Period integrals

Ex: $y^2 = x(x-1)(x-\lambda)$

$$\omega = \frac{dx}{y} \qquad f(\lambda) = \int_\gamma \frac{dx}{y} = \Pi_\gamma .$$

$$\lambda(1-\lambda) f'' + (1-2\lambda) f' = \frac{1}{4} f$$

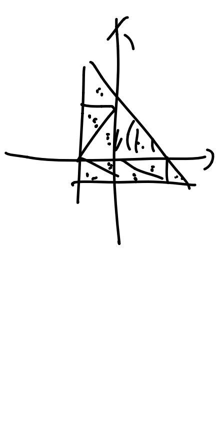
$$\sum \binom{-\frac{1}{2}}{n} \lambda^n = f_1 = {}_2F_1(1/2, 1/2; 1; \lambda) \quad \text{holomorphic.}$$

$$f_2 = (\log \lambda) f_1 + F_1(1/2, 1/2, \lambda)$$

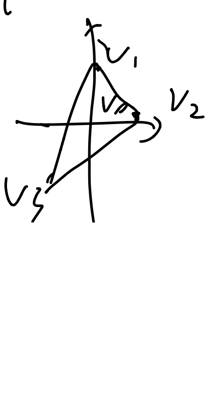
Hypersurfaces in $\mathbb{P}^n$.

Dwork-Griffiths residue.

Gelfand-Kapranov-Zelevinsky: (Toric hypersurfaces)



$$f = \sum_{i=0}^{p} a_i t^{v_i} . \qquad v_i \in \Delta \cap \mathbb{Z}^n .$$



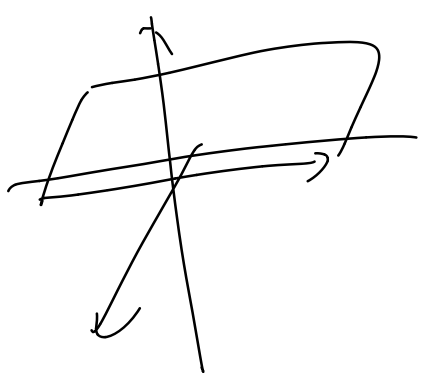
$\mathbb{P}_\Delta$

$$\Pi_\gamma = \int_\gamma \mathrm{Res} \frac{w}{F} = \int_\gamma \frac{dt_1 \cdots dt_n}{\underline{t_1 \cdots t_n}\, f}$$

$$L \in \{ (l_0, \dots, l_p) \in \mathbb{Z}^{p+1} :$$

$$\sum l_i v_i = 0, \quad \sum l_i = 0.$$

$$\begin{cases} \left(\prod_{l_i > 0} \left(\frac{\partial}{\partial a_i}\right)^{l_i} - \prod_{l_j < 0} \left(\frac{\partial}{\partial a_i}\right)^{-l_j} \right) \Pi_\gamma = 0 \\ \left(\sum_{i=0}^{p} v_i^j \, a_i \frac{\partial}{\partial a_i} \right) \Pi_\gamma = 0 \quad \forall j. \\ \left(\sum a_i \frac{\partial}{\partial a_i} + 1 \right) \Pi_\gamma = 0 \end{cases}$$



In GKZ.

$$w_i = (v_i, 1)$$

$$a_i = v_{w_i} \qquad \sum W_i^j a_i \frac{\partial}{\partial a_i} = \beta^j \Pi_\gamma.$$

$$\gamma = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$$

(γ resonant.) geometric

γ non resonant generic. GKZ.

Euler type

"Sol (GKZ) = sheaf of holomorphic soln.

generic pt $(a_0, \ldots a_p)$.

(GKZ)! rk Sol $= n!\,\mathrm{Vol}(\Delta)$

$\gamma = (0, 0 \ldots 0, 1)$

$\dim{}^{cl}\mathrm{Sol}_a = H_n((\mathbb{C}^*)^n - (f_a = 0))$

$$\gamma \longmapsto \int_\gamma \frac{dt_1 \wedge \cdots dt_n}{t_1 \cdots t_n f}$$

$H_n \longmapsto \mathrm{Sol}$

Q1: Explicit solutions expansions at 0.

(Hosono - Lian - Yau, B series)

(non resonant, GKZ) (Frobenius method)

Q2: Which are periods? (Hyperplane conj.)

($\mathbb{P}^n$, Lian - Minxian Zhu)

Hypersurface in G-variety X

X smooth, $L = K_X^{-1}$ very ample.

$f \in H^0(X, L) = V$. determines Y_s

$$K_Y \cong K_X \otimes \mathcal{O}(Y_s) \cong \mathcal{O}_X$$

$$\omega_f = \mathrm{Res}\,\frac{\Omega}{f} \mapsfrom 1$$

$$f = \sum a_i e_i.$$

$$\Pi_\Gamma = \int \omega_s$$

$$\begin{array}{c} P \\ \pi\downarrow \\ X \end{array}$$

P principle H-bundle over X.

$\chi : H \to \mathbb{C}^*$, s.t.

$$P \times_H \mathbb{C}_\chi \cong K_X^{-1}.$$

$$0 \to \ker \pi^* = P \times_{\mathrm{ad}} \mathfrak{h} \to TP \to \pi^* TX \to 0$$

$$K_P \cong (P \times \mathbb{C}_\chi) \otimes (\det \mathfrak{h}^*)$$

so ν top form on P.

s.t. $h^* \nu = (\chi \chi_{\det \mathfrak{h}}^{-1}(h))\, \nu$

$$\frac{\partial}{\partial a_i}\left(\frac{\Omega}{\sum a_i e_i}\right) = -\frac{e_i \Omega}{f^2}.$$

Q polynomial of e_i. and

$$Q(e_i) = 0.$$

then $\Omega(\partial a_i)\Pi_r = 0$.

G equivariant. $g^*\left(\frac{\Omega}{f}\right) = \psi^{-1}\frac{\Omega}{f}$.

ψ character of g.

$d\psi = \beta$; $\mathfrak{g} \to \mathbb{C}$.

$(\exp tz)^* \Pi_r = \psi^{-1}(\exp tz)\Pi_r$.

$\mathfrak{g} \to \mathrm{End}(V)$

$z \mapsto \sum z_{ij} a_i \frac{\partial}{\partial a_j}$.

$z_{ij} a_i \frac{\partial}{\partial a_j} + \beta(z) = 0$

$\sum a_i \frac{\partial}{\partial a_i} + 1 = 0$

$\mathcal{D}_V = \mathbb{C}[a_i, \frac{\partial}{\partial a_i}] \,/\, (\qquad)$

① When is this holonomic regular.
(finite rk sol'n)

G action on X has finitely many orbits

G semisimple

② $\mathrm{Sol} := (\mathcal{D}_V / (\quad), \mathcal{O}_V)$ Hunov

(Huang-Lian-Xinwen Zhu) $X = G/Q$.

Q parabolic X homogeneous.

Corollary: The map

a' is close to a.

$H_n(X - V(f_a)) \to H_n(X - V(f_{a'}))$

is injective

$|K_X^{-1}| : X \to \mathbb{P}(V^r)$

$I(X)$ is given by Kostant-Lichtenstein.
quadratic quadratic
polynomials.

Derived category of holonomic D-module regular   perverse sheaf

f_+ f_*.

$\mathbb{L}$ total space of $\mathcal{O}(1)$.

$\mathbb{L}^\circ$ complement of zero.

$V \times \mathbb{P} \to \mathbb{L}$ $\mathbb{L}^\perp = \ker(ev)$

$\boxed{\text{universal zero section.}}$

$$U = V \times \mathbb{P} - \mathbb{L}^\perp.$$

$$\begin{array}{c} U \\ \pi \downarrow \\ V \end{array} \qquad \mathcal{N} = \mathcal{O}_V \boxtimes \underline{\mathcal{D}_{\mathbb{P},\beta}}$$

$$\mathcal{D}_{\mathbb{P},\beta} = (\mathcal{D}_{\mathbb{P}} \boxtimes \mathbb{C}_\beta) \otimes_{\mathcal{g}} \mathbb{C}.$$

$\tau = \mu^{0} \pi_{+}^{\vee}(\mathcal{N}|_{U})$.

If G action on $\mathbb{P}$ is transitive. $\beta = 0$.

then $\mathcal{D}_{\mathbb{P},\beta} \cong \mathcal{O}_{\mathbb{P}}$.

$a : \mathbb{P} \times G \to \mathbb{P}$.

$da : \mathfrak{g} \to T_{\mathbb{P}}, \ \mathfrak{g} \to \mathbb{D}_{\mathbb{P}}$.

$$\mathbb{D}_{\mathbb{P},\beta} = \mathbb{D}_{\mathbb{P}} \Big/ \mathbb{D}_{\mathbb{P}}\big(da(\xi) + \beta(\xi)\big)$$

a transitive $\Rightarrow$ $\mathbb{D}_{\mathbb{P},\beta} = \mathcal{O}_{\mathbb{P}}$.

$\beta = 0$

$$\begin{array}{c} \mathbb{P} = \mathbb{P}(E^\vee) \\ \downarrow \\ X \end{array}$$

$\det E \cong K_X^{-1}$

$K_{Y_f} \cong \det E \otimes K_X|_{Y_f}$

$K_{Y_f} \longleftarrow ||$

$f \in H^0(X, E) \to \tilde{f} \in H^0(\mathbb{P}, \mathcal{O}(1))$

$\{f = 0\} = Y_f \qquad \{\tilde{f} = 0\} = \tilde{Y}_f$

Y_f smooth $\iff$ $\tilde{Y}_f$ smooth with ordinary overlap

$H^{n+r-1}(X - Y_f) \to H^{n+r-1}(\mathbb{P} - \tilde{Y}_f)$

$H^{n+r-1}(X - Y_f) \to H^d_{van}(Y_f)(n) \to 0$

$\uparrow$

$H^{n+r-1}(X)$

$\uparrow$

$H^{n-r-1}(X)$

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \to H^0(\mathbb{P}, K_{\mathbb{P}} \otimes \mathcal{O}(r))$$

$$\to F^{n} H^{n+r-1}(\mathbb{P} - Y_f) \qquad \frac{\Omega}{f^n}$$

$$\to F^{n-r} H^{n-r}(Y_f)$$

$$1 \longmapsto \operatorname{Res} \frac{\Omega}{f^r}.$$

$f = a_i e_i$

$Q(\partial a_i)$ $\quad$ Q is in ideal of

$|\mathcal{O}(1)| : \mathbb{P} \to \mathbb{P}(V^\vee)$

$\sum z_{ij}\, a_i \frac{\partial}{\partial a_j}$

$\sum a_i \frac{\partial}{\partial a_i} + r$

(Fano hypersurface with CY type
Hodge structure.

(cubic 4-fold)