Moduli of symmetric cubic 4-folds
and singular sextic curves.

Moduli spaces

1. GIT
2. Hodge Theory } cpctification.

GIT.

$G \curvearrowright (P, L)$ ← ample

$$G \backslash\backslash P = \mathrm{Proj}\left(\bigoplus_k H^0(P, L^{\otimes k})^G\right)$$

do we need this part?

$x \in P$ → stable P^s
→ semistable P^{ss}
→ unstable

If $P^s = P^{ss}$, $G\backslash\backslash P$ cpct.

Example cubic curves

$SL(3)\backslash\backslash P^9$

$$\begin{array}{ccc} SL(3)\backslash P^s & \xrightarrow{j} & SL(2,\mathbb{Z})\backslash \mathbb{H} \\ \downarrow & & \downarrow \\ SL(3)\backslash\backslash P^{ss} & \xrightarrow{\cong} & \overline{\left(SL(2,\mathbb{Z})\backslash \mathbb{H}\right)}^{BB} \cong \mathbb{P}^1. \end{array}$$

(smooth).

both isomorphic to $\mathbb{P}^1$

also isomorphism between invariants and modular forms.

Compare GIT with BB (Looijenga).

$$G \backslash\backslash P \dashrightarrow \overline{\Gamma \backslash \mathbb{D}} \qquad \mathbb{P}$$

deg 2 K3.

(X, H). $|H| : X \xrightarrow{2:1} \mathbb{P}^2$.

branched along plane sextic

$$p : \overline{M} \dashrightarrow \left(\Gamma \backslash \mathbb{D}\right)^{BB}$$

$\uparrow$

GIT for plane sextic.

$\Gamma \subset \mathrm{Aut}(H^{\perp})$

$\uparrow$

Spinor norm: 1.

p has one indeterminacy pt.

$w = [Q^3]$ Q quadric.

$\downarrow$

corresponds to elliptic K3.

Thm (Shah). $\hat{M} = Bl_w \overline{M}$ in GIT sense. $\hat{M} \to \overline{\Gamma \backslash \mathbb{D}}^{BB}$

contracting 2-dim boundary to 1-dim strata.

$\partial \, \Gamma \backslash \mathbb{D}$ pts. modular curves.

Looijenga.

In a lot of cases

$$\begin{array}{ccc} \overline{M} & \dashrightarrow & \left(\overline{\Gamma \backslash \mathbb{D}}\right)^{*} \\ \cup & & \\ M & \xrightarrow{p} & \left(\Gamma \backslash \mathbb{D}\right) \end{array}$$

- $\operatorname{Im} p$ is the complement of hyperplane arrangement $\mathcal{H}_\infty$
- $\operatorname{codim}(\partial \overline{M}) \geq 2$.
- p also identity GIT polarization with automorphic bundle on $\Gamma \backslash \mathbb{D}$

$\left(\overline{\Gamma \backslash \mathbb{D}}\right)_{\mathcal{H}_\infty}$ constructed from BB.

(1) $\overline{\Gamma \backslash \mathcal{H}_\infty} \subset \overline{\Gamma \backslash \mathbb{D}}^{BB}$

$\overline{\Gamma \backslash \mathbb{D}}^{\text{semitoric}} \to \overline{\Gamma \backslash \mathbb{D}}$

BB captures graded pieces of limiting mixed hodge.

Mumford's Toroidal captures extension depend on a choice.

12) successive blow up strata in $\mathcal{H}_\infty$

13) blow-down in other direction

$\mathrm{Proj}\left(\bigoplus H^0(\mathbb{D}, L^k \otimes (\mathcal{H}_\infty))^{\Gamma}\right)$

cubic 4-folds. $X \subset \mathbb{P}^5$.

$H^4(X)$ 0 1 21 1 0.

$H^4(X)_0 \cong E_8^2 \oplus U^2 \oplus A_2$

(Voisin) $p: \mathcal{M} \longrightarrow \Gamma \backslash \mathbb{D}^{20}$

$\Gamma \subset \mathrm{Aut}(\Lambda_0)$ with index 4.

$\Gamma \subset \Gamma_0 \subset \mathrm{Aut}(\Lambda_0)$

keep the discriminant form 3'. spinor norm $= 1$.

GIT of smooth cubic 4-folds

p open embedding.

(Radu, Looijenga)

p extends to isomorphism.

$\mathcal{M} \xrightarrow{\cong} \overline{(\Gamma \backslash \mathbb{D})}_{\mathcal{H}_\infty}$.

almost all known cases can be considered as cubic fourfolds with certain symmetry or degeneration (including deg 2 K3).

$\dim V_{\mathbb{C}} = 6.$ $A \subset SL(V).$
finite subgroup.

$\lambda : A \to \mathbb{C}^*$

$V_\lambda \subset \mathrm{Sym}^3 V^*$ character subspace.

$\exists F \in V_\lambda.$ X_F smooth cubic 4-fold.

$N = \{ g \mid gAg^{-1} = A.\ \lambda^g = \lambda \}$

$N \backslash\backslash \mathbb{P}(V_\lambda) \longrightarrow$ GIT moduli of (A, λ)-symmetric cubic 4-fold.

Ex: $A = \langle \mathrm{diag}(\omega, 1, 1, 1, 1, 1) \rangle$ (Allcock - Toledo - Carlson)

$\mathbb{P}(V_\lambda) = \{ y^3 + F(x_0, x_1, x_2, x_3, x_4) = 0 \}.$

$N \backslash\backslash \mathbb{P}(V_\lambda) =$ GIT moduli of cubic 3-fold.

A acts on $H^4(X_F)$.

0 1 20 1. 0.

$H^4(X_F)_\chi$ 1 m 1.

or 1 m 0

according to χ real of not real. K3-type

type A

(In ACT, $H^4_\chi = 1, 10, 0$

Thm: $p: \overline{M} \dashrightarrow \Gamma_A \backslash \mathbb{D}_A$

$\mathbb{D}$ type IV or cplx hyperbolic ball

extends to $\overline{M}_A \longrightarrow \overline{(\Gamma_A \backslash \mathbb{D}_A)}_{H_\infty}$.

(There is a criterion on (A, λ) to determine $H_\infty = \phi$).

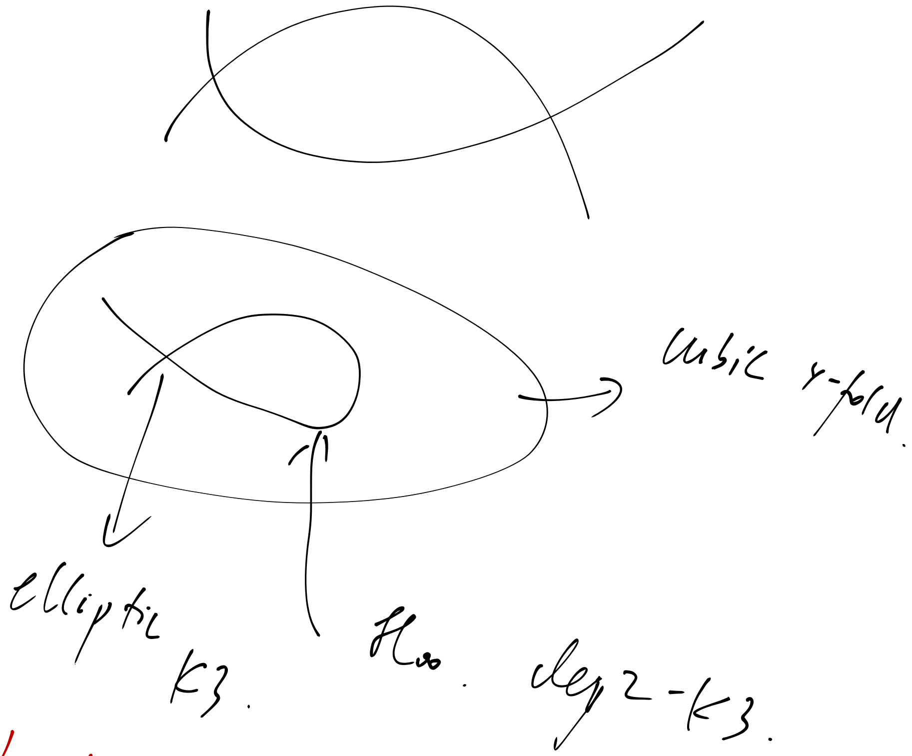
Ex: (Pearlstein-Laza-Zhang)

$F = y^2 H + F(x_0, \dots x_4)$

pair of Hypersurface and cubic 3-fold.

$\mathbb{P}^4 \times \mathbb{P}^{34}$. $\mathcal{O}(1) \boxtimes \mathcal{O}(3)$ $/\!/ SL(5)$

$\cong (\Gamma \backslash \mathbb{D})^{BB}$



should I mention the example of Hilbert Modular surface ??

classification of symplectic symmetry of cubic 4-folds (Laza-Zheng).

Pf:
$$\begin{tikzcd}
\overline{M}_A \arrow[r, dashed] \arrow[d, "①"] & \overline{(\Gamma_A \backslash \mathbb{D}_A)}_{H_\infty} \arrow[d, "②"] \\
\overline{M} \arrow[r] & \overline{(\Gamma \backslash \mathbb{D})}_{H_\infty}.
\end{tikzcd}$$

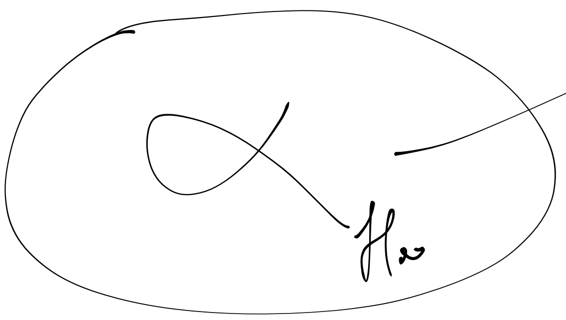
① Luna's Theorem.

② functoriality of semi-toric cptification

$G'_{\mathbb{Q}} \hookrightarrow G_{\mathbb{Q}}$ BB

Toroidal M. Harris.

BB criterion.



Blow-up, Blow-down

parametrize by

$\mathbb{P}^1$

$$g(x_1, \ldots x_5) = \begin{vmatrix} x_0 & x_1 & x_2+2ax_5 \\ x_1 & x_2-ax_5 & x_3 \\ x_2+2ax_5 & x_3 & x_4 \end{vmatrix} + b\,x_5^3$$

$(a:b) \in W\mathbb{P}(1,3)$

$b=0$. determinantal $X_{(1,0)}$

$\mathrm{Sing}(X_{(1,0)}) = \mathbb{P}(\mathbb{C}^3) \to \mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^3))$

$GL(3) \longrightarrow GL(V)$

$b \neq 0$, $\mathrm{Sing}(X(a,b)) =$

$$\mathbb{P}(\mathbb{C}^2) \to \mathbb{P}(\mathrm{Sym}^2(\mathbb{C}^2) \oplus \mathbb{C})$$

$$GL(2) \times \mathbb{C}^* \longrightarrow GL(V)$$

Not BB $(\Rightarrow)$ (A, λ) factor through

$GL(2) \times \mathbb{C}^*$, or $GL(3)$

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deg 2 K3.

$W = \{$ sextic with exactly n-nodes $\}_0 \subset \mathbb{P}^{27}$.

$$\begin{array}{c} \widetilde{W} \\ \downarrow \text{normalization} \\ W \end{array}$$

$\forall C \in W$.

$$\tilde{X} \longrightarrow X$$
$$X \xrightarrow{2:1} \mathbb{P}^2 \qquad C = \cup C_i, \quad E_1 \dots E_n$$

$$M = \{ [H], [\tilde{C}_i], [E_i] \}$$

Prop: $M \longrightarrow \mathbb{L}_{K3}$ primitive sublattice.

M-polarized $K3$.

Thm (Yf): $\overline{W} /\!/ SL(2) \xrightarrow{\cong} (\Gamma \backslash \mathbb{D})_{\text{Huo}}$

If $[Q^3] \notin \overline{W}$, then $(\Gamma \backslash \mathbb{D})_{\text{Huo}} \cong (\Gamma \backslash \mathbb{D})^{BB}$.

Ex: (Matsumoto-Sasaki-Yoshida)
$$W = \{ F = l_1 l_2 \cdots l_6,\ l_i \text{ line} \}.$$
$$\overline{W} = (\mathbb{P}^1)^6 / S_6$$
$$\Rightarrow (\mathbb{P}^1)^6 /\!/_{\mathcal{O}(1)^{\boxtimes 6}} SL(2) \times S_6 \cong \overline{\Gamma \backslash \mathbb{D}^4}^{BB}.$$

partition of 6, 11 families. 9 B.B.

Same proof applies to $\mathcal{H}_\infty$ (elliptic K3)

$\{$GIT of Weierstrass model$\} \cong (\mathbb{D}^{18}/\Gamma)^{BB}$.

Any finite group. Looijenga compactification

functorial property