MAT334 Notes Final

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$\boxed{z = r\cdot e^{i\theta}}$ modulus $|z| = r$
argument $Arg(z) = \theta$

$|z_1 z_2| = |z_1||z_2|$ $|z^n| = |z|^n$
$arg(z_1 z_2) = arg(z_1) + arg(z_2)$
$arg(z^n) = n\, arg(z)$

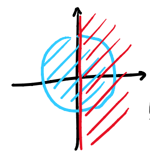
Roots $z^n = \rho e^{i\varphi} \Rightarrow z = \rho^{\frac{1}{n}} \cdot e^{i\frac{\varphi}{n}}$ $|e^z| = e^{Re(z)}$

All standard operations allowed

trig and hyperbolic

$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ $\sinh(z) = \frac{e^z - e^{-z}}{2}$

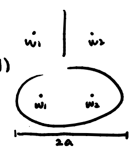
$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$ $\cosh(z) = \frac{e^z + e^{-z}}{2}$



$\frac{1+z}{1-z}$

Geometry

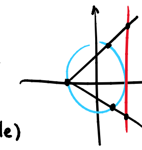
line: $|z - w_1| = |z - w_2|$
circle: $|z - w_1| = \rho|z - w_2|$ $(\rho \neq 1)$
ellipse: $|z - w_1| + |z - w_2| = 2a$
hyperbola: $|z - w_1| - |z - w_2| = d$



Transformations

$z \to z + a$ is translation
$z \to \lambda z$ is scaling
$z \to z e^{i\varphi}$ is rotation
} doesnot change geometry

$z \to z^{-1}$ is inversion (circles → lines, lines → circle)



$\frac{1}{z}$

Topology

closed open

simple not simple

smooth: contiously differentiable

Smooth vs piecewise smooth

D is open ✗ D is an open subset of $\mathbb{C}$ ✓

A function is from an open subset D of $\mathbb{C}$ to $\mathbb{C}$
important

Jordan Curve Theorem: (important)
simple closed curve separate inside and outside.
inside is bounded, outside is bounded

Line integral: $\int_\gamma f(z)dz = \int_{t_1}^{t_2} f(\gamma(t))\gamma'(t)dt$ ($\gamma(t)$ is parametrization
∗ line integral does not depend on parametrization.

Green's formula : $\oint_\gamma (Mdx + Ndy) = 0$ for simple connected ↺ γ

$\oint_\gamma f\, dz = 2i \iint_D \frac{\partial f}{\partial \bar{z}}\, dx\, dy$

Limit

• $\{z_n\} \to L$
$\forall \varepsilon > 0 .\ \exists N > 0 .\ n \geq N \Rightarrow |z_n - L| < \varepsilon$
$z_n \to A \Rightarrow \bar{z}_n \to \bar{A}$ $|z_n| \to |A|$ $Re(z_n) \to Re(A)$
$Im(z_n) \to Im(A)$

• $f(z) \to L$
$\forall \varepsilon > 0 .\ \exists \delta > 0 .\ |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$
$\forall \varepsilon > 0\ \exists R > 0,\ |z| > R \Rightarrow |f(z) - L| < \varepsilon$ $\frac{\bar{z}}{z}$ has no limit at 0 or ∞
$f(z) \to L \Rightarrow \overline{f(z)} \to \bar{L}$ $|f(z_n)| \to |A|$ $Re(f(z)) \to Re(L)$ $Im(f(z)) \to Im(L)$

• $\lim_{z \to z_0} f(z) = f(z_0)$
$f(z)$ cts. at $z_0 \Rightarrow \overline{f(z)}$ $|f(z_n)|$ $Re(f(z))$ $Im(f(z))$ cts. at z_0

$\sum a z^n = \frac{a}{1-z}$ $\sum \frac{z^n}{n!} = e^z$

analytic = holomorphic
entire = analytic in $\mathbb{C}$ e^z is entire
$\cos(z)$ $\sin(z)$

Cauchy Riemann Equation $f = u + iv$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow$ If a function is constant
so is $Re(f)$ and $Im(f)$

Cauchy's Integral Formula

$\int_\gamma \frac{f(z)}{z - z_0} dz = 2\pi i \cdot f(z_0)$

Convergence

Three types of convergence for $\sum a_n (z - z_0)^n$
① converges for $z = z_0$
② converges on a disk $\{z : |z - z_0| < R\}$
③ converges everywhere

Note: convergence on the boundary is unknown.

f converge in a disk $\Rightarrow F$ and f' converge in a disk.

Radius: $\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

Singularities

isolated : f is analytic a small neighborhood.
removable : $\lim_{z\to z_0} < \infty$
pole : $\lim_{z\to z_0} = \infty$ / exist $g(z)$ analytic, $f(z) = \frac{g(z)}{(z - z_0)^m}$
essential : $\lim_{z\to z_0}$ approach all possible values.
non-isolated: not covered.

Examples:
removable: $\frac{z^3}{z}$, $\frac{\sin(z)}{z}$
pole: $\frac{z}{z^3}$, $\frac{\sin(z)}{\cos(z)}$
essential: $\exp(\frac{1}{z})$, $\sin(\frac{1}{z})$

Residues:

removable: $Res(f; z_0) = 0$
pole: simple pole - $f(z) = \frac{g(z)}{h(z)}$, $Res = \frac{g(z_0)}{h'(z_0)}$
$f(z) = \frac{H(z)}{(z - z_0)^m}$, $Res = \frac{H^{m-1}(z_0)}{(m-1)!}$
non-isolated: undefined.

Argument Principle : $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \#\text{zeros} - \#\text{poles}$
counting multiplicity

Rouché's Theorem: f, g analytic inside γ,
$|f(z) + g(z)| \leq |f(z)| \Rightarrow f\ g$ has same # of zeros inside γ

Maximum modulus principle:
$f(z)$ non-constant analytic on $\mathbb{D} \Rightarrow f(z)$ has no local max on $\mathbb{D}$
$\Rightarrow Im(f), Re(f)$ has no local max.
$f(z)$ analytic on $\mathbb{D}$ bounded, cts. on $\mathbb{D} \cup \partial\mathbb{D}$
$\Rightarrow |f|, Re(f), Im(f)$ attain max on $\partial\mathbb{D}$

Schwartz Lemma:
$f(z)$ analytic in unit disk, assume $f(0) = 0$, $|f(z)| \leq 1$ $\forall z \in \mathbb{D}$
Then $|f(z)| \leq |z|$ $\forall z \in \mathbb{D}$.