7.7.1.

$$u(r,\theta,t) = \phi(r,\theta)\cdot G(t).$$

$$\frac{G''(t)}{c^2 G(t)} = \frac{\Delta\phi}{\phi} = -\lambda.$$

$$\begin{cases} \Delta\phi + \lambda\phi = 0 \\ \phi\big|_{\partial\Omega} = 0. \end{cases}$$

So $\phi = J_n\left(\frac{z_{nm}}{a} r\right)\cdot \begin{cases} \cos n\theta \\ \sin n\theta \end{cases}$

$n = 0, 1, 2, \ldots$

$m = 1, 2, \ldots$

$$\lambda = \left(\frac{z_{nm}}{a}\right)^2$$

$G''(t) = -\lambda c^2 G(t),\quad G(0) = 0.$

$G(t) = \sin(\sqrt{\lambda}\, c t)$

So $u(r,\theta,t) = \sum_{m=1}^{+\infty}\sum_{n=0}^{+\infty} A_{nm} J_n\left(\frac{z_{nm}}{a} r\right)\cos n\theta.$
$\sin\left(\frac{z_{nm}}{a}\cdot c t\right)$

$$+ \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} B_{nm} J_n\left(\frac{z_{nm}}{a} \cdot r\right) \sin n\theta \cdot \sin\left(\frac{z_{nm}}{a} \cdot ct\right)$$

$$\frac{\partial}{\partial t} u(r, \theta, 0) = \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} A_{nm} \left(\frac{z_{nm}}{a} \cdot c\right) \cdot J_n\left(\frac{z_{nm}}{a} r\right) \cos n\theta$$

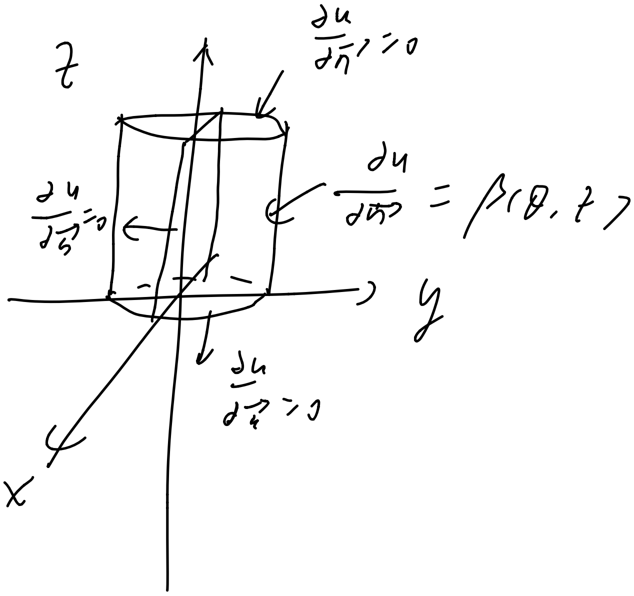
$$+ \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} B_{nm} \left(\frac{z_{nm}}{a} \cdot c\right) J_n\left(\frac{z_{nm} r}{a}\right) \sin n\theta$$

$$= \alpha(r) \cdot \sin 3\theta$$

so
$$\begin{cases} A_{nm} = 0 \\ B_{3m} = \dfrac{\int_0^a J_3\left(\frac{z_{3m}}{a} r\right) \cdot \alpha(r)\, r\, dr}{\left(\frac{z_{3m}}{a} \cdot c\right) \int_0^a J_3\left(\frac{z_{3m}}{a} r\right) \cdot J_3\left(\frac{z_{3m}}{a} \cdot r\right) r\, dr} \end{cases}$$

$B_{nm} = 0$ if $n \neq 3$.

7.9.2. c)



(BCs for z and θ are homogeneous.
BC for r is not.)

$u(r, \theta, z) = \phi(r, \theta) \cdot G(z)$.

$$\frac{G''(z)}{G(z)} = -\frac{\Delta \phi}{\phi} = -\lambda .$$

$G'(0) = G'(H) = 0$

so $G(z) = \cos \frac{n\pi z}{H}$, $n = 0, 1, 2 \ldots$

$\lambda = \left(\frac{n\pi}{H}\right)^2$

$$\frac{\Delta\phi}{\phi} = \left(\frac{m}{l-l}\right)^2 . \qquad \phi(r,\theta) = f(r)\cdot g(\theta).$$

$$\frac{\frac{1}{r}(rf')'}{f} + \frac{1}{r^2}\cdot\frac{g''}{g} = \left(\frac{n\pi}{l-l}\right)^2 .$$

$$\frac{g''}{g} = -\mu , \quad \text{and} \quad g'(0) = g'(\pi) = 0 .$$

$$\mu = \left(\frac{m\pi}{\pi}\right)^2 = m^2 . \quad m = 0, 1, 2, \cdots$$

$$g(\theta) = \cos m\theta$$

① If $n \neq 0$,

change of variable $w = \frac{n\pi}{l-l} r$

gives $w^2 f'' + w f' + (-w^2 - m^2) f = 0.$

$$f = I_m\left(\frac{n\pi}{l-l} r\right).$$

② If $n = 0$. $\frac{1}{r}(rf')' + \frac{1}{r^2}(-m^2) = 0.$

$$r(rf')' - m^2 = 0 .$$

$$f = r^p \Rightarrow p^2 = m^2.$$

$$f(r) = r^m. \quad \left(f(r) \neq r^{-m} \text{ or } \log r \text{ because } |f(0)| < +\infty \right)$$

So

$$u(r, \theta, z) = \sum_{m=0}^{+\infty} A_m (\cos m\theta) \cdot r^m$$

$$+ \sum_{n=1}^{+\infty} \sum_{m=0}^{+\infty} B_{mn} (\cos m\theta) \cdot I_m\left(\frac{n\pi}{H} r\right) \cdot \cos\left(\frac{n\pi}{H} z\right)$$

$$u_r(a, \theta, z) = \beta(\theta, z).$$

$$\Rightarrow \sum_{m=0}^{+\infty} m A_m (\cos m\theta) a^{m-1}$$

$$+ \sum_{n=1}^{+\infty} \sum_{m=0}^{+\infty} \frac{n\pi}{H} B_{mn} (\cos m\theta) \cdot I_m\left(\frac{n\pi}{H} a\right) \cos\left(\frac{n\pi}{H} z\right).$$

$$= \beta(\theta, z).$$

A_0 can be any number.

$$B_{mn} = \frac{\int_0^H \int_0^\pi \beta(\theta,z)\cdot \cos\left(\frac{n\pi}{H}z\right)\cos m\theta \, d\theta \, dz}{I_m\left(\frac{n\pi}{H}a\right)\int_0^H \int_0^\pi \cos^2\left(\frac{n\pi}{H}z\right)\cos^2 m\theta \, d\theta \, dz}$$

$$= \int_0^H \int_0^\pi \beta(\theta,z)\cdot \cos\frac{n\pi}{H}z \cos m\theta \, d\theta \, dz \cdot \begin{cases} \dfrac{1}{I_0\left(\frac{n\pi}{H}a\right)} \cdot \dfrac{2}{H\pi} & \text{if } m=0 \\ \dfrac{1}{I_m\left(\frac{n\pi}{H}a\right)} \cdot \dfrac{4}{H\pi} & \text{if } m\neq 0. \end{cases}$$

$$A_m = \frac{\int_0^H \int_0^\pi \beta(\theta,z)\cdot \cos m\theta \, d\theta \, dz}{m a^{m-1} \int_0^H \int_0^\pi (\cos m\theta)^2 \, d\theta \, dz}$$

$$= \frac{2}{m a^{m-1}\cdot H\pi} \cdot \int_0^H \int_0^\pi \beta(\theta,z) \cos m\theta \, d\theta \, dz.$$

Condition on <u>$\beta(\theta,z)$</u>

Consider $\langle u_r(a,\theta,z), \quad 1 \rangle$

eigen fct $\cos(0\cdot\theta)$

$\cos\left(\frac{0\cdot\pi}{H} z\right)$

Then $0 \cdot A_0 = 0 \Rightarrow$

$$\int_0^{H} \int_0^{\pi} \beta(\theta, z)\, d\theta\, dz = 0 \,. \quad (*)$$

Physics explanation:

$\Delta u = 0$, Then u is equilibrium solution to heat equation.

So $\displaystyle\iint_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} = 0.$

So $(*)$ holds