Math 241 Homework#8

due 10/31 Thursday in class

Heat equation

Read *A*pplied PDE by Haberman (5th edition) Chapter 7, 7.1-7.6. Heat and wave equations on a rectangle, see Chapter 7.3. See the main theorem on page 283.

1. *A*pplied PDE by Haberman, chapter 7.3, exercise 7.3.4 a).
2. *A*pplied PDE by Haberman, chapter 7.3, exercise 7.3.5.
3. *A*pplied PDE by Haberman, chapter 7.3, exercise 7.3.7 a).
4. *A*pplied PDE by Haberman, chapter 7.4, exercise 7.4.1.
5. *A*pplied PDE by Haberman, chapter 7.5, exercise 7.5.8.
6. Can you hear the shape of a rectangle?
   (a) Find the lengths of the sides of the rectangle $R = [0, L] \times [0, H]$ such that $\lambda = 2$ and $\lambda = 5$ are the smallest eigenvalues of the problem
   $$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } R \\ \phi = 0 & \text{on } \partial R \end{cases}$$
   (b) How about the rectangle $R' = [0, L'] \times [0, H']$ such that the smallest eigenvalues of the same problem above are $\lambda = \frac{13}{36}$ and $\lambda = \frac{25}{36}$?
7. [See Section 7.3 in Haberman] In the square $\Omega = [0, L] \times [0, L]$ in the plane, a population of bacteria is evolving according to a diffusion equation. The bacteria also grows at a rate proportional to the concentration. It satisfies the equation
   $$u_t = k\Delta u + \alpha u$$
   where k and α are constants. Assume the sides of the square are coated in penicillin, so $u = 0$ there. What is the condition on k and α so that the bacteria's concentration does not grow without bound?
8. Let region R be the unit disc $\{(x, y) | x^2 + y^2 \leq 1\}$. Consider the eigenvalue problem
   $$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } R \\ \phi = 0 & \text{on } \partial R \end{cases}$$
   Find an upper bound of the first eigenvalue by
   (a) Test function $f(x, y) = 1 - r^2$,
   (b) Comparing with a square inside the disc.

   Which upper bound is better?