

Algebraic Curves homework 1, due 9/28 in class

1. Prove that when $\omega_1, \omega_2 \in \mathbb{C}$ are \mathbb{R} -linearly independent, then

1. $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is discrete.
2. $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is Hausdorff.
3. $\mathbb{C} \rightarrow \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ is a covering map.

2. Let V be a complex vector space of dimension n , with \mathbb{C} -basis e_1, \dots, e_n , and $T: V \rightarrow V$ is a \mathbb{C} -linear transformation. Suppose T has matrix representation $X = A + Bi$ where $A, B \in M_n(\mathbb{R})$ under (complex) basis e_1, \dots, e_n . Prove

1. $e_1, \dots, e_n, ie_1, \dots, ie_n$ is an \mathbb{R} -basis of V .
2. T has matrix

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

under the \mathbb{R} -basis above when T is viewed as an \mathbb{R} -linear transformation.

3.

$$\det \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = |\det X|^2.$$

3. Complete the proof of implicit function theorem: Assume $f(z, w)$ is holomorphic w.r.t $(z, w) \in U \subset \mathbb{C}^2$, and $\frac{\partial f}{\partial z} \neq 0$ for all $(z, w) \in U$, and

$$\{f = 0\} \cap U = \{(g(w), w) \mid w \in D\}$$

for some open subset $D \subset \mathbb{C}$.

Prove g is holomorphic w.r.t w by showing $\frac{\partial g}{\partial \bar{w}} = 0$.

4. Let x_1, \dots, x_n be distinct points on \mathbb{C} and

$$f(x, y) = y^d - (x - x_1) \cdots (x - x_n).$$

Prove that $C = \{f(x, y) = 0\}$ defines a Riemann surface in \mathbb{C}^2 . (Question to think about: what is the topological shape of C ?)

5. (Extra exercise. You get extra credit for this exercise) Prove the implicit function theorem for several variables.

Let $f_1(z_1, \dots, z_n), \dots, f_m(z_1, \dots, z_n)$ be holomorphic functions w.r.t $(z_1, \dots, z_n) \in U \subset \mathbb{C}^n$, where $m \leq n$. Assume $(a_1, \dots, a_n) \in U$ satisfies $f(a_1, \dots, a_n) = 0$ and

$$\left(\frac{\partial f_i}{\partial z_j} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

is nondegenerate at (a_1, \dots, a_n) .

Then there exists a neighborhood V of (a_1, \dots, a_n) and holomorphic functions

$$g_1(z_1, \dots, z_m), \dots, g_{n-m}(z_1, \dots, z_m)$$

defined on a neighborhood D of $(a_1, \dots, a_n) \in \mathbb{C}^n$ such that

$$\{f = 0\} \cap V = \{(z_1, \dots, z_m, g_1(z_1, \dots, z_m), \dots, g_{n-m}(z_1, \dots, z_m)) \mid (z_1, \dots, z_m) \in D\}.$$