Improved Robust Control for Multi-link Flexible Manipulator with Mismatched Uncertainties

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***Abstract*—We consider a class of flexible joint manipulator system which is nonlinear and contains uncertainty. The inertial matrix can be used to construct Lyapunov function, even though some of the joints are prismatic. We construct subsystems via backstepping method and suggest a state transformation under which implanted control is used. From this state transformation, a robust controller, which is only based on the possible bound of uncertainty and the uniform positive definiteness of the inertial matrix, is designed to guarantee the practical stability of the system. A numerical example illustrates our conclusions.**

Keywords—Robust control; uncertain system; flexible joint; robot; uniform ultimate boundedness.

I. INTRODUCTION

Since there are always unknown various uncertainties and unmeasurable external disturbances in both modeling and control design of flexible manipulators, it is difficult to address these issues to achieve high performance. In literature, there have been many control schemes proposed in the past decades. With singular perturbation analysis in [1], controllers which were developed for rigid manipulators could be extended for weak joint flexibility cases. Feedback linearization method was also used in the controller design, but it required exact dynamic model of the robot and the measurements of its joint accelerations or jerks(See [2–4]). To deal with the uncertainties, adaptive methodology was applied, but the use of a regressor matrix is quite common in the conventional approach for analysis and design of an adaptive control system for robot manipulator (See [5]). Furthermore, in [6, 7], the model-based adaptive control law was highly sensitive to unmodeled dynamics such as unstructured uncertainties. Treating a flexible joint manipulator as a cascading system, joint torque feedback and backstepping approaches were also applied in the controller design [8, 9]. They required the measurement of joint torques and their derivatives.

Robust control is regarded as another basic control strategy in mechanical system, and has been investigated for manipulator control in [10–12]. Robust controller has a fixed-structure that guarantees stability and performance for uncertain systems. It is capable of compensating for both structured and unstructured uncertainties and simpler to be implemented. However, as it is shown in [9], one may construct a legitimate Lyapunov function by assuming the inertial matrix is positive definite and uniformly ultimately bounded. Actually, there are cases that the inertia matrix of the model may be positive semi-definite or not uniformly ultimately bounded.

In this paper, we consider the control design for the flexible joint manipulator system which is nonlinear and contains uncertainty. It has been shown in [13] that the flexibility of the joint has a significant influence on the system performance, therefore we introduce uncertainties into both of joint flexibility and inertial matrix. Firstly, we divide the overall system into two subsystems by implanting a fictitious control, which is similar to the backstepping method; then, based on the second method of Lyapunov, we construct a robust controller with a more genetic upper boundedness condition of inertial matrix.

The rest parts of paper are organized as follows: Section 2 introduces the dynamics model of the flexible joint manipulator, and some properties are assumed for the control design. A robust controller, which is based on the possible boundedness of the uncertainty, is presented in section 3. The control renders the transformed system practically stable, we prove this by Lyapunov minimax approach. Section 4 is devoted to give an illustrative example of the proposed control law. Conclusions are given in Section 5. Unless otherwise stated, the norms in this paper are Euclidean.

II. SYSTEM DESCRIPTION

We consider a flexible manipulator system described by the following equation [14]:

$$\begin{bmatrix} D(q_1) & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} C(q_1, \dot{q}_1)\dot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} G(q_1) \\ 0 \end{bmatrix} + \begin{bmatrix} K(q_1 - q_2) \\ -K(q_1 - q_2) \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix}, \tag{1}$$

where $q_1 = [q^2 \; q^4 \; \cdots \; q^{2n-2} \; q^{2n}]^T$ is link angles vector and $q_2 = [q^1 \; q^2 \; \cdots \; q^{2n-3} \; q^{2n-1}]^T$ is joint angles vector, let $q = [q_1^T \quad q_2^T]^T$ be a $2n-$ dimension vector representing the generalized coordinate for the system. The joint flexibility is denoted by a linear torsional spring whose elasticity coefficient is K, $K = diag[K_i]_{n\times n}$, with $K_i > 0$, $i = 1, 2, \cdots, n$. $D(q_1)$ is the link inertial matrix and J is a diagonal matrix representing the inertial of actuator, $C(q_1, \dot{q}_1)\dot{q}_1$ represents the Coriolis and centrifugal force, $G(q_1)$ is the gravitation force, and u denotes the input force from the actuators.

Assumption 1: The inertial matrix $D(q_1)$ is uniformly positive definite. That is, there exists a constant $\underline{\sigma} > 0$, such that

$$D(q_1) \geq \underline{\sigma} I, \quad \forall q_1 \in \mathbf{R}^n, \tag{2}$$

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where I denotes the identity matrix. We state this as an assumption other than a fact.

In many other studies of flexible manipulators, the control scheme is constructed by assuming that the inertial matrix is uniformly ultimately upper bounded by a positive constant(i.e. $\|D(q_1)\| \leq \bar{\sigma}$, where $\bar{\sigma} > 0$). However, this doesn't hold for arbitrary manipulator. One example is the 2-link manipulator (with one revolute joint and one prismatic joint) in [15], the inertial matrix is given by

$$M(z) = \begin{bmatrix} (m_1 {l_1}^2 + I_1 + I_2 + m_2 {d_2}^2) & 0 \\ 0 & m_2 \end{bmatrix}, \tag{3}$$

where $z = [\theta_1 \quad d_2]^T$ is the generalized coordinate. Obviously, inertial matrix is not uniformly upper bounded due to the ${d_2}^2$ term. Reference [16] explored and proved a more general property of the inertial matrix in any serial manipulator. Conclusion is that the upper bound of inertial matrix is a quadratic polynomial of the generalized coordinate. We take this assertion as a fact in this paper, that is there exist scalar constants $\bar{\eta}_1 > 0$, $\bar{\eta}_2 \geq 0$, $\bar{\eta}_3 \geq 0$, such that

$$\|D(q_1)\| \leq \bar{\eta}_1 + \bar{\eta}_2\|q_1\| + \bar{\eta}_3\|q_1\|^2, \quad \forall q_1 \in \mathbf{R}^n. \tag{4}$$

This upper bound property is generic. In a special case that the manipulator joints are all revolute, the property is reduced to $\bar{\eta}_2 = \bar{\eta}_3 = 0$, such that

$$\|D(q_1)\| \leq \bar{\eta}_1, \quad \forall q_1 \in \mathbf{R}^n. \tag{5}$$

We propose two steps to design the robust control. Firstly, we rewrite the first part of (1) as

$$\begin{aligned} D(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + G(q_1) \\ + Kq_1 - K(q_2 - u_1) = Ku_1, \end{aligned} \tag{6}$$

where u_1 is a fictitious control implanted into the system, without changing the dynamics of original system. In the later control design procedure, it can be seen that u_1 is only used to formulate the real control u. With u_1 introduced, the system can be divided into two parts, which are: link angles subsystem; joint angles subsystem. The first subsystem is controlled by u_1, which is virtual. The second subsystem is controlled by u which is the real control.

Secondly, we transfer the system with new state variables. Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2 - u_1$, $x_4 = \dot{q}_2 - \dot{u}_1$, then

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, X_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

In practical situations, there always exist modeling uncertainties and extern noises which prevent one from using the precise knowledge of D, C, K and G. Therefore, we introduce time-varying uncertainties $\sigma_1(t)$ and $\sigma_2(t)$ which are uncertain parameter vectors into the system. Then, the dynamics of flexible joint manipulators can be expressed as following :

$$\begin{aligned} S_1 : D(x_1, \sigma_1)\dot{x}_2 &= -C(x_1, \dot{x}_1, \sigma_1)\dot{x}_1 - G(x_1, \sigma_1) - \\ &\quad K(\sigma_1)x_1 + K(\sigma_1)x_3 + K(\sigma_1)u_1, \\ S_2 : J(\sigma_2)\dot{x}_4 &= -J(\sigma_2)\ddot{u}_1 - K(\sigma_2)x_3 + K(\sigma_2)x_1 - \\ &\quad K(\sigma_2)u_1 + u. \end{aligned} \tag{7}$$

Assumption 2: The uncertain parameter vectors $\sigma_1(t) \in \Sigma_1$ and $\sigma_2(t) \in \Sigma_2$ are Lebesgue measurable, where Σ_1 and Σ_2 are prescribed and compact. Meanwhile, it is possible to estimate the bounds of $\sigma_1(t)$ and $\sigma_2(t)$.

The $K(\sigma_1)$ matrix can be decomposed into the nominal part and uncertain part which are denoted by $\hat{K}$ and $\Delta K(\sigma_1)$, respectively, such that:

$$K(\sigma_1) = \hat{K} + \Delta K(\sigma_1). \tag{8}$$

Furthermore, there exists a diagonal matrix $E(\sigma_1) \in \mathbf{R}^{n\times n}$ satisfying $\Delta K(\sigma_1) = \hat{K}E(\sigma_1)$, thus, the matrix $K(\sigma_1)$ can then be represented as

$$K(\sigma_1) = \hat{K}(I + E(\sigma_1)), \tag{9}$$

where $I \in \mathbf{R}^{n\times n}$ is a identity matrix.

Assumption 3: For the decomposed $\hat{K}$ and $E(\sigma_1)$, there exists an e_m such that

$$e_m = \min_i\{\min_{\sigma_1\in\Sigma_1}(e_i(\sigma_1))\} > -1, \quad i = 1, 2, \cdots, n, \tag{10}$$

where $e_i(\sigma_1)$ is the diagonal component of $E(\sigma_1)$.

III. Robust Control Design

Since the overall system is regarded as the cascades of two subsystems, we first give the virtual control u_1 for the first part of system (7). Let us choose a scalar function $\bar{\rho}_1 : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+$ such that for all $\sigma_1(t) \in \Sigma_1$,

$$\bar{\rho}_1(x_1, \dot{x}_1) \geq \|\Phi_1(x_1, \dot{x}_1, \sigma_1)\|, \tag{11}$$

where

$$\begin{aligned} \Phi_1(x_1, \dot{x}_1, \sigma_1) =& \frac{1}{2}\dot{D}(x_1, \dot{x}_1, \sigma_1)(\dot{x}_1 + S_1 x_1) \\ &- C(x_1, \dot{x}_1, \sigma_1)\dot{x}_1 - G(x_1, \sigma_1) \\ &- K(\sigma_1)x_1 + D(x_1, \sigma_1)S_1\dot{x}_1, \end{aligned} \tag{12}$$

$$S_1 = diag[S_{1i}]_{n\times n}, \; S_{1i} > 0, \; i = 1, 2, \cdots, n. \tag{13}$$

For a given positive scalar $\alpha_1 > 0$, a good choice of implanted fictitious control u_1 could be

$$\begin{aligned} u_1(t) =& \hat{K}^{-1}[-\alpha_1(\dot{x}_1(t) + S_1 x_1(t)){\rho_1}^2(x_1, \dot{x}_1) \\ &- \beta_1(\dot{x}_1(t) + S_1 x_1(t))], \end{aligned} \tag{14}$$

where

$$\rho_1(x_1, \dot{x}_1) \geq (1 + e_m)^{-1}\bar{\rho}_1(x_1, \dot{x}_1). \tag{15}$$

Here, the function $\rho_1 : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+$ is C^2 (which means 2-times continuously differentiable). The proper choice of the designed parameter β_1 will be given later. To give the control torque u with a given $S_2 = diag[S_{2i}]_{n\times n}$, $S_{2i} > 0$, $i = 1, 2, \cdots n$, we first choose a scalar function $\rho_2 : \mathbf{R}^{2n} \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+$ such that

$$\rho_2(X_1, x_3, x_4) \geq \|\Phi_2(X, \sigma_1(t), \sigma_2(t))\|, \tag{16}$$

where

$$\begin{aligned} \Phi_2(X, \sigma_1, \sigma_2) =& -J(\sigma_2)\ddot{u}_1(X, \sigma_1, \sigma_2) - K(\sigma_2)x_3 \\ &+ K(\sigma_2)x_1 - K(\sigma_2)u_1(X_1) \\ &+ J(\sigma_2)S_2\dot{x}_3 + \frac{1}{2}\dot{J}(\sigma_2)(x_4 + S_2 x_3). \end{aligned} \tag{17}$$

The input torque u can be constructed as

$$u(t) = -\alpha_2(\dot{x}_3(t) + S_2 x_3(t))\rho_2{}^2(X_1, x_3, x_4) \\ -\beta_2(\dot{x}_3(t) + S_2 x_3(t)), \tag{18}$$

where α_2 and β_2 are designed parameters. From now on, we omit the arguments on uncertainty in $D(x_1, \sigma_1), C(x_1, \dot{x}_1, \sigma_1)$ and $K(\sigma_1)$ etc. if no confusion arises. Otherwise it will be denoted.

Theorem 1: Subject to Assumption (1-3), the control (18) renders the systems (7) practically stable, furthermore, the size of uniform ultimate boundedness ball can be made arbitrarily small by a suitable choice of α_1 and α_2.

Proof : Choose the Lyapunov function candidate as

$$V(X) = V_1(X_1) + V_2(X_2), \tag{19}$$

where

$$V_1(X_1) = \tfrac{1}{2}(x_2 + S_1 x_1)^T D(x_2 + S_1 x_1), \tag{20}$$
$$V_2(X_2) = \tfrac{1}{2}(x_4 + S_2 x_3)^T J(x_4 + S_2 x_3). \tag{21}$$

To show $V(X)$ is a legitimate Lyapunov function candidate for any flexible joint manipulator, we need to prove that $V(X)$ is positive definite and decrescent. Let

$$\Psi_1 = \begin{bmatrix} S_1^2 & S_1 \\ S_1 & I \end{bmatrix}, \bar{S}_1 = \lambda_{\max}(\Psi_1), \underline{S}_1 = \lambda_{\min}(\Psi_1), \tag{22}$$

based on Assumption 1,

$$\begin{aligned} V_1 &\geq \frac{1}{2}\underline{\sigma}_1\|\dot{x}_1 + S_1 x_1\|^2 \\ &= \frac{1}{2}\underline{\sigma}_1 [x_1 \quad \dot{x}_1] \begin{bmatrix} S_1^2 & S_1 \\ S_1 & I \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} \\ &\geq \frac{1}{2}\underline{\sigma}_1 \underline{S}_1 \|X_1\|^2 =: \eta_0^1 \|X_1\|^2, \end{aligned}$$

where $\eta_0^1 = \frac{1}{2}\underline{\sigma}_1\underline{S}_1$. This means V_1 is positive definite.

For the upper bound condition (4) of inertial matrix,

$$V_1 \leq (\eta_1^1 + \eta_2^1\|x_1\| + \eta_3^1\|x_1\|^2)\|\dot{x}_1 + S_1 x_1\|^2. \tag{23}$$

Since

$$\|x_1\| \leq \|X_1\|, \tag{24}$$
$$\|x_1\|^2 \leq \|x_1\|^2 + \|\dot{x}_1\|^2 = \|X_1\|^2, \tag{25}$$

one has

$$V_1(X_1) \leq (\eta_1^1\bar{S}_1\|X_1\|^2 + \eta_2^1\bar{S}_1\|X_1\|^3 + \eta_3^1\bar{S}_1\|X_1\|^4). \tag{26}$$

Similarly to V_1, we can compute the bound of V_2 as

$$\eta_0^2\|X_2\|^2 \leq V_2(X_2) \leq \eta_1^2\|X_2\|^2, \tag{27}$$

where

$$\begin{aligned} &\eta_0^2 = \frac{1}{2}\underline{\theta}\lambda_{\min}(\Psi_2), \qquad \eta_1^2 = \frac{1}{2}\bar{\theta}\lambda_{\max}(\Psi_2), \\ &\Psi_2 = \begin{bmatrix} S_2^2 & S_2 \\ S_2 & I \end{bmatrix}, \\ &\underline{\theta} = \lambda_{\min}(J), \qquad \bar{\theta} = \lambda_{\max}(J). \end{aligned} \tag{28}$$

By combining (26) and (27), we have

$$\eta_0\|X\|^2 \leq V \leq \eta_1\|X\|^2 + \eta_2\|X\|^3 + \eta_3\|X\|^4, \tag{29}$$

where $\eta_0 = \min\{\eta_0^1, \eta_0^2\}, \eta_1 = \max\{\eta_1^1\bar{S}_1, \eta_1^2\}, \eta_2 = \eta_2^1\bar{S}_1, \eta_3 = \eta_3^1\bar{S}_1$.

That is, V is decrescent for all $X \in \mathbf{R}^{4n}$. Therefore, we have proved that V is a legitimate Lyapunov function candidate.

The derivative of V_1 along the trajectory of the controlled system is given by

$$\begin{aligned} \dot{V}_1 =&(\dot{x}_1 + S_1 x_1)^T D(\ddot{x}_1 + S_1\dot{x}_1) \\ &+ \frac{1}{2}(\dot{x}_1 + S_1 x_1)^T \dot{D}(\dot{x}_1 + S_1 x_1) \\ =&(\dot{x}_1 + S_1 x_1)^T(\frac{1}{2}\dot{D}\dot{x}_1 + \frac{1}{2}\dot{D}S_1 x_1 - C\dot{x}_1 \\ &- G - Kx_1 + DS_1\dot{x} + Ku_1 + Kx_3). \end{aligned} \tag{30}$$

From (9) and (12) it can be seen that

$$\begin{aligned} \dot{V}_1 =&(\dot{x}_1 + S_1 x_1)^T\Phi_1 + (\dot{x}_1 + S_1 x_1)^T K u_1 \\ &+ (\dot{x}_1 + S_1 x_1)^T K x_3 \\ =&(\dot{x}_1 + S_1 x_1)^T\Phi_1 + (\dot{x}_1 + S_1 x_1)^T \hat{K} u_1 \\ &+ (\dot{x}_1 + S_1 x_1)^T \hat{K} E u_1 + (\dot{x}_1 + S_1 x_1)^T K x_3. \end{aligned} \tag{31}$$

We substitute the control (14), then

$$\begin{aligned} \dot{V}_1 =&(\dot{x}_1 + S_1 x_1)^T\Phi_1 + (\dot{x}_1 + S_1 x_1)^T[-\alpha_1(\dot{x}_1 + S_1 x_1)\rho_1^2 \\ &- \beta_1(\dot{x}_1 + S_1 x_1)] + (\dot{x}_1 + S_1 x_1)^T\hat{K}E\hat{K}^{-1}[-\alpha_1(\dot{x}_1 \\ &+ S_1 x_1)\rho_1^2 - \beta_1(\dot{x}_1 + S_1 x_1)] + (\dot{x}_1 + S_1 x_1)^T K x_3 \\ \leq&\|\dot{x}_1 + S_1 x_1\|\bar{\rho}_1 - \alpha_1(1 + e_m)\|\dot{x}_1 + S_1 x_1\|^2\rho_1^2 \\ &- \beta_1(1 + e_m)\|\dot{x}_1 + S_1 x_1\|^2 + (\dot{x}_1 + S_1 x_1)^T K x_3. \end{aligned} \tag{32}$$

From the choosing of function ρ_1 in (15), we have

$$\begin{aligned} \dot{V}_1 \leq&\|\dot{x}_1 + S_1 x_1\|(1 + e_m)\rho_1 - \alpha_1(1 + e_m)\|\dot{x}_1 + S_1 x_1\|^2\rho_1^2 \\ &- \beta_1(1 + e_m)\|\dot{x}_1 + S_1 x_1\|^2 + (\dot{x}_1 + S_1 x_1)^T K x_3 \\ \leq&\frac{1 + e_m}{4\alpha_1} - \beta_1(1 + e_m)\|\dot{x}_1 + S_1 x_1\|^2 \\ &+ (\dot{x}_1 + S_1 x_1)^T K x_3. \end{aligned} \tag{33}$$

According to the inequality $ab \leq \frac{1}{2}(a^2 + b^2),\ a, b \in \mathbf{R}$, we have

$$\begin{aligned} (\dot{x}_1 + S_1 x_1)^T K x_3 &\leq \|\dot{x}_1 + S_1 x_1\|\|K\|\|x_3\| \\ &\leq \frac{1}{2}(\omega_1\|\dot{x}_1 + S_1 x_1\|^2 + \omega_1^{-1}\|x_3\|^2)\|K\| \\ &\leq \frac{1}{2}(\omega_1\|\dot{x}_1 + S_1 x_1\|^2 + \omega_1^{-1}\|x_3\|^2)\lambda_k, \end{aligned} \tag{34}$$

where $\omega_1 > 0$ is a constant, $\lambda_k \geq \|K\|$. Therefore,

$$\begin{aligned} \dot{V}_1 \leq&\frac{1 + e_m}{4\alpha_1} - \beta_1(1 + e_m)\|\dot{x}_1 + S_1 x_1\|^2 \\ &+ (\frac{1}{2}\omega_1\|\dot{x}_1 + S_1 x_1\|^2 + \frac{1}{2}\omega_1^{-1}\|x_3\|^2)\lambda_k \\ =&\frac{1 + e_m}{4\alpha_1} - [\beta_1(1 + e_m) - \frac{1}{2}\omega_1\lambda_k]\|\dot{x}_1 + S_1 x_1\|^2 \\ &+ \frac{1}{2}\omega_1^{-1}\lambda_k\|x_3\|^2. \end{aligned} \tag{35}$$

Next, based on the second part of (7), the derivative of V_2 is given by

$$\begin{aligned}\dot{V}_2 =&(\dot{x}_3 + S_2 x_3)^T J(\ddot{x}_3 + S_2\dot{x}_3)\\ &+ \frac{1}{2}(\dot{x}_3 + S_2 x_3)^T \dot{J}(\dot{x}_3 + S_2 x_3)\\ =&(\dot{x}_3 + S_2 x_3)^T [-J\ddot{u}_1 - Kx_3 + Kx_1 - Ku_1\\ &+ JS_2\dot{x}_3 + \frac{1}{2}\dot{J}(\dot{x}_3 + S_2 x_3) + u].\end{aligned} \tag{36}$$

Based on the (16), (17) and the control (18),

$$\begin{aligned}\dot{V}_2 =&(\dot{x}_3 + S_2 x_3)^T \Phi_2 + (\dot{x}_3 + S_2 x_3)^T u\\ =&(\dot{x}_3 + S_2 x_3)^T \Phi_2 + (\dot{x}_3 + S_2 x_3)^T\\ &\times [-\alpha_2(\dot{x}_3 + S_2 x_3)\rho_2^2 - \beta_2(\dot{x}_3 + S_2 x_3)]\\ \le&\|\dot{x}_3 + S_2 x_3\|\rho_2 - \alpha_2\|\dot{x}_3 + S_2 x_3\|^2\rho_2^2\\ &- \beta_2\|\dot{x}_3 + S_2 x_3\|^2\\ \le&\frac{1}{4\alpha_2} - \beta_2\|\dot{x}_3 + S_2 x_3\|^2.\end{aligned} \tag{37}$$

Now, with (35),(37) and the inequality $\|x_3\|^2 \le \|X_2\|^2$, it can be seen that

$$\begin{aligned}\dot{V}(X) =&\dot{V}_1(X_1) + \dot{V}_2(X_2)\\ \le&\frac{1+e_m}{4\alpha_1} - [\beta_1(1+e_m) - \frac{1}{2}\omega_1\lambda_k]\|\dot{x}_1 + S_1 x_1\|^2\\ &+ \frac{1}{2}\omega_1^{-1}\lambda_k\|x_3\|^2 + \frac{1}{4\alpha_2} - \beta_2\|\dot{x}_3 + S_2 x_3\|^2\\ \le&\frac{1+e_m}{4\alpha_1} - [\beta_1(1+e_m) - \frac{1}{2}\omega_1\lambda_k]\underline{S}_1\|X_1\|^2\\ &+ \frac{1}{2}\omega_1^{-1}\lambda_k\|X_2\|^2 + \frac{1}{4\alpha_2} - \beta_2\underline{S}_2\|X_2\|^2\\ =&\frac{1+e_m}{4\alpha_1} + \frac{1}{4\alpha_2} - [\beta_1(1+e_m) - \frac{1}{2}\omega_1\lambda_k]\underline{S}_1\|X_1\|^2\\ &- (\beta_2\underline{S}_2 - \frac{1}{2}\omega_1^{-1}\lambda_k)\|X_2\|^2\\ =:&\, h_1 + h_2 - h_3\|X_1\|^2 - h_4\|X_2\|^2,\end{aligned} \tag{38}$$

where

$$\begin{aligned}&\underline{S}_2 = \lambda_{\min}(\Psi_2), h_1 = \frac{1+e_m}{4\alpha_1}, h_2 = \frac{1}{4\alpha_2},\\ &h_3 = \beta_1(1+e_m) - \frac{1}{2}\omega_1\lambda_k,\\ &h_4 = \beta_2\underline{S}_2 - \frac{1}{2}\omega_1^{-1}\lambda_k.\end{aligned} \tag{39}$$

If we choose suitable β_1, β_2 to satisfy $\beta_1(1+e_m) - \frac{1}{2}\omega_1\lambda_k > 0, \beta_2\underline{S}_2 - \frac{1}{2}\omega_1^{-1}\lambda_k > 0$, then we have

$$\dot{V} \le -\gamma\|X\|^2 + \bar{h}, \tag{40}$$

where $\gamma = \min\{h_3, h_4\}, \bar{h} = h_1 + h_2$. The uniform boundedness performance follows [17]. That is, given a $r > 0$, with $\|X(t_0)\| \le r$, where t_0 is the initial time, there is a $d(r)$ given by

$$\begin{aligned}d(r) &= \begin{cases} r\left[\frac{\eta_1+\eta_2 r+\eta_3 r^2}{\eta_0}\right]^{\frac{1}{2}} & \text{if} \quad r > R\\ R\left[\frac{\eta_1+\eta_2 R+\eta_3 R^2}{\eta_0}\right]^{\frac{1}{2}} & \text{if} \quad r \le R\end{cases},\\ R &= \sqrt{\bar{h}/\gamma},\end{aligned} \tag{41}$$

such that $\|X(t)\| \le d(r)$ for all $t \ge t_0$. Uniform ultimate boundedness also follows. That is, given any $\bar{d}$ such that

$$d(r) > R\left[\frac{\eta_1 + \eta_2 R + \eta_3 R^2}{\eta_0}\right]^{\frac{1}{2}}, \tag{42}$$

then $\|X(t)\| \le \bar{d}$ for all $t \ge t_0 + T(\bar{d}, r)$, where

$$\begin{aligned}T(\bar{d}, r) &= \begin{cases} 0 & \text{if} \quad r \le R\\ \frac{\eta_1 r^2+\eta_2 r^3+\eta_3 r^4-\eta_0\bar{R}^2}{\gamma\bar{R}^2-\bar{h}} & \text{otherwise},\end{cases}\\ \bar{R} &= \xi^{-1}(\eta_0\bar{d}^2),\end{aligned} \tag{43}$$

where the function $\xi(\cdot)$ is given by

$$\xi(\theta) = \eta_1\theta^2 + \eta_2\theta^3 + \eta_3\theta^4. \tag{44}$$

Obviously, $\bar{d}$ determines the size of the uniform ultimate boundedness ball and the uniform stability ball. As it is shown above, $\bar{d}$ approaches to 0 when R approaches to 0, which means both of α_1^{-1} and α_2^{-1} are close to 0. Thus, if both of α_1^{-1} and $\alpha_2^{-1} \to 0$, then $\bar{d} \to 0$. Q.E.D.

There are no restrictions about (positive) gain parameters S_1 and S_2 which we used in the control design procedure, they can be arbitrary. In a practical case, one may choose these according to the specific situation such as the actuator saturation limits. The choices of β_1 and β_2 are to guarantee that the coefficient of $\|X\|^2$ (i.e. $\min\{h_3,\ h_4\}$) is positive. Firstly, we need to compute the value for λ_k where $\max_{\sigma_1\in\Sigma_1}\|K\| < \lambda_k$; secondly, we choose ω_1 and compute e_m, $\underline{S}_1$; then, β_1 and β_2 can be chosen based on $h_3 > 0$ and $h_4 > 0$, respectively.

As previously mentioned, in many other Lyapunov-based control designs, the controllers are limited to address the revolute joint case due to the upper bound assumption of inertia matrix. The current work, judged from this, can also be extended to prismatic revolute cases by introducing the new upper bound condition.

IV. Illustrative Example

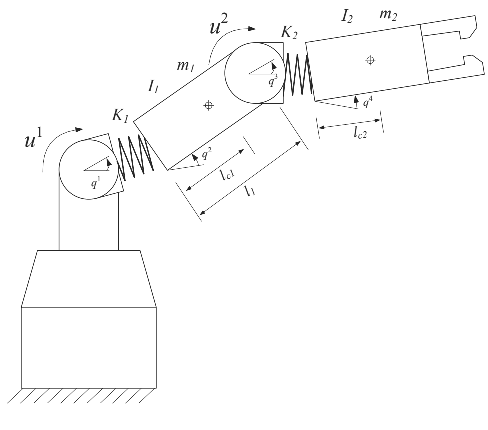

Fig. 1. 2-link flexible joint manipulator mechanism

In Figure (1), we consider a 2-link flexible joint manipulator to show the feasibility of the control proposed in this paper. Let link angle vector $q_1 = [q^2 \quad q^4]^T$, joint angle vector

$q_2 = [q^1 \quad q^3]^T$. The system model of the illustrative example is given by

$$
\begin{aligned}
D(q_1) &= \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \\
C(q_1, \dot{q}_1) &= \begin{bmatrix} -m_1 l_1 l_{c2} \sin q^4 \dot{q}^4 & -m_2 l_1 l_{c2} \sin q^4 (\dot{q}^4 + \dot{q}^2) \\ m_2 l_1 l_{c2} \sin q^4 \dot{q}^2 & 0 \end{bmatrix}, \\
G(q_1) &= \begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \sin q^2 + m_2 l_{c2} g \sin(q^2 + q^4) \\ m_2 l_{c2} g \sin(q^2 + q^4) \end{bmatrix}, \\
J &= \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}, \\
S_1 &= \begin{bmatrix} S_{11} & 0 \\ 0 & S_{12} \end{bmatrix}, S_2 = \begin{bmatrix} S_{21} & 0 \\ 0 & S_{22} \end{bmatrix}, \\
K &= \begin{bmatrix} \hat{K}_{11} & 0 \\ 0 & \hat{K}_{22} \end{bmatrix} (I + \begin{bmatrix} \Delta K_1 & 0 \\ 0 & \Delta K_2 \end{bmatrix}),
\end{aligned}
\tag{45}
$$

where

$$
\begin{aligned}
d_{11} &= m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos q^4) + m_1 l_{c1}^2 + I_1 + I_2, \\
d_{12} &= m_2(l_{c2}^2 + l_1 l_{c2} \cos q^4) + I_2, \\
d_{21} &= d_{12}, \\
d_{22} &= m_2 l_{c2}^2 + I_2.
\end{aligned}
\tag{46}
$$

All elements of inertial matrix are bounded by following

$$
\begin{aligned}
|d_{11}| &\le m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}) + m_1 l_{c1}^2 + I_1 + I_2, \\
|d_{12}| &\le m_2(l_{c2}^2 + l_1 l_{c2}) + I_2, \\
|d_{22}| &\le m_2 l_{c2}^2 + I_2.
\end{aligned}
\tag{47}
$$

To formulate the fictitious control u_1, based on (11) and (12) we can get Φ_1 as

$$
\begin{aligned}
\Phi_1 &= \frac{1}{2} \dot{D}(\dot{q}_1 + S_1 q_1) - C \dot{q}_1 - G - K q_1 + D S_1 \dot{q}_1 \\
&= \frac{1}{2} \begin{bmatrix} -2 m_2 l_1 l_{c2} \sin q^4 \dot{q}^4 & -m_2 l_1 l_{c2} \sin q^4 \dot{q}^4 \\ -m_2 l_1 l_{c2} \sin q^4 \dot{q}^4 & 0 \end{bmatrix} \\
&\times \begin{bmatrix} \dot{q}^2 + S_{11} q^2 \\ \dot{q}^4 + S_{12} q^4 \end{bmatrix} \\
&- \begin{bmatrix} -m_1 l_1 l_2 \sin q^4 \dot{q}^4 & -m_2 l_1 l_2 \sin q^4 (\dot{q}^4 + \dot{q}^2) \\ m_2 l_1 l_2 \sin q^4 \dot{q}^2 & 0 \end{bmatrix} \\
&\times \begin{bmatrix} \dot{q}^2 \\ \dot{q}^4 \end{bmatrix} \\
&- \begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \sin q^2 + m_2 l_{c2} g \sin(q^2 + q^4) \\ m_2 l_{c2} g \sin(q^2 + q^4) \end{bmatrix} \\
&- \begin{bmatrix} -\hat{K}_{11}(1 + e_{11}) & 0 \\ 0 & -\hat{K}_{22}(1 + e_{22}) \end{bmatrix} \begin{bmatrix} q^2 \\ q^4 \end{bmatrix} \\
&+ \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} S_{11} & 0 \\ 0 & S_{12} \end{bmatrix} \begin{bmatrix} \dot{q}^2 \\ \dot{q}^4 \end{bmatrix}.
\end{aligned}
\tag{48}
$$

A simple scalar boundedness function $\bar{\rho}_1$ can be got as

$$\bar{\rho}_1 = \|\Phi_1\|. \tag{49}$$

We can get Φ_2 and ρ_2 by the similar way.

For simulation, we choose $m_1 = m_2 = 1, l_1 = 1, l_{c1} = l_{c2} = 0.5, \hat{K}_{11} = \hat{K}_{22} = 1, I_1 = I_2 = 1, J_{11} = J_{22} = 1, g = 9.81, S_{11} = S_{12} = 1, S_{21} = S_{22} = 2, \omega_1 = 1, \beta_1 = \beta_2 = 2.$ From the control design, we know that $\alpha_{1,2}$ are designed parameters to against the uncertainties, thus, we compare the performances of controlled system by choosing $\alpha_{1,2} = 10$ and $\alpha_{1,2} = 0$, respectively. The simulations are performed by setting uncertainties in mass as $\Delta m_1 = \Delta m_2 = 0.4 \sin(20t)$, uncertainties in joint stiffness as $\Delta K_1 = \Delta K_2 = 0.4 \sin(20t)$. Figure (2) shows the state norm (i.e. $\|q(t)\|$) trajectory under

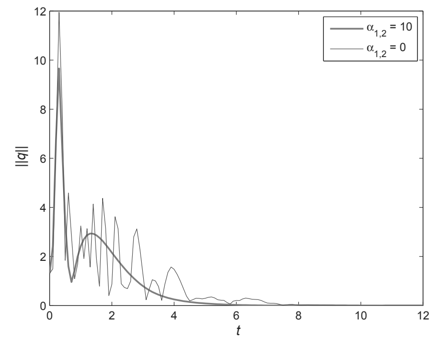

Fig. 2. Response histories of the norms of state

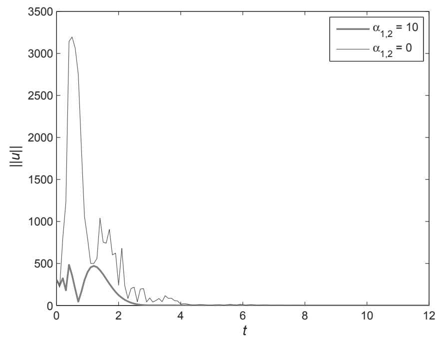

Fig. 3. The corresponding control efforts of the system

the proposed control. The controlled trajectory $\|q(t)\|$ enters a small region around 0 after some time (hence ultimately bounded). Figure 3 shows the corresponding histories of control efforts $\|u(t)\|$. Obviously, comparing to the trajectories of system which doesn't against the uncertainty($\alpha_{1,2} = 0$), both of the trajectories under the robust control ($\alpha_{1,2} = 10$) have less vibration.

In order to describe the magnitude of the uncertainty bound in this case, we may define two parameters as $\hat{m} = \max_t \Delta m_{1,2}(t)$, $\hat{K} = \max_t \Delta K_{1,2}(t)$, the relationship between the average control effort $\bar{u}$ and the upper bounds of uncertainties is shown in Figure(4), with $\bar{u}$ given by

$$\bar{u} = \frac{\int_0^T u(t) dt}{T} \Big|_{(\hat{m}, \hat{K})}. \tag{50}$$

V. Conclusions

We developed a robust control for the uncertain flexible manipulators. The overall system is transformed by defining new state variables, and then be divided into two subsystems

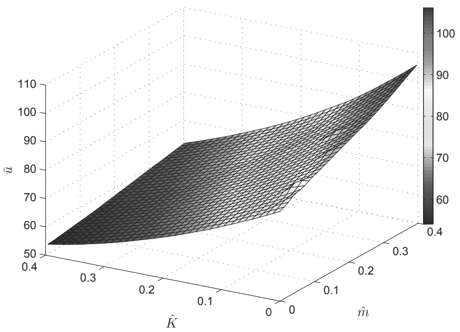

Fig. 4. Average control efforts under different bounds of uncertainty

by implanting a fictitious control for the first subsystem. The transformation is only based on the possible bound of the uncertainty. This control scheme renders practical stability for the system, furthermore, the uniform ultimate boundedness ball and uniform stability ball can be made arbitrarily small by choosing suitable designed parameters. The simulation results show the feasibility for high frequency uncertainty.

The control only depends on the characteristics of inertial matrix, which needs to be positive definite. Many other control designs, which are Lyapunov-based, are limited to revolute joint case due to the uniformly ultimate boundedness of inertial matrix. From the analysis of this work, we can extend the control design procedure into both revolute and prismatic cases. This in turn helps these control designs to extend their applications.

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