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What is $\mathcal{L}_1$ Adaptive Control

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Recently, a class of adaptive control schemes called L_1 Adaptive Control (L_1-AC) has been proposed and widely advertised in aerospace control for achieving fast and robust adaptation and better performance than the existing Model Reference Adaptive Control (MRAC) Schemes. The L_1-AC scheme is designed mainly for plants with full state measurement even though the name L_1-AC has been used as an umbrella name for more general classes of plants. In this paper, we show that the L_1-AC for plants with measured states is simply a standard MRAC with a low pass filter inserted in front of the control input. The analysis of the scheme is almost identical to that of MRAC as the same Lyapunov function is used to establish stability. The motivation for using the filter is the fact that for this class of adaptive schemes i.e. MRAC for plants with full state measurement the tracking error can be made arbitrarily small during transient by increasing the adaptive gain. A high adaptive gain however makes the differential equation of the adaptive law or estimator very stiff and leads to numerical problems that cause high oscillations in the estimated parameters leading to loss of adaptivity and deviations from what the theoretical properties dictate. The L_1-AC approach mistook these numerical oscillations as properties of the adaptive scheme and inserted an input low pass filter in order to filter them out. While the filter helps reduce the frequency of these oscillations in the control law the price paid is high. First the numerical instability does not go away and the estimated parameters continue to oscillate without converging to the true parameters even in the presence of sufficiently rich signals. Second, due to the filter the tracking error is no longer guaranteed to converge to zero and the transient bounds for the tracking error also depend on the filter. As a result, the tracking properties of the L_1-AC scheme are worse than what a simple MRAC scheme can generate with adaptive gains that could be high but away from the region of numerical instability. In addition, the presence of the filter reduces the robust stability margins in the presence of unmodeled dynamics and provides literally no advantage as simple robust MRAC techniques can solve the same problem achieving much better properties. The authors of L_1-AC often compare the properties of a numerically unstable MRAC due to extremely high adaptive gains something that is prohibited by robust adaptive control with those of the filtered MRAC aka L_1-AC to show that L_1-AC performs better. Such comparisons are not only misleading but do not reveal what causes what giving the reader a false impression of a new theory that results to new performance and robustness improvements. The use of filters in adaptive control is not new and they are used to improve the performance and robustness of certain adaptive control schemes without destroying their ideal tracking properties. In this paper we present such a scheme that guarantees good transient performance and robustness and reveals the trade off between transient performance and robust stability.

I. Introduction

Adaptive control as a powerful control design tool for handling large parametric uncertainties has been attracting the interest of researchers. Since the early 60's, there have been many accomplishments on the design, stability analysis, robustness, and performance improvement of adaptive control schemes.[1–9] During past decades, several attempts were made to design adaptive control schemes for aerospace applications and in particular longitudinal adaptive controllers for aircraft among which a research group emerged with a new class of adaptive control schemes named L_1 Adaptive Controllers (L_1-AC) with claims of arbitrary tracking

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error performance and "separation of performance improvement from robustness" achieved by simply filtering the plant input and boosting the adaptive gain of the estimated parameters to very large values.[10–15]

In this paper, we investigate the results and claims regarding the performance and robustness of the L_1-AC and compare these properties of well-known robust model reference adaptive control (MRAC) schemes with those of the L_1-AC. In addition, we bring up another MRAC control design that is developed with attention to performance for not just plants with full state measurement but also plants with arbitrary relative degree and with only output measurement. We focus on a class of plants considered by the authors of L_1-AC[10, 15] with full-state measurable, known input vector, and an unknown parameter vector with a large uncertainty such that the control objective cannot be achieved with robust non-adaptive techniques.

II. Problem Formulation

We consider the following class of plants considered by the L_1-AC authors:[10, 15]

$$\dot{x}(t) = A_m x(t) + b\theta^{*T} x(t) + bu(t), \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector which is assumed to be measurable, $u \in \mathbb{R}$ is the control input, $\theta^* \in \mathbb{R}^n$ is an unknown parameters vector belongs to a known compact convex set $\Omega \subset \mathbb{R}^n$, $A_m \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ are known, A_m is Hurwitz, the pair (A_m, b) is controllable (if the eigenvalues of A_m are not in desired locations, they can be arbitrarily assigned by a simple state feedback gain).

The objective is to choose $u(t)$ such that all signals in the closed-loop system are uniformly bounded and $x(t)$ tracks the state vector of a desired reference model

$$\dot{x}_m(t) = A_m x_m(t) + b_m r(t), \quad x_m(0) = x_0 \tag{2}$$

both in transient and in steady-state for any bounded reference signal $r(t)$, where $b_m = k_0 b$, for some $k_0 \in \mathbb{R}$. This paper examines and compares properties of the L_1-AC proposed in Ref. 10, 15 for this class of plants with the standard MRAC with/without normalization and the modified MRAC for performance improvement.[2, 16–18]

III. Adaptive Control Schemes

This section studies some adaptive control schemes as solutions to the problem formulated above and compares their stability and performance properties. It is assumed that there is no plant unmodeled dynamics and that the equation (1) describes the plant dynamics precisely. The robustness analysis for the plant in the presence of unknown unmodeled dynamics will be discussed subsequently in Section **??**.

A. Standard MRAC vs. L_1-AC

The standard MRAC solution to the problem is given by[2]

$$u(t) = -\hat{\theta}^T(t)x(t) + k_0 r(t), \tag{3}$$

where $\hat{\theta}(t)$ is the estimate of θ^* at time t obtained by using the adaptive law

$$\dot{\hat{\theta}}(t) = \operatorname{proj}\left(\Gamma e^T(t) P b x(t)\right), \quad \hat{\theta}(0) = \hat{\theta}_0 \in \Omega \tag{4}$$

where $\Gamma > 0$ is the adaptive gain, $e(t) = x(t) - x_m(t)$ is the tracking error, and $P = P^T > 0$ is the solution of the Lyapunov equation $PA_m + A_m^T P = -I$. The projection operator $\operatorname{proj}(\cdot)$ is used to constrain $\hat{\theta}(t)$ to be within Ω for all t. The L_1-AC solution to the same problem is given by[10, 15]

$$u(t) = C(s)[-\hat{\theta}^T(t)x(t) + k_0 r(t)], \tag{5}$$

where $C(s)$ is a stable strictly proper transfer function with $C(0) = 1$. A first-order low-pass filter of the form

$$C(s) = \frac{1}{\kappa s + 1} \tag{6}$$

is the simplest choice for the input filter, where $\kappa > 0$ is a design parameter. In general $C(s)$ must be chosen so that $\|W_b(s)(C(s)-1)\|_1\theta_{\max} < 1$, where $W_b(s) = (sI - A_m)^{-1}b$ and $\theta_{\max}$ is an upper bound for $\|\hat{\theta}\|_1$. The vector $\hat{\theta}(t)$ is the estimate of θ^* which is given by

$$\dot{\hat{\theta}} = \mathrm{proj}\left(\Gamma\tilde{x}^T(t)Pbx(t)\right), \quad \hat{\theta}(0) = \hat{\theta}_0 \in \Omega \tag{7}$$

$$\dot{\hat{x}}(t) = A_m\hat{x}(t) + b\left(\hat{\theta}^T(t)x(t) + u(t)\right), \quad \hat{x}(0) = x_0, \tag{8}$$

where $\tilde{x}(t) \triangleq x(t) - \hat{x}(t)$, $\Gamma > 0$ is the adaptive gain, $P = P^T > 0$ satisfies $A_m^TP + PA_m = -I$, and the projection operator $\mathrm{proj}(\cdot)$ ensures $\hat{\theta}(t) \in \Omega$, $\forall t$. Note that if filter $C(s)$ is removed i.e., $C(s) = 1$ in (5), then $\hat{x}(t) = x_m(t)$, $\forall t$, and hence $\tilde{x}(t) = e(t)$. Therefore the only difference between (3)-(4) and (5)-(8) is the insertion of $C(s)$. Theorem 1 states what can be analytically proved for the MRAC scheme and the L_1-AC scheme with the filter of the form (6).

Theorem 1 *Consider plant (1) and the desired reference model (2):*
(i) The standard MRAC scheme given by (3)-(4) guarantees that all signals are uniformly bounded and for any reference signal $r(t)$ the tracking error $e(t) = x(t) - x_m(t)$ converges to zero as $t \to \infty$ and if $e(0) = 0$ it satisfies

$$\|e(t)\|_\infty \leq \frac{\nu_0}{\sqrt{\Gamma}}, \tag{9}$$

where $\nu_0 = \frac{\|\tilde{\theta}(0)\|_2}{\sqrt{\lambda_{min}\{P\}}}$ is a constant independent of Γ and $\tilde{\theta}(0) = \hat{\theta}(0) - \theta^*$ is the initial parameter error.
(ii) Assume $\|W_b(s)(C(s)-1)\|_1\theta_{max} < 1$. Then, the L_1-AC given by (5)-(8) guarantees that there exists a $\kappa_{max} > 0$ so that for any $\kappa \in [0, \kappa_{max})$ all signals are uniformly bounded and if $e(0) = 0$, the tracking error satisfies

$$\|e(t)\|_\infty \leq \kappa\,\nu_1 + \frac{\nu_2}{\sqrt{\Gamma}}, \tag{10}$$

where $\kappa_{max} = 1/(\theta_{max}\|sW_b(s)\|_1)$, $W_b(s) \triangleq (sI - A_m)^{-1}b$, θ_{max} is a known upper bound for $\|\theta\|_1$, and ν_1 and $\nu_2 = \frac{\nu_0}{1-\kappa/\kappa_{max}}$ are positive constants independent of Γ (these constants can be explicitly calculated).
(iii) In the case of constant reference signal ($r(t) =$ constant), both schemes guarantee convergence of the tracking error to zero provided the dc gain of the reference model is equal to one.

Proof: (i) The proof follows by choosing the Lyapunov function $V = e^TPe + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta}$, where $P > 0$ is a solution of $A_m^TP + PA_m = -I$ and $\tilde{\theta}(t) = \hat{\theta}(t) - \theta^*$, and using Barbalat's lemma. (ii) Using the same Lyapunov function with e is replaced by $\tilde{x}$ and the small gain theorem the proof follows. (iii) The proof of part (iii) is a direct consequence of previous parts. □

Theorem 1 clearly states that in (10) the best bound for the tracking error is obtained by setting $\kappa = 0$, i.e., removing the filter $C(s)$. In such case we can also guarantee that the tracking error is not only made small by choosing a large adaptive gain but also it goes to zero with time. The presence of the input filter destroys the property of zero tracking error at steady state and introduces a constant bias that is proportional to κ. In addition, the presence of the filter puts a limitation on the size of the unknown parameters.

Table 1 summarizes the properties of the standard MRAC scheme (3)-(4) and the L_1-AC scheme (5)-(8) a more representative name of which is "filtered MRAC" (where ν is a generic symbol for a positive constant).

The authors of L_1-AC were motivated from the bound (9) and tried to boost the adaptive gain to very high values in order to make the tracking error very small even during transient. For large adaptive gains however the differential equation associated with the adaptive law becomes very stiff and eventually leads to numerical instability which often produces high frequency oscillations in the time response of the estimated parameters and other signals responses not consistent with the theory. In an effort to deal with such oscillations the L_1-AC authors proposed the use of a low pass input filter in order to cut down the excitation of the plant by the high frequency oscillations without recognizing the true origin of the oscillations. The filter causes certain signal bounds to become different from those of MRAC which motivated the authors of L_1-AC to call it a new theory in adaptive control. In fact the use of filters in adaptive control as we show in the next sections has been quite popular in adaptive control designs but they have to be inserted in the right place in order to avoid bias and negative impact on performance and robustness.

Table 1. Properties of the standard MRAC and the $\mathbf{L_1}$-AC schemes

	Standard MRAC	*L_1-AC*
Stability condition	*None*	$\|W_b(s)(C(s)-1)\|_1\theta_{\max} < 1$
Convergence of tracking error to zero	*Yes*	*No*
Tracking error bound	$\|e\|_\infty \leq \frac{\nu}{\sqrt{\Gamma}}$	$\|e\|_\infty \leq \kappa\nu + \frac{\nu}{\sqrt{\Gamma}}$

In the presence of unmodeled dynamics both the MRAC and L_1-AC above cannot be proven to have globally bounded solutions unless the adaptive gain has a reasonable size and further modifications in the adaptive law are imposed. The negative effect of high adaptive gains on robustness is well documented in the literature of adaptive control.[2] The negative effect of the input filter on robustness is shown in Ref. 19. In the following subsection we present a scheme that meets the transient performance and robustness criteria while at the same time preserving the ideal tracking properties of MRAC.

B. MRAC with Transient Performance Improvement

In this subsection, we show how the control input can be modified to improve both transient performance and robustness. It is to be noted that the robustness analysis of the L_1-AC with respect to unmodeled dynamics has been considered as an open problem by the authors of L_1-AC (see Ref. 15, p. 76). What they often referred to as robustness analysis requires an adequate knowledge of the unmodeled dynamics which makes them no longer unmodeled. Therefore, the claim of guaranteed robustness in the presence of fast adaptation made by the authors of L_1-AC as the key feature of the L_1-AC scheme is misleading. In the sequel, a robust MRAC scheme is presented for the problem formulated in Section II in the presence of unmodeled dynamics.

We consider the plant with a multiplicative uncertainty term as

$$\dot{x}(t) = A_m x(t) + b\theta^{*T}x(t) + b(1+\Delta_m(s))[u(t)], \quad x(0) = x_0, \tag{11}$$

where $\Delta_m(s)$ is the multiplicative uncertainty which is unknown but small in some norm sense in the low frequency range and may be large at high frequencies. An example of such uncertainties is $\Delta_m(s) = -\mu s$, where $0 < \mu \ll 1$ is a small constant may represent the approximation of a time delay μ in the input, i.e., $u(t-\mu)$ whose Laplace transform is $e^{-\mu s}[u(t)] \approx (1-\mu s)[u(t)]$.

Equation (11) can be written as

$$x(t) = W_b(s)[\theta^{*T}x(t)] + W_b(s)[u(t)] + W_b(s)\Delta_m(s)[u(t)] + w(t), \tag{12}$$

where $W_b(s) = (sI - A_m)^{-1}b$ and $w(t) = \mathcal{L}^{-1}((sI-A_m)^{-1}x_0)$ is a known exponentially decaying term. Since (A_m, b) is controllable, there exists a vector $c_0 \in \mathbb{R}^n$ so that $W_m(s) \triangleq c_0^T W_b(s)$ is a strictly proper minimum-phase transfer function. Then, the parametric model of the plant can be expressed as

$$z(t) = \theta^{*T}\phi(t) + \eta(t), \tag{13}$$

where $z(t) = c_0^T x(t) - W_m(s)[u(t)] - c_0^T w(t)$ and $\phi(t) = W_m(s)[x(t)]$ are available for measurement, and $\eta(t) = W_m(s)\Delta_m(s)[u(t)]$ is the modeling error term. We assume that $\Delta(s) = W_m(s)\Delta_m(s)$ is strictly proper and is analytic in $\mathrm{Re}[s] \geq -\frac{\delta_0}{2}$, for some known $\delta_0 > 0$. No knowledge about the structure and parameters of the unmodeled dynamics is assumed to be known.

We consider the following adaptive control law with an auxiliary input $u_a(t)$ that involves filtering of a number of signals. The filter is similar to the one used in L_1-AC but appears in a different place and operates on different signals.[2,16–18]

$$\begin{aligned} u(t) &= -\hat{\theta}^T(t)x(t) + k_0 r(t) + u_a(t) \\ u_a(t) &= -Q(s)\left[\varepsilon m_s^2 + W_{c0}(s)\left[W_b(s)[x^T(t)]\dot{\hat{\theta}}(t)\right]\right], \end{aligned} \tag{14}$$

where $W_{c0}(s) = -c_0^T(sI - A_m)^{-1}$, $Q(s) = W_m(s)^{-1}/(\tau s+1)^{n^*}$, n^* is the relative degree of $W_m(s)$, $\tau > 0$ is a design parameter, and the estimated parameter vector $\hat{\theta}(t)$ is obtained by using a robust adaptive law such as pure least-squares algorithm with projection

$$\begin{aligned} \dot{\hat{\theta}}(t) &= \mathrm{proj}\left(P(t)\varepsilon(t)\phi(t)\right), \quad \hat{\theta}(0) = \hat{\theta}_0 \\ \dot{P}(t) &= -P(t)\frac{\phi(t)\phi^T(t)}{m_s^2(t)}P(t), \quad P(0) = P_0, \end{aligned} \tag{15}$$

where $\varepsilon(t) = (z(t) - \hat{\theta}^T(t)\phi(t))/m_s^2(t)$ is the estimation error, the projection operator proj(.) ensures $\hat{\theta}(t)$ remains in a known predefined convex set, and the normalizing signal

$$\begin{aligned} m_s^2(t) &= 1 + \phi^T(t)\phi(t) + \alpha_0 n_d(t) \\ \dot{n}_d(t) &= -\delta_0 n_d(t) + |u(t)|^2, \quad n_d(0) = 0 \end{aligned} \tag{16}$$

is designed to guarantee that $|\phi(t)/m_s(t)|$ and $|\eta(t)/m_s(t)|$ are bounded independent of the boundedness of $\phi(t)$ and $\eta(t)$. Note that the dynamic normalizing signal $n_d(t)$ is used to ensure $|\eta(t)/m_s(t)| \in \mathcal{L}_\infty$ and in the absence of modeling error ($\eta(t) = 0$), the design constant α_0 can be set to be zero. The following lemma describes the properties of the robust adaptive law (15).

Lemma 1 (Ref. 6) *Consider the parametric model (13). The adaptive law (15) guarantees that:*
(i) $\varepsilon(t)$, $\varepsilon(t)m_s(t)$, $\dot{\hat{\theta}}(t)$, *and* $\hat{\theta}(t) \in \mathcal{L}_\infty$
(ii) $\varepsilon(t)$, $\varepsilon(t)m_s(t)$, *and* $\dot{\hat{\theta}}(t) \in \mathcal{S}(\frac{\eta^2}{m_s^2})$
where for a given constant $\mu \geq 0$, $x(t) \in \mathcal{S}(\mu)$ *means* $\frac{1}{T}\int_t^{t+T} x^T(\tau)x(\tau)d\tau \leq \nu\mu + \frac{\nu}{T}$, *for any* $t, T \geq 0$, *where* ν *is a positive constant independent of* μ.

The properties of this modified control scheme are given in the following theorem.

Theorem 2 *Consider plant (11) and the desired reference model (2). The robust MRAC given by (14), (15) with the normalizing signal (16) guarantees that for any* $\tau \in (\tau_{min}, \frac{1}{\delta_0})$, *where* $0 < \tau_{min} < \frac{1}{\delta_0}$, *there exists a positive scalar* $\Delta^*(\tau_{min})$ *(a function of* τ_{min}*) such that if the size of* $\Delta(s)$ *is smaller than* $\Delta^*(\tau_{min})$, *then all signals in the closed-loop system are uniformly bounded and if* $e(0) = 0$ *the tracking error satisfies*

$$\|e(t)\|_\infty \leq \tau\nu + \Delta_2\nu, \tag{17}$$

where $\Delta_2 > 0$ *is a constant proportional to some norm of* $\Delta(s)$, *and* ν *is a positive constant independent of* τ *and the modeling error. In addition, in the absence of modeling error (i.e.,* $\Delta_m(s) = 0$*), the tracking error converges to zero as* $t \to \infty$. *The function* $\Delta^*(\tau_{min})$ *is such that as* $\tau_{min} \to 0$, $\Delta^*(\tau_{min}) \to 0$, *that is for a given size of modeling error, the value of* τ *cannot be made arbitrarily small.*
In addition, in the absence of modeling error ($\Delta_m(s) = 0$*), the adaptive control law guarantees that for any bounded reference signal* $r(t)$ *the tracking error* $e(t) = x(t) - x_m(t)$ *converges to zero as* $t \to \infty$ *with a (theoretical) quantifiable upper bound for the zero-state tracking error.*

Proof: The proof follows from Lemma 1 and the proofs of Theorem 6.1 and Theorem 6.2 in Ref. 18. □

Theorem 2 states that in the presence of modeling error, the value of τ cannot be made arbitrarily small as reducing τ has adverse effects on robust stability and it has to meet a lower bound. This demonstrates the trade-off between improving performance and ensuring robust stability.

Although in the absence of unmodeled dynamics there is a quantifiable bound $\|e(t)\|_\infty \leq \tau\nu$ for the tracking error, in practice due to possible numerical instability we are not allowed to reduce τ arbitrarily to improve the transient performance. A very small τ implies a high gain equal to $\frac{1}{\tau}$ in the control law which may lead to numerical instability even in the absence of any disturbances or modeling errors.

In the MRAC scheme discussed above, with $\Delta_m(s) = 0$ the convergence of the tracking error to zero for any reference signal $r(t)$ is guaranteed; moreover, in the presence of sufficiently rich signals the exponential convergence of $\hat{\theta}(t)$ to θ^* is ensured. In such adaptive control schemes, no practical engineer will suggest large adaptive gains or very small τ as such parameters will lead to possible numerical instability and degradation of robustness.

IV. Extension to SISO Plants

The robust MRAC scheme discussed above is applicable to SISO plants with only the output signal available for measurement in the presence of bounded disturbances and unmodeled dynamics (see Ref. 18 and Ref. 2, sec. 9.4).

The authors of L$_1$-AC often use the name "adaptive" and "L$_1$" to label schemes that are clearly non-adaptive as the plant parameters are assumed to be known. We present one such non-adaptive scheme referred to as L$_1$-AC and show that the scheme is nothing more than a fixed gain LTI controller with an integral action. It is therefore very confusing to call such schemes adaptive. Adaptive schemes are nonlinear and are designed to handle large parametric uncertainties that non-adaptive schemes cannot handle.

Considered the plant

$$y(t) = G(s)[u(t) + d(t)], \tag{18}$$

where $u(t)$ and $y(t)$ are the plant input and output, respectively, and $d(t) \triangleq f(t, y(t))$ is due to unknown nonlinear uncertainties and disturbances. It is assumed that there exist positive constants L, L_0, such that inequalities $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$, $|f(t, y)| \leq L|y| + L_0$ hold uniformly in t; also there exist positive constants L_1, L_2, L_3, so that $|\dot{d}(t)| \leq L_1|\dot{y}(t)| + L_2|y(t)| + L_3$, for any $t \geq 0$.

A first order reference system of the form

$$y_m(t) = G_m(s)[r(t)] = \frac{m}{s+m}[r(t)] \tag{19}$$

is considered, where $m > 0$ is a constant. It is assumed that a strictly proper filter $C(s)$ with dc gain of one can be found so that

$$\|H(s)(1 - C(s))\|_1 L < 1, \quad H(s) \triangleq \frac{G(s)G_m(s)}{C(s)G(s) + (1 - C(s))G_m(s)} \text{ is stable.} \tag{20}$$

This assumption essentially implies that any uncertainty in the plant parameter values is accommodated by using a stabilizing controller $C(s)$ that satisfies (20). How to find such a controller for an unknown plant with large parametric uncertainties which is the crucial question in adaptive control is not addressed in general (see Ref. 15, sec. 4.1.4). The controller proposed in Ref. 14, 15 for this plant is of the form

$$\begin{aligned} u(t) &= C(s)[r(t) - \sigma(t)] \\ \dot{\sigma}(t) &= \text{proj}\left(\gamma(y(t) - \hat{y}(t))\right), \quad \sigma(0) = 0 \\ \dot{\hat{y}}(t) &= -m\hat{y}(t) + m(u(t) + \sigma(t)), \quad \hat{y}(0) = 0, \end{aligned} \tag{21}$$

and is illustrated in Figure 1. It is clear from Figure 1 that the nonadaptive controller $C(s)$ is augmented with an integral action that involves an anti-wind up modification in the form of projection.

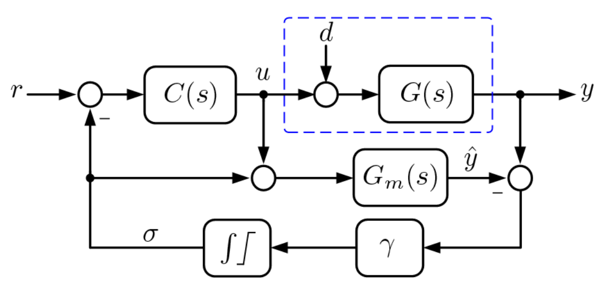

Figure 1. The structure of the closed-loop system with L$_1$-output feedback controller.

The nonadaptive nature of the L$_1$-AC scheme in this case has been pointed out by other authors independently.[20,21] It should be also noted that the fact that an integral controller can be derived using a similar analysis as in parameter estimation, is well known (see eq. (5.181) in Ref. 6).

Given the fact that the underlying controller in Ref. 14 is LTI within projection bounds in the integral action part, and given that robustness of LTI systems has been investigated exhaustively using several powerful methods including $\mathcal{H}_\infty$ and μ-synthesis, what is absent in the L$_1$-AC papers is a comparison of their (linear) controllers with others based on these robust control methods. Two extensive applications of L_1-AC to flight platforms are reported in Ref. 22,23. Both of these cases have employed the linear controller described above (see eq. (31) in Ref. 22, and eq. (3) in Ref. 23). In Ref. 24, simulation studies using a flight platform are used to compare L_1-Ac with other adaptive algorithms. These studies did not indicate any improvements of L_1-AC over MRAC. On the contrary, the performance of L_1-AC in comparison to MRAC was mediocre.

V. Numerical Example

In this section, simulation results are given to show the tracking performance of L_1-AC scheme (5)-(8) and that of the MRAC scheme (14), (15).

Consider plant (1) and the desired reference model (2) with

$$A_m = \begin{bmatrix} 0 & 1 \\ -0.3 & -1 \end{bmatrix},\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ b_m = \begin{bmatrix} 0 \\ 4 \end{bmatrix},\ x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ r(t) = 50\sin(0.5t),$$

and the unknown vector $\theta^* = [\theta_1^*, \theta_2^*]^T$ is such that $-5 \leq \theta_i^* \leq 5$, for $i = 1, 2$; we assume the true value is $\theta^* = [4, -3]^T$. In the simulations we assume $\hat{\theta}(0) = [0, 0]^T$, the sampling rate is 10 KHz (fixed sampling period of 10^{-4} sec) and the Matlab Simulink ODE solver is the forth-order Runge-Kutta ('ode4(Runge-Kutta)').

In the L_1-AC scheme (5)-(8), we have $\theta_{\max} = 10$, then for $\kappa < 0.05$ the stability condition $\|W_b(s)(C(s) - 1)\|_1 \theta_{\max} < 1$ is satisfied. We choose $\kappa = 0.02$ and a very large adaptive gain $\Gamma = 10^4$ (as suggested by the L_1-AC authors). We compare the performance of the L_1-AC scheme with that of the MRAC scheme given by (14), (15) with $c_0 = [1, 1]^T$, $P(0) = 500I$, $\alpha_0 = 0$, and $\tau = 0.5$. In both schemes, the projection operator is used to constrain $-5 \leq \hat{\theta}_i(t) \leq 5$, $\forall t$, for $i = 1, 2$. Figure 2, 3 show the simulation results.

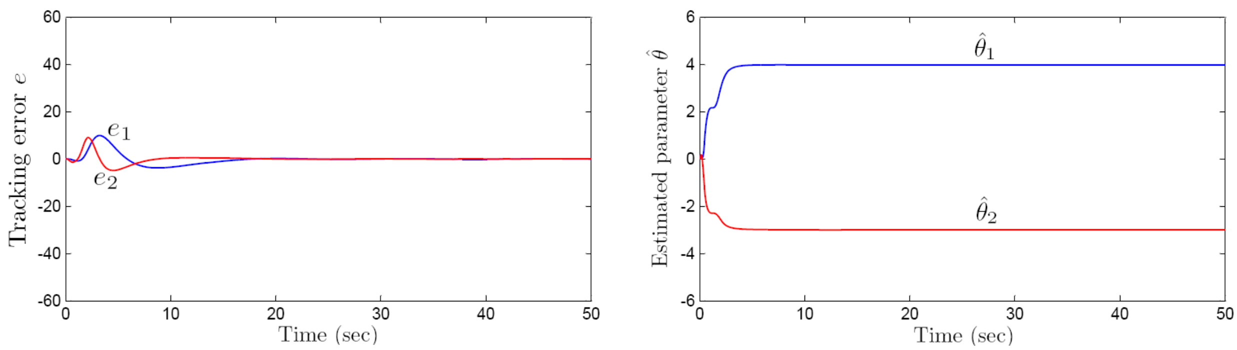

Figure 2. The performance of the MRAC scheme (14), (15) for $\tau = 0.5$, $P(0) = 500I$: (Left) The tracking error $e(t) = [e_1(t), e_2(t)]^T$ for any reference signal converges to zero with improved transient performance; (Right) Since the signals are persistently exciting, the estimated parameter $\hat{\theta}(t) = [\hat{\theta}_1(t), \hat{\theta}_2(t)]^T$ converges to the true value $\theta^* = [4, -3]^T$.

It is clear from Figure 3 that the parameter estimates oscillate with high frequency oscillations due to the use of a high adaptive gain that turned the adaptive law into a stiff differential equation. It is clearly a numerical instability problem as if we keep all parameters the same and reduce the sampling period to 10^{-5} sec, the oscillations go away as demonstrated in Figure 4 . If we then keep increasing the adaptive gain it will again lead to a numerical instability as demonstrated in Figure 5. The fact that there is still boundedness is due to the use of projection. If the projection is removed all signals go unbounded due to numerical instability. Even with a more computationally expensive solver designed for stiff problems, we cannot avoid such phenomena and possible singularity in the solution in general. These results are very consistent with what has been known for decades in robust adaptive control and numerical analysis. The use of an ad hoc input filter in L_1-AC to filter the oscillations observed in simulations without realizing that their origin is numerical led to wrong conclusions and claims. The numerical instability and resulting oscillations cannot be avoided by such an ad hoc filter but by reducing the adaptive gain to reasonable values as suggested by the robust adaptive control theory and basic numerical analysis considerations.

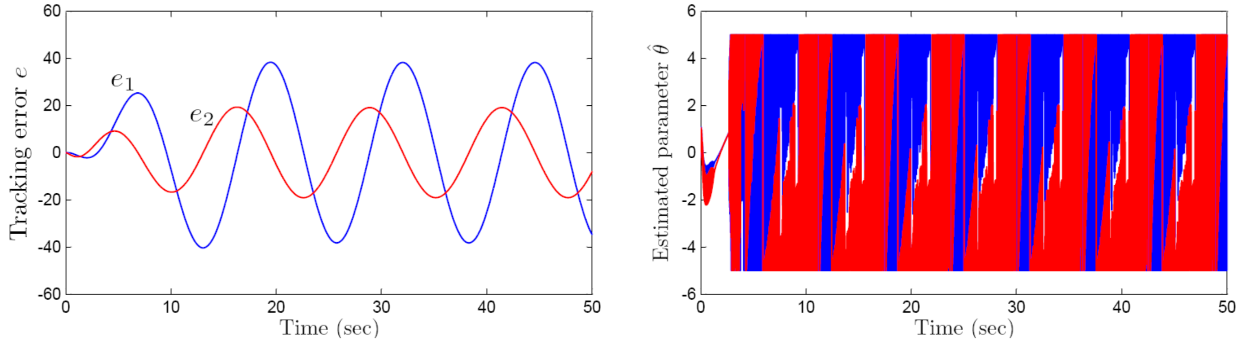

Figure 3. The performance of L$_1$-AC scheme (5)-(8) with $\kappa = 0.02$, $\Gamma = 10^4$: (Left) The tracking error $e(t) = [e_1(t), e_2(t)]^T$ for a non-constant reference signal does not converge to zero and its steady-state amplitude is proportional to the value of κ; (Right) The use of a large adaptive gain $\Gamma = 10^4$ leads to numerical instability and the loss of learning even in the presence of persistently exciting signals as the estimated parameters $\hat{\theta}(t) = [\hat{\theta}_1(t), \hat{\theta}_2(t)]^T$ oscillate between the projection bounds. The boundedness of the signals is due to the projection. In the absence of projection all signals in the closed-loop system go unbounded due to numerical instability.

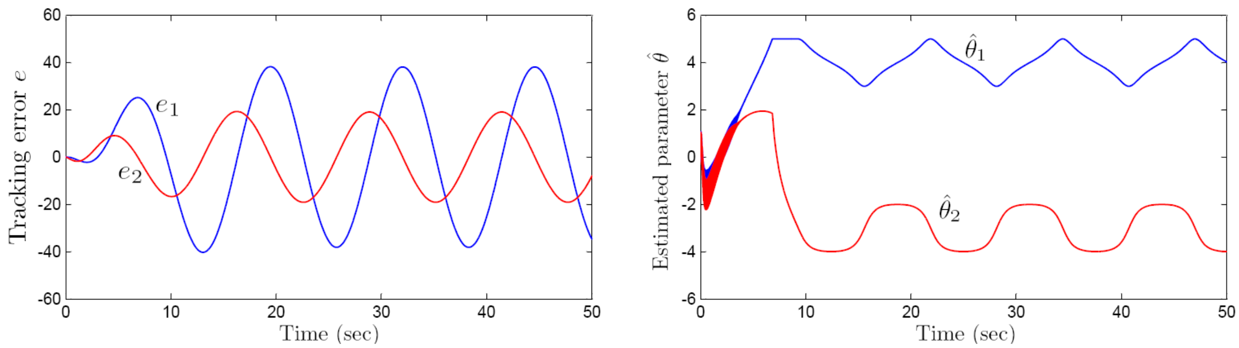

Figure 4. The performance of L$_1$-AC scheme (5)-(8) with $\kappa = 0.02$, $\Gamma = 10^4$, and a smaller step size of 10^{-5} sec: (Left) The tracking error; (Right) With a smaller sampling period the estimated parameters $\hat{\theta}(t) = [\hat{\theta}_1(t), \hat{\theta}_2(t)]^T$ will no longer permanently oscillate between the projection bounds. Non-zero steady state tracking error remains but the high frequency oscillations are gone due to smaller step size.

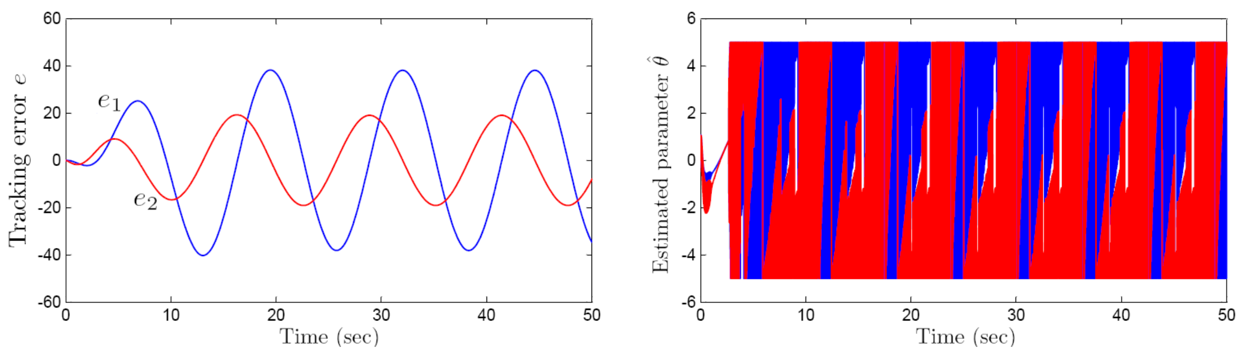

Figure 5. The performance of L$_1$-AC scheme (5)-(8) with $\kappa = 0.02$, a small step size 10^{-5} sec and a larger adaptive gain $\Gamma = 10^6$: (Left) The tracking error; (Right) Choosing a larger adaptive gain $\Gamma = 10^6$ leads again to numerical instability, loss of learning and permanent high frequency oscillations of the estimated parameters between the projection bounds. In the absence of projection all signals in the closed-loop system go unbounded.

VI. Conclusion

In this paper we use analysis to compare the properties of L$_1$-AC with those of MRAC. We show that L$_1$-AC is not based on a new theory but it is an ad hoc modification of a standard MRAC for plants whose full state is available by inserting a low pass filter at its input. The filter was motivated from the observation of oscillations in the estimated parameters due to the use of very large adaptive gains. These oscillations are due to numerical instability as a result of using large adaptive gains that make the adaptive law differential equation very stiff. Reducing the adaptive gain eliminates these oscillations and there is no need for a filter. The input filter introduces a bias in performance and reduces the robust stability margins no matter what size adaptive gains are used. We also presented a scheme that uses the right kind of filtering to achieve transient performance as well as robustness without destroying the ideal properties of the adaptive scheme and without causing any numerical problems during implementation.

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