RT HW1
Due 7/25 please submit your solutions to the TAs in tutorial session

July 22, 2025

Problem 1. *Let p be a prime number. Prove that the set of nonzero residue classes modulo p forms a group under usual multiplication of residue classes. Is it still true if p is not prime?*

Problem 2. *Let G be a group. Prove that the map $\varphi : G \to G$ defined by $\varphi(g) = g^{-1}$ for all $g \in G$ is a bijection.*

Problem 3. *Let G be a group and $g \in G$. If h satisfies $gh = e$, prove that h is the inverse of g.*

Problem 4. *Consider the following elements (permutations) of the symmetric group S_5:*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

1. *Compute the products $\sigma\tau$ and $\tau\sigma$.*

2. *Compute the inverses of σ and τ.*

Problem 5. *What are the possible numbers of symmetries of a quadrilateral? What does each number of symmetries correspond to in terms of the shape of the quadrilateral? (In class, we discussed the case of a triangle, and found the numbers of symmetries to be 6, 2, and 1, corresponding to the equilateral triangle, isosceles triangle, and a general triangle, respectively.)*

Problem 6. *Consider the symmetric group S_n and the following elements: s_i for $1 \leq i < n$, defined as*

$$s_i = \begin{pmatrix} 1 & 2 & \dots & i & i+1 & \dots & n \\ 1 & 2 & \dots & i+1 & i & \dots & n \end{pmatrix}.$$

1. *Show that every element in S_n can be written as products of elements in $\{s_1, s_2, \dots, s_{n-1}\}$.*

2. Prove that the elements satisfies the following equalities:

$$\begin{aligned} s_i s_j &= s_j s_i \quad \textit{if } |i-j| > 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i^2 &= e \quad \textit{for all } i. \end{aligned}$$