# Math 241 Homework#3

due 9/19 Thursday in class

# Heat equation

Read *A*pplied PDE by Haberman (5th edition) Chapter 2.1-2.4.

1. *A*pplied PDE by Haberman, chapter 2.2, exercise 2.2.2.
2. *A*pplied PDE by Haberman, chapter 2.3, exercise 2.3.1 (a), (b), (c), (f).
3. *A*pplied PDE by Haberman, chapter 2.3, exercise 2.3.3 (b).
4. *A*pplied PDE by Haberman, chapter 2.4, exercise 2.4.1 (b).
5. *A*pplied PDE by Haberman, chapter 2.4, exercise 2.4.7 (b).
6. Prove the uniqueness of solutions to heat equations :

$$u_t = u_{xx} + Q(x)$$

   with with Neumann boundary conditions $u_x(0,t) = a_1$ and $u_x(L,t) = a_2$ and initial condition $u(x,0) = f(x)$. (Hint: use the same method proving uniqueness with Dirichlet conditions.)

7. In this question, you will solve inhomogeneous heat equation with nonzero Neumann boundary conditions. Consider the heat equation (in exercise 1.4.7)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = 1, \quad \frac{\partial u}{\partial x}(L,t) = \beta.$$

   (a) If $\beta = 1 - L$, find the solution. (Hint: in this case, equilibrium solution exists)

   (b) If $\beta = 1 + L$, find one solution in the form $u(x,t) = u_0(x) + ct$ ignoring initial condition

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1, \quad \frac{\partial u}{\partial x}(0,t) = 1, \quad \frac{\partial u}{\partial x}(L,t) = \beta.$$

   (c) If $\beta = 1 + L$, find the solution to heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial x}(0,t) = 1, \quad \frac{\partial u}{\partial x}(L,t) = \beta.$$

   (d) (Extra credit) If the heat source is replaced by arbitrary function $Q(x)$,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q(x), \quad \frac{\partial u}{\partial x}(0,t) = 1, \quad \frac{\partial u}{\partial x}(L,t) = \beta,$$

   how to find one special solution $u(x,t) = u_0(x) + ct$.

8. (Extra credit)

(a) Let $f(x, y)$ and $g(x, y)$ be smooth functions defined on a bounded region $D$. If $f$ and $g$ are equal to zero on the boundary of $D$, prove that

$$\int_D f \cdot \Delta g = \int_D g \cdot \Delta f.$$

(b) If $f$ and $g$ are eigenfunctions to the boundary value problem

$$\Delta\phi = -\lambda\phi, \quad \phi|_{\partial D} = 0$$

corresponding to distinct eigenvalues $\lambda$, prove that

$$\int_D f \cdot g = 0.$$