1. a). $|G| = p^2$. p prime

then $G \cong C_p \times C_p$ or C_{p^2}.

so $G \cong C_3 \times C_3$ or C_9

b). $|G| = 45 = 3^2 \times 5$

Let S_3 be the number of Sylow 3-groups.

S_5 be the number of Sylow 5-groups.

$S_3 | 5$. $S_3 \equiv 1 \mod 3$

$S_5 | 9$. $S_5 \equiv 1 \mod 5$.

so $S_3 = S_5$

Let H, K be the Sylow 3-group and 5-group,

then $H \cap K = \{1\}$. $HK = G$

$\Rightarrow G \cong H \times K$

$\cong C_3 \times C_3 \times C_5$

or $\cong C_9 \times C_5$

2. Let S_2 be the number of Sylow 2-group.
S_3 be the number of Sylow 3-group.

$S_2 \mid 3$, $S_2 \equiv 1 \bmod 3$. $S_2 = 1$ or 3

$S_3 \mid 4$. $S_3 \equiv 1 \bmod 4$ $S_3 = 1$ or 4.

We need to exclude $S_2 = 3$, $S_3 = 4$.

Assume this is the case.

Let H_1 H_2 H_3 be Sylow 2-groups.
K_1, K_2, K_3, K_4 be Sylow 3-groups.

Then $K_i \cap K_j = \{1\}$. $i \neq j$

$H_i \cap K_j = \{1\}$ (But $H_i \cap H_j$ may not equal to $\{1\}$)

So $\left|\bigcup_i K_i \cup H_j\right| = 1 + 2 \times 4 + 3$

$= 12 = |G|$.

So $H_j = (\bigcup K_i)^c$, and $H_1 = H_2 = H_3$.

3. a). $S_3 = \{(1), (123), (132), (12), (13), (23)\}$

subgroups (1),

$\langle (123) \rangle$

$\langle (12) \rangle$

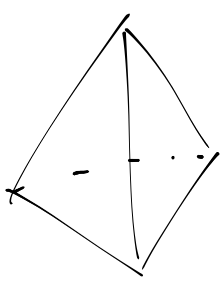
$\langle (13) \rangle$

$\langle (23) \rangle$

S_3.

b) (1). $\langle (123) \rangle$, S_3.

4. (a) $|G| = |S| \cdot |G_x|$

(b). 

consider G-action on the four vertices.

G_x = rotations of base triangle $= C_3$

$|G| = 4 \cdot 3 = 12$

Rings and Fields

1. a). 1, 5, 7, 11,

b). $a+b\sqrt{-5}$, $a, b \in \mathbb{Z}$

$$(a+b\sqrt{-5})(c+d\sqrt{-5}) = 1.$$

Hint: consider $|\cdot|^2$.

$\Rightarrow$ units are ± 1

Hint:

c). $\mathbb{Z}[i]/(i+3) = \mathbb{Z}[x]/(x^2+1, x+3)$

$$\cong \mathbb{Z}/10\mathbb{Z}.$$

2. Fields: a) b). c). d).

Hints: a). $|2+3i|^2 = 9+4 = 13$ is a prime number.

So $2+3i$ is a prime element in $\mathbb{Z}[i]$.

b) $\mathbb{Z}[\sqrt{-2}]/(7) = \mathbb{Z}[x]/(x^2+2, 7)$

$= \mathbb{F}_7[x]/(x^2+2)$

$x = 0, 1\ \ 2\ \ 3\ \ 4, 5\ \ 6$

$x^2 = 0, 1, 4, 2, 2, 4, 1.$

So x^2+2 irreducible in $\mathbb{F}_7[x]$.

c). Eisenstein.

d) mod 2.

3. $x, \quad x+1$

x^2+x+1

$x^3+x+1, \quad x^3+x^2+1$

$x^4+x^3+1, \quad x^4+x+1, \quad x^4+x^3+x^2+x+1.$

4. $\mathbb{Z}[x]/(x^2+3x+3) \cong \mathbb{Z}[x]/(x+1)^2+(x+1)+1.$

$\mathbb{Z}[x]/(x^2+x+1) \cong \mathbb{Z}[w]$ because $w^2+w+1=0.$

$x \longmapsto w.$

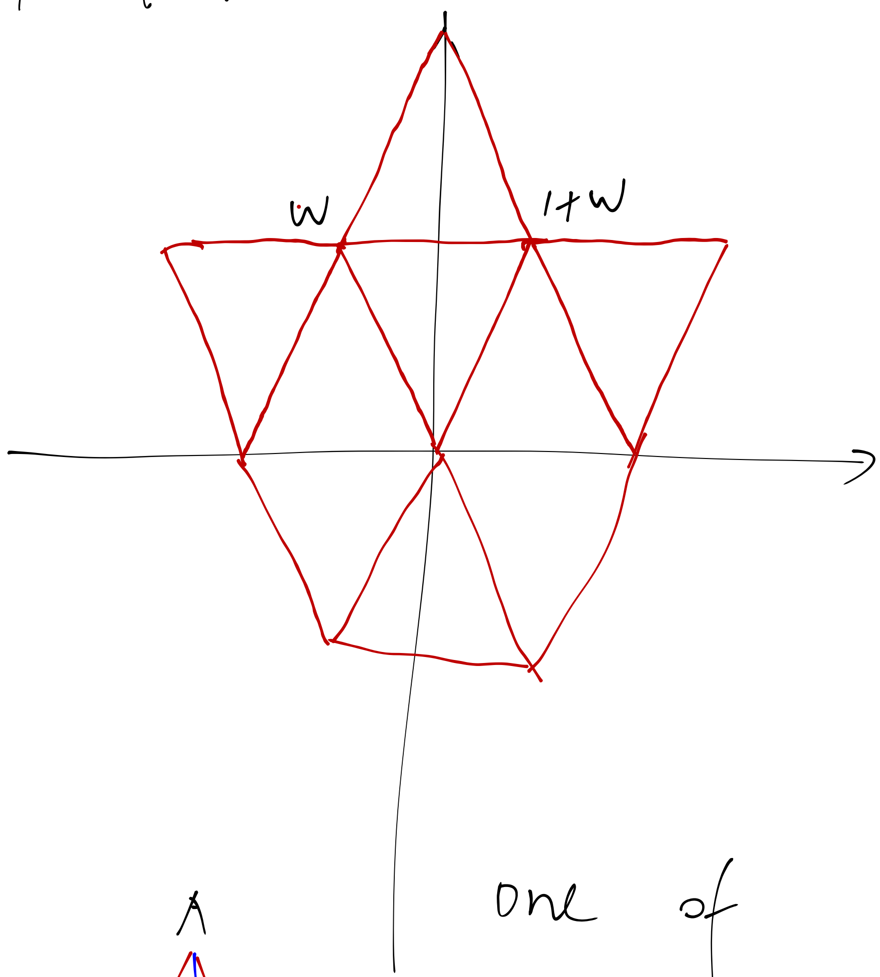
$\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x]$

$x \longmapsto x+1.$

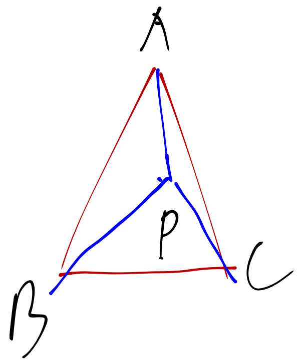
$\Rightarrow \mathbb{Z}[x]/(x^2+x+1)$

$\cong \mathbb{Z}[x]/(x+1)^2+(x+1)+1$

5. Hint.



algebraic solution at the end.

In  one of the distances PA, PB, PC is less than $|AB|$

6. a). $[\mathbb{Q}(w,\sqrt{2}), \mathbb{Q}(\sqrt{2})] \le 2$

because $w^2+w+1=0$.

and $[\mathbb{Q}(w,\sqrt{2}) : \mathbb{Q}(\sqrt{2})] = 2$

because $w \notin \mathbb{Q}(\sqrt{2})$.

$\Rightarrow [\mathbb{Q}(w,\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4$

b). $x^6 - 6$ is irreducible in $\mathbb{Q}$.

(from Eisenstein)

$$\Rightarrow [\mathbb{Q}(\sqrt[6]{6}), \mathbb{Q}] = 6$$

c). $[\mathbb{Q}(\sqrt{6}), \mathbb{Q}] = 2$.

$$\Rightarrow [\mathbb{Q}(\sqrt[6]{6}), \mathbb{Q}(\sqrt{6})] = 3$$

Hint:

7. a). $f(x) = \prod_{i=1}^{4} (x - \alpha_i)$

$$\alpha_i = \pm\sqrt{2} \pm \sqrt{3}.$$

b). $f(x) = (x - \sqrt{2+\sqrt{2}})$

$(x + \sqrt{2+\sqrt{2}})$

$(x - \sqrt{2-\sqrt{2}})$

$(x + \sqrt{2+\sqrt{2}})$

8. a) Homework 12 deg = 4.

b) $K = \mathbb{Q}(\sqrt[5]{2}, \eta)$

$$\eta = e^{\frac{2\pi i}{5}}$$

Since $\eta^5 - 1 = 0$, $\eta^4 + \eta^3 + \eta^2 + \eta + 1 = 0$.

$f(x) = x^4 + x^3 + x + x + 1$ irreducible over $\mathbb{Q}$ (cyclotomic polynomial Thm 12.4.9)

so

$$\begin{tikzcd}
 & K \arrow[dl, dash, "\leq 4"'] \arrow[dr, dash, "\leq 5"] & \\
\mathbb{Q}(\sqrt[5]{2}) \arrow[dr, dash, "5"'] & & \mathbb{Q}(\eta) \arrow[dl, dash, "4"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

so $[K : \mathbb{Q}] = 20$

c) deg = 4.

9. a) $G(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{1\}$.

Hint: $f(x) = x^3 - 2$.

$\sigma \in G$, $\sigma(\sqrt[3]{2})$ is also a root of $f(x)$ but other roots are not in $\mathbb{Q}(\sqrt[3]{2})$

b). $C_2 \times C_2$ Homework 12 or notes $\not\beta_0$ example 2.

10. $\alpha = \sqrt{2} + \sqrt{3}$.

11. see 9 b)

12. $x = 0, 1, 2,$

$x^3 + x^2 = 0, 2, 0$.

so $a = 2$.

5. Let $\sigma(\alpha) = |\alpha|^2$. for $\alpha \in \mathbb{Z}[w]$.

If $\beta \neq 0$.

$\frac{\alpha}{\beta} = a + bw$. $a. b \in \mathbb{Q}$

since $\mathbb{Q}[w]$ is a field

choose $m, n \in \mathbb{Z}$. such that

$|m-a| \le \frac{1}{2}$. $|n-b| \le \frac{1}{2}$.

so $\alpha = (m+nw)\beta + ((a-m)+(b-n)w)\beta$.

Let $q = m+nw \in \mathbb{Z}[w]$

$r = \alpha - (m+nw)\beta \in \mathbb{Z}[w]$

Denote $a' = a-m$. $|a'| \le \frac{1}{2}$

$b' = b-n$. $|b'| \le \frac{1}{2}$.

Then $|a'+b'w|^2 = |a' + \frac{-1+\sqrt{-3}}{2} b'|^2$

$$= (a' + \tfrac{1}{2} b')^2 + 3\left(\tfrac{b'}{2}\right)^2$$

$$= (a')^2 + a'b' + (b')^2$$

$$\leq \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} < 1.$$

So $|r|^2 < |\beta|^2$

because $r = \beta \cdot (a' + b'w)$