IMSC 2048 HW6
Due 2026/2/26

February 12, 2026

1 Exercises

1.1 Mandatory part

In this homework, we always work over the complex field $\mathbb{C}$ and finite dimensional representations of finite groups.

Excercise 1. *Let V be an irreducible representation of a finite group G over $\mathbb{C}$ and χ be the character of V. Prove that V is irreducible if and only if $\langle \chi, \chi \rangle = 1$. Use this to reprove that the dual representation of an irreducible representation is also irreducible.*

Excercise 2. *Let G_1 and G_2 be two groups and V_i is an representation of G_i over $\mathbb{C}$ for $i = 1, 2$. Define the representation $V_1 \otimes V_2$ of $G_1 \times G_2$ by*

$$(g_1, g_2) \cdot (v_1 \otimes v_2) = (g_1 \cdot v_1) \otimes (g_2 \cdot v_2)$$

for $g_i \in G_i$ and $v_i \in V_i$. Prove that $V_1 \otimes V_2$ is irreducible if and only if both V_1 and V_2 are irreducible. How do you express the character of $V_1 \otimes V_2$ in terms of the characters of V_1 and V_2?

Excercise 3. *Let G be a finite group and operates on a finite set X. Let V be the vector space over $\mathbb{C}$ with basis X, i.e., $V = \mathbb{C}^X$. Then the G-action on X induces a linear representation of G on V. Show that number of orbits of G on X is equal to multiplicity of the trivial representation in the irreducible decomposition of V.*

Excercise 4. *(Artin Algebra Chapter 10, 7.1) Prove a converse to Schur's Lemma: Let ρ be a complex representation of a finite group G on a vector space V. If the only G-invariant linear operators on V are scalar multiplications (i.e., $\mathrm{Hom}_G(V, V) = \mathbb{C} \cdot \mathrm{Id}_V$), then ρ is irreducible.*

Excercise 5. *Let S_n be the symmetric group on $[n] = \{1, 2, \cdots, n\}$. Prove the following combinatorial identity by characters*

$$\sum_{\sigma \in S_n} (\textit{number of elements in } [n] \textit{ fixed by } \sigma)^2 = 2(n!)^2$$

1. *Let V be the representation of S_n on $\mathbb{C}^n$ induced by the action of S_n on standard basis $e_1, \cdots, e_n$ of $\mathbb{C}^n$. Show that the character χ_V of V is given by*
$$\chi_V(\sigma) = \textit{number of elements in } [n] \textit{ fixed by } \sigma$$
for $\sigma \in S_n$.

2. *In class, we showed that the trivial representation of S_n is contained in the representation V given by subspace spanned by $e_1 + e_2 + \cdots + e_n$. The G-invariant compliment W is defined by $W = \{v = (v_1 \cdots v_n)^T \in \mathbb{C}^n \mid \sum_{i=1}^n v_i = 0\}$. Show that W is irreducible by the following method. Let V' be a nonzero G-invariant subspace of W and $v \in V'$ be a non-zero vector. If any two component v_i and v_j of v are not equal, then use a permutation $\sigma \in S_n$ to prove that $e_i - e_j$ is also in V'. Then prove that V' must be all of W.*

3. *Use character of V to prove the combinatorial identity stated in the beginning.*

1.2 Optional problems

Excercise 6. *In this exercise, we work on representation of finite group G over field F whose characteristic does not divide the order of the group. Define the convolution of two functions on G by $\phi, \psi \in Map(G, F)$ by*

$$(\phi * \psi)(g) = \frac{1}{|G|} \sum_{h \in G} \phi(h)\psi(h^{-1}g).$$

Prove the following:

1. *The convolution is associative, i.e., $(\phi * \psi) * \theta = \phi * (\psi * \theta)$ for any $\phi, \psi, \theta \in Map(G, F)$.*

2. *The space of class functions $C(G, F)$ is closed under convolution.*

3. *If ϕ is a function on G such that $\phi * \psi = \psi * \phi$ for any $\psi \in Map(G, F)$, then ϕ is a class function.*