Math 241 Homework#1

due 9/5 Thursday in class

Heat equation

Read *A*pplied PDE by Haberman (5th edition) Chapter 1.1-1.3.

1. *A*pplied PDE by Haberman, chapter 1.2, exercise 1.2.1
2. *A*pplied PDE by Haberman, chapter 1.3, exercise 1.3.1
3. Derive the heat equation for a one-dimensional rod assuming that the cross-sectional area $A(x)$ is a non-constant function of x, where $0 < x < L$. Assume all other thermal properties are constant, and there is no heat source.
4. Let $u(x,t)$ be the temperature in a one-dimensional rod, and satisfy the following heat equation with insulated boundary conditions.

   $$u_t = 3u_{xx},\ t \geq 0, 0 \leq x \leq 5.$$

   Denote the total energy by

   $$E(t) = \int_{x=0}^{x=5} u(x,t)\,dx.$$

   Prove that $E(t)$ is a constant by computing $E'(t)$. Explain this from the equation.
5. Let $u(x,t)$ be the temperature in a one-dimensional rod, and satisfy the following heat equation with insulated boundary conditions.

   $$u_t = 3u_{xx} + 2,\ t \geq 0, 0 \leq x \leq 5.$$

   Denote the total energy by

   $$E(t) = \int_{x=0}^{x=5} u(x,t)\,dx.$$

   Assume $E(0) = 4$. Compute $E(t)$.

Calculus

1. Let f, g be functions in $3D$ space and X be a vector field in $3D$.

   (a) Prove the product rule

   $$\operatorname{div}(fX) = \langle \nabla f, X \rangle + f \operatorname{div} X.$$

   (Remark: $2D$ version is the same.)

(b) Let Ω be a bounded region in $3D$ space and n be the outward unit vector on $\partial\Omega$. Use product rule and divergence theorem to show

$$\int_{\Omega} f\Delta g = \int_{\partial\Omega} f\frac{\partial g}{\partial n} - \int_{\Omega}\langle \nabla f, \nabla g\rangle$$

Here $\frac{\partial g}{\partial n}$ is the directional derivative of g in direction n.

2. Show that if $u(x,t)$ is a solution to the transport equation $u_t = u_x$, then it is also a solution of wave equation $u_{tt} = u_{xx}$. Verify that $u(x,t) = f(x+t)$ is always a solution to the transport equation $u_t = u_x$.

3. Show that the general solutions to the transport equation $u_t = u_x$ has the form $u(x,t) = f(x+t)$. (Hint: use change of variables $X = x+t$, $T = x-t$ and multivariable chain rule.)

4. Find general solutions to the PDEs

   (a) $u_t = 2u_x$

   (b) $u_t = 2u_x + x$

Linear algebra

1. Find the general solutions to ODEs

$$\begin{array}{lll} y' + 5y = 0 & y' - 5y = 0 & xy' + 3x^3y = 0 \\ y'' + 4y = 0 & y'' - 4y = 0 & y'' - 4y' + 4y = 0 \end{array}$$

2. Let V be a vector space equipped with an inner ("dot") product $\langle , \rangle$ and a basis $\mathcal{B} = \{v_1, \cdots, v_n\}$. (When $n = 3$, you can think of V as $\mathbb{R}^3$ with the usual dot product. Higher dimensional case is similar.)The basis vectors are *orthogonal* if $\langle v_i, v_j\rangle = 0$ for any $i \neq j$. Let $w \in V$ have the expansion $w = c_1v_1 + \cdots + c_nv_n$. In general, solving for the c_i requires row reduction.

   (a) If the basis vectors are *orthogonal*, there is an explicit formula for the c_i. Prove that

$$c_i = \frac{\langle w, v_i\rangle}{\langle v_i, v_i\rangle}, \quad i = 1, \dots n$$

   (b) In the case where $V = \mathbb{R}^3, \mathcal{B} = \{v_1, v_2, v_3\} = \{(1,0,0), (0,1,1), (0,1,-1)\}$, use this to find the coefficients c_1, c_2, c_3, where $w = (3,4,5)$.

3. The following table gives the inner ("dot") product of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

	$\mathbf{u}$	$\mathbf{v}$	$\mathbf{w}$
$\mathbf{u}$	9	0	6
$\mathbf{v}$	0	1	3
$\mathbf{w}$	6	3	38

For example, $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} = 3$.

(a) Find a unit vector in the same direction as $\mathbf{u}$.

(b) Compute $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$

(c) Compute the length $\|\mathbf{v} + \mathbf{w}\|$

(d) Find the orthogonal projection of $\mathbf{w}$ into the plane E spanned by $\mathbf{u}$ and $\mathbf{v}$. (Express your solution as linear combinations of $\mathbf{u}$ and $\mathbf{v}$.)

(e) Find a unit vector orthogonal to the plane E. (Express your solution as linear combinations of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.)

(f) Find an orthonormal basis of the three dimensional spanned by $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$. Here *orthonormal basis* means the basis vectors are orthogonal unit vectors. (Express your solution as linear combinations of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.)