Chapter 8

The acoustic-electromagnetic analogy

Mathematical analysis is as extensive as nature itself; it defines all perceptible relations , measures times, spaces, forces, temperatures; this difficult science is formed slowly, but it preserves every principle, which it has once acquired; it grows and strengthens itself incessantly in the midst of the many variations and errors of the human mind. Its chief attribute is clearness; it has no marks to express confined notions. It brings together phenomena the most diverse, and discovers the hidden analogies which unite them.

Joseph Fourier (Fourier, 1822).

Many of the great scientists of the past have studied the theory of wave motion. Throughout this development there has been an interplay between the theory of light waves and the theory of material waves. In 1660 Robert Hooke formulated stress-strain relationships which established the elastic behavior of solid bodies. Hooke believed light to be a vibratory displacement of the medium, through which it propagates at finite speed. Significant experimental and mathematical advances came in the nineteenth century. Thomas Young was one of the first to consider shear as an elastic strain, and defined the elastic modulus that was later named Young's modulus. In 1809 Etienne Louis Malus discovered polarization of light by reflection, which at the time David Brewster correctly described as "a memorable epoch in the history of optics". In 1815 Brewster discovered the law that regulated the polarization of light. Augustus Jean Fresnel showed that if light were a transverse wave, then it would be possible to develop a theory accommodating the polarization of light. George Green (Green, 1838; 1842) made extensive use of the analogy between elastic waves and light waves, and an analysis of his developments illustrates the power of the use of mathematical analogies.

Later, in the second part of the nineteenth century, James Clerk Maxwell and Lord Kelvin used physical and mathematical analogies to study wave phenomena in elastic theory and electromagnetism. In fact, the displacement current introduced by Maxwell into the electromagnetic equations is analogous to the elastic displacements. Maxwell assumed his equations were valid in an absolute system regarded as a medium (called the ether) that filled the whole of space. The ether was in a state of stress and would only transmit transverse waves. With the advent of the theory of relativity, the concept of the ether was abandoned. However the fact that electromagnetic waves are transverse waves is important. This situation is in contrast to a fluid, which can only transmit longitudinal waves. A viscoelastic body transmits both longitudinal waves and transverse waves. It is also possible to recast the viscoelastic equations into a form that closely parallels Maxwell's equations. In many cases this formal analogy becomes a complete mathematical equivalence such that the same equations can be used to solve problems in both disciplines.

In this chapter, it is shown that the 2-D Maxwell's equations describing propagation of the TEM mode in anisotropic media is completely analogous to the SH-wave equation based on the Maxwell anisotropic-viscoelastic solid. This equivalence was probably known to Maxwell, who was aware of the analogy between the process of conduction (static induction through dielectrics) and viscosity (elasticity). Actually, Maxwell's electromagnetic theory of light, including the conduction and displacement currents, was already completed in his paper "On physical lines of force" published in two parts in 1861 and 1862 (Hendry, 1986). On the other hand, the viscoelastic model was proposed in 1867 (Maxwell, 1867, 1890). He seems to have arrived to the viscoelastic rheology from a comparison with Thomson's telegraphy equations (Bland, 1988), which describe the process of conduction and dissipation of electric energy through cables. We use this theory to obtain a complete mathematical analogy for the reflection-transmission problem.

Furthermore, the analogy can be used to get insight into the proper definition of energy. The concept of energy is important in a large number of applications where it is necessary to know how the energy transferred by the electromagnetic field is related to the strength of the field. This context involves the whole electrical, radio, and optical engineering, where the medium can be assumed dielectrically and magnetically linear. Energy-balance equations are important for characterizing the energy stored and the transport properties in a field. However, the definition of stored (free) energy and energy dissipation rate is controversial, both in electromagnetism (Oughstun and Sherman, 1984) and viscoelasticity (Caviglia and Morro, 1992). The problem is particularly intriguing in the time domain, since different definitions may give the same time-average value for harmonic fields. This ambiguity is not present when the constitutive equation can be described in terms of springs and dashpots. That is, when the system can be defined in terms of internal variables and the relaxation function has an exponential form. In Chapter 2 we gave a general expression of the viscoelastic energy densities which is consistent with the mechanical model description. In this chapter, the electric, dielectric and magnetic energies are defined in terms of the viscoelastic expressions by using the analogy. The theory is applied to a simple dielectric-relaxation process – the Debye model – that is mathematically equivalent to the viscoelastic Zener model. The Debye model has been applied to bio-electromagnetism in the analysis of the response of biological tissues (Roberts and Petropolous, 1996), and to geophysics in the simulation of ground-penetrating-radar wave propagation through wet soils (Turner and Siggins, 1994, Carcione, 1996c).

The 3-D Maxwell's equations are generalized to describe realistic wave propagation by using mechanical viscoelastic models. A set of Zener elements describe several magnetic and dielectric-relaxation mechanisms, and a single Kelvin-Voigt element incorporates the out-of-phase behaviour of the electric conductivity (any deviation from Ohm's law). We assume that the medium has orthorhombic symmetry, that the principal systems of the three material tensors coincide and that a different relaxation function is associated with each principal component. A brief derivation of the Kramers-Kronig dispersion relations by using the Cauchy integral formula follows, and the equivalence with the acoustic case is shown. Moreover, the averaging methods use in elasticity (Backus, 1962) can be used in electromagnetism. We derive the constitutive equation for a layered medium, where each single layer is anisotropic, homogeneous and thin compared to the wavelength of the electromagnetic wave. Assuming that the layer interfaces are flat, we obtain the dielectric-permittivity and conductivity matrices of the composite medium. Other mathematical analogies include the high-frequency time-average and CRIM equations, the reciprocity principle, Babinet's principle and Alford rotation. Finally, a formal analogy can be established between the diffusion equation corresponding to the slow compressional wave described by Biot's theory (see Section 7.7.1) and Maxwell's equations at low frequencies. A common analytical solution is obtained for both problems and a numerical method is outlined in Chapter 9.

The use of mathematical analogies is extensively used in many fields of physics (e.g., Tonti, 1976). For instance, the Laplace equation describes different physical processes such as thermal conduction, electric conduction and stationary irrotational flow in hydrodynamics. On the other hand, the static constitutive equations of poroelasticity and thermoelasticity are formally the same if we identify the pore-fluid pressure with the temperature and the fluid compression with entropy (Norris, 1991, 1994).

The analogy can be exploited in several ways. In first place, existing acoustic modeling codes can be easily modified to simulate electromagnetic propagation. Secondly, the set of solutions of the acoustic problem, obtained from the correspondence principle, can be used to test electromagnetic codes. Moreover, the theory of propagation of plane harmonic waves in acoustic media also applies to electromagnetic propagation.

8.1 Maxwell's equations

In 3-D vector notation, Maxwell's equations are

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} + \mathbf{M} \tag{8.1}$$

and

$$\nabla \times \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J}' \tag{8.2}$$

(Born and Wolf, 1964, p. 1), where $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{J}'$ and $\mathbf{M}$ are the electric vector, the magnetic vector, the electric displacement, the magnetic induction, the electric-current density (including an electric-source current) and the magnetic-source current density, respectively. In general, they depend on the Cartesian coordinates (x, y, z) and the time variable t.

Additional constitutive equations are needed to relate $\mathbf{D}$ and $\mathbf{B}$ to the field vectors. For anisotropic lossy media including dielectric relaxation and magnetic loss, $\mathbf{D}$ and $\mathbf{B}$ can be written as,

$$\mathbf{D} = \hat{\boldsymbol{\epsilon}} * \partial_t \mathbf{E} \tag{8.3}$$

and

$$\mathbf{B} = \hat{\boldsymbol{\mu}} * \partial_t \mathbf{H}, \tag{8.4}$$

where $\hat{\boldsymbol{\epsilon}}(\mathbf{x}, t)$ is the dielectric-permittivity tensor and $\hat{\boldsymbol{\mu}}(\mathbf{x}, t)$ is the magnetic-permeability tensor. The electric-current density is given by the generalized Ohm's law

$$\mathbf{J}' = \hat{\boldsymbol{\sigma}} * \partial_t \mathbf{E} + \mathbf{J}, \tag{8.5}$$

where $\hat{\boldsymbol{\sigma}}(\mathbf{x}, t)$ is the conductivity tensor; the convolution accounts for out-of-phase components of the conduction-current density with respect to the electric vector, and $\mathbf{J}$ is the electric-source current density[1]. Substituting the constitutive equations (8.3) and (8.4) and the current-density (equation (8.5)) into (8.1) and (8.2), and using properties of the convolution, gives

$$\nabla \times \mathbf{E} = -\hat{\boldsymbol{\mu}} * \partial_{tt}^2 \mathbf{H} + \mathbf{M} \tag{8.6}$$

and

$$\nabla \times \mathbf{H} = \hat{\boldsymbol{\sigma}} * \partial_t \mathbf{E} + \hat{\boldsymbol{\epsilon}} * \partial_{tt}^2 \mathbf{E} + \mathbf{J}, \tag{8.7}$$

which are a system of six scalar equations in six scalar unknowns.

The time-dependent tensors, which are symmetric and positive definite, describe various electromagnetic relaxation processes of the material, like dielectric relaxation and out-of-phase behavior of the conduction current at high frequencies. The time dependence is not arbitrary; it is assumed for each tensor that its eigenvectors are invariant in time, so that in a coordinate system coincident with these fixed eigenvectors, the time dependence of the tensor is fully specified by three time functions on the main diagonal which serve as the time-dependent eigenvalues of the matrix. These equations also include paramagnetic losses through the time-dependent magnetic-permeability tensor $\hat{\boldsymbol{\mu}}$.

In lossless media, the material tensors are replaced by

$$\begin{array}{lll} \hat{\boldsymbol{\mu}}(\mathbf{x}, t) & \rightarrow & \hat{\boldsymbol{\mu}}(\mathbf{x}) H(t) \\ \hat{\boldsymbol{\sigma}}(\mathbf{x}, t) & \rightarrow & \hat{\boldsymbol{\sigma}}(\mathbf{x}) H(t) \\ \hat{\boldsymbol{\epsilon}}(\mathbf{x}, t) & \rightarrow & \hat{\boldsymbol{\epsilon}}(\mathbf{x}) H(t), \end{array} \tag{8.8}$$

where $H(t)$ is Heaviside's function, and the classical Maxwell's equations for anisotropic media are obtained from equations (8.6) and (8.7):

$$\nabla \times \mathbf{E} = -\hat{\boldsymbol{\mu}} \cdot \partial_t \mathbf{H} + \mathbf{M} \tag{8.9}$$

and

$$\nabla \times \mathbf{H} = \hat{\boldsymbol{\sigma}} \cdot \mathbf{E} + \hat{\boldsymbol{\epsilon}} \cdot \partial_t \mathbf{E} + \mathbf{J}. \tag{8.10}$$

In general, each of the 3×3 symmetric and positive definite tensors $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\epsilon}}$ and $\hat{\boldsymbol{\sigma}}$ have a set of mutually perpendicular eigenvectors. If there is no eigenvector in common for all three tensors, the medium is said to be triclinic. If there is a single eigenvector common to all three tensors, the medium is said to be monoclinic and has a mirror plane of symmetry perpendicular to the common eigenvector.

8.2 The acoustic-electromagnetic analogy

In order to establish the mathematical analogy between electromagnetism and acoustics, we recast the acoustic equations in the particle-velocity/stress formulation. The conservation equation (1.28) and use of (1.44) give

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho \partial_t \mathbf{v}, \tag{8.11}$$

[1]Note the difference between magnetic permeability, dielectric permittivity and conductivity ($\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\epsilon}}$ and $\hat{\boldsymbol{\sigma}}$) and shear modulus, strain and stress (μ, $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$) defined in previous chapters.

and equations (1.26) and (1.44) combine to give the relation between strain and particle velocity

$$\nabla^\top \cdot \mathbf{v} = \partial_t \mathbf{e}. \tag{8.12}$$

Auld (1990a, p. 101) establishes the acoustic-electromagnetic analogy by using a 3-D Kelvin-Voigt model:

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \mathbf{e} + \boldsymbol{\eta} \cdot \partial_t \mathbf{e}, \tag{8.13}$$

where $\mathbf{C}$ and $\boldsymbol{\eta}$ are the elasticity and viscosity matrices, respectively. (Compare this relation to the 1-D Kelvin-Voigt stress-strain relation in equation (2.159)). Taking the first-order time derivative of (8.13), multiplying the result by $\mathbf{C}^{-1}$, and using equation (8.12), we get

$$\nabla^\top \cdot \mathbf{v} + \mathbf{C}^{-1} \cdot \boldsymbol{\eta} \cdot \nabla^\top \cdot \partial_t \mathbf{v} = \mathbf{C}^{-1} \cdot \partial_t \boldsymbol{\sigma}. \tag{8.14}$$

Auld establishes a formal analogy between (8.11) and (8.14) with Maxwell's equations (8.9) and (8.10), where $\boldsymbol{\sigma}$ corresponds to $\mathbf{E}$ and $\mathbf{v}$ corresponds to $\mathbf{H}$.

A better correspondence can be obtained by introducing, instead of (8.13), a 3-D Maxwell constitutive equation:

$$\partial_t \mathbf{e} = \mathbf{C}^{-1} \cdot \partial_t \boldsymbol{\sigma} + \boldsymbol{\eta}^{-1} \cdot \boldsymbol{\sigma}. \tag{8.15}$$

(Compare this relation to the 1-D Maxwell stress-strain relation (2.145).) Eliminating the strain, by using equation (8.12), gives an equation analogous to (8.10):

$$\nabla^\top \cdot \mathbf{v} = \boldsymbol{\eta}^{-1} \cdot \boldsymbol{\sigma} + \mathbf{C}^{-1} \cdot \partial_t \boldsymbol{\sigma}. \tag{8.16}$$

Defining the compliance matrix

$$\mathbf{S} = \mathbf{C}^{-1} \tag{8.17}$$

and the fluidity matrix

$$\boldsymbol{\tau} = \boldsymbol{\eta}^{-1}, \tag{8.18}$$

equation (8.16) becomes

$$\nabla^\top \cdot \mathbf{v} = \boldsymbol{\tau} \cdot \boldsymbol{\sigma} + \mathbf{S} \cdot \partial_t \boldsymbol{\sigma}. \tag{8.19}$$

In general, the analogy does not mean that the acoustic and electromagnetic equations represent the same mathematical problem. In fact, $\boldsymbol{\sigma}$ is a 6-D vector and $\mathbf{E}$ is a 3-D vector. Moreover, acoustics involves 6×6 matrices (for material properties) and electromagnetism 3×3 matrices. The complete equivalence can be established in the 2-D case by using the Maxwell model, as can be seen in the following.

A realistic medium is described by symmetric dielectric-permittivity and conductivity tensors. Assume an isotropic magnetic-permeability tensor

$$\hat{\boldsymbol{\mu}} = \hat{\mu} \mathbf{I}_3 \tag{8.20}$$

and

$$\hat{\boldsymbol{\epsilon}} = \begin{pmatrix} \hat{\epsilon}_{11} & 0 & \hat{\epsilon}_{13} \\ 0 & \hat{\epsilon}_{22} & 0 \\ \hat{\epsilon}_{13} & 0 & \hat{\epsilon}_{33} \end{pmatrix} \tag{8.21}$$

and

$$\hat{\boldsymbol{\sigma}} = \begin{pmatrix} \hat{\sigma}_{11} & 0 & \hat{\sigma}_{13} \\ 0 & \hat{\sigma}_{22} & 0 \\ \hat{\sigma}_{13} & 0 & \hat{\sigma}_{33} \end{pmatrix} \tag{8.22}$$

where $\mathbf{I}_3$ is the 3×3 identity matrix. Tensors (8.21) and (8.22) correspond to a monoclinic medium with the y-axis perpendicular to the plane of symmetry. There always exists a coordinate transformation that diagonalizes these symmetric matrices. This transformation is called the principal system of the medium, and gives the three principal components of these tensors. In cubic and isotropic media, the principal components are all equal. In tetragonal and hexagonal materials, two of the three parameters are equal. In orthorhombic, monoclinic, and triclinic media, all three components are unequal.

Now, let us assume that the propagation is in the (x, z)-plane, and that the material properties are invariant in the y-direction. Then, E_1, E_3 and H_2 are decoupled from E_2, H_1 and H_3. In the absence of electric-source currents, the first three fields obey the TM (transverse-magnetic) differential equations:

$$\partial_1 E_3 - \partial_3 E_1 = \hat{\mu}\partial_t H_2 + M_2, \tag{8.23}$$

$$-\partial_3 H_2 = \hat{\sigma}_{11} E_1 + \hat{\sigma}_{13} E_3 + \hat{\epsilon}_{11}\partial_t E_1 + \hat{\epsilon}_{13}\partial_t E_3, \tag{8.24}$$

$$\partial_1 H_2 = \hat{\sigma}_{13} E_1 + \hat{\sigma}_{33} E_3 + \hat{\epsilon}_{13}\partial_t E_1 + \hat{\epsilon}_{33}\partial_t E_3, \tag{8.25}$$

where we have used equations (8.7) and (8.10). On the other hand, in acoustics, uniform properties in the y-direction imply that one of the shear waves has its own (decoupled) differential equation, known in the literature as the SH-wave equation (see Section (1.2.1)). This is strictly true in the plane of mirror symmetry of a monoclinic medium. Propagation in this plane implies pure cross-plane strain motion, and it is the most general situation for which pure shear waves exist at all propagation angles. Pure shear-wave propagation in hexagonal media is a degenerate case. A set of parallel fractures embedded in a transversely isotropic formation can be represented by a monoclinic medium. When the plane of mirror symmetry of this medium is vertical, the pure cross-plane strain waves are SH waves. Moreover, monoclinic media include many other cases of higher symmetry. Weak tetragonal media, strong trigonal media and orthorhombic media are subsets of the set of monoclinic media.

In a monoclinic medium, the elasticity and viscosity matrices and their inverses have the form (1.37). It is assumed that any kind of symmetry possessed by the attenuation follows the symmetry of the crystallographic form of the material. This statement, which has been used in Chapter 4, can be supported by an empirical law known as Neumann's principle (Neumann, 1885).

The SH-wave differential equations equivalent to equations (1.46), corresponding to the Maxwell viscoelastic model represented by equation (8.15), are

$$\partial_1 \sigma_{12} + \partial_3 \sigma_{23} = \rho \partial_t v_2 - f_2, \tag{8.26}$$

$$-\partial_3 v_2 = -\tau_{44}\sigma_{23} - \tau_{46}\sigma_{12} - s_{44}\partial_t \sigma_{23} - s_{46}\partial_t \sigma_{12}, \tag{8.27}$$

$$\partial_1 v_2 = \tau_{46}\sigma_{23} + \tau_{66}\sigma_{12} + s_{46}\partial_t \sigma_{23} + s_{66}\partial_t \sigma_{12}, \tag{8.28}$$

where

$$\tau_{44} = \eta_{66}/\bar{\eta}, \quad \tau_{66} = \eta_{44}/\bar{\eta}, \quad \tau_{46} = -\eta_{46}/\bar{\eta}, \quad \bar{\eta} = \eta_{44}\eta_{66} - \eta_{46}^2, \tag{8.29}$$

and

$$s_{44} = c_{66}/c, \quad s_{66} = c_{44}/c, \quad s_{46} = -c_{46}/c, \quad c = c_{44}c_{66} - c_{46}^2, \tag{8.30}$$

where the stiffnesses c_{IJ} and the viscosities η_{IJ}, $(I, J = 4, 6)$ are the components of matrices $\mathbf{C}$ and $\boldsymbol{\eta}$, respectively.

Equations (8.23)-(8.25) are converted into equations (8.26)-(8.28) and vice versa, under the following substitutions:

$$\underline{\mathbf{v}} \equiv \begin{pmatrix} v_2 \\ \sigma_{23} \\ \sigma_{12} \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} H_2 \\ -E_1 \\ E_3 \end{pmatrix} \tag{8.31}$$

$$f_2 \quad \Leftrightarrow \quad -M_2 \tag{8.32}$$

$$\mathbf{S} \equiv \begin{pmatrix} s_{44} & s_{46} \\ s_{46} & s_{66} \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} \hat{\epsilon}_{11} & -\hat{\epsilon}_{13} \\ -\hat{\epsilon}_{13} & \hat{\epsilon}_{33} \end{pmatrix} \equiv \hat{\boldsymbol{\epsilon}}' \tag{8.33}$$

$$\boldsymbol{\tau} \equiv \begin{pmatrix} \tau_{44} & \tau_{46} \\ \tau_{46} & \tau_{66} \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} \hat{\sigma}_{11} & -\hat{\sigma}_{13} \\ -\hat{\sigma}_{13} & \hat{\sigma}_{33} \end{pmatrix} \equiv \hat{\boldsymbol{\sigma}}' \tag{8.34}$$

$$\rho \quad \Leftrightarrow \quad \hat{\mu}, \tag{8.35}$$

where $\mathbf{S}$ and $\boldsymbol{\tau}$ are redefined here as 2×2 matrices for simplicity. Introducing the 2×2 stiffness and viscosity matrices

$$\mathbf{C} = \begin{pmatrix} c_{44} & c_{46} \\ c_{46} & c_{66} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\eta} = \begin{pmatrix} \eta_{44} & \eta_{46} \\ \eta_{46} & \eta_{66} \end{pmatrix}, \tag{8.36}$$

we obtain the 2-D identities $\mathbf{S} = \mathbf{C}^{-1}$ and $\boldsymbol{\tau} = \boldsymbol{\eta}^{-1}$, which are similar to the 3-D equations (8.17) and (8.18), respectively. Then, the SH-wave equation for anisotropic media, based on a Maxwell stress-strain relation, is mathematically equivalent to the TM equations whose "forcing term" is a magnetic current.

The mathematical analogy also holds for the TE equations under certain conditions. If we consider the dielectric permittivity an scalar quantity, the conductivity tensor equal to zero, and the magnetic permeability a tensor, we obtain the following TE differential equations:

$$-\left(\partial_1 H_3 - \partial_3 H_1\right) = \hat{\epsilon}\partial_t E_2 + J_2, \tag{8.37}$$

$$\partial_3 E_2 = \hat{\mu}_{11}\partial_t H_1 + \hat{\mu}_{13}\partial_t H_3, \tag{8.38}$$

$$-\partial_1 E_2 = \hat{\mu}_{13}\partial_t H_1 + \hat{\mu}_{33}\partial_t H_3. \tag{8.39}$$

Then, the TM equations (8.23)-(8.25) and the preceding TE equations are equivalent for the following correspondence: $H_2 \Leftrightarrow -E_2$, $E_1 \Leftrightarrow H_1$, $E_3 \Leftrightarrow H_3$, $M_2 \Leftrightarrow J_2$, $\hat{\mu} \Leftrightarrow \hat{\epsilon}$, $\hat{\epsilon}_{11} \Leftrightarrow \hat{\mu}_{11}$, $\hat{\epsilon}_{13} \Leftrightarrow \hat{\mu}_{13}$ and $\hat{\epsilon}_{33} \Leftrightarrow \hat{\mu}_{33}$. In the frequency domain, the zero conductivity restriction can be relaxed and the correspondence for the properties becomes $\hat{\mu} \Leftrightarrow \hat{\sigma} + \mathrm{i}\omega\hat{\epsilon}$, $\hat{\sigma}_{11} + \mathrm{i}\omega\hat{\epsilon}_{11} \Leftrightarrow \hat{\mu}_{11}$, $\hat{\sigma}_{13} + \mathrm{i}\omega\hat{\epsilon}_{13} \Leftrightarrow \hat{\mu}_{13}$ and $\hat{\sigma}_{33} + \mathrm{i}\omega\hat{\epsilon}_{33} \Leftrightarrow \hat{\mu}_{33}$, where ω is the angular frequency.

To get a more intuitive idea of the analogy, and to introduce the concept of quality factor, we develop the following considerations, which lead to Figures 8.1 and 8.2. For instance, equation (8.28) with $c_{46} = \eta_{46} = 0$ can be constructed from the model displayed in Figure 8.1, where γ_1 and γ_2 are the strains on the dashpot and on the spring, respectively. In fact,

$$\sigma_{12} = \eta_{44}\partial_t\gamma_1 \quad \text{and} \quad \sigma_{12} = c_{44}\gamma_2,$$

and

$$\partial_t(\gamma_1 + \gamma_2) = \partial_1 v_2,$$

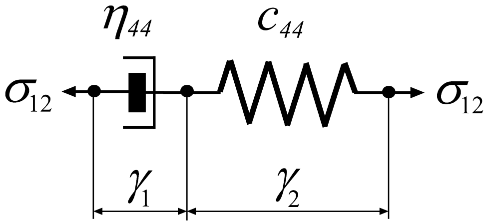

Figure 8.1: Maxwell viscoelastic model corresponding to the xy-component of the stress-strain constitutive equation, with $c_{46} = c_{66} = 0$. The strains acting on the dashpot and spring are γ_1 and γ_2, respectively.

imply (8.28); indeed, if $c_{46} = \eta_{46} = 0$, then $s_{44} = 1/c_{44}$ and $\tau_{44} = 1/\eta_{44}$.

Obtaining a pictorial representation of the electromagnetic field equations is not so easy. However, if, instead of the distributed-parameter system (8.24) and (8.25), we consider the corresponding lumped-parameter system (electric circuit), then such an interpretation becomes straightforward. Indeed, if we consider, for example, equation (8.24) and assume, for simplicity, that $\hat{\sigma}_{13} = \hat{\epsilon}_{13} = 0$, then its right-hand side becomes

$$\hat{\sigma}_{11} E_1 + \hat{\epsilon}_{11} \partial_t E_1$$

or, in terms of circuit elements,

$$\frac{1}{R} V + C \frac{dV}{dt} \equiv I_1 + I_2 \equiv I,$$

which corresponds to a parallel connection of a capacitor and a resistor as shown in Figure 8.2, where R and C are the resistance and the capacitor, respectively, V is the voltage (i.e., the integral of the electric field) and I_1 and I_2 are the electric currents (V/R corresponds to $\hat{\sigma} E$).

An important parameter of the circuit represented in Figure 8.2 is the loss tangent of the capacitor. The circuit can be considered as a real capacitor whose losses are modeled by the resistor R. Under the action of a harmonic voltage of frequency ω, the total current I is not in quadrature with the voltage, but makes an angle $\pi/2 - \delta$ with it (I_1 is in phase with V, while I_2 is in quadrature). As a consequence, the loss tangent is given by

$$\tan \delta = \frac{I_1}{I_2} = \frac{I \cos(\pi/2 - \delta)}{I \sin(\pi/2 - \delta)}. \tag{8.40}$$

Multiplying and dividing (8.40) by V gives the relation between the dissipated power in the resistor and the reactive power in the capacitor

$$\tan \delta = \frac{VI \cos(\pi/2 - \delta)}{VI \sin(\pi/2 - \delta)} = \frac{V^2/R}{\omega C V^2} = \frac{1}{\omega C R}. \tag{8.41}$$

The quality factor of the circuit is the inverse of the loss tangent. In terms of dielectric permittivity and conductivity it is given by

$$Q = \frac{\omega \hat{\epsilon}}{\hat{\sigma}}. \tag{8.42}$$

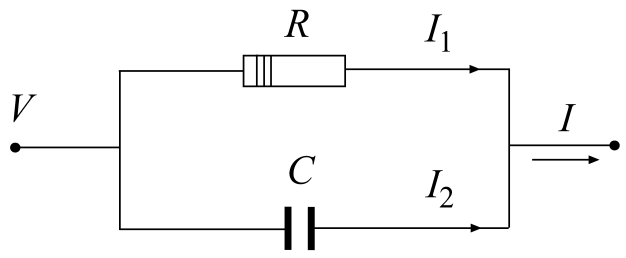

Figure 8.2: Electric-circuit equivalent to the viscoelastic model shown in Figure 8.1, where R and C are the resistance and capacitor, V is the voltage, and I_1 and I_2 are the electric currents. The analogy implies that the energy dissipated in the resistor is equivalent to the energy loss in the dashpot, and the energy stored in the capacitor is equivalent to the potential energy stored in the spring. The magnetic energy is equivalent to the elastic kinetic energy.

The kinetic- and strain-energy densities are associated with the magnetic- and electric-energy densities. In terms of circuit elements, the kinetic, strain and dissipated energies represent the energies stored in inductances, capacitors and the dissipative ohmic losses, respectively. A similar analogy, used by Maxwell, can be established between particle mechanics and circuits (Hammond, 1981).

8.2.1 Kinematics and energy considerations

The kinematic quantities describing wave motion are the slowness, and the phase-velocity and attenuation vectors. The analysis is carried out for the acoustic case, and the electromagnetic case is obtained by applying the equivalence (8.31)-(8.35). For a harmonic plane wave of angular frequency ω, equation (8.11) – in absence of body forces – becomes

$$\nabla \cdot \boldsymbol{\sigma} - \mathrm{i}\omega\rho\mathbf{v} = 0. \tag{8.43}$$

On the other hand, the generalized Maxwell stress-strain relation (8.15) takes the form

$$\boldsymbol{\sigma} = \mathbf{P} \cdot \mathbf{e}, \tag{8.44}$$

where $\mathbf{P}$ is the complex stiffness matrix given by

$$\mathbf{P} = \left(\mathbf{S} - \frac{\mathrm{i}}{\omega}\boldsymbol{\tau}\right)^{-1}. \tag{8.45}$$

All the matrices in this equation have dimension six. However, since the SH mode is pure, a similar equation can be obtained for matrices of dimension three. In this case, the stress and strain simplify to

$$\boldsymbol{\sigma} = (\sigma_{32},\ \sigma_{12}) \quad \text{and} \quad \mathbf{e} = (\partial_3 u_2,\ \partial_1 u_2), \tag{8.46}$$

respectively, where u_2 is the displacement field.

The displacement associated to a homogeneous viscoelastic SH plane wave has the form (4.107):

$$\mathbf{u} = u_2 \hat{\mathbf{e}}_2, \quad u_2 = U_0 \exp[\mathrm{i}(\omega t - \mathbf{k} \cdot \mathbf{x})], \tag{8.47}$$

where $\mathbf{x} = (x, z)$ is the position vector and

$$\mathbf{k} = (\kappa - \mathrm{i}\alpha)\hat{\boldsymbol{\kappa}} = k\hat{\boldsymbol{\kappa}} \tag{8.48}$$

is the complex wavevector, with $\hat{\boldsymbol{\kappa}} = (l_1, l_3)^\top$, defining the propagation direction through the direction cosines l_1 and l_3. Replacing the stress-strain equation (8.44) into equation (8.43) yields the dispersion relation

$$p_{66} l_1^2 + 2p_{46} l_1 l_3 + p_{44} l_3^2 - \rho \left(\frac{\omega}{k}\right)^2 = 0, \tag{8.49}$$

which is equivalent to equation (6.5). The relation (8.49) defines the complex velocity (see equation (4.28)),

$$v_c = \frac{\omega}{k} = \sqrt{\frac{p_{66} l_1^2 + 2p_{46} l_1 l_3 + p_{44} l_3^2}{\rho}}. \tag{8.50}$$

The phase-velocity, slowness and attenuation vectors can be expressed in terms of the complex velocity and are given by equations (4.29), (4.33) and (4.34), respectively. The energy velocity is obtained by the same procedure used in Section 4.4.1. We obtain

$$\mathbf{v}_e = \frac{v_p}{\mathrm{Re}(v_c)} \left\{ \mathrm{Re} \left(\frac{1}{\rho v_c} [(p_{66} l_1 + p_{46} l_3)\hat{\mathbf{e}}_1 + (p_{44} l_3 + p_{46} l_1)\hat{\mathbf{e}}_3] \right) \right\}, \tag{8.51}$$

which generalizes equation (4.115). The quality factor is given by equation (4.92).

From equation (8.45), in virtue of the acoustic-electromagnetic equivalence (8.31)-(8.35), it follows that $\mathbf{P}$ corresponds to the inverse of the complex dielectric-permittivity matrix $\bar{\boldsymbol{\epsilon}}$, namely:

$$\mathbf{P}^{-1} \quad \Leftrightarrow \quad \bar{\boldsymbol{\epsilon}} \equiv \hat{\boldsymbol{\epsilon}}' - \frac{\mathrm{i}}{\omega} \hat{\boldsymbol{\sigma}}'. \tag{8.52}$$

Then, the electromagnetic phase velocity, slowness, attenuation, energy velocity and quality factor can be obtained from equations (4.29), (4.33), (4.34), (8.51) and (4.92), respectively, by applying the analogy.

In orthorhombic media, the 46-components vanish; therefore the complex stiffness matrix is diagonal, with components

$$(c_{II}^{-1} - \mathrm{i}\omega^{-1}\eta_{II}^{-1})^{-1} \tag{8.53}$$

in the acoustic case, where $I = 4$ or 6, and

$$(\hat{\epsilon}_{ii} - \mathrm{i}\omega^{-1}\hat{\sigma}_{ii})^{-1} \tag{8.54}$$

in the electromagnetic case, where $i = 1$ or 3. In isotropic media, where 44-components equal the 66-components, the complex velocity becomes

$$v_c = \left[(\mu^{-1} - \mathrm{i}\omega^{-1}\eta^{-1})\rho\right]^{-1/2} \tag{8.55}$$

in the acoustic case, and

$$v_c = \left[(\hat{\epsilon} - \mathrm{i}\omega^{-1}\hat{\sigma})\hat{\mu}\right]^{-1/2}, \tag{8.56}$$

in the electromagnetic case, where μ is the shear modulus.

In the isotropic case, the acoustic and electromagnetic quality factors are

$$Q = \frac{\omega\eta}{\mu} \tag{8.57}$$

and equation (8.42), respectively. If $\eta \to 0$ and $\hat{\sigma} \to \infty$, then the behaviour is diffusive; while conditions $\eta \to \infty$ and $\hat{\sigma} \to 0$ correspond to the lossless limit. Note that η/μ and $\hat{\epsilon}/\hat{\sigma}$ are the relaxation times of the respective wave processes.

The analogy allows the use of the transient analytical solution obtained in Section 4.6 for the electromagnetic case (Ursin, 1983). The most powerful application of the analogy is the use of the same computer code to solve acoustic and electromagnetic propagation problems in general inhomogeneous media. The finite-difference program shown in Section 9.9.2 can easily be adapted to simulate electromagnetic wave propagation based on the Debye model, as we shall see in Section 8.3.2. Examples of simulations using the acoustic-electromagnetic analogy can be found in Carcione and Cavallini (1995b).

8.3 A viscoelastic form of the electromagnetic energy

The electromagnetic Umov-Poynting theorem can be re-interpreted in the light of the theory of viscoelasticity in order to define the stored and dissipated energy densities in the time domain. A simple dielectric-relaxation model equivalent to a viscoelastic mechanical model illustrates the analogy, that identifies electric field with stress, electric displacement with strain, dielectric permittivity with reciprocal bulk modulus, and resistance with viscosity.

For time-harmonic fields with time dependence $\exp(\mathrm{i}\omega t)$, equations (8.1) and (8.2) read

$$\nabla \times \mathbf{E} = -\mathrm{i}\omega\mathbf{B}, \tag{8.58}$$

$$\nabla \times \mathbf{H} = \mathrm{i}\omega\mathbf{D} + \mathbf{J}', \tag{8.59}$$

respectively, where $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{B}$ are the corresponding time-harmonic fields and we have neglected the magnetic source. For convenience, the field quantities, source and medium properties are denoted by the same symbols, in both, the time and the frequency domains.

For harmonic fields, the constitutive equations (8.3), (8.4) and (8.5) read

$$\mathbf{D} = \mathcal{F}\left[\partial_t \hat{\boldsymbol{\epsilon}}\right] \cdot \mathbf{E} \equiv \hat{\boldsymbol{\epsilon}} \cdot \mathbf{E}, \tag{8.60}$$

$$\mathbf{B} = \mathcal{F}\left[\partial_t \hat{\boldsymbol{\mu}}\right] \cdot \mathbf{H} \equiv \hat{\boldsymbol{\mu}} \cdot \mathbf{H}, \tag{8.61}$$

and

$$\mathbf{J}' = \mathcal{F}\left[\partial_t \hat{\boldsymbol{\sigma}}\right] \cdot \mathbf{E} + \mathbf{J} \equiv \hat{\boldsymbol{\sigma}} \cdot \mathbf{E} + \mathbf{J}, \tag{8.62}$$

where $\mathcal{F}[\ \cdot\ ]$ is the Fourier-transform operator.

8.3.1 Umov-Poynting's theorem for harmonic fields

The scalar product of the complex conjugate of equation (8.59) with $\mathbf{E}$, use of div $(\mathbf{E}\times\mathbf{H}^*)$ $= (\nabla\times\mathbf{E})\cdot\mathbf{H}^* - \mathbf{E}\cdot(\nabla\times\mathbf{H}^*)$, and substitution of equation (8.58) gives Umov-Poynting's theorem for harmonic fields

$$-\text{div }\mathbf{p} = \frac{1}{2}\mathbf{J}'^* \cdot \mathbf{E} - 2\mathrm{i}\omega\left(\frac{1}{4}\mathbf{E}\cdot\mathbf{D}^* - \frac{1}{4}\mathbf{B}\cdot\mathbf{H}^*\right), \tag{8.63}$$

where

$$\mathbf{p} = \frac{1}{2}\mathbf{E}\times\mathbf{H}^* \tag{8.64}$$

is the complex Umov-Poynting vector.

Without loss of generality regarding the energy problem, we consider an isotropic medium, for which, $\hat{\boldsymbol{\epsilon}} = \hat{\epsilon}\ \mathbf{I}_3$, $\hat{\boldsymbol{\mu}} = \hat{\mu}\ \mathbf{I}_3$ and $\hat{\boldsymbol{\sigma}} = \hat{\sigma}\ \mathbf{I}_3$. Then, substitution of the constitutive equations (8.60), (8.61) and (8.62) into equation (8.63) yields

$$\text{div }\mathbf{p} = 2\mathrm{i}\omega\left(\frac{1}{4}\bar{\epsilon}^*|\mathbf{E}|^2 - \frac{1}{4}\hat{\mu}|\mathbf{H}|^2\right), \tag{8.65}$$

where

$$\bar{\epsilon} \equiv \hat{\epsilon} - \frac{\mathrm{i}}{\omega}\hat{\sigma}, \tag{8.66}$$

and we have assumed $\mathbf{J} = 0$. Each term has a precise physical meaning on a time-average basis:

$$\frac{1}{4}\mathrm{Re}(\bar{\epsilon}^*)|\mathbf{E}|^2 = \frac{1}{4}\mathrm{Re}(\bar{\epsilon})|\mathbf{E}|^2 \equiv \langle E_e\rangle \tag{8.67}$$

is the time-averaged electric-energy density,

$$\frac{\omega}{2}\mathrm{Im}(\bar{\epsilon}^*)|\mathbf{E}|^2 = -\frac{\omega}{2}\mathrm{Im}(\bar{\epsilon})|\mathbf{E}|^2 \equiv \langle \dot{D}_e\rangle \tag{8.68}$$

is the time-averaged rate of dissipated electric-energy density,

$$\frac{1}{4}\mathrm{Re}(\hat{\mu})|\mathbf{H}|^2 \equiv \langle E_m\rangle \tag{8.69}$$

is the time-averaged magnetic-energy density, and

$$-\frac{\omega}{2}\mathrm{Im}(\hat{\mu})|\mathbf{H}|^2 \equiv \langle \dot{D}_m\rangle \tag{8.70}$$

is the time-averaged rate of dissipated magnetic-energy density. Substituting the preceding expressions into equation (8.65), yields the energy-balance equation

$$\text{div }\mathbf{p} - 2\mathrm{i}\omega(\langle E_e\rangle - \langle E_m\rangle) + \langle \dot{D}_e\rangle + \langle \dot{D}_m\rangle = 0. \tag{8.71}$$

This equation is equivalent to (4.57) for viscoelastic media, and, particularly, to (7.490) for poro-viscoelastic media, since the magnetic-energy loss is equivalent to the kinetic-energy loss of Biot's theory. The minus sign in equation (8.70) and the condition that $\omega\langle D_m\rangle \equiv \langle \dot{D}_m\rangle > 0$, where $\langle D_m\rangle$ is the time-averaged dissipated-energy density, implies $\mathrm{Im}(\hat{\mu}) < 0$.

Using (8.66), equation (8.71) can be rewritten in terms of the dielectric and conductive energies as

$$\text{div } \mathbf{p} - 2\mathrm{i}\omega(\langle E_\epsilon + E_\sigma\rangle - \langle E_m\rangle) + \langle \dot{D}_\epsilon\rangle + \langle \dot{D}_\sigma\rangle + \langle \dot{D}_m\rangle = 0, \tag{8.72}$$

where

$$\frac{1}{4}\mathrm{Re}(\hat{\epsilon}^*)|\mathbf{E}|^2 = \frac{1}{4}\mathrm{Re}(\hat{\epsilon})|\mathbf{E}|^2 \equiv \langle E_\epsilon\rangle \tag{8.73}$$

is the time-averaged dielectric-energy density,

$$\frac{\omega}{2}\mathrm{Im}(\hat{\epsilon}^*)|\mathbf{E}|^2 = -\frac{\omega}{2}\mathrm{Im}(\hat{\epsilon})|\mathbf{E}|^2 \equiv \langle \dot{D}_\epsilon\rangle \tag{8.74}$$

is the time-averaged rate of dissipated dielectric-energy density,

$$\frac{1}{4\omega}\mathrm{Im}(\hat{\sigma})|\mathbf{E}|^2 \equiv \langle E_\sigma\rangle \tag{8.75}$$

is the time-averaged conductive-energy density, and

$$\frac{1}{2}\mathrm{Re}(\hat{\sigma})|\mathbf{E}|^2 \equiv \langle \dot{D}_\sigma\rangle \tag{8.76}$$

is the time-averaged rate of dissipated conductive-energy density, with

$$\langle E_e\rangle = \langle E_\epsilon\rangle + \langle E_\sigma\rangle \quad \text{and} \quad \langle \dot{D}_e\rangle = \langle \dot{D}_\epsilon\rangle + \langle \dot{D}_\sigma\rangle. \tag{8.77}$$

The Umov-Poynting theorem provides a consistent formulation of energy flow, but this does not preclude the existence of alternative formulations. For instance, Jeffreys (1993) gives an alternative energy balance, implying a new interpretation of the Umov-Poynting vector (see also the discussion in Robinson (1994) and Jeffreys (1994)).

8.3.2 Umov-Poynting's theorem for transient fields

As we have seen in the previous section, time-averaged energies for harmonic fields are precisely defined. The definition of stored and dissipated energies is particularly controversial in the time domain (Oughstun and Sherman, 1984), since different definitions may give the same time-averaged value for harmonic fields (Caviglia and Morro, 1992). We present in this section a definition, based on viscoelasticity theory, where energy can be separated between stored and dissipated in the time domain. The energy expressions are consistent with the mechanical-model description of constitutive equations.

Let us consider an arbitrary time dependence and the difference between the scalar product of equation (8.1) with $\mathbf{H}$ and (8.2) with $\mathbf{E}$. We obtain the Umov-Poynting theorem for transient fields:

$$-\text{div } \mathbf{p} = \mathbf{J}' \cdot \mathbf{E} + \mathbf{E} \cdot \partial_t \mathbf{D} + \mathbf{H} \cdot \partial_t \mathbf{B}. \tag{8.78}$$

Since dielectric energy is analogous to strain energy, let us consider a stored dielectric-(free-) energy density of the form (2.7),

$$E_\epsilon(t) = \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} K(t-\tau_1, t-\tau_2)\partial_{\tau_1}\mathbf{D}(\tau_1) \cdot \partial_{\tau_2}\mathbf{D}(\tau_2) d\tau_1 d\tau_2. \tag{8.79}$$

Note that the electric displacement $\mathbf{D}$ is equivalent to the strain field, since the electric field is equivalent to the stress field and the dielectric permittivity is equivalent to the compliance (see equations (8.31)-(8.35)). The underlying assumptions are that the dielectric properties of the medium do not vary with time (non-aging material), and, as in the lossless case, the energy density is quadratic in the electric field. Moreover, the expression includes a dependence on the history of the electric field.

Differentiating E_ϵ yields

$$\partial_t E_\epsilon = \partial_t \mathbf{D} \cdot \int_{-\infty}^{t} K(t-\tau_2, 0)\partial_{\tau_2}\mathbf{D}(\tau_2) d\tau_2$$

$$+\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} \partial_t K(t-\tau_1, t-\tau_2)\partial_{\tau_1}\mathbf{D}(\tau_1)\cdot\partial_{\tau_2}\mathbf{D}(\tau_2) d\tau_1 d\tau_2. \tag{8.80}$$

The constitutive equation (8.3) for isotropic media can be rewritten as

$$\mathbf{E} = \beta * \partial_t \mathbf{D}, \tag{8.81}$$

where $\beta(t)$ is the dielectric-impermeability function, satisfying

$$\partial_t \hat{\epsilon} * \partial_t \beta = \delta(t), \qquad \hat{\epsilon}^\infty \beta_\infty = \hat{\epsilon}^0 \beta_0 = 1, \qquad \hat{\epsilon}(\omega)\beta(\omega) = 1, \tag{8.82}$$

with the subindices ∞ and 0 corresponding to the limits $t \to 0$ and $t \to \infty$, respectively. If $\beta(t)$ has the form

$$\beta(t) = K(t, 0)H(t), \tag{8.83}$$

where $H(t)$ is Heaviside's function, then,

$$\int_{-\infty}^{t} K(t-\tau_2, 0)\partial_{\tau_2}\mathbf{D}(\tau_2) d\tau_2 = \mathbf{E}(t), \tag{8.84}$$

and (8.80) becomes

$$\mathbf{E} \cdot \partial_t \mathbf{D} = \partial_t E_\epsilon + \dot{D}_\epsilon, \tag{8.85}$$

where

$$\dot{D}_\epsilon(t) = -\frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} \partial_t K(t-\tau_1, t-\tau_2)\partial_{\tau_1}\mathbf{D}(\tau_1)\cdot\partial_{\tau_2}\mathbf{D}(\tau_2) d\tau_1 d\tau_2 \tag{8.86}$$

is the rate of dissipated dielectric-energy density. Note that the relation (8.83) does not determine the stored energy, i.e., this can not be obtained from the constitutive equation. However, if we assume that

$$K(t, \tau_1) = \breve{\beta}(t+\tau_1), \tag{8.87}$$

such that $\breve{\beta}$ is defined by the relation

$$\beta(t) = \breve{\beta}(t)H(t), \tag{8.88}$$

this choice will suffice to determine K, and

$$E_\epsilon(t) = \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} \breve{\beta}(2t-\tau_1-\tau_2)\partial_{\tau_1}\mathbf{D}(\tau_1)\cdot\partial_{\tau_2}\mathbf{D}(\tau_2) d\tau_1 d\tau_2, \tag{8.89}$$

$$\dot{D}_\epsilon(t) = -\int_{-\infty}^{t}\int_{-\infty}^{t} \partial\breve{\beta}(2t - \tau_1 - \tau_2)\partial_{\tau_1}\mathbf{D}(\tau_1) \cdot \partial_{\tau_2}\mathbf{D}(\tau_2) d\tau_1 d\tau_2, \tag{8.90}$$

where ∂ denotes differentiation with respect to the argument of the corresponding function. Equation (8.87) is consistent with the theory implied by mechanical models (Christensen, 1982). Breuer and Onat (1964) discuss some realistic requirements from which $K(t, \tau_1)$ must have the reduced form $\breve{\beta}(t + \tau_1)$.

Let us calculate the time average of the stored energy density for harmonic fields using equation (8.89). Although $\mathbf{D}(-\infty)$ does not vanish, the transient contained in (8.89) vanishes for sufficiently large times, and this equation can be used to compute the average of time-harmonic fields. The change of variables $\tau_1 \to t - \tau_1$ and $\tau_2 \to t - \tau_2$ yields

$$E_\epsilon(t) = \frac{1}{2}\int_0^{\infty}\int_0^{\infty} \breve{\beta}(\tau_1 + \tau_2)\partial\mathbf{D}(t - \tau_1) \cdot \partial\mathbf{D}(t - \tau_2) d\tau_1 d\tau_2. \tag{8.91}$$

Using (1.105), the time average of equation (8.91) is

$$\langle E_\epsilon \rangle = \frac{1}{4}\omega^2|\mathbf{D}|^2 \int_0^{\infty}\int_0^{\infty} \breve{\beta}(\tau_1 + \tau_2)\cos[\omega(\tau_1 - \tau_2)] d\tau_1 d\tau_2. \tag{8.92}$$

A new change of variables $u = \tau_1 + \tau_2$ and $v = \tau_1 - \tau_2$ gives

$$\langle E_\epsilon \rangle = \frac{1}{8}\omega^2|\mathbf{D}|^2 \int_0^{\infty}\int_{-u}^{u} \breve{\beta}(u)\cos(\omega v) du dv = \frac{1}{4}\omega|\mathbf{D}|^2 \int_0^{\infty} \breve{\beta}(u)\sin(\omega u) du. \tag{8.93}$$

From equation (8.88) and using integration by parts, we have that

$$\mathrm{Re}\left[\mathcal{F}\left[\partial_t\beta\right]\right] = \mathrm{Re}[\beta(\omega)] = \breve{\beta}(\infty) + \omega\int_0^{\infty} [\breve{\beta}(t) - \breve{\beta}(\infty)]\sin(\omega t) dt. \tag{8.94}$$

Using the property

$$\omega\int_0^{\infty} \sin(\omega t) dt = 1, \tag{8.95}$$

we obtain

$$\mathrm{Re}[\beta(\omega)] = \omega\int_0^{\infty} \breve{\beta}(t)\sin(\omega t) dt. \tag{8.96}$$

Substituting (8.96) into equation (8.93), and using $\mathbf{E} = \beta(\omega)\mathbf{D}$ ($\beta(\omega) = \mathcal{F}[\partial_t\beta(t)]$, see equation (8.81)), and equation (8.82), we finally get

$$\langle E_\epsilon \rangle = \frac{1}{4}|\mathbf{D}|^2\mathrm{Re}[\beta(\omega)] = \frac{1}{4}\mathrm{Re}(\hat{\epsilon})|\mathbf{E}|^2, \tag{8.97}$$

which is the expression (8.73). A similar calculation shows that $\langle \dot{D}_\epsilon \rangle$ is equal to the expression (8.74).

Similarly, the magnetic term on the right-hand side of equation (8.78) can be recasted as

$$\mathbf{H} \cdot \partial_t\mathbf{B} = \partial_t E_m + \dot{D}_m, \tag{8.98}$$

where

$$E_m(t) = \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{t} \breve{\gamma}(2t - \tau_1 - \tau_2)\partial_{\tau_1}\mathbf{B}(\tau_1) \cdot \partial_{\tau_2}\mathbf{B}(\tau_2) d\tau_1 d\tau_2, \tag{8.99}$$

$$\dot{D}_m(t) = -\int_{-\infty}^{t}\int_{-\infty}^{t} \partial\breve{\gamma}(2t-\tau_1-\tau_2)\partial_{\tau_1}\mathbf{B}(\tau_1)\cdot\partial_{\tau_2}\mathbf{B}(\tau_2)d\tau_1 d\tau_2, \tag{8.100}$$

are the stored magnetic-energy density and rate of dissipated magnetic-energy density, respectively, such that

$$\mathbf{H} = \gamma * \partial_t \mathbf{B}, \qquad \gamma(t) = \breve{\gamma}(t)H(t), \tag{8.101}$$

with γ the magnetic-impermeability function.

The rate of dissipated conductive-energy density can be defined as

$$\dot{D}_\sigma(t) = -\int_{-\infty}^{t}\int_{-\infty}^{t} \breve{\hat{\sigma}}(2t-\tau_1-\tau_2)\partial_{\tau_1}\mathbf{E}(\tau_1)\cdot\partial_{\tau_2}\mathbf{E}(\tau_2)d\tau_1 d\tau_2. \tag{8.102}$$

Formally, the stored energy density due to the electric currents out-of-phase with the electric field, E_σ, satisfies

$$\partial_t E_\sigma = \mathbf{J}'\cdot\mathbf{E} - \dot{D}_\sigma, \tag{8.103}$$

where

$$\mathbf{J}' = \hat{\sigma} * \partial_t \mathbf{E}, \qquad \hat{\sigma}(t) = \breve{\hat{\sigma}}(t)H(t). \tag{8.104}$$

In terms of the energy densities, equation (8.78) becomes

$$-\mathrm{div}\ \mathbf{p} = \partial_t(E_\epsilon + E_\sigma + E_m) + \dot{D}_\epsilon + \dot{D}_\sigma + \dot{D}_m, \tag{8.105}$$

which is analogous to equation (2.95). The correspondence with time-averaged quantities are given in the previous section.

Note that $\langle \mathbf{J}'\cdot\mathbf{E}\rangle$ is equal to the rate of dissipated energy density $\langle \dot{D}_\sigma\rangle$, and that

$$\langle \partial_t E_\epsilon \rangle = 0. \tag{8.106}$$

The same property holds for the stored electric- and magnetic-energy densities.

There are other alternative time-domain expressions for the energy densities whose time-average values coincide with those given in Section 8.3.1, but fail to match the energy in the time domain. For instance, the following definition

$$E'_\epsilon = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}, \tag{8.107}$$

as the stored dielectric-energy density, and

$$\dot{D}'_\epsilon = \frac{1}{2}\left(\mathbf{E}\cdot\partial_t\mathbf{D} - \mathbf{D}\cdot\partial_t\mathbf{E}\right) \tag{8.108}$$

as the rate of dissipation, satisfy equation (8.105) and $\langle E'_\epsilon\rangle = \langle E_\epsilon\rangle$ and $\langle \dot{D}'_\epsilon\rangle = \langle \dot{D}_\epsilon\rangle$. However, E'_ϵ is not equal to the energy stored in the capacitors for the Debye model given in the next section (see equations (8.120) and (8.122)). In the viscoelastic case (see Chapter 2), the definition of energy is consistent with the theory of mechanical models. In electromagnetism, the theory should be consistent with the theory of circuits, i.e., with the energy stored in the capacitors and the energy dissipated in the resistances.

The Debye-Zener analogy

It is well known that the Debye model used to describe the behaviour of dielectric materials (Hippel, 1962) is mathematically equivalent to the Zener or standard-linear-solid model used in viscoelasticity (Zener, 1948). The following example uses this equivalence to illustrate the concepts presented in the previous section.

Let us consider a capacitor C_2 in parallel with a series connection between a capacitor C_1 and a resistance R. This circuit obeys the following differential equation:

$$U + \tau_U \partial_t U = \frac{1}{C} \left(I + \tau_I \partial_t I \right), \tag{8.109}$$

where $U = \partial V / \partial t$, I is the current, V is the voltage,

$$C = C_1 + C_2, \qquad \tau_U = R \left(\frac{1}{C_1} + \frac{1}{C_2} \right)^{-1}, \qquad \tau_I = C_1 R. \tag{8.110}$$

From the point of view of a pure dielectric process, we identify U with $\mathbf{E}$ and I with $\mathbf{D}$ (see Figure 8.3).

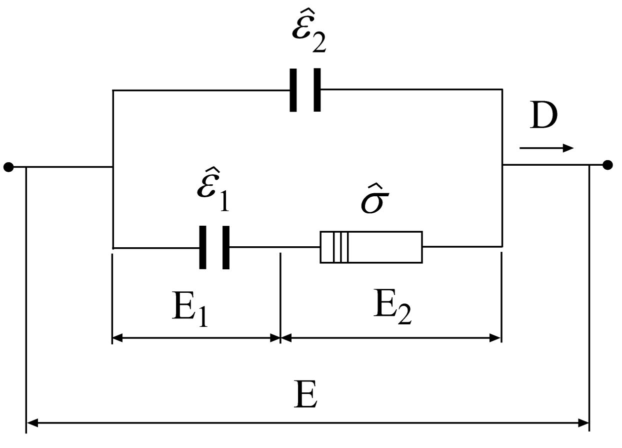

Figure 8.3: This electric circuit is equivalent to a purely dielectric-relaxation process, where $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ are the capacitors, $\hat{\sigma}$ is the conductivity, $\mathbf{E}$ is the electric field, and $\mathbf{D}$ is the electric displacement.

Hence, the dielectric-relaxation model is

$$\mathbf{E} + \tau_{\mathcal{E}} \partial_t \mathbf{E} = \frac{1}{\hat{\epsilon}^0} \left(\mathbf{D} + \tau_{\mathcal{D}} \partial_t \mathbf{D} \right), \tag{8.111}$$

where

$$\hat{\epsilon}^0 = \hat{\epsilon}_1 + \hat{\epsilon}_2, \qquad \tau_{\mathcal{E}} = \frac{1}{\hat{\sigma}} \left(\frac{1}{\hat{\epsilon}_1} + \frac{1}{\hat{\epsilon}_2} \right)^{-1}, \qquad \tau_{\mathcal{D}} = \hat{\epsilon}_1 / \hat{\sigma}, \tag{8.112}$$

with $\hat{\sigma}$ the conductivity. Note that $\hat{\epsilon}^0$ is the static (low-frequency) dielectric permittivity and $\hat{\epsilon}^\infty = \hat{\epsilon}^0 \tau_{\mathcal{E}} / \tau_{\mathcal{D}} = \hat{\epsilon}_2 < \hat{\epsilon}^0$ is the optical (high-frequency) dielectric permittivity.

We have that

$$\hat{\epsilon}(t) = \hat{\epsilon}^0 \left[1 - \left(1 - \frac{\tau_{\mathcal{E}}}{\tau_{\mathcal{D}}}\right) \exp(-t/\tau_{\mathcal{D}})\right] H(t), \tag{8.113}$$

$$\beta(t) = \breve{\beta}(t)H(t) = \frac{1}{\hat{\epsilon}^0} \left[1 - \left(1 - \frac{\tau_{\mathcal{D}}}{\tau_{\mathcal{E}}}\right) \exp(-t/\tau_{\mathcal{E}})\right] H(t) \tag{8.114}$$

and

$$\hat{\epsilon}(\omega) = \beta^{-1}(\omega) = \hat{\epsilon}^0 \left(\frac{1 + \mathrm{i}\omega\tau_{\mathcal{E}}}{1 + \mathrm{i}\omega\tau_{\mathcal{D}}}\right). \tag{8.115}$$

Equation (8.115) can be rewritten as

$$\hat{\epsilon}(\omega) = \hat{\epsilon}^\infty + \frac{\hat{\epsilon}^0 - \hat{\epsilon}^\infty}{1 + \mathrm{i}\omega\tau_{\mathcal{D}}}. \tag{8.116}$$

The dielectric permittivity (8.116) describes the response of polar molecules, such as water, to the electromagnetic field (Debye, 1929; Turner and Siggins, 1994).

Substituting (8.114) into equation (8.81) and defining the internal variable

$$\boldsymbol{\xi}(t) = \hat{\phi} \exp(-t/\tau_{\mathcal{E}})H(t) * \mathbf{D}(t), \quad \hat{\phi} = \frac{1}{\hat{\epsilon}^0\tau_{\mathcal{E}}} \left(1 - \frac{\tau_{\mathcal{D}}}{\tau_{\mathcal{E}}}\right), \tag{8.117}$$

yields

$$\mathbf{E} = \frac{1}{\hat{\epsilon}^\infty}\mathbf{D} + \boldsymbol{\xi}, \tag{8.118}$$

where $\boldsymbol{\xi}$ satisfies

$$\partial_t \boldsymbol{\xi} = \hat{\phi}\mathbf{D} - \frac{\boldsymbol{\xi}}{\tau_{\mathcal{E}}}. \tag{8.119}$$

The dielectric-energy density is that stored in the capacitors:

$$E_\epsilon = \frac{1}{2\hat{\epsilon}_1}\mathbf{D}_1 \cdot \mathbf{D}_1 + \frac{1}{2\hat{\epsilon}_2}\mathbf{D}_2 \cdot \mathbf{D}_2, \tag{8.120}$$

where $\mathbf{D}_1$ and $\mathbf{D}_2$ are the respective electric displacements. Since $\mathbf{D}_2 = \hat{\epsilon}_2\mathbf{E}$, $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_2$ and $\hat{\epsilon}^\infty = \hat{\epsilon}_2$, we obtain

$$\mathbf{D}_1 = -\hat{\epsilon}^\infty \boldsymbol{\xi}, \tag{8.121}$$

where equation (8.118) has been used. Note that the internal variable $\boldsymbol{\xi}$ is closely related to the electric field acting on the capacitor in series with the dissipation element. Substitution of $\mathbf{D}_1$ and $\mathbf{D}_2$ into equation (8.120) and after some calculations yields

$$E_\epsilon = \frac{\hat{\epsilon}^\infty}{2} \left[\left(\frac{\hat{\epsilon}^\infty}{\hat{\epsilon}^0 - \hat{\epsilon}^\infty}\right) \boldsymbol{\xi} \cdot \boldsymbol{\xi} + \mathbf{E} \cdot \mathbf{E}\right]. \tag{8.122}$$

Let us verify that equation (8.122) is in agreement with equation (8.89). From equations (8.114) and (8.117) we have

$$\breve{\beta}(t) = \frac{1}{\hat{\epsilon}^0} - \hat{\phi}\tau_{\mathcal{E}} \exp(-t/\tau_{\mathcal{E}}). \tag{8.123}$$

Replacing (8.123) into equation (8.89) and after some algebra yields

$$E_\epsilon = \frac{1}{2\hat{\epsilon}^0}\mathbf{D}\cdot\mathbf{D} - \frac{1}{2}\hat{\phi}\tau_{\mathcal{E}}\left[\exp(-t/\tau_{\mathcal{E}})H(t) * \partial_t\mathbf{D}(t)\right]^2, \tag{8.124}$$

where the exponent 2 means the scalar product. Using equations (8.117) and (8.119) gives

$$E_\epsilon = \frac{1}{2\hat{\epsilon}^0}\mathbf{D}\cdot\mathbf{D} - \frac{1}{2\hat{\phi}\tau_{\mathcal{E}}}(\hat{\phi}\tau_{\mathcal{E}}\mathbf{D} - \boldsymbol{\xi})\cdot(\hat{\phi}\tau_{\mathcal{E}}\mathbf{D} - \boldsymbol{\xi}). \tag{8.125}$$

Since $\hat{\epsilon}^\infty\tau_{\mathcal{D}} = \hat{\epsilon}^0\tau_{\mathcal{E}}$, and a few calculations show that the expression in (8.125) is equal to the stored energy density (8.122). This equivalence can also be obtained by avoiding the use of internal variables. However, the introduction of these variables is a requirement to obtain a complete differential formulation of the electromagnetic equations. This formulation is the basis of most simulation algorithms (Carcione, 1996c; Xu and McMechan, 1997).

The rate of dissipated dielectric-energy density is

$$\dot{D}_\epsilon = \frac{1}{\hat{\sigma}}\partial_t\mathbf{D}_1\cdot\partial_t\mathbf{D}_1, \tag{8.126}$$

which from equation (8.119) and (8.121) becomes

$$\dot{D}_\epsilon = \frac{1}{\hat{\sigma}}\left(\frac{\hat{\epsilon}^\infty}{\tau_{\mathcal{E}}}\right)^2(\hat{\phi}\tau_{\mathcal{E}}\mathbf{D} - \boldsymbol{\xi})\cdot(\hat{\phi}\tau_{\mathcal{E}}\mathbf{D} - \boldsymbol{\xi}). \tag{8.127}$$

Taking into account the previous calculations, it is easy to show that substitution of equation (8.123) into (8.90) gives equation (8.127).

The Zener model has been introduced in Sections 2.4.3 and 2.7.3. In this case, the free (stored) energy density can be uniquely determined (Cavallini and Carcione, 1994). The relaxation function and complex modulus are given in equations (2.173) and (2.170), respectively:

$$\psi(t) = M_R\left[1 - \left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right)\exp(-t/\tau_\sigma)\right]H(t), \tag{8.128}$$

and

$$M(\omega) = M_R\left(\frac{1 + \mathrm{i}\omega\tau_\epsilon}{1 + \mathrm{i}\omega\tau_\sigma}\right), \tag{8.129}$$

where M_R, τ_ϵ and τ_σ are defined in equations (2.168) and (2.169), respectively. Equation (8.129) can be rewritten as

$$M^{-1}(\omega) = M_U^{-1} + \frac{M_R^{-1} - M_U^{-1}}{1 + \mathrm{i}\omega\tau_\epsilon}, \tag{8.130}$$

where $M_U = M_R\tau_\epsilon/\tau_\sigma$. The memory variable is given in equation (2.283):

$$\xi(t) = \varphi_0[\exp(-t/\tau_\sigma)H(t)] * \epsilon(t), \qquad \varphi_0 = \frac{M_R}{\tau_\sigma}\left(1 - \frac{\tau_\epsilon}{\tau_\sigma}\right), \tag{8.131}$$

and the field variables satisfy equation (2.283),

$$\sigma = M_U\epsilon + \xi, \tag{8.132}$$

and equation (2.286),

$$\partial_t \xi = \varphi_0 \epsilon - \frac{\xi}{\tau_\sigma}. \tag{8.133}$$

Assuming that the strain energy is stored in the springs, we have that

$$V = \frac{1}{2}(k_1 \epsilon_1^2 + k_2 \epsilon_2^2), \tag{8.134}$$

where ϵ_1 and ϵ_2 are the dilatations of the springs (see Figure 2.8), and k_1 and k_2 can be expressed from equations (2.168) and (2.169) as

$$k_1 = M_U \quad \text{and} \quad k_2 = -\frac{M_U M_R}{\varphi_0 \tau_\sigma}. \tag{8.135}$$

Since $\sigma = k_1 \epsilon_1$ and $\epsilon = \epsilon_1 + \epsilon_2$, and using (2.283), we obtain

$$\epsilon_1 = \epsilon + \frac{\xi}{M_U} \quad \text{and} \quad \epsilon_2 = -\frac{\xi}{M_U}. \tag{8.136}$$

Note that the memory variable ξ is closely related to the dilatation on the spring that is in parallel with the dashpot. Substitution of the dilatations into equation (8.134) yields

$$V = \frac{M_U}{2}\left[\left(\epsilon + \frac{\xi}{M_U}\right)^2 - \frac{M_R}{\varphi_0 \tau_\sigma}\left(\frac{\xi}{M_U}\right)^2\right], \tag{8.137}$$

which, after some calculations, can be rewritten as

$$V = \frac{1}{2} M_R \epsilon^2 - \frac{1}{2\varphi_0 \tau_\sigma}(\varphi_0 \tau_\sigma \epsilon - \xi)^2. \tag{8.138}$$

On the other hand, the rate of energy density dissipated in the dashpot of viscosity η is

$$\dot{D} = \eta \left(\frac{\partial \epsilon_2}{\partial t}\right)^2, \tag{8.139}$$

which from equations (8.133) and (8.136) becomes

$$\dot{D} = \frac{\eta}{(\tau_\sigma M_U)^2}(\varphi_0 \tau_\sigma \epsilon - \xi)^2. \tag{8.140}$$

The mathematics of the viscoelastic problem is the same as for the dielectric relaxation model previously introduced, since equations (8.114)-(8.119) are equivalent to (8.128)-(8.133) and equations (8.125) and (8.127) are equivalent to (8.138) and (8.140), respectively. The mathematical equivalence identifies electric vector **E** with stress σ and electric displacement **D** with strain ϵ. The complete correspondence between the dielectric and the viscoelastic models is

$$\begin{array}{ccccccc}
 & \text{Fields} & & & \text{Properties} & \\
\mathbf{E} & \Leftrightarrow & \sigma & \hat{\epsilon}^0 & \Leftrightarrow & M_R^{-1} \\
\mathbf{D} & \Leftrightarrow & \epsilon & \hat{\epsilon}^\infty & \Leftrightarrow & M_U^{-1} \\
\mathbf{E}_1 & \Leftrightarrow & \sigma_1 & \tau_{\mathcal{D}} & \Leftrightarrow & \tau_\epsilon \\
\mathbf{E}_2 & \Leftrightarrow & \sigma_2 & \tau_{\mathcal{E}} & \Leftrightarrow & \tau_\sigma \\
\mathbf{D}_1 & \Leftrightarrow & \epsilon_1 & \hat{\epsilon}_1 & \Leftrightarrow & k_1^{-1} \\
\mathbf{D}_2 & \Leftrightarrow & \epsilon_2 & \hat{\epsilon}_2 & \Leftrightarrow & k_2^{-1} \\
\boldsymbol{\xi} & \Leftrightarrow & \xi & \hat{\sigma} & \Leftrightarrow & \eta^{-1},
\end{array} \tag{8.141}$$

where some of the symbols can be identified in Figures 2.8 and 8.3.

The Cole-Cole model

Equation (8.116) can be generalized as

$$\hat{\epsilon}(\omega) = \hat{\epsilon}^\infty + \frac{\hat{\epsilon}^0 - \hat{\epsilon}^\infty}{1 + (\mathrm{i}\omega\tau_\mathcal{D})^q}, \tag{8.142}$$

where $q = m/n$, with m and n positive, integer and prime, and $m < n$. This model has been introduced by Cole and Cole (1941). The corresponding frequency- and time-domains constitutive equations are

$$\mathbf{D} = \left[\frac{\hat{\epsilon}^0 + \hat{\epsilon}^\infty(\mathrm{i}\omega\tau_\mathcal{D})^q}{1 + (\mathrm{i}\omega\tau_\mathcal{D})^q}\right]\mathbf{E} \tag{8.143}$$

and

$$\mathbf{E} + \tau_\mathcal{E}^q \frac{\partial^q \mathbf{E}}{\partial t^q} = \frac{1}{\hat{\epsilon}^0}\left(\mathbf{D} + \tau_\mathcal{D}^q \frac{\partial^q \mathbf{D}}{\partial t^q}\right), \tag{8.144}$$

where $\tau_\mathcal{E}^q = \hat{\epsilon}^\infty \tau_\mathcal{D}^q / \hat{\epsilon}^0$, and $\partial^q/\partial t^q$ is the fractional derivative of order q (see Section 2.5.2). Equation (8.144) is a generalization of (8.111).

The rational power of the imaginary unit $(\mathrm{i})^q$ in equations (8.142) and (8.143) is a multi-valued function and implies a number n of different physically accepted values of the dielectric permittivity. As a consequence, a time-harmonic wave is split into a set of waves with the same frequency and slightly different wavelengths which interfere and disperse (Caputo, 1998; Belfiore and Caputo, 2000). The expression (8.142) is also called the generalized Debye form of the dielectric permittivity, and the Debye-Zener analogy (8.141) can also be applied to the Cole-Cole model.

The fractional derivative is a generalization of the derivative of natural order by using Cauchy's well-known formula. For a given function $f(t)$, the fractional derivative is given by

$$\frac{\partial^q f}{\partial t^q} = f(t) * \Phi_{-q}(t), \quad \text{where} \quad \Phi_q(t) = \frac{t_+^{q-1}}{\Gamma(q)}, \qquad t_+^{q-1} = \begin{cases} t^{q-1}, & t > 0, \\ 0, & t \le 0, \end{cases} \tag{8.145}$$

and Γ is Euler's Gamma function (Caputo and Mainardi, 1971). If $\Phi_{-j} = \delta^{(j)}(t)$, $j = 0, 1, 2, \ldots$, where δ is Dirac's function, equation (8.145) gives the j-order derivative of $f(t)$. Caputo and Mainardi (1971) have shown that

$$\hat{\epsilon}(t) = \{\hat{\epsilon}^0 + (\hat{\epsilon}^\infty - \hat{\epsilon}^0) E_q[-(t/\tau_D)^q]\} H(t), \tag{8.146}$$

where

$$E_q(\tau) = \sum_{k=0}^{\infty} \frac{\tau^k}{\Gamma(qk+1)} \tag{8.147}$$

is the Mittag-Leffler function of order q, introduced by Gösta Mittag-Leffler in 1903 (note the similarity with the Wright function (3.212)). It is a generalization of the exponential function, with $E_1(\tau) = \exp(\tau)$ (e.g., Podlubny, 1999). Equation (8.146) becomes equation (8.113) for q =1.

8.4 The analogy for reflection and transmission

In this section, we obtain a complete parallelism for the reflection and refraction (transmission) problem, considering the most general situation, that is the presence of anisotropy and attenuation – viscosity in the acoustic case and conductivity in the electromagnetic case (Carcione and Robinson, 2002). The analysis of the elastic-solid theory of reflection applied by George Green to light waves (Green, 1842), and a brief historical review of wave propagation through the ether, further illustrate the analogy.

Let us assume that the incident, reflected and refracted waves are identified by the superscripts I, R and T. The boundary separates two linear viscoelastic and monoclinic media. The upper medium is defined by the stiffnesses p_{IJ} and density ρ and the complex permittivities $\bar{\epsilon}_{ij}$ and magnetic permeability $\hat{\mu}$. The lower medium is defined by the corresponding primed quantities. Let us denote by θ and δ the propagation and attenuation angles, and by ψ the Umov-Poynting vector (energy) direction, as indicated in Figure 6.1. The propagation and energy directions do not necessarily coincide.

The analogy can be extended to the boundary conditions at a surface of discontinuity, say, the (x, z)-plane, because according to equation (8.31) continuity of

$$\sigma_{32} \quad \text{and} \quad v_2 \tag{8.148}$$

in the acoustic case, is equivalent to continuity of

$$E_1 \quad \text{and} \quad H_2 \tag{8.149}$$

in the electromagnetic case. The field variables in (8.149) are precisely the tangential components of the electric and magnetic vectors. In the absence of surface current densities at the interface, the boundary conditions impose the continuity of those components (Born and Wolf, 1964, p. 4).

The SH reflection-transmission problem is given in Section 6.1, where the Zener model is used to describe the attenuation properties. In the case of an incident inhomogeneous plane wave and a general stiffness matrix $\mathbf{P}$, the relevant equations are summarized in the following section.

8.4.1 Reflection and refraction coefficients

The particle velocities of the reflected and refracted waves are given by

$$v_2^R = \mathrm{i}\omega R \exp[\mathrm{i}\omega(t - s_1 x - s_3^R z)] \tag{8.150}$$

and

$$v_2^T = \mathrm{i}\omega T \exp[\mathrm{i}\omega(t - s_1 x - s_3^T z)], \tag{8.151}$$

respectively, and the reflection and refraction (transmission) coefficients are

$$R = \frac{Z^I - Z^T}{Z^I + Z^T}, \qquad T = \frac{2Z^I}{Z^I + Z^T}, \tag{8.152}$$

where

$$Z^I = p_{46} s_1 + p_{44} s_3^I, \qquad Z^T = p'_{46} s_1 + p'_{44} s_3^T, \tag{8.153}$$

with

$$s_1^R = s_1^T = s_1^I = s_1 \quad \text{(Snell's law)}, \tag{8.154}$$

$$s_3^R = -\left(s_3^I + \frac{2p_{46}}{p_{44}} s_1\right), \tag{8.155}$$

and

$$s_3^T = \frac{1}{p'_{44}}\left(-p'_{46}s_1 + \mathrm{pv}\sqrt{\rho' p'_{44} - p'^2 s_1^2}\right), \tag{8.156}$$

with

$$p'^2 = p'_{44}p'_{66} - {p'_{46}}^2. \tag{8.157}$$

(For the principal value, the argument of the square root lies between $-\pi/2$ and $+\pi/2$.) As indicated by Krebes (1984), special care is needed when choosing the sign, since a wrong choice may lead to discontinuities of the vertical wavenumber as a function of the incidence angle.

Propagation, attenuation and ray angles

For each plane wave,

$$\tan\theta = \frac{\mathrm{Re}(s_1)}{\mathrm{Re}(s_3)}, \qquad \tan\delta = \frac{\mathrm{Im}(s_1)}{\mathrm{Im}(s_3)}, \qquad \tan\psi = \frac{\mathrm{Re}(X)}{\mathrm{Re}(Z)}, \tag{8.158}$$

where

$$\begin{aligned} X^I &= p_{66}s_1 + p_{46}s_3^I \\ X^R &= p_{66}s_1 + p_{46}s_3^R \\ X^T &= p'_{66}s_1 + p'_{46}s_3^T. \end{aligned} \tag{8.159}$$

The ray angle denotes the direction of the power-flow vector $\mathrm{Re}(\mathbf{p})$, where $\mathbf{p}$ is the Umov-Poynting vector (6.9).

Energy-flux balance

The balance of energy flux involves the continuity of the normal component of the Umov-Poynting vector across the interface. This is a consequence of the boundary conditions that impose continuity of normal stress σ_{32} and particle velocity v_2. The balance of power flow at the interface, on a time-average basis, is given in Section 6.1.7. The equation are

$$\langle p^I\rangle + \langle p^R\rangle + \langle p^{IR}\rangle = \langle p^T\rangle, \tag{8.160}$$

where

$$\langle p^I\rangle = -\frac{1}{2}\mathrm{Re}(\sigma_{32}^I v_2^{I*}) = \frac{1}{2}\omega^2\mathrm{Re}(Z^I)\exp[2\omega\mathrm{Im}(s_1)x] \tag{8.161}$$

is the incident flux,

$$\langle p^R\rangle = -\frac{1}{2}\mathrm{Re}(\sigma_{32}^R v_2^{R*}) = \frac{1}{2}\omega^2|R|^2\mathrm{Re}(Z^R)\exp[2\omega\mathrm{Im}(s_1)x] \tag{8.162}$$

is the reflected flux,

$$\langle p^{IR}\rangle = -\frac{1}{2}\mathrm{Re}(\sigma_{32}^I v_2^{R*} + \sigma_{32}^R v_2^{I*}) = \omega^2\mathrm{Im}(R)\mathrm{Im}(Z^I)\exp[2\omega\mathrm{Im}(s_1)x] \tag{8.163}$$

is the interference between the incident and reflected normal fluxes, and

$$\langle p^T \rangle = -\frac{1}{2}\mathrm{Re}(\sigma_{32}^T v_2^{T*}) = \frac{1}{2}\omega^2 |T|^2 \mathrm{Re}(Z^T) \exp[2\omega \mathrm{Im}(s_1)x] \tag{8.164}$$

is the refracted flux. In the lossless case, Z^I is real and the interference flux vanishes.

8.4.2 Application of the analogy

On the basis of the solution of the SH-wave problem, we use the analogy to find the solution in the electromagnetic case. For every electromagnetic phenomenon – using the electromagnetic terminology – we analyze its corresponding mathematical and physical counterpart in the acoustic case. Maxwell (1891, p. 65), who used this approach, writes: *The analogy between the action of electromotive intensity in producing the displacement of an elastic body is so obvious that I have ventured to call the ratio of electromotive intensity to the corresponding electric displacement the coefficient of electric elasticity of the medium.*

Refraction index and Fresnel's formulae

Let us assume a lossless, isotropic medium. Isotropy implies $c_{44} = c_{66} = \mu$ and $c_{46} = 0$ and $\hat{\epsilon}_{11} = \hat{\epsilon}_{33} = \hat{\epsilon}$, and $\hat{\epsilon}_{13} = 0$. It is easy to show that, in this case, the reflection and refraction coefficients (8.152) reduce to

$$R = \frac{\sqrt{\rho\mu}\cos\theta^I - \sqrt{\rho'\mu'}\cos\theta^T}{\sqrt{\rho\mu}\cos\theta^I + \sqrt{\rho'\mu'}\cos\theta^T} \quad \text{and} \quad T = \frac{2\sqrt{\rho'\mu'}\cos\theta^I}{\sqrt{\rho\mu}\cos\theta^I + \sqrt{\rho'\mu'}\cos\theta^T}, \tag{8.165}$$

respectively. From the analogy (equation (8.33)) and equation (8.30) we have

$$\mu^{-1} \Leftrightarrow \hat{\epsilon}, \tag{8.166}$$

The refraction index is defined as the velocity of light in vacuum, c_0, divided by the phase velocity in the medium, where the phase velocity is the reciprocal of the real slowness. For lossless, isotropic media, the refraction index is

$$n = sc_0 = \sqrt{\frac{\hat{\mu}\hat{\epsilon}}{\hat{\mu}_0\hat{\epsilon}_0}}, \tag{8.167}$$

where $s = \sqrt{\hat{\mu}\hat{\epsilon}}$ is the slowness, and $c_0 = 1/\sqrt{\hat{\mu}_0\hat{\epsilon}_0}$, with $\hat{\epsilon}_0 = 8.85\ 10^{-12}$ F/m and $\hat{\mu}_0 = 4\pi\ 10^{-7}$ H/m, the dielectric permittivity and magnetic permeability of free space. In acoustic media there is not a limit velocity, but using the analogy we can define a refraction index

$$n_a = \nu\sqrt{\frac{\rho}{\mu}}, \tag{8.168}$$

where ν is a constant with the dimensions of velocity. Assuming $\rho = \rho'$ in (8.165) and using (8.166), the electromagnetic coefficients are

$$R = \frac{\sqrt{\hat{\epsilon}'}\cos\theta^I - \sqrt{\hat{\epsilon}}\cos\theta^T}{\sqrt{\hat{\epsilon}'}\cos\theta^I + \sqrt{\hat{\epsilon}}\cos\theta^T} \quad \text{and} \quad T = \frac{2\sqrt{\hat{\epsilon}}\cos\theta^I}{\sqrt{\hat{\epsilon}'}\cos\theta^I + \sqrt{\hat{\epsilon}}\cos\theta^T}. \tag{8.169}$$

In terms of the refraction index (8.167) we have

$$R = \frac{n' \cos\theta^I - n\cos\theta^T}{n'\cos\theta^I + n\cos\theta^T} \quad \text{and} \quad T = \frac{2n\cos\theta^I}{n'\cos\theta^I + n\cos\theta^T}. \tag{8.170}$$

Equations (8.170) are Fresnel's formulae, corresponding to the electric vector in the plane of incidence (Born and Wolf, 1964, p. 40). Hence, Fresnel's formulae are mathematically equivalent to the SH-wave reflection and transmission coefficients for lossless, isotropic media, with no density contrast at the interface.

Brewster (polarizing) angle

Fresnel's formulae can be written in an alternative form, which may be obtained from (8.170) by using Snell's law

$$\frac{\sin\theta^I}{\sin\theta^T} = \sqrt{\frac{\mu}{\mu'}} = \frac{n'_a}{n_a} = \sqrt{\frac{\hat{\epsilon}'}{\hat{\epsilon}}} = \frac{n'}{n}. \tag{8.171}$$

It yields

$$R = \frac{\tan(\theta^I - \theta^T)}{\tan(\theta^I + \theta^T)} \quad \text{and} \quad T = \frac{2\sin\theta^T\cos\theta^I}{\sin(\theta^I + \theta^T)\cos(\theta^I - \theta^T)}. \tag{8.172}$$

The denominator in $(8.172)_1$ is finite, except when $\theta^I + \theta^T = \pi/2$. In this case the reflected and refracted rays are perpendicular to each other and $R = 0$. It follows from Snell's law that the incidence angle, $\theta_B \equiv \theta^I$, satisfies

$$\tan\theta_B = \cot\theta^T = \sqrt{\frac{\mu}{\mu'}} = \frac{n'_a}{n_a} = \sqrt{\frac{\hat{\epsilon}'}{\hat{\epsilon}}} = \frac{n'}{n}. \tag{8.173}$$

The angle θ_B is called the Brewster angle, first noted by Étienne Malus and David Brewster (Brewster, 1815)(see Section 6.1.5). It follows that the Brewster angle in elasticity can be obtained when the medium is lossless and isotropic, and the density is constant across the interface. This angle is also called polarizing angle, because, as Brewster states, *When a polarised ray is incident at any angle upon a transparent body, in a plane at right angles to the plane of its primitive polarisation, a portion of the ray will lose its property of being reflected, and will entirely penetrate the transparent body. This portion of light, which has lost its reflexibility, increases as the angle of incidence approaches to the polarising angle, when it becomes a maximum.* Thus, at the polarizing angle, the electric vector of the reflected wave has no components in the plane of incidence.

The restriction about the density can be removed and the Brewster angle is given by

$$\tan\theta_B = \sqrt{\frac{\rho\mu/\mu' - \rho'}{\rho' - \rho\mu'/\mu}}, \tag{8.174}$$

but $\theta^I + \theta^T \neq \pi/2$ in this case. The analogies (8.34) and (8.35) imply

$$\tan\theta_B = \sqrt{\frac{\hat{\mu}\hat{\epsilon}'/\hat{\epsilon} - \hat{\mu}'}{\hat{\mu}' - \hat{\mu}\hat{\epsilon}/\hat{\epsilon}'}} \tag{8.175}$$

in the electromagnetic case.

In the anisotropic and lossless case, the angle is obtained from

$$\cot \theta_B = (-b \pm \sqrt{b^2 - 4ac})/(2a), \tag{8.176}$$

where

$$a = c_{44}(\rho c_{44} - \rho' c'_{44})/\rho, \qquad b = 2ac_{46}/c_{44}, \tag{8.177}$$

and

$$c = c_{46}^2 - {c'_{46}}^2 - c'_{44}(\rho' c_{66} - \rho c'_{66})/\rho \tag{8.178}$$

(see Section 6.1.5). If $c_{46} = c'_{46} = 0$, we obtain

$$\tan \theta_B = \sqrt{\frac{c_{44}(\rho c_{44} - \rho' c'_{44})}{c'_{44}(\rho' c_{66} - \rho c'_{66})}}, \tag{8.179}$$

or, using the analogy,

$$\begin{array}{ccc} c_{44}^{-1} & \Leftrightarrow & \hat{\epsilon}_{11} \\ c_{66}^{-1} & \Leftrightarrow & \hat{\epsilon}_{33} \\ \rho & \Leftrightarrow & \hat{\mu}, \end{array} \tag{8.180}$$

the Brewster angle is given by

$$\tan \theta_B = \frac{1}{\hat{\epsilon}_{11}} \sqrt{\frac{\hat{\epsilon}_{33} \hat{\epsilon}'_{33} (\hat{\mu} \hat{\epsilon}'_{11} - \hat{\mu}' \hat{\epsilon}_{11})}{\hat{\mu}' \hat{\epsilon}'_{33} - \hat{\mu} \hat{\epsilon}_{33}}}. \tag{8.181}$$

In the lossy case, $\tan \theta_B$ is complex, in general, and there is no Brewster angle. However let us consider equation (8.175) and incident homogeneous plane waves. According to the correspondence (8.52), its extension to the lossy case is

$$\tan \theta_B = \sqrt{\frac{\hat{\mu} \bar{\epsilon}' / \bar{\epsilon} - \hat{\mu}'}{\hat{\mu}' - \hat{\mu} \bar{\epsilon} / \bar{\epsilon}'}}. \tag{8.182}$$

The Brewster angle exists if $\bar{\epsilon}'$ is proportional to $\bar{\epsilon}$, for instance, if the conductivity of the refraction medium satisfies $\hat{\sigma}' = (\hat{\epsilon}'/\hat{\epsilon})\hat{\sigma}$ ($\eta' = (\mu'/\mu)\eta$ in the acoustic case). This situation is unlikely to occur in reality, unless the interface is designed for this purpose.

Critical angle. Total reflection

In isotropic, lossless media, total reflection occurs when Snell's law

$$\sin \theta^T = \sqrt{\frac{\rho \mu'}{\rho' \mu}} \sin \theta^I = \sqrt{\frac{\hat{\mu} \hat{\epsilon}}{\hat{\mu}' \hat{\epsilon}'}} \sin \theta^I \tag{8.183}$$

does not give a real value for the refraction angle θ^T. When the angle of incidence exceeds the critical angle θ_C defined by

$$\sin \theta^I = \sin \theta_C = \sqrt{\frac{\rho' \mu}{\rho \mu'}} = \frac{n'_a}{n_a} = \sqrt{\frac{\hat{\mu}' \hat{\epsilon}'}{\hat{\mu} \hat{\epsilon}}} = \frac{n'}{n}, \tag{8.184}$$

all the incident wave is reflected back into the incidence medium (Born and Wolf, 1964, p. 47). Note from equations (8.173) and (8.184) that $\tan\theta_B = \sin\theta_C$ when $\rho' = \rho$ and $\hat{\mu}' = \hat{\mu}$.

The critical angle is defined as the angle of incidence beyond which the refracted Umov-Poynting vector is parallel to the interface. The condition $\mathrm{Re}(Z^T)=0$ (see Section 6.1.5) yields the critical angle θ_C. For the anisotropic, lossless case, with $c_{46} = c'_{46} = 0$, we obtain

$$\tan\theta_C = \sqrt{\frac{\rho' c_{44}}{\rho c'_{66} - \rho' c_{66}}} = \sqrt{\frac{\hat{\mu}'\hat{\epsilon}_{33}\hat{\epsilon}'_{33}}{\hat{\epsilon}_{11}(\hat{\mu}\hat{\epsilon}_{33} - \hat{\mu}'\hat{\epsilon}'_{33})}}, \tag{8.185}$$

where we have used the correspondence (8.180).

In the isotropic and lossy case we have

$$\tan\theta_C = \sqrt{\frac{\hat{\mu}'\bar{\epsilon}'}{\hat{\mu}\bar{\epsilon} - \hat{\mu}'\bar{\epsilon}'}}. \tag{8.186}$$

The critical angle exists if $\bar{\epsilon}'$ is proportional to $\bar{\epsilon}$, i.e., when the conductivity of the refraction medium satisfies $\hat{\sigma}' = (\hat{\epsilon}'/\hat{\epsilon})\hat{\sigma}$.

Example: The acoustic properties of the incidence and refraction media are

$$c_{44} = 9.68 \text{ GPa}, \quad c_{66} = 12.5 \text{ GPa}, \quad \eta_{44} = 20\ c_{44}/\omega, \quad \eta_{66} = \eta_{44}, \quad \rho = 2000 \text{ kg/m}^3$$

and

$$c'_{44} = 25.6 \text{ GPa}, \quad c'_{66} = c'_{44}, \quad \eta'_{44} = \eta'_{66} = \infty, \quad \rho = 2500 \text{ kg/m}^3,$$

respectively, where $\omega = 2\pi f$, with $f = 25$ Hz. The refraction medium is isotropic and lossless. The absolute value of the acoustic reflection and refraction coefficients – solid and dashed lines – are shown in Figure 8.4 for the lossless (a) and lossy (b) cases, respectively. The Brewster and critical angles are $\theta_B = 42.61^\circ$ and $\theta_C = 47.76^\circ$ (see Figure 8.4a), which can be verified from equations (8.179) and (8.185), respectively.

The electromagnetic properties of the incidence and refraction media are

$$\hat{\epsilon}_{11} = 3\ \hat{\epsilon}_0, \quad \hat{\epsilon}_{33} = 7\ \hat{\epsilon}_0, \quad \hat{\sigma}_{11} = \hat{\sigma}_{33} = 0.15 \text{ S/m}, \quad \hat{\mu} = 2\hat{\mu}_0$$

and

$$\hat{\epsilon}_{11} = \hat{\epsilon}_{33} = \hat{\epsilon}_0, \quad \hat{\sigma}_{11} = \hat{\sigma}_{33} = 0, \quad \hat{\mu} = \hat{\mu}_0,$$

respectively, where we consider a frequency of 1 GHz. The refraction medium is vacuum. We apply the analogy

$$\begin{array}{rcl} c_{44}^{-1} & \Leftrightarrow & \hat{\epsilon}_{11} \\ c_{66}^{-1} & \Leftrightarrow & \hat{\epsilon}_{33} \\ \eta_{44}^{-1} & \Leftrightarrow & \hat{\sigma}_{11} \\ \eta_{66}^{-1} & \Leftrightarrow & \hat{\sigma}_{33} \\ \rho & \Leftrightarrow & \hat{\mu}, \end{array} \tag{8.187}$$

and use the same computer code used to obtain the acoustic reflection and refraction coefficients. The absolute value of the electromagnetic reflection and refraction coefficients – solid and dashed lines – are shown in Figure 8.5 for the lossless (a) and lossy (b) cases, respectively. The Brewster and critical angles are $\theta_B = 13.75^\circ$ and $\theta_C = 22.96^\circ$, which can be verified from equations (8.181) and (8.185), respectively.

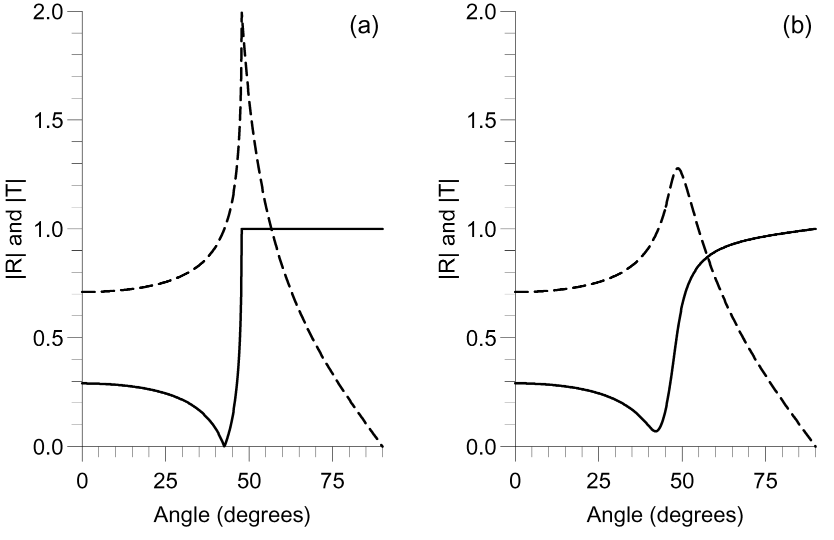

Figure 8.4: Reflection and transmission coefficients (solid and dashed lines) for elastic media: (a) lossless case and (b) lossy case.

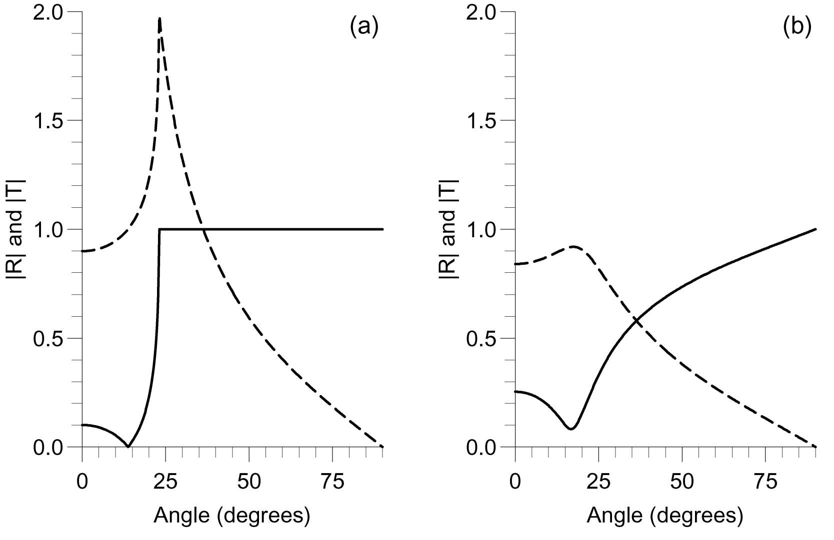

Figure 8.5: Reflection and transmission coefficients (solid and dashed lines) for electromagnetic media: (a) lossless case and (b) lossy case.

Reflectivity and transmissivity

Equation (8.160) is the balance of energy flux across the interface. After substitution of the fluxes (8.161)-(8.164), we obtain

$$\mathrm{Re}(Z^I) = -\mathrm{Re}(Z^R)|R|^2 + \mathrm{Re}(Z^T)|T|^2 - 2\mathrm{Im}(Z^I)\mathrm{Im}(R). \tag{8.188}$$

Let us consider the isotropic and lossy case and an incident homogeneous plane wave. Thus, $p_{46} = 0$, $p_{44} = p_{66} = \mu$, where μ is complex, and equations $(6.10)_2$ and (6.28) imply $Z = \sqrt{\rho\mu}\cos\theta$. Then, equation (8.188) becomes

$$\mathrm{Re}(\sqrt{\rho\mu})\cos\theta^I = |R|^2\mathrm{Re}\,(\sqrt{\rho\mu})\cos\theta^I + |T|^2\mathrm{Re}\left[\mathrm{pv}\left(\sqrt{\rho'\mu'}\,\sqrt{1-\frac{\rho\mu'}{\rho'\mu}\sin^2\theta^I}\right)\right]$$

$$-2\mathrm{Im}(R)\mathrm{Im}(\sqrt{\rho\mu})\cos\theta^I, \tag{8.189}$$

where we have used equations (6.36) and (8.153)-(8.157). For lossless media, the interference flux – the last term on the right-hand side – vanishes, because μ is real. Moreover, using Snell's law (8.183) we obtain

$$1 = \mathcal{R} + \mathcal{T}, \tag{8.190}$$

where

$$\mathcal{R} = |R|^2 \quad \text{and} \quad \mathcal{T} = \sqrt{\frac{\rho'\mu'}{\rho\mu}}\frac{\cos\theta^T}{\cos\theta^I}|T|^2 \tag{8.191}$$

are called the reflectivity and transmissivity, respectively. Using the analogy (8.166) and assuming $\rho = \rho'$ and $\hat{\mu}' = \hat{\mu}$, we obtain

$$\mathcal{T} = \frac{n_a'}{n_a}\frac{\cos\theta^T}{\cos\theta^I}|T|^2 = \frac{n'}{n}\frac{\cos\theta^T}{\cos\theta^I}|T|^2 \tag{8.192}$$

(Born and Wolf, 1964, p. 41), where n and n_a are defined in equations (8.167) and (8.168), respectively.

Dual fields

The reflection and refraction coefficients that we have obtained above correspond to the particle-velocity field or, to be more precise, to the displacement field (due to the factor $\mathrm{i}\omega$ in equations (8.150) and (8.151)). In order to obtain the reflection coefficients for the stress components, we should make use of the constitutive equations, which for the plane wave are

$$\sigma_{12} = -Xv_2, \quad \text{and} \quad \sigma_{32} = -Zv_2 \tag{8.193}$$

(see equations (6.4)), where Z and X are defined in equations (8.153) and (8.159), respectively. Let us consider the reflected wave. Combining equation (8.150) and (8.193) we obtain

$$\begin{aligned}\sigma_{12}^R &= R_{12}\exp[\mathrm{i}\omega(t - s_1x - s_3^Rz)],\\ \sigma_{32}^R &= R_{32}\exp[\mathrm{i}\omega(t - s_1x - s_3^Rz)],\end{aligned} \tag{8.194}$$

where

$$R_{12} = -\mathrm{i}\omega X^R R \quad \text{and} \quad R_{32} = -\mathrm{i}\omega Z^R R \tag{8.195}$$

are the stress reflection coefficients.

In isotropic and lossless media, we have

$$R_{12} = -\mathrm{i}\omega\sqrt{\rho\mu}\sin\theta^I R \quad \text{and} \quad R_{32} = \mathrm{i}\omega\sqrt{\rho\mu}\cos\theta^I R. \tag{8.196}$$

where we have used equations $(8.153)_1$, $(8.159)_2$, $s_1 = \sin\theta^I\sqrt{\rho/\mu}$ (see equation $(6.27)_1$), and $Z^R = -Z^I$ (see equation (6.36)).

The analogies (8.31), (8.35) and (8.166) imply

$$E_3 = -\mathrm{i}\omega\sqrt{\frac{\hat{\mu}}{\hat{\epsilon}}}\sin\theta^I R \quad \text{and} \quad E_1 = -\mathrm{i}\omega\sqrt{\frac{\hat{\mu}}{\hat{\epsilon}}}\cos\theta^I R \tag{8.197}$$

(Born and Wolf, 1964, p. 39).

Sound waves

There is a mathematical analogy between the TM equations and a modified version of the acoustic wave equation for fluids. Denoting the pressure field by p, the modified acoustic equations can be written as

$$\partial_1 v_1 + \partial_3 v_3 = -\kappa_f \partial_t p, \tag{8.198}$$

$$-\partial_1 p = \gamma v_1 + \rho\partial_t v_1, \tag{8.199}$$

$$-\partial_3 p = \gamma v_3 + \rho\partial_t v_3, \tag{8.200}$$

where κ_f is the fluid compressibility, and $\gamma = 0$ yields the standard acoustic equations of motion. Equations (8.198)-(8.200) correspond to a generalized density of the form

$$\tilde{\rho}(t) = \gamma I(t) + \rho H(t), \tag{8.201}$$

where $H(t)$ is Heaviside's function and $I(t)$ is the integral operator. The acceleration term for, say, the x-component is

$$\gamma v_1 + \rho\partial_t v_1 = \partial_t\tilde{\rho}(t) * \partial_t v_1. \tag{8.202}$$

Equations (8.198)-(8.200) are mathematically analogous to the isotropic version of the electromagnetic equations (8.23)-(8.25) for the following correspondence

$$\begin{array}{ccr} \text{TM} & \Leftrightarrow & \text{Fluid} \\ H_2 & \Leftrightarrow & -p \\ E_3 & \Leftrightarrow & v_1 \\ E_1 & \Leftrightarrow & -v_3 \\ \hat{\epsilon} & \Leftrightarrow & \rho \\ \hat{\sigma} & \Leftrightarrow & \gamma \\ \hat{\mu} & \Leftrightarrow & \kappa_f, \end{array} \tag{8.203}$$

where $M_2 = 0$ has been assumed. Let us assume a lossless electromagnetic medium, and consider Snell's law (8.183) and the analogy between the SH and TM waves. That is, transform equation (8.165) to the TM equations by using the analogies $\mu^{-1} \Leftrightarrow \hat{\epsilon}$ and $\rho \Leftrightarrow \hat{\mu}$. In order to apply the mathematical analogies correctly, we need to recast the reflection coefficients as a function of the material properties and incidence angle. We obtain

$$R = \left(\sqrt{\frac{\hat{\mu}}{\hat{\epsilon}}}\cos\theta^I - \sqrt{\frac{\hat{\mu}'}{\hat{\epsilon}'}}\sqrt{1 - \frac{\hat{\mu}\hat{\epsilon}}{\hat{\mu}'\hat{\epsilon}'}\sin^2\theta^I}\right)\left(\sqrt{\frac{\hat{\mu}}{\hat{\epsilon}}}\cos\theta^I + \sqrt{\frac{\hat{\mu}'}{\hat{\epsilon}'}}\sqrt{1 - \frac{\hat{\mu}\hat{\epsilon}}{\hat{\mu}'\hat{\epsilon}'}\sin^2\theta^I}\right)^{-1}. \tag{8.204}$$

If $\kappa_f^{-1} = \rho c^2$, where c is the sound-wave velocity, application of the analogy (8.203) to equation (8.204) implies

$$R = \frac{\rho' c' \cos\theta^I - \rho c \cos\theta^T}{\rho' c' \cos\theta^I + \rho c \cos\theta^T}, \tag{8.205}$$

where we have used Snell's law for acoustic media

$$\frac{\sin\theta^I}{c} = \frac{\sin\theta^T}{c'}. \tag{8.206}$$

If we assume $\rho = \rho'$ and use Snell's law again, we obtain

$$R = \frac{\sin(\theta^T - \theta^I)}{\sin(\theta^T + \theta^I)}, \tag{8.207}$$

which is the reflection coefficient for light polarized perpendicular to the plane of incidence (the electric vector perpendicular to the plane of incidence), as we shall see in the next section. Note that we started with the TM equation, corresponding to the electric vector lying in the plane of incidence.

8.4.3 The analogy between TM and TE waves

The TE (transverse-electric) differential equations for an isotropic and lossless medium are

$$\partial_3 H_1 - \partial_1 H_3 = \hat{\epsilon}\partial_t E_2, \tag{8.208}$$

$$\partial_3 E_2 = \hat{\mu}\partial_t H_1, \tag{8.209}$$

$$-\partial_1 E_2 = \hat{\mu}\partial_t H_3. \tag{8.210}$$

The isotropic version of equations (8.23)-(8.25) and (8.208)-(8.210) are mathematically analogous for the following correspondence

$$\begin{array}{ccc} \mathrm{TM} & \Leftrightarrow & \mathrm{TE} \\ H_2 & \Leftrightarrow & -E_2 \\ E_1 & \Leftrightarrow & H_1 \\ E_3 & \Leftrightarrow & H_3 \\ \hat{\epsilon} & \Leftrightarrow & \hat{\mu} \\ \hat{\mu} & \Leftrightarrow & \hat{\epsilon}. \end{array} \tag{8.211}$$

From equation (8.204), and using the analogy (8.211) and Snell's law (8.183), the TE reflection coefficient is

$$R = \left(\sqrt{\frac{\hat{\epsilon}}{\hat{\mu}}}\cos\theta^I - \sqrt{\frac{\hat{\epsilon}'}{\hat{\mu}'}}\cos\theta^T\right)\left(\sqrt{\frac{\hat{\epsilon}}{\hat{\mu}}}\cos\theta^I + \sqrt{\frac{\hat{\epsilon}'}{\hat{\mu}'}}\cos\theta^T\right)^{-1}. \tag{8.212}$$

Assuming $\hat{\mu}' = \hat{\mu}$ and using again Snell's law, we obtain

$$R = \frac{\sin(\theta^T - \theta^I)}{\sin(\theta^T + \theta^I)}. \tag{8.213}$$

This is the reflection coefficient for the electric vector-component E_2, i.e., light polarized perpendicular to the plane of incidence. Note that R for H_2 (equation (8.172)) and R for E_2 (equation (8.213)) have different functional dependences in terms of the incidence and refraction angles.

From equation (8.175) and using the analogy (8.211), the TE Brewster angle is

$$\tan\theta_B = \sqrt{\frac{\hat{\epsilon}\hat{\mu}'/\hat{\mu} - \hat{\epsilon}'}{\hat{\epsilon}' - \hat{\epsilon}\hat{\mu}/\hat{\mu}'}}. \tag{8.214}$$

In the case of non-magnetic media, $\hat{\mu} = \hat{\mu}' = 1$, there is no TE Brewster angle.

Green's analogies

On December 11, 1837, Green read two papers to the Cambridge Philosophical Society. The first paper (Green, 1838) makes the analogy between sound waves and light waves polarized in the plane of incidence. To obtain his analogy, we establish the following correspondence between the acoustic equations (8.198)-(8.200) and the TE equations (8.208)-(8.210):

$$\begin{array}{ccr} \text{TE} & \Leftrightarrow & \text{Fluid} \\ E_2 & \Leftrightarrow & -p \\ H_1 & \Leftrightarrow & v_3 \\ H_3 & \Leftrightarrow & -v_1 \\ \hat{\epsilon} & \Leftrightarrow & \kappa_f \\ \hat{\mu} & \Leftrightarrow & \rho, \end{array} \tag{8.215}$$

where we have assumed that $\gamma = 0$. Using Snell's law (8.183), the TE reflection coefficient (8.212) can be rewritten as

$$R = \left(\sqrt{\frac{\hat{\epsilon}}{\hat{\mu}}}\cos\theta^I - \sqrt{\frac{\hat{\epsilon}'}{\hat{\mu}'}}\sqrt{1 - \frac{\hat{\mu}\hat{\epsilon}}{\hat{\mu}'\hat{\epsilon}'}\sin^2\theta^I}\right)\left(\sqrt{\frac{\hat{\epsilon}}{\hat{\mu}}}\cos\theta^I - \sqrt{\frac{\hat{\epsilon}'}{\hat{\mu}'}}\sqrt{1 - \frac{\hat{\mu}\hat{\epsilon}}{\hat{\mu}'\hat{\epsilon}'}\sin^2\theta^I}\right)^{-1}. \tag{8.216}$$

If we apply the analogy (8.215) to this equation and Snell's law (8.206), we obtain equation (8.205). Green obtained the reflection coefficient for the potential field, and assumed $\kappa_f = \kappa_f'$ or

$$\frac{\rho c}{\rho' c'} = \frac{c'}{c}. \tag{8.217}$$

Using this condition, Snell's law (8.206) and equation (8.205), we obtain

$$R = \frac{\sin\theta^I\cos\theta^I - \sin\theta^T\cos\theta^T}{\sin\theta^I\cos\theta^I + \sin\theta^T\cos\theta^T} = \frac{\tan(\theta^I - \theta^T)}{\tan(\theta^I + \theta^T)}, \tag{8.218}$$

which is the same ratio as for light polarized in the plane of incidence. Green (1838) has the opposite convention for describing the polarization direction. i.e., his convention is to denote R as given by equation (8.218) as the reflection coefficient for light polarized perpendicular to the plane of incidence.

Conversely, he considers the reflection coefficient (8.207) to correspond to light polarized in the plane of incidence. This is a convention dictated probably by the experiments performed by Malus, Brewster (1815) and Faraday, since Green did not know that light is a phenomenon related to the electric and magnetic fields – a relation that was discovered by Maxwell nearly 30 years later (Maxwell, 1865). Note that different assumptions lead to the different electromagnetic reflection coefficients. Assuming $\rho = \rho'$, we obtain the reflection coefficient for light polarized perpendicular to the plane of incidence (equation (8.207)), and assuming $\kappa_f = \kappa'_f$, we obtain the reflection coefficient for light polarized in the plane of incidence (equation (8.218)).

Green's second paper (Green, 1842) is an attempt to obtain the electromagnetic reflection coefficients by using the equations of elasticity (isotropic case). Firstly, he considers the SH-wave equation (Green's equations (7) and (8)) and the boundary conditions for the case $\mu = \mu'$ (his equation (9)). He obtains equation $(8.165)_1$ for the displacement reflection coefficient. If we use the condition (8.217) and Snell's law (8.206), we obtain precisely equation (8.207). i.e., the reflection coefficient for light polarized perpendicular to the plane of incidence – in the plane of incidence according to Green.

Secondly, Green considers the P-SV equation of motion in terms of the potential fields (Green's equations (14) and (16)), and makes the following assumptions

$$\rho c_P^2 = \rho' {c'_P}^2, \qquad \rho c_S^2 = \rho' {c'_S}^2, \tag{8.219}$$

that is, the P- and S-wave moduli are the same for both media. This condition implies

$$\frac{c_P}{c_S} = \frac{c'_P}{c'_S}, \tag{8.220}$$

which means that both media have the same Poisson ratio. Conversely, relation (8.220) implies that the P- and S-wave velocity contrasts are similar:

$$\frac{c_P}{c'_P} = \frac{c_S}{c'_S} \equiv w. \tag{8.221}$$

Green is aware – on the basis of experiments – that light waves with polarization perpendicular to the wave front were not observed experimentally. He writes: *But in the transmission of light through a prism, though the wave which is propagated by normal vibrations were incapable itself of affecting the eye, yet it would be capable of giving rise to an ordinary wave of light propagated by transverse vibrations....* He is then constrained to assume that $c_P \gg c_S$, that is, according to his own words, *that in the luminiferous ether, the velocity of transmission of waves propagated by normal vibrations, is very great compared with that of ordinary light.* The implications of this constraint will be clear below.

The reflection coefficient obtained by Green (1842), for the shear potential and an incident shear wave, has the following expression using our notation:

$$R^2 = \frac{r_-}{r_+}, \qquad r_\pm = (w^2+1)^2 \left(w^2 \pm \frac{s_{3S}^T}{s_{3S}^I} \right)^2 + (w^2-1)^4 \frac{s_1^2}{{s_{3S}^I}^2} \tag{8.222}$$

(Green's equation (26)), where s_{3S}^I and s_{3S}^T are the vertical components of the slowness vector corresponding to the S wave. On the basis of the condition $c_P \gg c_S$, Green assumed that the vertical components of the slowness vector corresponding to the incident, reflected and refracted P waves satisfy

$$\mathrm{i}s_{3P}^I = -\mathrm{i}s_{3P}^R = \mathrm{i}s_{3P}^T = s_1. \tag{8.223}$$

These relations can be obtained from the dispersion relation $s_1^2 + s_3^2 = \omega/c_P^2$ of each wave assuming $c_P \to \infty$. This assumption gives an incompressible medium and inhomogeneous P waves confined at the interface. The complete expression for the SS reflection coefficients are given, for instance, in Pilant (1979, p. 137) [2]. He defines $a = c_S/c_P$ and $c = c_S/c'_P$. Green's solution (8.222) is obtained for $a = c = 0$.

The vertical components of the shear slowness vector are given by

$$s_{3S}^I = \sqrt{\frac{1}{c_S^2} - s_1^2}, \qquad s_{3S}^T = \sqrt{\frac{1}{{c'_S}^2} - s_1^2}. \tag{8.224}$$

However, equation (8.222) is not Fresnel's reflection coefficient. To obtain this equation, Green assumed that $w \approx 1$; in his own words: *When the refractive power in passing from the upper to the lower medium is not very great, w (*μ using his notation*) does not differ much from 1.* The result of applying this approximation to equation (8.222) is

$$R = \left(w^2 - \frac{s_{3S}^T}{s_{3S}^I}\right)\left(w^2 + \frac{s_{3S}^T}{s_{3S}^I}\right)^{-1}. \tag{8.225}$$

If θ^I is the incidence angle of the shear wave and θ^T is the angle of the refracted shear wave, equation (8.221), Snell's law and the relation

$$\frac{s_{3S}^T}{s_{3S}^I} = \frac{\cot\theta^T}{\cot\theta^I} \tag{8.226}$$

(which can be obtained by using equation (8.224) and Snell's law), yield

$$R = \left(\frac{\sin^2\theta^I}{\sin^2\theta^T} - \frac{\cot\theta^T}{\cot\theta^I}\right)\left(\frac{\sin^2\theta^I}{\sin^2\theta^T} + \frac{\cot\theta^T}{\cot\theta^I}\right)^{-1} = \frac{\sin 2\theta^I - \sin 2\theta^T}{\sin 2\theta^I + \sin 2\theta^T} = \frac{\tan(\theta^I - \theta^T)}{\tan(\theta^I + \theta^T)}, \tag{8.227}$$

which is the reflection coefficient for light polarized in the plane of incidence. Green considers that equation (8.227) is an approximation of the observed reflection coefficients. He claims, on the basis of experimental data, that *the intensity of the reflected light never becomes absolutely null, but attains a minimum value.* Moreover, he calculates the minimum value of the reflection coefficient and obtains

$$R_{\min}^2 = \frac{(w^2-1)^4}{4w^2(w^2+1)^2 + (w^2-1)^4}, \tag{8.228}$$

which using the approximation $w \approx 1$ gives zero reflection coefficient. This minimum value corresponds to the Brewster angle when using the Fresnel coefficient (8.227). Green

[2]Note a mistake in Pilant's equation (12-21): the (43)-coefficient of matrix Δ_s should be $-2\sin\theta_{S1}\sqrt{c^2 - \sin^2\theta_{S1}}/(b^2d)$ instead of $-2\sin\theta_{S1}\sqrt{a^2 - \sin^2\theta_{S1}}/(b^2d)$.

assumed $w = 4/3$ for the air-water interface. The absolute values of the reflection coefficient R given by equations (8.222) and (8.227) are shown in Figure 8.6. The dashed line correspond to equation (8.222). We have assumed $c_S = 30$ cm/ns and $c'_S = c_S/w$. At the Brewster angle ($\theta = \text{atan}(w)$), Green obtained a minimum value $R_{\min} = 0.08138$.

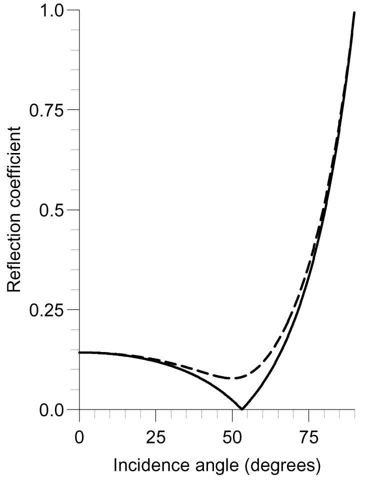

Figure 8.6: Green's reflection coefficient for light polarized in the plane of incidence (dashed line) and corresponding Fresnel's reflection coefficient (solid line).

The non-existence of the Brewster angle (zero reflection coefficient), can be explained by the presence of dissipation (ionic conductivity effects), as can be seen in Figure 8.5. Green attributes this to the fact that the refraction medium is highly refracting. Quoting him: *This minimum value* [$R_{\min}$] *increases rapidly, as the index of refraction increases, and thus the quantity of light reflected at the polarizing* [Brewster] *angle, becomes considerable for highly refracting substances, a fact which has been long known to experimental philosophers* (Green, 1842). For instance, fresh water is almost lossless and is a less refracting medium than salt water, which has a higher conductivity.

8.4.4 Brief historical review

We have seen in the previous section that Green's theory of refraction does not provide an exact parallel with the phenomenon of light propagation. MacCullagh (Trans. Roy. Irish. Acad., xxi, 1848; Whittaker, 1987, p. 141) presented an alternative approach to the Royal Irisih Academy in 1839. He devised an isotropic medium, whose potential energy is only based on rotation of the volume elements, thus ignoring pure dilatations from the beginning. The result is a rotationally elastic ether and the wave equation for shear waves. The corresponding reflection and refraction coefficients coincide with Fresnel's formulae.

Green (1842) assumed the P-wave velocity to be infinite and dismissed a zero P-wave velocity on the basis that the medium would be unstable (the potential energy must be positive). Cauchy (Comptes Rendus, ix (25 Nov. 1839), p. 676, and (2 Dec. 1839), p. 726; Whittaker, 1987, p. 145), neglecting this fact, considered that P waves have zero velocity, and obtained the sine law and tangent law of Fresnel. He assumed the shear modulus to be the same for both media. Cauchy's ether is known as the *contractile* or *labile aether*. It corresponds to an elastic medium of negative compressibility. The P-wave dispersion relation for this medium is $s_1^2 + s_3^2 = 0$, which leads to an infinite vertical slowness. This condition confines the propagation direction of the compressional waves to be normal to the interface. The energy carried away by the P waves is negligible, since no work is required to generate a dilatational displacement, due to the negative value of the compressibility. If we assume the shear modulus of both media to be the same (the differences depend on density contrasts only), we obtain Fresnel's formulae. The advantage of the labile ether is that it overcomes the difficulty of requiring continuity of the normal component of the displacement at the interface. Light waves do not satisfy this condition, but light waves plus dilatational vibrations, taken together, do satisfy the condition.

8.5 3-D electromagnetic theory and the analogy

We cannot establish a complete mathematical analogy in three-dimensional space, but we can extend Maxwell's equations to include magnetic and dielectric-relaxation processes and out-of-phase electric currents using viscoelastic models. The approach, based on the introduction of memory or hidden variables, uses the analogy between the Zener and Debye models (see Section 8.3.2), and a single Kelvin-Voigt element to describe the out-of-phase behaviour of the electric conductivity (any deviation from Ohm's law). We assume that the medium is orthorhombic; i.e., that the principal systems of the three material tensors coincide and that a different relaxation function is associated with each principal component. The physics is investigated by probing the medium with a *uniform* (homogeneous) plane wave. This analysis gives the expressions of measurable quantities, like the energy velocity and the quality factor, as a function of propagation direction and frequency.

In orthorhombic media, $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\epsilon}}$ and $\hat{\boldsymbol{\sigma}}$ have coincident eigenvectors. Rotating to a coordinate system defined by those common eigenvectors, allows the tensors to be written as

$$\hat{\boldsymbol{\mu}} = \begin{pmatrix} \hat{\mu}_1 & 0 & 0 \\ 0 & \hat{\mu}_2 & 0 \\ 0 & 0 & \hat{\mu}_3 \end{pmatrix}, \quad \hat{\boldsymbol{\epsilon}} = \begin{pmatrix} \hat{\epsilon}_1 & 0 & 0 \\ 0 & \hat{\epsilon}_2 & 0 \\ 0 & 0 & \hat{\epsilon}_3 \end{pmatrix} \quad \text{and} \quad \hat{\boldsymbol{\sigma}} = \begin{pmatrix} \hat{\sigma}_1 & 0 & 0 \\ 0 & \hat{\sigma}_2 & 0 \\ 0 & 0 & \hat{\sigma}_3 \end{pmatrix}. \tag{8.229}$$

The following symmetries are embraced by the term *orthotropy*: i) *Orthorhombic*, for which there are no two eigendirections for which all three tensors have equal eigenvalues. Crystals of this kind are said to be optically *biaxial*; ii) *Transverse isotropy*, for which there are two eigendirections, and only two, for which all three tensors have equal eigenvalues, e.g., if the two directions are the x- and y-directions, then $\hat{\mu}_1 = \hat{\mu}_2$, $\hat{\epsilon}_1 = \hat{\epsilon}_2$ and $\hat{\sigma}_1 = \hat{\sigma}_2$. This electromagnetic symmetry includes that of hexagonal, tetragonal and trigonal crystals. These are said to be optically *uniaxial*; iii) *Isotropy*, for which all three tensors have three equal eigenvalues, i.e., they are all isotropic tensors. Crystals of cubic symmetry are electromagnetically isotropic.

For the sake of simplicity in the evaluation of the final equations, we consider a Cartesian system that coincides with the principal system of the medium. The electromagnetic equations (8.6) and (8.7) in Cartesian components are

$$
\begin{aligned}
\partial_3 E_2 - \partial_2 E_3 &= \hat{\mu}_1 * \partial^2_{tt} H_1 + M_1 \\
\partial_1 E_3 - \partial_3 E_1 &= \hat{\mu}_2 * \partial^2_{tt} H_2 + M_2 \\
\partial_2 E_1 - \partial_1 E_2 &= \hat{\mu}_3 * \partial^2_{tt} H_3 + M_3 \\
\partial_2 H_3 - \partial_3 H_2 &= \hat{\sigma}_1 * \partial_t E_1 + \hat{\epsilon}_1 * \partial^2_{tt} E_1 + J_1 \\
\partial_3 H_1 - \partial_1 H_3 &= \hat{\sigma}_2 * \partial_t E_2 + \hat{\epsilon}_2 * \partial^2_{tt} E_2 + J_2 \\
\partial_1 H_2 - \partial_2 H_1 &= \hat{\sigma}_3 * \partial_t E_3 + \hat{\epsilon}_3 * \partial^2_{tt} E_3 + J_3 .
\end{aligned}
\qquad (8.230)
$$

8.5.1 The form of the tensor components

The principal components of the dielectric-permittivity tensor can be expressed as

$$
\hat{\epsilon}_i(t) = \hat{\epsilon}_i^0 \left[1 - \frac{1}{L_i} \sum_{l=1}^{L_i} \left(1 - \frac{\lambda_{il}}{\tau_{il}} \right) \exp(-t/\tau_{il}) \right] H(t), \qquad i = 1, \ldots, 3, \qquad (8.231)
$$

where $\hat{\epsilon}_i^0$ is the static dielectric permittivity, λ_{il} and τ_{il} are relaxation times ($\lambda_{il} \leq \tau_{il}$), and L_i is the number of Debye relaxation mechanisms. The condition $\lambda_{il} \leq \tau_{il}$ makes the relaxation function (8.231) analogous to the viscoelastic creep function corresponding to Zener elements connected in series (see Section 2.4.5 and Casula and Carcione (1992)). The optical (or high-frequency) dielectric permittivity

$$
\hat{\epsilon}_i^\infty = \frac{\hat{\epsilon}_i^0}{L_i} \sum_{l=1}^{L_i} \frac{\lambda_{il}}{\tau_{il}} \qquad (8.232)
$$

is obtained as $t \to 0$. Note that $\hat{\epsilon}_i^\infty \leq \hat{\epsilon}_i^0$.

Similarly, the principal components of the magnetic-permeability tensor can be written as

$$
\hat{\mu}_i(t) = \hat{\mu}_i^0 \left[1 - \frac{1}{N_i} \sum_{n=1}^{N_i} \left(1 - \frac{\gamma_{in}}{\theta_{in}} \right) \exp(-t/\theta_{in}) \right] H(t), \qquad n = 1, \ldots, 3, \qquad (8.233)
$$

where $\hat{\mu}_i^0$ is the static permeability, γ_{il} and θ_{il} are relaxation times ($\gamma_{il} \leq \theta_{il}$), and N_i is the number of Debye relaxation mechanisms.

On the other hand, the conductivity components are represented by a Kelvin-Voigt mechanical model (see Section 2.4.2):

$$
\hat{\sigma}_i(t) = \hat{\sigma}_i^0 [H(t) + \chi_i \delta(t)], \qquad i = 1 \ldots 3, \qquad (8.234)
$$

where $\hat{\sigma}_i^0$ is the static conductivity, χ_i is a relaxation time and $\delta(t)$ is Dirac's function. The out-of-phase component of the conduction current is quantified by the relaxation time χ_i. This choice implies a component of the conduction current 90° out-of-phase with respect to the electric field.

8.5.2 Electromagnetic equations in differential form

Equations (8.230) could be the basis for a numerical solution algorithm. However, the numerical evaluation of the convolution integrals is prohibitive when solving the differential equations with grid methods and explicit time-evolution techniques. The conductivity terms pose no problems, since conductivity does not involve time convolution. To circumvent the convolutions in the dielectric-permittivity and magnetic-permeability components, a new set of field variables is introduced, following the same approach as in Section 3.9.

The dielectric internal (hidden) variables, which are analogous to the memory variables of viscoelastic media, are defined as

$$e_{il} = -\frac{1}{\tau_{il}}\phi_{il} * E_i, \qquad l = 1, \ldots, L_i, \tag{8.235}$$

where $i = 1, \ldots, 3$, and

$$\phi_{il}(t) = \frac{H(t)}{L_i \tau_{il}}\left(1 - \frac{\lambda_{il}}{\tau_{il}}\right)\exp(-t/\tau_{il}), \qquad l = 1, \ldots, L_i. \tag{8.236}$$

Similarly, the magnetic hidden variables are

$$d_{in} = -\frac{1}{\theta_{in}}\varphi_{in} * H_i, \qquad l = 1, \ldots, N_i, \tag{8.237}$$

where

$$\varphi_{in}(t) = \frac{H(t)}{N_i \theta_{in}}\left(1 - \frac{\gamma_{in}}{\theta_{in}}\right)\exp(-t/\theta_{in}), \qquad n = 1, \ldots, N_i \tag{8.238}$$

(there is no implicit summation in equations (8.235)-(8.238)).

Following the same procedure as in Section 3.9, the electromagnetic equations in differential form become

$$\begin{aligned}
\partial_3 E_2 - \partial_2 E_3 &= \hat{\mu}_1^\infty \partial_t H_1 + \hat{\mu}_1^0 \left[\Psi_1 H_1 + \textstyle\sum_{n=1}^{N_1} d_{1n}\right] + M_1 \\
\partial_1 E_3 - \partial_3 E_1 &= \hat{\mu}_2^\infty \partial_t H_2 + \hat{\mu}_2^0 \left[\Psi_2 H_2 + \textstyle\sum_{n=1}^{N_2} d_{2n}\right] + M_2 \\
\partial_2 E_1 - \partial_1 E_2 &= \hat{\mu}_3^\infty \partial_t H_3 + \hat{\mu}_3^0 \left[\Psi_3 H_3 + \textstyle\sum_{n=1}^{N_3} d_{3n}\right] + M_3 \\
\partial_2 H_3 - \partial_3 H_2 &= \hat{\sigma}_{e1}^\infty E_1 + \hat{\epsilon}_{e1}^\infty \partial_t E_1 + \hat{\epsilon}_1^0 \textstyle\sum_{l=1}^{L_1} e_{1l} + J_1 \\
\partial_3 H_1 - \partial_1 H_3 &= \hat{\sigma}_{e2}^\infty E_2 + \hat{\epsilon}_{e2}^\infty \partial_t E_2 + \hat{\epsilon}_2^0 \textstyle\sum_{l=1}^{L_2} e_{2l} + J_2 \\
\partial_1 H_2 - \partial_2 H_1 &= \hat{\sigma}_{e3}^\infty E_3 + \hat{\epsilon}_{e3}^\infty \partial_t E_3 + \hat{\epsilon}_3^0 \textstyle\sum_{l=1}^{L_3} e_{3l} + J_3,
\end{aligned} \tag{8.239}$$

where

$$\hat{\epsilon}_{ei}^\infty = \hat{\epsilon}_i^\infty + \hat{\sigma}_i^0 \chi_i \tag{8.240}$$

and

$$\hat{\sigma}_{ei}^\infty = \hat{\sigma}_i^0 + \hat{\epsilon}_i^0 \Phi_i \tag{8.241}$$

are the effective optical dielectric-permittivity and conductivity components, respectively, with

$$\Psi_i = \sum_{n=1}^{N_i} \varphi_{in}(0) \quad \text{and} \quad \Phi_i = \sum_{l=1}^{L_i} \phi_{il}(0). \tag{8.242}$$

The first two terms on the right side of the last three of equations (8.239) correspond to the instantaneous response of the medium, as can be inferred from the relaxation functions (8.231) and (8.234). Note that the terms containing the conductivity relaxation time χ_i are in phase with the instantaneous polarization response. The third terms in each equation involve the relaxation processes through the hidden variables.

The set of equations is completed with the differential equations corresponding to the hidden variables. Time differentiation of equations (8.235) and (8.237), and the use of convolution properties, yield

$$\partial_t e_{il} = -\frac{1}{\tau_{il}} \left[e_{il} + \phi_{il}(0) E_i \right], \qquad l = 1, \ldots, L_i, \tag{8.243}$$

and

$$\partial_t d_{in} = -\frac{1}{\theta_{in}} \left[d_{in} + \varphi_{in}(0) H_i \right], \qquad n = 1, \ldots, N_i. \tag{8.244}$$

Equations (8.239), (8.243) and (8.244) give the electromagnetic response of a conducting anisotropic medium with magnetic and dielectric-relaxation behaviour and out-of-phase conduction currents. These equations are the basis of numerical algorithms for obtaining the unknown vector field

$$\bar{\mathbf{v}} = [H_1, H_2, H_3, E_1, E_2, E_3, \{e_{il}\}, \{d_{in}\}]^\top, \quad i = 1, \ldots, 3, \quad l = 1, \ldots, L_i, \quad n = 1, \ldots, N_i. \tag{8.245}$$

8.6 Plane-wave theory

The plane-wave analysis gives the expressions of measurable quantities, such as the slowness vector, the energy-velocity vector and the quality factor as a function of frequency. Assume *non-uniform* (inhomogeneous) harmonic plane waves with a phase factor

$$\exp[\mathrm{i}\omega(t - \mathbf{s} \cdot \mathbf{x})], \tag{8.246}$$

where $\mathbf{s}$ is the complex slowness vector. We use the following correspondences between time and frequency domains:

$$\nabla \times \to -\mathrm{i}\omega \mathbf{s} \times \quad \text{and} \quad \partial_t \to \mathrm{i}\omega. \tag{8.247}$$

Substituting the plane wave (8.246) into Maxwell's equations (8.6) and (8.7), in the absence of sources, and using (8.247) gives

$$\mathbf{s} \times \mathbf{E} = \hat{\boldsymbol{\mu}} \cdot \mathbf{H}, \tag{8.248}$$

and

$$\mathbf{s} \times \mathbf{H} = -\bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}, \tag{8.249}$$

where

$$\mathcal{F}[\partial_t \hat{\boldsymbol{\mu}}] \to \hat{\boldsymbol{\mu}} \tag{8.250}$$

and

$$\mathcal{F}[\partial_t \hat{\boldsymbol{\epsilon}}] - \frac{\mathrm{i}}{\omega} \mathcal{F}[\partial_t \hat{\boldsymbol{\sigma}}] \to \hat{\boldsymbol{\epsilon}} - \frac{\mathrm{i}}{\omega} \hat{\boldsymbol{\sigma}} \equiv \bar{\boldsymbol{\epsilon}}. \tag{8.251}$$

For convenience, the medium properties are denoted by the same symbols, in both the time and frequency domains.

Note that $\bar{\boldsymbol{\epsilon}}$ can alternatively be written as

$$\bar{\boldsymbol{\epsilon}} = \hat{\boldsymbol{\epsilon}}_e - \frac{\mathrm{i}}{\omega}\hat{\boldsymbol{\sigma}}_e, \tag{8.252}$$

where

$$\hat{\boldsymbol{\epsilon}}_e = \mathrm{Re}(\hat{\boldsymbol{\epsilon}}) + \frac{1}{\omega}\mathrm{Im}(\hat{\boldsymbol{\sigma}}) \tag{8.253}$$

and

$$\hat{\boldsymbol{\sigma}}_e = \mathrm{Re}(\hat{\boldsymbol{\sigma}}) - \omega\mathrm{Im}(\hat{\boldsymbol{\epsilon}}) \tag{8.254}$$

are the real effective dielectric-permittivity and conductivity matrices, respectively. The components of $\hat{\boldsymbol{\epsilon}}$ and $\hat{\boldsymbol{\sigma}}$ from equations (8.231) and (8.234) are

$$\hat{\epsilon}_i = \mathcal{F}\left(\partial_t \hat{\epsilon}_i\right) = \frac{\hat{\epsilon}_i^0}{L_i}\sum_{l=1}^{L_i}\frac{1+\mathrm{i}\omega\lambda_{il}}{1+\mathrm{i}\omega\tau_{il}} \tag{8.255}$$

and

$$\hat{\sigma}_i = \mathcal{F}\left(\partial_t \hat{\sigma}_i\right) = \hat{\sigma}_i^0(1+\mathrm{i}\omega\chi_i). \tag{8.256}$$

The dielectric-permittivity component (8.255) can be rewritten as equation (8.116:

$$\hat{\epsilon}_i = \hat{\epsilon}_i^{\infty} + \frac{1}{L_i}\sum_{l=1}^{L_i}\frac{\hat{\epsilon}_i^0 - \hat{\epsilon}_{il}^{\infty}}{1+\mathrm{i}\omega\tau_{il}}, \tag{8.257}$$

where $\hat{\epsilon}_{il}^{\infty} = \hat{\epsilon}_i^0\lambda_{il}/\tau_{il}$ is the infinite-frequency (optical) dielectric permittivity of the l-th relaxation mechanism. A similar expression is used in bio-electromagnetism (Petropoulos, 1995).

Similarly, from equation (8.233),

$$\hat{\mu}_i = \mathcal{F}\left(\partial_t \hat{\mu}_i\right) = \frac{\hat{\mu}_i^0}{N_i}\sum_{n=1}^{N_i}\frac{1+\mathrm{i}\omega\gamma_{in}}{1+\mathrm{i}\omega\theta_{in}}. \tag{8.258}$$

Since $\lambda_{il} \leq \tau_{il}$ implies $\mathrm{Im}(\hat{\epsilon}_i) \leq 0$ and $\mathrm{Re}(\hat{\sigma}_i) \geq 0$, the two terms on the right side of equation (8.254) have the same sign and the wave propagation is always dissipative. The importance of the effective matrices $\hat{\boldsymbol{\epsilon}}_e$ and $\hat{\boldsymbol{\sigma}}_e$ is that their components are the quantities that are measured in experiments. The coefficients multiplying the electric field and the time derivative of the electric field in equations (8.239) correspond to the components of $\hat{\boldsymbol{\sigma}}_e^{\infty}$ and $\hat{\boldsymbol{\epsilon}}_e^{\infty}$, respectively.

Taking the vector product of equation (8.248) with $\mathbf{s}$, gives

$$\mathbf{s} \times (\hat{\boldsymbol{\mu}}^{-1} \cdot \mathbf{s} \times \mathbf{E}) = \mathbf{s} \times \mathbf{H}, \tag{8.259}$$

which, with equation (8.249), becomes

$$\mathbf{s} \times (\hat{\boldsymbol{\mu}}^{-1} \cdot \mathbf{s} \times \mathbf{E}) + \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E} = 0, \tag{8.260}$$

for three equations for the components of $\mathbf{E}$. Alternatively, the vector product of equation (8.249) with $\mathbf{s}$ and use of (8.248) yields

$$\mathbf{s} \times [(\bar{\boldsymbol{\epsilon}})^{-1} \cdot \mathbf{s} \times \mathbf{H}] + \hat{\boldsymbol{\mu}} \cdot \mathbf{H} = 0, \tag{8.261}$$

for three equations for the components of $\mathbf{H}$.

From equation (8.260), the equivalent of the 3×3 Kelvin-Christoffel equations (see Sections 1.3 and 4.2), for the electric-vector components, are

$$(\epsilon_{ijk} s_j \hat{\mu}_{kl}^{-1} \epsilon_{lpq} s_p + \bar{\epsilon}_{iq}) E_q = 0, \quad i = 1, \ldots, 3, \tag{8.262}$$

where ϵ_{ijk} are the components of the Levi-Civita tensor.

Similarly, the equations for the magnetic-vector components are

$$(\epsilon_{ijk} s_j (\bar{\epsilon}_{kl})^{-1} \epsilon_{lpq} s_p + \hat{\mu}_{iq}) H_q = 0, \quad i = 1, \ldots, 3, \tag{8.263}$$

Both dispersion relations (8.262) and (8.263) are identical. Getting one relation from the other implies an interchange of $\bar{\epsilon}_{ij}$ and $\hat{\mu}_{ij}$ and vice versa.

So far, the dispersion relations correspond to a general triclinic medium. Consider the orthorhombic case given by equations (8.229). Then, the analogue of the Kelvin-Christoffel equation for the electric vector is

$$\boldsymbol{\Gamma} \cdot \mathbf{E} = \mathbf{0}, \tag{8.264}$$

where the Kelvin-Christoffel matrix is

$$\Gamma = \begin{pmatrix} \bar{\epsilon}_1 - \left(\dfrac{s_2^2}{\hat{\mu}_3} + \dfrac{s_3^2}{\hat{\mu}_2}\right) & \dfrac{s_1 s_2}{\hat{\mu}_3} & \dfrac{s_1 s_3}{\hat{\mu}_2} \\ \dfrac{s_1 s_2}{\hat{\mu}_3} & \bar{\epsilon}_2 - \left(\dfrac{s_1^2}{\hat{\mu}_3} + \dfrac{s_3^2}{\hat{\mu}_1}\right) & \dfrac{s_2 s_3}{\hat{\mu}_1} \\ \dfrac{s_1 s_3}{\hat{\mu}_2} & \dfrac{s_2 s_3}{\hat{\mu}_1} & \bar{\epsilon}_3 - \left(\dfrac{s_1^2}{\hat{\mu}_2} + \dfrac{s_2^2}{\hat{\mu}_1}\right) \end{pmatrix}. \tag{8.265}$$

After defining

$$\eta_i = \bar{\epsilon}_i \hat{\mu}_i, \quad \zeta_i = \bar{\epsilon}_j \hat{\mu}_k + \bar{\epsilon}_k \hat{\mu}_j, \quad j \neq k \neq i, \tag{8.266}$$

the 3-D dispersion relation (i.e. the vanishing of the determinant of the Kelvin-Christoffel matrix), becomes,

$$(\bar{\epsilon}_1 s_1^2 + \bar{\epsilon}_2 s_2^2 + \bar{\epsilon}_3 s_3^2)(\hat{\mu}_1 s_1^2 + \hat{\mu}_2 s_2^2 + \hat{\mu}_3 s_3^2) - (\eta_1 \zeta_1 s_1^2 + \eta_2 \zeta_2 s_2^2 + \eta_3 \zeta_3 s_3^2) + \eta_1 \eta_2 \eta_3 = 0. \tag{8.267}$$

There are only quartic and quadratic terms of the slowness components in the dispersion relation of an orthorhombic medium.

8.6.1 Slowness, phase velocity and attenuation

The slowness vector $\mathbf{s}$ can be split into real and imaginary vectors such that $\omega \mathrm{Re}(t - \mathbf{s} \cdot \mathbf{x})$ is the phase and $-\omega \mathrm{Im}(\mathbf{s} \cdot \mathbf{x})$ is the attenuation. Assume that propagation and attenuation directions coincide to produce a uniform plane wave, which is equivalent to a homogeneous plane wave in viscoelasticity. The slowness vector can be expressed as

$$\mathbf{s} = s(l_1, l_2, l_3)^\top \equiv s\hat{\mathbf{s}}, \tag{8.268}$$

where s is the complex slowness and $\hat{\mathbf{s}} = (l_1, l_2, l_3)^\top$ is a real unit vector, with l_i the direction cosines. We obtain the real wavenumber vector and the real attenuation vector as

$$\mathbf{s}_R = \mathrm{Re}(\mathbf{s}) \quad \text{and} \quad \boldsymbol{\alpha} = -\omega \mathrm{Im}(\mathbf{s}), \tag{8.269}$$

respectively. Substituting equation (8.268) into the dispersion relation (8.267) yields

$$As^4 - Bs^2 + \eta_1\eta_2\eta_3 = 0, \tag{8.270}$$

where

$$A = (\bar{\epsilon}_1 l_1^2 + \bar{\epsilon}_2 l_2^2 + \bar{\epsilon}_3 l_3^2)(\hat{\mu}_1 l_1^2 + \hat{\mu}_2 l_2^2 + \hat{\mu}_3 l_3^2)$$

and

$$B = \eta_1\zeta_1 l_1^2 + \eta_2\zeta_2 l_2^2 + \eta_3\zeta_3 l_3^2.$$

In terms of the complex velocity $v_c \equiv 1/s$, the magnitudes of the phase velocity and attenuation vectors are

$$v_p = \left[\mathrm{Re}\left(\frac{1}{v_c}\right)\right]^{-1} \quad \text{and} \quad \alpha = -\omega \mathrm{Im}\left(\frac{1}{v_c}\right), \tag{8.271}$$

respectively.

Assume, for instance, propagation in the (x, y)-plane. Then, $l_3 = 0$ and the dispersion relation (8.270) is factorizable, giving

$$[s^2(\bar{\epsilon}_1 l_1^2 + \bar{\epsilon}_2 l_2^2) - \bar{\epsilon}_1\bar{\epsilon}_2\hat{\mu}_3][s^2(\hat{\mu}_1 l_1^2 + \hat{\mu}_2 l_2^2) - \bar{\epsilon}_3\hat{\mu}_1\hat{\mu}_2] = 0. \tag{8.272}$$

These factors give the TM and TE modes with complex velocities

$$v_c(\mathrm{TM}) = \sqrt{\frac{1}{\hat{\mu}_3}\left(\frac{l_1^2}{\bar{\epsilon}_2} + \frac{l_2^2}{\bar{\epsilon}_1}\right)} \tag{8.273}$$

and

$$v_c(\mathrm{TE}) = \sqrt{\frac{1}{\bar{\epsilon}_3}\left(\frac{l_1^2}{\hat{\mu}_2} + \frac{l_2^2}{\hat{\mu}_1}\right)}. \tag{8.274}$$

In the TM (TE) case the magnetic (electric) vector is perpendicular to the propagation plane. For obtaining the slowness and complex velocities for the other planes, simply make the following subindex substitutions:

$$\begin{array}{lccc} \text{from the } (x,y)\text{−plane to the } (x,z)\text{−plane} & (1,2,3) & \rightarrow & (3,1,2), \\ \text{from the } (x,y)\text{−plane to the } (y,z)\text{−plane} & (1,2,3) & \rightarrow & (2,3,1). \end{array} \tag{8.275}$$

The analysis of all three planes of symmetry gives the slowness sections represented in Figure 8.7, where the values on the axes refer to the square of the complex slowness. There exists a single conical point given by the intersection of the TE and TM modes, as can be seen in the (x, z)-plane of symmetry. The location of the conical point depends on the values of the material properties. At the orthogonal planes, the waves are termed *ordinary* (circle) and *extraordinary* (ellipse). For the latter, the magnitude of the slowness vector is a function of the propagation direction. The result of two waves propagating at different velocities is called birefringence or double refraction (e.g., Kong, 1986), This phenomenon is analogous to shear-wave splitting in elastic wave propagation (see Section 1.4.4).

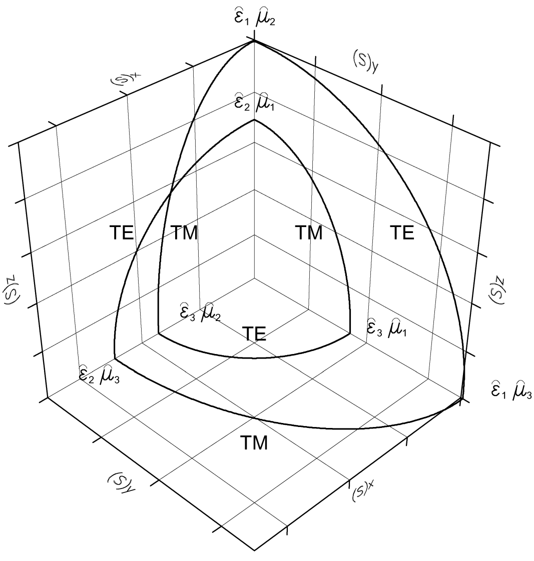

Figure 8.7: Intersection of the slowness surface with the principal planes. The corresponding waves are either transverse electric (TE) or transverse magnetic (TM). The values at the axes refer to the square of the complex slowness.

8.6.2 Energy velocity and quality factor

The scalar product of the complex conjugate of equation (8.249) with $\mathbf{E}$, use of the relation $2\mathrm{Im}(\mathbf{s}) \cdot (\mathbf{E} \times \mathbf{H}^*) = (\mathbf{s} \times \mathbf{E}) \cdot \mathbf{H}^* + \mathbf{E} \cdot (\mathbf{s} \times \mathbf{H})^*$ (that can be deduced from div $(\mathbf{E} \times \mathbf{H}^*) = (\nabla \times \mathbf{E}) \cdot \mathbf{H}^* - \mathbf{E} \cdot (\nabla \times \mathbf{H}^*)$ and equation (8.247)), and substitution of equation (8.248), gives Umov-Poynting's theorem for plane waves

$$-2\mathrm{Im}(\mathbf{s}) \cdot \mathbf{p} = 2\mathrm{i}(\langle E_e \rangle - \langle E_m \rangle) - \langle \dot{D}_e \rangle - \langle \dot{D}_m \rangle \tag{8.276}$$

where

$$\mathbf{p} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* \tag{8.277}$$

is the complex Umov-Poynting vector,

$$\langle E_e \rangle = \frac{1}{4}\mathrm{Re}[\mathbf{E} \cdot (\bar{\boldsymbol{\epsilon}} \cdot \mathbf{E})^*] \tag{8.278}$$

is the time-averaged electric-energy density,

$$\langle \dot{D}_e \rangle = \frac{\omega}{2}\mathrm{Im}[\mathbf{E} \cdot (\bar{\boldsymbol{\epsilon}} \cdot \mathbf{E})^*] \tag{8.279}$$

is the time-averaged rate of dissipated electric-energy density,

$$\langle E_m \rangle = \frac{1}{4}\mathrm{Re}[(\hat{\boldsymbol{\mu}} \cdot \mathbf{H}) \cdot \mathbf{H}^*] \tag{8.280}$$

is the time-averaged magnetic-energy density and

$$\langle \dot{D}_m \rangle = -\frac{\omega}{2} \mathrm{Im}[(\hat{\boldsymbol{\mu}} \cdot \mathbf{H}) \cdot \mathbf{H}^*] \tag{8.281}$$

is the time-averaged rate of dissipated magnetic-energy density. These expressions are generalizations to the anisotropic case of the equations given in Section 8.3.1.

The energy-velocity vector, $\mathbf{v}_e$, is given by the energy power flow, $\mathrm{Re}(\mathbf{p})$, divided by the total stored energy density,

$$\mathbf{v}_e = \frac{\mathrm{Re}(\mathbf{p})}{\langle E_e + E_m \rangle}. \tag{8.282}$$

As in the acoustic case, the relation (4.78) holds, i.e., $\hat{\mathbf{s}} \cdot \mathbf{v}_e = v_p$, where $\hat{\mathbf{s}}$ and v_p are defined in equations (8.268) and $(8.271)_1$, respectively.

The quality factor quantifies energy dissipation in matter from the electric-current standpoint. As stated by Harrington (1961, p. 28), the quality factor is defined as the magnitude of reactive current density to the magnitude of dissipative current density. In visco-elastodynamics, a common definition of quality factor is that it is twice the ratio between the averaged strain energy density and the dissipated energy density. The kinetic and strain energy densities are associated with the magnetic- and electric-energy densities. Accordingly, and using the acoustic-electromagnetic analogy, the quality factor is defined here as twice the time-averaged electric-energy density divided by the time-averaged dissipated electric-energy density, where we consider the dissipation due to the magnetic permeability, as in poroelasticity we consider the dissipation due to the kinetic energy (see Sections 7.14.3 and 7.14.4). Then,

$$Q = \frac{2\langle E_e \rangle}{\langle D_e \rangle + \langle D_m \rangle}, \tag{8.283}$$

where

$$\langle D_e \rangle = \omega^{-1} \langle \dot{D}_e \rangle \quad \text{and} \quad \langle D_m \rangle = \omega^{-1} \langle \dot{D}_m \rangle \tag{8.284}$$

are the time-averaged electric and magnetic dissipated-energy densities, respectively.

Consider the TE mode propagating in the (x, y)-plane. Then,

$$\mathbf{E} = E_0 (0, 0, 1)^\top \exp(-\mathrm{i}\mathbf{s} \cdot \mathbf{x}), \tag{8.285}$$

where E_0 is a complex amplitude. By equation (8.248),

$$\mathbf{H} = \hat{\boldsymbol{\mu}}^{-1} \cdot \mathbf{s} \times \mathbf{E} = s E_0 \left(\frac{l_2}{\hat{\mu}_1}, -\frac{l_1}{\hat{\mu}_2}, 0 \right)^\top \exp(-\mathrm{i}\mathbf{s} \cdot \mathbf{x}), \tag{8.286}$$

where we have assumed uniform plane waves. Substituting the electric and magnetic vectors into the energy densities (8.278)-(8.281) yields

$$\langle E_e \rangle = \frac{1}{4} \mathrm{Re}(\bar{\epsilon}_3) |E_0|^2 \exp(-2\boldsymbol{\alpha} \cdot \mathbf{x}), \tag{8.287}$$

$$\langle E_m \rangle = \frac{1}{4} \mathrm{Re} \left(\bar{\epsilon}_3 \frac{v_c}{v_c^*} \right) |E_0|^2 \exp(-2\boldsymbol{\alpha} \cdot \mathbf{x}), \tag{8.288}$$

$$\langle \dot{D}_e \rangle = -\frac{\omega}{2}\mathrm{Im}(\bar{\epsilon}_3)|E_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}) \tag{8.289}$$

and

$$\langle \dot{D}_m \rangle = \frac{\omega}{2}\mathrm{Im}\left(\bar{\epsilon}_3 \frac{v_c}{v_c^*}\right)|E_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}), \tag{8.290}$$

where the complex velocity v_c is given by equation (8.274).

Summing the electric and magnetic energies gives the total stored energy

$$\langle E_e + E_m \rangle = \frac{1}{2}\mathrm{Re}\left(\bar{\epsilon}_3 \frac{v_c}{v_p}\right)|E_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}), \tag{8.291}$$

where v_p is the phase velocity $(8.271)_1$. The TE power-flow vector is

$$\mathrm{Re}(\mathbf{p}) = \frac{1}{2}\mathrm{Re}\left[\frac{1}{v_c}\left(\hat{\mathbf{e}}_1\frac{l_1}{\hat{\mu}_2} + \hat{\mathbf{e}}_2\frac{l_2}{\hat{\mu}_1}\right)\right]|E_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}). \tag{8.292}$$

From equations (8.291) and (8.292), we obtain the energy velocity for TE waves propagating in the (x, y)-plane as

$$\mathbf{v}_e(\mathrm{TE}) = \frac{v_p}{\mathrm{Re}(\bar{\epsilon}_3 v_c)}\left[l_1\mathrm{Re}\left(\frac{1}{v_c\hat{\mu}_2}\right)\hat{\mathbf{e}}_1 + l_2\mathrm{Re}\left(\frac{1}{v_c\hat{\mu}_1}\right)\hat{\mathbf{e}}_2\right]. \tag{8.293}$$

Performing similar calculations, the energy densities, power-flow vector (8.277) and energy velocity for TM waves propagating in the (x, y)-plane are

$$\langle E_e \rangle = \frac{1}{4}\mathrm{Re}\left(\hat{\mu}_3 \frac{v_c}{v_c^*}\right)|H_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}), \tag{8.294}$$

$$\langle E_m \rangle = \frac{1}{4}\mathrm{Re}(\hat{\mu}_3)|H_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}), \tag{8.295}$$

$$\langle \dot{D}_e \rangle = \frac{\omega}{2}\mathrm{Im}\left(\hat{\mu}_3 \frac{v_c}{v_c^*}\right)|H_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}) \tag{8.296}$$

and

$$\langle \dot{D}_m \rangle = -\frac{\omega}{2}\mathrm{Im}(\hat{\mu}_3)|H_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}), \tag{8.297}$$

where the complex velocity v_c is given by equation (8.273).

The TM total stored energy and power-flow vector are

$$\langle E_e + E_m \rangle = \frac{1}{2}\mathrm{Re}\left(\hat{\mu}_3 \frac{v_c}{v_p}\right)|H_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}) \tag{8.298}$$

and

$$\mathrm{Re}(\mathbf{p}) = \frac{1}{2}\mathrm{Re}\left[\frac{1}{v_c}\left(\hat{\mathbf{e}}_1\frac{l_1}{\bar{\epsilon}_2} + \hat{\mathbf{e}}_2\frac{l_2}{\bar{\epsilon}_1}\right)\right]|H_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}), \tag{8.299}$$

and the energy velocity is

$$\mathbf{v}_e(\mathrm{TM}) = \frac{v_p}{\mathrm{Re}(\hat{\mu}_3 v_c)}\left[l_1\mathrm{Re}\left(\frac{1}{v_c\bar{\epsilon}_2}\right)\hat{\mathbf{e}}_1 + l_2\mathrm{Re}\left(\frac{1}{v_c\bar{\epsilon}_1}\right)\hat{\mathbf{e}}_2\right]. \tag{8.300}$$

Calculation of the total time-averaged rate of dissipated energy for the TE and TM waves yields

$$\langle \dot{D}_e + \dot{D}_m \rangle(\mathrm{TE}) = \frac{1}{2}\mathrm{Re}\,(\bar{\epsilon}_3 \alpha v_c)\,|E_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}) \tag{8.301}$$

and

$$\langle \dot{D}_e + \dot{D}_m \rangle(\mathrm{TM}) = \frac{1}{2}\mathrm{Re}\,(\hat{\mu}_3 \alpha v_c)\,|H_0|^2 \exp(-2\boldsymbol{\alpha}\cdot\mathbf{x}), \tag{8.302}$$

where we have used equations (8.287)-(8.290) and (8.294)-(8.297).

Let us consider the TE mode. Substitution of (8.287) and (8.301) into equations (8.283) and (8.284) gives

$$Q = \frac{\mathrm{Re}(\bar{\epsilon}_3)}{\alpha \mathrm{Im}(\bar{\epsilon}_3 v_c)}. \tag{8.303}$$

If we neglect the magnetic losses, for instance, by assuming that $\hat{\boldsymbol{\mu}}$ is real, we obtain

$$Q = -\frac{\mathrm{Re}(\bar{\epsilon}_3)}{\mathrm{Im}(\bar{\epsilon}_3)} = \frac{\mathrm{Re}(v_c^2)}{\mathrm{Im}(v_c^2)}, \tag{8.304}$$

which is the viscoelastic expression (e..g. see equation (4.92)).

Another definition of quality factor, which considers the total energy, is a generalization of equation (2.124),

$$\mathcal{Q} = \frac{\langle E_e + E_m \rangle}{\langle D_e + D_m \rangle}. \tag{8.305}$$

In this case, the quality factor takes the simple form

$$\mathcal{Q} = \frac{\omega}{2\alpha v_p}. \tag{8.306}$$

The form (8.306) coincides with the relation between quality factor and attenuation for low-loss media (see equation (2.123)), although we did not invoke such a restriction here.

The quality factor (8.283) for TM waves is

$$Q = \frac{\omega \mathrm{Re}(\hat{\mu}_3 v_c / v_c^*)}{\alpha \mathrm{Re}(\hat{\mu}_3 v_c)}, \tag{8.307}$$

and $\mathcal{Q}$ has the same form (8.306) but using the phase velocity and attenuation factor corresponding to the TM wave.

An application of this theory to ground-penetrating-radar wave propagation is given in Carcione and Schoenberg (2000).

8.7 Analytical solution for anisotropic media

We can derive a closed-form frequency-domain analytical solution for electromagnetic waves propagating in a 3-D lossy orthorhombic medium, for which the dielectric-permittivity tensor is proportional to the magnetic-permeability tensor. Although this solution has limited practical value, it can be used to test simulation algorithms.

Maxwell's equations (8.6) and (8.7) for a time-harmonic magnetic field propagating in an inhomogeneous anisotropic medium can be written as

$$\nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}\right) - \omega^2 \hat{\boldsymbol{\mu}} \cdot \mathbf{H} = \nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}\right), \tag{8.308}$$

where the dielectric-permittivity tensor $\bar{\boldsymbol{\epsilon}}$ is given by equation (8.251). Maxwell's equations are symmetric by interchanging $\mathbf{H}$ and $\mathbf{E}$. The equivalence – or duality – is given by

$$\mathbf{H} \Leftrightarrow \mathbf{E}, \quad \mathbf{J} \Leftrightarrow -\mathbf{M}, \quad \bar{\boldsymbol{\epsilon}} \Leftrightarrow -\hat{\boldsymbol{\mu}}, \quad \hat{\boldsymbol{\mu}} \Leftrightarrow -\bar{\boldsymbol{\epsilon}}. \tag{8.309}$$

The equivalent of the vector equation (8.308) is

$$\nabla \times \left(\hat{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}\right) - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \mathbf{E} = \nabla \times \left(\hat{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}\right). \tag{8.310}$$

We assume now that the medium is homogeneous. However, note that even in this situation, the tensors $\bar{\boldsymbol{\epsilon}}^{-1}$ and $\hat{\boldsymbol{\mu}}^{-1}$ do not commute with the curl operator. We further assume that the medium is orthorhombic and that its principal system coincides with the Cartesian system where the problem is solved. In orthorhombic media, the eigenvectors of the material tensors coincide, allowing these tensors to have a diagonal form (see equation (8.229)). In Cartesian coordinates, the vector term $\nabla \times (\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H})$ consists of three scalar terms:

$$(\bar{\epsilon}_3)^{-1}(\partial_1\partial_2 H_2 - \partial_2^2 H_1) - (\bar{\epsilon}_2)^{-1}(\partial_3^2 H_1 - \partial_1\partial_3 H_3), \tag{8.311}$$

$$(\bar{\epsilon}_1)^{-1}(\partial_2\partial_3 H_3 - \partial_3^2 H_2) - (\bar{\epsilon}_3)^{-1}(\partial_1^2 H_2 - \partial_1\partial_2 H_1) \tag{8.312}$$

and

$$(\bar{\epsilon}_2)^{-1}(\partial_1\partial_3 H_1 - \partial_1^2 H_3) - (\bar{\epsilon}_1)^{-1}(\partial_2^2 H_3 - \partial_2\partial_3 H_2). \tag{8.313}$$

In the absence of magnetic-current densities, we have $\nabla \cdot \mathbf{B} = 0$, where $\mathbf{B} = \hat{\boldsymbol{\mu}} \cdot \mathbf{H}$, and then

$$\hat{\mu}_1\partial_1 H_1 + \hat{\mu}_2\partial_2 H_2 + \hat{\mu}_3\partial_3 H_3 = 0. \tag{8.314}$$

Using (8.311)-(8.314) and multiplying the three components of (8.308) by $-\bar{\epsilon}_2\bar{\epsilon}_3$, $-\bar{\epsilon}_1\bar{\epsilon}_3$ and $-\bar{\epsilon}_1\bar{\epsilon}_2$, respectively, yields

$$\frac{\hat{\mu}_1}{\hat{\mu}_2}\bar{\epsilon}_2\partial_1^2 H_1 + \bar{\epsilon}_2\partial_2^2 H_1 + \bar{\epsilon}_3\partial_3^2 H_1 - \left(\bar{\epsilon}_3 - \frac{\hat{\mu}_3}{\hat{\mu}_2}\bar{\epsilon}_2\right)\partial_1\partial_3 H_3 + \omega^2\hat{\mu}_1\bar{\epsilon}_2\bar{\epsilon}_3 H_1 = \bar{\epsilon}_3\partial_3 J_2 - \bar{\epsilon}_2\partial_2 J_3, \tag{8.315}$$

$$\bar{\epsilon}_1\partial_1^2 H_2 + \frac{\hat{\mu}_2}{\hat{\mu}_1}\bar{\epsilon}_1\partial_2^2 H_2 + \bar{\epsilon}_3\partial_3^2 H_2 - \left(\bar{\epsilon}_3 - \frac{\hat{\mu}_3}{\hat{\mu}_1}\bar{\epsilon}_1\right)\partial_2\partial_3 H_3 + \omega^2\hat{\mu}_2\bar{\epsilon}_1\bar{\epsilon}_3 H_2 = \bar{\epsilon}_1\partial_1 J_3 - \bar{\epsilon}_3\partial_3 J_1, \tag{8.316}$$

$$\bar{\epsilon}_1\partial_1^2 H_3 + \bar{\epsilon}_2\partial_2^2 H_3 + \frac{\hat{\mu}_3}{\hat{\mu}_1}\bar{\epsilon}_1\partial_3^2 H_3 - \left(\bar{\epsilon}_2 - \frac{\hat{\mu}_2}{\hat{\mu}_1}\bar{\epsilon}_1\right)\partial_2\partial_3 H_2 + \omega^2\hat{\mu}_3\bar{\epsilon}_1\bar{\epsilon}_2 H_3 = \bar{\epsilon}_2\partial_2 J_1 - \bar{\epsilon}_1\partial_1 J_2. \tag{8.317}$$

The system of equations (8.315)-(8.317) can be solved in closed form by assuming that the general dielectric permittivity tensor is proportional to the magnetic-permeability tensor:

$$\bar{\boldsymbol{\epsilon}} \propto \hat{\boldsymbol{\mu}}. \tag{8.318}$$

This particular class of orthorhombic media satisfies

$$\hat{\mu}_1\bar{\epsilon}_2 = \hat{\mu}_2\bar{\epsilon}_1, \quad \hat{\mu}_1\bar{\epsilon}_3 = \hat{\mu}_3\bar{\epsilon}_1, \quad \hat{\mu}_2\bar{\epsilon}_3 = \hat{\mu}_3\bar{\epsilon}_2. \tag{8.319}$$

This assumption is similar to one proposed by Lindell and Olyslager (1997). Using these relations, equations (8.315)-(8.317) become three Helmholtz equations,

$$\Delta_\epsilon H_1 + \omega^2\eta H_1 = \bar{\epsilon}_3\partial_3 J_2 - \bar{\epsilon}_2\partial_2 J_3, \tag{8.320}$$

$$\Delta_\epsilon H_2 + \omega^2 \eta H_2 = \bar{\epsilon}_1 \partial_1 J_3 - \bar{\epsilon}_3 \partial_3 J_1, \tag{8.321}$$

$$\Delta_\epsilon H_3 + \omega^2 \eta H_3 = \bar{\epsilon}_2 \partial_2 J_1 - \bar{\epsilon}_1 \partial_1 J_2, \tag{8.322}$$

where

$$\eta = \hat{\mu}_1 \bar{\epsilon}_2 \bar{\epsilon}_3 \tag{8.323}$$

and

$$\Delta_\epsilon = \bar{\epsilon}_1 \partial_1^2 + \bar{\epsilon}_2 \partial_2^2 + \bar{\epsilon}_3 \partial_3^2. \tag{8.324}$$

The equations for the electric-vector components can be obtained from equations (8.320)-(8.322) using the duality (8.309):

$$\Delta_\mu E_1 + \omega^2 \chi E_1 = \hat{\mu}_2 \partial_2 M_3 - \hat{\mu}_3 \partial_3 M_2, \tag{8.325}$$

$$\Delta_\mu E_2 + \omega^2 \chi E_2 = \hat{\mu}_3 \partial_3 M_1 - \hat{\mu}_1 \partial_1 M_3, \tag{8.326}$$

$$\Delta_\mu E_3 + \omega^2 \chi E_3 = \hat{\mu}_1 \partial_1 M_2 - \hat{\mu}_2 \partial_2 M_1, \tag{8.327}$$

where

$$\Delta_\mu = \hat{\mu}_1 \partial_1^2 + \hat{\mu}_2 \partial_2^2 + \hat{\mu}_3 \partial_3^2 \tag{8.328}$$

and

$$\chi = \bar{\epsilon}_1 \hat{\mu}_2 \hat{\mu}_3. \tag{8.329}$$

Note that the relations (8.319) are not modified by duality.

8.7.1 The solution

The following change of coordinates

$$x \to \alpha\sqrt{\bar{\epsilon}_1}, \quad y \to \beta\sqrt{\bar{\epsilon}_2}, \quad z \to \gamma\sqrt{\bar{\epsilon}_3} \tag{8.330}$$

transforms Δ_ϵ into a pure Laplacian differential operator. Using equation $(8.330)_1$, equation (8.320) becomes

$$\Delta H_\alpha + \omega^2 \eta H_\alpha = \sqrt{\bar{\epsilon}_3} \partial_\gamma J_\beta - \sqrt{\bar{\epsilon}_2} \partial_\beta J_\gamma, \tag{8.331}$$

where

$$\Delta = \partial_\alpha^2 + \partial_\beta^2 + \partial_\gamma^2 \tag{8.332}$$

and analogously for equations (8.321) and (8.322).

Consider equation (8.331) for the Green function

$$(\Delta + \omega^2 \eta) g = -\delta(\hat{\rho}), \tag{8.333}$$

whose solution is

$$g(\hat{\rho}) = \frac{1}{4\pi\hat{\rho}} \exp(-\mathrm{i}\omega\hat{\rho}\sqrt{\eta}), \tag{8.334}$$

where

$$\hat{\rho} = \sqrt{\alpha^2 + \beta^2 + \gamma^2} \tag{8.335}$$

(Pilant, 1979, p. 64). The spatial derivatives of the electric currents in (8.331) imply the differentiation of the Green function. Assume, for instance, that the electric currents J_β and J_γ are delta functions: $J_\beta = \mathcal{J}_\beta \delta(\hat{\rho})$ and $J_\gamma = \mathcal{J}_\gamma \delta(\hat{\rho})$. Since the solution of (8.331) is the convolution of the Green function with the source term, it can be obtained as the β spatial derivative of the Green function. Then, for impulsive electric currents, the solution is

$$H_\alpha = -\left(\sqrt{\bar{\epsilon}_3}\mathcal{J}_\beta \partial_\gamma g - \sqrt{\bar{\epsilon}_2}\mathcal{J}_\gamma \partial_\beta g\right). \tag{8.336}$$

We have that

$$\partial_\beta g = \left(\frac{\beta}{\hat{\rho}}\right)\partial_\rho g, \qquad \partial_\gamma g = \left(\frac{\gamma}{\hat{\rho}}\right)\partial_\rho g, \tag{8.337}$$

where

$$\partial_\rho g = -\left(\frac{1}{\hat{\rho}} + \mathrm{i}\omega\sqrt{\eta}\right) g. \tag{8.338}$$

In terms of Cartesian coordinates, the solution is

$$H_1 = \frac{1}{4\pi\hat{\rho}^2}\left(z\mathcal{J}_2 - y\mathcal{J}_3\right)\left(\frac{1}{\hat{\rho}} + \mathrm{i}\omega\sqrt{\eta}\right)\exp(-\mathrm{i}\omega\hat{\rho}\sqrt{\eta}), \tag{8.339}$$

where

$$\hat{\rho} = \sqrt{\frac{x^2}{\bar{\epsilon}_1} + \frac{y^2}{\bar{\epsilon}_2} + \frac{z^2}{\bar{\epsilon}_3}}. \tag{8.340}$$

Similarly, the other components are given by

$$H_2 = \frac{1}{4\pi\hat{\rho}^2}\left(x\mathcal{J}_3 - z\mathcal{J}_1\right)\left(\frac{1}{\hat{\rho}} + \mathrm{i}\omega\sqrt{\eta}\right)\exp(-\mathrm{i}\omega\hat{\rho}\sqrt{\eta}) \tag{8.341}$$

and

$$H_3 = \frac{1}{4\pi\hat{\rho}^2}\left(y\mathcal{J}_1 - x\mathcal{J}_2\right)\left(\frac{1}{\hat{\rho}} + \mathrm{i}\omega\sqrt{\eta}\right)\exp(-\mathrm{i}\omega\hat{\rho}\sqrt{\eta}). \tag{8.342}$$

The three components of the magnetic vector are not functionally independent, since they must satisfy equation (8.314). When solving the problem with a limited-band wavelet source $f(t)$, the frequency-domain solution is multiplied by the Fourier transform $F(\omega)$. To ensure a real time-domain solution, we consider an Hermitian frequency-domain solution. Finally, the time-domain solution is obtained by an inverse transform. Examples illustrating this analytical solution can be found in Carcione and Cavallini (2001).

8.8 Finely layered media

The electromagnetic properties of finely plane-layered media can be obtained by using the same approach used in Section 1.5 for elastic media. Let us consider a plane-layered medium, where each layer is homogeneous, anisotropic and thin compared to the wavelength of the electromagnetic wave. If the layer interfaces are parallel to the (x, y)-plane, the properties are independent of x and y and may vary with z.

We follow Backus's approach (Backus, 1962) to obtain the properties of a finely layered medium. Let $w(z)$ be a continuous weighting function that averages over a length d. This function has the following properties:

$$\begin{aligned}
&w(z) \geq 0\\
&w(\pm\infty) = 0\\
&\textstyle\int_{-\infty}^{\infty} w(z')dz' = 1\\
&\textstyle\int_{-\infty}^{\infty} z'w(z')dz' = 0\\
&\textstyle\int_{-\infty}^{\infty} z'^2 w(z')dz' = d^2.
\end{aligned} \tag{8.343}$$

Then, the average of a function f over the length d around the location z is

$$\langle f\rangle(z) = \int_{-\infty}^{\infty} w(z'-z)f(z')dz'. \tag{8.344}$$

The averaging removes the wavelengths of f which are smaller than d. An important approximation in this context is

$$\langle fg\rangle = f\langle g\rangle, \tag{8.345}$$

where f is nearly constant over the distance d and g may have an arbitrary dependence as a function of z.

Let us consider first the dielectric-permittivity properties. The explicit form of the frequency-domain constitutive equation is obtained from equation (8.60),

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} \hat{\epsilon}_{11} & \hat{\epsilon}_{12} & \hat{\epsilon}_{13} \\ \hat{\epsilon}_{12} & \hat{\epsilon}_{22} & \hat{\epsilon}_{23} \\ \hat{\epsilon}_{13} & \hat{\epsilon}_{23} & \hat{\epsilon}_{33} \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \tag{8.346}$$

where the dielectric-permittivity components are complex and frequency dependent. The boundary conditions at the single interfaces impose the continuity of the following field components

$$D_3, \ E_1, \ \text{ and } \ E_2 \tag{8.347}$$

(Born and Wolf, 1964, p. 4), which vary very slowly with z. On the contrary, D_1, D_2 and E_3 vary rapidly from layer to layer. We express the rapidly varying fields in terms of the slowly varying fields. This gives

$$D_1 = \left(\hat{\epsilon}_{11} - \frac{\hat{\epsilon}_{13}^2}{\hat{\epsilon}_{33}}\right) E_1 + \left(\hat{\epsilon}_{12} - \frac{\hat{\epsilon}_{13}\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}}\right) E_2 + \frac{\hat{\epsilon}_{13}}{\hat{\epsilon}_{33}} D_3, \tag{8.348}$$

$$D_2 = \left(\hat{\epsilon}_{12} - \frac{\hat{\epsilon}_{13}\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}}\right) E_1 + \left(\hat{\epsilon}_{22} - \frac{\hat{\epsilon}_{23}^2}{\hat{\epsilon}_{33}}\right) E_2 + \frac{\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}} D_3 \tag{8.349}$$

and

$$E_3 = \frac{1}{\hat{\epsilon}_{33}}(D_3 - \hat{\epsilon}_{13}E_1 - \hat{\epsilon}_{23}E_2). \tag{8.350}$$

These equations contain no products of a rapidly varying field and a rapidly variable dielectric-permittivity component. Then, the average of equations (8.348)-(8.350) over the length d can be performed by using equation (8.345). We obtain

$$\langle D_1\rangle = \left\langle \hat{\epsilon}_{11} - \frac{\hat{\epsilon}_{13}^2}{\hat{\epsilon}_{33}}\right\rangle E_1 + \left\langle \hat{\epsilon}_{12} - \frac{\hat{\epsilon}_{13}\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}}\right\rangle E_2 + \left\langle \frac{\hat{\epsilon}_{13}}{\hat{\epsilon}_{33}}\right\rangle D_3, \tag{8.351}$$

$$\langle D_2\rangle = \left\langle \hat{\epsilon}_{12} - \frac{\hat{\epsilon}_{13}\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}}\right\rangle E_1 + \left\langle \hat{\epsilon}_{22} - \frac{\hat{\epsilon}_{23}^2}{\hat{\epsilon}_{33}}\right\rangle E_2 + \left\langle \frac{\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}}\right\rangle D_3 \tag{8.352}$$

and

$$\langle E_3\rangle = \left\langle \frac{1}{\hat{\epsilon}_{33}}\right\rangle D_3 - \left\langle \frac{\hat{\epsilon}_{13}}{\hat{\epsilon}_{33}}\right\rangle E_1 - \left\langle \frac{\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}}\right\rangle E_2. \tag{8.353}$$

Expressing the average electric-displacement components in terms of the averaged electric-vector components gives the constitutive equations of the medium,

$$\begin{pmatrix} \langle D_1 \rangle \\ \langle D_2 \rangle \\ D_3 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ E_2 \\ \langle E_3 \rangle \end{pmatrix}, \tag{8.354}$$

where

$$\varepsilon_{11} = \left\langle \hat{\epsilon}_{11} - \frac{\hat{\epsilon}_{13}^2}{\hat{\epsilon}_{33}} \right\rangle + \left\langle \frac{\hat{\epsilon}_{13}}{\hat{\epsilon}_{33}} \right\rangle^2 \left\langle \frac{1}{\hat{\epsilon}_{33}} \right\rangle^{-1}, \tag{8.355}$$

$$\varepsilon_{12} = \left\langle \hat{\epsilon}_{12} - \frac{\hat{\epsilon}_{13}\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}} \right\rangle + \left\langle \frac{\hat{\epsilon}_{13}}{\hat{\epsilon}_{33}} \right\rangle \left\langle \frac{\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}} \right\rangle \left\langle \frac{1}{\hat{\epsilon}_{33}} \right\rangle^{-1}, \tag{8.356}$$

$$\varepsilon_{13} = \left\langle \frac{\hat{\epsilon}_{13}}{\hat{\epsilon}_{33}} \right\rangle \left\langle \frac{1}{\hat{\epsilon}_{33}} \right\rangle^{-1}, \tag{8.357}$$

$$\varepsilon_{22} = \left\langle \hat{\epsilon}_{22} - \frac{\hat{\epsilon}_{23}^2}{\hat{\epsilon}_{33}} \right\rangle + \left\langle \frac{\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}} \right\rangle^2 \left\langle \frac{1}{\hat{\epsilon}_{33}} \right\rangle^{-1}, \tag{8.358}$$

$$\varepsilon_{23} = \left\langle \frac{\hat{\epsilon}_{23}}{\hat{\epsilon}_{33}} \right\rangle \left\langle \frac{1}{\hat{\epsilon}_{33}} \right\rangle^{-1} \tag{8.359}$$

and

$$\varepsilon_{33} = \left\langle \frac{1}{\hat{\epsilon}_{33}} \right\rangle^{-1}. \tag{8.360}$$

For isotropic layers, $\hat{\epsilon}_{12} = \hat{\epsilon}_{13} = \hat{\epsilon}_{23} = 0$, $\hat{\epsilon}_{11} = \hat{\epsilon}_{22} = \hat{\epsilon}_{33} = \hat{\epsilon}$, and we have

$$\varepsilon_{11} = \varepsilon_{22} = \langle \hat{\epsilon} \rangle, \tag{8.361}$$

$$\varepsilon_{33} = \left\langle \frac{1}{\hat{\epsilon}} \right\rangle^{-1}, \tag{8.362}$$

and $\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$.

The acoustic-electromagnetic analogy between the TM and SH cases is $\hat{\epsilon} \Leftrightarrow \mu^{-1}$ (see equation (8.33)), where μ is the shear modulus. Using the preceding equations, we obtain the following stiffness constants

$$c_{44} = \left\langle \frac{1}{\mu} \right\rangle^{-1} \tag{8.363}$$

and

$$c_{66} = \langle \mu \rangle, \tag{8.364}$$

respectively. These equations are equivalent to equations $(1.188)_5$ and $(1.188)_4$ for isotropic layers, respectively.

The same functional form is obtained for the magnetic-permeability and conductivity tensors of a finely layered medium if we apply the same procedure to equations (8.61) and (8.62), with $\mathbf{J} = 0$. In this case, continuity of B_3 , H_1, H_2 and J_3, E_1, E_2 is required, respectively.

8.9 The time-average and CRIM equations

The acoustic and electromagnetic wave velocities of rocks depends strongly on the rock composition. Assume a stratified model of n different media, each having a thickness h_i and a wave velocity v_i. The transit time t for a wave through the rock is the sum of the partial transit times:

$$t = \frac{h}{v} = \sum_{i=1}^{n} \frac{h_i}{v_i}, \tag{8.365}$$

where $h = \sum_{i=1}^{n} h_i$ and v is the average velocity. Defining the material proportions as $\phi_i = h_i/h$, the average velocity is

$$v = \left(\sum_{i=1}^{n} \frac{\phi_i}{v_i} \right)^{-1}. \tag{8.366}$$

For a rock saturated with a single fluid, we obtain the time-average equation:

$$v = \left(\frac{\phi}{v_f} + \frac{1-\phi}{v_s} \right)^{-1}, \tag{8.367}$$

where ϕ is the porosity[3], v_f is the fluid wave velocity and v_s is the wave velocity in the mineral aggregate (Wyllie, Gregory and Gardner, 1956).

The electromagnetic version of the time-average equation is the CRIM equation (complex refraction index model). If $\hat{\mu}_i(\omega)$ and $\bar{\epsilon}_i(\omega)$ are the magnetic permeability and dielectric permittivity of the single phases, the respective slownesses are given by $1/v_i = \sqrt{\hat{\mu}_i \bar{\epsilon}_i}$. Using equation (8.366), the equivalent electromagnetic equation is

$$\sqrt{\hat{\mu}\bar{\epsilon}} = \sum_{i=1}^{n} \phi_i \sqrt{\hat{\mu}_i \bar{\epsilon}_i}, \tag{8.368}$$

where $\hat{\mu}$ and $\bar{\epsilon}$ are the average permeability and permittivity, respectively. The CRIM equation is obtained for constant magnetic permeability. That is

$$\bar{\epsilon} = \left(\sum_{i=1}^{n} \phi_i \sqrt{\bar{\epsilon}_i} \right)^2 \equiv \hat{\epsilon}_e - \frac{\mathrm{i}}{\omega}\hat{\sigma}_e, \tag{8.369}$$

where $\hat{\epsilon}_e$ and $\hat{\sigma}_e$ are the real-valued effective permittivity and conductivity, respectively (see equation (8.252)). A useful generalization is the Lichtnecker-Rother formula:

$$\bar{\epsilon} = \left(\sum_{i=1}^{n} \phi_i (\bar{\epsilon}_i)^{1/\gamma} \right)^{\gamma}, \tag{8.370}$$

where γ is a fitting parameter (e.g., Guéguen and Palciauskas, 1994).

While Backus averaging yields the low-frequency elasticity constants, the time-average and CRIM equations are a high-frequency approximation, i.e., the limit known as *geometrical optics.*

[3]Note that ϕ is the linear porosity, which is equal to the volume porosity, ϕ_V, for planar pores (or cracks). For three intersecting, mutually perpendicular, planar cracks, the relation is $\phi_V = 1 - (1-\phi)^3$, with $\phi_V \approx 3\phi$ for $\phi \ll 1$.

8.10 The Kramers-Kronig dispersion relations

The Kramers-Kronig dispersion relations obtained in Section 2.2.4 for anelastic media were first derived as a relation between the real and imaginary parts of the frequency-dependent dielectric-permittivity function (Kramers, 1927; Kronig, 1926). Actually, the relations are applied to the electric susceptibility of the material,

$$\hat{\chi}(\omega) \equiv \hat{\epsilon}(\omega) - \hat{\epsilon}_0 = \hat{\epsilon}_1(\omega) + \mathrm{i}\hat{\epsilon}_2(\omega) - \hat{\epsilon}_0, \tag{8.371}$$

where $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ are the real and imaginary parts of the dielectric permittivity, and, here, $\hat{\epsilon}_0$ is the dielectric permittivity of free space. Under certain conditions, the linear response of a medium can be expressed by the electric-polarization vector $\mathbf{P}(t)$,

$$\mathbf{P}(t) = \hat{\chi} * \partial_t \mathbf{E} = \int_{-\infty}^{\infty} \hat{\chi}(t - t')\partial \mathbf{E}(t')dt' \tag{8.372}$$

(Born and Wolf, 1964, p. 76 and 84), where ∂ denotes the derivative with respect to the argument. A Fourier transform to the frequency domain gives

$$\mathbf{P}(\omega) = \hat{\chi}(\omega)\mathbf{E}(\omega), \tag{8.373}$$

where $\hat{\chi}(\omega)$ stands for $\mathcal{F}[\partial_t \hat{\chi}(t)]$ to simplify the notation. The electric-displacement vector is

$$\mathbf{D}(\omega) = \hat{\epsilon}_0 \mathbf{E}(\omega) + \mathbf{P}(\omega) = [\hat{\epsilon}_0 + \hat{\chi}(\omega)]\mathbf{E}(\omega) = \hat{\epsilon}(\omega)\mathbf{E}(\omega), \tag{8.374}$$

according to equation (8.371). The electric susceptibility $\hat{\chi}(\omega)$ is analytic and bounded in the lower half-plane of the complex frequency argument. This is a consequence of the causality condition, i.e., $\hat{\chi}(t - t') = 0$ for $t < t'$ (Golden and Graham, 1988, p. 48).

An alternative derivation of the Kramers-Kronig relations is based on Cauchy's integral formula applied to the electric susceptibility. Since this is analytic in the lower half-plane, we have

$$\hat{\chi}(\omega) = \frac{1}{\mathrm{i}\pi}\mathrm{pv}\int_{-\infty}^{\infty} \frac{\hat{\chi}(\omega')}{\omega - \omega'} d\omega'. \tag{8.375}$$

where pv is the principal value. Separating real and imaginary parts and using equation (8.371), we obtain the Kramers-Kronig relations,

$$\hat{\epsilon}_1(\omega) = \hat{\epsilon}_0 + \frac{1}{\pi}\mathrm{pv}\int_{-\infty}^{\infty} \frac{\hat{\epsilon}_2(\omega')}{\omega - \omega'} d\omega' \tag{8.376}$$

and

$$\hat{\epsilon}_2(\omega) = -\frac{1}{\pi}\mathrm{pv}\int_{-\infty}^{\infty} \frac{\hat{\epsilon}_1(\omega') - \hat{\epsilon}_0}{\omega - \omega'} d\omega'. \tag{8.377}$$

The acoustic-electromagnetic analogy (8.33) implies the mathematical equivalence between the dielectric permittivity and the complex creep compliance defined in equation (2.43), i. e., $\hat{\epsilon} \Leftrightarrow J \equiv J_1 + \mathrm{i}J_2$. Hence, we obtain

$$J_1(\omega) = \frac{1}{\pi}\mathrm{pv}\int_{-\infty}^{\infty} \frac{J_2(\omega')}{\omega - \omega'} d\omega' \tag{8.378}$$

and

$$J_2(\omega) = -\frac{1}{\pi}\mathrm{pv}\int_{-\infty}^{\infty} \frac{J_1(\omega')}{\omega - \omega'} d\omega', \tag{8.379}$$

which are mathematically equivalent to the Kramers-Kronig relations (2.70) and (2.72), corresponding to the viscoelastic complex modulus. The term equivalent to $\hat{\epsilon}_0$ is zero in the acoustic case, since there is not an upper-limit velocity equivalent to the velocity of light (M and J can be infinite and zero, respectively). Analogous Kramers-Kronig relations apply to the complex magnetic-permeability function.

8.11 The reciprocity principle

The reciprocity principle for acoustic waves is illustrated in detail in Chapter 5. In this section, we obtain the principle for electromagnetic waves in anisotropic lossy media.

We suppose that the source currents $\mathbf{J}_1$ and $\mathbf{J}_2$ give rise to fields $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. These fields satisfy equation (8.308):

$$\nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_1\right) - \omega^2 \hat{\boldsymbol{\mu}} \cdot \mathbf{H}_1 = \nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}_1\right) \tag{8.380}$$

and

$$\nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_2\right) - \omega^2 \hat{\boldsymbol{\mu}} \cdot \mathbf{H}_2 = \nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}_2\right). \tag{8.381}$$

The following scalar products are valid:

$$\mathbf{H}_2 \cdot \nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_1\right) - \omega^2 \mathbf{H}_2 \cdot \hat{\boldsymbol{\mu}} \cdot \mathbf{H}_1 = \mathbf{H}_2 \cdot \nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}_1\right) \tag{8.382}$$

and

$$\mathbf{H}_1 \cdot \nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_2\right) - \omega^2 \mathbf{H}_1 \cdot \hat{\boldsymbol{\mu}} \cdot \mathbf{H}_2 = \mathbf{H}_1 \cdot \nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}_2\right). \tag{8.383}$$

The second terms on the left-hand-side of equations (8.382) and (8.383) are equal if the magnetic-permeability tensor is symmetric, i.e., if $\hat{\boldsymbol{\mu}} = \hat{\boldsymbol{\mu}}^\top$. The first terms can be rewritten using the vector identity $\mathbf{B} \cdot \nabla \times \mathbf{A} = \nabla \cdot (\mathbf{A} \times \mathbf{B}) + \mathbf{A} \cdot (\nabla \times \mathbf{B})$. For instance, $\mathbf{H}_1 \cdot \nabla \times (\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_2) = \nabla \cdot [(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_2) \times \mathbf{H}_1] + (\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_2) \cdot (\nabla \times \mathbf{H}_1)$. Integrating this quantity over a volume Ω bounded by surface S, and using Gauss's theorem, we obtain

$$\int_S [(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_2) \times \mathbf{H}_1] \cdot \hat{\mathbf{n}}\, dS + \int_\Omega (\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_2) \cdot (\nabla \times \mathbf{H}_1)\, d\Omega, \tag{8.384}$$

where $\hat{\mathbf{n}}$ is a unit vector directed along the outward normal to S. The second term on the right-hand side of equation (8.384) is symmetric by interchanging $\mathbf{H}_2$ and $\mathbf{H}_1$ if $\bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}^\top$. Regarding the first term, we assume that the medium is isotropic and homogeneous when $S \to \infty$, with a dielectric permittivity equal to $\bar{\epsilon}$. Furthermore, the wave fields are plane waves in the far field, so that $\nabla \to -\mathrm{i}\mathbf{k}$, where $\mathbf{k}$ is the complex wavevector. Moreover, the plane-wave assumption implies $\mathbf{k} \times \mathbf{H} = 0$. Hence

$$(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}_2) \times \mathbf{H}_1 = \mathrm{i}\mathbf{k}\ (\bar{\epsilon})^{-1} (\mathbf{H}_2 \cdot \mathbf{H}_1) \tag{8.385}$$

(Chew, 1990). Thus, also the first term on the right-hand side of equation (8.384) is symmetric by interchanging $\mathbf{H}_2$ and $\mathbf{H}_1$.

Consequently, a volume integration and subtraction of equations (8.382) and (8.383) yields

$$\int_\Omega \left[\mathbf{H}_2 \cdot \nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}_1\right) - \mathbf{H}_1 \cdot \nabla \times \left(\bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}_2\right)\right] d\Omega = 0. \tag{8.386}$$

Using the vector identity indicated above, with $\mathbf{A} = \bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}$ and $\mathbf{B} = \mathbf{H}$, and using Maxwell's equation $\nabla \times \mathbf{H} = \mathrm{i}\omega\bar{\boldsymbol{\epsilon}} \cdot \mathbf{E}$, we obtain $\mathbf{H} \cdot \nabla \times (\bar{\boldsymbol{\epsilon}}^{-1} \cdot \mathbf{J}) = \mathrm{i}\omega\mathbf{E} \cdot \mathbf{J}$. Hence, equation (8.386) becomes

$$\int_\Omega (\mathbf{E}_2 \cdot \mathbf{J}_1 + \mathbf{E}_1 \cdot \mathbf{J}_2)\, d\Omega = 0. \tag{8.387}$$

This equation is equivalent to the acoustic version of the reciprocity (equation (5.3)). It states that the field generated by $\mathbf{J}_1$ measured by $\mathbf{J}_2$ is the same field generated by $\mathbf{J}_2$ measured by $\mathbf{J}_1$. Note that the principle holds if the magnetic permeability and dielectric permittivity are symmetric tensors.

8.12 Babinet's principle

Babinet's principle was originally used to relate the diffracted light fields by complementary thin screens (Jones, 1986). In electromagnetism, Babinet's principle for infinitely thin perfectly conducting complementary screens implies that the sum, beyond the screen plane, of the electric and the magnetic fields (adjusting physical dimensions) equals the incident (unscreened) electric field. A complementary screen is a plane screen with opaque areas where the original plane screen had transparent areas. Roughly speaking, the principle states that behind the diffracting plane, the sum of the fields associated with a screen and with its complementary screen is just the field that would exist in the absence of any screen; that is, the diffracted fields from the two complementary screens are the negative of each other and cancel when summed. The principle is also valid for electromagnetic fields and perfectly conducting plane screens or diffractors (Jones, 1986).

Consider a screen S and its complementary screen C and assume that the total field in the presence of S is $\mathbf{v}_S$ and that related to C is $\mathbf{v}_C$. Babinet's principle states that the total fields on the opposite sides of the screens from the source satisfy

$$\mathbf{v}_S + \mathbf{v}_C = \mathbf{v}_0, \tag{8.388}$$

where $\mathbf{v}_0$ is the field in the absence of any screen. Equation (8.388) states that the diffraction fields for the complementary screens will be the negative of each other. Moreover, the total fields on the source side must satisfy

$$\mathbf{v}_S + \mathbf{v}_C = 2\mathbf{v}_0 + \mathbf{v}_R, \tag{8.389}$$

where $\mathbf{v}_R$ is the reflected field by a screen composed of S and C.

Carcione and Gangi (1998) have investigated Babinet's principle for acoustic waves by using a numerical simulation technique. In elastodynamics, the principle holds for the same field (particle velocity or stress), but for complementary screens satisfying different types of boundary conditions, i.e, if the original screen is weak (stress-free condition), the complementary screen must be rigid. On the other hand, if the original screen is rigid, the complementary screen must be weak.

Babinet's principle holds for screens embedded in anisotropic media, both for SH and qP-qS waves. The simulations indicate that Babinet's principle is satisfied also in the case of shear-wave triplications (qS waves). Moreover, the numerical experiments show that Babinet's principle holds for the near and far fields, and for an arbitrary pulse waveform and frequency spectrum. However, as expected, lateral and interface waves (e.g., Rayleigh waves) do not satisfy the principle.

Babinet's principle is of value since it allow us to obtain the solution of the complementary problem from the solution of the original problem without any additional effort. Moreover, it provides a check of the solutions for problems that are self-complementary (e.g., the problem of a plane wave normally incident on a half-plane). Finally, it adds to our knowledge of the complex phenomena of elastic wave diffraction.

8.13 Alford rotation

The analogy between acoustic and electromagnetic waves also applies to multi-component data acquisition of seismic and ground-penetrating-radar (GPR) surveys. Alford (1986) developed a method, subsequently referred as to "Alford rotation", to determine the main axis of subsurface seismic anisotropy. Alford considered four seismic sections acquired by using two horizontal (orthogonal) sources and two orthogonal horizontal receivers. If we denote source and receiver by S and R and in-line and cross-line by I and C, respectively, the four seismic sections can be denoted by: S_IR_I, S_IR_C, S_CR_I and S_CR_C, where "line" refers to the orientation of the seismic section. Alford observed that the seismic events in the cross-component sections (S_IR_C and S_CR_I) were better than those of the principal components sections (S_IR_I and S_CR_C). The reason for this behavior is shear-wave splitting, which occurs in azimuthally anisotropic media (see Section 1.4.4); for instance, a transversely isotropic medium whose axis of symmetry is horizontal and makes an angle $\pi/2-\theta$ with the direction of the seismic line. If $\theta = 0$, the seismic energy in the cross-component sections should be minimum. Thus, Alford's method consist in a rotation of the data to minimize the energy in the cross-component surveys, obtaining in this way the orientation of the symmetry axis of the medium. An application is to find the orientation of a set of vertical fractures, whose planes are perpendicular to the symmetry axis. In addition, Alford rotation allows us to obtain the reflection amplitudes for every angle of orientation of transmitter and receiver without having to collect data for all configurations.

The equivalent acquisition configurations in GPR surveys are shown in Figure 8.8, where the xx-, xy-, yx and yy-configurations correspond to the seismic surveys S_IR_I, S_IR_C, S_CR_I and S_CR_C, respectively. In theory, the xy- and yx-configurations should give the same result because of reciprocity.

We consider the 1-D equations along the vertical z-direction and a lossless transversely isotropic medium whose axis of symmetry is parallel to the surface ((x,y)-plane). In this case, the slower and faster shear waves S1 and S2 waves, whose velocities are $\sqrt{c_{55}/\rho}$ (S1) and $\sqrt{c_{66}/\rho}$ (S2), are analogous to the TM and TE waves, whose velocities are $1/\sqrt{\mu_2\epsilon_1}$ (TM) and $1/\sqrt{\mu_1\epsilon_2}$ (TE) (see Section 1.3.1 for the acoustic case and the (x,z)-plane of Figure 8.7 for the lossless electromagnetic case). A rigorous seismic theory illustrating the physics involved in Alford rotation is given by Thomsen (1988).

When the source radiation directivities (seismic shear vibrators or dipole antennas)

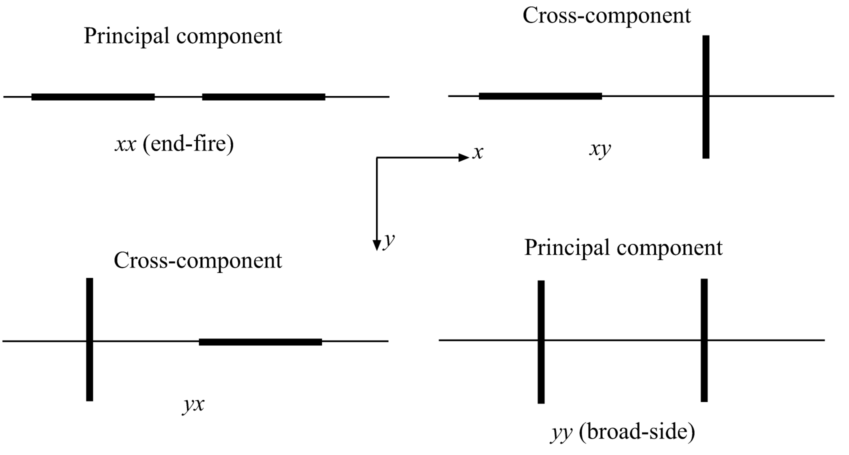

Figure 8.8: Different GPR transmitter-receiver antenna configurations, where S is transmitter and R is receiver. The survey line is oriented along the x-direction.

are aligned with the principal axes of the medium, the propagation equations can be written as

$$\begin{pmatrix} L_{11} & 0 \\ 0 & L_{22} \end{pmatrix} \cdot \begin{pmatrix} u_{11} & 0 \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} \delta(z)f(t) & 0 \\ 0 & \delta(z)f(t) \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (8.390)$$

where the L_{ij} are differential propagation operators, the u_{ij} are the recorded wave fields, and $f(t)$ is the source time history. The source term, in the right-hand side of equation (8.390), defines a set of two orthogonal sources aligned along the principal coordinates axes of the medium, such that the solutions u_{11} and u_{22} correspond to the seismic sections $S_I R_I$ and $S_C R_C$ or to the GPR configurations xx and yy, respectively.

Equation (8.390) can be expressed in matrix form as

$$\mathbf{L} \cdot \mathbf{U} = \mathbf{S} \cdot \mathbf{I}_2 = \mathbf{S}. \qquad (8.391)$$

The rotation matrix is given by

$$\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \mathbf{R} \cdot \mathbf{R}^\top = \mathbf{I}_2. \qquad (8.392)$$

Right-multiplying equation (8.391) by $\mathbf{R}$ rotates the sources counter-clockwise through an angle θ:

$$\mathbf{L} \cdot [\mathbf{U} \cdot \mathbf{R}] = \mathbf{S} \cdot \mathbf{R}. \qquad (8.393)$$

The term in square brackets is the solution for the new sources in the principal coordinates of the medium. The following operation corresponds to a counter-clockwise rotation of the receivers through an angle θ:

$$\mathbf{L} \cdot \mathbf{R} \cdot [\mathbf{R}^\top \cdot \mathbf{U} \cdot \mathbf{R}] = \mathbf{S} \cdot \mathbf{R}, \qquad (8.394)$$

where we have used equation (8.392).

Denoting by primed quantities the matrices in the acquisition coordinate system, equation (8.394) reads

$$\mathbf{L}' \cdot \mathbf{U}' = \mathbf{S}', \tag{8.395}$$

where $\mathbf{L}' = \mathbf{L} \cdot \mathbf{R}$, $\mathbf{S}' = \mathbf{S} \cdot \mathbf{R}$, and

$$\mathbf{U}' = \mathbf{R}^\top \cdot \mathbf{U} \cdot \mathbf{R}. \tag{8.396}$$

This equation allows the computation of the solutions in the principal system in terms of the solutions in the acquisition system:

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \mathbf{R} \cdot \mathbf{U}' \cdot \mathbf{R}^\top, \tag{8.397}$$

where

$$\begin{aligned} u_{11} &= u'_{11} \cos^2\theta + u'_{22} \sin^2\theta + 0.5(u'_{21} + u'_{12}) \sin 2\theta \\ u_{12} &= u'_{12} \cos^2\theta - u'_{21} \sin^2\theta + 0.5(u'_{22} - u'_{11}) \sin 2\theta \\ u_{21} &= u'_{21} \cos^2\theta - u'_{12} \sin^2\theta + 0.5(u'_{22} - u'_{11}) \sin 2\theta \\ u_{22} &= u'_{22} \cos^2\theta + u'_{11} \sin^2\theta - 0.5(u'_{21} + u'_{12}) \sin 2\theta. \end{aligned} \tag{8.398}$$

Minimizing the energy in the off-diagonal sections (u_{12} and u_{21}) as a function of the angle of rotation, we obtain the main orientation of the axis of symmetry. An example of application of Alford rotation to GPR data can be found in Van Gestel and Stoffa (2001).

8.14 Poro-acoustic and electromagnetic diffusion

Diffusion equations are obtained in poroelasticity and electromagnetism at low frequencies and under certain conditions, by which the inertial terms and displacement currents are respectively neglected. In this section, we derive the equations from the general theories, study the physics and obtain analytical solutions.

8.14.1 Poro-acoustic equations

The quasi-static limit of Biot's poroelastic equations, to describe the diffusion of the second (slow) compressional mode, is obtained by neglecting the accelerations terms in the equations of momentum conservation (7.210) and (7.211), and considering the constitutive equations (7.131) and (7.132), and Darcy's law (7.194). We obtain

$$p_f = M(\zeta - \alpha_{ij}\epsilon_{ij}^{(m)}), \tag{8.399}$$

$$\sigma_{ij} = c_{ijkl}^{(m)}\epsilon_{kl}^{(m)} - \alpha_{ij}p_f, \tag{8.400}$$

$$-\partial_i p_f = \frac{\eta}{\bar{\kappa}_{(i)}} \partial_t w_i \tag{8.401}$$

and

$$\partial_j \sigma_{ij} = 0, \tag{8.402}$$

where $i, j = 1, \ldots, 3$, and the parenthesis in (8.401) indicates that there is no implicit summation. Using $\zeta = -\partial_i w_i$ (see equation (7.173)), doing the operation ∂_t on (8.399), substituting (8.401) into the resulting equation, and combining (8.400) and (8.402) gives

$$\frac{1}{M}\partial_t p_f + \alpha_{ij}\partial_t \epsilon_{ij}^{(m)} = \partial_i \partial_i \left(\frac{\bar{\kappa}_i}{\eta} p_f\right) \tag{8.403}$$

and

$$\partial_j (c_{ijkl}^{(m)} \epsilon_{kl}^{(m)} - \alpha_{ij} p_f) = 0. \tag{8.404}$$

These equations and the strain-displacement relations $\epsilon_{ij}^{(m)} = (\partial_i u_j^{(m)} + \partial_j u_i^{(m)})/2$ is a set of four partial differential equations for $u_1^{(m)}$, $u_2^{(m)}$, $u_3^{(m)}$ and p_f. Equation (8.404) can be differentiated, and summing the three equations, we obtain

$$\partial_i \partial_j (c_{ijkl}^{(m)} \epsilon_{kl}^{(m)} - \alpha_{ij} p_f) = 0. \tag{8.405}$$

In the isotropic case, equations (8.403) and (8.405) become

$$\frac{1}{M}\partial_t p_f + \alpha \partial_t \theta_m = \Delta \left(\frac{\bar{\kappa}}{\eta} p_f\right) \tag{8.406}$$

and

$$\Delta(\lambda_m \theta_m - \alpha p_f) + 2\partial_i \partial_j (\mu_m \epsilon_{ij}^{(m)}) = 0, \tag{8.407}$$

where

$$\lambda_m = K_m - \frac{2}{3}\mu_m, \tag{8.408}$$

and we have used equation (7.28), $\Delta = \partial_i \partial_i$, $\theta_m = \epsilon_{ii}^{(m)}$, $\bar{\kappa}_i = \bar{\kappa}$, and $\alpha_{ij} = \alpha \delta_{ij}$. If we assume an homogeneous medium and use the property $\partial_j \partial_i \epsilon_{ij}^{(m)} = \theta_m$, we obtain

$$\frac{1}{M}\partial_t p_f + \alpha \partial_t \theta_m = \frac{\bar{\kappa}}{\eta} \Delta p_f \tag{8.409}$$

and

$$E_m \Delta \theta_m - \alpha \Delta p_f = 0, \tag{8.410}$$

where E_m is given by equation (7.291). Equation (7.299) is obtained if we take the Laplacian of equation (8.409) and combine the result with (8.410). An alternative diffusion equation can be obtained by doing a linear combination of equations (8.409) and (8.410),

$$\partial_t \left(\frac{1}{M} p_f + \alpha \theta_m\right) = d\Delta \left(\frac{1}{M} p_f + \alpha \theta_m\right), \tag{8.411}$$

where d is the hydraulic diffusivity constant defined in equation (7.300). Then, it is the quantity $M^{-1} p_f + \alpha \theta_m$ and not the fluid pressure, which satisfies the diffusion equation in Biot's poroelastic theory.

8.14.2 Electromagnetic equations

Maxwell's equations (8.9) and (8.10), neglecting the displacement-currents term $\hat{\boldsymbol{\epsilon}} * \partial_t \mathbf{E}$ and redefining the source terms, can be written as

$$\nabla \times \mathbf{E} = -\hat{\boldsymbol{\mu}} \cdot \partial_t (\mathbf{H} + \mathbf{M}) \tag{8.412}$$

and

$$\nabla \times \mathbf{H} = \hat{\boldsymbol{\sigma}} \cdot (\mathbf{E} + \mathbf{J}). \tag{8.413}$$

These equations can be expressed in terms of the electric vector or in terms of the magnetic vector as

$$\partial_t \mathbf{E} = -\hat{\boldsymbol{\sigma}}^{-1} \cdot \nabla \times (\hat{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}) - \hat{\boldsymbol{\sigma}}^{-1} \cdot \partial_t (\nabla \times \mathbf{M}) - \partial_t \mathbf{J}, \tag{8.414}$$

and

$$\partial_t \mathbf{H} = -\hat{\boldsymbol{\mu}}^{-1} \cdot \nabla \times (\hat{\boldsymbol{\sigma}}^{-1} \cdot \nabla \times \mathbf{H}) - \partial_t \mathbf{M} + \hat{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{J}, \tag{8.415}$$

respectively. Assuming a homogeneous and isotropic medium, equation (8.414) can be rewritten as

$$\partial_t \mathbf{E} = -(\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{\sigma}})^{-1} [\nabla(\nabla \cdot \mathbf{E}) - \Delta \mathbf{E}] - \partial_t \mathbf{J} = (\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{\sigma}})^{-1} \Delta \mathbf{E} - \partial_t \mathbf{J}, \tag{8.416}$$

where we have considered a region free of charges ($\nabla \cdot \mathbf{E} = 0$) and have neglected the magnetic source. Note that only for an isotropic medium, the tensors $\hat{\boldsymbol{\mu}}^{-1}$ and $\hat{\boldsymbol{\sigma}}^{-1}$ commute with the curl operator. In this case, $(\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{\sigma}})^{-1} = (\hat{\mu}\hat{\sigma})^{-1} \mathbf{I}_3$.

Similarly, equation (8.415) can be written as

$$\partial_t \mathbf{H} = (\hat{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{\sigma}})^{-1} \Delta \mathbf{H} + \hat{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{J}. \tag{8.417}$$

Equation (8.417) is a diffusion equation for $\mathbf{H}$, which is analogous to equation (8.411).

The TM and TE equations

If the material properties and the sources are invariant in the y-direction, the propagation can be described in the (x, z)-plane, and E_1, E_3 and H_2 are decoupled from E_2, H_1 and H_3, corresponding to the TM and TE equations, respectively.

Writing equations (8.414) and (8.415) in explicit Cartesian form, we obtain the TM equations

$$\hat{\sigma}\partial_t \begin{pmatrix} E_1 \\ E_3 \end{pmatrix} = \begin{pmatrix} \partial_3 \hat{\mu}^{-1} \partial_3 & -\partial_3 \hat{\mu}^{-1} \partial_1 \\ -\partial_1 \hat{\mu}^{-1} \partial_3 & \partial_1 \hat{\mu}^{-1} \partial_1 \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ E_3 \end{pmatrix} - \partial_t \begin{pmatrix} -\partial_3 M_2 \\ \partial_1 M_2 \end{pmatrix} - \hat{\sigma}\partial_t \begin{pmatrix} J_1 \\ J_3 \end{pmatrix} \tag{8.418}$$

and

$$\hat{\mu}\partial_t H_2 = \partial_1(\hat{\sigma}^{-1}\partial_1 H_2) + \partial_3(\hat{\sigma}^{-1}\partial_3 H_2) - \hat{\mu}\partial_t M_2 + (\partial_3 J_1 - \partial_1 J_3). \tag{8.419}$$

The respective TE equations are

$$\hat{\sigma}\partial_t E_2 = \partial_1(\hat{\mu}^{-1}\partial_1 E_2) + \partial_3(\hat{\mu}^{-1}\partial_3 E_2) - \partial_t(\partial_3 M_1 - \partial_1 M_3) - \hat{\sigma}\partial_t J_2 \tag{8.420}$$

and

$$\hat{\mu}\partial_t \begin{pmatrix} H_1 \\ H_3 \end{pmatrix} = \begin{pmatrix} \partial_3 \hat{\sigma}^{-1} \partial_3 & -\partial_3 \hat{\sigma}^{-1} \partial_1 \\ -\partial_1 \hat{\sigma}^{-1} \partial_3 & \partial_1 \hat{\sigma}^{-1} \partial_1 \end{pmatrix} \cdot \begin{pmatrix} H_1 \\ H_3 \end{pmatrix} - \hat{\mu}\partial_t \begin{pmatrix} M_1 \\ M_3 \end{pmatrix} + \begin{pmatrix} -\partial_3 J_2 \\ \partial_3 J_2 \end{pmatrix}. \tag{8.421}$$

Phase velocity, attenuation factor and skin depth

Let us consider an homogeneous isotropic medium. Then, the Green function corresponding to equation (8.416) and a source current

$$\mathbf{J}(x, y, z, t) = \boldsymbol{\iota}\delta(x)\delta(y)\delta(z)[1 - H(t)], \tag{8.422}$$

is the solution of

$$\partial_t \mathbf{E} = a\Delta\mathbf{E} + \boldsymbol{\iota}\delta(x)\delta(y)\delta(z)\delta(t), \tag{8.423}$$

where $\boldsymbol{\iota}$ defines the direction and the strength of the source, and

$$a = \frac{1}{\hat{\mu}\hat{\sigma}}. \tag{8.424}$$

In the frequency domain, the diffusion equation can then be written as a Helmholtz equation

$$\Delta\mathbf{E} + \left(\frac{\omega}{v_c}\right)^2 \mathbf{E} = -(\boldsymbol{\iota}/a)\delta(x)\delta(y)\delta(z), \tag{8.425}$$

where

$$v_c = \sqrt{\frac{a\omega}{2}}(1 + \mathrm{i}) \tag{8.426}$$

is the complex velocity. The same kinematic concepts used in wave propagation (acoustics and electromagnetism) are useful in this analysis. The phase velocity and attenuation factor can be obtained from the complex velocity as

$$v_p = \left[\mathrm{Re}\left(v_c^{-1}\right)\right]^{-1} \quad \text{and} \quad \alpha = -\omega\mathrm{Im}(v_c^{-1}), \tag{8.427}$$

respectively. The skin depth is the distance $\bar{d}$ for which $\exp(-\alpha\bar{d}) = 1/\mathrm{e}$, where e is Napier's number, i.e., the effective distance of penetration of the signal. Using equation (8.426) yields

$$v_p = 2\pi f\bar{d}, \quad \text{and} \quad \alpha = 1/\bar{d}, \tag{8.428}$$

$$\bar{d} = \sqrt{\frac{a}{\pi f}}, \tag{8.429}$$

where $f = \omega/2\pi$ is the frequency.

Analytical solutions

Equation (8.423) has the following solution (Green's function):

$$\mathbf{E}(r, t) = \left(\frac{\boldsymbol{\iota}}{4\pi a t}\right)\exp[-r^2/(4at)], \tag{8.430}$$

where

$$r = \sqrt{x^2 + y^2 + z^2} \tag{8.431}$$

(Carslaw and Jaeger, 1959; Polyanin and Zaitsev, 2004). The time-domain solution for a source $F(t)$, e.g., equation (2.233), is obtained by a numerical time convolution between the expression (8.430) and $F(t)$.

Equation (8.423) corresponding to the initial-value problem is

$$\partial_t \mathbf{E} = a\Delta \mathbf{E}. \tag{8.432}$$

Assume for each component E_i the initial condition $E_{i0} = E_i(x, y, z, 0) = \delta(x)\delta(y)\delta(z)$. A transform of (8.432) to the Laplace and wavenumber domains yields

$$E_i(k_1, k_2, k_3, p) = \frac{1}{p + a(k_1^2 + k_2^2 + k_3^2)}, \tag{8.433}$$

where p is the Laplace variable, and the properties $\partial_t E_i \rightarrow pE_i - E_0(k_1, k_2, k_3)$ and $E_{i0}(k_1, k_2, k_3) = 1$ have been used.

To obtain $E_i(k_1, k_2, k_3, t)$, we compute the inverse Laplace transform of (8.433),

$$E_i(k_1, k_2, k_3, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp(pt)dp}{p + a(k_1^2 + k_2^2 + k_3^2)}, \tag{8.434}$$

where $c > 0$. There is one pole,

$$p_0 = -a(k_1^2 + k_2^2 + k_3^2). \tag{8.435}$$

Use of the residue theorem gives the solution

$$E_i(k_1, k_2, k_3, t) = \exp[-a(k_1^2 + k_2^2 + k_3^2)t]H(t). \tag{8.436}$$

The solution for a general initial condition $E_{i0}(k_1, k_2, k_3)$ is given by

$$E_i(k_1, k_2, k_3, t) = E_{i0}(k_1, k_2, k_3)\exp[-a(k_1^2 + k_2^2 + k_3^2)t]H(t), \tag{8.437}$$

where we have used equation (8.436). In the space domain the solution is the spatial convolution between the expression (8.436) and the initial condition. The effect of the exponential on the right-hand side is to filter the higher wavenumbers. The solution in the space domain is obtained by a discrete inverse Fourier transform, using the fast Fourier transform. The three components of the electric vector are not functionally independent, since they must satisfy $\nabla \cdot \mathbf{E} = 0$ (in a region free of electric charges). These analytical solutions also describe the diffusion of the slow compressional mode, since equations (8.411) and (8.432) are mathematically equivalent.

8.15 Electro-seismic wave theory

The acoustic (poroelastic) and electromagnetic wave equations can be coupled to describe the so-called electro-seismic phenomenon. In porous materials, the grain surfaces have an excess (bound) charge that is balanced by free ions diffusing in the fluid layers. The bound and diffusive charges are called the "electric double layer". Acoustic waves generate a force which transports the diffuse charge of the double layer relative to the bound charge on the grain surfaces, resulting in a "streaming" electric current. This phenomenon is known as *electro-filtration*. On the other hand, an electric field induces a conduction current – according to Ohms's law – and a body force on the excess charge of the diffuse double layer, resulting in fluid filtration. This phenomenon is known as *electro-osmosis*.

There is experimental evidence that earthquake triggering is associated with fluid pressures gradients (Mizutani, Ishido, Yokokura and Ohnishi, 1976). The related fluid flow produces the motion of the fluid electrolyte and creates an electric field (*electrokinetic* effect) (Sill, 1983; Pride and Morgan, 1991). It has been reported that anomalous electromagnetic emissions were observed hours before the occurrence of major earthquakes and volcanic eruptions (Yamada, Masuda and Mizutani, 1989). Similarly, the electrokinetic phenomenon may play an important role in predicting rock fracturing in mines, and locating water and oil reservoirs (Wurmstich and Morgan, 1994).

The basic electro-seismic theory involves the coupling between Maxwell's equations and Biot's equations of dynamical poroelasticity. Frenkel (1944) was the first to have developed a theory to describe the phenomenon. Pride and Garambois (2005) analyzed Frenkel's equations and point out an error in developing his effective compressibility coefficients, preventing him to obtain a correct expression for Gassmann's modulus. A complete theory is given by Pride (1994), who obtained the coupled electromagnetic and poroelastic equations from first principles.

The general equations describing the coupling between mass and electric-current flows are obtained by including coupling terms in Darcy's and Ohm's laws (7.194) and (8.5), respectively. We obtain

$$\partial_t \mathbf{w} = -\frac{1}{\eta}\bar{\boldsymbol{\kappa}} * \partial_t[\mathrm{grad}(p_f)] + \mathbf{L} * \partial_t \mathbf{E}, \tag{8.438}$$

$$\mathbf{J}' = -\mathbf{L} * \partial_t[\mathrm{grad}(p_f)] + \hat{\boldsymbol{\sigma}} * \partial_t \mathbf{E} + \mathbf{J}, \tag{8.439}$$

where p_f is the pore pressure, $\mathbf{E}$ is the electric field, η is the fluid viscosity, $\bar{\boldsymbol{\kappa}}(t)$ is the global-permeability matrix, $\hat{\boldsymbol{\sigma}}(t)$ is the time-dependent conductivity matrix, $\mathbf{J}$ is an external electric source, and $\mathbf{L}(t)$ is the time-dependent electrokinetic coupling matrix. We have considered time-dependent transport properties (Pride, 1994); equations (7.194) and (8.5) are obtained by substitution of $\bar{\boldsymbol{\kappa}}(t)$ and $\mathbf{L}(t)$ with $\bar{\boldsymbol{\kappa}}H(t)$ and $\mathbf{L}H(t)$, where the time dependence is only in the Heaviside step function. For an electric flow deriving from a streaming potential U (Sill, 1983; Wurmstich and Morgan, 1994), $\mathbf{E} = -\mathrm{grad}(U)$, and the electromagnetic equations reduce to quasistatic equations similar to those describing piezoelectric wave propagation, i.e., the acoustic field is coupled with a quasi-static electric field.

The complete time-domain differential equations for anisotropic (orthorhombic) media are given by the poroelastic equations (7.255) and (7.256), and the electromagnetic equations (8.1)-(8.5), including the coupling terms according to equations (8.438) and (8.439). We obtain

$$\partial_j \sigma_{ij} = \rho \partial^2_{tt} u_i^{(m)} + \rho_f \partial^2_{tt} w_i, \tag{8.440}$$

$$-\partial_i p_f = \rho_f \partial^2_{tt} u_i^{(m)} + m_i \partial^2_{tt} w_i + \eta \chi_i * [\partial_{tt} w_i - (\mathbf{L} * \partial_t \mathbf{E})_i], \tag{8.441}$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B} + \mathbf{M} \tag{8.442}$$

and

$$\nabla \times \mathbf{H} = \partial_t \mathbf{D} - \mathbf{L} * \partial_t[\mathrm{grad}(p_f)] + \hat{\boldsymbol{\sigma}} * \partial_t \mathbf{E} + \mathbf{J}, \tag{8.443}$$

where there is no implicit summation in the last term of equation (8.441). Note the property $\partial_t \bar{\kappa}_i(t) * \partial_t \chi_i(t) = \delta(t)$, according to equation (2.41). Pride (1994) obtained analytical expressions for the transport coefficients as well as for the electric conductivity as a function of frequency.