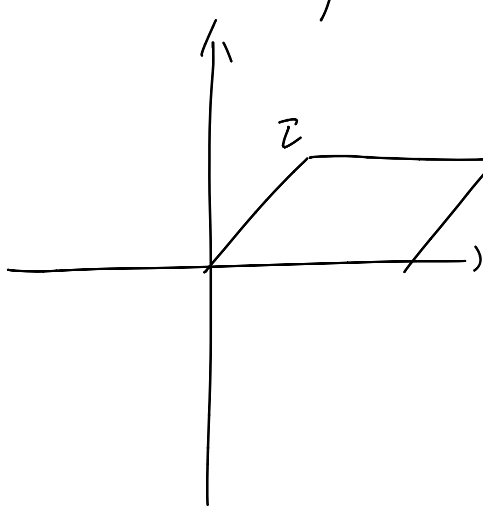
Joint w/ Zhiwei Zheng

Construction of moduli spaces.

- GIT.
- Hodge theory.

Elliptic curves / cubic curves

$$E \cong \mathbb{C} / \mathbb{Z} + \mathbb{Z}\tau .$$



$\tau \in \mathbb{H}$

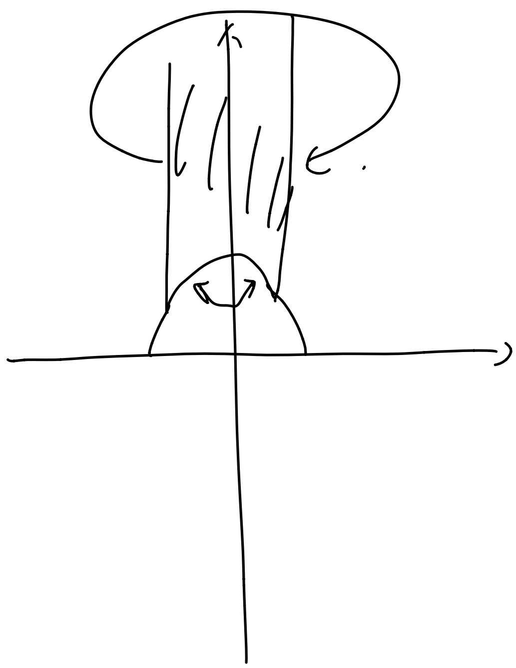
$= \{ x + yi \mid y > 0 \}.$

$= \{ \tau \mid \operatorname{Im} \tau > 0 \}.$

moduli $\mathcal{M}_1 = \mathbb{H} / SL(2,\mathbb{Z}) \cong \mathbb{A}^1$

$j(z)$.

j invariant.



Cptification $\bar{\mathcal{M}}_1 = \big(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})\big) / SL(2,\mathbb{Z})$

Satake-Baily-Borel.

$$\bar{\mathcal{M}}_1 = \mathcal{M}_1 \sqcup \{\infty\} \cong \mathbb{P}^1.$$

Algebraic description: $\mathrm{Proj}\big(\underbrace{H^0(\mathbb{H}, \mathcal{O}^k)^{SL(2,\mathbb{Z})}}_{\text{ring of modular}}\big)$, GIT construction: forms.

$E \hookrightarrow \mathbb{P}^2$ as cubic curve.

$F(x,y,z)=0$. $\deg F = 0$.

space of cubic forms $\mathbb{P}^9$

$$\dim \mathrm{Sym}^3(\mathbb{C}^3) = \binom{3+2}{2} = 10.$$

$P^s \subset \mathbb{P}^9$

$\downarrow$

smooth cubic forms.

$$P^s / SL(3,\mathbb{C}) \cong \mathbb{A}^1$$

$SL(3,\mathbb{C}) \curvearrowright \mathbb{P}^9$ Naive orbit space is not Hausdorff.

x^3 $H = \langle (\varepsilon, \varepsilon^{-1}, 1) \rangle$

rescale to $\varepsilon^3 x^3 \to 0$.

GIT quotient: (P, L) G reductive.
L: polarization.

$G \curvearrowright (P, L)$ equivariant action.

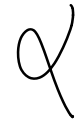
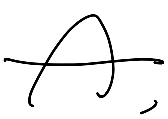
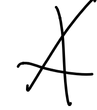
$$G \backslash\backslash P = \mathrm{Proj}\left(\bigoplus_k H^0(P, L^k)^G \right).$$

$x \in P$
- stable P^s G_x finite.
- semistable P^{ss} $\exists s \in H^0(P, L^k)^G$, $s(x) \neq 0$
- unstable.

$$G \backslash\!\backslash P \cong G \backslash\!\backslash P^{ss}$$ as orbit space.

cubic forms. $P = \mathbb{P}^9$.

$P^s = \{$ smooth cubic forms $\}$

$P^{ss} = \{$ smooth , ,  $\}$

$$\begin{tikzcd}
SL(3,\mathbb{C}) \backslash P^s \arrow[r, "\tau = \frac{\int_\alpha \omega}{\int_\beta \omega}"] \arrow[d] & SL(2,\mathbb{Z}) \backslash \mathbb{H} \arrow[d] \\
SL(3,\mathbb{C}) \backslash\!\backslash P^{ss} \arrow[r] & SL(2,\mathbb{Z}) \backslash \left(\mathbb{H} \sqcup \mathbb{P}^1(\mathbb{Q})\right) = \left(SL(2,\mathbb{Z}) \backslash \mathbb{H}\right)^*
\end{tikzcd}$$

Satake-Baily-Borel.

Not only on the space level.

but also w.r.t polarizations

$$\bigoplus H^0(\mathbb{P}^9, \mathcal{O}(1))^{SL(3,\mathbb{C})} \longrightarrow \underline{\bigoplus H^0(\mathbb{H}, \mathbb{C}^k)^{SL(2,\mathbb{Z})}}$$

(S, T)

aronhold's invariants.

space of modular forms

E_4, E_6

$$\boxed{j = 1728 \cdot (4S)^3 / (4S)^3 - T^2}$$

More examples: $C: y^m = (x-x_1)^{n_1} \cdots (x-x_s)^{n_s}$.

(Deligne-Mostow). $\widetilde{(\mathbb{P}^1, \mathcal{O}(n_1) \boxtimes \cdots \mathcal{O}(n_s))} \big/ SL(2) \cong (\Gamma \backslash \mathbb{B})^* \; BB.$

$H^1(C)_{\mathbb{C}} : g \qquad g \circlearrowleft \mathbb{Z}/m\mathbb{Z}$-action.

$H^1(C)_\chi : 1, \; s-3 \; := (V, h)$.

$H^{1,0}(C) \subset H^1(C)_\chi$.

$$\mathbb{B} = \{ x \in \mathbb{P}(V) \mid \langle x, x \rangle_h > 0 \}.$$

$$\overline{M} \xrightarrow[\text{Picard}]{} (\Gamma \backslash \mathbb{B})^* .$$

Lauricella functions

$$\int \frac{dx}{\sqrt{(x-x_1)^{n_1} \cdots (x-x_s)^{n_s}}}$$

Goal: compare GIT with Baily-Borel.

(Mumford's toroidal. Looijenga).

Hodge theory side.

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deg 2 K3. (X, H). H semiample.

$(H,H) = 2$, generic X, H.

$|H|: X \xrightarrow{2:1} \mathbb{P}^2$.

branched along $C \subset \mathbb{P}^2$.

$\deg C = 6$.

GIT: $\mathbb{P}(\mathrm{Sym}^6(\mathbb{C}^3)) // SL(3,\mathbb{C}) = \overline{M}$.

Hodge theory side.

$\mathbb{L}_{K3} = H^2(X, \mathbb{Z}) \ni [H]$

$T = [H]^\perp$ in $H^2(X, \mathbb{Z})$

$T, \langle , \rangle$ has signature $(2, 19)$

$$\begin{array}{ll} & H^{2,0} \quad H^{1,1} \quad H^{0,2} \\ H^2(X, \mathbb{C})_0 & \quad 1, \quad 19, \quad 1. \end{array}$$

$= T \otimes \mathbb{C}$.

$$\mathbb{D} = \left\{ x \in \mathbb{P}(T_{\mathbb{C}}) \;\middle|\; \begin{array}{l} \langle x, x \rangle = 0 \\ \langle x, \bar{x} \rangle > 0 \end{array} \right\}$$

$\mathbb{D}$ type IV Hermitian symmetric domain

$\mathbb{D} / \Gamma$.

Global Torelli: $\mathcal{M}^{sm} \hookrightarrow \mathbb{D} / \Gamma$.

Compare GIT with BB.

(Shah)

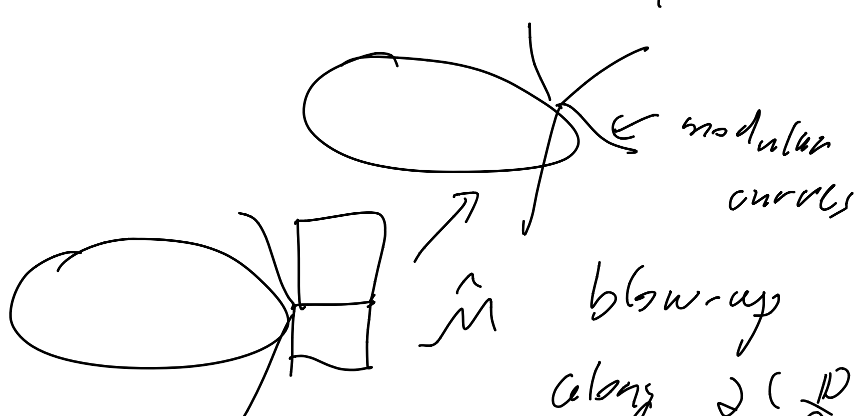
$$\begin{tikzcd} \overline{M} \arrow[rr, dashed] & & (\mathbb{P}^{10})^* \\ & \hat{M} \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

$\overline{M}$ has only one semi-stable point.

$w = [Q^3]$ Q quadric.

$\hat{M} = Bl_{[w]} \overline{M}$ (Kirwan blow-up).

$\pi_1^{-1}[w] =$ Hyperplane in $\mathbb{P}^{10}$.



$\hat{M}$ blow-up along $\partial (\mathbb{P}^{10})^*$.

General setting (Looijenga)

In many cases $\mathbb{D}$ = type IV.

or $\mathbb{D}$ = cplx hyperbolic ball

V, h Hermitian form $(1,n)$ ball.

$\mathbb{B}^n = \{x \in \mathbb{P}(V) \mid (x,x) > 0\}$.

$p : M \to \Gamma \backslash \mathbb{D}$.

- $\mathrm{Im}(M) = \Gamma \backslash \mathbb{D} - \bigcup_{H \in \mathcal{H}} H$

- $\mathrm{codim}(\bar{M} - M) \geq 2$.

- p also identifies polarizations.

$\overline{\Gamma \backslash \mathbb{D} - \mathcal{H}_\Delta} \overset{\text{Looijenga}}{:=} (\Gamma \backslash \mathbb{D})_{\mathcal{H}}$.

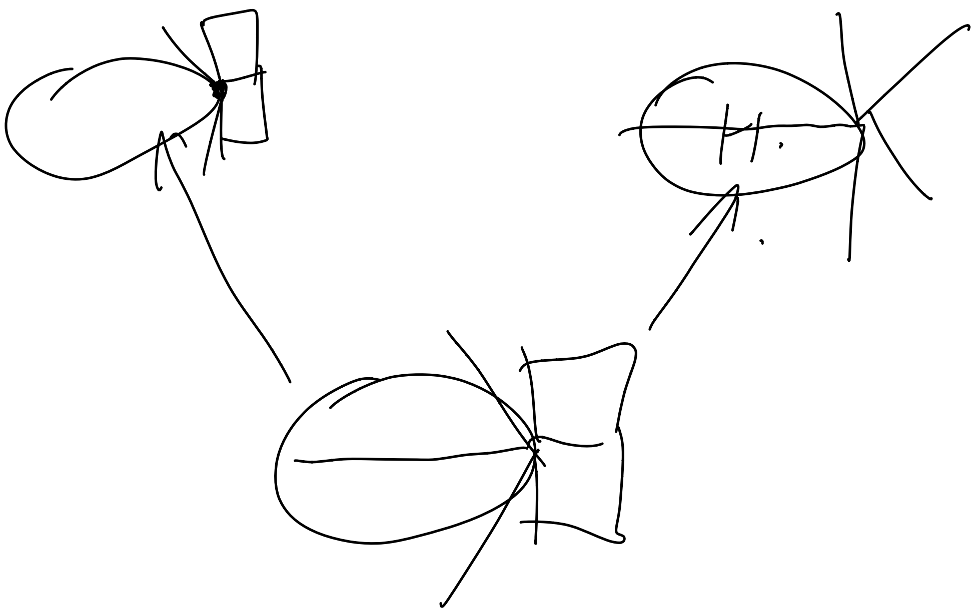
is constructed as follows.

① Blow-up of Baily-Borel along the bdry.

$$(\overline{\mathbb{D}/\Gamma})^{\text{semi-toric}} \longrightarrow (\overline{\mathbb{D}/\Gamma})^{*}.$$

② Blow-up intersections of $\mathcal{H}_{HG}$.

③ Blow-down in opposite direction.



(Hodge Theory interpretation).

cubic 4-folds, $\dim V_{\mathbb{C}} =$

$X \subset \mathbb{P}^5$.

12) successive blow up strata in $\mathcal{H}_\infty$

13) blow-down in other direction

$\mathrm{Proj}\left(\bigoplus H^0(\mathbb{D}-\mathcal{H}_\infty, \mathbb{L}^{\otimes k})^\Gamma\right)$

Cubic 4-folds. $X \subset \mathbb{P}^5$.

h^2 hyperplane class.

$H^4(X)$ 0 1 21 1 0.

Kirwan

$H^4(X)_0 \cong E_8^2 \oplus U^2 \oplus A_2$

(Voisin) $p: \mathcal{M}_0 \longrightarrow \Gamma \backslash \mathbb{D}^{20}$

$\Gamma \subset \mathrm{Aut}(\Lambda_0)$ with index 4.

$\mathcal{M}_0$ GIT of smooth cubic 4-folds

$\Gamma \subset \Gamma_0 \subset \mathrm{Aut}(\Lambda_0)$

Γ: keep the discriminant form 3'.

Γ_0: spinor norm $= 1$.

p open embedding.

(Radu, Looijenga)

p extends to isomorphism.

$\overline{\mathcal{M}} \xrightarrow{\cong} \overline{(\Gamma \backslash \mathbb{D})}_{\mathcal{H}_\infty}$.

almost all known cases can be considered as cubic fourfolds with certain symmetry or degeneration (including deg 2 K3).

$\dim V_{\mathbb{C}} = 6$. $A \subset SL(V)$.
finite subgroup.

$\lambda : A \to \mathbb{C}^*$

$V_\lambda \subset \mathrm{Sym}^3 V^*$ λ-eigenspace.

$\exists F \in V_\lambda$. X_F smooth cubic 4-fold.

$N = \{ g \mid gAg^{-1} = A,\ \lambda^g = \lambda \}$

$$\mathbb{P}(V_\lambda) /\!/ N \longrightarrow \text{GIT moduli of } (A,\lambda)\text{-symmetric cubic 4-fold.} \quad \overline{\mathcal{M}_A}.$$

Ex: $A = \langle \mathrm{diag}(\omega, 1, 1, 1, 1, 1) \rangle$ (Allcock-Carlson-Toledo)

$\mathbb{P}(V_\lambda) = \{ y^3 + F(x_0, x_1, x_2, x_3, x_4) = 0 \}$.

$\mathbb{P}(V_\lambda) /\!/ N$ = GIT moduli of cubic 3-fold.

A acts on $H^4(X_F)$.

0 1 20 1. 0.

$H^4(X_F)_\chi$ 1 m 1. or 1 m 0 — according to χ real or not real. K3-type, type A

ACT, $H^4_\chi = 1, 10, 0$

Thm: $p : \overline{\mathcal{M}}_A \dashrightarrow \Gamma_A \backslash \mathbb{D}_A$.

$\mathbb{D}$ type IV or cplx hyperbolic ball

extends to $\overline{\mathcal{M}}_A \longrightarrow \overline{(\Gamma_A \backslash \mathbb{D}_A)}^{\mathcal{H}_\infty}$.

(There is a criterion on (A, Λ) to determine $\mathcal{H}_\infty = \phi$).

Ex: (Pearlstein-Laza-Zheng)

$F = y^2 H + F(x_0, \dots x_4)$

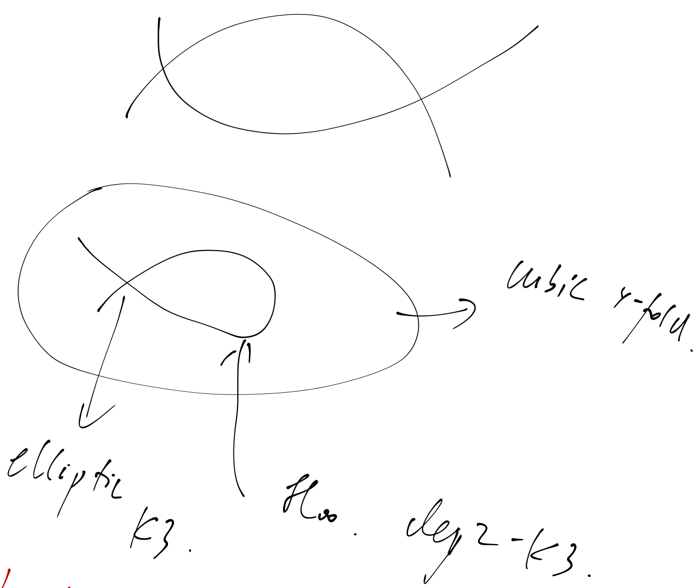
pair of Hypersurface and cubic 3-fold.

$\mathbb{P}^4 \times \mathbb{P}^{3*}$. $\mathcal{O}(1) \boxtimes \mathcal{O}(3)$

$/\!/ SL(5)$

$\cong \left(\Gamma \backslash \mathbb{D}^{14}\right)^{BB}$

Type IV.



should I mention the example of Hilbert Modular surface ??

classification of symplectic symmetry of cubic 4-folds (Laza-Zheng).

Pf:
$$\begin{array}{ccc}
\overline{\mathcal{M}}_A & \dashrightarrow & \overline{(\Gamma_A \backslash \mathbb{D}_A)}^{\mathcal{H}_\infty} \\
\text{①} \downarrow & & \text{②} \downarrow \\
\overline{\mathcal{M}} & \longrightarrow & \overline{(\Gamma \backslash \mathbb{D})}^{\mathcal{H}_\infty}.
\end{array}$$

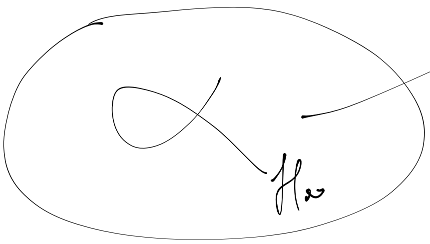
① Luna's Theorem. finite map

② functoriality of semi-toric compactification

$G'_{\mathbb{Q}} \hookrightarrow G_{\mathbb{Q}}$ BB (Kiernan-Kobayashi).

Toroidal M. Harris.

BB criterion.



Blow-up, Blow-down

parametrize by $\mathbb{P}^1$.

$$g(x_0, \ldots x_5) = \begin{vmatrix} x_0 & x_1 & x_2+2ax_5 \\ x_1 & x_2-ax_5 & x_3 \\ x_2+2ax_5 & x_3 & x_4 \end{vmatrix} + b x_5^3$$

$(a:b) \in W\mathbb{P}(1,3)$

$b=0$. determinantal $X_{(1,0)}$

$$\mathrm{Sing}(X_{(1,0)}) = \mathbb{P}(\mathbb{C}^3) \to \mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^3))$$

$$GL(3) \to GL(V)$$

$A = \langle w, w^2, w^3, w^4, w^5, w^6 \rangle$

$w = e^{2\pi i/7}$

$$F = x_0^2 x_4 + x_1^2 x_2 + x_0 x_2^2 + x_3^2 x_5 + x_3 x_4^2 + x_1 x_5^2 + a x_0 x_1 x_3 + b x_2 x_4 x_5$$

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ & & \downarrow 1:1 \\ & & \mathbb{P}^2 \end{array}$$

$C = \bigcup C_i$.

$E_1 \dots E_n$

$$M = \operatorname{span}\{[H], [C_i], [E_i]\}$$

Prop: $M \hookrightarrow \mathbb{L}_{K3}$ primitive sublattice

X is a M-polarized K3., period map by $M^{\perp}$

Thm (K-T):
$$SL(3)\backslash\backslash \overline{W} \xrightarrow{\cong} \left(\Gamma \backslash \mathbb{D}\right)_{\text{Looi}}$$

If $[Q^3] \notin \overline{W}$, then $\left(\Gamma\backslash\mathbb{D}\right)_{\text{Looi}} \cong \left(\Gamma\backslash\mathbb{D}\right)^{BB}$.

Ex: (Matsumoto-Sasaki-Yoshida)

$W = \{F = l_1 l_2 \cdots l_6,\ l_i \text{ line}\}$.

$$\overline{W} = S_6 \backslash (\mathbb{P}^1)^6,\ \mathcal{O}(1)^{\boxtimes 6}.$$

$$\Rightarrow \quad SL(2) \times S_6 \backslash\backslash (\mathbb{P}^1)^6,\ \mathcal{O}(1)^{\boxtimes 6} \cong \overline{\Gamma \backslash \mathbb{D}^4}^{BB}.$$

C-Y Donagi-Macerato-Sharpe.

Hodge

line bundle

torsion

Symplectic lattice. reflection.

not coming from geometry

why???