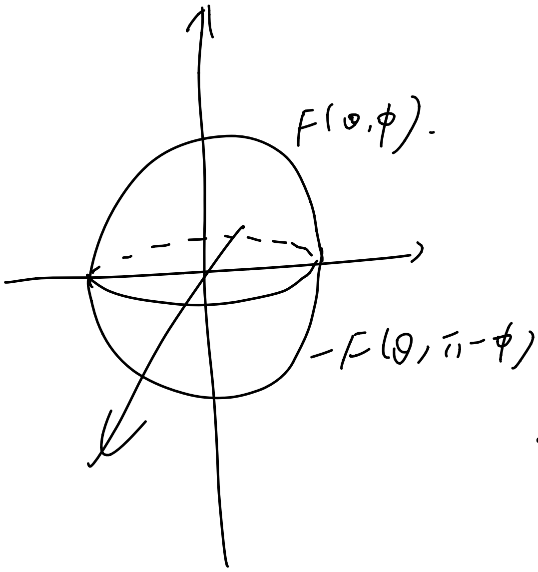
7.10, 12.



Use antisymmetric potential on the lower hemisphere.

Then solution to

$$\begin{cases} \Delta u = 0 \\ u(a,\theta,\phi) = G(\theta,\phi) = \begin{cases} F(\theta,\phi) & \text{if } 0 \le \phi \le \frac{\pi}{2}. \\ -F(\theta,\pi-\phi) & \text{if } \frac{\pi}{2} \le \phi \le \pi. \end{cases} \end{cases}$$

is equal to zero in the middle,

$$u(x,y,0) = 0.$$

This is the solution to the original problem

$$\begin{cases} \Delta u = 0. \\ u(x,y,0) = 0 \\ u(a,\theta,\phi) = F(\theta,\phi) \quad 0 \le \phi \le \frac{\pi}{2}. \end{cases}$$

So $$u(\rho,\theta,\phi) = \sum_{m=0}^{+\infty} \sum_{n=m}^{+\infty} \rho^n \left[A_{mn} \cos m\theta + B_{mn} \sin m\theta\right] \cdot P_n^m(\cos\phi)$$

$$B_{mn} = \frac{\int_{-\pi}^{\pi}\int_0^{\pi} G(\theta,\phi) \cdot P_n^m(\cos\phi) \sin\phi \, d\phi \, d\theta}{\int_{-\pi}^{\pi}\int_0^{\pi} \sin^2 m\theta \left(P_n^m(\cos\phi)\right)^2 \sin\phi \, d\phi \, d\theta}$$

$$= \frac{\int_{-\pi}^{\pi}\int_0^{\frac{\pi}{2}} F(\theta,\phi) \cdot P_n^m(\cos\phi) \sin\phi \, d\phi \, d\theta - \int_{-\pi}^{\pi}\int_{\frac{\pi}{2}}^{\pi} F(\theta,\pi-\phi) \cdot P_n^m(\cos\phi) \sin\phi \, d\phi \, d\theta}{\int_{-\pi}^{\pi}\int_0^{\pi} \sin^2 m\theta \left(P_n^m(\cos\phi)\right)^2 \sin\phi \, d\phi \, d\theta}$$

Similar formula for A_{mn}.

$$A_{mn} = \frac{\int_{-\pi}^{\pi}\int_{0}^{\frac{\pi}{2}} F(\theta,\phi)\cdot P_n^m(\cos\phi)\sin\phi \, d\phi \, d\theta - \int_{-\pi}^{\pi}\int_{\frac{\pi}{2}}^{\pi} F(\theta,\pi-\phi)\cdot P_n^m(\cos\phi)\sin\phi \, d\phi \, d\theta}{\int_{-\pi}^{\pi}\int_{0}^{\pi} \cos^2 m\theta \left(P_n^m(\cos\phi)\right)^2 \sin\phi \, d\phi \, d\theta}$$

You can further simplify by using.

substitution $s = \pi - \phi$,

$F(\theta, \pi-\phi) = F(\theta, s)$.

$\cos(\phi) = -\cos(s)$

$\sin(\phi) = \sin s$.

$$P_n^m(\cos\phi) = \begin{cases} P_n^m(\cos s) & \text{if } n-m \text{ is even.} \\ -P_n^m(\cos s) & \text{if } n-m \text{ is odd.} \end{cases}$$

8.2.3. ① Find equilibrium solution.

$u(r,t) = u_E(r)$.

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -Q(r)$$

$$k\,(r u_E')' = -r\,Q(r)$$

$$r u_E' = \int_0^r \bar{r}\,Q(\bar{r})\,d\bar{r}$$

$$u_E(r) = \int_0^r \frac{1}{s}\int_0^s \bar{r}\,Q(\bar{r})\,d\bar{r}\,ds + C.$$

Choose C such that $u_E(a) = T$.

$$C = T - \int_0^a \frac{1}{s}\int_0^s \bar{r}\,Q(\bar{r})\,d\bar{r}\,ds.$$

② $w = u - u_E$ solves

$$\begin{cases} w_t = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial w}{\partial r}\right) \\ w(a,t) = 0 \\ w(r,0) = f(r) - u_E(r). \end{cases}$$

So $w(r,t) = \sum_{n=1}^{+\infty} A_n \cdot J_{0n}\left(\frac{z_{0n}}{a} r\right) \cdot e^{-k\left(\frac{z_{0n}}{a}\right)^2 t}$

$$A_n = \frac{\int_0^a \left(f(r) - u_E(r)\right) \cdot J_{0n}\left(\frac{z_{0n} r}{a}\right) r \, dr}{\int_0^a \left(J_{0n}\left(\frac{z_{0n} r}{a}\right)\right)^2 r \, dr}.$$

$u(r,t) = u_E + w(r,t)$.

8.2.x. ① Solve $u_E(x,y)$

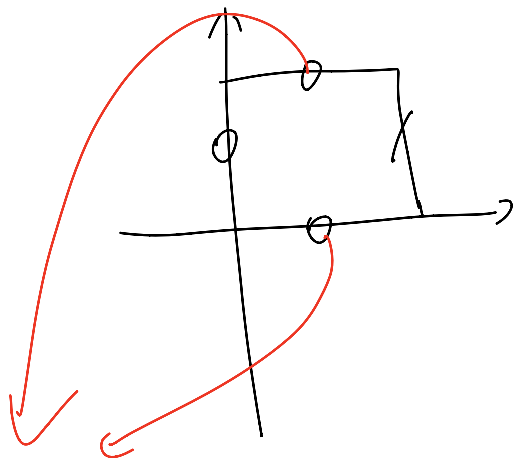
$$\begin{cases} \Delta u_E = 0 \\ \text{The same BCs of } u_E. \end{cases}$$

② Solve $w = u(x,y,t) - u_E(x,y)$.

$$\begin{cases} \Delta w = 0 \\ \text{homogeneous BCs.} \\ w(x,y,0) = f(x,y) - u_E(x,y) \end{cases}$$

7.3.6 see Lecture Notes 23.

7.6.1 b).



homogeneous BCs. in y direction.

Write $u(x,y) = \sum_{n=1}^{+\infty} A_n(x) \cdot \sin \frac{n\pi}{H} y$

eigen functions for

$$\begin{cases} \phi''(y) + \lambda \phi(y) = 0 \\ \phi(0) = \phi(H) = 0. \end{cases}$$

$$Q(x,y) = \sum_{n=1}^{+\infty} b_n(x) \sin \frac{n\pi}{H} y$$

$$b_n(x) = \frac{2}{H} \int_0^H Q(x,y) \sin \frac{n\pi}{H} y \, dy.$$

$\Delta u = Q(x,y)$

$\Rightarrow \quad A_n'' - \left(\frac{n\pi}{H}\right)^2 A_n = b_n(x)$

So $A_n(x) = C_1 e^{\frac{n\pi}{H}x} + C_2 e^{-\frac{n\pi}{H}x}$ + particular solution to the ODE

→ The general formula will be given if needed.

Use variation of parameters. Usually just guess the solution.

Particular solution is

$$a_n(x) = -e^{\frac{n\pi}{H}x} \int_0^x b_n(t) \cdot \frac{e^{\frac{n\pi}{H}t}}{-2\frac{n\pi}{H}} \, dt$$

$$+ e^{-\frac{n\pi}{H}x} \int_0^x b_n(t) \frac{e^{3\frac{n\pi}{H}t}}{-2\frac{n\pi}{H}} \, dt.$$

Expansion of $u(x,y)$ at $x=0$, $x=L$

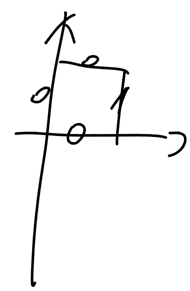
$$\begin{cases} A_n(0) = 0 \\ A_n(L) = \frac{2}{n\pi}\left(1-(-1)^n\right). \end{cases}$$

determines the coefficients C_1, C_2.

$C_1 = -C_2$,

$$2C_1\left(\sinh\left(\frac{n\pi}{H}L\right)\right) + a_n(L) = \frac{2}{n\pi}\left(1-(-1)^n\right).$$

3.6.1.c). ① Solve $\begin{cases} \Delta u_0 = 0 \\ \text{BCs of } u. \end{cases}$



separation of variables.

$$u_0(x,y) = \sum_{n=1}^{+\infty} A_n \sinh\frac{n\pi}{H}x \cdot \sin\frac{n\pi}{H}y.$$

$u_0(L,y) = 1$.

$$\Rightarrow A_n = \frac{2}{n\pi}\left(1-(-1)^n\right) \Big/ \sinh\frac{n\pi}{H}L.$$

② $w(x,y) = u(x,y) - u_0(x,y)$

$$\begin{cases} \Delta w = Q(x,y) \\ w|_{\partial D} = 0. \end{cases}$$

$$w = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} A_{mn} \sin \frac{m\pi x}{L} \sin \frac{n\pi}{H} y .$$

$$Q(x,y) = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} b_{mn} \sin \frac{m\pi}{L} x \sin \frac{n\pi}{H} y$$

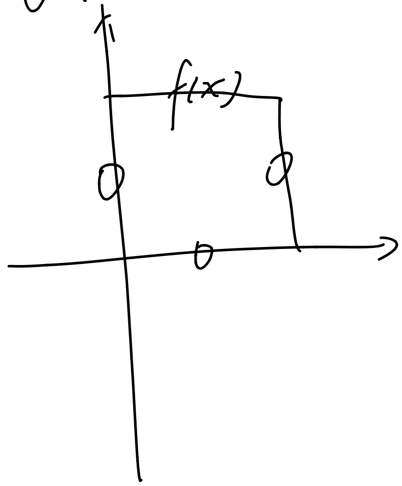
$$b_{mn} = \frac{\int_0^H \int_0^L Q(x,y) \cdot \sin \frac{m\pi}{L} x \sin \frac{n\pi}{H} y \, dx dy}{\frac{4}{LH}.}$$

$$\Delta w = \sum_n \sum_m A_{mn} \cdot (-\lambda_{mn}) \cdot \phi_{mn}$$

$$\text{So } A_{mn} = \frac{\int_0^H \int_0^L Q(x,y) \cdot \sin \frac{m\pi}{L} x \sin \frac{n\pi}{H} y \, dx dy}{\frac{4}{LH} \left[-\left(\left(\frac{m^2\pi^2}{L^2} \right) + \left(\frac{n^2\pi^2}{H^2} \right) \right) \right]}$$

$$u(x,y) = u_0(x,y) + w(x,y)$$

8.6.6.



$$u(x,y) = \sum_{n=1}^{+\infty} A_n(y) \cdot \sin nx$$

$$\Delta u = \sum_n \left(A_n''(y) - n^2 A_n(y)\right) \sin nx = e^{2y} \sin x$$

So $\quad A_n''(y) = n^2 A_n(y), \quad n \neq 1.$

$$A_1''(y) = A_1(y) + e^{2y}$$

Because $A_n(0) = 0$.

So $\quad A_n(y) = C_n \sinh(ny) \quad n \neq 1.$

$$A_1(y) = \underline{\frac{1}{3} \cdot e^{2y}} + B_1 \sinh y + B_2 \cosh y.$$

Get this by guessing the solution

$$A_1(y) = C \cdot e^{2y}.$$

$$A_1''(y) = C \cdot 4 e^{2y}$$

$$\Rightarrow C = \frac{1}{3}.$$

$$A_n(L) = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin nx \, dx.$$

$$n \neq 1, \quad C_n = \frac{2}{\pi \sinh(nL)} \int_0^{\pi} f(x) \cdot \sin nx \, dx.$$

$$n = 1, \quad A_1(0) = \frac{1}{3} + B_2 = 0, \quad B_2 = -\frac{1}{3}.$$

$$A_1(L) = \frac{1}{3} e^{2L} + B_2 \cosh L + B_1 \sinh L$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin x \, dx.$$

$$B_1 = \frac{\frac{2}{\pi} \int_0^{\pi} f(x) \sin x \, dx - B_2 \cosh L}{\sinh L}.$$