# 代数 1 H 班 作业 5

2022 年 10 月 15 日

**题 1.** ***Artin, Chapter 11, 1.7 (a)***
*Let $U$ be an arbitrary set and $R$ be the set of subsets in $U$. Addition and multiplication of elements of $R$ are defined by $A+B = A \cup B - A \cap B$ and $A \cdot B = A \cap B$. Prove that $R$ is a ring.*

**题 2.** *Determine whether the division with reminder $g(x) = f(x)q(x) + r(x)$ exists in $R[x]$ for the following $R, f(x), g(x)$. If it exists, find the $q(x), r(x)$.*

1. $R = \mathbb{Z}, f(x) = 2x^2 + x + 1, g(x) = 2x^3 + 7x^2 + 4x + 8$
2. $R = \mathbb{Z}/6\mathbb{Z}, f(x) = 2x + 1, g(x) = 2x^2 + 2x$
3. $R = \mathbb{Z}/6\mathbb{Z}, f(x) = 5x + 1, g(x) = 2x^2 + 2x$

**题 3.** ***Artin, Chapter 11, 1.8***
*Determine the units in $\mathbb{Z}/12\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$.*

**题 4.** *Let $R$ be a ring and $I$, $J$ ideals of $R$. Prove the following*

1. *$I \cap J$ is an ideal of $R$,*
2. *$I + J = \{a + b | a \in I, b \in J\}$ is an ideal of $R$,*
3. *$IJ = \{\sum_{i=0}^{n} a_i b_i | a_i \in I, b_i \in J, n \in \mathbb{Z}_{\geq 0}\}$ is an ideal of $R$.*

**题 5.** ***Artin Chapter 11, 3.4*** *Let $\phi\colon \mathbb{C}[x, y] \to \mathbb{C}[t]$ defined by $x \mapsto t + 1$ and $y \mapsto t^3 - 1$. Determine the kernel $K$ of $\phi$ and prove that every ideal $I$ of $\mathbb{C}[x, y]$ that contains $K$ can be generated by two elements.*

**题 6** (Nilpotent groups)**.**

**定义 1.** *Let $G_1 = G/C(G)$, and $G_{n+1} = G_n/C(G_n)$. We call a group $G$ unipotent if and only if $G_m$ has order 1 for some $m$.*

1. *Prove that p-groups have nontrivial center and hence nilpotent.*
2. *Prove that a finite group is nilpotent if and only if it is the product of its Sylow subgroups.*
3. *Prove that if $G$ is nilpotent, then the subgroups and quotient groups of $G$ are also nilpotent.*
4. *Prove that the group $U$ consisting of $n \times n$ upper triangular matrices with elements in a ring $R$ and diagonal elements being 1 is nilpotent.*
5. *Prove that a finite group $G$ is nilpotent if and only if it is a isomorphic to a subgroup of $U$ for some $n$ and $R$.*
6. *Prove that if $G$ is nilpotent and $[G : G] = G$, then $G$ has only one element.*

**题 7.** *Use the notation from homework 1, question 5. Prove the Bruhat decomposition.*

$$GL(n, F) = \sqcup_{w \in W} BwB. \tag{1}$$

**题 8** (Parabolic subgroup of $PSL(n, F)$)**.** *Let $n \geq 3$ and $F$ a field. Denote by $G = SL(n, F)$ the group of matrices with determinant 1.*

1. *Let $B$ be the subgroup of $G$ consisting of upper triangular matrices, $T$ the subgroup of $G$ consisting of diagonal matrices, $N$ the normalizer of $T$ in $G$. Prove that $B$ and $N$ generates $G$.*
2. *Let $e_i$ be the column vector in $F^n$ with $i$-th component 1 and other components 0. Prove that the multiplication of matrices with vectors induces an group operation of $W = N/T$ on the set $\{Span(e_1), \cdots, Span(e_n)\}$. Identify $Span(e_i)$ with $i \in [n]$. Prove this action induces an isomorphism $f \colon N/T \to S_n$.*
3. *Fixing $w \in S_n$, let $\tilde{w} \in N$ be a representative in $f^{-1}(w)$. Prove that $\tilde{w}B$, $B\tilde{w}$ and $B\tilde{w}B$ do not depend on the choice of $\tilde{w}$. So we can*

*denote by $BwB$ for $B\tilde{w}B$. Prove the following decomposition.*

$$G = \sqcup_{w \in W} BwB. \tag{2}$$

4. *Let $\{s_1, \cdots, s_{n-1}\}$ be the set of fundamental transpositions. Prove that $s_i$ are not in the normalizer of $B$ and $s_i Bw \subset BwB \sqcup Bs_i wB$. In general, when $S_n$ is replaced by other Coxeter groups (not necessarily finite), such a structure is called a $(B, N)$-pair.*

5. *Let $\pi$ be a subset of fundamental transpositions $\{s_1, \cdots, s_{n-1}\}$. Let $W_\pi$ be the subgroup of $S_n$ generated by $\pi$. Prove that*

$$P_\pi = \sqcup_{w \in W_\pi} BwB$$

   *is a subgroup of $G$.*

6. *Prove that any subgroup $P$ of $G$ containing $B$ is of the form $P_\pi$. We call them parabolic subgroups. (Hint: if $\tilde{\omega} = \tilde{s}_{i_1} \cdots \tilde{s}_{i_l} \in P$ with length $l(w) = l$, try to prove all $\tilde{s}_{i_j} \in P$. )*

7. *Count the number of parabolic subgroups.*

**题 9** (Simplicity of $PSL(n, F)$)**.** *Following the notation in last question. Let $U$ be the subgroup of $G$ consisting of upper triangular matrices with diagonal elements being $1$. Prove that*

1. *Show that the center of $G$ is*

$$Z = \{\lambda I | \lambda^n = 1\}.$$

2. *$G$ is generated by conjugates of $U$.*

3. *$G = [G, G]$.*

4. *The intersection of conjugates of $B$ is the center of $G$.*

5. *Let $H$ be a normal subgroup of $G$, then either $H \subset Z$ or $HU = G$. (Hint: 1. Use the classification of parabolic groups. 2. Take a look at the proof in the class. 3. Prove that if $s_i \in HU$, then the nearby $s_j \in HU$.)*

6. *Let $H$ be a normal subgroup of $G$, then either $H \subset Z$ or $H = G$.*

7. *Prove that $PSL(n, F)$ is simple for $n \geq 3$.*

**题 10.** *Let $V$ be a $n$-dimensional vector space over field $F$.*

**定义 2** (Flag)**.** *A flag $F$ is defined to be a chain of subspaces*

$$F : \{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_n = V$$

*Denote by $FL$ the set of flags.*

*Let $G = GL(V)$, and define an action of $G$ on the set of flags by*

$$g \cdot F \colon \{0\} = V_0 \subset g(V_1) \subset g(V_2) \cdots \subset V_n = V.$$

**定义 3** (Borel subgroup)**.** *Let $F$ be a flag, the stablizer of $F$ is denoted by $B$ and called the Borel subgroup of $G$.*

*Fixing a flag $F$ and the corresponding Borel subgroup $B$, prove that the linear action of $B$ on $V_i$ induces a linear action on quotient space $V_i/V_{i-1}$, or in other words, there is a group homomorphism*

$$B \to GL(V_n/V_{n-1}) \times GL(V_{n-1}/V_{n-2}) \cdots \times GL(V_1/V_0).$$

**定义 4** (Nilponent subgroup)**.** *Define $U$ to be the kernel of the above homomorphism.*

1. *Prove that the action of $G$ on $FL$ is transitive and hence the Borel subgroups are conjugate to each other.*

2. *Restrict the action of $G$ on $FL$ to $B$. Find a bijection between the set of $B$-orbits with $S_n$, and for each orbit corresponding to $\omega \in S_n$, find a bijection to $F^{l(\omega)}$.*

3. *Prove that $B$ is the normalizer of $U$ in $G$.*

4. *Prove that $U$ is a unipotent group.*

5. *Is it true that $U$ is the commutator subgroup of $B$?*

*6. Define a partial flag to be a chain of subspaces*

$$F : \{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_m = V.$$

*Find the relations between partial flags and parabolic subgroups.*

**题 11.** *Let $F$ be a field. Find the derived subgroup or the commutator subgroup of $GL(n, F)$.*