Conjugacy classes in

$$G = \left\{ \begin{bmatrix} x & y \\ & x^{-1} \end{bmatrix} \,\middle|\, x \in \mathbb{R},\ y \in \mathbb{R} \right\}.$$

a little different when $\mathbb{R}$ replaced by $\mathbb{C}$.

$$A = \begin{bmatrix} x & y \\ & x^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ & a^{-1} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} a^{-1} & -b \\ & a \end{bmatrix}$$

$$BAB^{-1}$$

$$= \begin{bmatrix} ax & ay + bx^{-1} \\ & a^{-1}x^{-1} \end{bmatrix} \begin{bmatrix} a^{-1} & -b \\ & a \end{bmatrix}$$

$$= \begin{bmatrix} x & -abx + (ay + bx^{-1})a \\ & x^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} x & -abx + a^2y + abx^{-1} \\ & x^{-1} \end{bmatrix} \quad (*)$$

① when $x \neq x^{-1}$,

choose $b = \dfrac{ay}{x - x^{-1}}$

Then $BAB^{-1} = \begin{bmatrix} x & \\ & x^{-1} \end{bmatrix}$

and $\begin{bmatrix} x & \\ & x^{-1} \end{bmatrix}$, $\begin{bmatrix} x^{-1} & \\ & x \end{bmatrix}$ are

not in the same conjugacy class

because of $(*)$

(The elements on the diagonal cannot change)

(2) when $x = x^{-1}$, $x = \pm 1$.

(2.1) $x = 1$,

$$BAB^{-1} = \begin{bmatrix} 1 & a^2 y \\ & 1 \end{bmatrix}$$

$y = 0$, $A = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$.

conjugacy class $\left\{ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \right\}$

$y \neq 0$, then. $a^2 y$ runs through all real numbers with the same sign

Two conjugacy classes

$$\left\{ \begin{bmatrix} 1 & y \\ & 1 \end{bmatrix} \middle| y > 0 \right\}, \left\{ \begin{bmatrix} 1 & y \\ & 1 \end{bmatrix} \middle| y < 0 \right\}.$$

(2.2) Same

$$\left\{ \begin{bmatrix} -1 & y \\ & -1 \end{bmatrix} \middle| \, y > 0 \right\}$$

$$\left\{ \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} -1 & y \\ & -1 \end{bmatrix} \middle| \, y < 0 \right\}.$$

In conclusion:

$x \neq \pm 1$ $\left\{ \begin{bmatrix} x & y \\ & x^{-1} \end{bmatrix} \right\}$

$x = 1$ $\left\{ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 & y \\ & 1 \end{bmatrix} \middle| \, y > 0 \right\}$

$$\left\{ \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \middle| \, y < 0 \right\}.$$

$x = -1$ $\left\{ \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} -1 & y \\ & -1 \end{bmatrix} \middle| \, y > 0 \right\}$

$\left\{ \begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \middle| \, y < 0 \right\}$.