Calculus problem 2. 3.4.

2. ① If $u(x,t)$ is a solution to

$u_t = u_x$, then

$$u_{tt} = (u_t)_t = (u_x)_t = u_{xt}$$

$$u_{xx} = (u_x)_x = (u_t)_x = u_{tx}.$$

By Clairaut's theorem. $u_{xt} = u_{tx}$.

so $u_{tt} = u_{xx}$

② $u(x,t) = f(x+t)$

By Chain rule: $u_x = f'(x+t) \cdot 1$

$= f'(x+t)$

$u_t = f'(x+t) \cdot 1 = f'(x+t)$

so $u_t = u_x$.

3. Use change of variables

$$\begin{cases} X = x+t \\ T = x-t. \end{cases} \Rightarrow \begin{cases} x = \dfrac{X+T}{2} \\ t = \dfrac{X-T}{2} \end{cases}$$

Then $$\frac{\partial u}{\partial T} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial T} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial T}$$

$$= u_x \cdot \frac{1}{2} + u_t \left(-\frac{1}{2}\right).$$

If $u(x,t)$ is a solution to

$$u_x = u_t,$$

then $u_T = 0$.

so $u = f(X) = f(x+t)$

4. a) (We can guess what the solution should look like.

$u(x,t) = f(x+2t)$

Because $u_x = f'(x+2t)$

$u_t = f'(x+2t) \cdot 2 = 2f'(x+2t)$)

So we have the following change of variables:

$X = x+2t.$

$T = t$ (You can use any $ax+bt$ here as long as $ax+bt$ is not a multiple of $x+2t$ Think about why).

$\Rightarrow \quad x = X - 2T.$
$\quad\quad t = T$

So $\frac{\partial u}{\partial T} = u_x \cdot \frac{\partial x}{\partial T} + u_t \cdot \frac{\partial t}{\partial T}$

$= u_x(-2) + u_t = 0$

$\Rightarrow u = f(X) = f(x+2t)$

b) Using the same change of variables

$u_T = u_x(-2) + u_t$

$= x = X - 2T$

So $u = \int X - 2T \, dT$

$= X \cdot T - T^2 + f(X)$

$= (x+2t) \cdot t - t^2 + f(x+2t)$

$$= xt + t^2 + f(x+2t)$$

The form of the answer is not unique. see another approach below.

Another method to solve 4 b)

(Linearity method)

We can guess one special solution

$$u_0(x,t) = u_0(x) = -\frac{1}{4}x^2.$$

Then for any solution $u(x,t)$ to $u_t = 2u_x + x$

we consider $v(x,t) = u - u_0$.

Check v satisfies the "homogeneous equation"

$v_t = 2v_x$

So $v(x,t) = f(x+2t)$ for some function f

(problem 4(a))

$\Rightarrow u(x,t) = -\frac{1}{4}x^2 + f(x+2t)$

You can check

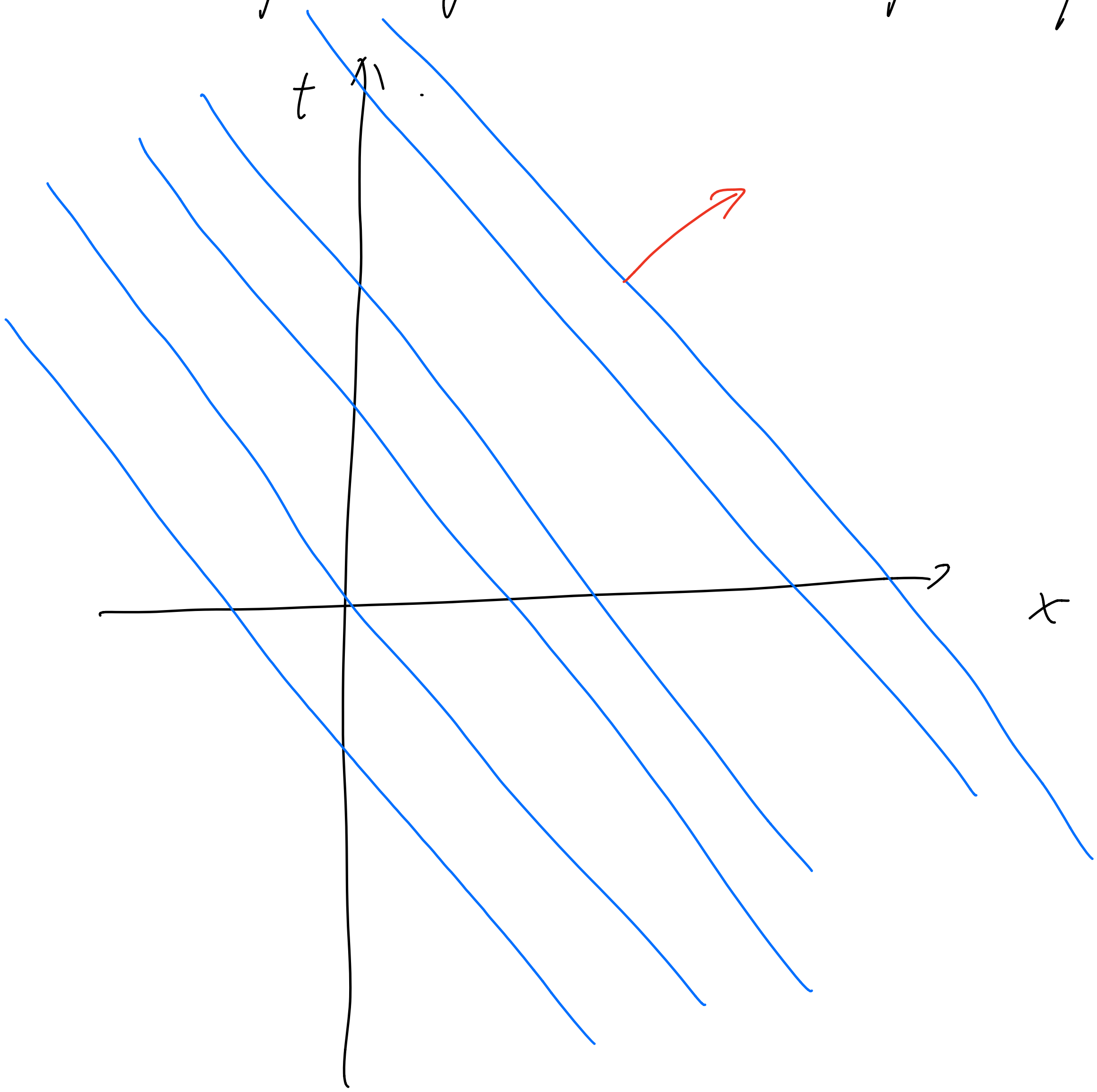
$$-\frac{1}{4}x^2 - (xt + t^2) = -\frac{1}{4}(x+2t)^2$$

so $-\frac{1}{4}x^2 + f(x+2t)$

$xt + t^2 + f(x+2t)$

represent the same set of functions $u(x,t)$

Some geometry in transport equation.



Look at $u_t - u_x = 0$.

This means the directional derivative

$$D_{\vec{w}}u = 0 \quad \text{for } \vec{w} = \frac{\langle -1, 1\rangle}{|\langle -1, 1\rangle|}$$

which means $u(x,t)$ is a constant along the lines with direction vector $\langle -1, 1\rangle$

which are $x+t=C$ (Blue lines)

$u(x,t)$ only depends on C.

so $u(x,t) = f(c) = f(x+t)$

The same idea works for general transport equation $u_t + cu_x = 0$.

$x - ct = d$ for different d are called characteristic lines.

The information are transported along characteristic lines in space time with velocity c.

For example if you know the vaule of the function at x_0 $u(x_0, 0)$ at time $t=0$.

then after some time t_0, You know the information at position x_0 is tranported to position $x_0 + ct$.