * M_1 = -\frac{1}{\pi} \text{pv} \int_{-\infty}^{\infty} \frac{M_1(\omega') d\omega'}{\omega - \omega'}. \quad (2.70)$$

Similarly, since $\dot{\psi}_e(t) = \text{sgn}(t)\dot{\psi}_o(t)$,

$$\dot{\psi}(t) = [\text{sgn}(t) + 1]\dot{\psi}_o(t) \quad (2.71)$$

and since $\mathcal{F}[\dot{\psi}_o] = iM_2$, we obtain

$$M_1 = \left(\frac{1}{\pi\omega} \right) * M_2 = \frac{1}{\pi} \text{pv} \int_{-\infty}^{\infty} \frac{M_2(\omega') d\omega'}{\omega - \omega'}. \quad (2.72)$$

Equations (2.70) and (2.72) are known as Kramers-Kronig dispersion relations (Kronig, 1926; Kramers, 1927). In mathematical terms, M_1 and M_2 are Hilbert transform pairs (Bracewell, 1965, p. 267-272). Causality also implies that M has no poles (or is analytic) in the lower half complex ω -plane (Golden and Graham, 1988, p. 48). In the case of dispersive lossless media, $M_1(\omega)$ can depend on ω only through functions of ω whose Hilbert transform is zero.

Equations (2.70) and (2.72) are a consequence of linearity, causality and square-integrability of $M(\omega)$ along the real axis of the ω -plane, i.e.,

$$\int_{-\infty}^{\infty} |M(\omega)|^2 d\omega < C, \quad (2.73)$$

where C is a constant (Weaver and Pao, 1981). Square-integrability is equivalent to $M(\omega) \rightarrow 0$, for $|\omega| \rightarrow \infty$ ($\pi \geq \arg(\omega) \geq 0$). In most cases, the square-integrability

condition cannot be satisfied, but rather the weaker condition that $|M(\omega)|$ is bounded is satisfied, i.e., $|M(\omega)|^2 < C$ is bounded. A lossless medium and the indexNiMaxwell-Maxwell and Zener models satisfy this weak condition (see Section 2.4.1, equation (2.147), and Section 2.4.3, equation (2.170)), but the elvin-Voigt and constant- Q models do not (see Section 2.4.2, equation (2.161), and Section 2.5, equation (2.212)). For models satisfying the weak condition, we may construct a new function

$$H(\omega) = \frac{M(\omega) - M(\omega_0)}{\omega - \omega_0}, \quad \text{Im}(\omega_0) \geq 0. \quad (2.74)$$

This function is square-integrable and has no poles in the upper half plane, and, hence, satisfies equations (2.70) and (2.72). Substituting $H(\omega)$ as defined above for $M(\omega)$ in equations (2.70) and (2.72) and taking ω_0 to be real, we obtain

$$M_1(\omega) = M_1(\omega_0) + \left(\frac{\omega - \omega_0}{\pi} \right) \text{pv} \int_{-\infty}^{\infty} \text{Im} \left[\frac{M(\omega') - M(\omega_0)}{\omega' - \omega_0} \right] \frac{d\omega'}{\omega - \omega'}, \quad (2.75)$$

$$M_2(\omega) = M_2(\omega_0) - \left(\frac{\omega - \omega_0}{\pi} \right) \text{pv} \int_{-\infty}^{\infty} \text{Re} \left[\frac{M(\omega') - M(\omega_0)}{\omega' - \omega_0} \right] \frac{d\omega'}{\omega - \omega'}. \quad (2.76)$$

(Weaver and Pao, 1981). These are known as dispersion relations for $M(\omega)$ with one subtraction. Further subtractions may be taken if $M(\omega)$ is bounded by a polynomial function of ω .

2.2.5 Summary of the main properties

Relaxation function

1. It is causal.
2. It is a positive real function.
3. It is a decreasing function of time.

Complex modulus

1. It is an Hermitian function of ω .
2. Its real and imaginary parts are greater than zero, since the strain-energy density and the rate of dissipated-energy density must be positive.
3. Its low- and high-frequency limits are real valued and coincide with the relaxed and instantaneous (unrelaxed) values of the relaxation function.
4. Its real and imaginary parts are Hilbert-transform pairs.
5. It is analytic in the lower half complex ω -plane.

2.3 Wave propagation concepts for 1-D viscoelastic media

We note that the frequency-domain stress-strain relation (2.30) has the same form as the elastic stress-strain relation (2.25), but the modulus is complex and frequency dependent. The implications for wave propagation can be made clear if we consider the displacement plane wave

$$u = u_0 \exp[i(\omega t - kx)], \quad (2.77)$$

where k is the complex wavenumber, and the balance between the surface and inertial forces is

$$\partial_1 \sigma = \rho \partial_{tt}^2 u \quad (2.78)$$

(see equation (1.23)). Assuming constant material properties, using $\epsilon = \partial_1 u$ and equations (2.28) and (2.31), we obtain the dispersion relation

$$Mk^2 = \rho\omega^2, \quad (2.79)$$

which, for propagating waves (k complex, ω real), gives the complex velocity

$$v_c(\omega) = \frac{\omega}{k} = \sqrt{\frac{M(\omega)}{\rho}}. \quad (2.80)$$

Expressing the complex wavenumber as

$$k = \kappa - i\alpha, \quad (2.81)$$

we can rewrite the plane wave (2.77) as

$$u = u_0 \exp(-\alpha x) \exp[i(\omega t - \kappa x)], \quad (2.82)$$

meaning that κ is the wavenumber and α is the attenuation factor. We define the phase velocity

$$v_p = \frac{\omega}{\kappa} = \left[\operatorname{Re} \left(\frac{1}{v_c} \right) \right]^{-1}, \quad (2.83)$$

the real slowness

$$s_R = \frac{1}{v_p} = \operatorname{Re} \left(\frac{1}{v_c} \right), \quad (2.84)$$

and the attenuation factor

$$\alpha = -\omega \operatorname{Im} \left(\frac{1}{v_c} \right). \quad (2.85)$$

We have seen in Chapter 1 (equation (1.125)) that the velocity of the modulation wave is the derivative of the frequency with respect to the wavenumber. In this case, we should consider the real wavenumber κ ,

$$v_g = \frac{\partial \omega}{\partial \kappa} = \left(\frac{\partial \kappa}{\partial \omega} \right)^{-1} = \left[\operatorname{Re} \left(\frac{\partial k}{\partial \omega} \right) \right]^{-1}. \quad (2.86)$$

Let us assume for the moment that $u(t)$ is not restricted to the form (2.77). Since the particle velocity is $v = \partial_t u$, we multiply equation (2.78) on both sides by v to obtain

$$v\partial_1\sigma = \rho v\partial_t v. \quad (2.87)$$

Multiplying $\partial_1 v = \partial_t \epsilon$ by σ and using equation (2.28), we have

$$\sigma\partial_1 v = (\partial_t\psi * \epsilon)\partial_t\epsilon. \quad (2.88)$$

In the lossless case, $\psi = M_e H(t)$ and

$$\sigma\partial_1 v = M_e\epsilon\partial_t\epsilon. \quad (2.89)$$

Adding equations (2.87) and (2.89), we obtain the energy-balance equation for dynamic elastic fields

$$-\partial_1 p = \partial_t(T + V) = \partial_t E, \quad (2.90)$$

where

$$p = -\sigma v \quad (2.91)$$

is the Umov-Poynting power flow,

$$T = \frac{1}{2}\rho v^2 \quad (2.92)$$

is the kinetic-energy density, and

$$V = \frac{1}{2}M_e\epsilon^2 \quad (2.93)$$

is the strain-energy density.

The balance equation in the lossy case is obtained by adding equations (2.87) and (2.88),

$$-\partial_1 p = \partial_t T + (\partial_t\psi * \epsilon)\partial_t\epsilon. \quad (2.94)$$

In general, the partition of the second term in the right-hand side in terms of the rates of strain and dissipated energies is not unique (Caviglia and Morro, 1992, p. 56). The splitting (2.12) is one choice, consistent with the mechanical-model description of viscoelasticity – this is shown in Section 2.4.1 for the Maxwell model. A more general demonstration is given by Carcione (1999a) for the Zener model and Hunter (1983, p. 542) for an arbitrary array of springs and dashpots. We then can write

$$-\partial_1 p = \partial_t(T + V) + \dot{D}, \quad (2.95)$$

where, from equations (2.16) and (2.17),

$$V(t) = \frac{1}{2} \int_{-\infty}^t \int_{-\infty}^t \check{\psi}(2t - \tau_1 - \tau_2) \partial_{\tau_1}\epsilon(\tau_1) \partial_{\tau_2}\epsilon(\tau_2) d\tau_1 d\tau_2, \quad (2.96)$$

$$\dot{D}(t) = - \int_{-\infty}^t \int_{-\infty}^t \partial\check{\psi}(2t - \tau_1 - \tau_2) \partial_{\tau_1}\epsilon(\tau_1) \partial_{\tau_2}\epsilon(\tau_2) d\tau_1 d\tau_2. \quad (2.97)$$

Let us consider again the form (2.77) for the displacement field. In order to compute the balance equation for average quantities, we obtain the complex versions of equations (2.87) and (2.88) by multiplying (2.78) by v^* and $(\partial_1 v)^* = (\partial_t\epsilon)^*$ by σ . We obtain

$$v^*\partial_1\sigma = \rho v^*\partial_t v, \quad (2.98)$$

$$\sigma(\partial_1 v)^* = (\partial_t \psi * \epsilon)(\partial_t \epsilon)^*. \quad (2.99)$$

Because for the harmonic plane wave (2.77), $\partial_t \rightarrow i\omega$ and $\partial_1 \rightarrow -ik$, we can use equation (2.30) – omitting the tildes – to obtain

$$-kv^* \sigma = \omega \rho |v|^2, \quad (2.100)$$

and

$$-k^* \sigma v^* = \omega M |\epsilon|^2. \quad (2.101)$$

Now, using equations (1.105) and (1.106), we introduce the complex Umov-Poynting energy flow

$$p = -\frac{1}{2} \sigma v^*, \quad (2.102)$$

the time-averaged kinetic-energy density

$$\langle T \rangle = \frac{1}{2} \langle \rho [\text{Re}(v)]^2 \rangle = \frac{1}{4} \rho \text{Re}(vv^*) = \frac{1}{4} \rho |v|^2, \quad (2.103)$$

the time-averaged strain-energy density

$$\langle V \rangle = \frac{1}{2} \langle \text{Re}(\epsilon) \text{Re}(M) \text{Re}(\epsilon) \rangle = \frac{1}{4} \text{Re}(\epsilon M \epsilon^*) = \frac{1}{4} |\epsilon|^2 M_1, \quad (2.104)$$

and the time-averaged rate of dissipated-energy density

$$\langle \dot{D} \rangle = \omega \langle \text{Re}(\epsilon) \text{Im}(M) \text{Re}(\epsilon) \rangle = \frac{1}{2} \omega \text{Im}(\epsilon M \epsilon^*) = \frac{1}{2} \omega |\epsilon|^2 M_2, \quad (2.105)$$

in agreement with equations (2.48) and (2.49). We can, alternatively, define the time-averaged dissipated-energy density $\langle D \rangle$ as

$$\langle D \rangle = \omega^{-1} \langle \dot{D} \rangle, \quad \omega > 0 \quad (2.106)$$

(there is no loss at zero frequency). Thus, in terms of the energy flow and energy densities, equations (2.100) and (2.101) become

$$kp = 2\omega \langle T \rangle, \quad (2.107)$$

and

$$k^* p = 2\omega \langle V \rangle + i \langle \dot{D} \rangle. \quad (2.108)$$

Because the right-hand side of (2.107) is real, kp is also real. Adding equations (2.107) and (2.108) and using $k + k^* = 2\kappa$ (see equation (2.81)), we have

$$\kappa p = \omega \langle E \rangle + \frac{i}{2} \langle \dot{D} \rangle, \quad (2.109)$$

where

$$\langle E \rangle = \langle T + V \rangle \quad (2.110)$$

is the time-averaged energy density. Separating equation (2.109) into real and imaginary parts, we obtain

$$\kappa \langle p \rangle = \omega \langle E \rangle \quad (2.111)$$

and

$$\kappa \operatorname{Im}(p) = \frac{1}{2} \langle \dot{D} \rangle, \quad (2.112)$$

where

$$\langle p \rangle = \operatorname{Re}(p) \quad (2.113)$$

is the time-averaged power-flow density. The energy velocity is defined as

$$v_e = \frac{\langle p \rangle}{\langle E \rangle}. \quad (2.114)$$

Now, note from equations (2.102)-(2.104) that

$$\begin{aligned} \langle p \rangle &= \operatorname{Re}(p) = -\frac{1}{2} \operatorname{Re}(v^* \sigma) = -\frac{1}{2} \operatorname{Re}[(-i\omega u^*)(M\epsilon)] \\ &= -\frac{1}{2} \operatorname{Re}[(-i\omega u^*)(-ikMu)] = \frac{1}{2} \omega |u|^2 \operatorname{Re}(kM), \end{aligned} \quad (2.115)$$

and

$$\langle E \rangle = \langle T \rangle + \langle V \rangle = \frac{1}{4} \rho |v|^2 + \frac{1}{4} M_1 |\epsilon|^2 = \frac{1}{4} \rho \omega^2 |u|^2 + \frac{1}{4} M_1 |k|^2 |u|^2. \quad (2.116)$$

Substituting these expression into equation (2.114) and using (2.80), we obtain

$$v_e = \frac{2 \operatorname{Re}(M v_c^{-1}) |v_c|^2}{\rho |v_c|^2 + M_1}. \quad (2.117)$$

Because $M_1 = \operatorname{Re}(M)$, $M = \rho v_c^2$, and using properties of complex numbers (in particular, $[\operatorname{Re}(v_c)]^2 = (|v_c|^2 + \operatorname{Re}(v_c^2))/2$), we finally obtain

$$v_e = |v_c|^2 [\operatorname{Re}(v_c)]^{-1} = v_p, \quad (2.118)$$

where v_p is the phase velocity (2.83). We then have the result that the energy velocity is equal to the phase velocity in 1-D viscoelastic media. (Note that this result confirms the relation (2.111)). In the next chapter, we show that this is also the case for homogeneous viscoelastic plane waves in 2-D and 3-D isotropic media. We have seen in Chapter 1 that phase and energy velocities differ in anisotropic elastic media, and that the group velocity is equal to the energy velocity. This result proved very useful in the computation of the wave-front surfaces. However, the group velocity loses its physical meaning in viscoelastic media due to the dispersion of the harmonic components of the signal. We investigate this in some detail in Section 2.6 (1-D case), and Section 4.4.5, where we discuss wave propagation in anisotropic viscoelastic media.

Dissipation can also be quantified by the quality factor Q , whose inverse, Q^{-1} , is called the dissipation factor. Here, we define the quality factor as twice the time-averaged strain-energy density divided by the time-averaged dissipated-energy density (2.106). Hence, we have

$$Q = \frac{2 \langle V \rangle}{\langle D \rangle}, \quad (2.119)$$

which, by virtue of equations (2.80), (2.104) and (2.105) becomes

$$Q = \frac{M_1}{M_2} = \frac{\operatorname{Re}(M)}{\operatorname{Im}(M)} = \frac{\operatorname{Re}(v_c^2)}{\operatorname{Im}(v_c^2)}. \quad (2.120)$$

Another form of the quality factor can be obtained from the definition of complex velocity (2.80). It can be easily shown that

$$Q = -\frac{\operatorname{Re}(k^2)}{\operatorname{Im}(k^2)}. \quad (2.121)$$

Since $k^2 = \kappa^2 - \alpha^2 - 2i\kappa\alpha$, it follows from (2.85) and (2.121) that the quality factor is related to the magnitudes of the attenuation factor and the wavenumber by

$$\alpha = \left(\sqrt{Q^2 + 1} - Q \right) \kappa. \quad (2.122)$$

For low-loss solids, it is $Q \gg 1$, and using (2.83) and $f = \omega/(2\pi)$, we note that a Taylor expansion yields

$$\alpha = \frac{\kappa}{2Q} = \frac{\omega}{2Qv_p} = \frac{\pi f}{Qv_p}. \quad (2.123)$$

Another common definition of quality factor is

$$\mathcal{Q} = \frac{\langle E \rangle}{\langle D \rangle} \quad (2.124)$$

(Buchen, 1971a). It can be easily shown that this definition leads to a relation similar to (2.123), without approximations; that is $\alpha = \pi f/(\mathcal{Q}v_p)$.

2.3.1 Wave propagation for complex frequencies

The analysis of wave propagation can also be performed for complex frequencies and real wavenumbers. Let us consider the 1-D case and the displacement plane wave

$$u = u_0 \exp[i(\Omega t - \kappa x)], \quad (2.125)$$

where $\Omega = \omega + i\omega_I$ is the complex frequency, and κ is the real wavenumber. It is clear that the phase velocity is equal to ω/κ .

The balance between the surface and inertial forces is given by

$$\partial_1 \sigma = \rho \partial_{tt} u, \quad (2.126)$$

where σ is the stress. Since $\sigma = M\epsilon = M\partial_1 u$, where ϵ is the strain, we obtain the dispersion relation

$$M\kappa^2 = \rho\Omega^2, \quad (2.127)$$

which gives the complex velocity

$$v_c(\kappa) = \frac{\Omega}{\kappa} = \sqrt{\frac{M[\Omega(\kappa)]}{\rho}}. \quad (2.128)$$

The phase velocity is

$$v_p = \frac{\operatorname{Re}(\Omega)}{\kappa} = \frac{\omega}{\kappa} = \operatorname{Re}(v_c). \quad (2.129)$$

In order to compute the balance equation for average quantities, we note that

$$-\kappa v^* \sigma = \Omega \rho |v|^2 \quad (2.130)$$

and

$$-\kappa v^* \sigma = \Omega^* M |\epsilon|^2, \quad (2.131)$$

where the asterisk indicates complex conjugate. These equations were obtained by multiplying (2.126) by v^* and $(\partial_t v)^* = (\partial_t \epsilon)^*$ by σ , respectively.

We introduce the complex Umov-Poynting energy flow

$$p = -\frac{1}{2} \sigma v^*, \quad (2.132)$$

the time-averaged kinetic-energy density

$$\langle T \rangle = \frac{1}{2} \langle \rho [\text{Re}(v)]^2 \rangle = \frac{1}{4} \rho \text{Re}(vv^*) = \frac{1}{4} \rho |v|^2, \quad (2.133)$$

the time-averaged strain-energy density

$$\langle V \rangle = \frac{1}{2} \langle \text{Re}(\epsilon) \text{Re}(M) \text{Re}(\epsilon) \rangle = \frac{1}{4} \text{Re}(\epsilon M \epsilon^*) = \frac{1}{4} |\epsilon|^2 \text{Re}(M) = \frac{1}{4} |\epsilon|^2 M_1, \quad (2.134)$$

and the time-averaged dissipated-energy density

$$\langle D \rangle = \langle \text{Re}(\epsilon) \text{Im}(M) \text{Re}(\epsilon) \rangle = \frac{1}{2} \text{Im}(\epsilon M \epsilon^*) = \frac{1}{2} |\epsilon|^2 \text{Im}(M) = \frac{1}{2} |\epsilon|^2 M_2. \quad (2.135)$$

Thus, in terms of the energy flow and energy densities, equations (2.130) and (2.131) become

$$\frac{\kappa p}{\Omega} = 2\langle T \rangle \quad (2.136)$$

and

$$\frac{\kappa p}{\Omega^*} = 2\langle V \rangle + i\langle D \rangle. \quad (2.137)$$

Adding these equations, we have

$$2\kappa p \text{Re}\left(\frac{1}{\Omega}\right) = 2\omega \langle E \rangle + i\langle D \rangle, \quad (2.138)$$

where

$$\langle E \rangle = \langle T + V \rangle \quad (2.139)$$

is the time-averaged energy density.

Separating equation (2.138) into real and imaginary parts and using equation (2.113), the energy velocity is given by

$$v_e = \frac{\langle p \rangle}{\langle E \rangle} = \left[\kappa \text{Re}\left(\frac{1}{\Omega}\right) \right]^{-1} = \left[\text{Re}\left(\frac{1}{v_c}\right) \right]^{-1}, \quad (2.140)$$

i.e., the energy velocity has the same expression as a function of the complex velocity, irrespective of the fact that the frequency is complex or the wavenumber is complex. On the contrary, the phase velocity is given by equation (2.83) for real frequencies and by equation (2.129) for real wavenumbers and complex frequencies.

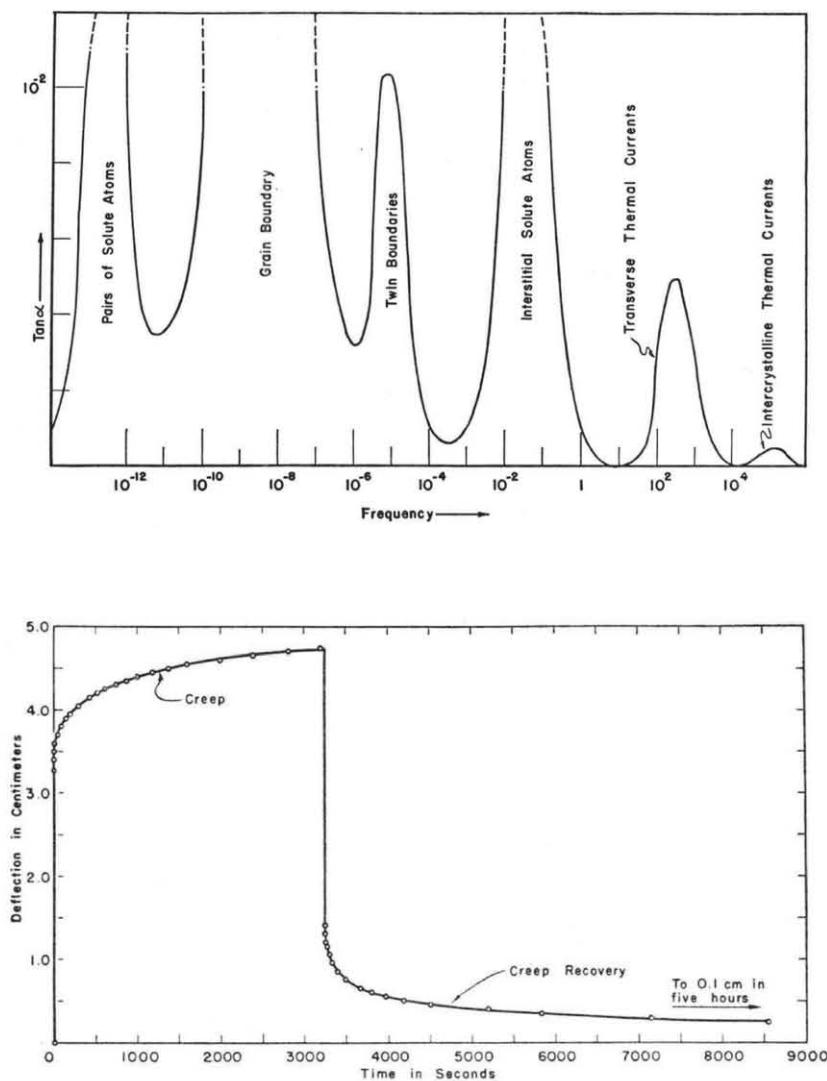


Figure 2.1: Creep function of aluminum and typical relaxation spectrum (after Zener, 1948).

2.4 Mechanical models and wave propagation

A typical creep function versus time, as well as a dissipation factor versus frequency are shown in Figure 2.1. These behaviors can be described by using viscoelastic constitutive equations based on mechanical models. To construct a mechanical model, two types of basic elements are required: weightless springs – no inertial effects are present – that represent the elastic solid, and dashpots, consisting of loosely fitting pistons in cylinders filled with a viscous fluid. The simplest are the Maxwell and Kelvin-Voigt models. The Maxwell model was introduced by Maxwell (1867) when discussing the nature of viscosity in gases. Meyer (1874) and Voigt (1892) obtained the so-called Voigt stress-strain relation by generalizing the equations of classical elasticity. The mechanical model representation of the Voigt solid (the Kelvin-Voigt model) was introduced by Lord Kelvin (Kelvin, 1875).

The relaxation function can be obtained by measuring the stress after imposing a rapidly constant unit strain in a relaxed sample of the medium, i.e., $\epsilon = H(t)$, such that (2.28) becomes

$$\sigma(t) = \partial_t \psi(t) * H(t) = \psi(t) * \delta(t) = \psi(t). \quad (2.141)$$

A constant state of stress instantaneously applied to the sample ($\sigma = H(t)$), with the resulting strain being measured as a function of time, describes the creep experiment. The resulting time function is the creep function. That is

$$\epsilon(t) = \partial_t \chi(t) * H(t) = \chi(t) * \delta(t) = \chi(t). \quad (2.142)$$

There are materials for which creep continues indefinitely as time increases. If the limit $\partial_t \chi(t = \infty)$ is finite, permanent deformation occurs after the application of a stress field. Such behavior is akin to that of viscoelastic fluids. If that quantity is zero, the material is referred to as a viscoelastic solid. If χ increases indefinitely, the relaxation function ψ must tend to zero, according to (2.41). This is another criterion to distinguish between fluid and solid behavior: that is, for fluid-like materials ψ tends to zero; for solid-like materials, ψ tends to a finite value.

2.4.1 Maxwell model

The simplest series combination of mechanical models is the Maxwell model depicted in Figure 2.2. A given stress σ applied to the model produces a deformation ϵ_1 on the spring and a deformation ϵ_2 on the dashpot. The stress-strain relation in the spring is

$$\sigma = M_U \epsilon_1, \quad (2.143)$$

where M_U is the elasticity constant of the spring (M_e in equation (2.25)). The subindex U denotes “unrelaxed”. Its meaning will become clear in the following discussion. The stress-strain relation in the dashpot is

$$\sigma = \eta \partial_t \epsilon_2, \quad \eta \geq 0, \quad (2.144)$$

where η is the viscosity. Assuming that the total elongation of the system is $\epsilon = \epsilon_1 + \epsilon_2$, the stress-strain relation of the Maxwell element is

$$\frac{\partial_t \sigma}{M_U} + \frac{\sigma}{\eta} = \partial_t \epsilon. \quad (2.145)$$

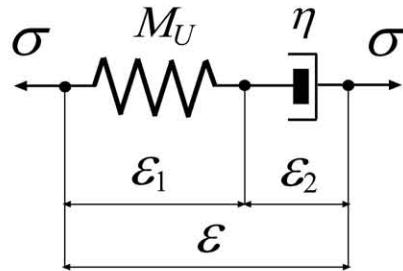


Figure 2.2: Mechanical model for a Maxwell material. The force on both elements is the same, but the elongation (strain) is different.

The Fourier transform of equation (2.145), or equivalently, the substitution of a harmonic wave $[\cdot] \exp(i\omega t)$, yields

$$\sigma = M\epsilon, \quad (2.146)$$

where

$$M(\omega) = \frac{\omega\eta}{\omega\tau - i} \quad (2.147)$$

is the complex modulus, with

$$\tau = \frac{\eta}{M_U} \quad (2.148)$$

being a relaxation time.

The corresponding relaxation function is

$$\psi(t) = M_U \exp(-t/\tau) H(t). \quad (2.149)$$

This can be verified by performing the Boltzmann operation (2.6),

$$\partial_t \psi = \psi \odot \delta = M_U \delta(t) - \frac{M_U}{\tau} \exp(-t/\tau) H(t), \quad (2.150)$$

and calculating the complex modulus (2.31),

$$\mathcal{F}[\partial_t \psi] = \int_{-\infty}^{\infty} \partial_t \psi \exp(-i\omega t) dt = M_U - \frac{M_U}{1 + i\omega\tau} = \frac{\omega\eta}{\omega\tau - i}. \quad (2.151)$$

The complex modulus (2.147) and the relaxation function (2.149) can be shown to satisfy all the requirements listed in Section 2.2.5. Using equations (2.41) and (2.42), we note that the creep function of the Maxwell model is

$$\chi(t) = \frac{1}{M_U} \left(1 + \frac{t}{\tau} \right) H(t). \quad (2.152)$$

The creep and relaxation functions are depicted in Figure 2.3a-b, respectively. As can be seen, the creep function is not representative of the real creep behavior in real solids. Rather, it resembles the creep function of a viscous fluid. In the relaxation

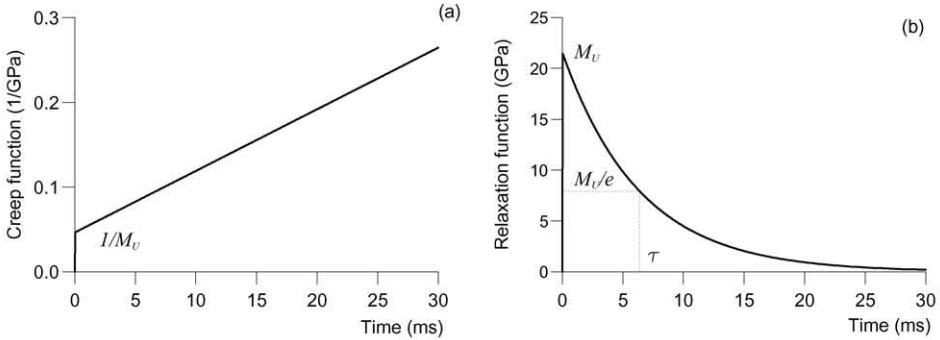


Figure 2.3: Creep (a) and relaxation (b) functions of the Maxwell model ($M_U = 2.16 \text{ GPa}$, $\tau = 1/(2\pi f)$, $f = 25 \text{ Hz}$). The creep function resembles the creep function of a viscous fluid. The system does not present an asymptotical residual stress as in the case of real solids.

experiment, both the spring and the dashpot experience the same force, and because it is not possible to have an instantaneous deformation in the dashpot, the extension is initially in the spring. The dashpot extends and the spring contracts, such that the total elongation remains constant. At the end, the force in the spring relaxes completely and the relaxation function does not present an asymptotical residual stress, as in the case of real solids. In conclusion, the Maxwell model appears more appropriate for representing a viscoelastic fluid. We can see from Figure 2.3a that M_U represents the instantaneous response of the system, hence, the name unrelaxed modulus.

We have seen in Section 2.3 that the partition of the second term in the right-hand side of equation (2.94) in terms of the rate of strain-energy density and rate of dissipated-energy density is, in general, not unique. We have claimed that the splitting (2.12) is consistent with the mechanical-model description of viscoelasticity. As an example, we verify the correctness of the general form (2.16) (or (2.96)) for the Maxwell model. Substituting the relaxation function (2.149) into that equation, we obtain

$$V(t) = \frac{1}{2M_U} \left\{ \int_{-\infty}^t M_U \exp[-(t - \tau_1)/\tau] \partial_{\tau_1} \epsilon(\tau_1) d\tau_1 \right\}^2 = \\ \frac{1}{2M_U} \left\{ \int_{-\infty}^{\infty} \psi(t - \tau_1) \partial_{\tau_1} \epsilon(\tau_1) d\tau_1 \right\}^2 = \frac{1}{2M_U} (\psi * \partial_t \epsilon)^2 = \frac{\sigma^2}{2M_U}. \quad (2.153)$$

But this is precisely the energy stored in the spring, since, using (2.143) and the form (2.93), we obtain

$$V = \frac{1}{2} M_U \epsilon_1^2 = \frac{\sigma^2}{2M_U}. \quad (2.154)$$

Note that because $\psi = \check{\psi} H$, the second term in the right-hand side of (2.94) can be written as

$$(\partial_t \psi * \epsilon) \partial_t \epsilon = \psi(0) \epsilon \partial_t \epsilon + (\partial_t \check{\psi} * \epsilon) \partial_t \epsilon. \quad (2.155)$$

This is one possible partition and one may be tempted to identify the first term with the rate of strain-energy density. However, a simple calculation using the Maxwell model shows that this choice is not consistent with the energy stored in the spring.

The wave propagation properties are described by the phase velocity (2.83), the attenuation factor (2.85) and the quality factor (2.120). The quality factor has the simple expression

$$Q(\omega) = \omega\tau. \quad (2.156)$$

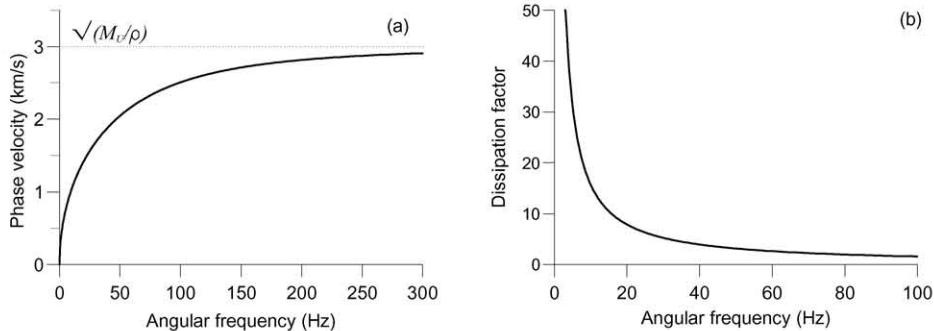


Figure 2.4: Phase velocity (a) and dissipation factor (b) of the Maxwell model ($M_U = \rho c^2$, $\rho = 2.4$ gr/cm³, $c = 3$ km/s, $\tau = 1/(2\pi f)$, $f = 25$ Hz). The system acts as a high-pass filter because low-frequency modes dissipate completely. The velocity for lossless media is obtained at the high-frequency limit. At low frequencies there is no propagation.

The phase velocity and dissipation factors are shown in Figures 2.4a-b, respectively. When $\omega \rightarrow 0$, then $v_p \rightarrow 0$, and $\omega \rightarrow \infty$ implies $v_p \rightarrow \sqrt{M_U/\rho}$, i.e., the velocity in the unrelaxed state. This means that a wave in a Maxwell material travels slower than a wave in the corresponding elastic material – if this is represented by the spring. The dissipation is infinite at zero frequency and the medium is lossless at high frequencies.

2.4.2 Kelvin-Voigt model

A viscoelastic model commonly used to describe anelastic effects is the Kelvin-Voigt stress-strain relation, which consists of a spring and a dashpot connected in parallel (Figure 2.5).

The total stress is composed of an elastic stress

$$\sigma_1 = M_R \epsilon, \quad (2.157)$$

where M_R is the spring constant – the subindex R denotes “relaxed” – and a viscous stress

$$\sigma_2 = \eta \partial_t \epsilon, \quad (2.158)$$

where ϵ is the total strain of the system. The stress-strain relation becomes

$$\sigma = \sigma_1 + \sigma_2 = M_R \epsilon + \eta \partial_t \epsilon. \quad (2.159)$$

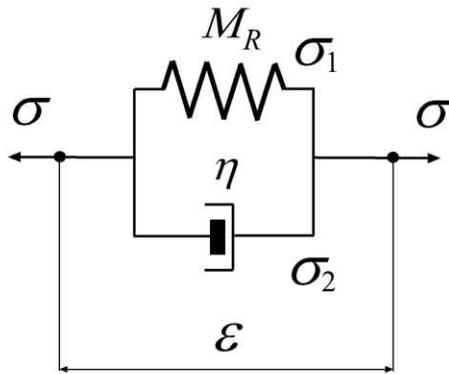


Figure 2.5: Mechanical model for a Kelvin-Voigt material. The strain on both elements is the same, but the forces are different.

The Fourier transform of (2.159) yields

$$\sigma = (M_R + i\omega\eta)\epsilon, \quad (2.160)$$

which identifies the complex modulus

$$M(\omega) = M_R + i\omega\eta. \quad (2.161)$$

The relaxation and creep functions are

$$\psi(t) = M_R H(t) + \eta\delta(t), \quad (2.162)$$

and

$$\chi(t) = \frac{1}{M_R} [1 - \exp(-t/\tau)] H(t), \quad (2.163)$$

where $\tau = \eta/M_R$.

The calculation of the relaxation function from (2.159) is straightforward, and the creep function can be obtained by using (2.41) and (2.42) and Fourier-transform methods. The two functions are represented in Figure 2.6a-b, respectively.

The relaxation function does not show any time dependence. This is the case of pure elastic solids. The delta function implies that, in practice, it is impossible to impose an instantaneous strain on the medium. In the creep experiment, initially the dashpot extends and begins to transfer the stress to the spring. At the end, the entire stress is on the spring. The creep function does not present an instantaneous strain because the dashpot cannot move instantaneously. This is not the case of real solids. The creep function tends to the relaxed modulus M_R at infinite time.

The quality factor (2.120) is

$$Q(\omega) = (\omega\tau)^{-1}. \quad (2.164)$$

Comparing this equation to equation (2.156) shows that the quality factors of the Kelvin-Voigt and Maxwell models are reciprocal functions.

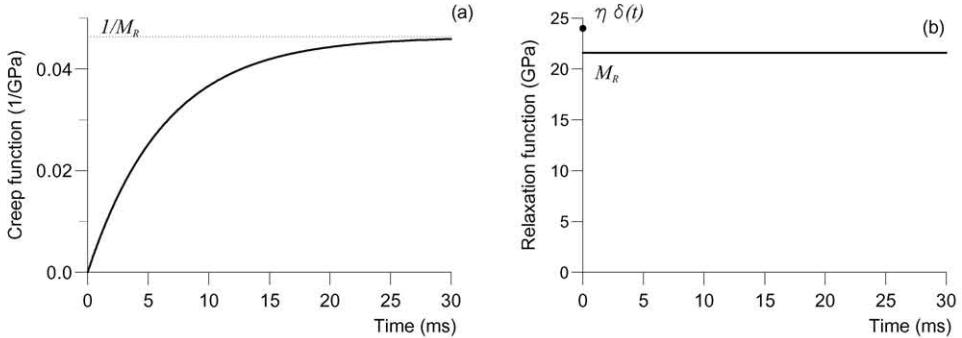


Figure 2.6: Creep (a) and relaxation (b) functions of the Kelvin-Voigt model ($M_R = 2.16 \text{ GPa}$, $\tau = 1/(2\pi f)$, $f = 25 \text{ Hz}$). The creep function lacks the instantaneous response of real solids. The relaxation function presents an almost elastic behavior.

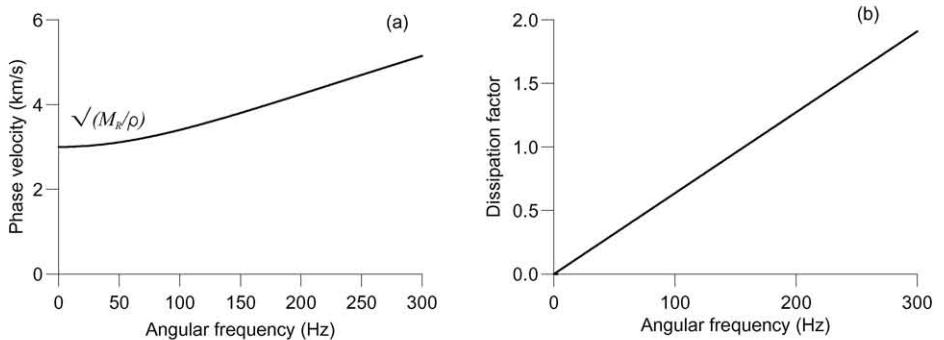


Figure 2.7: Phase velocity (a) and dissipation factor (b) of the Kelvin-Voigt model ($M_R = \rho c^2$, $\rho = 2.4 \text{ gr/cm}^3$, $c = 3 \text{ km/s}$, $\tau = 1/(2\pi f)$, $f = 25 \text{ Hz}$). The system acts as a low-pass filter because high-frequency modes dissipate completely. The elastic (lossless) velocity is obtained at the low-frequency limit. High frequencies propagate with infinite velocity.

The phase velocity and dissipation factor are displayed in Figure 2.7a-b.

The Kelvin-Voigt model can be used to approximate the left slope of a real relaxation peak (see Figure 2.1). The phase velocity $v_p \rightarrow \sqrt{M_R/\rho}$ for $\omega \rightarrow 0$, and $v_p \rightarrow \infty$ for $\omega \rightarrow \infty$, which implies that a wave in a Kelvin-Voigt material travels faster than a wave in the corresponding elastic material.

2.4.3 Zener or standard linear solid model

A series combination of a spring and a Kelvin-Voigt model gives a more realistic representation of material media, such as rocks, polymers and metals. The resulting system, called the Zener model (Zener, 1948) or standard linear solid, is shown in Figure 2.8. This model was introduced by Poynting and Thomson (1902).

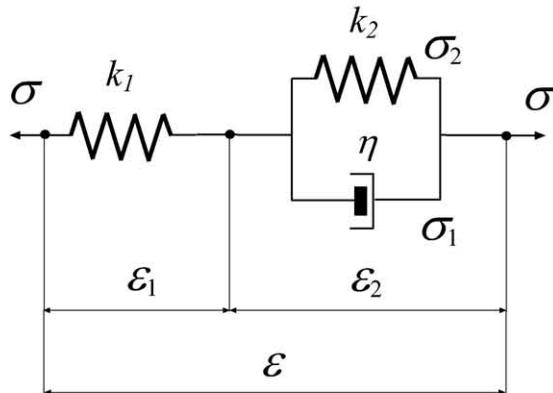


Figure 2.8: Mechanical model for a Zener material.

The stress-strain relations for the single elements are

$$\begin{aligned}\sigma &= k_1\epsilon_1, \\ \sigma_1 &= \eta\partial_t\epsilon_2, \\ \sigma_2 &= k_2\epsilon_2,\end{aligned}\tag{2.165}$$

with $k_1 \geq 0$, $k_2 \geq 0$ and $\eta \geq 0$. Moreover,

$$\sigma = \sigma_1 + \sigma_2, \quad \epsilon = \epsilon_1 + \epsilon_2.\tag{2.166}$$

The solution of these equations for σ and ϵ gives the stress-strain relation

$$\sigma + \tau_\sigma \partial_t \sigma = M_R(\epsilon + \tau_\epsilon \partial_t \epsilon),\tag{2.167}$$

where

$$M_R = \frac{k_1 k_2}{k_1 + k_2},\tag{2.168}$$

is the relaxed modulus, and

$$\tau_\sigma = \frac{\eta}{k_1 + k_2}, \quad \tau_\epsilon = \frac{\eta}{k_2} \geq \tau_\sigma\tag{2.169}$$

are the relaxation times.

As in the previous models, the complex modulus is obtained by performing a Fourier transform of the stress-strain relation (2.167),

$$M(\omega) = M_R \left(\frac{1 + i\omega\tau_\epsilon}{1 + i\omega\tau_\sigma} \right). \quad (2.170)$$

The relaxed modulus M_R is obtained for $\omega = 0$, and the unrelaxed modulus

$$M_U = M_R \left(\frac{\tau_\epsilon}{\tau_\sigma} \right), \quad (M_U \geq M_R) \quad (2.171)$$

for $\omega \rightarrow \infty$.

The stress-strain and strain-stress relations are

$$\sigma = \psi * \partial_t \epsilon, \quad \epsilon = \chi * \partial_t \sigma, \quad (2.172)$$

where the relaxation and creep functions are

$$\psi(t) = M_R \left[1 - \left(1 - \frac{\tau_\epsilon}{\tau_\sigma} \right) \exp(-t/\tau_\sigma) \right] H(t) \quad (2.173)$$

and

$$\chi(t) = \frac{1}{M_R} \left[1 - \left(1 - \frac{\tau_\sigma}{\tau_\epsilon} \right) \exp(-t/\tau_\epsilon) \right] H(t). \quad (2.174)$$

(As an exercise, the reader may obtain the complex modulus (2.170) by using equations (2.31) and (2.173)). Note that by the symmetry of the strain-stress relation (2.167), exchanging the roles of τ_σ and τ_ϵ and substituting M_R for M_R^{-1} in equation (2.173), the creep function (2.174) can be obtained.

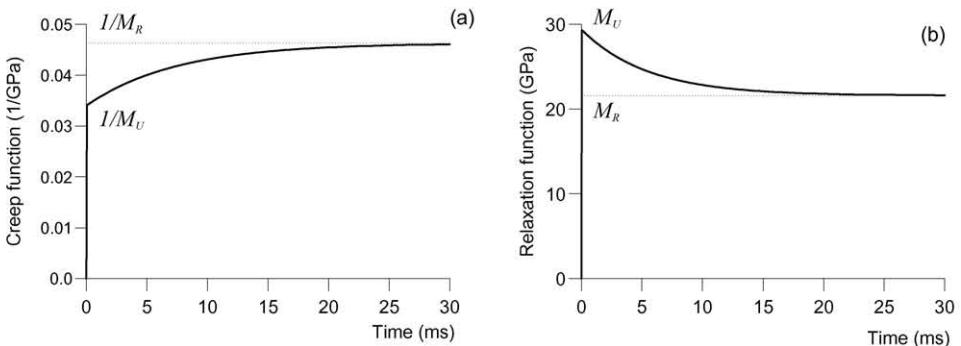


Figure 2.9: Creep (a) and relaxation (b) functions of the Zener model ($M_R = 2.16$ GPa, $M_U = 29.4$ GPa, $\tau_0 = 1/(2\pi f)$, $f = 25$ Hz). The creep function presents an instantaneous response and a finite asymptotic value as in real solids. The relaxation function presents an instantaneous unrelaxed state, and at the end of the process, the system has relaxed completely to the relaxed modulus M_R . The curve in (a) is similar to the experimental creep function shown in 2.1.

The relaxation and creep functions are represented in Figure 2.9a-b, respectively. In the creep experiment, there is an instantaneous initial value $\chi(0^+) = M_U^{-1}$, and an asymptotic strain $\chi(\infty) = M_R^{-1}$, determined solely by the spring constants. After the first initial displacement, the force across the dashpot is gradually relaxed by deformation therein, resulting in a gradual increase in the observed overall deformation; finally, the asymptotic value is reached. Similarly, the relaxation function exhibits an instantaneous unrelaxed state of magnitude M_U . At the end of the process, the system has relaxed completely to the relaxed modulus M_R . Such a system, therefore, manifests the general features of the experimental creep function illustrated in Figure 2.1a. The relaxation function and complex modulus can be shown to satisfy all the requirements listed in Section 2.2.5.

The quality factor (2.120) is

$$Q(\omega) = \frac{1 + \omega^2 \tau_\epsilon \tau_\sigma}{\omega(\tau_\epsilon - \tau_\sigma)}, \quad (2.175)$$

where we have used equation (2.170).

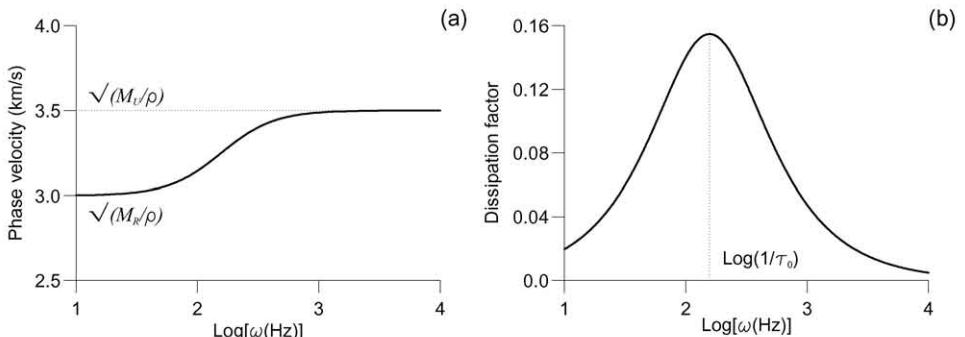


Figure 2.10: Phase velocity (a) and dissipation factor (b) of the Zener model. ($M_R = \rho c_R^2$, $\rho = 2.4$ gr/cm³, $c_R = 3$ km/s, $M_U = \rho c_U^2$, $c_U = 3.5$ km/s, $\tau_0 = 1/(2\pi f)$, $f = 25$ Hz).

The phase velocity and dissipation factor Q^{-1} are shown in Figure 2.10a-b. The model has a relaxation peak at $\omega_0 = 1/\tau_0$, where

$$\tau_0 = \sqrt{\tau_\epsilon \tau_\sigma}. \quad (2.176)$$

The phase velocity increases with frequency. (The same happens for the Maxwell and Kelvin-Voigt models). The type of dispersion in which this happens is called anomalous dispersion in the electromagnetic terminology. In electromagnetism, the index of refraction – defined as the velocity of light in a vacuum divided by the phase velocity – decreases with frequency for anomalously dispersive media (Born and Wolf, 1964, p. 18; Jones, 1986, p. 644).

The Zener model is suitable to represent relaxation mechanisms such as those illustrated in Figure 2.8b. Processes such as grain-boundary relaxation have to be explained by a distribution of relaxation peaks. This behavior is obtained by considering several

Zener elements in series or in parallel, a system which is described in the next section. The phase velocity ranges from $\sqrt{M_R/\rho}$ at the low-frequency limit to $\sqrt{M_U/\rho}$ at the high-frequency limit, and the system exhibits a pure elastic behavior ($Q^{-1} = 0$) at both limits.

2.4.4 Burgers model

A unique model to describe both the transient and steady-state creep process is given by the Burgers model, which is formed with a series connection of a Zener element and a dashpot, or equivalently, a series connection of a Kelvin-Voigt element and a Maxwell element (Klausner, 1991). The model is shown in Figure 2.11, and the constitutive equations of the single elements are

$$\begin{aligned}\sigma_1 &= k_2 \epsilon_2 \\ \sigma_2 &= \eta_2 \partial_t \epsilon_2 = i\omega \eta_2 \epsilon_2 \\ \sigma &= \eta_1 \partial_t \epsilon_3 = i\omega \eta_1 \epsilon_3 \\ \sigma &= k_1 \epsilon_1,\end{aligned}\tag{2.177}$$

where a time Fourier transform is implicit.

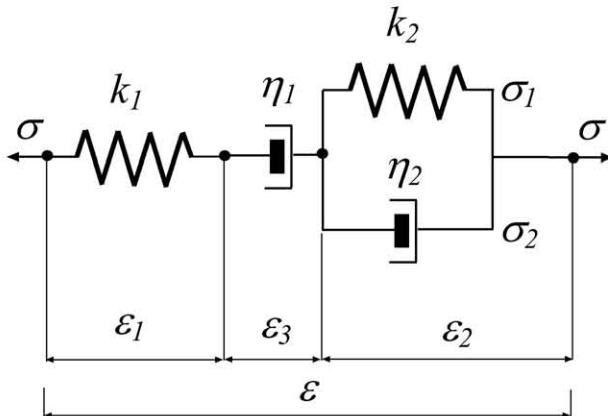


Figure 2.11: Burgers's viscoelastic model. The response of the Burgers model is instantaneous elasticity, delayed elasticity (or viscoelasticity) and viscous flow, the latter described by the series dashpot. On removal of the perturbation, the instantaneous and delayed elasticity are recovered, and it remains the viscous flow. The viscoelastic creep – with steady-state creep – of rocksalt can be described by the Burgers model which includes the transient creep of the Zener model, which does not exhibit steady-state creep, and the steady-state creep of a Maxwell model. (Carcione, Helle and Gangi, 2006).

Since

$$\begin{aligned}\epsilon &= \epsilon_1 + \epsilon_2 + \epsilon_3 \\ \sigma &= \sigma_1 + \sigma_2,\end{aligned}\tag{2.178}$$

we have

$$\sigma = \sigma_1 + \sigma_2 = (k_2 + i\omega \eta_2) \epsilon_2\tag{2.179}$$

and

$$\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{\sigma}{k_1} + \frac{\sigma}{i\omega\eta_1} + \frac{\sigma}{k_2 + i\omega\eta_2} \equiv J(\omega)\sigma, \quad (2.180)$$

where

$$J(\omega) = \frac{1}{M(\omega)} = \frac{1}{k_1} + \frac{1}{i\omega\eta_1} + \frac{1}{k_2 + i\omega\eta_2} \quad (2.181)$$

is the complex creep compliance (2.43).

An inverse Fourier transforms of (2.181) and a time integration of the result leads to

$$\partial_t \chi(t) = \frac{\delta(t)}{k_1} + \frac{H(t)}{\eta_1} + \frac{1}{\eta_2} \exp(-t/\tau_\epsilon) H(t) \quad (2.182)$$

and

$$\chi(t) = \left\{ \frac{1}{k_1} + \frac{t}{\eta_1} + \frac{1}{k_2} [1 - \exp(-t/\tau_\epsilon)] \right\} H(t), \quad (2.183)$$

where τ_ϵ is given by equation (2.169)₂. Equation (2.183) can also be obtained by adding the creep functions of the Maxwell (M) and Kelvin-Voigt (KV) models (equations (2.152) and (2.163), respectively), because $\epsilon_2 = \chi_{KV} * \partial_t \sigma$ and $\epsilon_1 + \epsilon_3 = \chi_M * \partial_t \sigma$.

The calculation of the relaxation function is more tricky. The model obeys a time-domain differential equation, which can be obtained by combining equations (2.177) and (2.178):

$$\partial_{tt}^2 \sigma + \left(\frac{k_1}{\eta_1} + \frac{k_1}{\eta_2} + \frac{k_2}{\eta_2} \right) \partial_t \sigma + \frac{k_1 k_2}{\eta_1 \eta_2} \sigma = k_1 \partial_{tt}^2 \epsilon + \frac{k_1 k_2}{\eta_2} \partial_t \epsilon. \quad (2.184)$$

The relaxation function $\psi(t) = \sigma(t)$ is obtained for $\epsilon(t) = H(t)$. Then, factorizing the left-hand side, equation (2.184) can be rewritten as

$$(\omega_1 \delta - \delta') * (\omega_2 \delta - \delta') * \psi = k_1 \delta' + \frac{k_1 k_2}{\eta_2} \delta, \quad (2.185)$$

where $\delta' = \partial_t \delta$, and

$$(2\eta_1 \eta_2) \omega_{1,2} = -b \pm \sqrt{b^2 - 4k_1 k_2 \eta_1 \eta_2}, \quad b = k_1 \eta_1 + k_1 \eta_2 + k_2 \eta_1. \quad (2.186)$$

Hence, the relaxation function is

$$\psi = (\omega_1 \delta - \delta')^{-1} * (\omega_2 \delta - \delta')^{-1} * \left(k_1 \delta' + \frac{k_1 k_2}{\eta_2} \delta \right), \quad (2.187)$$

where here $(\)^{-1}$ denotes the inverse with respect to convolution. Since³

$$(\omega_{1,2} \delta - \delta')^{-1} = -\exp(\omega_{1,2} t) H(t), \quad (2.188)$$

we finally obtain

$$\psi(t) = [A_1 \exp(-t/\tau_1) - A_2 \exp(-t/\tau_2)] H(t), \quad (2.189)$$

where

$$\tau_{1,2} = -\frac{1}{\omega_{1,2}} \quad \text{and} \quad A_{1,2} = \frac{k_1 k_2 + \omega_{1,2} \eta_2 k_1}{\eta_2 (\omega_1 - \omega_2)}. \quad (2.190)$$

³Equation (2.188) is equivalent to $(\omega_{1,2} \delta - \delta') * [-\exp(\omega_{1,2} t) H(t)]$, i.e., $(\omega_{1,2} - \partial_t) [-\exp(\omega_{1,2} t) H(t)] = \delta$, which is identically true.

The models studied in the previous sections are limiting cases of the Burgers model. The Maxwell creep function (2.152) is obtained for $k_2 \rightarrow \infty$ and $\eta_2 \rightarrow 0$, where $M_U = k_1$, $\tau = \eta_1/k_1$ and $\tau_\epsilon = 0$. The Kelvin-Voigt creep function (2.163) is obtained for $k_1 \rightarrow \infty$ and $\eta_1 \rightarrow \infty$, where $M_R = k_2$ and $\tau = \tau_\epsilon$. The Zener creep function (2.174) is obtained for $\eta_1 \rightarrow \infty$, where $\tau_1 = \infty$, $\tau_2 = \tau_\sigma$, $A_1 = M_R$ and $A_2 = M_R(\tau_\epsilon/\tau_\sigma - 1)$.

An example of the use of the Burgers model to describe borehole stability is given in Carcione, Helle and Gangi (2006).

2.4.5 Generalized Zener model

As stated before, some processes, as for example, grain-boundary relaxation, have a dissipation factor that is much broader than a single relaxation curve. It seems natural to try to explain this broadening with a distribution of relaxation mechanisms. This approach was introduced by Liu, Anderson and Kanamori (1976) to obtain a nearly constant quality factor over the seismic frequency range of interest. Strictly, their model cannot be represented by mechanical elements, since it requires a spring of negative constant (Casula and Carcione, 1992). Here, we consider the parallel system shown in Figure 2.12, with L Zener elements connected in parallel. The stress-strain relation for each single element is

$$\sigma_l + \tau_{\sigma l} \partial_t \sigma_l = M_{Rl}(\epsilon + \tau_{\epsilon l} \partial_t \epsilon), \quad l = 1, \dots, L, \quad (2.191)$$

where the relaxed moduli are given by

$$M_{Rl} = \frac{k_{1l}k_{2l}}{k_{1l} + k_{2l}}, \quad (2.192)$$

and the relaxation times by

$$\tau_{\sigma l} = \frac{\eta_l}{k_{1l} + k_{2l}}, \quad \tau_{\epsilon l} = \frac{\eta_l}{k_{2l}}. \quad (2.193)$$

According to (2.170), each complex modulus is given by

$$M_l(\omega) = M_{Rl} \left(\frac{1 + i\omega\tau_{\epsilon l}}{1 + i\omega\tau_{\sigma l}} \right). \quad (2.194)$$

The total stress acting on the system is $\sigma = \sum_{l=1}^L \sigma_l$. Therefore, the stress-strain relation in the frequency domain is

$$\sigma = \sum_{l=1}^L M_l \epsilon = \sum_{l=1}^L M_{Rl} \left(\frac{1 + i\omega\tau_{\epsilon l}}{1 + i\omega\tau_{\sigma l}} \right) \epsilon. \quad (2.195)$$

We can choose $M_{Rl} = M_R/L$, and the complex modulus can be expressed as

$$M(\omega) = \sum_{l=1}^L M_l(\omega), \quad M_l(\omega) = \frac{M_R}{L} \left(\frac{1 + i\omega\tau_{\epsilon l}}{1 + i\omega\tau_{\sigma l}} \right), \quad (2.196)$$

thereby reducing the number of independent constants to $2L + 1$.

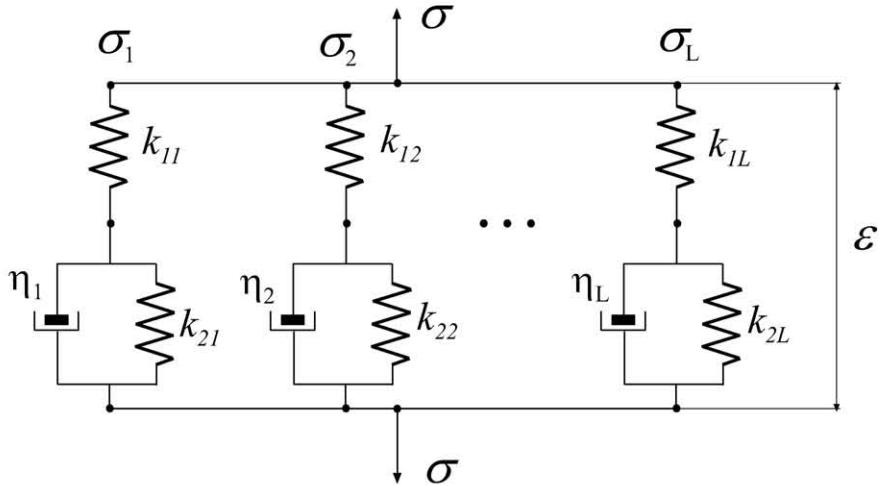


Figure 2.12: Mechanical model for a generalized Zener material.

The relaxation function is easily obtained from the time-domain constitutive equation

$$\sigma = \sum_{l=1}^L \sigma_l = \sum_{l=1}^L \psi_l * \partial_t \epsilon \equiv \psi * \partial_t \epsilon, \quad (2.197)$$

where ψ_l has the form (2.173), and

$$\psi(t) = M_R \left[1 - \frac{1}{L} \sum_{l=1}^L \left(1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \exp(-t/\tau_{\sigma l}) \right] H(t). \quad (2.198)$$

The unrelaxed modulus is obtained for $t = 0$,

$$M_U = M_R \left[1 - \frac{1}{L} \sum_{l=1}^L \left(1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \right] = \frac{M_R}{L} \sum_{l=1}^L \frac{\tau_{el}}{\tau_{\sigma l}}. \quad (2.199)$$

The relaxation function obtained by Liu, Anderson and Kanamori (1976) lacks the factor $1/L$.

Nearly constant Q

In oil prospecting and seismology, constant- Q models are convenient to parameterize attenuation in rocks, since the frequency dependence is usually not known. Moreover, there is physical evidence that attenuation is almost linear with frequency – therefore Q is constant – in many frequency bands (McDonald, Angona, Milss, Sengbush, van Nostrand and White, 1958). The technique to obtain a nearly constant Q over a given frequency range is to consider equispaced relaxation mechanisms in a $\log(\omega)$ scale (Liu, Anderson

and Kanamori, 1976). We show, in the following discussion, how to obtain a constant- Q model for low-loss solids by using a simple algorithm, without curve fitting of the Q factor.

A more physical parameterization of a single Zener element can be obtained with the center frequency $\omega_0 = \tau_0^{-1}$, and the value of the quality factor at this frequency,

$$Q_0 = \frac{2\tau_0}{\tau_\epsilon - \tau_\sigma}. \quad (2.200)$$

The quality factor (2.175) becomes

$$Q(\omega) = Q_0 \left(\frac{1 + \omega^2 \tau_0^2}{2\omega \tau_0} \right). \quad (2.201)$$

Solving for τ_σ and τ_ϵ in equations (2.176) and (2.200), we obtain

$$\tau_\epsilon = \frac{\tau_0}{Q_0} \left(\sqrt{Q_0^2 + 1} + 1 \right) \quad \text{and} \quad \tau_\sigma = \frac{\tau_0}{Q_0} \left(\sqrt{Q_0^2 + 1} - 1 \right). \quad (2.202)$$

Now, the problem is to find a set of relaxation times τ_{el} and $\tau_{\sigma l}$ that gives an almost constant quality factor Q in a given frequency band centered at $\omega_{0m} = 1/\tau_{0m}$. This is the location of the mechanism situated at the middle of the band, which, for odd L , has the index $m = L/2 - 1$. As mentioned above, single relaxation peaks should be taken equidistant in a $\log(\omega)$ scale. The quality factor of the system is

$$Q(\omega) = \frac{\text{Re}(M)}{\text{Im}(M)} = \frac{\text{Re}(\sum_{l=1}^L M_l)}{\text{Im}(\sum_{l=1}^L M_l)}, \quad (2.203)$$

where M_l is given in equation (2.196)₂. Since $Q_l = \text{Re}(M_l)/\text{Im}(M_l)$ is the quality factor of each element, equation (2.203) becomes

$$Q(\omega) = \frac{\sum_{l=1}^L Q_l \text{Im}(M_l)}{\sum_{l=1}^L \text{Im}(M_l)}, \quad (2.204)$$

where

$$Q_l(\omega) = Q_{0l} \left(\frac{1 + \omega^2 \tau_{0l}^2}{2\omega \tau_{0l}} \right). \quad (2.205)$$

Using equation (2.200) and assuming the low-loss approximation ($\tau_{\sigma l} \approx \tau_{0l}$), we have

$$\text{Im}(M_l) = \frac{M_R}{L} \left[\frac{\omega(\tau_{el} - \tau_{\sigma l})}{1 + \omega^2 \tau_{\sigma l}^2} \right] \approx \frac{M_R}{L} \left[\frac{2\omega \tau_{0l}}{Q_{0l}(1 + \omega^2 \tau_{0l}^2)} \right] = \frac{M_R}{LQ_l}. \quad (2.206)$$

We now choose $Q_{0l} = Q_0$, and substitute equation (2.206) into equation (2.204) to obtain

$$Q(\omega) = LQ_0 \left(\sum_{l=1}^L \frac{2\omega \tau_{0l}}{1 + \omega^2 \tau_{0l}^2} \right)^{-1}. \quad (2.207)$$

We choose τ_{0l} regularly distributed in the $\log(\omega)$ axis, and $Q(\omega_{0m}) = \bar{Q}$, the desired value of the quality factor.

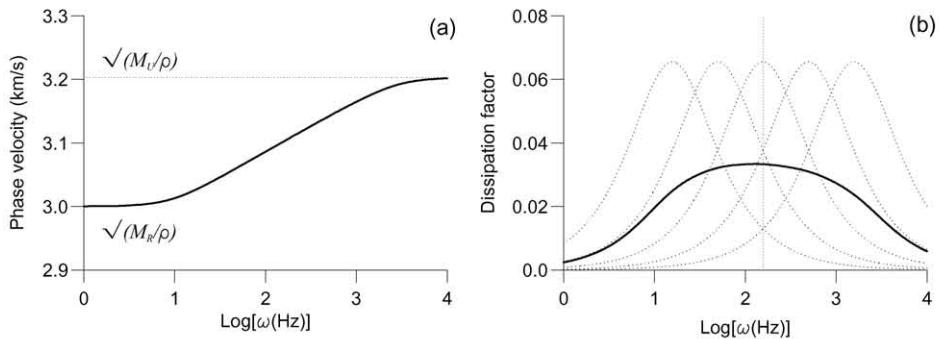


Figure 2.13: Phase velocity (a) and dissipation factor (b) of the generalized Zener model.

Thus, the choice

$$Q_0 = \frac{\bar{Q}}{L} \sum_{l=1}^L \frac{2\omega_{0m}\tau_{0l}}{1 + \omega_{0m}^2\tau_{0l}^2} \quad (2.208)$$

gives a constant Q (equal to \bar{Q}), as can be verified by substitution of (2.208) into (2.207).

Figure 2.13 shows the phase velocity (a) and the dissipation factor (b) versus frequency, for five dissipation mechanisms – each with a quality-factor parameter $Q_0 = 15$, such that $\bar{Q} = 30$. The dotted curves are the quality factor of each single mechanism, and the vertical dotted line indicates the location of the third relaxation peak. The relaxation function of the nearly constant- Q model is shown in Figure 2.14.

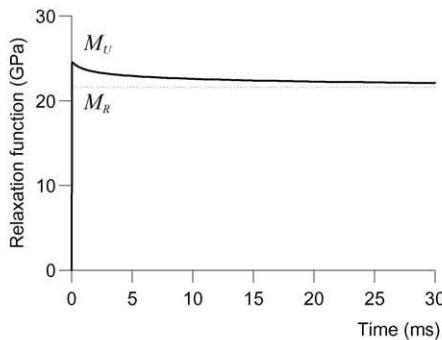


Figure 2.14: Relaxation function of the generalized Zener model.

2.4.6 Nearly constant- Q model with a continuous spectrum

A linear and continuous superposition of Zener elements, where each element has equal weight, gives a continuous relaxation spectrum with a constant quality factor over a given frequency band (Liu, Anderson and Kanamori, 1976; Ben-Menahem and Singh, 1981, p.

911). The resulting relaxation function exhibits elastic (lossless) behavior in the low- and high-frequency limits. Its frequency-domain form is

$$M(\omega) = M_R \left[1 + \frac{2}{\pi \bar{Q}} \ln \left(\frac{1 + i\omega\tau_2}{1 + i\omega\tau_1} \right) \right]^{-1}, \quad (2.209)$$

where τ_1 and τ_2 are time constants, with $\tau_2 < \tau_1$, and \bar{Q} defines the value of the quality factor, which remains nearly constant over the selected frequency band. The low-frequency limit of M is M_R , and we can identify this modulus with the elastic modulus. Alternatively, we may consider

$$M(\omega) = M_U \left[1 + \frac{2}{\pi \bar{Q}} \ln \left(\frac{\tau_2^{-1} + i\omega}{\tau_1^{-1} + i\omega} \right) \right]^{-1}, \quad (2.210)$$

whose high-frequency limit is the elastic modulus M_U . These functions give a nearly constant quality factor in the low-loss approximation. Figure 2.15 represents the dissipation factor $Q^{-1} = \text{Im}(M)/\text{Re}(M)$ for the two functions (2.209) and (2.210) (solid and dashed lines, respectively).

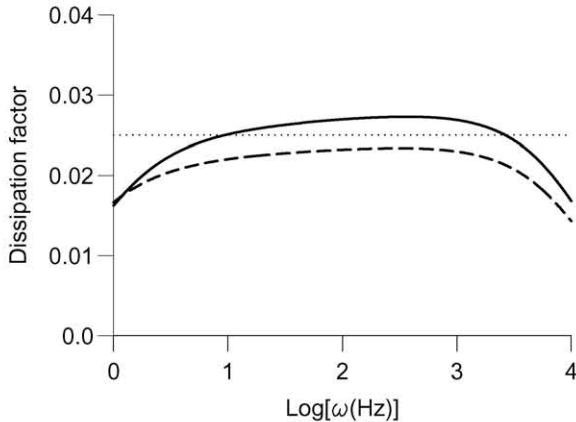


Figure 2.15: Dissipation factors for the nearly constant- Q model, corresponding to the two functions (2.209) and (2.210) (solid and dashed lines, respectively). The curves correspond to $\bar{Q} = 40$, $\tau_1 = 1.5$ s and $\tau_2 = 8 \times 10^{-5}$ s. The dotted line represents \bar{Q}^{-1} .

2.5 Constant- Q model and wave equation

A perfect constant- Q model can be designed for all frequencies. Bland (1960), Caputo and Mainardi (1971), Kjartansson (1979), Müller (1983) and Mainardi and Tomirotti (1998) discuss a linear attenuation model with the required characteristics, but the idea is much older (Nutting, 1921; Scott Blair, 1949). The so-called Kjartansson's constant- Q model – in seismic prospecting literature – is based on a creep function of the form $t^{2\gamma}$,

where t is time and $\gamma \ll 1$ for seismic applications. This model is completely specified by two parameters, i.e., phase velocity at a reference frequency and Q . Therefore, it is mathematically much simpler than any nearly constant Q , such as, for instance, a spectrum of Zener models (Carcione, Kosloff and Kosloff, 1988b,c,d). Due to its simplicity, Kjartansson's model is used in many seismic applications, mainly in its frequency-domain form.

The relaxation function is

$$\psi(t) = \frac{M_0}{\Gamma(1-2\gamma)} \left(\frac{t}{t_0}\right)^{-2\gamma} H(t), \quad (2.211)$$

where M_0 is a bulk modulus, Γ is Euler's Gamma function, t_0 is a reference time and γ is a dimensionless parameter. The parameters M_0 , t_0 and γ have precise physical meanings that will become clear in the following analysis.

Using equation (2.31) and after some calculations, we get the complex modulus,

$$M(\omega) = M_0 \left(\frac{i\omega}{\omega_0}\right)^{2\gamma}, \quad (2.212)$$

where $\omega_0 = 1/t_0$ is the reference frequency.

2.5.1 Phase velocity and attenuation factor

The complex velocity is given by equation (2.80),

$$v_c = \sqrt{\frac{M}{\rho}}, \quad (2.213)$$

and the phase velocity can be obtained from equation (2.83),

$$v_p = c_0 \left| \frac{\omega}{\omega_0} \right|^\gamma \quad (2.214)$$

with

$$c_0 = \sqrt{\frac{M_0}{\rho}} \left[\cos\left(\frac{\pi\gamma}{2}\right) \right]^{-1}. \quad (2.215)$$

The attenuation factor (2.85) is given by

$$\alpha = \tan\left(\frac{\pi\gamma}{2}\right) \operatorname{sgn}(\omega) \frac{\omega}{v_p}, \quad (2.216)$$

and the quality factor, according to equation (2.120), is

$$Q = \frac{1}{\tan(\pi\gamma)}. \quad (2.217)$$

Firstly, we have from equation (2.214) that c_0 is the phase velocity at $\omega = \omega_0$ (the reference frequency), and that

$$M_0 = \rho c_0^2 \cos^2\left(\frac{\pi\gamma}{2}\right). \quad (2.218)$$

Secondly, it follows from equation (2.217) that Q is independent of frequency, so that

$$\gamma = \frac{1}{\pi} \tan^{-1} \left(\frac{1}{Q} \right) \quad (2.219)$$

parameterizes the attenuation level. Hence, we see that $Q > 0$ is equivalent to $0 < \gamma < 1/2$. Moreover, $v_p \rightarrow 0$ when $\omega \rightarrow 0$, and $v_p \rightarrow \infty$ when $\omega \rightarrow \infty$. It follows that very high frequencies of the signal propagate at almost infinite velocity, and the differential equation describing the wave motion is parabolic (e.g., Prüss, 1993).

2.5.2 Wave equation in differential form. Fractional derivatives.

Let us consider propagation in the (x, z) -plane and a 2-D wave equation of the form

$$\frac{\partial^\beta w}{\partial t^\beta} = b\Delta w + f_w, \quad (2.220)$$

where $w(x, z, t)$ is a field variable, β is the order of the time derivative, b is a positive parameter, Δ is the 2-D Laplacian operator

$$\Delta = \partial_1^2 + \partial_3^2, \quad (2.221)$$

and f_w is a forcing term. Consider a plane wave

$$\exp[i(\omega t - k_1 x - k_3 z)], \quad (2.222)$$

where ω is real and (k_1, k_3) is the complex wavevector. Substitution of the plane wave (2.222) in the wave equation (2.220) with $f_w = 0$ yields the dispersion relation

$$(i\omega)^\beta + bk^2 = 0, \quad (2.223)$$

where $k = \sqrt{k_1^2 + k_3^2}$ is the complex wavenumber. Equation (2.223) is the Fourier transform of equation (2.220). The properties of the Fourier transform when it acts on fractional derivatives are well established, and a rigorous treatment is available in the literature (e.g., Dattoli, Torre and Mazzacurati, 1998). Since $k^2 = \rho\omega^2/M$, a comparison of equations (2.223) and (2.212) gives

$$\beta = 2 - 2\gamma, \quad \text{and} \quad b = \left(\frac{M_0}{\rho} \right) \omega_0^{-2\gamma}. \quad (2.224)$$

Equation (2.220), together with (2.224), is the wave equation corresponding to Kjartansson's stress-strain relation (Kjartansson, 1979). In order to obtain realistic values of the quality factor, which correspond to wave propagation in rocks, $\gamma \ll 1$ and the time derivative in equation (2.220) has a fractional order.

Kjartansson's wave equation (2.220) is a particular version of a more general wave equation for variable material properties. The convolutional stress-strain relation (2.28) can be written in terms of fractional derivatives. In fact, it is easy to show, using equations (2.212) and (2.224), that it is equivalent to

$$\sigma = \rho b \frac{\partial^{2-\beta} \epsilon}{\partial t^{2-\beta}}. \quad (2.225)$$

Coupled with the stress-strain relation (2.225) are the momentum equations

$$\partial_1 \sigma = \rho \partial_{tt}^2 u_1, \quad (2.226)$$

$$\partial_3 \sigma = \rho \partial_{tt}^2 u_3, \quad (2.227)$$

where u_1 and u_3 are the displacement components. By redefining

$$\epsilon = \partial_1 u_1 + \partial_3 u_3 \quad (2.228)$$

as the dilatation field, differentiating and adding equations (2.226) and (2.227), and substituting equation (2.225), we obtain

$$\Delta_\rho \left(\rho b \frac{\partial^{2-\beta} \epsilon}{\partial t^{2-\beta}} \right) = \frac{\partial^2 \epsilon}{\partial t^2}, \quad (2.229)$$

where

$$\Delta_\rho = \partial_1 \rho^{-1} \partial_1 + \partial_3 \rho^{-1} \partial_3. \quad (2.230)$$

Multiplying by $(i\omega)^{\beta-2}$ the Fourier transform of equation (2.229), we have, after an inverse Fourier transform, the inhomogeneous wave equation

$$\frac{\partial^\beta \epsilon}{\partial t^\beta} = \Delta_\rho (\rho b \epsilon) + f_\epsilon, \quad (2.231)$$

where we included the source term f_ϵ . This equation is similar to (2.220) if the medium is homogeneous.

A more general stress-strain relation is considered by Müller (1983), where the quality factor is proportional to ω^a , with $-1 \leq a \leq 1$. The cases $a = -1$, $a = 0$ and $a = 1$ correspond to the Maxwell, Kelvin-Voigt and constant- Q models, respectively (see equations (2.156), (2.164) and (2.217)). Müller derives the viscoelastic modulus using the Kramers-Kronig relations, obtaining closed-form expressions for the cases $a = \pm 1/n$, with n a natural number. Other stress-strain relations involving derivatives of fractional order are the Cole-Cole models (Cole and Cole, 1941; Bagley and Torvik, 1983, Caputo; 1998; Bano, 2004), which are used to describe dispersion and energy loss in dielectrics (see Section 8.3.2), anelastic media and electric networks.

Propagation in Pierre shale

Attenuation measurements in a relatively homogeneous medium (Pierre shale) were made by McDonal, Angona, Milss, Sengbush, van Nostrand and White (1958) near Limon, Colorado. They reported a constant- Q behavior with attenuation $\alpha = 0.12f$, where α is given in dB per 1000 ft and the frequency f in Hz. Conversion of units implies α (dB/1000 ft) = 8.686α (nepers/1000 ft) = 2.6475α (nepers/km). For low-loss solids, the quality factor is, according to (2.123),

$$Q = \frac{\pi f}{\alpha v_p},$$

with α given in nepers per unit length (Toksoz and Johnston, 1981). Since c is approximately 7000 ft/s (2133.6 m/s), the quality factor is $Q \simeq 32.5$. We consider a reference frequency $f_0 = \omega_0/(2\pi) = 250$ Hz, corresponding to the dominant frequency of the seismic

source used in the experiments. Then, $\gamma = 0.0097955$, $\beta = 1.980409$, and $c_0 = \sqrt{M_0/\rho} = 2133.347$ m/s. The phase velocity (2.214) and attenuation factor (2.216) versus frequency $f = \omega/2\pi$ are shown in Figures 2.16a-b, respectively, where the open circles are the experimental points. Carcione, Cavallini, Mainardi and Hanyga (2002) solve the wave equation by using a numerical method and compute synthetic seismograms in inhomogeneous media. (The dotted and dashed lines in Figures 2.16a-b correspond to finite-difference approximations of the differential equations.) This approach finds important applications for porous media as well, since fractional derivatives appear in Biot's theory, which are related to memory effects at seismic frequencies (Gurevich and Lopatnikov, 1995; Hanyga and Seredyńska, 1999).

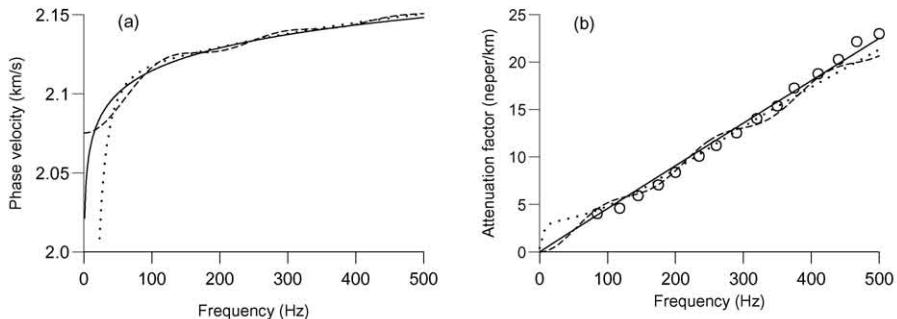


Figure 2.16: Phase velocity and attenuation factor versus frequency in Pierre shale (solid line). The open circles are the experimental data reported by McDonal, Angona, Milss, Sengbush, van Nostrand and White (1958).

2.6 The concept of centrovelocity

The velocity of a pulse in an absorbing and dispersive medium is a matter of controversy. The concept of velocity, which is relevant in the field of physics of materials and Earth sciences, has been actively studied under the impetus provided by the atomic theory on the one hand, and by radio and sound on the other (Eckart, 1948). In seismology, the concept of velocity is very important, because it provides the spatial location of an earthquake hypocenter and geological strata (Ben-Menahem and Singh, 1981). Similarly, ground-penetrating-radar applications are based on the interpretation of radargrams, where the travel times of the reflection events provide information about the dielectric permittivity and ionic conductivity of the shallow geological layers (Daniels, 1996; Carcione, 1996c).

The three velocities, strictly defined for a plane harmonic wave, are the phase velocity (2.83), the group velocity (2.86) and the energy velocity (2.114). As we have seen in Section 2.3, the latter is equal to the phase velocity in 1-D media. Sommerfeld and Brillouin (Brillouin, 1960) clearly show the breakdown of the group-velocity concept, which may exceed the velocity of light in vacuum and even become negative. They introduced the concept of signal velocity, which has been analyzed in detail for the Lorentz model. For

non-periodic (non-harmonic) waves with finite energy, the concept of centrovelocity has been introduced (Vainshtein, 1957; Smith, 1970; Gurwich, 2001). Smith (1970) defines the centrovelocity as the distance travelled divided by the centroid of the time pulse. van Groesen and Mainardi (1989), Derkx and van Groesen (1992) and Gurwich (2001) define the centrovelocity as the velocity of the “mass” center, where the integration is done over the spatial variable instead of the time variable. That is, on the “snapshot” of the wave field instead of the pulse time history. Unlike the phase (energy) and group velocities, the centrovelocity depends on the shape of the pulse, which changes as a function of time and travel distance. Therefore, an explicit analytical expression in terms of the medium properties cannot be obtained.

In order to investigate the concept of wave velocity in the presence of attenuation, we consider a 1-D medium and compare the energy (phase) and group velocities of a harmonic wave to the velocity obtained as the distance divided by the travel time of the centroid of the energy, where by energy we mean the square of the absolute value of the pulse time history. This concept is similar to the centrovelocity introduced by Smith (1970), in the sense that it is obtained in the time domain. Smith’s definition is an instantaneous centrovelocity, as well as Gurwich’s velocity (Gurwich, 2001), which is defined in the space domain. The travel times corresponding to the “theoretical” energy and group velocities are evaluated by taking into account that the pulse dominant frequency decreases with increasing travel distance. Thus, the dominant frequency depends on the spatial variable and is obtained as the centroid of the power spectrum. A similar procedure is performed in the spatial domain by computing a centroid wavenumber.

2.6.1 1-D Green’s function and transient solution

The 1-D Green’s function (impulse response) of the medium is

$$G(\omega) = \exp(-ikx) \quad (2.232)$$

(e.g., Eckart, 1948; Pilant, 1979, p. 52), where k is the complex wavenumber and x is the travel distance. We consider that the time history of the source is

$$f(t) = \exp\left[-\frac{\Delta\omega^2(t-t_0)^2}{4}\right] \cos[\bar{\omega}(t-t_0)], \quad (2.233)$$

whose frequency spectrum is

$$F(\omega) = \frac{\sqrt{\pi}}{\Delta\omega} \left\{ \exp\left[-\left(\frac{\omega+\bar{\omega}}{\Delta\omega}\right)^2\right] + \exp\left[-\left(\frac{\omega-\bar{\omega}}{\Delta\omega}\right)^2\right] \right\} \exp(-i\omega t_0), \quad (2.234)$$

where t_0 is a delay, $\bar{\omega}$ is the central angular frequency, and $2\Delta\omega$ is the width of the pulse, such that $F(\bar{\omega} \pm \Delta\omega) = F(\bar{\omega})/e$. ($\Delta\omega = \bar{\omega}/2$ in the example below).

Then, the frequency-domain response is

$$U(\omega) = F(\omega)G(\omega) = F(\omega) \exp(-ikx) \quad (2.235)$$

and its power spectrum is

$$P(\omega) = |U(\omega)|^2 = \frac{\pi}{\Delta\omega^2} \left\{ \exp\left[-\left(\frac{\omega+\bar{\omega}}{\Delta\omega}\right)^2\right] + \exp\left[-\left(\frac{\omega-\bar{\omega}}{\Delta\omega}\right)^2\right] \right\}^2 \exp(-2\alpha x), \quad (2.236)$$

where we have used equations (2.80) and (2.81), and α is given by equation (2.85). A numerical inversion by the discrete Fourier transform yields the desired time-domain (transient) solution.

2.6.2 Numerical evaluation of the velocities

In this section, we obtain expressions of the energy and group velocities and two different centrovelocities.

The energy of a signal is defined as

$$E = \int_0^\infty |u(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^\infty |U(\omega)|^2 d\omega, \quad (2.237)$$

where $u(t)$ is the Fourier transform of $U(\omega)$, and Parseval's theorem has been used (Bracewell, 1965, p. 112).

We define “location of energy” as the time t_c corresponding to the centroid of the function $|u|^2$ in the time domain (time history) (Bracewell, 1965, p. 139). That is

$$t_c(x) = \frac{\int_0^\infty t |u(x, t)|^2 dt}{\int_0^\infty |u(x, t)|^2 dt}. \quad (2.238)$$

Then, the first centrovelocity, defined here as the mean velocity from 0 to x , is

$$\bar{c}_1(x) = \frac{x}{t_c(x)}. \quad (2.239)$$

Smith's centrovelocity is

$$c_1(x) = \left(\frac{dt_c(x)}{dx} \right)^{-1} \quad (2.240)$$

(Smith, 1970).

The group and energy velocities (2.86) and (2.114) are evaluated at the centroid ω_c of the power spectrum. Since the medium is lossy, frequency ω_c depends on the position x' , where $0 \leq x' \leq x$. We have

$$\omega_c(x') = \frac{\int_0^\infty \omega P(\omega, x') d\omega}{\int_0^\infty P(\omega, x') d\omega} = \frac{\int_0^\infty \omega |F|^2 \exp(-2\alpha x') d\omega}{\int_0^\infty |F|^2 \exp(-2\alpha x') d\omega}, \quad (2.241)$$

where we have used equations (2.235) and (2.236).

The energy and group travel times are then obtained as

$$t_e(x) = \int_0^x \frac{dx'}{v_e[\omega_c(x')]} \quad \text{and} \quad t_g(x) = \int_0^x \frac{dx'}{v_g[\omega_c(x')]}, \quad (2.242)$$

and the respective mean velocities are

$$\bar{v}_e(x) = \frac{x}{t_e(x)} \quad \text{and} \quad \bar{v}_g(x) = \frac{x}{t_g(x)}. \quad (2.243)$$

We define a second centrovelocity as the mean velocity computed from the snapshots of the field, from 0 to time t ,

$$\bar{c}_2(t) = \frac{x_c(t)}{t}, \quad (2.244)$$

where the “location of energy” is

$$x_c(t) = \frac{\int_0^\infty x |u(x, t)|^2 dx}{\int_0^\infty |u(x, t)|^2 dx}, \quad (2.245)$$

i.e., the centroid of the function $|u|^2$ in the space domain (snapshot). Gurwich’s centrovelocity is

$$c_2(t) = \frac{dx_c(t)}{dt} \quad (2.246)$$

(Gurwich, 2001). In this case, it is possible to compute the energy and group velocities if we assume a complex frequency $\Omega = \omega + i\omega_I$ and a real wavenumber, as in Section 2.3.1. The dispersion relation is given by equation (2.128). Generally, this equation has to be solved numerically for Ω to obtain $\omega(\kappa) = \text{Re}(\Omega)$. Then, the energy and group velocities are evaluated at the centroid κ_c of the spatial power spectrum. As before, the centroid wavenumber κ_c depends on the snapshot time t' , where $0 \leq t' \leq t$. We have

$$\kappa_c(t') = \frac{\int_0^\infty \kappa P(\kappa, t') d\kappa}{\int_0^\infty P(\kappa, t') d\kappa}, \quad (2.247)$$

where $P(\kappa, t')$ is the spatial power spectrum obtained by an inverse spatial Fourier transform. The phase, energy and group locations are then obtained as

$$x_p(t) = \int_0^t \frac{dt'}{v_p[\omega(\kappa_c(t'))]}, \quad x_e(t) = \int_0^t \frac{dt'}{v_e[\omega(\kappa_c(t'))]} \quad \text{and} \quad x_g(t) = \int_0^t \frac{dt'}{v_g[\omega(\kappa_c(t'))]}, \quad (2.248)$$

where

$$v_p[\omega(\kappa)] = \frac{\omega(\kappa)}{\kappa} = \text{Re}(v_c), \quad v_e[\omega(\kappa)] = \left[\text{Re} \left(\frac{1}{v_c} \right) \right]^{-1} \quad \text{and} \quad v_g[\omega(\kappa)] = \text{Re} \left[\frac{\partial \Omega(\kappa)}{\partial \kappa} \right]. \quad (2.249)$$

An energy velocity that differs from the phase velocity arises from the energy balance (see Section 2.3.1). The respective mean velocities are

$$\bar{v}_p(t) = \frac{x_p(t)}{t}, \quad \bar{v}_e(t) = \frac{x_e(t)}{t} \quad \text{and} \quad \bar{v}_g(t) = \frac{x_g(t)}{t}. \quad (2.250)$$

In the next section, we consider an example of the first centrovelocity concept (equation (2.239)).

2.6.3 Example

We consider a Zener model whose complex modulus is given by equation (2.170), and we use equations (2.200) and (2.202). We assume $\omega_0 = 1/\tau_0 = 157/\text{s}$ and $M_U = \rho c_U^2$, with $c_U = 2 \text{ km/s}$. (The value of the density is irrelevant for the calculations.)

Figure 2.17 shows the energy and group velocities as a function of frequency (a), the initial spectrum (dashed line) and the spectrum at $x = 50 \text{ m}$ (solid line) (b), and the absolute value of the pulse in a lossless medium (dashed line) and for $Q_0 = 5$ (solid line) (c) (the travel distance is $x = 1 \text{ km}$). The group velocity is greater than the energy

(phase) velocity, mainly at the location of the relaxation peak. The amplitude of the spectrum for $Q_0 = 5$ is much lower than that of the initial spectrum, and the dominant frequency has decreased. From (c), we may roughly estimate the pulse velocity by taking the ratio travel distance (1 km) to arrival time of the maximum amplitude. It gives 2 km/s (1 km/0.5 s) for $Q_0 = \infty$ (dashed-line pulse) and 1.67 km/s (1 km/0.6 s) for $Q_0 = 5$. More precise values are obtained by using the centrovelocity.

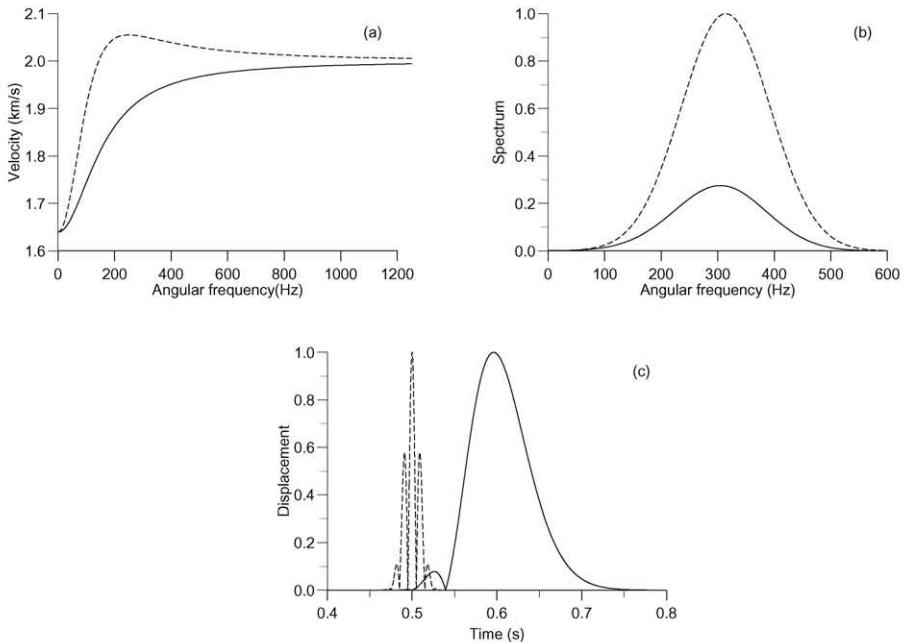


Figure 2.17: (a) Energy (solid line) and group (dashed line) velocities as a function of frequency for $Q_0 = 5$. (b) Initial spectrum (dashed line) and spectrum for $x = 50$ m (solid line). (c) Absolute value of the normalized displacement in a lossless medium (dashed line) and pulse for $Q_0 = 5$ (the travel distance is $x = 1$ km). The relation between the pulse maximum amplitudes is 194. The relaxation mechanism has a peak at $\omega_0 = 157/\text{s}$ ($f_0 = \omega_0/2\pi = 25$ Hz) and the source (initial) dominant frequency is $\bar{\omega} = 628/\text{s}$ ($\bar{f} = \bar{\omega}/2\pi = 50$ Hz).

The comparison between the energy and group velocities (see equation (2.243)) to the centrovelocity \bar{c}_1 (equation (2.239)) is shown in Figure 2.18. In this case $Q_0 = 10$. The relaxation mechanism has a peak at $f_0 = 25$ Hz and (a) and (b) correspond to source (initial) dominant frequencies of 50 Hz and 25 Hz, respectively. As can be seen, the centrovelocity is closer to the group velocity at short travel distances, where the wave packet keeps its shape. At a given distance, the centrovelocity equals the energy velocity and beyond that distance this velocity becomes a better approximation, particularly when the initial source central frequency is close to the peak frequency of the relaxation mechanism (case (b)). The problem is further discussed in Section 4.4.5.

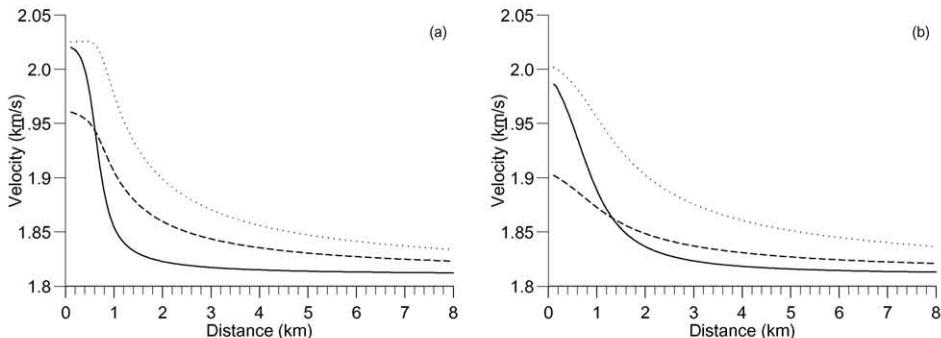


Figure 2.18: Centrovelocity (solid line), and energy (dashed line) and group (dotted line) velocities as a function of travel distance and $Q_0 = 10$. The relaxation mechanism has a peak at $f_0 = 25$ Hz and the source (initial) dominant frequency is $\bar{f} = 50$ Hz (a) and $\bar{f} = 25$ Hz (b).

2.7 Memory variables and equation of motion

It is convenient to recast the equation of motion for a viscoelastic medium in the particle-velocity/stress formulation. This allows the numerical calculation of wave fields without the explicit differentiation of the material properties, and the implementation of boundary conditions, such as free-surface boundary conditions. Moreover, the equation of motion is more efficiently solved in the time domain, since frequency-domain methods are expensive because they involve the solutions of many Helmholtz equations.

In order to avoid the calculation of convolutional integrals, which can be computationally expensive, the time-domain formulation requires the introduction of additional field variables. Applying the Boltzmann operation (2.6) to the stress-strain relation (2.28), we have

$$\sigma = \partial_t \psi * \epsilon = \psi \odot \epsilon = \psi(0^+) (\epsilon + \varphi * \epsilon), \quad (2.251)$$

where φ is the response function, defined as

$$\varphi = \check{\varphi} H, \quad \check{\varphi} = \frac{\partial_t \check{\psi}}{\psi(0^+)}. \quad (2.252)$$

2.7.1 Maxwell model

For the Maxwell model (see equation (2.149)),

$$\check{\psi} = M_U \exp(-t/\tau) \quad (2.253)$$

and

$$\check{\varphi} = -\frac{1}{\tau} \exp(-t/\tau). \quad (2.254)$$

Equation (2.251) yields

$$\sigma = M_U (\epsilon + e), \quad (2.255)$$

where

$$e = \varphi * \epsilon \quad (2.256)$$

is the strain memory variable. (The corresponding stress memory variable can be defined as $M_U e$ – the term memory variable to describe hidden field variables in viscoelasticity being introduced by Carcione, Kosloff and Kosloff (1988b,c,d)). Note that the response function obeys the following first-order equation

$$\partial_t \check{\varphi} = -\frac{1}{\tau} \check{\varphi}. \quad (2.257)$$

If we apply the Boltzmann operation to equation (2.256), we obtain a first-order differential equation in the time variable,

$$\partial_t e = \varphi(0)\epsilon + (\partial_t \check{\varphi} H) * \epsilon = \varphi(0)\epsilon - \frac{1}{\tau} \varphi * \epsilon, \quad (2.258)$$

or,

$$\partial_t e = -\frac{1}{\tau}(\epsilon + e) = -\frac{\sigma}{\tau M_U}. \quad (2.259)$$

The equation of motion (2.78), including a body-force term f_u , can be rewritten as

$$\partial_t v = \frac{1}{\rho} \partial_1 \sigma + f_u, \quad (2.260)$$

where we used $\partial_t u = v$. Differentiating (2.255) with respect to the time variable and using $\epsilon = \partial_1 u$, we obtain

$$\partial_t \sigma = M_U (\partial_1 v + e_1), \quad (2.261)$$

where $e_1 = \partial_t e$ obeys equation (2.259), that is

$$\partial_t e_1 = -\frac{1}{\tau}(\partial_1 v + e_1). \quad (2.262)$$

Equations (2.260), (2.261) and (2.262) can be recast as a first-order matrix differential equation of the form

$$\partial_t \underline{\mathbf{v}} = \mathbf{H} \cdot \underline{\mathbf{v}} + \underline{\mathbf{f}}, \quad (2.263)$$

where

$$\underline{\mathbf{v}} = (v, \sigma, e_1)^\top \quad (2.264)$$

is the unknown field 3×1 array,

$$\underline{\mathbf{f}} = (f_u, 0, 0)^\top \quad (2.265)$$

is the source 3×1 array, and

$$\mathbf{H} = \begin{pmatrix} 0 & \rho^{-1} \partial_1 & 0 \\ M_U \partial_1 & 0 & M_U \\ -\tau^{-1} \partial_1 & 0 & -\tau^{-1} \end{pmatrix}. \quad (2.266)$$

In this case, the memory variable can be avoided if we consider equations (2.259) and (2.260), and the stress-strain relation (2.261):

$$\underline{\mathbf{v}} = (v, \sigma)^\top, \quad (2.267)$$

$$\underline{\mathbf{f}} = (f_u, 0)^\top \quad (2.268)$$

as well as

$$\mathbf{H} = \begin{pmatrix} 0 & \rho^{-1} \partial_1 \\ M_U \partial_1 & -\tau^{-1} \end{pmatrix}. \quad (2.269)$$

2.7.2 Kelvin-Voigt model

In the Kelvin-Voigt model, the strain ϵ plays the role of a memory variable, since the strain-stress relation (2.39) and the creep function (2.163) yield

$$\epsilon = \varphi_\sigma * \sigma, \quad (2.270)$$

(note that $\chi(0^+) = 0$), with

$$\check{\varphi}_\sigma = \frac{1}{\tau M_R} \exp(-t/\tau). \quad (2.271)$$

Then, as with the Maxwell model, there is no need to introduce an additional field variable. To express the equation as a first-order differential equation in time, we recast equation (2.260) as

$$\partial_t \partial_1 v = \partial_1 \rho^{-1} \partial_1 \sigma + \partial_1 f_u = \Delta_\rho \sigma + \partial_1 f_u, \quad \Delta_\rho = \partial_1 \rho^{-1} \partial_1, \quad (2.272)$$

and redefine

$$\epsilon_1 = \partial_t \epsilon. \quad (2.273)$$

Noting that $\partial_1 v = \epsilon_1$, and using the stress-strain relation (2.160), we obtain

$$\partial_t \epsilon_1 = \Delta_\rho (M_R \epsilon + \eta \epsilon_1) + \partial_1 f_u. \quad (2.274)$$

The matrix form (2.263) is obtained for

$$\underline{\mathbf{v}} = (\epsilon_1, \epsilon)^\top, \quad (2.275)$$

$$\underline{\mathbf{f}} = (\partial_1 f_u, 0)^\top \quad (2.276)$$

and

$$\mathbf{H} = \begin{pmatrix} \Delta_\rho \eta & \Delta_\rho M_R \\ 1 & 0 \end{pmatrix}. \quad (2.277)$$

Another approach is the particle-velocity/stress formulation. Using $v = \partial_t u$, the time derivative of the stress-strain relation (2.160) becomes

$$\partial_t \sigma = M_R \partial_1 v + \eta \partial_1 \partial_t v. \quad (2.278)$$

Substituting (2.260) into (2.278) yields

$$\partial_t \sigma = M_R \partial_1 v + \eta \partial_1 \left(\frac{1}{\rho} \partial_1 \sigma + f_u \right). \quad (2.279)$$

This equation and (2.260) can be recast in the matrix form (2.263), where

$$\underline{\mathbf{v}} = (v, \sigma)^\top, \quad (2.280)$$

$$\underline{\mathbf{f}} = (0, \eta \partial_1 f_u)^\top \quad (2.281)$$

and

$$\mathbf{H} = \begin{pmatrix} 0 & \rho^{-1} \partial_1 \\ M_R \partial_1 & \eta \partial_1 \eta^{-1} \partial_1 \end{pmatrix}. \quad (2.282)$$

Carcione, Poletto and Gei (2004) generalize this approach to the 3-D case and develop a numerical algorithm to solve the differential equation for isotropic inhomogeneous media, including free-surface boundary conditions. The modeling simulates 3-D waves by using the Fourier and Chebyshev methods to compute the spatial derivatives along the horizontal and vertical directions, respectively (see Chapter 9). The formulation, based on one Kelvin-Voigt element, models a linear quality factor as a function of frequency.

2.7.3 Zener model

The stress-strain relation (2.251) is based on the relaxation function (2.173) and, after application of the Boltzmann operation (2.6), becomes

$$\sigma = M_U(\epsilon + e), \quad (2.283)$$

where M_U is given by equation (2.171),

$$e = \varphi * \epsilon, \quad (2.284)$$

is the strain memory variable, and

$$\check{\varphi} = \frac{1}{\tau_\epsilon} \left(1 - \frac{\tau_\epsilon}{\tau_\sigma} \right) \exp(-t/\tau_\sigma). \quad (2.285)$$

Equation (2.285) obeys a differential equation of the form (2.257). The memory variable satisfies

$$\partial_t e = \varphi(0)\epsilon - \frac{e}{\tau_\sigma} = -\frac{1}{\tau_\sigma} \left[\left(1 - \frac{\tau_\sigma}{\tau_\epsilon} \right) \epsilon + e \right]. \quad (2.286)$$

First-order differential equations of the form (2.286) were introduced by Day and Minster (1984) to simulate wave propagation in anelastic media. Defining $e_1 = \partial_t e$ and differentiating (2.283) and (2.286) with respect to the time variable, we obtain

$$\partial_t \sigma = M_U(\partial_1 v + e_1) \quad (2.287)$$

and

$$\partial_t e_1 = -\frac{1}{\tau_\sigma} \left[\left(1 - \frac{\tau_\sigma}{\tau_\epsilon} \right) \partial_1 v + e_1 \right]. \quad (2.288)$$

These equations and the equation of motion (2.260) can be written in the matrix form (2.263), with \mathbf{v} and \mathbf{f} given by equations (2.264) and (2.265), and

$$\mathbf{H} = \begin{pmatrix} 0 & \rho^{-1}\partial_1 & 0 \\ M_U\partial_1 & 0 & M_U \\ \varphi(0)\partial_1 & 0 & -\tau_\sigma^{-1} \end{pmatrix}. \quad (2.289)$$

2.7.4 Generalized Zener model

In this case, the stress-strain relation (2.251) (see (2.198)) is expressed in terms of L memory variables e_l ,

$$\sigma = M_U \left(\epsilon + \sum_{l=1}^L e_l \right), \quad (2.290)$$

which, after defining $e_{1l} = \partial_t e_l$ and differentiating with respect to the time variable, becomes

$$\partial_t \sigma = M_U \left(\partial_1 v + \sum_{l=1}^L e_{1l} \right). \quad (2.291)$$

The memory variables satisfy

$$\partial_t e_{1l} = \varphi_l(0)\partial_1 v - \frac{e_{1l}}{\tau_{\sigma l}}, \quad (2.292)$$

with

$$\check{\varphi}_l = \frac{1}{\tau_{\sigma l}} \left(\sum_{l=1}^L \frac{\tau_{el}}{\tau_{\sigma l}} \right)^{-1} \left(1 - \frac{\tau_{el}}{\tau_{\sigma l}} \right) \exp(-t/\tau_{\sigma l}). \quad (2.293)$$

The matrix differential equation (2.263) has

$$\underline{\mathbf{v}} = (v, \sigma, e_{11}, e_{12}, \dots, e_{1L})^\top, \quad (2.294)$$

$$\underline{\mathbf{f}} = (f_u, 0, 0, 0, \dots, 0)^\top \quad (2.295)$$

and

$$\mathbf{H} = \begin{pmatrix} 0 & \rho^{-1}\partial_1 & 0 & 0 & \dots & 0 \\ M_U\partial_1 & 0 & M_U & M_U & \dots & M_U \\ \varphi_1(0)\partial_1 & 0 & -\tau_{\sigma 1}^{-1} & 0 & \dots & 0 \\ \varphi_2(0)\partial_1 & 0 & 0 & -\tau_{\sigma 2}^{-1} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \varphi_L(0)\partial_1 & 0 & 0 & 0 & \dots & -\tau_{\sigma L}^{-1} \end{pmatrix}. \quad (2.296)$$

This formulation for the generalized Zener model is appropriate to simulate wave propagation in inhomogeneous viscoelastic media, with a general dependence of the quality factor as a function of frequency.

Alternatively, we can solve for the dilatation field (the strain ϵ in 1-D space) or the pressure field ($-\sigma$ in 1-D space). These formulations for the viscoacoustic equation of motion are convenient for 3-D problems where memory storage is demanding. We substitute the stress-strain relation (2.290) into equation (2.272) to obtain

$$\partial_t \epsilon_1 = \Delta_\rho \left[M_U \left(\epsilon + \sum_{l=1}^L e_l \right) \right] + \partial_1 f_u, \quad (2.297)$$

where $\epsilon_1 = \partial_t \epsilon$. Then, the unknown field in equation (2.263) is

$$\underline{\mathbf{v}} = (\epsilon, \epsilon_1, e_1, e_2, \dots, e_L)^\top, \quad (2.298)$$

the force term is

$$\underline{\mathbf{f}} = (0, \partial_1 f_u, 0, 0, \dots, 0)^\top \quad (2.299)$$

and

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ \Delta_\rho M_U & 0 & \Delta_\rho M_U & \Delta_\rho M_U & \dots & \Delta_\rho M_U \\ \varphi_1(0) & 0 & -\tau_{\sigma 1}^{-1} & 0 & \dots & 0 \\ \varphi_2(0) & 0 & 0 & -\tau_{\sigma 2}^{-1} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \varphi_L(0) & 0 & 0 & 0 & \dots & -\tau_{\sigma L}^{-1} \end{pmatrix}, \quad (2.300)$$

where $\Delta_\rho = \partial_i \rho^{-1} \partial_i$ in 3-D space (Carcione, Kosloff and Kosloff, 1988d).