线性代数 作业 5

2025 年 3 月 4 日

1 基础题

本部分题必做.

题 1. 假设 A 是可逆矩阵，u, v 是列向量。证明：

1. $A + uv^T$ 可逆当且仅当 $1 + v^T A^{-1} u \neq 0$。
2. $A + uv^T$ 可逆时有 $(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1+v^T A^{-1} u}$

题 2. 求下述 n 阶矩阵的逆：

$$\begin{pmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{pmatrix}$$

(其中 $a_i \neq 0$.)

题 3. 假设 $A \in M_{n\times m}(\mathbb{R}), B \in M_{m\times n}(\mathbb{R})$. 证明 $I_n + AB$ 可逆当且仅当 $I_m + BA$ 可逆.

题 4. *Let A, B be $2^n \times 2^n$ matrices, and we want to compute the product $C = AB$. From Wiki, "The Strassen algorithm partitions A, B and C into equally sized block matrices*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

with $A_{ij}, B_{ij}, C_{ij} \in \mathrm{M}_{2^{n-1}\times 2^{n-1}}(\mathbb{R})$*. The naive algorithm would be:*

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} \times B_{11} + A_{12} \times B_{21} & A_{11} \times B_{12} + A_{12} \times B_{22} \\ A_{21} \times B_{11} + A_{22} \times B_{21} & A_{21} \times B_{12} + A_{22} \times B_{22} \end{bmatrix}$$

This construction does not reduce the number of multiplications: 8 multiplications of matrix blocks are still needed to calculate the C_{ij} *matrices, the same number of multiplications needed when using standard matrix multiplication.*

The Strassen algorithm defines instead new values:

$$\begin{aligned} M_1 &= (A_{11} + A_{22}) \times (B_{11} + B_{22})\,; \\ M_2 &= (A_{21} + A_{22}) \times B_{11}; \\ M_3 &= A_{11} \times (B_{12} - B_{22})\,; \\ M_4 &= A_{22} \times (B_{21} - B_{11})\,; \\ M_5 &= (A_{11} + A_{12}) \times B_{22}; \\ M_6 &= (A_{21} - A_{11}) \times (B_{11} + B_{12})\,; \\ M_7 &= (A_{12} - A_{22}) \times (B_{21} + B_{22})\,, \end{aligned}$$

using only 7 multiplications (one for each M_k *) instead of 8 . We may now express the* C_{ij} *in terms of* M_k *:*

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{bmatrix},,$$

Do this inductively, compare how many multiplications of two numbers needed by the naive algorithm and Strassen algorithm. Convince yourself that Strassen algorithm is more efficient. (There is a faster one by Coppersmith-Winograd).

2 思考题，不用交

题 5. 设 A, B 是 $\mathbb{R}$ 上的 $m \times n$ 矩阵, $\text{rank}(A) = r, \text{rank}(B) = s$, 并且 $\text{rank}(A+B) = r+s$ 证明: 存在 m 阶可逆矩阵 P 与 n 阶可逆矩阵 Q 使得

$$PAQ = \begin{pmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \quad PBQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 0 \end{pmatrix}$$