10.4.8. a) $\hat{u}(w,t)$ = Fourier Transform of $u(x,t)$.

$$\hat{u}_t = k(-iw)^2\hat{u} - r\hat{u}$$

$$\hat{u}_t = (-kw^2 - r)\hat{u}$$

So $$\hat{u} = e^{(-kw^2-r)t} \cdot c(w)$$

$$\hat{u}(w,0) = \hat{f}(w) \Rightarrow c(w) = \hat{f}(w)$$

$$\hat{u}(w,t) = \hat{f}(w) \cdot e^{(-kw^2-r)t}$$

Inverse Fourier Transform of

$$e^{-ktw^2} \cdot e^{-rt} \quad \text{is}$$

$$e^{-rt} \cdot \sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}$$

So $$u(x,t) = f * e^{-rt}\sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}$$

$$= e^{-rt} \cdot \sqrt{\frac{\pi}{kt}} \cdot \int_{-\infty}^{+\infty} f(\bar{x}) \cdot e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$$

10.6.3. a) Use fourier transform in y-direction.

$$\hat{u}(x,w) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,y) \cdot e^{iwy} dy$$

$$\hat{u}_{xx} + (-iw)^2 \hat{u} = 0.$$

$$\hat{u}_{xx} = w^2 \hat{u}$$

$$\hat{u}(x,w) = e^{wx} A(w) + e^{-wx} B(w)$$

Since $u(x,y)$ is bounded when $(x,y) \to \infty$.

$x < 0$, so we only have

$$\hat{u}(x,w) = e^{|w|x} A(w).$$

Since $u(0,y)=g(y)$

$$\hat{u}(x,w)=e^{|w|x}\cdot\hat{g}(w).$$

So $u(x,y)=g(y)*$ IFT of $e^{|w|x}$.

IFT of $e^{|w|x}$ is $\frac{-2x}{x^2+y^2}$.

So $u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\bar{y})\cdot\frac{-2x}{x^2+(y-\bar{y})^2}\,d\bar{y}$.

5) $u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(\bar{y}) \frac{-2x}{x^2+(y-\bar{y})^2} d\bar{y}$

$$= \frac{-2x}{2\pi}\int_{-1}^{1} \frac{1}{x^2+(y-\bar{y})^2} d\bar{y}$$

$s=\bar{y}-y$

$$= -\frac{x}{\pi}\int_{-1-y}^{1-y} \frac{1}{s^2+x^2} ds$$

$$= -\frac{x}{\pi}\cdot\left(\frac{1}{x}\tan^{-1}\left(\frac{s}{x}\right)\Big|_{s=-1-y}^{s=1-y}\right)$$

$$= -\frac{1}{\pi}\tan^{-1}(1-y) + \frac{1}{\pi}\tan^{-1}(-1-y).$$

$$= \frac{1}{\pi}\tan^{-1}(y-1) - \frac{1}{\pi}\tan^{-1}(y+1).$$