

G finite group.

$GL_n = GL(n, \mathbb{C})$

V = vector space / \mathbb{C} .

①

Matrix rep'n.

$R : G \rightarrow GL_n$ group homomorphism

$\chi_R : G \rightarrow \mathbb{C}$ character function.

$$\chi_R(g) = \text{Tr}(R(g))$$

② G operates on V linearly

or V is a G -rep'n.

$$G \times V \rightarrow V$$

$$(g, v) \mapsto g \cdot v.$$

$$1. (g_1 g_2) \cdot v = g_1 (g_2 v) \quad g_1, g_2 \in G, \\ v \in V.$$

$$2. 1 \cdot v = v$$

$$3. g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2 \quad v_1, v_2 \in V$$

$$g(c \cdot v) = c \cdot (g \cdot v) \quad c \in \mathbb{C}, v \in V$$

③ $\rho: G \rightarrow GL(V)$
 $= \{ \text{invertible linear transformations } V \rightarrow V \}.$

② \subseteq ③

② \supseteq ③, $g \mapsto (m_g: V \rightarrow V \quad v \mapsto g \cdot v)$

③ \supseteq ②: $g \cdot v = \rho(g)v$

① \subseteq ②, ③ by choice of basis.

If $B = \{v_1, \dots, v_n\}$ is a basis of V ,

then $R_B(g)$ is defined to be

the matrix of linear transformation

$m_g: V \rightarrow V$ under basis B .
 $v \mapsto g \cdot v$

$$g(v_1, \dots, v_n) = (v_1, \dots, v_n) \cdot R_g(g)$$

$$R(g) = \begin{bmatrix} g_{11} & \cdots \\ g_{21} & \ddots \\ g_{31} & \ddots \\ \vdots & \ddots \\ g_{n1} & \ddots \end{bmatrix}$$

$$\begin{aligned} g v_1 &= (v_1, \dots, v_n) \cdot \begin{pmatrix} g_{11} \\ g_{21} \\ \vdots \\ g_{n1} \end{pmatrix} \\ &= \sum_{k=1}^n g_{k1} \cdot v_k \end{aligned}$$

$$R_g : G \rightarrow GL(n, \mathbb{C})$$

$\textcircled{1} \Rightarrow \textcircled{2}, \textcircled{3}$. by choosing $V = \mathbb{C}^n$

$$\text{and } g \cdot v = R(g) \cdot v.$$

Different choice of basis: $C = \{w_1, \dots, w_n\}$.

$$(w_1, \dots, w_n) = (v_1, \dots, v_n) \cdot P$$

P $n \times n$ invertible.

is $(P_B \leftarrow C)$ change of basis matrix.

$$g(w_1, \dots, w_n) = (w_1, \dots, w_n) \cdot R_C(g)$$

$$g(v_1, \dots, v_n) P = (v_1, \dots, v_n) \cdot P \cdot R_C(g).$$

So $g(v_1, \dots, v_n) = (v_1, \dots, v_n) \cdot P R_C(g) P^{-1}$

so $\boxed{R_B(g) = P R_C(g) \cdot P^{-1}}$

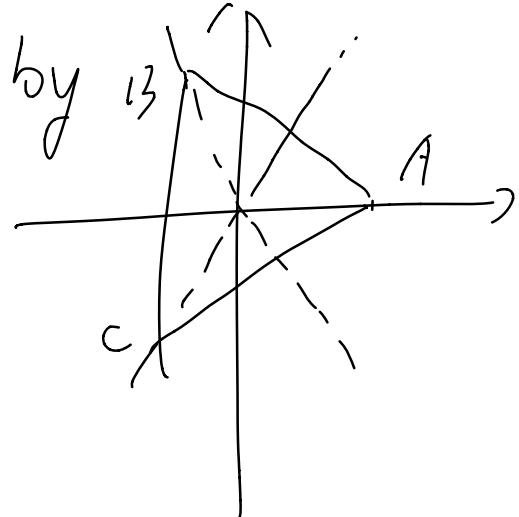
Prop: χ_P is well-defined

since $(\text{tr } R_B(g))$ does not depend on
choice of basis.

Prop: If $g_1 \sim g_2$ are in the same conjugacy class, then

$$\chi_p(g_1) = \chi_p(g_2).$$

Ex: $S_3 = D_3 \xrightarrow{R} GL_2(\mathbb{C})$.



X rotation by 120° .

Y reflection with respect to x-axis.

$$R(x) = \begin{bmatrix} \cos 120^\circ, -\sin 120^\circ \\ \sin 120^\circ, \cos 120^\circ \end{bmatrix}$$

$$R(y) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

$$\chi_{\rho}(x) = -1.$$

$$\chi_{\rho}(y) = 0.$$

V_1, V_2 rep's of G .

$f: V_1 \rightarrow V_2$ linear isomorphism.

If $f(g \cdot v) = g \cdot f(v)$, then f is called an isomorphism of G -repns.

Prop: ρ_1, ρ_2 are isomorphic, and choose $B = \{v_1, \dots, v_n\}$ a basis of V_1 .
and $B' = \{f(v_1), \dots, f(v_n)\}$ a basis of V_2 .
then $R_B = R_{B'}$. Hence $\chi_{\rho_1} = \chi_{\rho_2}$.

Invariant subspace.

Defn W is invariant subspace if

$g \cdot w \in W$, $\forall g \in G, w \in W$.

Prop: $gW = W$.

Trick: averaging under G -operations.

$v \in V$,

$$w = \frac{1}{|G|} \sum_{g \in G} g \cdot v.$$

then $g w = w \quad \forall g \in G$.

CW is an invariant subspace.

Direct sum:

(External) W_1, W_2 are two subspaces,

$$W_1 \overset{E}{\oplus} W_2 = W_1 \times W_2 = \{(w_1, w_2) \mid w_1 \in W_1, w_2 \in W_2\}$$

① addition $(w_1, w_2) + (v_1, v_2) = (w_1 + v_1, w_2 + v_2)$

② scalar multiplication $c \cdot (v_1, v_2) = (cv_1, cv_2)$

(Internal) $W_1, W_2 \subset V$ subspaces.

$$\textcircled{1} \quad W_1 \cap W_2 = \{0\}$$

$$\textcircled{2} \quad W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\} \\ = V.$$

$$\text{then } V = W_1 \overset{I}{\oplus} W_2.$$

Prop: $V = W_1 \overset{E}{\oplus} W_2$ then

$$V \subseteq W_1 \overset{E}{\oplus} W_2.$$

Prop: $V = W_1 \overset{E}{\oplus} W_2$, then $V = \overline{W}_1 \overset{I}{\oplus} \overline{W}_2$
 $\overline{W}_1 = \{(0, w_1) \mid w_1 \in W_1\}$, $\overline{W}_2 = \{(0, w_2) \mid w_2 \in W_2\}$.

If W_1, W_2 are both G -invariant,

then V is a direct sum of
 W_1, W_2 as G -repsn.

Ex: S_3 operates on \mathbb{C}^3 by

$$r \cdot e_i = e_{\sigma(i)}. \quad (e_1, e_2, e_3) \text{ standard basis}$$

$$W_1 = \frac{1}{3} (e_1 + e_2 + e_3)$$

$$W_2 = \left\{ ae_1 + be_2 + ce_3 \mid a+b+c=0 \right\}.$$

$$\text{then } \mathbb{C}^3 = W_1 \oplus W_2.$$

Irreducible repn.

Defn: If V has only trivial G -invariant subspaces $\text{sof. } V$, then V is called irreducible G -repn.

Ex: 1-dim'l repns.

Unitary rep'n.

If V is a G -rep'n. and

\langle , \rangle is a positive Hermitian form
definite.

and $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in G$,

$v, w \in V$, then we say

V is unitary rep'n.

Prop: If V is unitary, then V

W G -invariant. W^\perp is also G -invariant.

Prop: V is unitary, then V is

the direct sum of irreducible reps.
 $V = W_1 \oplus W_2 \oplus \dots \oplus W_n$.

Pf: If V is not irreducible.

V has G -invariant subspace
 W , $\dim W > 0$, $\dim W < \dim V$.

$$S = V = W \oplus W^\perp$$

and $\dim W^\perp < \dim V$.

use induction on \dim .

(Maschke's Thm).

Every G -repn V is the direct sum
of irreducible subspaces.

Pf.: Every V is unitary repn.

\langle , \rangle is a positive definite Hermitian
form.

$$\langle v, w \rangle_G = \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle$$

then $\langle \cdot, \cdot \rangle$ is preserved by g .

$$\langle g^v, g^w \rangle_G = \langle v, w \rangle_G.$$

Orthogonal relations.

$$G = \bigsqcup_{i=1}^r C_i.$$

C_i : conjugacy classes

\mathcal{H} space of class functions $G \rightarrow \mathbb{C}$

$$\dim \mathcal{H} = r.$$

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \psi(g)$$

$$= \frac{1}{|G|} \sum_{i=1}^r c_i \overline{\varphi(g_i)} \psi(g_i)$$

$$c_i = |C_i|, \quad g_i \in C_i.$$

Prop: ρ_1, ρ_2 two repn's with characters χ_1, χ_2 . irreducible,

$$\textcircled{1} \quad \langle \chi_1, \chi_2 \rangle = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ 1 & \text{if } \rho_1 \cong \rho_2 \end{cases}$$

$$\textcircled{2} \quad \text{The number of isomorphism classes of irreducible repns} \\ = r.$$

Denote by χ_1, \dots, χ_r the corresponding characters.

(1) $\Rightarrow \chi_1, \dots, \chi_r$ linearly independent,

+ (2) $\Rightarrow \chi_1, \dots, \chi_r$ is a basis of H^* .

(3) χ is irreducible iff

$$\langle \chi, \chi \rangle = 1$$

$$g_1, g_2 \text{ are in}$$

the same conjugacy class
if and only if

$\chi(g_1) = \chi(g_2)$
for all characters χ .

$$(4) f = n_1 p_1 \oplus n_2 p_2 \oplus \dots \oplus n_r p_r$$

$$\chi = n_1 \chi_1 + \dots + n_r \chi_r.$$

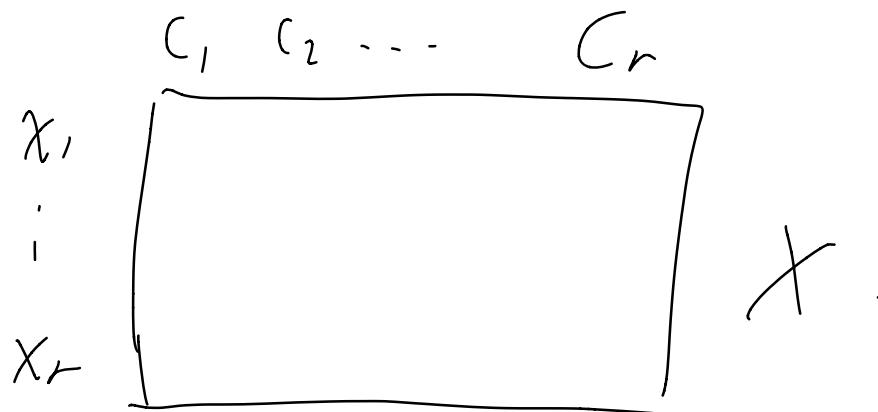
$$n_i = \langle \chi, \chi_i \rangle.$$

(5) $f \cong f'$ iff $\chi_p = \chi_{p'}$.

" \Rightarrow "

" \Leftarrow " $\langle \chi_p, \chi_i \rangle = n_i$

$$\textcircled{b} \quad \sum (d_i)^2 = |G|.$$



Matrix X .

$$\frac{1}{|G|} \cdot \bar{X} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} \cdot X^T = id$$

$$X^T \cdot \bar{X} \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} |G| \\ \vdots \\ |G| \end{bmatrix}.$$

Second orthogonal relations:

$$\underbrace{X^T \cdot X}_{\sum_{k=1}^r} \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} |G| \\ \vdots \\ |G| \end{bmatrix}$$

$$\underbrace{\sum_{k=1}^r \chi_k(g_i) \overline{\chi_k(g_j)}}_{\text{Orthogonal relations of columns vectors.}} = \begin{cases} |G| & i=j \\ 0 & i \neq j \end{cases}$$

First column. $\Rightarrow \sum_{i=1}^r (d_i)^2 = |G|$

G abelian group. $|G|=n$.

$$r=n, \quad \sum_{i=1}^n (d_i)^2 = n.$$

$$d_i = 1.$$

Prop: G is abelian iff all the irreducible repns are 1 dim'l.

Special example $\mathbb{Z}/3\mathbb{Z}$.

ℓ, a, α^2

$$\begin{array}{cccc} \chi_1 & 1 & 1 & 1 \\ \chi_2 & 1 & w & \sqrt{w} \\ \chi_3 & 1 & w^2 & w \end{array} \quad w = e^{\frac{2\pi i}{3}}$$

The product of any two characters is still a character.

Prop: If χ_ρ is a 1-dim'l character,
 χ is a character, then $\chi \cdot \chi_\rho$ is a character of G .

Pf: $\rho: G \rightarrow GL(V)$
 $\rho_\circ: G \rightarrow \mathbb{C}^*$

Check $\rho_\circ \cdot \rho(g) = \rho_\circ(g) \cdot \rho(g)$ is a group homomorphism and $\text{tr}(\rho_\circ \rho) = \chi_\circ(g) \cdot \chi(g)$

Prop: χ is irreducible iff $\chi \circ \chi$ is irreducible.

Pf: $\langle \chi \circ \chi, \chi \circ \chi \rangle$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_0(g) \chi(g)} \cdot \chi_0(g) \cdot \chi(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} \underbrace{\overline{\chi_0(g)} \cdot \chi_0(g)}_{=} \cdot \overline{\chi(g)} \cdot \chi(g)$$

$= 1$ because

$$\rho_0: G \rightarrow U(1)$$

$$= \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi(g)$$

$$= \langle \chi, \chi \rangle .$$

How to find character table for S_4 .

S_4 . 5 conjugacy classes.

	1	6	8	3	6
χ_1	1	1	1	1	1
↑ trivial repn.					

$\rho_{\text{perm}} : S_4 \rightarrow GL(4)$.

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\chi_{\text{perm}}^{(\sigma)} = \#\{i \in \{1, \dots, n\} \mid \sigma(i) = i\}.$$

	1	6	8	3	6
χ_{perm}	4	2	1	0	0

$$\begin{aligned}\langle \chi_{\text{perm}}, \chi_{\text{perm}} \rangle &= \frac{1}{24} (4 \cdot 4 + 6 \cdot 2 \cdot 2 + 8 \cdot 1 \cdot 1 + 3 \cdot \\ &\quad + 6 \cdot 0) \\ &= 2 = 1^2 + 1^2\end{aligned}$$

So χ_{perm} = two irreducible repn's.

$$\langle \chi_{\text{perm}}, \chi_1 \rangle = 1.$$

$$\text{So } \chi_{\text{perm}} = \chi_1 + \chi_2.$$

$$\begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ \chi_1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & \\ & & & & & \\ \chi_2 & 3 & 1 & 0 & -1 & -1 \\ & & & & & \\ & & & & & \\ \chi_{\text{perm}} & 4 & 6 & 8 & 3 & 6 \\ & & (12) & (123) & (12)(13x) & (123)_k \end{array}$$

def ρ_{perm} = sign of σ . we get χ_3 .

$$\begin{array}{cccccc} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ \chi_3 & 1 & -1 & 1 & 1 & -1 \\ & & & & & \\ & & & & & \\ \chi_{\text{perm}} & 4 & 6 & 8 & 3 & 6 \\ & & (12) & (123) & (12)(13x) & (123)_k \end{array}$$

$$x_3 \cdot x_2 \quad 3 \quad -1 \quad 0 \quad -1 \quad 1$$

	(1)	(12)	(123)	$(12)_{13x}$	$(12)_{13}$
x_1	1	1	1	1	1
x_2	3	1	0	-1	-1
x_3	1	-1	1	1	-1
$x_3 \cdot x_2$ $= x_4$	3	-1	0	-1	1
x_5	d_5	a	b	c	d

Second orthogonal relations.

$$(d_5)^2 + 1^2 + 3^2 + 1^2 + 1^2 = 24 \Rightarrow d_5 = 2.$$

$$1 \cdot 1 + 3 \cdot 1 + 1 \cdot (-1) + 3 \cdot (-1) + a \cdot d_5 = 0 \Rightarrow$$

$$a = 0, b = -1, c = 2, d = 0$$

$$\ker \rho = \{ g \mid \chi(g) = \chi(1) \} \quad \begin{matrix} \text{Homework.} \\ = \ker \chi \end{matrix}$$

We can read off normal subgroups of S_4 from the table.

$$\ker \chi_1 = S_k$$

$$\ker \chi_2 = \{ (1) \}$$

$$\begin{aligned} \ker \chi_3 &= \left\{ (1), \begin{matrix} (132) \\ (123), (134), (142) \\ (124), (234), (243), \\ (1434), (13)(14), (14)(23) \end{matrix} \right\} \\ &= A_4. \end{aligned}$$

$$\ker \chi_4 = \{ (1) \}$$

$$\ker \chi_5 = \{ (1), (1234), (14)(23), (13)(24) \}.$$

Actually all the normal subgroups arises
this way.

Any N a normal subgroup of G can be
written as $N = \ker \chi$ for some
character χ

(This will be proved after
regular rep'n)

Permutation representation.

Ex: $S_n \rightarrow GL(n)$ is induced by

S_n -operation on \mathbb{C}^n ,

$$r \cdot e_i = e_{\sigma(i)}.$$

More generally, we consider

G operation on a set S
with $|S| = n$.

$$G \times S \rightarrow S.$$

then $G \rightarrow \{\text{Bijections } S \rightarrow S\} \cong S_n$.

So we get a repn,

$$G \rightarrow S_n \rightarrow GL(n).$$

Another interpretation: (More intrinsic)

\mathbb{C}^S = vector space with elements
 $\sum_{i=1}^n a_i s_i \quad a_i \in \mathbb{C}, \quad s_i \in S.$

(formal linear combinations).

\mathbb{C}^S has basis s_1, \dots, s_n

G operates on \mathbb{C}^S linearly by

$$g \cdot \left(\sum_{i=1}^n a_i s_i \right) = \sum_{i=1}^n a_i g \cdot s_i$$

So we have a G repn \mathbb{C}^S, ρ .

Prop : $\chi_\rho(g) = \#\{s \in S \mid g \cdot s = s\}$.

If: The same as the midterm question.

regular rep'n.

G operates on G itself by

$$g * h = g \cdot h .$$

The induced G -rep'n \mathbb{C}^G is

called regular rep'n. ρ_{reg}

Prop : $\chi_{\text{reg}}(g) = \begin{cases} |G| & \text{if } g=e \\ 0 & \text{if } g \neq e . \end{cases}$

Pf: $\{h \mid g \cdot h = h\}$

$$= \begin{cases} G & \text{if } g=e \\ \emptyset & \text{if } g \neq e . \end{cases}$$

Cor: Let ρ_1, \dots, ρ_r be all the irreducible repn's.

$$\rho_{\text{reg}} = \sum_{i=1}^r n_i \rho_i, \quad n_i = \dim \rho_i$$

$$\begin{aligned} \text{Pf: } & \langle \chi_{\text{reg}}, \rho_i \rangle = \frac{1}{|G|} \cdot \sum_{g \in G} \chi_{\text{reg}}(g) \cdot \chi_i(g) \\ &= \frac{1}{|G|} \cdot \underbrace{\chi_{\text{reg}}(\ell)}_{1} \cdot \underbrace{\chi_i(\ell)}_{\dim \rho_i} \\ &= \dim \rho_i. \end{aligned}$$

Rmk: $\ker \rho_{\text{reg}} = \{e\}$.

Pf: $g \cdot h = h \wedge h$, then
 $g = e$.

Assume H is a normal subgroup of G .

G/H is the quotient group.

$$\rho : G/H \rightarrow GL(V)$$

induces $\tilde{\rho} : G \rightarrow G/H \rightarrow GL(V)$

Prop: $H = \bigcap_i^{\curvearrowleft} \ker \tilde{\rho}_i$
 $\tilde{\rho}_i$ irreducible
rep'n of G/H .

or $H = \bigcap_{\substack{H \subset \ker \tilde{\rho}_i \\ \tilde{\rho}_i \text{ irreducible}}} \ker \tilde{\rho}_i$
rep'n of G .

Pf: $H \subset \ker \tilde{\rho}_i$ so $H \subset \bigcap_i \ker \tilde{\rho}_i$.

$$\bigcap \ker \rho_i = \{eH\}$$

ρ_i irreducible rep's
of G/H

$$\text{So } \bigcap \ker \tilde{\rho}_i = H.$$

commutator group $[G, G] = G'$

$$\text{Defn: } G' = \bigcap H$$

H normal subgroup of G .

$$H \ni [g, h] \quad \forall g, h \in G$$

$$[g, h] = ghg^{-1}h^{-1}$$

Prop: G/G' is commutative.

Prop: If H normal subgroup and G/H is commutative. then $H \supset [G, G]$

$$\text{Thm (HW)} \quad G' = \bigcap \ker \chi_i$$

$$\dim \chi_i = 1.$$

dual, tensor product, Hom.

$$\text{Defn: } V^* = \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) \\ = \{ f: V \rightarrow \mathbb{C} \mid f \text{ linear} \}.$$

V^* has a G -rep'n structure by

$$(g \cdot f)(v) = f(g^{-1} \cdot v).$$

In terms of matrix rep'n. Let

$B = \{e_1, \dots, e_n\}$ be basis of V .

$B' = \{e_1^*, \dots, e_n^*\}$ be dual basis of V^*

defined by $e_i^*(e_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Then $R_{B'}(g) = [(R_B(g))^T]^{-1}$.

If V is unitary and e_1, \dots, e_n are orthonormal basis, then $(R_B(g)^T)^{-1} = \overline{R_B(g)}$. So $\boxed{\chi_{V^*}(g) = \overline{\chi_V(g)}}$

Tensor product. V, W are G -repn's.

$V \otimes W$ has a G -repn structure

by $g \cdot (v \otimes w) = (gv) \otimes (gw)$

Ex: V has basis $\{v_1, v_2, v_3\} = \mathcal{B}$

W has basis $\{w_1, w_2\} = \mathcal{C}$.

$V \otimes W$ has basis

$$\{v_1 \otimes w_1, v_1 \otimes w_2, v_2 \otimes w_1, v_2 \otimes w_2, v_3 \otimes w_1, v_3 \otimes w_2\} = \mathcal{D}.$$

$$g \cdot (v_1, v_2, v_3) = (v_1, v_2, v_3) \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \stackrel{A}{=} \quad \text{where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$g(w_1, w_2) = (w_1, w_2) \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \stackrel{B}{=} \quad \text{where } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\text{Then } g \cdot D = D \cdot \begin{bmatrix} a_{11}B & a_{12}B & a_{13}B \\ a_{21}B & a_{22}B & a_{23}B \\ a_{31}B & a_{32}B & a_{33}B \end{bmatrix}$$

Prop:

$$\chi_{V \otimes W}(g) = \chi_V(g) \cdot \chi_W(g)$$

The product of two characters is still a character, but usually not irreducible. if V, W are both irreducible.

Important to know how to decompose $V \otimes W$ into irreducible repn's.

(Clebsch-Gordan coefficients in quantum mechanism)

$\text{Hom}_{\mathbb{C}}(V, W) \subset \text{rep}^n$.

Defn: $\text{Hom}_{\mathbb{C}}(V, W) = \{ F: V \rightarrow W \text{ linear transformations}\}$.

$$(g \cdot F)(v) = gF(g^{-1}v)$$

Prop: $\text{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$.

as \mathbb{C} -algn's.

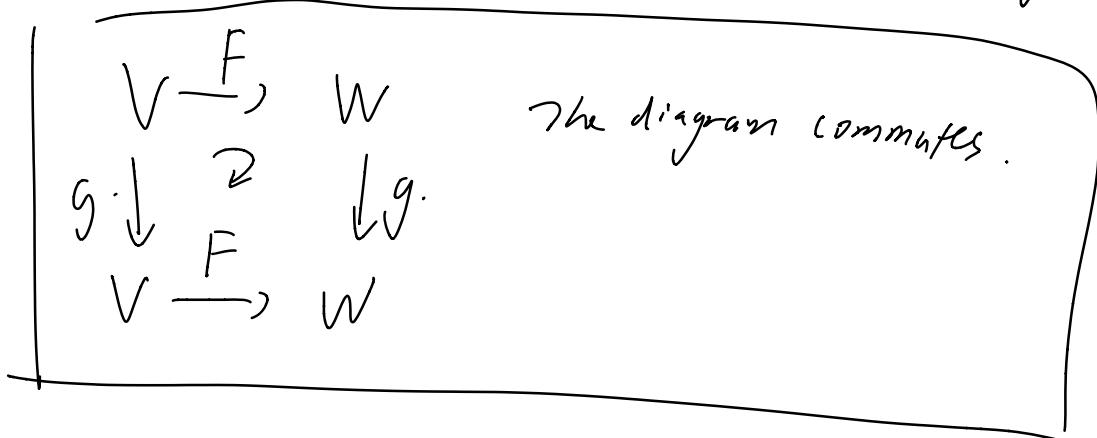
Pf: $V^* \otimes W \rightarrow \text{Hom}_{\mathbb{C}}(V, W)$
 $f \otimes w \mapsto (F: V \mapsto f(v) \otimes w)$.

Schur's lemma.

Defn : (G -invariant linear transformations)
or G -compatible

$$\text{Hom}_G(V, W) = \{ F: V \rightarrow W \text{ linear trans.} \mid$$

$$F(g \cdot v) = g \cdot F(v), \forall g \in G, \forall v \in V\}$$



Prop: $F \in \text{Hom}_G(V, W)$.

$\ker F$ is G -invariant subspace.

$\text{Im } F$ is G -invariant subspace.

Check this from definition.

(Schur's lemma) V, W irreducible.

① $\text{Hom}_G(V, W) = 0 \quad \text{if} \quad V \not\cong W.$

② $\text{Hom}_G(V, V) \cong \mathbb{C}.$

i.e. if $F \in \text{Hom}_G(V, V)$

F is scalar multiplication

$$F(v) = c \cdot v \quad \text{for some } c \in \mathbb{C}.$$

Pf: ① If $F \in \text{Hom}_G(V, W)$,

$\ker F = \{0\}$ or V .

If $\ker F = V$, then $F = 0$

If $\ker F = \{0\}$, then $V \hookrightarrow W$

$\text{Im } F \neq \{0\}$, so $\text{Im } F = W$,

$$F : V \xrightarrow{\cong} W$$

(2) Let $F \in \text{Hom}_G(V, V)$.

F has eigenvalue c .

Eigenspace $\ker(F - c\text{Id})$ is G -invariant, and $\neq \emptyset$

so $\ker(F - c\text{Id}) = V$.

so $F(v) = c \cdot v$.

Application: Center of G from
character table.

Defn: $Z(G) = \{g \in G \mid gh = hg \ \forall h \in G\}$.

Defn: $Z_X = \{g \in G \mid |X(g)| = X(e)\}$

$$\boxed{\begin{aligned} Z &= \{g \in G \mid \underbrace{f(g) = c \cdot \text{Id}}_{c = e^{\frac{2\pi i}{k}}} \} \\ &\text{from Hw.} \end{aligned}}$$

Prop: $\mathcal{Z}(G) = \bigcap_{X \text{ irreducible}} \mathcal{Z}_X$

Pf: " \subset " If $g \in \mathcal{Z}(G)$, ρ irred rep.

$$\rho(g) \cdot \rho(h) = \rho(h) \rho(g) \quad \forall h \in G.$$

$$\therefore \rho(g) \in \text{Hom}_G(V, V)$$

$$\therefore \rho(g) = c \cdot \text{Id}, (\text{Schur})$$

$$g \in \mathcal{Z}_X.$$

" \supset " if $g \in \bigcap \mathcal{Z}_X$,

$$\begin{aligned} \rho(g h g^{-1} h^{-1}) &= \rho(g) \cdot \rho(h) \rho(g)^{-1} \rho(h)^{-1} \\ &= (c \cdot \text{Id}) \rho(h) \cdot (c^{-1} \cdot \text{Id}) \rho(h)^{-1} \\ &= \text{Id}. \end{aligned}$$

$\therefore g h g^{-1} h^{-1} \in \ker \rho$, for all ρ irred.

$$\therefore g h g^{-1} h^{-1} = e.$$

D

Proof of orthogonal relations.

Defn: V G -rep'n. The invariant space

$$\text{is } \text{Inv}_G(V) = \{v \in V \mid g \cdot v = v, \forall g \in G\}.$$

Lemma: $\frac{1}{|G|} \sum_{g \in G} \chi_V(g) = \frac{1}{|G|} \sum_{g \in G} \chi_{\text{Inv}_G(v)}(g)$

$$= \frac{1}{|G|} \sum_{g \in G} \dim \text{Inv}_G(v)$$

$$= \dim \text{Inv}_G(v).$$

Pf: $\tau = \frac{1}{|G|} \sum_{g \in G} \rho(g)$

Then $\tau \cdot \rho(h) = \tau$ for all $h \in G$.

and $\rho(h) \cdot \tau = \tau$



$$\begin{aligned} \tau \cdot \rho(h) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \rho(h) \\ &= \frac{1}{|G|} \sum_{g \in G} \rho(gh) = \tau \end{aligned}$$

$$\text{so } \tau^2 = \tau$$

τ has two eigenspaces.

$$\lambda = 1 \quad V_1$$

$$\lambda = 0 \quad V_0$$

$$V = V_1 \oplus V_0 \quad \text{because}$$

$$v = \underbrace{(2d - \tau)v}_{} + \underbrace{\tau v}_{V_1}$$

$$(\text{aim : } V_1 = \text{Inv}_\tau(v))$$

" \subset " If $v \in V_1$, $\tau(v) = v$.

$$\begin{aligned} \text{then } g \cdot v &= \rho(g)(\tau(v)) \\ &= \tau(v) = v. \end{aligned}$$

" \supset " If $g \cdot v = v$, $\forall g \in G$.

$$\text{then } \tau(v) = v.$$

$$\begin{aligned} \text{So } \frac{1}{|G|} \sum_{g \in G} \chi_{V_1}(g) &= \text{Trace } \tau \\ &= \dim V_1 \\ &= \dim \text{Inv}_G(V). \end{aligned}$$

$$\begin{aligned} \text{Fact: } \text{Inv}_G(\text{Hom}_G(V, W)) \\ = \text{Hom}_G(V, W). \end{aligned}$$

Pf.: Plug in the defn of

Fr-rep'n structure on

$$\text{Hom}_G(V, W)$$

Pf of orthogonal relation.

compute $\frac{1}{|G|} \sum_{g \in G} \chi_{\text{Hom}_G(V, W)}(g)$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{V^* \otimes W}(g)$$

$$= \left(\frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \cdot \chi(g) \right)$$

Lemma $\Leftarrow \dim \text{Inv}_G(\text{Hom}_G(V, W))$

$$\dim \text{Hom}_G(V, W)$$

If V, W irreducible

$$= \begin{cases} 0 & V \not\cong W \\ 1 & V \cong W. \end{cases}$$