RT HW2
Due 7/29 please submit your solutions to the TAs in tutorial session

July 24, 2025

Problem 1. *Prove that the map $a \mapsto a^{-1}$ is a group isomorphism of G to itself if and only if G is abelian.*

Problem 2. *Find all subgroups of D_4 of order 4. (In class I listed two of them, but there is a third one that I missed. Thank the students who pointed it out to me.)*

Problem 3. *Prove that all the subgroups of $(\mathbb{Z}, +)$ has the form $n\mathbb{Z}$ for some integer $n \geq 0$.*

Problem 4. *Let G be the subset of $\mathrm{GL}(2, \mathbb{R})$ consisting of matrices*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

with $a^2 + b^2 \neq 0$. Prove that G is a subgroup of $\mathrm{GL}(2, \mathbb{R})$ and isomorphic to $\mathbb{C}^\times$. Here $\mathbb{C}^\times$ is the multiplicative group of nonzero complex numbers, and $\mathrm{GL}(2, \mathbb{R})$ is the group of invertible 2×2 matrices with real entries. Is G a normal subgroup of $\mathrm{GL}(2, \mathbb{R})$?

Furthermore, if a, b have the form, $a = \cos\theta$ and $b = \sin\theta$, prove these matrices also form a subgroup of $\mathrm{GL}(2, \mathbb{R})$ and it is isomorphic to $\mathrm{U}(1)$.

Problem 5. *Let G be a group and H is a subgroup of G. Assume the number of elements in G/H is 2. Prove that H is a normal subgroup of G.*

Problem 6. *Let $G = \{(a, b) | a, b \in \mathbb{R}, a \neq 0\}$. Define a binary operation of G as $(a, b) \cdot (c, d) = (ac, ad + b)$. Prove that G is a group with this operation.*

Problem 7. *In this question, you will explore when the group is a product group.*

1. *Let G_1 and G_2 be two groups with indentity elements e_1 and e_2, and G is the product group $G = G_1 \times G_2$. Prove that $G_1 \times \{e_2\}$ is a normal subgroup of G and isomorphic to G_1. Similarly, $\{e_1\} \times G_2$ is a normal subgroup of G and isomorphic to G_2. The intersection of these two normal subgroups is $\{(e_1, e_2)\}$, the identity element of G.*

2. *Let H and K be two normal subgroups of a group G such that $H \cap K = \{e\}$, the identity element of G. Prove that the subset $HK = \{hk | h \in H, k \in K\}$ is a subgroup of G and isomorphic to the product group $H \times K$. If we assume that $HK = G$, then we can say that G is isomorphic to the product group $H \times K$. (Hint: first you can show that the elements from H and K commute with each other, i.e., for any $h \in H$ and $k \in K$, we have $hk = kh$. Then you can use the first part of this question to show that the map*

$$H \times K \to HK, (h, k) \mapsto hk$$

is a group isomorphism from $H \times K$ to HK.)