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论文题目：Study on Control Design for Uncertain Flexible Joint Manipulator

学科专业：机械制造及其自动化

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导师姓名：韩江　　教授

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Study on Control Design for Uncertain Flexible Joint Manipulator

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申请学位：工学博士　　　　　　　培养单位：机械工程学院

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Study on Control Design for Uncertain Flexible Joint Manipulator

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Abstract

Flexible joint manipulator is an under-actuated, highly nonlinear, mismatched system and is always subject to unknown model uncertainty and external disturbance. In this dissertation, we consider the control problems of uncertain flexible joint manipulator system (unconstrained and constrained) via Lyapunov stability theory.

We first propose a robust control for uncertain flexible joint manipulator whose uncertainty bound is known a priori. In order to broaden the existing control schemes to more applications, a generic upper bound condition of inertia matrix is investigated so that the presented control can be used in prismatic joint manipulator. By using backstepping-like technique, a virtual control is implanted and the real control can then be formulated via state equivalent transformation. The control, which is only based on the possible upper bound of uncertainty, renders the system uniform boundedness and uniform ultimate boundedness.

Since the information of uncertainty is usually poorly known or totally unknown, we then propose an adaptive robust control for flexible joint manipulator whose uncertainty bound is unknown. This unknown bound is assumed to be in a known manner on an unknown parameter. That is, the bound may be governed by a function with known structure, but some arguments of the function are unknown. An adaptive law is then proposed to obtain the estimation of these unknown arguments such that the actual value of uncertainty bound is obtained accordingly. No knowledge of uncertainty bound is required other than its existence. The adaptive law here is of leakage type which means the adaptive parameter does not always grow with time, since once the system performance is satisfactory or the gradients of the boundary function are small. Furthermore, not only the stability of transformed system, containing adaptive laws, is proved, but also the stability of original system is proved theoretically. This part of work is neglected in many other studies.

Then, a novel fuzzy dynamical system approach to the control design of flexible joint manipulator with fuzzy uncertainty bound is proposed. Fuzzy set theory is employed to describe the information of uncertainty and system state rather than probability theory. The performance of system is assured by the proposed control which is deterministic and is not Takagi-Sugeno nor

Mamdani type nor other IF-THEN heuristic fuzzy control. The optimal control problem is equivalent to a performance index minimization problem in the consideration of the average control cost. The performance index is a combination of initial condition, deterministic performance and average fuzzy control effort. It is proved that the extreme solution to this minimization problem, which is obtained by the first-order necessary condition, is indeed the minimum solution, which is verified by the second-order sufficient condition, to the optimal problem.

For the constraint force servo control of uncertain flexible joint manipulator system, we firstly introduce a new modeling approach based on Udwadia-Kalaba theory. The fundamental equation of motion can be established via Newtonian or Lagrangian mechanics. The Moore-Penrose generalized inverse is employed to reveal the relation between impressed force and constraint force. Then the constraint force, by using Udwadia-Kalaba equation, is obtained in the closed-form so that it can be applied in the servo control. This method is feasible of dealing with both holonomic and nonholonomic constraints since they are utilized in second-order form (we also show that stabilization problem, trajectory following problem, optimization problem, etc., can be cast into this form). Under these, an adaptive robust control is proposed to generate appropriate constraint force such that the controlled system (maybe only a part of the system) approximately follows the given constraints. The uncertainty is regarded as fuzzy and there is no restriction on the initial condition. Meanwhile, the tracking error (and maybe also the rest of system) is guaranteed to be uniformly bounded and uniformly ultimately bounded.

Experimental results of a two-link flexible joint manipulator, in Chapter 2 and Chapter 3, are used to verify the theoretical contributions of this work. In Chapter 4 and Chapter 6, the contributions are verified by numerical simulations.

Keywords: Flexible joint manipulator; uncertainty; servo constraint; adaptive robust control; optimal control; fuzzy theory.

摘　要

柔性关节机器人是一个欠驱动的、高度非线性的不匹配系统，且总是受到未知的模型不确定性和外部扰动的影响。本文采用李雅普诺夫稳定性理论来研究（受约束的和不受约束的）不确定柔性关节机器人系统的控制问题。

对于不确定性边界已知的柔性关节机器人，本文首先提出了一个鲁棒控制器。为了推广现有控制策略的应用场合，给出了一个更具有一般性的惯性矩阵上界条件，从而使原本只适用于柔性旋转关节机器人的控制器可以应用到平移关节机器人中。通过采用反步法植入一个虚拟控制，使得可以通过状态变量的等效转换来得到真实的控制。所提出的控制策略仅仅是建立在不确定性可能的上界条件基础上，就能够保证系统的一致有界性和一致最终有界性。

由于关于不确定性的信息通常很难能够获得甚至完全不能获得，我们提出了对于不确定性边界未知的柔性关节机器人的自适应鲁棒控制器。假设这个未知的边界具有一个已知的形式，但是又含有未知的变量。即边界是由一个已知结构的边界函数决定，但边界函数中的某些参数是未知的。因而，可以设计一个自适应律来获得这个未知参数的估计值，从而可以获得相应的边界函数估计值。这种方法中，除了假定不确定性的边界的存在性以外，不需要其他任何关于不确定性的信息。给出的自适应律是渐亏类型，这意味着当系统性能满足要求或者边界函数的梯度值很小的时候，自适应参数便不再随着时间一直增长。此外，我们不仅证明了转换为含有自适应律的系统的稳定性，而且对原始系统的稳定性问题进行了理论上的证明，这一点在其他研究中一直是被忽视的。

对于不确定性的边界是模糊描述的柔性关节机器人，提出了一种新的模糊动态系统控制设计方法。系统中的不确定性和状态变量由模糊集来描述，而不是随机系统理论。所提出的控制可以保证系统的确定性性能。这种控制不同于Takagi-Sugeno类型或者Mamdani类型或者其他基于IF-THEN规则的启发式模糊控制。系统的最优控制问题可以等效为一个考虑平均控制消耗的性能指标最小化问题。这个性能指标综合了初始状态、确定性的系统表现和平均的模糊控制消耗。分析表明，通过一阶必要条件得到的这个最小化问题的极值解就是最优化问题的最小解，并可以通过二阶充分条件来加以验证。

对于不确定柔性关节机器人的约束力伺服控制问题，首先引入了基于Udwadia-

Kalaba理论的全新建模方法。系统不受约束运动的基本方程可以由牛顿或者拉格朗日动力学来建立。考虑约束条件时，运用Moore-Penrose广义逆矩阵揭示主动力和约束力之间的内在联系。然后便可以通过Udwadia-Kalaba方程获得约束力的解析解从而可以用在控制中。由于这个方法使用的是约束条件的二阶微分形式，因而同时适用于完整的和非完整约束。同时可以证明，稳定性问题、路径跟踪问题和最优化问题等均可以转化成二阶形式的约束。根据这些，提出了一个提供约束力的自适应鲁棒控制器从而可以使被控系统（或系统的一部分）跟随给定的约束。系统中的不确定性也是由模糊集描述，对于初始条件也没有限制。同时，可以保证系统的跟踪误差（或系统的其余部分）是一致有界和一致最终有界的。

对于本文的理论成果，第二、三章中在两连杆柔性关节机器人实验平台上进行了实验验证，第四、六章中进行了仿真实验验证。

关键词： 柔性关节机器人，不确定性，伺服约束，自适应鲁棒控制，最优控制，模糊理论。

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Chapter 1 Preface

1.1 Foreword

A manipulator is a multi-body system which has serial links connected in an open kinematic chain. Since Joseph F. Engelberger invented the first manipulator in 1959, the use and manufacturing technology of manipulator obtained a rapid development period. As an automation equipment, manipulator provides more flexibility and adaptability in the hostile environment and unfavorable conditions, thereby earning more advantages in industrial manufacturing areas. Figures 1.1 and 1.2 show the manipulators used in space station.

In general, most of transmissions of manipulators are harmonic gear reducers and RV gear reducers. These reducers have large transmission ratio, high payload capacity, small size and high efficiency. On the contrary, the flexible components are inevitably existed in the joint which reduce the precision of the movement and raise the difficulty of control. The system should be considered as flexible joint manipulator (FJM). In addition, accompanying the influences of inertia force and centrifugal force, the change of the meshing point of the gear leads to the nonlinear variation of the stiffness and damping of the articular surface. This cannot be exactly modeled and usually can be treated as the uncertainty of the system.

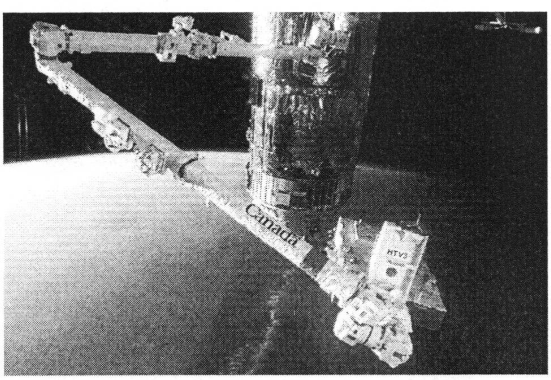

Figure 1.1: The Canadarm 1 manipulator.

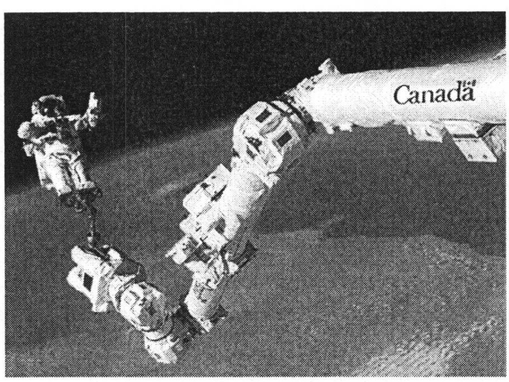

Figure 1.2: The Canadarm 2 manipulator.

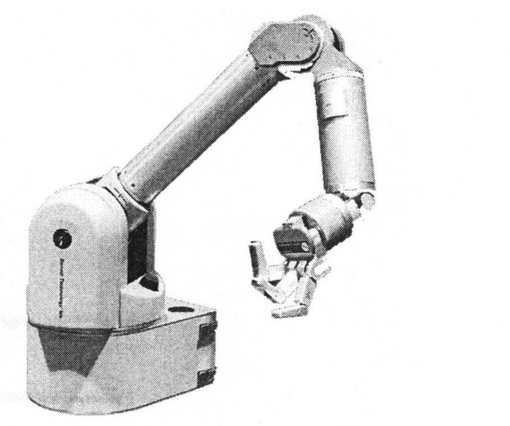

Figure 1.3: The Barett WAM arm.

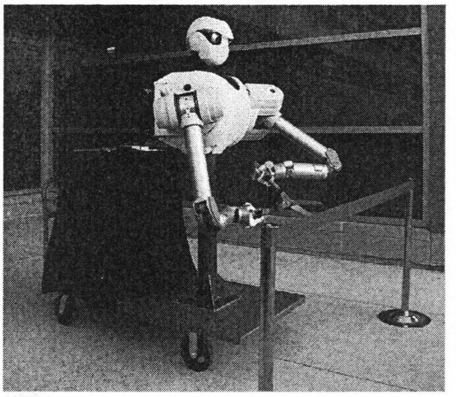

Figure 1.4: The upper-torso humanoid robot with WAM arm.

Engineers also developed new classes of manipulators whose joints were designed to be flexible in order to obtain faster motion, lighter structure, more flexibility and less energy consumption, so that they can satisfy the requirements of the aerospace and outer space industries [1–4], chemical material handling and semiconductor manufacturing [5], nuclear plants applications [6], medical care and operation [7], etc. Figure 1.3 shows a Barrett Whole Arm Manipulator (WAM) whose transmission is steel rope-pulley. Figure 1.4 is a upper-torso humanoid robot which contains two Barrett Whole Arm Manipulators. Each WAM has seven degrees of freedom (DOFs). In addition to that, each arm has a three-finger Barrett Hand which also has seven DOFs.

The control of FJM attracts many efforts because the performance is always limited. For one thing, the controlled object is nonlinear coupling system and always subject to unknown model uncertainty and external disturbance. This leads to undesirable vibrations at the end-tip during movement, and it becomes severe when moving at high speed. For another thing, since the flexibility exists in the joint (may be revolute or prismatic), the control can not reach certain modes directly. For a rigid joint manipulator (RJM) with n joints, its behavior can be described by an n dimensional vector. However, for a FJM, it needs a $2n$ dimensional vector (see Figure 1.5). In other words, the dimension of system is extended and the system turns to be under-actuated.

1.2 Research overview

Beginning in early 1980's, engineers gradually realized that the joint flexibility was an important factor which limited robot performance. To achieve high performance by applying advanced control techniques, an increased understanding of the dynamics of FJM is required. The work on

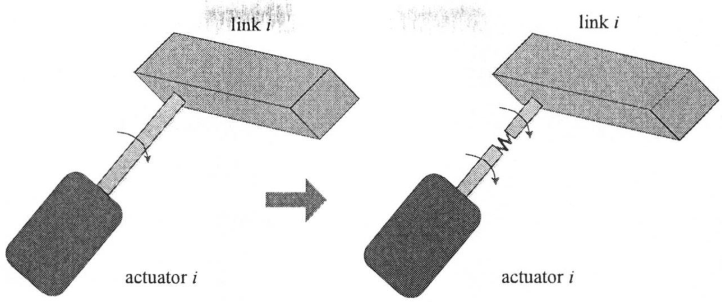

Figure 1.5: The difference between rigid joint and flexible joint.

dynamic models of FJMs using general Lagrangian mechanics was initiated by Nicosia et al. [8]. By regarding the joint flexibility as a linear torsional spring (see Figure 2.2), Spong proposed a simplified model of rigid link flexible-joint (RLFJ) robot under Lagrangian mechanics [9] and applied the globally feedback linearization technique with adaptive control. This RLFJ model is one of the most widely used models for the control design of FJM controllers by the robotic control research community. After that, Feliu et al. analyzed manipulators considering friction in the joints [10]. In 1990, based on Spong's model, a single link FJM model was proposed by Khorasani et al. with the consideration of effects of damping of harmonic drive [11]. By taking the friction force, nonlinear flexibility and kinematic error into account, Bridge et al. established a more precise model of FJM sysem [12]. In 2004, Macnab et al. discussed the modeling and control problem of highly flexible robots under unknown payload [13].

For the flexible joint type manipulators, the joints can be revolute joints or prismatic joints [14, 15], but most of studies focus on revolute joints types [16–24]. The control issue with flexible joints can be nonlinearities due to frictional torques [24] and damping at the joints or uncertainties introduced by the joint flexibilities [25].

Benosman and Vey point out that the control issues of a FJM can be divided into four main objectives which are, of increasing difficulty [26] : (a) end-effector position regulation problems; (b) end-effector motion from point to point in fixed time; (c) tracking control in the joint space (tracking of a prescribed trajectory of joint angular); (d) tracking control in the operational space (tracking of a prescribed trajectory of end-effector position). Since a FJM is a non-minimum phase system, the last problem turns to be the most difficult one of all.

In order to address the aforementioned problems in control of FJMs, many sophisticated control strategies have been explored during the latest decades. The control strategies adopted for

FJMs can be classified under two main categories: feed-forward (open loop) approach [27–30] and feedback (closed loop) approach [31]. In order to suppress the vibration of the end effector, the physical and vibrational properties of the FJM are carefully considered by the engineers in the feed-forward control design. This control method is easy since it does not need the signals of the position and velocity of FJM. The main drawback for feed-forward control is that the input from actuators does not consider the changes in the system once the control structure is conformed [16]. In contrast, feedback control always utilizes measurement equipment to obtain the system states and it can be fed to regulate the control input in such a way as to minimize the vibration. If some of the states are immensurable, the observer can be designed to obtain the state estimations. As the main drawback of this approach, the input delay in the feedback loop is unavoidable.

In [32], the authors improved the singular perturbation method so that it can be used in weak joint flexibility cases. The authors of [33, 34] used linear velocity feedback control to design a controller, and, in order to stabilize nonlinear modes of the system, linear quadratic Gaussian (LQG) [35] and linear quadratic regulator (LQR) [36] are used for the fast terms. In [37], a robust control is designed to counteract the joint stiffness uncertainty, but the uncertainty bound needs to be known a priori. A robust finite-time control is proposed in [38] to stabilize the tracking errors, but the desired joint position is required to be bounded constant vector. To estimate the unknown parameters in system model or control scheme, the adaptive control technique is applied in [39–42] and most of these approaches are single-estimation-based. A multiestimation-based adaptive controller architecture is proposed to improve the tracking performance and the robustness of robotic manipulators [43, 44]. However, the regressor matrix is hard to be found for constructing an adaptive law [45]. To overcome this drawback, reference [46] proposed an impedance controller design approach for FJMs without computing the regressor matrix. But there are still some model-based adaptive control laws which are highly sensitive to unmodeled dynamics [47, 48]. Since many feedback controllers usually need full-state measurement, in [49–51], the disturbance observer and state observer are used to estimate the system information and the use of sensors is reduced.

Other control strategies are also investigated such as positive position feedback (PPF) or negative position feedback (NPF) control [52–54], end-point acceleration feedback control [55, 56], repetitive control [57, 58], input shaping or command preshaping method [59–63], integral resonant control (IRC) [27], passive theorem control [52, 64], PID (Proportional-Integral-Derivative) control [65, 66], boundary control [67], fractional order control [68], sliding-mode control [69, 70], neural networks (NN) or artificial neural networks [50, 71] (ANN) based control, fuzzy logic control [14, 60, 72, 73], predictive controller [74], etc. Some other reports of intelligent control strategies for FJMs, focusing on regulating control or tracking control, can be found in [75, 76].

At a glance, it seems that all kinds of control methods have been considered for FJMs. However, from a practical point of view, a controller is acceptable only when it can meet practical requirements. In the first place, it should be stable. Then, the uncertainties which need adaptive or robust controllers to overcome should be considered. At the end, when the controller is finally implemented, measurements should be feasible.

1.3 Description manners to system uncertainties

In practical situations, the uncertainty always exists in the FJM system due to unknown modeling parameters, irregular disturbances, varying payload on tips, nonlinear friction in joints and so on. All these factors prevent us from using precise information of the system so that the aforementioned control schemes can not be applied. Generally, without considering the classification of control schemes, uncertainties (may be stochastic or deterministic, matched or mismatched) in FJM system are treated in two manners: one probability, one fuzzy.

The fundamental work on system containing uncertainty under probability theory was initiated by Kalman [77, 78]. Kalman looked into the estimation and control problem of system under stochastic noise in state space. Since the Kalman filter was first presented to the public in 1959, there are more than 20,0000 papers, technical reports, books, summer courses, etc., concerned with applications of Kalman filter theory. Although these literatures showed the probability theory was quite self-evident and an exciting area of control design, concerns on its validity in describing the physical world never stopped. Or, in other words, researchers doubted that the bridge between the stochastic theory (used as a mathematical tool) and the true physical world might not be as tight as it seems. Despite his early contributions to the probability theory, in 1994, Kalman claimed randomness was a fact of the real world while probability was only an intellectual construct. The probability, which was not quantifiable, measurable, concrete, did not exist. Meanwhile, Kalman filter was not a triumph of applied probability. It was only a slight inheritance from probability theory and was used as a pillar of system theory [79].

Fuzzy theory was initially adopted by L. A. Zadeh in 1965 with his seminal paper "Fuzzy sets" [80]. Originally, fuzzy theory was used to describe the information which only has a rough boundary with others. For example, the information of human's height can be described with "high", but there is no boundary of what can be regarded as "high". As a drawback of probability manner, uncertainty in practical case is usually obtained via observed data while the data are always limited by nature. This is because the uncertainty in system is always unrepeatable. On the contrary, fuzzy manner describes the uncertainty via the degree of occurrence in certain applications. The designers enjoyed its advantage that it is model free and devoted to turn control parameters based on linguistic reasoning. It has been proved to be a rather useful tool for many problems. For some applications, fuzzy theory can solve them equally well or better than probability theory. More studies on the relative advantages between fuzzy approach versus probability approach can be found in Kosko [81], Bezdek [82] and their bibliographies.

Fuzzy theory can be roughly classified into five major branches: (i) fuzzy mathematics, where classical mathematical concepts are extended by replacing classical sets with fuzzy sets; (ii) fuzzy logic and artificial intelligence, where approximations to classical logic are introduced and expert systems are developed based on fuzzy information and approximate reasoning; (iii) fuzzy systems, which include fuzzy control and fuzzy approaches in signal processing and communications; (iv) uncertainty and information, where different kinds of uncertainties are analyzed and (v) fuzzy decision making, which considers optimization problems with soft constraints. Surely, these five branches are not independent and there are strong interconnections among them. For example, fuzzy control uses concepts from fuzzy mathematics and fuzzy logic. So far, most interests in fuzzy theory were focused on fuzzy reasoning, estimation, decision-making, etc., and significant success had been made. However, the incorporation process between fuzzy theory and system theory (so-called fuzzy dynamical system) was not sufficiently investigated. Some of the basic work about the modeling of fuzzy dynamical system and controls can be found in Hanss [83] and Bede [84]. This so-called fuzzy dynamical system approach is different from the well known Takagi-Sugeno (T-S) fuzzy model or other IF-THEN rules-heuristic based strategies. From a quite different angle, with the fuzzy description of the system state and uncertainty, fuzzy dynamical system approach endeavors to explore applications of the original intention of fuzzy theory.

1.4 Control approach for constrained flexible joint manipulator

The FJM is sometimes designed to follow certain constraints. In order to achieve this performance, the ideal case is to design a controller which can provide the desired constraint force. However, research was blocked in obtaining this constraint force. As we know, the constraint has different forms and may be nonholonomic which makes the constraint equations unintegrable. Furthermore, the flexible joint manipulator system has fewer control inputs than its degrees of freedom, in other words, it is under-actuated. Therefore, it is hard to have precise description of constraint force. Conventional method to find the dynamics of constrained system is in Lagrangian mechanics, including Maggi equation [85, 86], the Boltzmann and Hamel equation [87, 88], the

Gibbs and Appell equation [89–91], etc. However, the efforts of all these achievements mainly focused on the passive constraint (also known as physical constraint) problem in which the constraint is met in a passive manner. This in turn means the environment (including the structure requirements and physical limitations) can generate the required constraint force to restrain the motion [92]. For example, the planar robot can only move in a surface under the constraint force from the structure. The designer is only responsible for providing force to follow the given trajectory. Based on this precondition, many outstanding investigations have been made to assure that the constraints are obeyed (see [41, 42, 93]).

The situation has been turned recently. Under the frame work of Udwadia and Kalaba [94, 95], the required constraint force can be obtained by adopting the Moore-Penrose generalized inverse to explore the geometric structure of the constraint [96]. The significant progress is that the constraint force is presented in the closed-form so that it can be applied in control design easily. That is to say, the engineer can imitate what the Nature does so that the constraints are followed [92]. This is so-called constraint force servo control. Furthermore, several classes of control problems can be formulated as constraint force servo control [97].

1.5 Research origination and motivation

Control of uncertain FJM systems has always been a vibrant area in control theory and engineering application. In order to obtain the usable description of uncertainty, two main approaches, stochastic and deterministic, are introduced. Historically, the incorporation of probability theory and system theory, which can be traced back to the 50s of 20th century, has made significant progress. The uncertainty of FJM system can be regarded from a stochastic point of view. The probabilistic method which is one of the most widely used endeavors in this regard. In the very beginning, Kalman initiated this work to analyze a system under stochastic noise [77, 78]. Although it is proved to be successful in thousands of applications, the criticisms of its validity in describing the real physical world never stop. In 1994, Kalman claimed that probability theory was only an intellectual construct and might not be all that suitable for describing the randomness in the real world. In a sense, the link between a rather sophisticated mathematical tool and the physical world might be loose [98]. It should be noted, however, that Kalman's comments on probability does not automatically assume him an advocate for fuzzy theory. His skeptical view on the latter [99] has been unchanged.

Another way of dealing with uncertainty is the deterministic approach. The salient feature, in comparing with the stochastic approach, is that it considers the bound of uncertainty instead of studying the uncertainty itself. In other words, the uncertainty is treated in a deterministic way. The essential knowledge about the uncertain elements is their possible size; that is, only the sets in which the values of the uncertain quantities can range are presumed to be known[100]. The objective is to guarantee, for every possible uncertainty, that the state will eventually end up and remain within some pre-specified region [74, 101–110]. The deterministic approach for control system design in the recent past has been mainly concerned with deterministic uncertainty bound type (see e.g.[105, 106, 111]). In other words, the control is designed assuming that the uncertainty bound is known a priori, or, even if the exact information of bound is not given, it is believed that it does exist with a fixed value [107–109]. This has the disadvantage that if the bound information can not be clearly identified, the designed control will result in impracticality.

Besides, there is another method adapted to describe uncertainty and is quite different from the aforementioned two, i.e., fuzzy approach. Probabilistic approach focuses on the frequency (how often) of an event happens, while fuzzy theory studies the degree of an element belongs to a set (i.e., membership function). An element may belong to a set to a certain degree, ranging from 0 to 1. The fuzzy approach was proposed by Zadeh [80] in 1965. From another point of view, he looked into the degree of occurrence. The merging of fuzzy theory, system theory, control theory, and thereby to the fuzzy dynamical system theory, on the other hand, has been less straightforward. One reason is that perhaps most interest in fuzzy theory has been attracted to the ever successful decision-making-related fields [112–116]. Since the origination of fuzzy theory, the fuzzy set theory has yet been actively applied to describing uncertainty. Among the limited amount of work in this attempt, Chen proposed a control system analysis and a design framework based on fuzzy uncertainty [117–119]. The framework originated by Chen obtains the property that the only available information of the uncertainty is prescribed via fuzzy set theory.

For the FJM system, which is high nonlinear, under-actuated, the precise model of the system is often considered not feasible to obtain. There is generally some degree of uncertainty. Adaptive control and robust control of FJM system have been important research areas for system theorists in recent past. These uncertainties, which are in general time-varying, may include unknown parameters and input disturbances via the system model and its environment. The limitations of designing control for uncertain FJM system with typical methods do exist. Therefore, constructing a framework of control design for the uncertain FJM system based on the new method is meaningful and theoretically valuable. Among others, such kind of fuzzy applications have never been used before. In this work, the attempt to pursue a possible use of fuzzy description of uncertainty in adaptive control and robust control design has achieved. This may be viewed as an alternative proposal to combine the fuzzy theory and control theory. Further exploration on fuzzy method in describing uncertainty will be proposed in this paper.

1.6 Significance of this paper

The upper bound property of the inertia matrix of uncertain FJM system is analyzed. It is shown that over-simplified model sometimes may lead to positive semi-definite inertia matrix. Furthermore, the uniform upper boundedness of inertia matrix does not always hold for arbitrary manipulators. In order to deal with mismatched uncertainty of the system, a virtual control and a state transformation are proposed to make the system satisfy the matching condition. A robust controller is presented to counteract the influence of uncertainty (whose bound is assumed to be known a priori) in FJM systems by assuming the inertia matrix is uniform positive definiteness. The control is only based on the possible bound of uncertainty and renders uniform boundedness and uniform ultimate boundedness. Furthermore, the radius of uniform stability ball can be made arbitrarily small by a suitable choice of control parameters.

For the uncertain FJM system whose uncertainty bound is unknown, we know that there does exist the bound and this bound may in form of an known function with respect to an unknown parameter. Then, an adaptive law is constructed to estimate this unknown parameter so that the uncertainty bound can be approximately approached. By employing fuzzy dynamical system approach, an adaptive robust control, which is deterministic and is not the well-known IF-THEN rules-based fuzzy control, is proposed. The stability of the controlled mechanical/adaptive system is analyzed via the Lyapunov minimax approach. Besides, under the fuzzy description of uncertainty and system states, the optimal control problem is addressed in the consideration of average fuzzy performance.

Since the FJM system is under-actuated and the uncertainty is mismatched, which in turn means the control input can not reach certain modes directly, we creatively implant a fictitious control to transform the system into a cascade of two subsystems. Then both of the subsystems meet the matching condition [120]. This is a backstepping-like control design method. For a long time of using backstepping method in control design area, the original system is necessarily transformed into another system described by new state variables. The other authors only can prove the stability of the transformed system. Then, they infer the original system is also stable based on the equivalent transformation. In this work, we prove the stability of the original system theoretically.

A fuzzy performance index, which is in accordance with the fuzzy uncertainty bound, is proposed. The optimal design for the control parameter is formulated as a tractable semi-infinite constrained optimization problem. The solution to this problem can be obtained by solving two scalar quartic algebraic equations. It is proven that the extreme solution to the optimization problem, which is solved by a first order necessary condition, is indeed the minimum solution, which is verified by a second-order sufficient condition.

We then focus on the constraint force servo control issue of constrained FJM system. The explicit closed-form equation of motion of constrained FJM systems is formulated by using Udwadia and Kalaba's new theory. Udwadia-Kalaba equations are uncoupled and Lagrangian multipliers are not required. They lead to a new and fundamental understanding of constrained motion and point towards new directions in modeling of FJM system. No transformation or elimination of coordinates is undertaken when constraints are present. The coordinates in which the constrained FJM system is described are the same as those used to describe the unconstrained FJM system, which is responsible for the simplicity of the explicit equation and the fundamental insights about the nature of constrained motion. By applying the proposed control, the system is guaranteed to obey the given constraints approximately and the rest of the system is uniformly ultimately bounded.

Chapter 2 Robust control of uncertain flexible joint manipulator

2.1 Introduction

For a rigid joint manipulator, see Figure 2.1, the position and velocity of link are linearly proportional to those of actuator. But for a flexible joint manipulator (FJM), flexibility always exists in the joint so that the relation of link and actuator is no longer linear. Spong systematically investigated the modeling and control problem of FJM system. By regarding the joint flexibility as a linear torsional spring (see Figure 2.2), Spong proposed a simplified model of FJM under Lagrangian mechanics [9] and applied the feedback linearization technique with adaptive control. The stability of the closed-loop is proved. In 1990, based on Spong's model, a single link FJM model was proposed by Khorasani et al. with the consideration of effects of damping of harmonic drive [11]. By taking the friction force, nonlinear flexibility and kinematic error into account, Bridge et al. established a more precise model of FJM sysem [12]. In 2004, Macnab et al. discussed the modeling and control problem of highly flexible robots under unknown payload [13].

Robust control, which is used as a basic control scheme for mechanical system, has been explored for manipulator control during past decades and has achieved prominent success [121–123]. The structure of robust controller is fixed so that it is simpler to implement. Besides, it is capable of compensating for both structured and unstructured uncertainties. However, as it is shown in [124], one may construct a legitimate Lyapunov function by assuming the inertia matrix is positive definite and uniformly ultimately bounded. Actually, there are cases that the inertia matrix of the model may be positive semi-definite or not uniformly ultimately bounded.

2.2 Mechanical system with uncertainty

In practical situation, there always exists modeling uncertainty and extern noise which prevent one from using the precise knowledge of the system. We consider a class of mechanical system with uncertainty described as following

$$\dot{x}(t) = f(x(t), \sigma(t), t) + B(x(t), \sigma(t), t)\tau(t), \tag{2.1}$$

where $t \in \mathbf{R}$ is the time (note that, t is independent), $x \in \mathbf{R}^n$ is the state vector of system , $\tau(t) \in \mathbf{R}^r$ is the input control torque, $\sigma(t) \in \mathbf{R}^o$ is the time-varying uncertainty, $f(x(t), \sigma(t), t)$ is the vector of system and $B(x(t), \sigma(t), t)$ represents the input matrix. n, r, o are positive integers representing the dimensions of corresponding vectors. In this thesis, the norms are Euclidean, otherwise it will be stated.

Definition 2.1. A feedback control $\tau(t) = p(x(t), t)$ renders the uncertain dynamical system (2.1) practically stable if and only if there exists constant $\underline{d} > 0$ such that for any initial time $t_0 \in \mathbf{R}$ and any initial state $x(t_0) \in \mathbf{R}^n$, the following properties are hold.

1. **Existence and continuation**: Given any $(x_0, t_0) \in \mathbf{R}^n \times \mathbf{R}$, the closed-loop system

$$\dot{x}(t) = f(x(t), \sigma(t), t) + B(x(t), \sigma(t), t)p(x(t), t) \tag{2.2}$$

processes a solution $x(\cdot) : [t_0, t_1) \to \mathbf{R}^n$, $x(t_0) = x_0, t_1 > t_0$. Furthermore, every solution $x(\cdot) : [t_0, t_1) \to \mathbf{R}^n$ can be continued over $[t_0, \infty)$.

2. **Uniform boundedness**: Given any constant $r > 0$ and any solution $x(\cdot) : [t_0, t_1) \to \mathbf{R}^n, x(t_0) = x_0$ with $\|x_0\| \leq r$, there exists a $d(r) > 0$ such that $\|x(t)\| \leq d(r)$ for all $t \in [t_0, \infty)$.

3. **Uniform ultimate boundedness**: Given any $\bar{d} > \underline{d}$ and any $r \geq 0$, there exists a finite time $T(\bar{d}, r)$ such that if the initial status $\|x_0\| \leq r$, then $\|x(t)\| \leq \bar{d}$ for all $t \geq t_0 + T(\bar{d}, r)$.

4. **Uniform stability**: Given any $\bar{d} > \underline{d}$, there exists a $\delta(\bar{d}) > 0$ such that $\|x_0\| \leq \delta(\bar{d})$ implies $\|x(t)\| \leq \bar{d}$ for all $t > t_0$.

Remark 2.1. Uniform ultimate boundedness means that the trajectory of the system enters into an extremely small region near the equilibrium position after some finite time and it remains close thereafter. In the later system performance analysis, it can been seen that the sizes of uniform boundedness ball and uniform stability ball are determined by control parameters. Better performance always requires higher control effort. The optimization problem associated with the control could be solved via a novel fuzzy dynamical system approach.

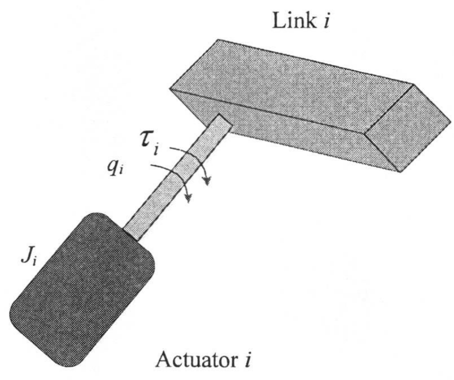

Figure 2.1: Rigid joint of mechanical manipulator.

2.3 Dynamical model of flexible joint manipulator with uncertainty

Establishing a proper mathematical model of system is a crucial stage for the development of control strategy. For this reason, we first explore the problem of dynamical model of the flexible joint manipulator (FJM). If there is no uncertainty, assuming that

Assumption 2.1. The motion of the rotor which is fixed on the link is a pure rotation, the kinetic energy of the rotor is mainly brought up due to this motion.

Assumption 2.2. The rotor/gear inertia is symmetric about the rotor axis of rotation so that the gravitational potential of the system and also the velocity of the rotor center of mass are both independent of the rotor position.

These two assumptions hardly need any justification. In fact, most of existing manipulator models are derived under these same assumptions [9].

Consider an n serial link mechanical manipulator. The links of manipulator are assumed to be rigid, then the position of each joint or link can be described by n generalized coordinates representing by q. For the i-th joint, as it is shown in Figure 2.1, if joint is assumed to be rigid, the position of the actuator i is the same as the position of link i and can be denoted as q_i. However, as shown in Figure 2.2, there exists flexibility (named K_i) in the joint which means the positions of the actuator i and link i are different. They should be described by $q_{1,i}$ and $q_{2,i}$, respectively.

Let us define generalized coordinate $q = [q_1, q_2]$, where $q_1 = [q_{1,1}, q_{1,2}, \cdots, q_{1,n}]$ representing

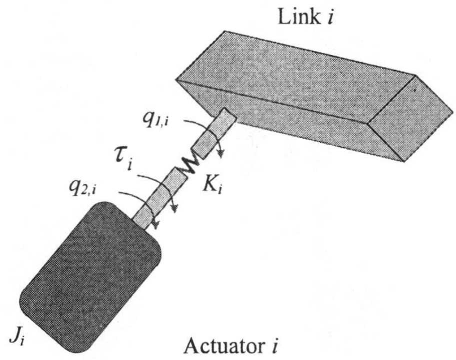

Figure 2.2: Flexible joint of mechanical manipulator.

the link position, $q_2 = [q_{2,1}, q_{2,2}, \cdots, q_{2,n}]$ representing the joint position. Under our Assumption 2.1 and Assumption 2.2, the kinetic energy of the system is

$$K_E = \frac{1}{2}\dot{q}_1^T M(q_1)\dot{q}_1 + \frac{1}{2}\dot{q}_2^T J\dot{q}_2, \tag{2.3}$$

where $M(q_1)$ is the inertia matrix of rigid joint robot and J is the inertia matrix of actuators. Since the inertia of the rotor is symmetric with respect to its axis of rotation, the system's entire potential energy, including the gravitational potential P_1 and the elastic potential P_2, is given by

$$P_E = P_1(q_1) + P_2(q_1 - q_2). \tag{2.4}$$

The gravitational potential P_1 is obtained in standard form of rigid joint manipulator. The elastic potential P_2 is

$$P_2 = \frac{1}{2}(q_1 - q_2)^T K(q_1 - q_2), \tag{2.5}$$

where K is a diagonal matrix representing the joint stiffness with its elements $K_i > 0$. Then we have the Lagrangian $L = K_E - P_E$ as follows

$$L = \frac{1}{2}\dot{q}_1^T M(q_1)\dot{q}_1 + \frac{1}{2}\dot{q}_2^T J\dot{q}_2 - P_1(q_1) - \frac{1}{2}(q_1 - q_2)^T K(q_1 - q_2). \tag{2.6}$$

By applying Euler-Lagrange equation, we have the equations of motion as

$$\begin{cases} M(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + G(q_1) + K(q_1 - q_2) = 0 \\ J\ddot{q}_2 - K(q_1 - q_2) = \tau. \end{cases} \tag{2.7}$$

Here, τ is the input torque, function $C(q_1, \dot{q}_1)\dot{q}_1$ contains coriolis and centripetal forces, $G(q_1)$ represents the gravitational force, and they can be expressed as

$$\begin{aligned} C(q_1, \dot{q}_1) =& \dot{M} - \frac{1}{2}\dot{q}_1^T \frac{\partial M}{\partial q_1}, \\ G(q_1) =& -\frac{\partial P_1}{\partial q_1}. \end{aligned} \tag{2.8}$$

Remark 2.2. The model shown in (2.8) is similar to the model shown in [125]. It also can be treated as an extension to the widely used rigid body models. To illustrate this point, we assume that there is no elastic deformation between the joints which means the elastic coefficient $K \to \infty$. Then we have $q_1 = q_2$ and $\dot{q}_1 = \dot{q}_2$, the potential energy P_2 in (2.5) satisfies

$$\frac{1}{2}(q_1 - q_2)^T K(q_1 - q_2) \to 0. \tag{2.9}$$

This leads to the well known equation of motion of rigid body manipulator shown as

$$(M(q_1) + J)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + G(q_1) = \tau. \tag{2.10}$$

Since there always exists uncertainty in modeling and external noise which prevents us from using precise knowledge of M, C, G, K, we introduce (maybe fast) time varying uncertainties $\sigma_1 \in \Sigma_1 \subset \mathbf{R}^{n_1}$ and $\sigma_2 \in \Sigma_2 \subset \mathbf{R}^{n_2}$ with Σ_1 and Σ_2 prescribed and compact, into the system, this yields

$$\begin{aligned} \begin{bmatrix} M(q_1, \sigma_1) & 0 \\ 0 & J(\sigma_2) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} C(q_1, \dot{q}_1, \sigma_1)\dot{q}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} G(q_1, \sigma_1) \\ 0 \end{bmatrix} \\ + \begin{bmatrix} K(\sigma_1)(q_1 - q_2) \\ -K(\sigma_2)(q_1 - q_2) \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix}. \end{aligned} \tag{2.11}$$

We note that certain entries of the uncertain parameter vectors σ_1 and σ_2 may represent the same physical parameters because these two subsystems are connected.

Remark 2.3. As shown by Sweet and Good [126], for electrical motors actuated robot manipulators with harmonic drive transmissions, joint stiffness has relatively large effects in comparison to other parameters in the system. Meanwhile, the damping of joint is small. Moreover, in heavy payloads applications, effects caused by joint flexibility become much more important and may lead to significant dynamic interactions. Thus, it is reasonable for us to omit the damping items in this system model. Besides, even if the damping and nonlinear friction are taken into account, only an additional term would be added into the bounding fucntion and the control design procedure would not change.

2.4 Fundamental properties of inertia matrix

Consider the n-link serial manipulator model in (2.7), it is often believed that the inertia matrix $M(q_1)$ is positive definite and uniformly upper bounded, i.e., there exist constants $\underline{\sigma} > 0$ and $\bar{\sigma} >$ such that

$$\underline{\sigma} \leq \|M(q_1)\| \leq \bar{\sigma}. \tag{2.12}$$

Researchers may use this "fact" to construct a legitimate Lyapunov function. However, this property is incorrect in some special cases. Let us review an example in McKerrow [127], the inertia matrix is given by

$$M(q) = \begin{bmatrix} ml_2^2 \cos^2\theta_2 & 0 \\ 0 & ml_2^2 \end{bmatrix}, \tag{2.13}$$

$q = [\theta_1, \theta_2]^T$ is the vector of coordinates. Obviously, the matrix is singular when $\theta_2 = (2n + 1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2, \cdots$. Under this condition, the kinetic energy $\frac{1}{2}\dot{q}^T M(q)\dot{q} = 0$ for all θ_1. This leads to a paradox: two particle masses are moving without bringing up kinetic energy. Although this problem is due to the an over-simplified model and can be avoided by employing more "realistic" model whose dimensions are non-negligible, this issue should be addressed. The particle model is widely employed in the control design of mechanical manipulator.

The inertia matrix is not uniformly upper bounded either. Consider a 2-DOF RP robot shown in Craig [128] with

$$M(q) = \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 d_2^2) & 0 \\ 0 & m_2 \end{bmatrix}, \tag{2.14}$$

$q = [\theta_1, d_2]^T$. Obviously, the inertia matrix $M(q)$ is not bounded due to the d_2 term. Chen [129] studied this issue and obtained a more generic conclusion, i.e., there exist constant scalars $\bar{\lambda}_0 > 0, \bar{\lambda}_1 \geq 0, \bar{\lambda}_2 \geq 0$ such that

$$\|M(q)\| \leq \bar{\lambda}_0 + \bar{\lambda}_1\|q\| + \bar{\lambda}_2\|q\|^2. \tag{2.15}$$

This conclusion holds for any serial link manipulator even some of the joints are prismatic. We take this as nature property of the manipulator in this study. In the special case that all of the joints are revolute, then this statement is reduced to $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$, and

$$\|M(q)\| \leq \bar{\lambda}_0, \qquad \forall q \in \mathbf{R}^n. \tag{2.16}$$

2.5 The proposed robust controller

We propose two steps to design the robust control. Firstly, let us rewrite the first part of (2.11) as

$$M(q_1,\sigma_1)\ddot{q}_1 + C(q_1,\dot{q}_1,\sigma_1)\dot{q}_1 + G(q_1,\sigma_1) + K(\sigma_1)q_1 - K(\sigma_1)(q_2 - \tilde{\tau}) = K\tilde{\tau}, \tag{2.17}$$

where $\tilde{\tau}$ is a fictitious control implanted into the system, without changing the dynamics of original system. In the later sections, we show that the real control τ can be obtained with the help of $\tilde{\tau}$. With $\tilde{\tau}$ introduced, the system could be divided into two parts, which are: (i) link position subsystem, (ii) joint position subsystem. The first subsystem is controlled by $\tilde{\tau}$, which is virtual. The second subsystem is controlled by τ which is the real control.

Secondly, we transfer the system with new state variables. Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2 - \tilde{\tau}$, $x_4 = \dot{q}_2 - \dot{\tilde{\tau}}$, then

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, X_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{2.18}$$

Thus, the dynamics of FJM can be expressed as follows by using new state variables:

$$\begin{aligned} M(x_1,\sigma_1)\dot{x}_2 &= -C(x_1,x_2,\sigma_1)x_2 - G(x_1,\sigma_1) - K(\sigma_1)x_1 \\ &\qquad + K(\sigma_1)x_3 + K(\sigma_1)\tilde{\tau}, \qquad (2.19) \\ J(\sigma_2)\dot{x}_4 &= -J(\sigma_2)\ddot{\tilde{\tau}} - K(\sigma_2)x_3 + K(\sigma_2)x_1 - K(\sigma_2)\tilde{\tau} + \tau. \qquad (2.20) \end{aligned}$$

Assumption 2.3. The inertia matrix $M(q_1,\sigma_1)$ is positive definite. That is, there exists a constant scalar $\underline{\eta} > 0$ such that

$$\|M(q)\| \geq \underline{\eta}. \tag{2.21}$$

Since the inertia matrix may be positive semi-definite, the positive definite property is stated as an assumption rather than a fact.

Property 2.1. The inertia matrix $M(q_1, \sigma_1)$ is upper bounded by a quadratic polynomial of the generalized coordinate. That is, there exist constant scalars $\eta_1^1 > 0$, $\eta_2^1 \geq 0$, $\eta_3^1 \geq 0$ such that

$$\|M(q_1)\| \leq \eta_1^1 + \eta_2^1\|q_1\| + \eta_3^1\|q_1\|^2. \tag{2.22}$$

This upper bound condition is stated as a fact rather than an assumption since it has been sufficiently investigated in Chen [129].

Assumption 2.4. The uncertain parameter vectors $\sigma_1(t) \in \Sigma_1$ and $\sigma_2(t) \in \Sigma_2$ are Lebesgue measurable, where Σ_1 and Σ_2 are prescribed and compact. Meanwhile, it is possible to estimate the bounds of $\sigma_1(t)$ and $\sigma_2(t)$.

The $K(\sigma_1)$ matrix is decomposable and is consisted of nominal part and uncertain part which are denoted by $\bar{K}$ and $\Delta K(\sigma_1)$, respectively, such that

$$K(\sigma_1) = \bar{K} + \Delta K(\sigma_1). \tag{2.23}$$

Furthermore, the matrix $K(\sigma_1)$ can then be represented as

$$K(\sigma_1) = \bar{K}(I + E(\sigma_1)), \tag{2.24}$$

where $I \in \mathbf{R}^{n\times n}$ is a identity matrix, and $E(\sigma_1) \in \mathbf{R}^{n\times n}$ is a diagonal matrix satisfying $\Delta K(\sigma_1) = \bar{K}E(\sigma_1)$.

Assumption 2.5. For the decomposed $\bar{K}$ and $E(\sigma_1)$, there exists an e_m such that

$$\begin{aligned} e_m &= \min_i\{\min_{\sigma_1\in\Sigma_1}(e_i(\sigma_1))\} > -1, \quad i = 1, 2, \cdots, n, \\ \lambda_E &= \min_{\sigma_1\in\Sigma_1}\{\lambda_{\min}(I + \bar{K}E(\sigma_1)\bar{K}^{-1})\} > 0, \end{aligned} \tag{2.25}$$

where $e_i(\sigma_1)$ is the diagonal component of $E(\sigma_1)$.

Since the overall system is regarded as the cascades of two subsystems, then we can design the control for each subsystem respectively. Firstly, for the virtual control $\tilde{\tau}$ of the link position subsystem (2.11), let us choose a scalar function $\bar{\rho}_1 : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+$ such that for all $\sigma_1(t) \in \Sigma_1$,

$$\bar{\rho}_1(x_1, x_2) \geq \|\Phi_1(x_1, x_2, \sigma_1)\|, \tag{2.26}$$

where

$$\begin{aligned}\Phi_1(x_1,x_2,\sigma_1) =&\frac{1}{2}\dot{M}(x_1,x_2,\sigma_1)(x_2+S_1x_1)-C(x_1,x_2,\sigma_1)x_2-G(x_1,\sigma_1)\\&-K(\sigma_1)x_1+M(x_1,\sigma_1)S_1x_2,\end{aligned} \tag{2.27}$$

$$S_1=diag[S_{1i}]_{n\times n},\ S_{1i}>0,\ i=1,2,\cdots,n. \tag{2.28}$$

For a given positive gain $\alpha_1>0$, the implanted virtual control $\tilde{\tau}$ could be given as

$$\begin{aligned}\tilde{\tau}(t) =&\bar{K}^{-1}\left[-\alpha_1(x_2(t)+S_1x_1(t))\rho_1{}^2(x_1(t),x_2(t))-\beta_1(x_2(t)+S_1x_1(t))\right]\\&+\bar{K}^{-1}(-K_{p1}x_1(t)-K_{d1}x_2(t)),\end{aligned} \tag{2.29}$$

where

$$\rho_1(x_1,x_2)\geq(1+e_m)^{-1}\bar{\rho}_1(x_1,x_2). \tag{2.30}$$

Here, the scalar function $\rho_1:\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}_+$ is 2-times continuously differentiable (C^2). The proper choice of the designed parameter β_1 will be given later. To give the control torque τ with a given $S_2=diag[S_{2i}]_{n\times n}$, $S_{2i}>0,\ i=1,2,\cdots n$, we first choose a scalar function ρ_2 : $\mathbf{R}^{2n}\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}_+$ such that

$$\rho_2(X_1,x_3,x_4)\geq\|\Phi_2(X,\sigma_1,\sigma_2)\|, \tag{2.31}$$

where

$$\begin{aligned}\Phi_2(X,\sigma_1,\sigma_2)=&-J(\sigma_2)\ddot{\tilde{\tau}}(X,\sigma_1,\sigma_2)-K(\sigma_2)x_3+K(\sigma_2)x_1-K(\sigma_2)\tilde{\tau}(X_1)\\&+J(\sigma_2)S_2x_4+\frac{1}{2}\dot{J}(\sigma_2)(x_4+S_2x_3).\end{aligned} \tag{2.32}$$

Then the real input control τ can be constructed as

$$\tau(t)=-\alpha_2(x_4(t)+S_2x_3(t))\rho_2{}^2(X_1(t),x_3(t),x_4(t))-\beta_2(x_4(t)+S_2x_3(t)), \tag{2.33}$$

where α_2 and β_2 are parameters chosen by designer. From now on, the arguments on uncertainty in $D(x_1,\sigma_1),C(x_1,x_2,\sigma_1)$ and $K(\sigma_1)$ etc., are omitted if no confusion arises.

The control strategy could be summarized as Figure 2.3, where q_{1d} and q_{2d} are the desired trajectories. It can be seen that both $\tilde{\tau}$ and τ have robust control parts and PD control parts. This work considers the regulation problem, the reference signal is set to be 0. For any non-zero regulation set point, a simple change of coordinate will suffice.

Theorem 2.1. Subject to Assumptions 2.3–2.5, the control (2.33) renders the systems (2.19) and (2.20) uniform boundedness and uniform ultimate boundedness. Furthermore, the radius of uniform ultimate boundedness ball approaches to 0 (i.e., arbitrarily small) by a suitable choice of α_1 and α_2.

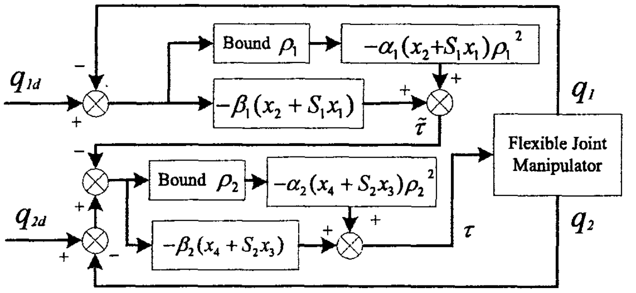

Figure 2.3: The block diagram of the robust control.

2.6 Theoretical proof of the proposed robust controller

Choose the Lyapunov function candidate as

$$V(X) = V_1(X_1) + V_2(X_2), \tag{2.34}$$

where

$$\begin{aligned} V_1(X_1) &= \frac{1}{2}(x_2 + S_1x_1)^T M(x_2 + S_1x_1) + \frac{1}{2}x_1^T(\bar{K}_{p1} + S_1\bar{K}_{d1})x_1, \\ V_2(X_2) &= \frac{1}{2}(x_4 + S_2x_3)^T J(x_4 + S_2x_3) + x_3^T\beta_2 S_2 x_3, \end{aligned} \tag{2.35}$$

and

$$\bar{K}_{p1} = K_{p1} + \bar{K}E\bar{K}^{-1}K_{p1}, \tag{2.36}$$

$$\bar{K}_{d1} = K_{d1} + \bar{K}E\bar{K}^{-1}K_{d1}. \tag{2.37}$$

To analyze the stability, we should prove $V(X)$ is a legitimate Lyapunov function candidate (means $V(X)$ is positive definite and decrescent). Let

$$\Psi_1 = \begin{bmatrix} S_1^2 & S_1 \\ S_1 & I \end{bmatrix}, \bar{S}_1 = \lambda_{\max}(\Psi_1), \tag{2.38}$$

based on Assumption 2.3,

$$
\begin{aligned}
V_1 \geq & \frac{1}{2}\underline{\eta}\|x_2 + S_1 x_1\|^2 + \frac{1}{2}x_1^T(\bar{K}_{p1} + S_1\bar{K}_{d1})x_1 \\
= & \frac{1}{2}\sum_{i=1}^{n}\underline{\eta}\,[x_{1i} \quad x_{2i}]\begin{bmatrix} S_{1i}^2 & S_{1i} \\ S_{1i} & 1 \end{bmatrix}\begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} \\
& + \frac{1}{2}\sum_{i=1}^{n}[x_{1i} \quad x_{2i}]\begin{bmatrix} \bar{K}_{p1i} + S_{1i}\bar{K}_{d11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix} \\
\geq & \frac{1}{2}\min_i\{\lambda_{\min}(\Omega_{1i})\}\|X_1\|^2,
\end{aligned}
\tag{2.39}
$$

where

$$
\Omega_{1i} = \begin{bmatrix} \underline{\eta}S_{1i}^2 + (\bar{K}_{p1i} + S_{1i}\bar{K}_{d1i}) & \underline{\eta}S_{1i} \\ \underline{\eta}S_{1i} & \underline{\eta} \end{bmatrix}. \tag{2.40}
$$

Since Ω_{1i} is positive definite, V_{X_1} is positive definite. For the upper bound condition of inertia matrix,

$$
V_1 \leq \frac{1}{2}(\eta_1^1 + \eta_2^1\|x_1\| + \eta_3^1\|x_1\|^2)\|x_2 + S_1 x_1\|^2 + \frac{1}{2}x_1^T(\bar{K}_{p1} + S_1\bar{K}_{d1})x_1. \tag{2.41}
$$

Since

$$
\|x_1\| \leq \|X_1\|, \tag{2.42}
$$

$$
\|x_1\|^2 \leq \|x_1\|^2 + \|x_2\|^2 = \|X_1\|^2, \tag{2.43}
$$

one has

$$
V_1(X_1) \leq \frac{1}{2}\left((\eta_1^1\bar{S}_1 + \lambda_{\max}(\bar{K}_{p1} + S_1\bar{K}_{d1}))\|X_1\|^2 + \eta_2^1\bar{S}_1\|X_1\|^3 + \eta_3^1\bar{S}_1\|X_1\|^4\right). \tag{2.44}
$$

Similar to V_1, we can compute the bound of V_2 as

$$
\eta_0^2\|X_2\|^2 \leq V_2(X_2) \leq \eta_1^2\|X_2\|^2, \tag{2.45}
$$

where

$$
\begin{aligned}
\eta_0^2 &= \min\{\frac{1}{2}\lambda_{\min}(\underline{\Omega}_{2i})\}, \\
\eta_1^2 &= \max\{\frac{1}{2}\lambda_{\max}(\bar{\Omega}_{2i})\}, \\
\underline{\Omega}_{2i} &= \begin{bmatrix} \underline{\theta}S_{2i}^2 + 2\beta_2 S_{2i} & \underline{\theta}S_{2i} \\ \underline{\theta}S_{2i} & \underline{\theta} \end{bmatrix}, \\
\bar{\Omega}_{2i} &= \begin{bmatrix} \bar{\theta}S_{2i}^2 + 2\beta_2 S_{2i} & \bar{\theta}S_{2i} \\ \bar{\theta}S_{2i} & \bar{\theta} \end{bmatrix}, \\
\underline{\theta} &= \lambda_{\min}(J), \\
\bar{\theta} &= \lambda_{\max}(J).
\end{aligned}
\tag{2.46}
$$

By combining (2.44) and (2.45), we have

$$\eta_0\|X\|^2 \leq V \leq \eta_1\|X\|^2 + \eta_2\|X\|^3 + \eta_3\|X\|^4, \tag{2.47}$$

where $\eta_0 = \min\{\eta_0^1, \eta_0^2\}$, $\eta_1 = \max\{\eta_1^1\bar{S}_1 + \lambda_{\max}(\bar{K}_{p1} + S_1\bar{K}_{d1}), \eta_1^2\}, \eta_2 = \frac{1}{2}\eta_2^1\bar{S}_1$, $\eta_3 = \frac{1}{2}\eta_3^1\bar{S}_1$.

This in turn means V is decrescent for all $X \in \mathbf{R}^{4n}$, or, we can say that $V(X)$ is a legitimate Lyapunov function candidate for all FJM system.

Taking first derivative of $V_1(X_1)$ along the trajectory of the controlled system yields

$$\begin{aligned}\dot{V}_1 =&(x_2+ S_1x_1)^T M(\dot{x}_2+ S_1x_2)+ \frac{1}{2}(x_2+ S_1x_1)^T\dot{M}(x_2+ S_1x_1)+ x_1^T(\bar{K}_{p1}+ S_1\bar{K}_{d1})x_2\\ =&(x_2 + S_1x_1)^T(\frac{1}{2}\dot{M}x_2 + \frac{1}{2}\dot{M}S_1x_1 - Cx_2 - G - Kx_1 + MS_1\dot{x} + K\tilde{\tau} + Kx_3)\\ &+ x_1^T(\bar{K}_{p1} + S_1\bar{K}_{d1})x_2.\end{aligned} \tag{2.48}$$

From (2.24) and (2.27) it can be seen that

$$\begin{aligned}\dot{V}_1 =&(x_2+ S_1x_1)^T\Phi_1+ (x_2+ S_1x_1)^T K\tilde{\tau}+ (x_2+ S_1x_1)^T Kx_3+ x_1^T(\bar{K}_{p1}+ S_1\bar{K}_{d1})x_2\\ =&(x_2 + S_1x_1)^T\Phi_1 + (x_2 + S_1x_1)^T\bar{K}\tilde{\tau} + (x_2 + S_1x_1)^T\bar{K}E\tilde{\tau}\\ &+ x_1^T(\bar{K}_{p1} + S_1\bar{K}_{d1})x_2 + (x_2 + S_1x_1)^T Kx_3.\end{aligned} \tag{2.49}$$

We substitute the control (2.29), then

$$\begin{aligned}\dot{V}_1 =&(x_2+S_1x_1)^T\Phi_1+(x_2+S_1x_1)^T[-\alpha_1(x_2+S_1x_1)\rho_1^2-K_{p1}x_1-K_{d1}x_2-\beta_1(x_2+S_1x_1)]\\ &+ (x_2 + S_1x_1)^T\bar{K}E\bar{K}^{-1}[-\alpha_1(x_2 + S_1x_1)\rho_1^2 - K_{p1}x_1 - K_{d1}x_2 - \beta_1(x_2 + S_1x_1)]\\ &+ (x_2 + S_1x_1)^T Kx_3 + x_1^T(\bar{K}_{p1} + S_1\bar{K}_{d1})x_2\\ \leq&\|x_2 + S_1x_1\|\bar{\rho}_1 - \alpha_1(1 + e_m)\|x_2 + S_1x_1\|^2\rho_1^2 - \underline{\lambda}_1\|X_1\|^2\\ &- \beta_1(1 + e_m)\|x_2 + S_1x_1\|^2 + (x_2 + S_1x_1)^T Kx_3,\end{aligned} \tag{2.50}$$

where

$$\underline{\lambda}_1 = \lambda_E \min\{\lambda_{\min}(\bar{K}_{d1}), \lambda_{\min}(S_1\bar{K}_{p1})\}. \tag{2.51}$$

From the choosing of function ρ_1 in (2.30), we have

$$\begin{aligned}\dot{V}_1 \leq&\|x_2 + S_1x_1\|(1 + e_m)\rho_1 - \alpha_1(1 + e_m)\|x_2 + S_1x_1\|^2\rho_1^2 - \underline{\lambda}_1\|X_1\|^2\\ &- \beta_1(1 + e_m)\|x_2 + S_1x_1\|^2 + (x_2 + S_1x_1)^T Kx_3\\ \leq&\frac{1 + e_m}{4\alpha_1} - \beta_1(1 + e_m)\|x_2 + S_1x_1\|^2 - \underline{\lambda}_1\|X_1\|^2\\ &+ (x_2 + S_1x_1)^T Kx_3.\end{aligned} \tag{2.52}$$

According to the inequality $ab \le \frac{1}{2}(a^2+b^2),\ a, b \in \mathbf{R}$, we have

$$\begin{aligned}(x_2+S_1x_1)^TKx_3 &\le \|x_2+S_1x_1\|\|K\|\|x_3\| \\ &\le \frac{1}{2}(\omega_1\|x_2+S_1x_1\|^2+\omega_1^{-1}\|x_3\|^2)\|K\| \\ &\le \frac{1}{2}(\omega_1\|x_2+S_1x_1\|^2+\omega_1^{-1}\|x_3\|^2)\lambda_k,\end{aligned} \tag{2.53}$$

where $\omega_1 > 0$ is a constant, $\lambda_k \ge \|K\|$. Therefore,

$$\begin{aligned}\dot{V}_1 \le &-\underline{\lambda}_1\|X_1\|^2+\frac{1+e_m}{4\alpha_1}-\beta_1(1+e_m)\|x_2+S_1x_1\|^2 \\ &+(\frac{1}{2}\omega_1\|x_2+S_1x_1\|^2+\frac{1}{2}\omega_1^{-1}\|x_3\|^2)\lambda_k \\ = &-\underline{\lambda}_1\|X_1\|^2+\frac{1+e_m}{4\alpha_1}-[\beta_1(1+e_m)-\frac{1}{2}\omega_1\lambda_k]\|x_2+S_1x_1\|^2 \\ &+\frac{1}{2}\omega_1^{-1}\lambda_k\|x_3\|^2.\end{aligned} \tag{2.54}$$

Next, based on the subsystem (2.20), the derivative of V_2 is given by

$$\begin{aligned}\dot{V}_2 =&(x_4+S_2x_3)^TJ(\dot{x}_4+S_2x_4)+\frac{1}{2}(x_4+S_2x_3)^T\dot{J}(x_4+S_2x_3)+2x_3^T\beta_2S_2x_4 \\ =&(x_4+S_2x_3)^T[-J\ddot{\tilde{\tau}}-Kx_3+Kx_1-K\tilde{\tau}+JS_2x_4+\frac{1}{2}\dot{J}(x_4+S_2x_3)+\tau] \\ &+2x_3^T\beta_2S_2x_4.\end{aligned} \tag{2.55}$$

Based on the (2.31), (2.32) and the control (2.33),

$$\begin{aligned}\dot{V}_2 =&(x_4+S_2x_3)^T\Phi_2+(x_4+S_2x_3)^T\tau+2x_3^T\beta_2S_2x_4 \\ =&(x_4+S_2x_3)^T\Phi_2+(x_4+S_2x_3)^T[-\alpha_2(x_4+S_2x_3)\rho_2^2-\beta_2(x_4+S_2x_3)] \\ &+2x_3^T\beta_2S_2x_4 \\ \le&\|x_4+S_2x_3\|\rho_2-\alpha_2\|x_4+S_2x_3\|^2\rho_2^2-\underline{\lambda}_2\|X_2\|^2 \\ \le&-\underline{\lambda}_2\|X_2\|^2+\frac{1}{4\alpha_2},\end{aligned} \tag{2.56}$$

where

$$\underline{\lambda}_2=\min\{\beta_2, \lambda_{\min}(\beta_2S_2^2)\}. \tag{2.57}$$

Now, with (2.54), (2.56) and the inequality $\|x_3\|^2 \le \|X_2\|^2$, it can be seen that

$$\begin{aligned}\dot{V}(X) =&\dot{V}_1(X_1)+\dot{V}_2(X_2) \\ \le&-\underline{\lambda}_1\|X_1\|^2+\frac{1+e_m}{4\alpha_1}-[\beta_1(1+e_m)-\frac{1}{2}\omega_1\lambda_k]\|x_2+S_1x_1\|^2+\frac{1}{2}\omega_1^{-1}\lambda_k\|x_3\|^2 \\ &-\underline{\lambda}_2\|X_2\|^2+\frac{1}{4\alpha_2} \\ \le&-\underline{\lambda}_1\|X_1\|^2+\frac{1+e_m}{4\alpha_1}-[\beta_1(1+e_m)-\frac{1}{2}\omega_1\lambda_k]\|x_2+S_1x_1\|^2 \\ &-(\underline{\lambda}_2-\frac{1}{2}\omega_1^{-1}\lambda_k)\|X_2\|^2+\frac{1}{4\alpha_2}.\end{aligned} \tag{2.58}$$

If we choose suitable β_1, β_2 which satisfies $\beta_1(1+e_m) - \frac{1}{2}\omega_1\lambda_k > 0, \underline{\lambda}_2 - \frac{1}{2}\omega_1^{-1}\lambda_k > 0$, we have

$$\begin{aligned} \dot{V} \leq & \kappa_1 + \kappa_2 - \underline{\lambda}_1\|X_1\|^2 - \kappa_3\|X_2\|^2 \\ \leq & -\gamma\|X\|^2 + \bar{\kappa}, \end{aligned} \tag{2.59}$$

where

$$\begin{aligned} \kappa_1 &= \frac{1+e_m}{4\alpha_1}, \kappa_2 = \frac{1}{4\alpha_2}, \\ \kappa_3 &= \underline{\lambda}_2 - \frac{1}{2}\omega_1^{-1}\lambda_k, \end{aligned} \tag{2.60}$$

$\gamma = \min\{\kappa_3, \underline{\lambda}_1\}, \bar{\kappa} = \kappa_1 + \kappa_2$.

With the standard form shown in [130], we have the uniform boundedness performance. That is, given a constant $r > 0$, with the initial condition $\|X(t_0)\| \leq r$ (t_0 is the initial time), there is a $d(r)$ given by

$$\begin{aligned} d(r) &= \begin{cases} r\left[\frac{\eta_1+\eta_2 r+\eta_3 r^2}{\eta_0}\right]^{\frac{1}{2}} & \text{if} \quad r > R \\ R\left[\frac{\eta_1+\eta_2 R+\eta_3 R^2}{\eta_0}\right]^{\frac{1}{2}} & \text{if} \quad r \leq R \end{cases}, \\ R &= \sqrt{\bar{\kappa}/\gamma}, \end{aligned} \tag{2.61}$$

such that the trajectory $\|X(t)\| \leq d(r)$ for all $t \geq t_0$. The uniform ultimate boundedness performance also follows [130]. That is, let

$$\underline{d} = R\left[\frac{\eta_1 + \eta_2 R + \eta_3 R^2}{\eta_0}\right]^{\frac{1}{2}}, \tag{2.62}$$

for any given $\bar{d} \geq \underline{d}$, then $\|X(t)\| \leq \bar{d}$ for all $t \geq t_0 + T(\bar{d}, r)$, where

$$\begin{aligned} T(\bar{d}, r) &= \begin{cases} 0 & \text{if} \quad r \leq \bar{R} \\ \frac{\eta_1 r^2+\eta_2 r^3+\eta_3 r^4-\eta_0\bar{R}^2}{\gamma\bar{R}^2-\bar{\kappa}} & \text{otherwise}, \end{cases} \\ \bar{R} &= \xi^{-1}(\eta_0\bar{d}^2), \end{aligned} \tag{2.63}$$

where the function $\xi(\cdot)$ is given by

$$\xi(\theta) = \eta_1\theta^2 + \eta_2\theta^3 + \eta_3\theta^4. \tag{2.64}$$

From the analysis, it can be seen that the uniform ultimate boundedness ball and the uniform stability ball is determined by $\underline{d}$. As it is shown above, $\underline{d}$ approaches to 0 when R approaches to 0, which means both α_1^{-1} and α_2^{-1} are close to 0. Thus, if both α_1^{-1} and $\alpha_2^{-1} \to 0$, then $\underline{d} \to 0$. Q.E.D.

Remark 2.4. The positive gain parameters S_1 and S_2 in $\tilde{\tau}$ and τ can be selected based on the specific situation such as the physical limit of the system. No more restrictions exist in the control design.

The procedure of the control design can be summarized as following steps:

Step 1: Compute proper value for λ_k where $\max\limits_{\sigma_1\in\Sigma_1}\|K\| < \lambda_k$ and compute λ_E in (2.25), then we have $\underline{\lambda}_1$.

Step 2: Select a ω_1, then we choose a β_1 such that $\beta_1 > \frac{\omega_1\lambda_k}{2(1+e_m)}$;

Step 3: Select $\underline{\lambda}_2$ such that $\underline{\lambda}_2 > \frac{1}{2}\omega_1^{-1}\lambda_k$;

Step 4: Select β_2 according to (2.57);

Step 5: With given $\alpha_1, \alpha_2, S_1, S_2$, formulate $\tilde{\tau}$ and τ according to (2.29) and (2.33).

Remark 2.5. As previously mentioned, in many other Lyapunov-based control schemes, the controllers can only be applied on the revolute joint type FJM since the inertia matrix is assumed to be uniformly upper bounded. The current work, at this point, extends the existing controls to prismatic joint type FJM by introducing the new upper bound condition.

2.7 Experimental verification

The experiments have been performed on a two-link FJM in the vertical plane. As shown in Figure 2.4, it is modified from a six-axis manipulator (the first, fourth, fifth and sixth joints are fixed) in the Laboratory of Mechanical Engineering. Two Panasonic A5 servo motors supply the driving torque. Each joint contains a harmonic reducer which is used to transmit the torque. There are four optical encoders to measure the link positions and motor positions. Two encoders are installed on the motors to measure the angles of the servo motors. The other two encoders, which are fixed on the steel supports, are used to measure the actual angles of the links. Angular velocities are obtained from the Euler method $\dot{\theta}(k+1) = [\theta(k+1) - \theta(k)]/t_s$, where t_s is sampling period, $\theta(k)$ is the feedback position of the k-th sampling period, $\dot{\theta}(k)$ is the corresponding velocity.

Let link angle vector $q_1 = [q_{1,1} \quad q_{1,2}]^T$, joint angle vector $q_2 = [q_{2,1} \quad q_{2,2}]^T$. The two-link FJM can be described as Figure 2.5 and the system model is given by

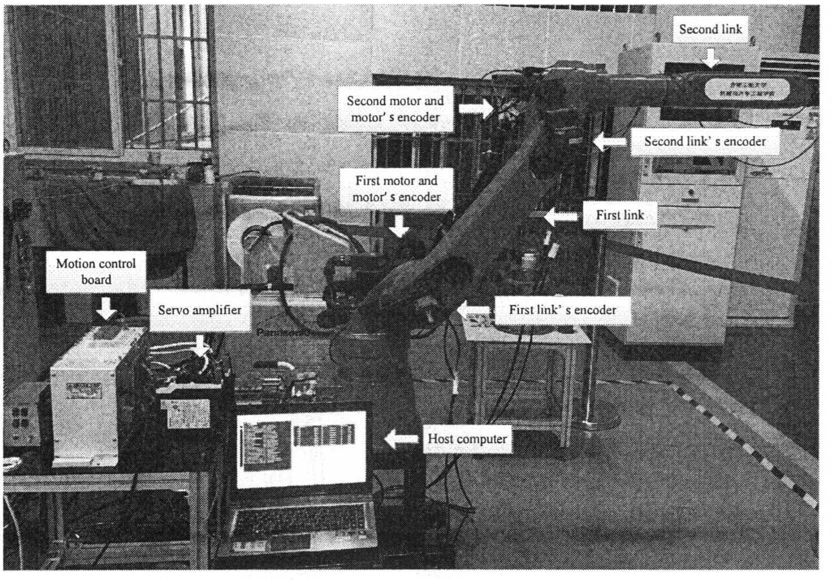

Figure 2.4: Photograph of the two-link flexible joint manipulator in the laboratory.

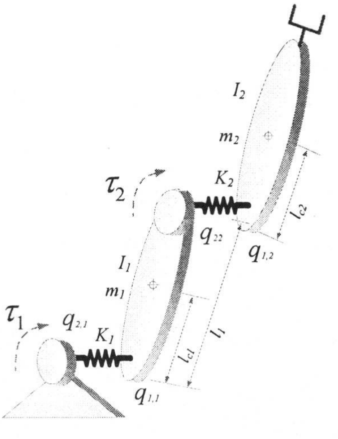

Figure 2.5: Two-link flexible joint manipulator mechanism.

$$
\begin{aligned}
M(q_1) &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \\
C(q_1, \dot{q}_1) &= \begin{bmatrix} -m_1 l_1 l_{c2} \sin q_{1,2} \dot{q}_{1,2} & -m_2 l_1 l_{c2} \sin q_{1,2}(\dot{q}_{1,1} + \dot{q}_{1,2}) \\ m_2 l_1 l_{c2} \sin q_{1,2} \dot{q}_{1,1} & 0 \end{bmatrix}, \\
G(q_1) &= \begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \sin q_{1,1} + m_2 l_{c2} g \sin(q_{1,1} + q_{1,2}) \\ m_2 l_{c2} g \sin(q_{1,1} + q_{1,2}) \end{bmatrix}, \\
J &= \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}, \\
S_1 &= \begin{bmatrix} S_{11} & 0 \\ 0 & S_{12} \end{bmatrix}, S_2 = \begin{bmatrix} S_{21} & 0 \\ 0 & S_{22} \end{bmatrix}, \\
K &= \begin{bmatrix} \bar{K}_1 & 0 \\ 0 & \bar{K}_2 \end{bmatrix} (I + \begin{bmatrix} \Delta K_1 & 0 \\ 0 & \Delta K_2 \end{bmatrix}),
\end{aligned} \tag{2.65}
$$

where

$$
\begin{aligned}
M_{11} &= m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos q_{1,2}) + m_1 l_{c1}^2 + I_1 + I_2, \\
M_{12} &= m_2(l_{c2}^2 + l_1 l_{c2} \cos q_{1,2}) + I_2, \\
M_{21} &= M_{12}, \\
M_{22} &= m_2 l_{c2}^2 + I_2.
\end{aligned} \tag{2.66}
$$

It can be seen that the elements in inertia matrix are bounded:

$$
\begin{aligned}
|M_{11}| &\le m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}) + m_1 l_{c1}^2 + I_1 + I_2, \\
|M_{12}| &\le m_2(l_{c2}^2 + l_1 l_{c2}) + I_2, \\
|M_{22}| &\le m_2 l_{c2}^2 + I_2.
\end{aligned} \tag{2.67}
$$

It should be noted that, the friction and damping items are not included in this model explicitly. Even if a certain amount of damping always exists, it was demonstrated experimentally in Sweet and Good [126] that the problems of joint flexibility are most evident in the case the joint damping is small. Furthermore, even if damping is present, the damping force terms do not influence our current control design procedure. Those can be included as additional terms in estimating bounding functions.

According to the motor manual and the manipulator design files, we have the nominal values of manipulator's parameters which are listed in Table 2.1.

Table 2.1: The nominal values of manipulator's parameters

Symbol	Meaning	Values	Unit
J_{11}	inertia of first actuator	3.17×10^{-4}	$kg \cdot m^2$
J_{22}	inertia of second actuator	0.97×10^{-4}	$kg \cdot m^2$
I_1	inertia of first link	0.96	$kg \cdot m^2$
I_2	inertia of second link	0.71	$kg \cdot m^2$
l_1	length of the first link	0.6	m
l_{c1}	mass centroid length of first link	0.3	m
l_{c2}	mass centroid length of second link	0.5	m
$\bar{m}_1$	mass of first link	19.31	kg
$\bar{m}_2$	mass of second link	13.73	kg
$\bar{K}_1$	elastic coefficient of first joint	1.06×10^3	$N \cdot m/arcmin$
$\bar{K}_2$	elastic coefficient of second joint	0.75×10^3	$N \cdot m/arcmin$
g	acceleration of gravity	9.8	m/s^2

Figure 2.6 depicts the overall experimental setup. There is a desktop computer acting as host computer and a PMAC motion control board acting as target. The main software, PEWin32PRO, which is used to interface with users, runs on the host computer. Through PEWin32PRO, we can monitor the activities of whole system, such as encode signals feedback, controller signals output, target memory reading and program uploading. The PMAC motion control board contains a data acquisition unit, which is used to receive the position signals from the encoders, and a position control unit, which is used to send the controller output signals to the power amplifiers. Each encoder transmits two signals – the continuous trains of two square waves called channel A and channel B. The phase difference of the two signals is used to tell the direction of the rotation, whether clockwise or counterclockwise. The two signals connect to the data acquisition unit through the digital input port. A software program is written to convert the two signals into angular position. The servo motor works under the Torque Control Model and motion control board is set correspondingly by adjusting parameters. The desired input torque is decomposed into motion command in each servo period, and then transformed into voltage command to actuate servo motor. The sampling time is set to be 2.5 ms.

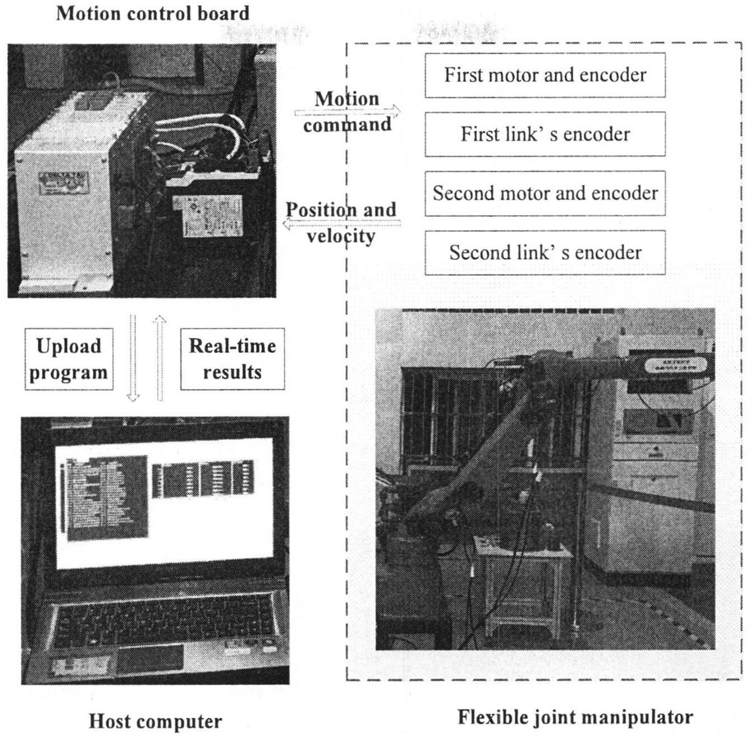

Figure 2.6: Diagram showing overall experimental setup.

To formulate the fictitious control $\tilde{\tau}$, based on (2.26) and (2.27) we can get Φ_1 as

$$\begin{aligned}
\Phi_1 =& \frac{1}{2}\dot{M}(\dot{q}_1 + S_1 q_1) - C\dot{q}_1 - G - Kq_1 + MS_1\dot{q}_1 \\
=& \frac{1}{2}\begin{bmatrix} -2m_2 l_1 l_{c2}\sin q_{1,2}\dot{q}_{1,2} & -m_2 l_1 l_{c2}\sin q_{1,2}\dot{q}_{1,2} \\ -m_2 l_1 l_{c2}\sin q_{1,1}\dot{q}_{1,2} & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_{1,1} + S_{11}q_{1,1} \\ \dot{q}_{1,2} + S_{12}q_{1,2} \end{bmatrix} \\
& - \begin{bmatrix} -m_1 l_1 l_2 \sin q_{1,2}\dot{q}_{1,2} & -m_2 l_1 l_2 \sin q_{1,2}(\dot{q}_{1,2} + \dot{q}_{1,1}) \\ m_2 l_1 l_2 \sin q_{1,2}\dot{q}_{1,1} & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_{1,1} \\ \dot{q}_{1,2} \end{bmatrix} \\
& - \begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \sin q_{1,1} + m_2 l_{c2} g \sin(q_{1,1} + q_{1,2}) \\ m_2 l_{c2} g \sin(q_{1,1} + q_{1,2}) \end{bmatrix} \\
& - \begin{bmatrix} \bar{K}_1(1 + \Delta K_1) & 0 \\ 0 & \bar{K}_2(1 + \Delta K_2) \end{bmatrix}\begin{bmatrix} q_{1,1} \\ q_{1,2} \end{bmatrix} \\
& + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\begin{bmatrix} S_{11} & 0 \\ 0 & S_{12} \end{bmatrix}\begin{bmatrix} \dot{q}_{1,1} \\ \dot{q}_{1,2} \end{bmatrix}.
\end{aligned} \tag{2.68}$$

A simple scalar bounding function $\bar{\rho}_1$ can be obtained as

$$\bar{\rho}_1 = \|\Phi_1\|. \tag{2.69}$$

We can get Φ_2 and ρ_2 by the similar way. For the experiment, the control parameters are chosen as $S_{11} = S_{12} = 1$, $S_{21} = S_{22} = 2$, $\omega_1 = 1$, $\beta_1 = \beta_2 = 2$, $\alpha_1 = \alpha_2 = 10$, $K_{p1} = 100$, $K_{d1} = 20$. Then we obtain the virtual control $\tilde{\tau}$ and actual control τ as

$$\begin{aligned} \tilde{\tau} =& \bar{K}^{-1}\left[-\alpha_1(\dot{q}_1 + S_1 q_1){\rho_1}^2 - \beta_1(\dot{q}_1 + S_1 q_1)\right] + \bar{K}^{-1}(-K_{p1}q_1 - K_{d1}\dot{q}_1), \\ \tau =& -\alpha_2\left((\dot{q}_2 - \dot{\tilde{\tau}}) + S_2(q_2 - \tilde{\tau})\right){\rho_2}^2 - \beta_2\left((\dot{q}_2 - \dot{\tilde{\tau}}) + S_2(q_2 - \tilde{\tau})\right), \end{aligned} \tag{2.70}$$

Let $e(t)$ denote the following error, $\tau(t) = [\tau_1(t), \tau_2(t)]^T$ denote the input torque, the performance indices are defined as:

1. $e_{\max} = \max\{|e(t)|\}$, representing the maximum following error, reflects the transitional performance.
2. $\bar{e} = \left(\int_0^T \|e(t)\|^2 dt/T\right)^{1/2}$, where T is the total tracking time, reflects the average tracking performance.
3. $\bar{\tau} = \left(\int_0^T (\tau_1(t)^2 + \tau_2(t)^2)\, dt/T\right)^{1/2}$, reflects the average control input.
4. $\tau_v = \Delta\tau/\bar{\tau}$, reflects the fluctuation of control input, where $\Delta\tau = \sqrt{\frac{1}{N}\sum_{k=1}^{N}|\tau(k) - \tau(k-1)|^2}$ is the average control increment, $\tau(k)$ is the control input of k-th sampling.

To testify the proposed controller, two experimental validations are conducted.

Square Trajectory Tracking: Movement of end-tip is in $X - Y$ plane and described by (x, y) in Cartesian position. A 100 mm $\times$ 100 mm square trajectory is required to be tracked in 1 s. The tracking procedure is (unit is mm):

1. Move the end-tip to start point (850,0);
2. Begin data acquisition;
3. Move the end-tip to points (900,0), (900,100), (800,100), (800,0), (850,0), continuously;
4. Stop data acquisition and upload data.

While this high-speed trajectory might not be used practically, it was still selected for nonlinearities and flexibility effects were significantly shown under this setting. In other words, its

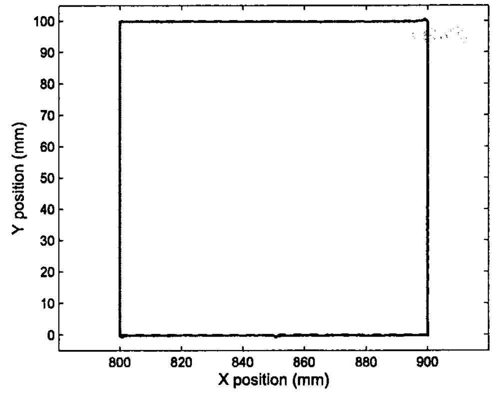

Figure 2.7: End-tip tracking results : Square trajectory, robust control.

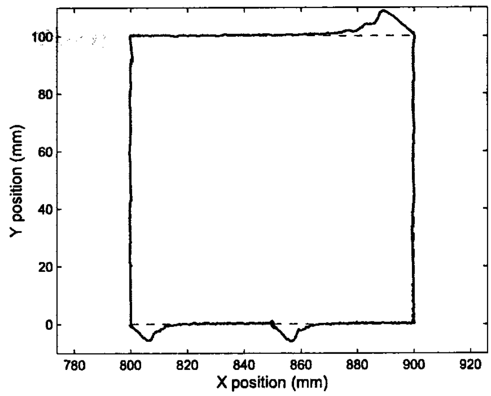

Figure 2.8: End-tip tracking results : Square trajectory, PD control.

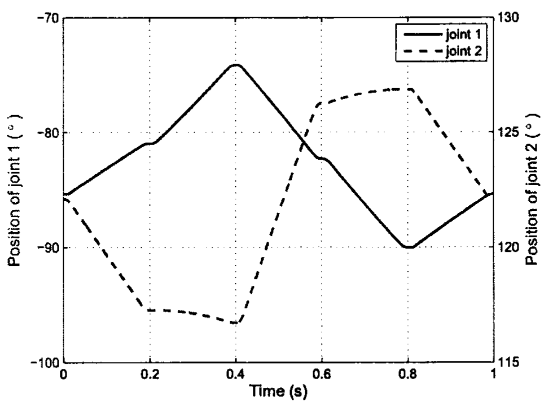

Figure 2.9: Joint angle tracking results : Square trajectory, robust control.

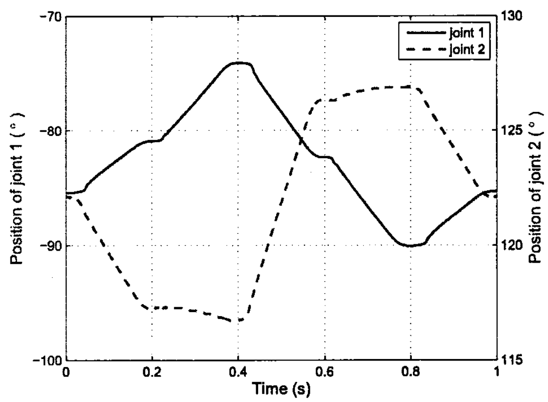

Figure 2.10: Joint angle tracking results : Square trajectory, PD control.

connatural control difficulties make it a challenging control task, as the transient changes in moving direction at each corners of trajectory and the joint vibrations are likely to be induced when the end-tip is moving at a high speed. In this context, the control task consists of guaranteeing the minimum tracking error along each side of square and minimizing the overshoot at each corner. For comparison, the well-known PD controller is also applied to this FJM system. The results of the investigation are shown from Figure 2.7 to Figure 2.14. The comparison of tracking performance indices under proposed robust control and PD control are listed in Table 2.2.

From the comparison of the experimental results, it can be seen that the performance under proposed robust is much better than that under PD control. Specifically, the large contour errors

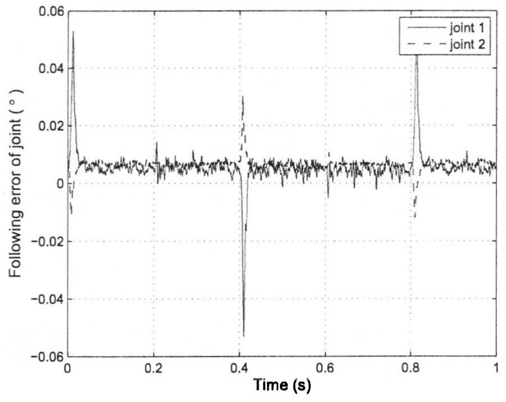

Figure 2.11: Joint angle following errors : Square trajectory, robust control.

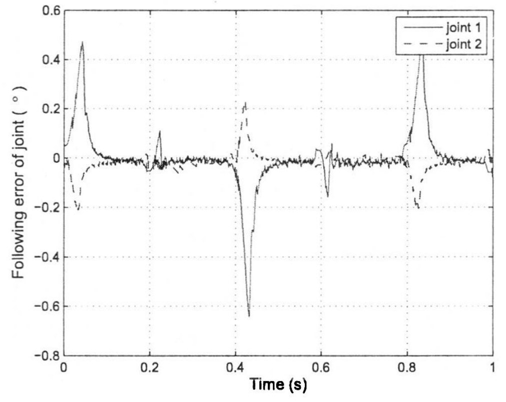

Figure 2.12: Joint angle following errors : Square trajectory, PD control.

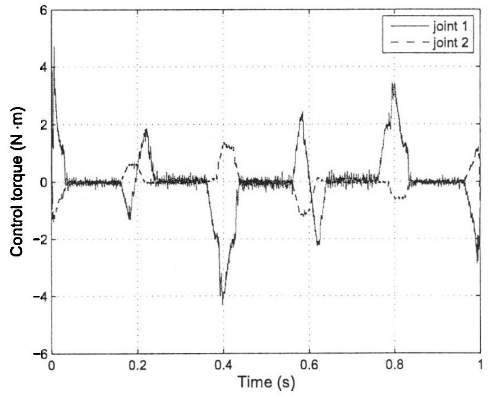

Figure 2.13: Input torque histories : Square trajectory, robust control.

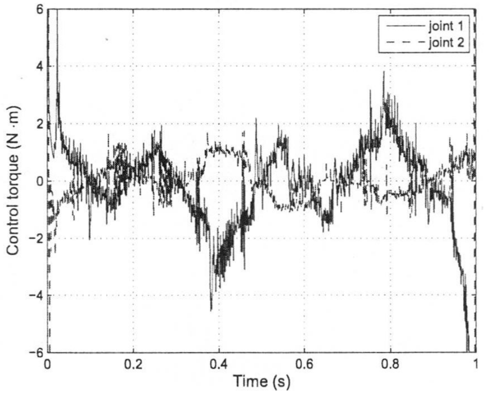

Figure 2.14: Input torque histories : Square trajectory, PD control.

Table 2.2: Comparison of index under robust control and PD control: Square trajectory

	joint 1		joint 2		end-tip	
	robust	PD	robust	PD	robust	PD
$e_{\max}$	0.0562°	0.6402°	0.0306°	0.2331°	0.7523 mm	8.9740 mm
$\bar{e}$	0.0063°	0.0587°	0.0066°	0.0299°	0.1908 mm	0.8777 mm
$\bar{\tau}$	0.6143 N·m	1.5976 N·m	0.2046 N·m	0.5884 N·m	–	–
τ_v	0.6496	4.6932	0.3963	5.7361	–	–

occur at the start/stop point, upper right corner and bottom left corner. With robust control, the maximum contour error (may not occur when following error is maximum) at the corner is only 0.7523 mm, while that is 8.9740 mm (almost 12 times) with PD control, see Figures 2.7 and 2.8. The tracking trajectory of joint and the corresponding following error are provided in Figure 2.9 to Figure 2.12. Large following error arises when the sudden change of direction of motion occurs. The maximum following errors exist in the first joint for the whole mass of the body is supported by this joint. The maximum value of following error, under robust control, is 0.0562° which is only one in ten (0.6402°) while PD control applies. When it comes to the input torque history shown in Figures 2.13 and 2.14, PD controller costs more than robust control and has an obvious oscillation, not only at corners, but also when the end-tip is moving along the sides of square. This point also can be described by performance index comparison ($\bar{\tau}$, τ_v) shown in Table 2.2.

Circle Trajectory Tracking: Movement of end-tip is in $X-Y$ plane and described by (x, y) in Cartesian position. A circle trajectory, whose radius is equal to 50 mm, is required to be tracked in 1 s. The tracking procedure is given by following (unit is mm):

1. Move the end-tip to start point (800,0);

2. Begin data acquisition;

3. Move the end-tip clockwise along a complete circle whose center is (750,0) and radius is 50 mm;

4. Stop data acquisition and upload data.

This kind of trajectory is selected for the practical applications, such as welding, laser cutting, spatial measurement, etc., where diverse trajectories are required. To realize circle trajectory, each joint should follow a sin wave movement. For comparison, the well-known PD controller is also applied to this FJM system. The experimental results are shown from Figure 2.15 to Figure 2.22.

Similar to the results of square trajectory tracking, in Figures 2.15 and 2.16, the contour errors along the trajectory are much smaller when robust control is applied. Figure 2.17 to 2.20 provide joint tracking trajectory and joint following error under robust control and PD control, respectively. It can be seen that, sudden changes of motion arise on 0.08 s, 0.49 s and 0.95 s under PD control (see Figure 2.18), while the trajectory of robust controlled is almost a smooth curve. In Figure 2.19, the maximum following error of robust control is 0.0755° while that of PD control is as much as 0.7388°. The input control torque histories are shown in Figures 2.21 and 2.22. Although the average input torque is close to each other, robust controller earns less oscillations and smaller

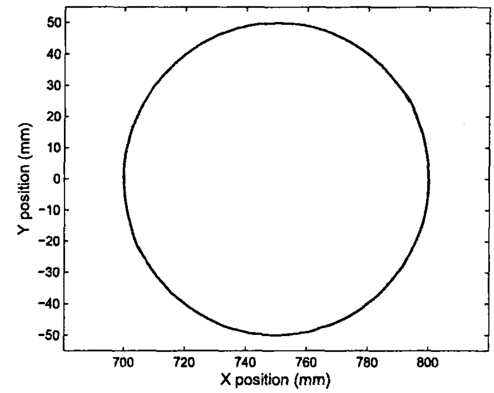

Figure 2.15: End-tip tracking results : Circle trajectory, robust control.

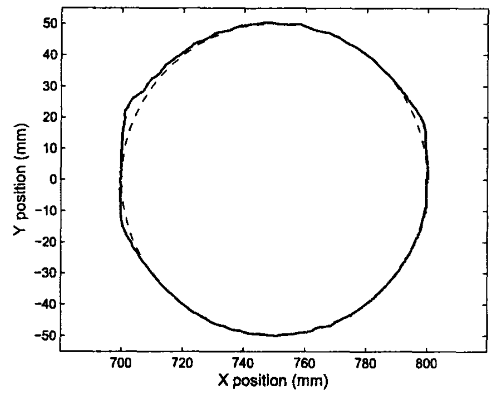

Figure 2.16: End-tip tracking results : Circle trajectory, PD control.

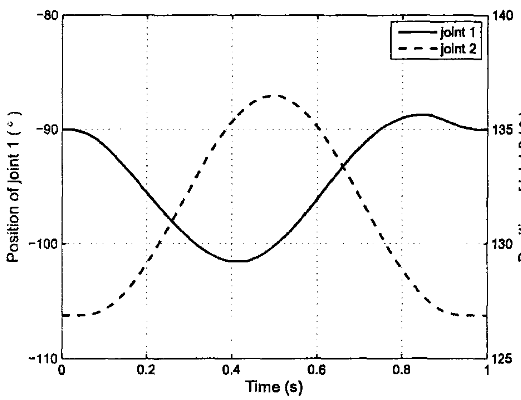

Figure 2.17: Joint angle tracking results : Circle trajectory, robust control.

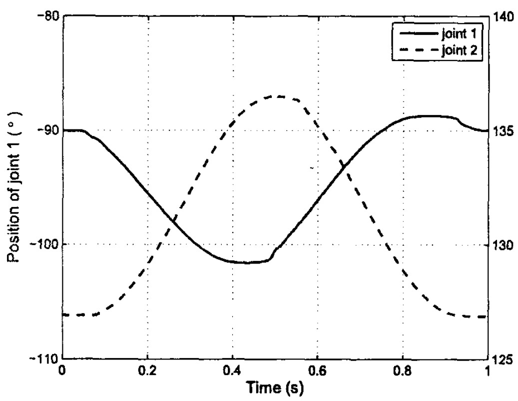

Figure 2.18: Joint angle tracking results : Circle trajectory, PD control.

overshoots. Detailed comparison of other performance indices under proposed robust control and PD control are provided in Table 2.3.

2.8 Conclusions

With the model developed by Spong, a robust control for the FJM systems, which is uncertain and mismatched, is proposed. The overall system is transformed by defining new state variables, and then be divided into two subsystems (link and joint) by implanting a fictitious control for the link position subsystem. Through doing this, the difficulty on designing control for mismatched

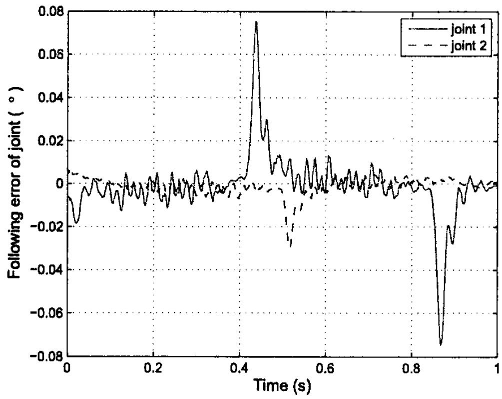

Figure 2.19: Joint angle following errors : Circle trajectory, robust control.

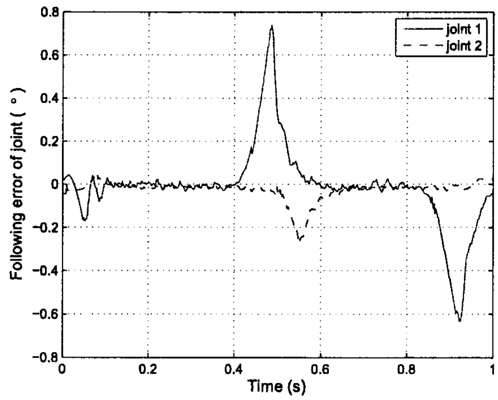

Figure 2.20: Joint angle following errors : Circle trajectory, PD control.

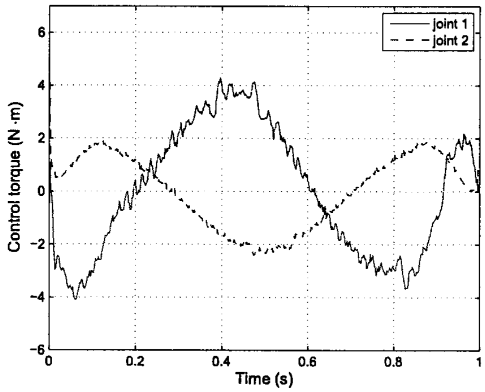

Figure 2.21: Input torque histories : Circle trajectory, robust control.

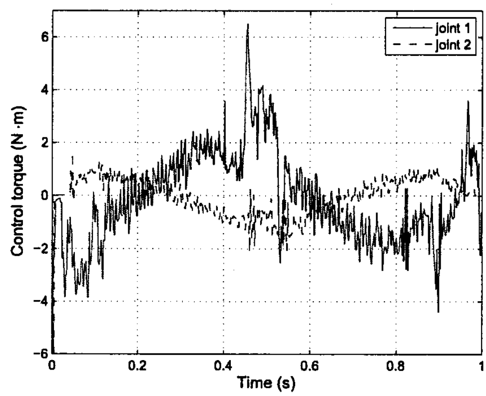

Figure 2.22: Input torque histories : Circle trajectory, PD control.

Table 2.3: Comparison of index under robust control and PD control: Circle trajectory

	joint 1		joint 2		end-tip	
	robust	PD	robust	PD	robust	PD
$e_{\max}$	0.0755°	0.7388°	0.0298°	0.2740°	0.4459 mm	3.8480 mm
$\bar{e}$	0.0075°	0.0922°	0.0027°	0.0292°	0.1109 mm	1.4527 mm
$\bar{\tau}$	2.1098 N·m	1.9237 N·m	1.1864 N·m	1.0599 N·m	–	–
τ_v	0.0925	0.5892	0.1509	1.3605	–	–

system is overcome. No more information of uncertainty is required for the proposed control other than its possible bound. This control scheme renders the transformed system uniform boundedness and uniform ultimate boundedness. Furthermore, the radius of uniform ultimate boundedness ball can approach to 0 by choosing suitable design parameters. The experimental results show the feasibility of proposed robust control.

In comparing with many other Lyapunov-based control designs, which are limited to revolute joint case, the proposed control employed a generic characteristic of inertia matrix, which makes it feasible to both revolute and prismatic cases.

However, since we can not use the skew-symmetric property (i.e., $q_1^T(\dot{D} - 2C)q_1 = 0$) if time-varying uncertainty arises, a heavy computation in estimating bounding functions will be encountered. Furthermore, the control is sensitive to the choice of uncertainty bound. Inappropriate choice of uncertainty bounding function may lead to extra control effort. This difficulty can be solved partly by design an improved control scheme, which will be shown later in Chapter 3.

Chapter 3 Adaptive robust control of uncertain flexible joint manipulator

3.1 Introduction

Although robust controller is capable of compensating uncertainties and contributes to the control of flexible joint manipulator (FJM) system significantly, as shown in [38, 131, 132], which are all Lyapunov-based approaches , knowledge of bound of uncertainty is needed a priori. In practical situations, the uncertainty is generally unknown or poorly known, it is impossible to have accurate information about the bound of uncertainty. This results in excessive control effort or degradation of performance, even instability.

By applying adaptive methodology, the gains of controller can be tuned automatically for achieving good tip trajectory tracking while suppressing tip deflection with varied payloads [133] or unknown disturbances. In [134], a backstepping-like procedure incorporating the model reference adaptive control is employed to circumvent the difficulty introduced by its cascade structure and various uncertainties. The drawback of this class of adaptive control is that they often suffer from heavy computational burden and the model-based adaptive laws are sometimes highly sensitive to unstructured uncertainties [47, 48]. Reference [135] proposed a class of control consisting of an adaptive fuzzy logic controller, an independent modal space adaptive controller for motion and vibration suppression of a single-link flexible active composite manipulator driven by piezoelectric actuators with dead zone, but fuzzy logic-based adaptive control requires proper formulation of control rules. Reference [136] proposed a hybrid neuro-fuzzy-based adaptive controller for a flexible two-link robotic manipulator, but it also needs a priori information about the input-output relation via off-line learning.

In this chapter, we consider the control design for the FJMs which are nonlinear and contain uncertainties. The possible bound of the uncertainty does not need to be given for the control design a priori. An adaptive algorithm is constructed for the estimation of the possible bound of the uncertainty. The compensation of uncertainty is described in a deterministic manner. We give two steps to design the control: firstly, the overall system is divided into link position subsystem and joint position subsystem by implanting a virtual control. This is a backstepping-like method; then we construct an adaptive robust controller by approximating to the bound of uncertainty which is unknown. The stability analysis is given by second method of Lyapunov with a more generic upper boundedness condition of inertia matrix.

3.2 Flexible joint manipulator with unknown uncertainty bound

Consider a FJM system described as follows

$$\begin{aligned}\begin{bmatrix} M(q_1,\sigma_1) & 0 \\ 0 & J(\sigma_2) \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\begin{bmatrix} C(q_1,\dot{q}_1,\sigma_1)\dot{q}_1 \\ F(q_2,\dot{q}_2,\sigma_2)\dot{q}_2 \end{bmatrix}+\begin{bmatrix} G(q_1,\sigma_1) \\ 0 \end{bmatrix} \\ +\begin{bmatrix} K(\sigma_1)(q_1-q_2) \\ -K(\sigma_2)(q_1-q_2) \end{bmatrix}=\begin{bmatrix} 0 \\ \tau \end{bmatrix},\end{aligned} \tag{3.1}$$

where $q_1 = [q_{1,1}\ q_{1,2}\ \cdots\ q_{1,n}]^T$ is link position vector and $q_2 = [q_{2,1}\ q_{2,2}\ \cdots\ q_{2,n}]^T$ is joint position vector. Let $q = [q_1^T \quad q_2^T]^T$ be a $2n-$ dimensional vector and represent the system's generalized coordinate. The joint flexibility is regarded as a linear torsional spring and elasticity coefficient of the spring is represented by a diagonal positive matrix $K(\sigma_{1,2})$. The link inertia matrix is denoted by $M(q_1,\sigma_1)$. The inertia of actuator is denoted as $J(\sigma_2)$ which is a diagonal matrix. Vectors $C(q_1,\dot{q}_1,\sigma_1)\dot{q}_1$ and $F(q_2,\dot{q}_2,\sigma_2)\dot{q}_2$ represent the Coriolis and centrifugal forces of links and actuators, respectively. $G(q_1,\sigma_1)$ is the gravitation force vector, and τ denotes the input control force from the actuators.

Remark 3.1. Comparing to the system model given in [9], we add an additional Coriolis and centrifugal force term $F(q_2,\dot{q}_2,\sigma_2)\dot{q}_2$ in 3.1 which makes this model become more general. For a specific application, there might not exist this term, but this does not influence our control design procedure by setting this term as 0. Detailed discussions of the modeling can be found in, e.g., [9] and [137].

Assumption 3.1. The mapping $\sigma_1(t) \in \Sigma_1$ and $\sigma_2(t) \in \Sigma_2$ are Lebesgue measurable, where Σ_1 and Σ_2 are prescribed and compact. Besides, $\|\sigma_1(t)\| \le \gamma_1$, $\|\sigma_2(t)\| \le \gamma_2$ with $\gamma_1, \gamma_2 > 0$ are *unknown* scalars.

Remark 3.2. The Assumption 3.1 means that we do not need to know the uncertainty, nor the bound of uncertainty exactly, there dose exist such bound for uncertainty under consideration. In the control design, we utilize a known function to approximate this unknown bound.

Assumption 3.2. The inertia matrix $M(q_1, \sigma_1)$ is uniformly positive definite. That is, there exists a constant $\underline{\eta} > 0$, such that

$$M(q_1, \sigma_1) \geq \underline{\eta} I, \quad \forall q_1 \in \mathbf{R}^n, \tag{3.2}$$

where I denotes the identity matrix.

Since the inertia matrix may be positive semi-definite, the positive definite property is stated as an assumption rather than a fact.

Property 3.1. There exist scalar constants $\bar{\eta}_1 > 0$, $\bar{\eta}_2 \geq 0$, $\bar{\eta}_3 \geq 0$, such that

$$\|M(q_1, \sigma_1)\| \leq \bar{\eta}_1 + \bar{\eta}_2\|q_1\| + \bar{\eta}_3\|q_1\|^2, \quad \forall q_1 \in \mathbf{R}^n. \tag{3.3}$$

This upper bound condition is stated as a fact rather than an assumption since it has been sufficiently investigated in Chen [129]. In a special case that the manipulator joints are all revolute, the property is reduced to $\bar{\eta}_2 = \bar{\eta}_3 = 0$, such that

$$\|M(q_1, \sigma_1)\| \leq \bar{\eta}_1, \quad \forall q_1 \in \mathbf{R}^n. \tag{3.4}$$

Here, we adopt the same implanted virtual control $\tilde{\tau}$ in (2.17) and the new state transformation (2.18). Thus, by multiplying $K^{-1}(\sigma_1)$ on both side of (2.19) the dynamics of FJMs can be expressed as follows by using new state variables:

$$\hat{S}_1 : \hat{M}(x_1, \sigma_1)\dot{x}_2 = -\hat{C}(x_1, x_2, \sigma_1)x_2 - \hat{G}(x_1, \sigma_1) - x_1 + x_3 + \tilde{\tau}, \tag{3.5}$$

$$\begin{aligned}\hat{S}_2 : J(\sigma_2)\dot{x}_4 = &- J(\sigma_2)\ddot{\tilde{\tau}} - F(x_1, x_2, x_3, x_4, \sigma_2)x_4 - F(x_1, x_2, x_3, x_4, \sigma_2)\dot{\tilde{\tau}} \\ &- K(\sigma_2)x_3 + K(\sigma_2)x_1 - K(\sigma_2)\tilde{\tau} + \tau,\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}\hat{M}(x_1, \sigma_1) &= K^{-1}(\sigma_1)M(x_1, \sigma_1), \\ \hat{C}(x_1, x_2, \sigma_1) &= K^{-1}(\sigma_1)C(x_1, x_2, \sigma_1), \\ \hat{G}(x_1, \sigma_1) &= K^{-1}(\sigma_1)G(x_1, \sigma_1).\end{aligned} \tag{3.7}$$

3.3 Adaptive robust control design

Since uncertainty is introduced, the overall system dose not meet the matching condition [120]. We divide system into the cascades of two subsystems as shown in (3.5), (3.6). Therefore, both of the two systems have "inputs". The problem is to develop a control τ which renders the system $\hat{S}_1$ and $\hat{S}_2$ to accomplish a good performance. We firstly define the implanted virtual control $\tilde{\tau}$ for the system (3.5). We know that there exists a scalar uncertain function $\rho_1 : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}_+$ such that for all $x_1 \in \mathbf{R}^n, x_2 \in \mathbf{R}^n, \sigma_1(t) \in \Sigma_1$,

$$\rho_1(x_1, x_2) \geq \|\Phi_1(x_1, x_2, \sigma_1)\|, \tag{3.8}$$

where

$$\begin{aligned}\Phi_1(x_1, x_2, \sigma_1) = & \frac{1}{2}\dot{\hat{M}}(x_1, x_2, \sigma_1)(x_2 + S_1 x_1) - \hat{C}(x_1, x_2, \sigma_1)x_2 \\ & - \hat{G}(x_1, \sigma_1) - x_1 + \hat{D}(x_1, \sigma_1)S_1 x_2,\end{aligned} \tag{3.9}$$

$$S_1 = diag[S_{1i}]_{n\times n},\ S_{1i} > 0,\ i = 1, 2, \cdots, n. \tag{3.10}$$

Assumption 3.3. [138]

(1) There exists an unknown constant k-dimensional vector $\psi_1 \in (0, \infty)^k$ and a known function $\Pi_1 : \mathbf{R}^n \times \mathbf{R}^n \times (0, \infty)^k \to \mathbf{R}_+$ such that for all $x_1 \in \mathbf{R}^n, x_2 \in \mathbf{R}^n$,

$$\rho_1(x_1, x_2) = \Pi_1(x_1, x_2, \psi_1). \tag{3.11}$$

(2) The function $\Pi_1(x_1, x_2, \cdot) : (0, \infty)^k \to \mathbf{R}_+$ is C^2 (i.e., 2-times continuously differentiable) and concave (i.e., $-\Pi_1(x_1, x_2, \cdot)$ is convex), and it is nondecreasing with respect to each ψ_1.
(3) The function $\Pi_1(x_1, x_2, \cdot)$ and $\frac{\partial \Pi_1}{\partial \psi_1}(\cdot)$ are both continuous.

For given constant positive scalars α_1, β_1, we construct the fictitious control $\tilde{\tau}$ for the subsystem $\hat{S}_1$ as follows:

$$\tilde{\tau}(t) = -\alpha_1(x_2(t) + S_1 x_1(t)){\Pi_1}^2(x_1, x_2, \hat{\psi}_1) - \beta_1(x_2(t) + S_1 x_1(t)), \tag{3.12}$$

where $\hat{\psi}_1$ is the adaptation law with the following form:

$$\dot{\hat{\psi}}_1(t) = \begin{cases} T_1^{-1}\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1, x_2, \hat{\psi}_1)\|x_2 + S_1 x_1\| - T_2\hat{\psi}_1, & \text{if} \|\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1, x_2, \hat{\psi}_1)\|\|x_2 + S_1 x_1\| > \varepsilon_1, \\ -T_2\hat{\psi}_1, & \text{if} \|\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1, x_2, \hat{\psi}_1)\|\|x_2 + S_1 x_1\| \leq \varepsilon_1, \end{cases}$$
$$\hat{\psi}_1(t_0) \in (0, \infty)^k. \tag{3.13}$$

Here, $T_1 = diag[T_{1i}]_{k\times k}, T_{1i} > 0$, $T_2 = diag[T_{2i}]_{k\times k}, T_{2i} > 0, i = 1, 2, \cdots, k, \varepsilon_1 > 0$. The α_1 part in (3.12) is designed against the uncertainty, the proper choice of α_1 and β_1 will be given later. Notice that, the adaptive law (3.13) is of leakage type with dead-zone.

To give the real control input τ with a given $S_2 = diag[S_{2i}]_{n\times n}$, $S_{2i} > 0,\ i = 1, 2, \cdots n$, we also know there exists a scalar function $\rho_2 : \mathbf{R}^{2n} \times \mathbf{R}^{2n} \to \mathbf{R}_+$ such that

$$\rho_2(X_1, X_2) \geq \|\Phi_2(X, \sigma_1, \sigma_2)\|, \tag{3.14}$$

where

$$\begin{aligned}\Phi_2(X,\sigma_1,\sigma_2) =& - J(\sigma_2)\ddot{\bar{\tau}}(X,\sigma_1,\sigma_2) - F(X,\sigma_2)(x_4 + \dot{\bar{\tau}}(x_1,x_2,\sigma_1)) \\ & - K(\sigma_2)x_3 + K(\sigma_2)x_1 - K(\sigma_2)\bar{\tau}(X_1) + J(\sigma_2)S_2x_4 \\ & + \frac{1}{2}\dot{J}(\sigma_2)(x_4 + S_2x_3).\end{aligned} \tag{3.15}$$

Assumption 3.4. [138]

(1) There exists an unknown constant j-dimensional vector $\psi_2 \in (0,\infty)^j$ and a known function $\Pi_2 : \mathbf{R}^{2n} \times \mathbf{R}^{2n} \times (0,\infty)^j \to \mathbf{R}_+$ such that for all $X_1 \in \mathbf{R}^{2n}, X_2 \in \mathbf{R}^{2n}$,

$$\rho_2(X_1, X_2) = \Pi_2(X_1, X_2, \psi_2). \tag{3.16}$$

(2) The function $\Pi_2(X_1, X_2, \cdot) : (0,\infty)^j \to \mathbf{R}_+$ is C^2 (i.e., 2-times continuously differentiable) and concave (i.e., $-\Pi_2(X_1, X_2, \cdot)$ is convex), and it is nondecreasing with respect to each ψ_2.
(3) The function $\Pi_2(X_1, X_2, \cdot)$ and $\frac{\partial \Pi_2}{\partial \psi_2}(\cdot)$ are both continuous.

Then, the input torque τ can be constructed as

$$\tau(t) = -\alpha_2(x_4(t) + S_2x_3(t)){\Pi_2}^2(X, \hat{\psi}_2(t)) - \beta_2(x_4(t) + S_2x_3(t)). \tag{3.17}$$

where $\hat{\psi}_2$ is the adaptation law with the following form:

$$\dot{\hat{\psi}}_2(t) = \begin{cases} T_3^{-1}\frac{\partial \Pi_2^T}{\partial \psi_2}(X,\hat{\psi}_2)\|x_4 + S_2x_3\| - T_4\hat{\psi}_2, & \text{if} \|\frac{\partial \Pi_2^T}{\partial \psi_2}(X,\hat{\psi}_2)\|\|x_4 + S_2x_3\| > \varepsilon_2, \\ -T_4\hat{\psi}_2, & \text{if} \|\frac{\partial \Pi_2^T}{\partial \psi_2}(X,\hat{\psi}_2)\|\|x_4 + S_2x_3\| \leq \varepsilon_2, \end{cases} \tag{3.18}$$

$$\hat{\psi}_2(t_0) \in (0,\infty)^j.$$

Here, $T_3 = diag[T_{3i}]_{j\times j}, T_{3i} > 0$, $T_4 = diag[T_{4i}]_{j\times j}, T_{ji} > 0, i = 1, 2, \cdots, j, \varepsilon_2 > 0$.

Remark 3.3. The adaptive schemes (3.13) and (3.18) have leakage terms. It means that the adaptive parameters $\hat{\psi}_1$ and $\hat{\psi}_2$ do not always grow with time, since once the system performance is

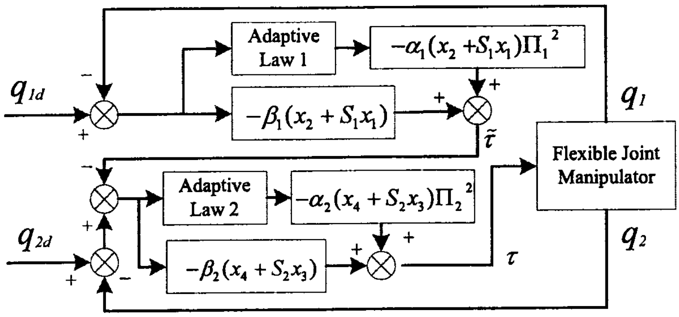

Figure 3.1: The block diagram of the adaptive robust control.

satisfactory in the sense that X_1 and X_2 are "small" and the gradients of $\Pi_1(X_1, \hat{\psi}_1)$ and $\Pi_2(X, \hat{\psi}_2)$ are small, then the second term in (3.13) and (3.18) becomes dominant. Therefore, $\hat{\psi}_1$ and $\hat{\psi}_2$ decrease. This result is dedicated to decreasing the size of uniform boundedness ball of the states and parameters to be estimated if we set T_2 and T_4 sufficiently large respectively. Moreover, the adaptive schemes are dead-zone type which means when the system states are small, the update laws will be in a simple form. Then the computation will decrease significantly.

The control strategy could be summarized as Figure 3.1, where q_{1d} and q_{2d} are the desired trajectories. It can be seen that both $\tilde{\tau}$ and τ have robust control parts and PD control parts. This work considers the regulation problem. The reference signal is set to be 0. For any non-zero regulation set point, a simple change of coordinate will suffice.

Assumption 3.3 and Assumption 3.4 state that, although the functions ρ_1 and ρ_2 which are related to the bounds of uncertainties are totally unknown, there exist known functions Π_1 and Π_2 with unknown parameters ψ_1 and ψ_2 as their arguments, so that they have the same values as those of ρ_1 and ρ_2. Here, ψ_1 and ψ_2 can be interpreted as the bounds of uncertainties. We do not know their values, but we do know that they exist. We can utilize (3.13) and (3.18) to estimate these two parameters.

Theorem 3.1. For the adaptation laws (3.13) and (3.18), if the initial condition $\hat{\psi}_1(t_0), \hat{\psi}_2(t_0) > 0$, then $\hat{\psi}_1(t), \hat{\psi}_2(t) > 0$, for all $t \geq t_0$, respectively.

Proof: Firstly, let us prove the $\hat{\psi}_1$ part of the theorem, the $\hat{\psi}_2$ part can be obtained in a similar way. For $\|\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1, x_2, \hat{\psi}_1)\|\|x_2 + S_1 x_1\| > \varepsilon_1$, Let U_1 be an open set in $\mathbf{R}^{2n+k}$, an element

$(X_1, \hat{\psi}_1) \in U_1$. We rewrite (3.13) as

$$\dot{\hat{\psi}}_1(t) = L_1(X_1(t), \hat{\psi}_1(t)) - T_2\hat{\psi}_1(t), \tag{3.19}$$

where

$$L_1(X_1(t), \hat{\psi}_1(t))) = T_1^{-1}\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1(t), x_2(t), \hat{\psi}_1(t))\|x_2(t) + S_1x_1(t)\|. \tag{3.20}$$

According to the continuity property of $\frac{\partial \Pi_1}{\partial \psi_1}(\cdot)$ in Assumption 3.3, $L_1(\cdot)$ is continuous on U_1, and there exists at least one solution of (3.19) passing through $(X_1(t_0), \hat{\psi}_1(t_0))$ with the given initial condition. Then, the solution of (3.19) can be given as

$$\begin{aligned} \hat{\psi}_1(t) &= e^{-T_2t}e^{T_2t_0}\hat{\psi}_1(t_0) + e^{-T_2t}\int_{t_0}^{t} e^{T_2s}L_1(X_1(s), \hat{\psi}_1(s))ds \\ &\geq e^{-T_2(t-t_0)}\hat{\psi}_1(t_0) + e^{-T_2t}\bar{L}_1\int_{t_0}^{t} e^{T_2s}ds > 0, \end{aligned} \tag{3.21}$$

where

$$\bar{L}_1 = \inf_{(X_1, \hat{\psi}_1)\in U_1} L_1(X_1, \hat{\psi}_1) \geq 0. \tag{3.22}$$

For $\|\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1, x_2, \hat{\psi}_1)\|\|x_2 + S_1x_1\| \leq \varepsilon_1$, we have

$$\dot{\hat{\psi}}_1(t) = -T_2\hat{\psi}_1(t), \tag{3.23}$$

the solution should be

$$\hat{\psi}_1(t) = e^{-T_2(t-t_0)}\hat{\psi}_1(t_0). \tag{3.24}$$

Therefore, if $\hat{\psi}_1(t_0) > 0$, we have $\hat{\psi}_1(t) > 0$ for all $t \geq t_0$. From the same point of view, it can be concluded that $\hat{\psi}_2(t) > 0$ for all $t \geq t_0$, if $\hat{\psi}_2(t_0) > 0$. Q.E.D.

Theorem 3.2. Suppose that Assumption 3.1–3.4 are satisfied, the control (3.17) renders the combined systems (3.5), (3.6), (3.13) and (3.18) practically stable.

3.4 The proof of the stability

From now on, we omit the arguments on uncertainty in $\hat{M}(x_1, \sigma_1)$, $\hat{C}(x_1, x_2, \sigma_1)$ and $\hat{G}(x_1, \sigma_1)$ etc. if no confusion arises. Otherwise it will be denoted.

Let $\tilde{\psi}_1 = \hat{\psi}_1 - \psi_1$, $\tilde{\psi}_2 = \hat{\psi}_2 - \psi_2$, $\psi = [\tilde{\psi}_1^T \quad \tilde{\psi}_2^T]^T$, choose the Lyapunov function candidates for the system as following:

$$V(X, \psi) = V_1(X_1, \tilde{\psi}_1) + V_2(X_2, \tilde{\psi}_2), \tag{3.25}$$

where

$$
\begin{aligned}
V_1(X_1,\tilde{\psi}_1) &= V_{X_1}(X_1) + V_{\psi_1}(\tilde{\psi}_1),\\
V_{X_1}(X_1) &= \frac{1}{2}(x_2+S_1x_1)^T\hat{M}(x_2+S_1x_1) + x_1^T\beta_1S_1x_1,\\
V_{\psi_1}(\tilde{\psi}_1) &= \frac{1}{2}\tilde{\psi}_1^T T_1\tilde{\psi}_1,\\
V_2(X_2,\tilde{\psi}_2) &= V_{X_2}(X_2) + V_{\psi_2}(\tilde{\psi}_2),\\
V_{X_2}(X_2) &= \frac{1}{2}(x_4+S_2x_3)^T J(x_4+S_2x_3) + x_3^T\beta_2S_2x_3,\\
V_{\psi_2}(\tilde{\psi}_2) &= \frac{1}{2}\tilde{\psi}_2^T T_3\tilde{\psi}_2.
\end{aligned}
\tag{3.26}
$$

To analyze the stability, we should prove $V(X)$ is a legitimate Lyapunov function candidate ($V(X)$ is positive definite and decrescent). Based on Assumption 3.2,

$$
\begin{aligned}
V_{X_1} &\geq \frac{1}{2}\underline{\eta}_1\|x_2+S_1x_1\|^2 + x_1^T\beta_1S_1x_1\\
&= \frac{1}{2}\sum_{i=1}^{n}[x_{1i} \quad \dot{x}_{1i}]\begin{bmatrix}\underline{\eta}_1S_{1i}^2+2\beta_1S_{1i} & \underline{\eta}_1S_{1i}\\ \underline{\eta}_1S_{1i} & \underline{\eta}_1\end{bmatrix}\begin{bmatrix}x_{1i}\\ \dot{x}_{1i}\end{bmatrix}\\
&=: \frac{1}{2}\sum_{i=1}^{n}[x_{1i} \quad \dot{x}_{1i}]\Omega_{1i}\begin{bmatrix}x_{1i}\\ \dot{x}_{1i}\end{bmatrix},
\end{aligned}
\tag{3.27}
$$

here,

$$
\Omega_{1i} = \begin{bmatrix}\underline{\eta}_1S_{1i}^2+2\beta_1S_{1i} & \underline{\eta}_1S_{1i}\\ \underline{\eta}_1S_{1i} & \underline{\eta}_1\end{bmatrix}.
\tag{3.28}
$$

Since Ω_{1i} is positive definite, V_1 is positive definite. Thus, we have

$$
V_{X_1} \geq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\min}(\Omega_{1i})(x_{1i}^2+x_{2i}^2).
\tag{3.29}
$$

Moreover,

$$
V_{\psi_1} \geq \frac{1}{2}\lambda_{\min}(T_1)\|\tilde{\psi}_1\|^2.
\tag{3.30}
$$

By combining (3.29) and (3.30), we have

$$
\begin{aligned}
V_1 &= V_{X_1} + V_{\psi_1}\\
&\geq \eta_0^{(1)}(\|X_1\|^2 + \|\tilde{\psi}_1\|^2)\\
&=: \eta_0^{(1)}\|Z_1\|^2,
\end{aligned}
\tag{3.31}
$$

where $\eta_0^{(1)} = \min\{\min_i\{\frac{1}{2}\lambda_{\min}(\Omega_{1i})\}, \frac{1}{2}\lambda_{\min}(T_1)\}, i = 1,2,\cdots,n$, $Z_1 = [X_1^T \quad \tilde{\psi}_1^T]^T$.

For the upper bound condition (3.3) of inertia matrix,

$$V_{x1} \leq (\bar{\eta}_1^{(1)} + \bar{\eta}_2^{(1)}\|x_1\| + \bar{\eta}_3^{(1)}\|x_1\|^2)\|x_2 + S_1 x_1\|^2 + x_1^T \beta_1 S_1 x_1. \tag{3.32}$$

Since

$$\|x_1\| \leq \|X_1\|, \tag{3.33}$$

$$\|x_1\|^2 \leq \|x_1\|^2 + \|x_2\|^2 = \|X_1\|^2, \tag{3.34}$$

let

$$\Psi_1 = \begin{bmatrix} S_1^2 & S_1 \\ S_1 & I \end{bmatrix}, \tag{3.35}$$

one has

$$\begin{aligned}\|x_2 + S_1 x_1\|^2 &= \begin{bmatrix} x_1^T & x_2^T \end{bmatrix} \begin{bmatrix} S_1^2 & S_1 \\ S_1 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\leq \lambda_{\max}(\Psi_1)(\|x_1\|^2 + \|x_2\|^2) \\ &=: \bar{S}_1 \|X_1\|^2,\end{aligned} \tag{3.36}$$

where $\bar{S}_1 = \lambda_{\max}(\Psi_1)$. Therefore, we have

$$V_{X_1} \leq \left(\bar{\eta}_1^{(1)} \bar{S}_1 + \beta_1 \lambda_{\max}(S_1)\right) \|X_1\|^2 + \bar{\eta}_2^{(1)} \bar{S}_1 \|X_1\|^3 + \bar{\eta}_3^{(1)} \bar{S}_1 \|X_1\|^4. \tag{3.37}$$

For

$$V_{\psi_1} \leq \lambda_{\max}(T_1)\|\tilde{\psi}_1\|^2, \tag{3.38}$$

we have

$$\begin{aligned} V_1 &= V_{X_1} + V_{\psi_1} \\ &\leq \left(\bar{\eta}_1^{(1)} \bar{S}_1 + \beta_1 \lambda_{\max}(S_1)\right) \|X_1\|^2 + \bar{\eta}_2^{(1)} \bar{S}_1 \|X_1\|^3 + \bar{\eta}_3^{(1)} \bar{S}_1 \|X_1\|^4 + \lambda_{\max}(T_1)\|\tilde{\psi}_1\|^2 \\ &\leq \eta_1^{(1)}\|Z_1\|^2 + \eta_2^{(1)}\|Z_1\|^3 + \eta_3^{(1)}\|Z_1\|^4, \end{aligned} \tag{3.39}$$

where $\eta_1^{(1)} = \max\{\left(\bar{\eta}_1^{(1)} \bar{S}_1 + \beta_1 \lambda_{\max}(S_1)\right), \lambda_{\max}(T_1)\}, \eta_2^{(1)} = \bar{\eta}_2^{(1)} \bar{S}_1, \eta_3^{(1)} = \bar{\eta}_3^{(1)} \bar{S}_1$. Similar to V_1, we can prove that V_2 is also positive definite and decrescent, and it follows

$$V_{X_2} \geq \frac{1}{2} \sum_{i=1}^{n} \lambda_{\min}(\underline{\Omega}_{2i})(x_{3i}^2 + x_{4i}^2), \tag{3.40}$$

$$\begin{aligned} \underline{\Omega}_{2i} &= \begin{bmatrix} \underline{\theta} S_{2i}^2 + 2\beta_2 S_{2i} & \underline{\theta} S_{2i} \\ \underline{\theta} S_{2i} & \underline{\theta} \end{bmatrix}, i = 1, 2, \cdots, n, \\ \underline{\theta} &= \lambda_{\min}(J), \end{aligned} \tag{3.41}$$

$$V_{\psi_2} \geq \frac{1}{2}\lambda_{\min}(T_3)\|\tilde{\psi}_2\|^2. \tag{3.42}$$

By combining (3.40) and (3.42), we have

$$\begin{aligned} V_2 &= V_{X_2} + V_{\psi_2} \\ &\geq \eta_0^{(2)}(\|X_2\|^2 + \|\tilde{\psi}_2\|^2) \\ &=: \eta_0^{(2)}\|Z_2\|^2, \end{aligned} \tag{3.43}$$

where $\eta_0^{(2)} = \min\{\min_i\{\frac{1}{2}\lambda_{\min}(\underline{\Omega}_{2i})\}, \frac{1}{2}\lambda_{\min}(T_3)\}, i = 1, 2, \cdots, n$, $Z_2 = [X_2^T \quad \tilde{\psi}_2^T]^T$.

$$V_{X_2} \leq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\max}(\bar{\Omega}_{2i})(x_{3i}^2 + x_{4i}^2), \tag{3.44}$$

$$\begin{aligned} \bar{\Omega}_{2i} &= \begin{bmatrix} \bar{\theta}S_{2i}^2 + 2\beta_2 S_{2i} & \bar{\theta}S_{2i} \\ \bar{\theta}S_{2i} & \bar{\theta} \end{bmatrix}, i = 1, 2, \cdots, n, \\ \bar{\theta} &= \lambda_{\max}(J), \end{aligned} \tag{3.45}$$

$$V_{\psi_2} \leq \frac{1}{2}\lambda_{\max}(T_3)\|\tilde{\psi}_2\|^2. \tag{3.46}$$

By combining (3.44) and (3.46), we have

$$\begin{aligned} V_2 &= V_{X_2} + V_{\psi_2} \\ &\leq \eta_1^{(2)}(\|X_2\|^2 + \|\tilde{\psi}_2\|^2) \\ &=: \eta_1^{(2)}\|Z_2\|^2, \end{aligned} \tag{3.47}$$

where $\eta_1^{(2)} = \max\{\max_i\{\frac{1}{2}\lambda_{\max}(\bar{\Omega}_{2i}), \frac{1}{2}\lambda_{\max}(T_3)\}, i = 1, 2, \cdots, n$. Therefore, we have

$$\eta_0\|Z\|^2 \leq V \leq \eta_1\|Z\|^2 + \eta_2\|Z\|^3 + \eta_3\|Z\|^4, \tag{3.48}$$

where $\eta_0 = \min\{\eta_0^{(2)}, \eta_0^{(2)}\}, \eta_1 = \max\{\eta_1^{(2)}, \eta_1^{(2)}\}, \eta_2 = \eta_2^{(2)}, \eta_3 = \eta_3^{(2)}, Z = [Z_1^T \quad Z_2^T]^T$. This in turn means $V(X)$ is decrescent for all $X \in \mathbf{R}^{4n+j+k}$, or, we can say that $V(X)$ is a legitimate Lyapunov function candidate for all FJM system.

Taking first derivative of $V_1(X_1)$ along the trajectory of the controlled system yields

$$\begin{aligned} \dot{V}_{X_1} =& (x_2 + S_1 x_1)^T \hat{M}(\dot{x}_2 + S_1 x_2) + \frac{1}{2}(x_2 + S_1 x_1)^T \dot{\hat{M}}(x_2 + S_1 x_1) + 2x_1^T \beta_2 S_1 x_2 \\ =& (x_2 + S_1 x_1)^T (\frac{1}{2}\dot{\hat{M}} x_2 + \frac{1}{2}\dot{\hat{M}} S_1 x_1 - \hat{C} x_2 - \hat{G} - x_1 + \hat{D} S_1 \dot{x} + \tilde{\tau} + x_3) \\ &+ 2x_1^T \beta_2 S_1 x_2. \end{aligned} \tag{3.49}$$

From (3.9), it can be seen that

$$\dot{V}_{X_1} = (x_2 + S_1x_1)^T\Phi_1 + (x_2 + S_1x_1)^T\tilde{\tau} + (x_2 + S_1x_1)^T x_3 + 2x_1^T\beta_2S_1x_2. \tag{3.50}$$

According to (3.8) and (3.11), we substitute the control (3.12), then

$$\begin{aligned}
\dot{V}_{X_1} =&(x_2 + S_1x_1)^T\Phi_1 + (x_2 + S_1x_1)^T[-\alpha_1(x_2 + S_1x_1)\Pi_1^2(X_1, \hat{\psi}_1) - \beta_1(x_2 + S_1x_1)]\\
&+ (x_2 + S_1x_1)^T x_3 + 2x_1^T\beta_2S_1x_2\\
\leq&\|x_2 + S_1x_1\|\Pi_1(X_1, \psi_1) + (x_2 + S_1x_1)^T[-\alpha_1(x_2 + S_1x_1)\Pi_1(X_1, \hat{\psi}_1)^2\\
&- \beta_1(x_2 + S_1x_1)] + (x_2 + S_1x_1)^T x_3 + 2x_1^T\beta_2S_1x_2\\
=&\|x_2 + S_1x_1\|\Pi_1(X_1, \psi_1) - \alpha_1\|x_2 + S_1x_1\|^2\Pi_1^2(X_1, \hat{\psi}_1)\\
&- x_2^T\beta_1x_2 - x_1^T\beta_1S_1^2x_1 + (x_2 + S_1x_1)^T x_3\\
\leq& - \lambda_{x1}\|X_1\|^2 + \|x_2 + S_1x_1\|\Pi_1(X_1, \psi_1) - \alpha_1\|x_2 + S_1x_1\|^2\Pi_1^2(X_1, \hat{\psi}_1)\\
&+ (x_2 + S_1x_1)^T x_3
\end{aligned} \tag{3.51}$$

For $\|\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1, x_2, \hat{\psi}_1)\|\|x_2 + S_1x_1\| > \varepsilon_1$, based on (3.13), the derivative of V_{ψ_1} follows

$$\begin{aligned}
\dot{V}_{\psi_1} &= (\hat{\psi}_1 - \psi_1)^T T_1\dot{\hat{\psi}}_1\\
&= (\hat{\psi}_1 - \psi_1)^T\frac{\partial \Pi_1^T}{\partial \psi_1}(X_1, \hat{\psi}_1)\|x_2 + S_1x_1\| - (\hat{\psi}_1 - \psi_1)^T T_1T_2\hat{\psi}_1.
\end{aligned} \tag{3.52}$$

From Assumption 3.3, we know that $-\Pi_1(X_1, \cdot)$ is convex for all $X_1 \in \mathbf{R}^{2n}$, this leads to

$$\frac{\partial \Pi_1}{\partial \psi_1}(X_1, \hat{\psi}_1)(\hat{\psi}_1 - \psi_1) \leq \Pi_1(X_1, \hat{\psi}_1) - \Pi_1(X_1, \psi_1), \tag{3.53}$$

therefore, we have

$$\dot{V}_{\psi_1} \leq (\Pi_1(X_1, \hat{\psi}_1) - \Pi_1(X_1, \psi_1))\|x_2 + S_1x_1\| - (\hat{\psi}_1 - \psi_1)^T T_1T_2\hat{\psi}_1. \tag{3.54}$$

By combining (3.51) and (3.54), we have

$$\begin{aligned}
\dot{V}_1 =&\|x_2 + S_1x_1\|\Pi_1(X_1, \psi_1) - \alpha_1\|x_2 + S_1x_1\|^2\Pi_1^2(X_1, \hat{\psi}_1) - \lambda_{x1}\|X_1\|^2 + (x_2 + S_1x_1)^T x_3\\
&+ (\Pi_1(X_1, \hat{\psi}_1) - \Pi_1(X_1, \psi_1))\|x_2 + S_1x_1\| - (\hat{\psi}_1 - \psi_1)^T T_1T_2\hat{\psi}_1\\
=&\|x_2 + S_1x_1\|\Pi_1(X_1, \hat{\psi}_1) - \alpha_1\|x_2 + S_1x_1\|^2\Pi_1^2(X_1, \hat{\psi}_1) - \lambda_{x1}\|X_1\|^2\\
&+ (x_2 + S_1x_1)^T x_3 - (\hat{\psi}_1 - \psi_1)^T T_1T_2\hat{\psi}_1\\
\leq&\frac{1}{4\alpha_1} - \lambda_{x1}\|X_1\|^2 + (x_2 + S_1x_1)^T x_3 - (\hat{\psi}_1 - \psi_1)^T T_1T_2(\hat{\psi}_1 - \psi_1 + \psi_1).\\
\leq&\frac{1}{4\alpha_1} - \lambda_{x1}\|X_1\|^2 + (x_2 + S_1x_1)^T x_3 - \lambda_{\min}(T_1T_2)\|\tilde{\psi}_1\|^2 + \|T_1T_2\psi_1\|\|\tilde{\psi}_1\|
\end{aligned} \tag{3.55}$$

For $\|\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1, x_2, \hat{\psi}_1)\|\|x_2 + S_1 x_1\| \le \varepsilon_1$,

$$
\begin{aligned}
\dot{V}_{X_1} \le & \|x_2 + S_1 x_1\|\Pi_1(X_1, \psi_1) - \alpha_1\|x_2 + S_1 x_1\|^2 \Pi_1^2(X_1, \hat{\psi}_1) - \lambda_{x1}\|X_1\|^2 \\
& + (x_2 + S_1 x_1)^T x_3 \\
= & \|x_2 + S_1 x_1\|\Pi_1(X_1, \psi_1) - \|x_2 + S_1 x_1\|\Pi_1(X_1, \hat{\psi}_1) + \|x_2 + S_1 x_1\|\Pi_1(X_1, \hat{\psi}_1) \\
& - \alpha_1\|x_2 + S_1 x_1\|^2 \Pi_1^2(X_1, \hat{\psi}_1) - \lambda_{x1}\|X_1\|^2 + (x_2 + S_1 x_1)^T x_3 \\
\le & \|x_2 + S_1 x_1\|\frac{\partial \Pi_1^T}{\partial \psi_1}(X_1, \hat{\psi}_1)(\psi_1 - \hat{\psi}_1) + \frac{1}{4\alpha_1} - \lambda_{x1}\|X_1\|^2 + (x_2 + S_1 x_1)^T x_3 \\
\le & \varepsilon_1\|\tilde{\psi}_1\| + \frac{1}{4\alpha_1} - \lambda_{x1}\|X_1\|^2 + (x_2 + S_1 x_1)^T x_3.
\end{aligned}
\tag{3.56}
$$

Then, according to (3.13), the derivative of V_{ψ_1} could be

$$
\begin{aligned}
\dot{V}_{\psi_1} &= (\hat{\psi}_1 - \psi_1)^T T_1 \dot{\hat{\psi}}_1 \\
&= -(\hat{\psi}_1 - \psi_1)^T T_1 T_2 (\hat{\psi}_1 - \psi_1 + \psi_1) \\
&= -(\hat{\psi}_1 - \psi_1)^T T_1 T_2 (\hat{\psi}_1 - \psi_1) - (\hat{\psi}_1 - \psi_1)^T T_1 T_2 \psi_1 \\
&\le -\lambda_{\min}(T_1 T_2)\|\tilde{\psi}_1\|^2 + \|T_1 T_2 \psi_1\|\|\tilde{\psi}_1\|.
\end{aligned}
\tag{3.57}
$$

According to the inequalities $ab \le \frac{1}{2}(a^2 + b^2),\ a, b \in \mathbf{R}$ and $\|x_3\|^2 \le \|X_2\|^2$, we have

$$
\begin{aligned}
(x_2 + S_1 x_1)^T x_3 &\le \|x_2 + S_1 x_1\|\|x_3\| \\
&\le \frac{1}{2}\omega_1\|x_2 + S_1 x_1\|^2 + \frac{1}{2}\omega_1^{-1}\|x_3\|^2 \\
&\le \frac{1}{2}\omega_1\|x_2 + S_1 x_1\|^2 + \frac{1}{2}\omega_1^{-1}\|X_2\|^2,
\end{aligned}
\tag{3.58}
$$

where $\omega_1 > 0$ is a constant.

Let

$$
\begin{aligned}
& c_1 = \lambda_{x1} - \frac{1}{2}\omega_1 \bar{S}_1, d_1 = \lambda_{\min}(T_1 T_2), e_1 = \|T_1 T_2 \psi_1\| + \varepsilon_1, \\
& A_1 = \begin{bmatrix} c_1 & 0 \\ 0 & d_1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & e_1 \end{bmatrix}.
\end{aligned}
\tag{3.59}
$$

Combining (3.55), (3.56), (3.57) and (3.58), one gets

$$\begin{aligned}
\dot{V}_1 \leq& \varepsilon_1\|\tilde{\psi}_1\| + \frac{1}{4\alpha_1} - \lambda_{x1}\|X_1\|^2 + (\frac{1}{2}\omega_1\|x_2 + S_1x_1\|^2 + \frac{1}{2}\omega_1^{-1}\|X_2\|^2) \\
&- \lambda_{\min}(T_1T_2)\|\tilde{\psi}_1\|^2 + \|T_1T_2\psi_1\|\|\tilde{\psi}_1\| \\
=& \frac{1}{4\alpha_1} - (\lambda_{x1} - \frac{1}{2}\omega_1\bar{S}_1)\|X_1\|^2 + \frac{1}{2}\omega_1^{-1}\|X_2\|^2 - \lambda_{\min}(T_1T_2)\|\tilde{\psi}_1\|^2 \\
&+ (\|T_1T_2\psi_1\| + \varepsilon_1)\|\tilde{\psi}_1\| \\
=:& \frac{1}{4\alpha_1} - c_1\|X_1\|^2 - d_1\|\tilde{\psi}_1\|^2 + e_1\|\tilde{\psi}_1\| + \frac{1}{2}\omega_1^{-1}\|X_2\|^2 \\
=& \frac{1}{4\alpha_1} - \begin{bmatrix} \|X_1\| & \|\tilde{\psi}_1\| \end{bmatrix}\begin{bmatrix} c_1 & 0 \\ 0 & d_1 \end{bmatrix}\begin{bmatrix} \|X_1\| \\ \|\tilde{\psi}_1\| \end{bmatrix} + \begin{bmatrix} 0 & e_1 \end{bmatrix}\begin{bmatrix} \|X_1\| \\ \|\tilde{\psi}_1\| \end{bmatrix} + \frac{1}{2}\omega_1^{-1}\|X_2\|^2 \\
\leq& \frac{1}{4\alpha_1} - \lambda_{\min}(A_1)\|Z_1\|^2 + \|B_1\|\|Z_1\| + \frac{1}{2}\omega_1^{-1}\|X_2\|^2.
\end{aligned} \tag{3.60}$$

Next, the derivative of V_2 is given by

$$\dot{V}_2 = \dot{V}_{X_2} + \dot{V}_{\psi_2}. \tag{3.61}$$

According to (3.6) and (3.15),

$$\begin{aligned}
\dot{V}_{X_2} =& (x_4 + S_2x_3)^T J(\dot{x}_4 + S_2x_4) + \frac{1}{2}(x_4 + S_2x_3)^T \dot{J}(x_4 + S_2x_3) + 2x_3^T\beta_2S_2x_4 \\
=& (x_4 + S_2x_3)^T(-J\ddot{\tilde{\tau}} - Fx_4 - F\dot{\tilde{\tau}} - Kx_3 + Kx_1 - K\tilde{\tau} \\
&+ JS_2x_4 + \frac{1}{2}\dot{J}x_4 + \frac{1}{2}\dot{J}S_2x_3 + \tau) + x_3^T\beta_2S_2x_4 \\
=& (x_4 + S_2x_3)^T\Phi_2 + (x_4 + S_2x_3)^T\tau + 2x_3^T\beta_2S_2x_4.
\end{aligned} \tag{3.62}$$

For $\|\frac{\partial\Pi_2^T}{\partial\psi_2}(X, \hat{\psi}_2)\|\|x_4 + S_2x_3\| > \varepsilon_2$, with Assumption 3.4 and substituting the control (3.17), we have

$$\begin{aligned}
\dot{V}_{X_2} \leq& \|x_4 + S_2x_3\|\rho_2 + (x_4 + S_2x_3)^T\tau + 2x_3^T\beta_2S_2x_4 \\
=& \|x_4 + S_2x_3\|\Pi_2(X, \psi_2) + (x_4 + S_2x_3)^T[-\alpha_2(x_4 + S_2x_3)\Pi_2{}^2(X, \hat{\psi}_2) \\
&- \beta_2(x_4 + S_2x_3)] + 2x_3^T\beta_2S_2x_4 \\
=& \|x_4 + S_2x_3\|\Pi_2(X, \psi_2) - \alpha_2\|x_4 + S_2x_3\|^2\Pi_2{}^2(X, \hat{\psi}_2) - x_4^T\beta_2x_4 - x_3^T\beta_2S_2^2x_3 \\
\leq& \|x_4 + S_2x_3\|\Pi_2(X, \psi_2) - \alpha_2\|x_4 + S_2x_3\|^2\Pi_2{}^2(X, \hat{\psi}_2) - \lambda_{x3}\|X_2\|^2,
\end{aligned} \tag{3.63}$$

where $\lambda_{x3} = \min\{\beta_2, \lambda_{\min}(\beta_2S_2^2)\}$.

Concerning $\dot{V}_{\psi_2}$, it follows from (3.18)

$$\begin{aligned}
\dot{V}_{\psi_2} &= (\hat{\psi}_2 - \psi_2)^T T_3\dot{\hat{\psi}}_2 \\
&= (\hat{\psi}_2 - \psi_2)^T\frac{\partial\Pi_2^T}{\partial\psi_2}(X, \hat{\psi}_2)\|x_4 + S_1x_3\| - (\hat{\psi}_2 - \psi_2)^T T_3T_4\hat{\psi}_2.
\end{aligned} \tag{3.64}$$

Since $-\Pi_2(X,\cdot)$ is convex with all $X \in \mathbf{R}^{4n}$, this leads to

$$\frac{\partial \Pi_2}{\partial \psi_2}(X,\hat{\psi}_2)(\hat{\psi}_2-\psi_2) \le \Pi_2(X,\hat{\psi}_2)-\Pi_2(X,\psi_2), \tag{3.65}$$

therefore, we have

$$\dot{V}_{\psi_2} \le (\Pi_2(X,\hat{\psi}_2-\Pi_2(X,\psi_2))\|x_4+S_2x_3\|-(\hat{\psi}_2-\psi_2)^T T_3T_4\hat{\psi}_2. \tag{3.66}$$

By combining (3.63) and (3.66), we obtain

$$\begin{aligned}
\dot{V}_2 \le& \|x_4+S_1x_3\|\Pi_2(X,\psi_2)-\alpha_2\|x_4+S_2x_3\|^2\Pi_2^2(X,\hat{\psi}_2)-\lambda_{x3}\|X_2\|^2\\
&+(\Pi_2(X,\hat{\psi}_2)-\Pi_2(X,\psi_2))\|x_4+S_2x_3\|-(\hat{\psi}_2-\psi_2)^T T_3T_4\hat{\psi}_2\\
=&\|x_4+S_2x_3\|\Pi_2(X,\hat{\psi}_2)-\alpha_2\|x_4+S_2x_3\|^2\Pi_2^2(X,\hat{\psi}_2)-\lambda_{x3}\|X_2\|^2-(\hat{\psi}_2-\psi_2)^T T_3T_4\hat{\psi}_2\\
\le&\frac{1}{4\alpha_2}-\lambda_{x3}\|X_2\|^2-(\hat{\psi}_2-\psi_2)^T T_3T_4(\hat{\psi}_2-\psi_2+\psi_2)\\
\le&\frac{1}{4\alpha_2}-\lambda_{x3}\|X_2\|^2-\lambda_{\min}(T_3T_4)\|\tilde{\psi}_2\|^2+\|T_3T_4\psi_2\|\|\tilde{\psi}_2\|.
\end{aligned} \tag{3.67}$$

For $\|\frac{\partial \Pi_2^T}{\partial \psi_2}(X,\hat{\psi}_2)\|\|x_4+S_2x_3\| \le \varepsilon_2$, according to (3.65), one has

$$\begin{aligned}
\dot{V}_{X_2} \le& \|x_4+S_2x_3\|\Pi_2(X,\psi_2)-\alpha_2\|x_4+S_2x_3\|^2{\Pi_2}^2(X,\hat{\psi}_2)-\lambda_{x3}\|X_2\|^2\\
=&\|x_4+S_2x_3\|\Pi_2(X,\psi_2)-\|x_4+S_2x_3\|\Pi_2(X,\hat{\psi}_2)+\|x_4+S_2x_3\|\Pi_2(X,\hat{\psi}_2)\\
&-\alpha_2\|x_4+S_2x_3\|^2{\Pi_2}^2(X,\hat{\psi}_2)-\lambda_{x3}\|X_2\|^2\\
\le&\|x_4+S_2x_3\|\frac{\partial \Pi_2^T}{\partial \psi_2}(X,\hat{\psi}_2)(\psi_2-\hat{\psi}_2)+\frac{1}{4\alpha_2}-\lambda_{x3}\|X_2\|^2\\
\le&\varepsilon_2\|\tilde{\psi}_2\|+\frac{1}{4\alpha_2}-\lambda_{x3}\|X_2\|^2.
\end{aligned} \tag{3.68}$$

Concerning the derivative of V_{ψ_2}, we have

$$\begin{aligned}
\dot{V}_{\psi_2} =& (\hat{\psi}_2-\psi_2)^T T_3\dot{\hat{\psi}}_2\\
=&-(\hat{\psi}_2-\psi_2)^T T_3T_4(\hat{\psi}_2-\psi_2+\psi_2)\\
\le&-\lambda_{\min}(T_3T_4)\|\tilde{\psi}_2\|^2+\|T_3T_4\psi_2\|\|\tilde{\psi}_2\|
\end{aligned} \tag{3.69}$$

With the results of (3.67), (3.68) and (3.69), we have

$$\begin{aligned}
\dot{V}_2 \le& \varepsilon_2\|\tilde{\psi}_2\|+\frac{1}{4\alpha_2}-\lambda_{x3}\|X_2\|^2-\lambda_{\min}(T_3T_4)\|\tilde{\psi}_2\|^2+\|T_3T_4\psi_2\|\|\tilde{\psi}_2\|\\
=&\frac{1}{4\alpha_2}-\lambda_{x3}\|X_2\|^2-\lambda_{\min}(T_3T_4)\|\tilde{\psi}_2\|^2+(\|T_3T_4\psi_2\|+\varepsilon_2)\|\tilde{\psi}_2\|.
\end{aligned} \tag{3.70}$$

Therefore, the total derivative of V is given by

$$
\begin{aligned}
\dot{V} =&\dot{V}_1 + \dot{V}_2 \\
=&\frac{1}{4\alpha_1} - \lambda_{\min}(A_1)\|Z_1\|^2 + \|B_1\|\|Z_1\| + \frac{1}{2}\omega_1^{-1}\|X_2\|^2 \\
&+ \frac{1}{4\alpha_2} - \lambda_{x3}\|X_2\|^2 - \lambda_{\min}(T_3T_4)\|\tilde{\psi}_2\|^2 + (\|T_3T_4\psi_2\| + \varepsilon_2)\|\tilde{\psi}_2\| \\
\leq&\frac{1}{4\alpha_1} + \frac{1}{4\alpha_2} - \lambda_{\min}(A_1)\|Z_1\|^2 + \|B_1\|\|Z_1\| - (\lambda_{x3} - \frac{1}{2}\omega_1^{-1})\|X_2\|^2 \\
&- \lambda_{\min}(T_3T_4)\|\tilde{\psi}_2\|^2 + (\|T_3T_4\psi_2\| + \varepsilon_2)\|\tilde{\psi}_2\| \\
=&\frac{1}{4\alpha_1} + \frac{1}{4\alpha_2} - \lambda_{\min}(A_1)\|Z_1\|^2 + \|B_1\|\|Z_1\| \\
&- \begin{bmatrix} \|X_2\| & \|\tilde{\psi}_2\| \end{bmatrix} \begin{bmatrix} c_2 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} \|X_2\| \\ \|\tilde{\psi}_2\| \end{bmatrix} + \begin{bmatrix} 0 & e_2 \end{bmatrix} \begin{bmatrix} \|X_2\| \\ \|\tilde{\psi}_2\| \end{bmatrix} \\
\leq&\frac{1}{4\alpha_1} + \frac{1}{4\alpha_2} - \lambda_{\min}(A_1)\|Z_1\|^2 + \|B_1\|\|Z_1\| - \lambda_{\min}(A_2)\|Z_2\|^2 + \|B_2\|\|Z_2\|,
\end{aligned}
\tag{3.71}
$$

where

$$
\begin{aligned}
c_2 &= \lambda_{x3} - \frac{1}{2}\omega_1^{-1}, \\
d_2 &= \lambda_{\min}(T_3T_4), \\
e_2 &= \|T_3T_4\psi_2\| + \varepsilon_2, \\
A_2 &= \begin{bmatrix} c_2 & 0 \\ 0 & d_2 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 0 & e_2 \end{bmatrix}.
\end{aligned}
\tag{3.72}
$$

Then we have

$$
\begin{aligned}
\dot{V} \leq& - \begin{bmatrix} \|Z_1\| & \|Z_2\| \end{bmatrix} \begin{bmatrix} \lambda_{\min}(A_1) & 0 \\ 0 & \lambda_{\min}(A_2) \end{bmatrix} \begin{bmatrix} \|Z_1\| \\ \|Z_2\| \end{bmatrix} + \begin{bmatrix} \|B_1\| & \|B_2\| \end{bmatrix} \begin{bmatrix} \|Z_1\| \\ \|Z_2\| \end{bmatrix} + \alpha \\
\leq& -\lambda_{\min}(A)\|Z\|^2 + \|B\|\|Z\| + \alpha,
\end{aligned}
\tag{3.73}
$$

where,

$$
\begin{aligned}
A &= \begin{bmatrix} \lambda_{\min}(A_1) & 0 \\ 0 & \lambda_{\min}(A_2) \end{bmatrix}, \\
B &= \begin{bmatrix} \|B_1\| & \|B_2\| \end{bmatrix}, \\
\alpha &= \frac{1}{4\alpha_1} + \frac{1}{4\alpha_2}.
\end{aligned}
\tag{3.74}
$$

If we choose suitable β_1, β_2 such that $c_1 > 0, c_2 > 0$, then A is positive definite, thereby

$$\dot{V} < 0, \tag{3.75}$$

for all $\|Z\|$ such that

$$\lambda_{\min}(A)\|Z\|^2 - \|B\|\|Z\| - \alpha > 0. \tag{3.76}$$

With the standard form shown in [130], we have the uniform boundedness performance. That is, given any $r > 0$ with $\|Z(t_0)\| \leq r$, where t_0 is the initial time, there exists a $d(r)$ given by

$$\begin{aligned} d(r) &= \begin{cases} r\sqrt{\frac{\eta_1+\eta_2 r+\eta_3 r^2}{\eta_0}} & \text{if} \quad r > R \\ R\sqrt{\frac{\eta_1+\eta_2 R+\eta_3 R^2}{\eta_0}} & \text{if} \quad r \leq R \end{cases}, \\ R &= \frac{\|B\| + \sqrt{\|B\|^2 + 4\lambda_{\min}(A)\alpha}}{2\lambda_{\min}(A)}, \end{aligned} \tag{3.77}$$

such that $\|Z(t)\| \leq d(r)$ for all $t \geq t_0$. Uniform ultimate boundedness also follows. That is, given any $\bar{d}$ with

$$\bar{d} > R\sqrt{\frac{\eta_1 + \eta_2 R + \eta_3 R^2}{\eta_0}}, \tag{3.78}$$

we have $\|Z(t)\| \leq \bar{d}$ for all $t \geq t_0 + T(\bar{d}, r)$, with

$$\begin{aligned} T(\bar{d}, r) &= \begin{cases} 0 & \text{if} \quad r \leq \bar{R} \\ \frac{\eta_1 r^2+\eta_2 r^3+\eta_3 r^4-\eta_0 \bar{R}^2}{\lambda_{\min}(A)\bar{R}^2-\alpha} & \text{if} \quad r > \bar{R} \end{cases} \\ \bar{R} &= \xi^{-1}(\eta_0 \bar{d}^2), \end{aligned} \tag{3.79}$$

where the function $\xi(\cdot)$ is given by

$$\xi(\theta) = \eta_1\theta^2 + \eta_2\theta^3 + \eta_3\theta^4. \tag{3.80}$$

Q.E.D.

3.5 Performance analysis of original system

So far we have proved the uniform boundedness and uniform ultimate boundedness properties of the transformed system. However, original system is described by $q, \dot{q}_1, q_2, \dot{q}_2$ rather than Z. It is necessary to analyze the performance of the original system. For convenience, let $\tilde{q}_1 = [q_1^T \ \dot{q}_1^T]^T$, $\tilde{q}_2 = [q_2^T \ \dot{q}_2^T]^T$, $\tilde{q} = [\tilde{q}_1^T \ \tilde{q}_2^T]^T$. Suppose Z is bounded by a constant δ,

$$\|Z\| \leq \delta. \tag{3.81}$$

Then we have

$$\|Z\|^2 = \|X_1\|^2 + \|\tilde{\psi}_1\|^2 + \|X_2\|^2 + \|\tilde{\psi}_2\|^2 \leq \delta^2. \tag{3.82}$$

This implies

$$\|X_1\| \leq \delta,\ \|\tilde{\psi}_1\| \leq \delta,\ \|X_2\| \leq \delta,\ \|\tilde{\psi}_2\| \leq \delta. \tag{3.83}$$

By (2.18),

$$\|\tilde{q}_1\| = \|X_1\| \leq \delta, \quad \|\tilde{q}_2 - \begin{bmatrix}\tilde{\tau}^T & \dot{\tilde{\tau}}^T\end{bmatrix}^T\| \leq \delta, \tag{3.84}$$

$$\tilde{q}_2 = X_2 + \begin{bmatrix}\tilde{\tau} \\ \dot{\tilde{\tau}}\end{bmatrix}. \tag{3.85}$$

According to (3.12), we have

$$\|\tilde{\tau}\| \leq \alpha_1 \|x_2 + S_1 x_1\| \Pi_1(x_1, x_2, \hat{\psi}_1)^2 + \beta_1 \|x_2 + S_1 x_1\|. \tag{3.86}$$

By the continuity property of $\Pi_1(x_1, x_2, \hat{\psi}_1)$, if $\|X_1\| \leq \delta$, $\|\tilde{\psi}_1\| \leq \delta$, there exists a constant $\hat{\rho}_1(\delta)$ such that

$$\Pi_1(x_1, x_2, \hat{\psi}_1) \leq \hat{\rho}_1(\delta). \tag{3.87}$$

Therefore,

$$\begin{aligned}\|\tilde{\tau}\| \leq& \alpha_1 \|x_2 + S_1 x_1\| \hat{\rho}_1^2(\delta) + \beta_1 \|x_2 + S_1 x_1\| \\ \leq& \alpha_1 \hat{\rho}_1^2(\delta)(1 + \|S_1\|)\delta + \beta_1(1 + \|S_1\|)\delta \\ =:& p_1(\delta) + h_1\delta,\end{aligned} \tag{3.88}$$

where

$$\begin{aligned}p_1(\delta) =& \alpha_1 \hat{\rho}_1^2(\delta)(1 + \|S_1\|)\delta, \\ h_1 =& \beta_1(1 + \|S_1\|).\end{aligned} \tag{3.89}$$

From (3.5) we obtain

$$\begin{aligned}\|\dot{x}_2\| &= \|\hat{M}^{-1}(-\hat{C}x_2 - \hat{G} - x_1 + x_3 + \tilde{\tau})\| \\ &\leq \|\hat{M}^{-1}\hat{C}\|\|x_2\| + \|\hat{M}^{-1}\hat{G}\| + \|\hat{M}^{-1}\|\|x_1 + x_3 + \tilde{\tau}\| \\ &\leq \sup_{\|X_1\|\leq\delta} \|\hat{M}^{-1}\hat{C}\|\delta + \sup_{\|x_1\|\leq\delta} \|\hat{M}^{-1}\hat{G}\| + \sup_{\|x_1\|\leq\delta} \|\hat{M}^{-1}\|(\delta + \delta + p_1(\delta) + h_1\delta) \\ &=: h_2\delta + h_3 + h_4(2\delta + p_1(\delta) + h_1\delta) \\ &= h_4 p_1(\delta) + (h_1 h_4 + h_2 + 2h_4)\delta + h_3 \\ &=: p_2(\delta) + h_5\delta + h_3,\end{aligned} \tag{3.90}$$

here

$$\begin{aligned}h_2 = &\sup_{\|X_1\|\le\delta} \|\hat{M}^{-1}\hat{C}\| < \infty,\\ h_3 = &\sup_{\|x_1\|\le\delta} \|\hat{M}^{-1}\hat{G}\| < \infty,\\ h_4 = &\sup_{\|x_1\|\le\delta} \|\hat{M}^{-1}\| < \infty,\\ h_5 =&h_1h_4 + h_2 + 2h_4,\\ p_2(\delta) = &h_4 p_1(\delta).\end{aligned} \tag{3.91}$$

The derivative of $\Pi_1(x_1, x_2, \hat{\psi}_1)$ is bounded by

$$\begin{aligned}\dot{\Pi}_1(x_1, x_2, \hat{\psi}_1) \le &\|\frac{\partial\Pi_1^T}{\partial x_1}x_2\| + \|\frac{\partial\Pi_1^T}{\partial x_2}\dot{x}_2\| + \|\frac{\partial\Pi_1^T}{\partial\psi_1}\dot{\hat{\psi}}_1\|\\ \le &\sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial x_1}\|\|x_2\| + \sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial x_2}\|\|\dot{x}_2\| + \sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial\psi_1}\|\|\dot{\hat{\psi}}_1\|.\end{aligned} \tag{3.92}$$

According to the adaptive law given in (3.13),

$$\begin{aligned}&\sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial\psi_1}\|\|\dot{\hat{\psi}}_1\|\\ &\le \sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial\psi_1}\|(\|T_1^{-1}\| \sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial\psi_1}\|\|x_2 + S_1x_1\| + \|T_2\|\|\hat{\psi}_1\|)\\ &\le h_6\delta + h_7,\end{aligned} \tag{3.93}$$

with

$$\begin{aligned}h_6 =&(\sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial\psi_1}\|)^2\|T_1^{-1}\|(1 + \|S_1\|) < \infty,\\ h_7 = &\sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial\psi_1}\|\|T_2\|\|\hat{\psi}_1\| < \infty.\end{aligned} \tag{3.94}$$

Then we have

$$\begin{aligned}\dot{\Pi}_1(x_1, x_2, \hat{\psi}_1) \le &h_8\delta + h_9(p_2(\delta) + h_5\delta + h_3) + h_6\delta + h_7\\ = &h_9p_2(\delta) + (h_6 + h_8 + h_5h_9)\delta + (h_3h_9 + h_7),\end{aligned} \tag{3.95}$$

$$\begin{aligned}h_8 = &\sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial x_1}\| < \infty,\\ h_9 = &\sup_{\substack{\|X_1\|\le\delta\\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial\Pi_1^T}{\partial x_2}\| < \infty.\end{aligned} \tag{3.96}$$

Furthermore, derivative of $\tilde{\tau}$ is bounded by

$$
\begin{aligned}
\|\dot{\tilde{\tau}}\| \leq & \alpha_1 \|\dot{x}_2 + S_1 x_2\| \Pi_1(x_1, x_2, \hat{\psi}_1)^2 + 2\alpha_1 \|x_2 + S_1 x_1\| \Pi_1(x_1, x_2, \hat{\psi}_1) \dot{\Pi}_1(x_1, x_2, \hat{\psi}_1) \\
& + \beta_1 \|\dot{x}_2 + S_1 x_2\| \\
\leq & \alpha_1 (p_2(\delta) + h_5 \delta + h_3 + \|S_1\| \delta) \hat{\rho}_1^2(\delta) \\
& + 2\alpha_1 (\delta + \|S_1\| \delta) \hat{\rho}_1(\delta) \left(h_9 p_2(\delta) + (h_6 + h_8 + h_5 h_9) \delta + (h_3 h_9 + h_7) \right) \\
& + \beta_1 (p_2(\delta) + h_5 \delta + h_3 + \|S_1\| \delta) \\
=: & p_3(\delta) + h_{10} \delta + h_{11},
\end{aligned}
\tag{3.97}
$$

where

$$
\begin{aligned}
p_3(\delta) = & \alpha_1 (p_2(\delta) + h_5 \delta + h_3 + \|S_1\| \delta) \hat{\rho}_1^2(\delta) \\
& + \alpha_1 (\delta + \|S_1\| \delta) 2 \hat{\rho}_1(\delta) (h_9 p_2(\delta) + (h_6 + h_8 + h_5 h_9) \delta + (h_3 h_9 + h_7)) \\
& + \beta_1 p_2(\delta), \\
h_{10} = & \beta_1 (h_5 + \|S_1\|), \\
h_{11} = & \beta_1 h_3.
\end{aligned}
\tag{3.98}
$$

Combining (3.85), (3.88) and (3.97) we have

$$
\begin{aligned}
\|\tilde{q}_2\| & \leq \|X_2\| + \|\tilde{\tau}\| + \|\dot{\tilde{\tau}}\| \\
& \leq \delta + p_1(\delta) + h_1 \delta + p_3(\delta) + h_{10} \delta + h_{11} \\
& =: H_1(\delta) + H_2 \delta + h_{11},
\end{aligned}
\tag{3.99}
$$

where

$$
\begin{aligned}
H_1(\delta) = & p_1(\delta) + p_3(\delta), \\
H_2 = & 1 + h_1 + h_{10}.
\end{aligned}
\tag{3.100}
$$

This implies that

$$
\begin{aligned}
\|\tilde{q}\| & = \sqrt{\|\tilde{q}_1\|^2 + \|\tilde{q}_2\|^2} \\
& \leq \sqrt{\delta^2 + (H_1(\delta) + H_2 \delta + h_{11})^2} \\
& =: \zeta(\delta) < \infty.
\end{aligned}
\tag{3.101}
$$

The uniform boundedness and uniform ultimate boundedness performance of original system follows the following statements:

Uniform boundedness: Given any $r > 0$ with $\|Z(t_0)\| \leq r$, there exists a $r_q = d_q^{-1}(\zeta(\delta))$, if $\|\tilde{q}(t_0)\| \leq r_q$, then $\|\tilde{q}(t)\| \leq d_q(r_q)$ with

$$d_q(r_q) = \begin{cases} r_q \left[\frac{\eta_1+\eta_2 r_q+\eta_3 r_q^2}{\eta_0}\right]^{\frac{1}{2}} & \text{if } r_q > R_q \\ R_q \left[\frac{\eta_1+\eta_2 R_q+\eta_3 R_q^2}{\eta_0}\right]^{\frac{1}{2}} & \text{if } r_q \leq R_q \end{cases}, \tag{3.102}$$
$$R_q = \zeta(R).$$

Uniform ultimate boundedness: Let

$$\underline{d}_q = R_q \left[\frac{\eta_1 + \eta_2 R_q + \eta_3 R_q^2}{\eta_0}\right]^{\frac{1}{2}}, \tag{3.103}$$

given any $\bar{d}_q \geq \underline{d}_q$, $\|\tilde{q}(t)\| \leq \bar{d}_q$ for all $t \geq t_0 + T_q(\bar{d}_q, r_q)$ with $T_q(\bar{d}_q, r_q)$ given by

$$T_q(\bar{d}_q, r_q) = \begin{cases} 0 & \text{if } r_q \leq \bar{R}_q \\ \frac{\eta_1 r_q^2+\eta_2 r_q^3+\eta_3 r_q^4-\eta_0 \bar{R}_q^2}{\lambda_{\min}(A)\bar{R}_q^2-\alpha} & \text{otherwise} \end{cases}, \tag{3.104}$$
$$\bar{R}_q = \xi^{-1}(\eta_0 \bar{d}_q^2).$$

Theorem 3.3. Subject to Assumptions 3.1–3.4, the control (3.17) renders the system (3.1) uniform boundedness and uniform ultimate boundedness properties.

Proof: The system performance analysis has been shown above. Q.E.D.

Remark 3.4. For a long time of using backstepping method in control design area, the original system is necessarily transformed into another system described by new state variables. The other authors only can prove the stability of the transformed system. Then, they infer the original system is also stable based on the equivalent transformation without giving specific proof. They always skip over the proof of the stability of the original system directly. In this work, we prove the stability of the original system theoretically.

3.6 Experimental verification

The experiments are performed on the two-link FJM experimental setup given in Chapter 2. Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2 - \tilde{\tau}$, $x_4 = \dot{q}_2 - \dot{\tilde{\tau}}$, we choose bounding function as

$$\begin{aligned} \Pi_1(q_1, \dot{q}_1, \hat{\psi}_1) =& \hat{\psi}_{11}(q_{1,1}^2 + q_{1,2}^2) + \hat{\psi}_{12}(\dot{q}_{1,2}^2 + \dot{q}_{1,2}^2), \\ \Pi_2(q_1, \dot{q}_1, q_2, \dot{q}_2, \hat{\psi}_2) =& \hat{\psi}_{21}(q_{1,1}^2 + q_{1,2}^2) + \hat{\psi}_{22}(q_{2,1}^2 + q_{2,2}^2) + \hat{\psi}_{23}(\dot{q}_{1,1}^2 + \dot{q}_{1,2}^2) \\ &+ \hat{\psi}_{24}(\dot{q}_{2,1}^2 + \dot{q}_{2,2}^2), \end{aligned} \tag{3.105}$$

where $\hat{\psi}_1$ and $\hat{\psi}_2$ are given by

$$\dot{\hat{\psi}}_1 = \begin{bmatrix} \dot{\hat{\psi}}_{11} \\ \dot{\hat{\psi}}_{12} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \|x_2 + S_1 x_1\|(q_{1,1}^2 + q_{1,2}^2) \\ \|x_2 + S_1 x_1\|(\dot{q}_{1,1}^2 + \dot{q}_{1,2}^2) \end{bmatrix} - 0.5 \begin{bmatrix} \hat{\psi}_{11} \\ \hat{\psi}_{12} \end{bmatrix},$$
$$\dot{\hat{\psi}}_2 = \begin{bmatrix} \dot{\hat{\psi}}_{21} \\ \dot{\hat{\psi}}_{22} \\ \dot{\hat{\psi}}_{23} \\ \dot{\hat{\psi}}_{24} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \|x_4 + S_2 x_3\|(q_{1,1}^2 + q_{1,2}^2) \\ \|x_4 + S_2 x_3\|(q_{2,1}^2 + q_{2,2}^2) \\ \|x_4 + S_2 x_3\|(\dot{q}_{1,1}^2 + \dot{q}_{1,2}^2) \\ \|x_4 + S_2 x_3\|(\dot{q}_{2,1}^2 + \dot{q}_{2,2}^2) \end{bmatrix} - 0.5 \begin{bmatrix} \hat{\psi}_{21} \\ \hat{\psi}_{22} \\ \hat{\psi}_{23} \\ \hat{\psi}_{24} \end{bmatrix}. \tag{3.106}$$

Then we have the controllers as

$$\begin{aligned} \tilde{\tau} &= -\alpha_1(x_2 + S_1 x_1)\Pi_1{}^2 - \beta_1(x_2 + S_1 x_1), \\ \tau &= -\alpha_2(x_4 + S_2 x_3)\Pi_2{}^2 - \beta_2(x_4 + S_2 x_3), \end{aligned} \tag{3.107}$$

For the experiment, the control parameters are chosen as $S_{11} = S_{12} = 1$, $S_{21} = S_{22} = 2$, $\omega_1 = 1$, $\alpha_1 = \alpha_2 = 10$, $\beta_1 = \beta_2 = 2$.

Let $e(t)$ denote the following error, $\tau(t) = [\tau_1(t), \tau_2(t)]^T$ denote the input torque, The performance indices are defined as:

1. $e_{\max} = \max\{|e(t)|\}$, representing the maximum following error, reflects the transitional performance.

2. $\bar{e} = \left(\int_0^T \|e(t)\|^2 dt/T\right)^{1/2}$, where T is the total tracking time, reflects the average tracking performance.

3. $\bar{\tau} = \left(\int_0^T (\tau_1(t)^2 + \tau_2(t)^2)\, dt/T\right)^{1/2}$, reflects the average control input.

4. $\tau_v = \Delta\tau/\bar{\tau}$, reflects the fluctuation of control input, where $\Delta\tau = \sqrt{\frac{1}{N}\sum_{k=1}^{N} |\tau(k) - \tau(k-1)|^2}$ is the average control increment, $\tau(k)$ is the control input of k-th sampling.

To testify the proposed controller, two experimental validations are conducted.

Square Trajectory Tracking: Movement of end-tip is in $X - Y$ plane and described by (x, y) in Cartesian position. A 100 mm $\times$ 100 mm square trajectory is required to be tracked in 1 s. The tracking procedure is (unit is mm):

1. Move the end-tip to start point (850,0);

2. Begin data acquisition;

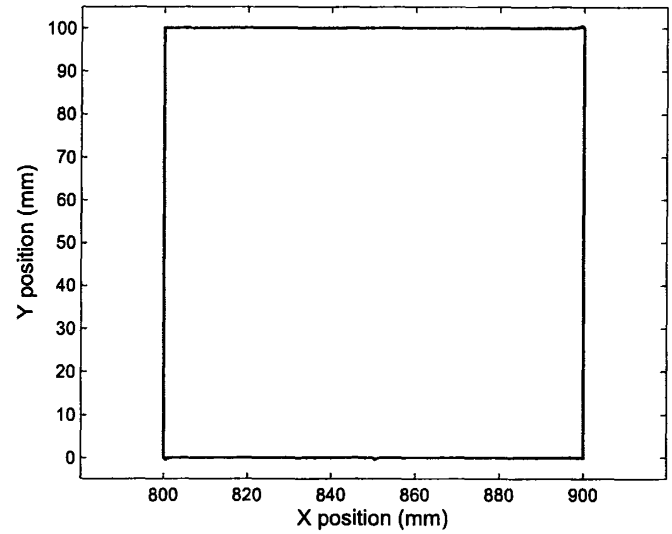

Figure 3.2: Square trajectory tracking performance under adaptive robust control.

3. Move the end-tip to points (900,0), (900,100), (800,100), (800,0), (850,0), continuously;

4. Stop data acquisition and upload data.

The results of the experiments are shown from Figures 3.2 to 3.6. The complete tracking trajectory, provided in Figure 3.2, well coincides to the desired square. Although large following errors arise at the points where the motors change their directions (see Figure 3.3), input torques do not change acutely except at the start point and stop point. The corresponding histories of adaptive parameters are shown in Figure 3.6 (the initial condition is $\hat{\psi}_1(0) = [0.8, 0.6]^T$, $\hat{\psi}_2(0) = [0.2, 0.2, 0.5, 0.5]^T$). Detailed tracking performance index comparison between robust control, proposed in Chapter 2, and adaptive robust control is provided in Table 3.1.

We stress that, although the maximum following errors of robust control in each joint are close to those of adaptive robust control, the average following errors (reflected by $\bar{e}$) are much better when adaptive robust control is applied. Furthermore, the average control input $\bar{\tau}$ and fluctuation of torque τ_v under adaptive robust control are smaller than those under non-adaptive control, which means the extra control efforts caused by inappropriate choice of bounding function in robust control are saved.

Circle Trajectory Tracking: Movement of end-tip is in $X-Y$ plane and described by (x, y) in Cartesian position. A circle trajectory, whose radius is equal to 50 mm, is required to be tracked in 1 s. The tracking procedure is given by following (unit is mm):

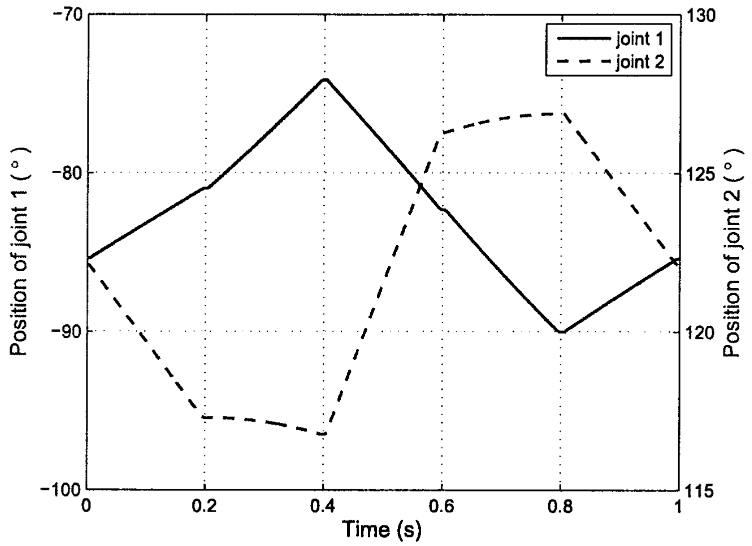

Figure 3.3: Position of each joint under adaptive robust control: Square trajectory.

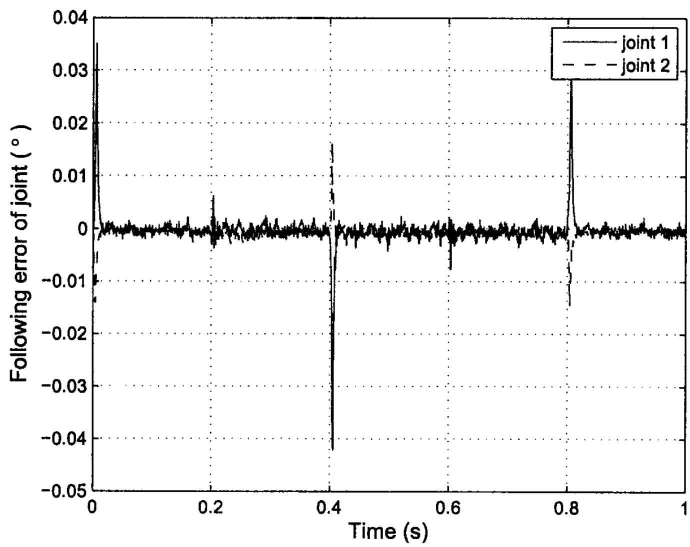

Figure 3.4: Following error of each joint under adaptive robust control: Square trajectory.

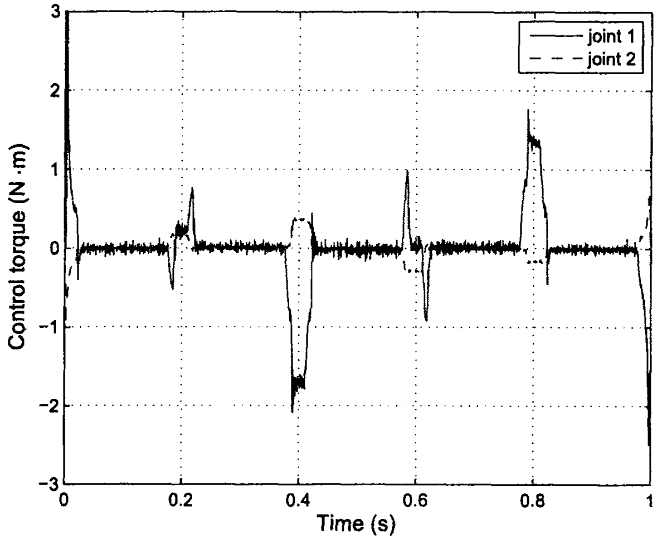

Figure 3.5: Input torque of each joint under adaptive robust control: Square trajectory.

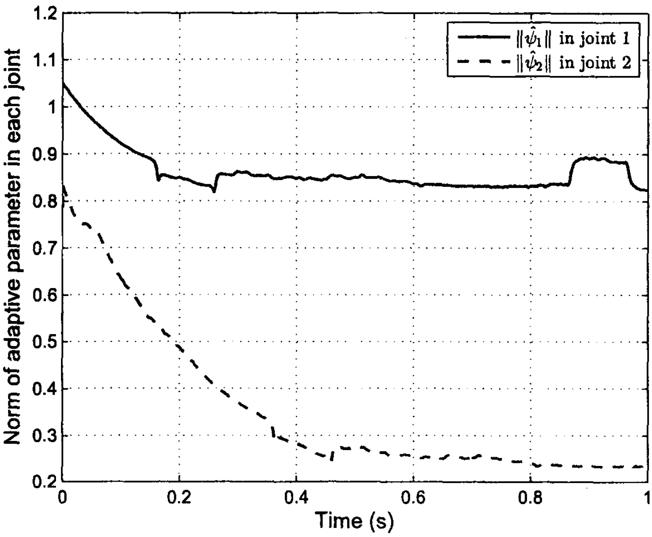

Figure 3.6: The adaptive parameters of joints: Square trajectory.

Table 3.1: Comparison of index under robust control and adaptive robust control: Square trajectory

	joint 1		joint 2		end-tip	
	robust	adaptive robust	robust	adaptive robust	robust	adaptive robust
$e_{\max}$	0.0562°	0.0420°	0.0306°	0.0164°	0.7523 mm	0.5669 mm
$\bar{e}$	0.0063°	0.0012°	0.0066°	0.0008°	0.1855 mm	0.0240 mm
$\bar{\tau}$	0.6143 N·m	0.2023 N·m	0.2046 N·m	0.0525 N·m	–	–
τ_v	0.6496	0.5607	0.3963	0.2855	–	–

1. Move the end-tip to start point (800,0);
2. Begin data acquisition;
3. Move the end-tip clockwise along a complete circle whose center is (750,0) and radius is 50 mm;
4. Stop data acquisition and upload data.

The experimental results are shown from Figure 3.7 to Figure 3.11. Similar to the results of square trajectory tracking, in Figure 3.7, after applying adaptive control, the controlled trajectory well coincides to the desired trajectory. Figures 3.8 and 3.9 provide the joint tracking trajectory and following errors under adaptive robust control, respectively. The maximum following error still arises in joint 1 and as much as 0.0389°. Be different from robust control, here, input torque of joint 1 has jerks at the start point. This is caused by the "learning" process of adaptive parameter. After a certain time, the control torque changes slowly. The corresponding histories of adaptive parameters are shown in Figure 3.6 (the initial condition is $\hat{\psi}_1(0) = [0.8, 0.6]^T$, $\hat{\psi}_2(0) = [0.2, 0.2, 0.5, 0.5]^T$). Detailed tracking performance index comparison between robust control, proposed in Chapter 2, and adaptive robust control is provided in Table 3.2. It can be seen that, by using adaptive laws to estimate the uncertainty bound, the maximum following error $e_{\max}$ and average following error $\bar{e}$ are improved. Particularly, the average control input torque $\bar{\tau}$ and fluctuation of torque τ_v are enhanced significantly.

3.7 Conclusions

In this chapter, an adaptive robust control for the FJM systems, which are nonlinear, time-varying and mismatched, is developed. The overall system is transformed by defining new state

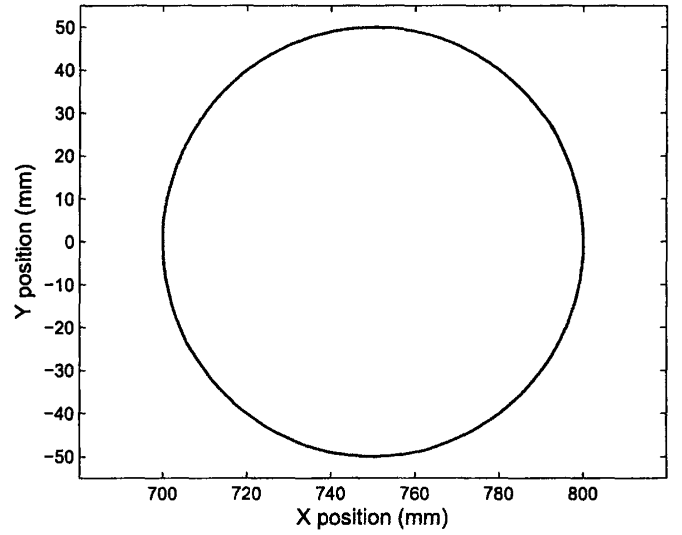

Figure 3.7: Circle trajectory tracking performance under adaptive robust control.

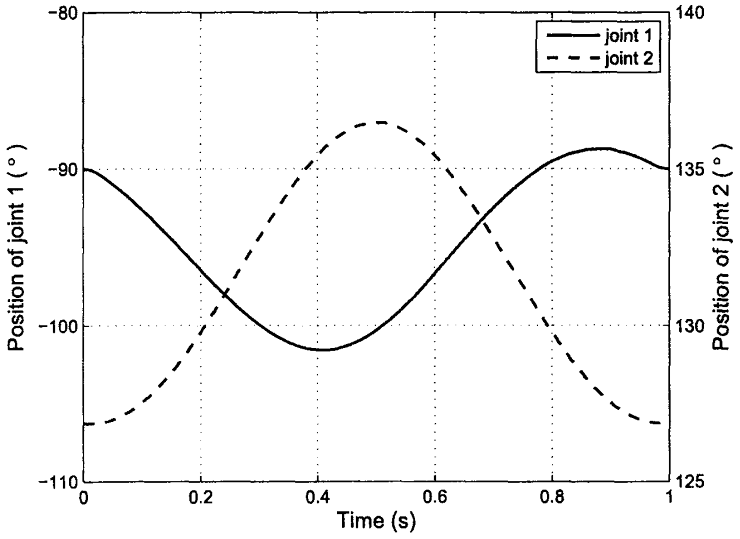

Figure 3.8: Position of each joint under adaptive robust control: Circle trajectory.

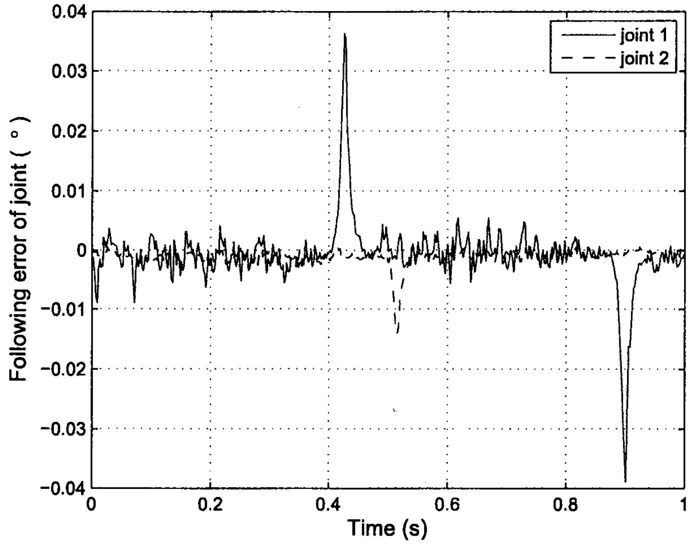

Figure 3.9: Following error of each joint under adaptive robust control: Circle trajectory.

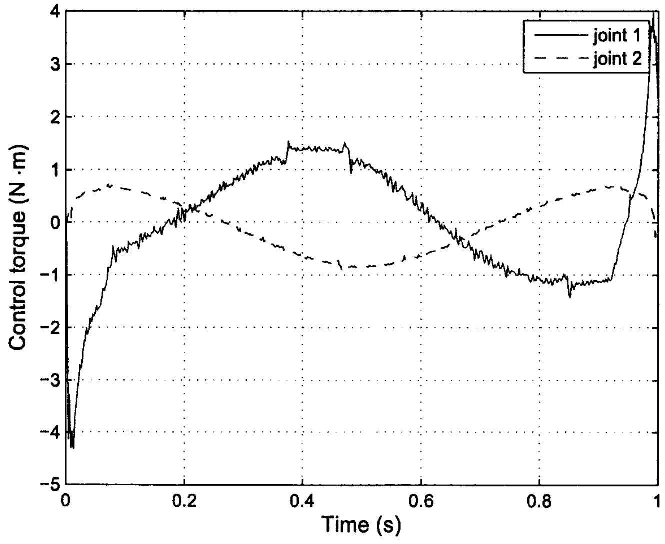

Figure 3.10: Input torque of each joint under adaptive robust control: Circle trajectory.

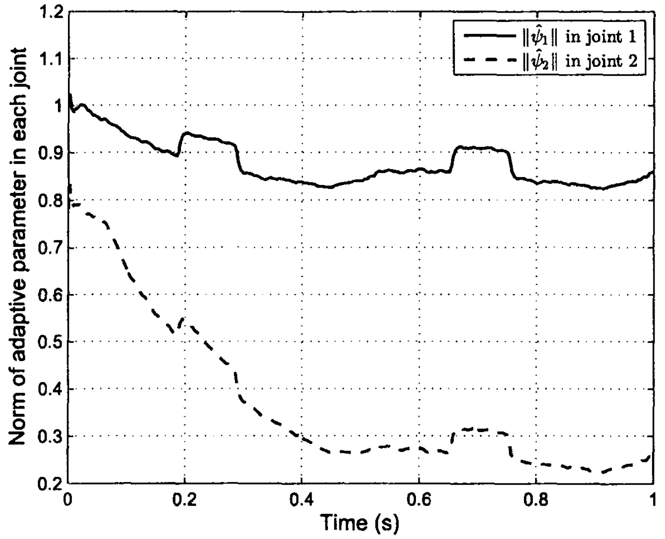

Figure 3.11: The adaptive parameters of joints: Circle trajectory.

Table 3.2: Comparison of index under robust control and adaptive robust control: Circle trajectory

	joint 1		joint 2		end-tip	
	robust	adaptive robust	robust	adaptive robust	robust	adaptive robust
$e_{\max}$	0.0755°	0.0389°	0.0298°	0.0142°	0.4459 mm	0.2643 mm
$\bar{e}$	0.0075°	0.0027°	0.0027°	0.0011°	0.1109 mm	0.0477 mm
$\bar{\tau}$	2.1098 N·m	0.9384 N·m	1.1864 N·m	0.4667 N·m	–	–
τ_v	0.0925	0.0239	0.1509	0.0905	–	–

variables, and then divided into two subsystems by implanting a fictitious control for the link position subsystem. Thereby, the matching condition is satisfied. The control is only based on assuming the existence of uncertainty bound, although the bound is not given. No other statistical properties of uncertainty is assumed and utilized. The presented control scheme guarantees practical stability for the system by combining states and parameters to be estimated. Moreover, the stability of original system is proved theoretically which is usually taken for granted by other studies. Although the uniform boundedness ball cannot be made arbitrary small by adjusting α_1 and α_2, the leakage terms in update laws provide smaller uniform boundedness ball in states and parameters than that without leakage terms. The constructed update laws are not in a complex form so that can be applied in real-time applications. When the following errors are small, the update laws become even more simpler. In comparing with robust control (non-adaptive) proposed in Chapter 2, experiment results show advantages of adaptive robust control proposed in this chapter.

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Chapter 4 Fuzzy dynamical system approach to the optimal control for flexible joint manipulator

4.1 Introduction

In the previous chapters, it can be seen that exploring the uncertainty is crucial important to our control design. Distinguish the known portion and unknown portion so that the pre-mentioned deterministic approaches (e.g., robust control, adaptive robust control) can be applied. However, the known portion sometimes cannot be clearly isolated, or in other words, the bound of uncertainty cannot be explicitly identified. Kalman initiated the work of describing the unknown uncertainty with the [77, 78]. He looked into the estimation and control problem of system under stochastic noise in state space (Kalman Filter). Although probability theory obtained prominent successes in control for uncertain mechanical system, in 1994, Kalman claimed that probability is only an intellectual construct. The probability is not quantifiable, measurable, concrete and does not exist [79].

Therefore, a fuzzy perspective via the degree of occurrence of uncertainty may be considered as an alternative. Fuzzy theory was initially adopted by L. A. Zadeh to describe the information which only has a rough boundary with others [80]. In the past literatures, most of the fuzzy control focused on the rule-based reasoning for control, estimation, decisionmaking, etc. This study, on the other hand, propose a rather different angle. With the fuzzy description of the system state and uncertainty (so called fuzzy dynamical system), it endeavors to design a fuzzy approach for the control of flexible joint manipulator (FJM) system.

The main contributions of this work are threefold. First, we consider uncertainty in the initial condition and the system parameter. The uncertainty is mismatched (i.e., it does not meet the matching condition). They are assumed to lie within prescribed fuzzy sets. Then the model is cast as the *fuzzy dynamical system*. Second, by implanting a fictitious control, the overall system is divided into two subsystems. The mismatched uncertainty is converted to be the matched uncertainty. The proposed control, including the PD control portion and the adaptive robust control portion, renders the system uniform boundedness and uniform ultimate boundedness performance. Third, we not only guarantee the deterministic performance, but also explore fuzzy description of system performance under the fuzzy described uncertainty. A performance index, which includes average fuzzy system performance and transient performance, is proposed based on the fuzzy description of the uncertainty. The optimal design problem associated with the control can then be formulated as a performance index minimizing problem. As a result, the controlled system performance is both guaranteed (in the deterministic sense) and optimal (in the fuzzy sense).

4.2 Preliminary: fuzzy mathematics

We briefly review some preliminaries regarding fuzzy numbers and their operations (see, e.g., [139]).

1. Fuzzy number: Let G be a fuzzy set in $\mathbf{R}$, the real number. G is called a fuzzy number if: (i) G is normal, (ii) G is convex, (iii) the support of G is bounded, and (iv) all α-cuts are closed intervals in $\mathbf{R}$.

Throughout, we shall always assume that the universe of discourse of a fuzzy number to be its 0-cut.

2. Fuzzy arithmetic: Let G and H be two fuzzy numbers and $G_\alpha = [g_\alpha^-, g_\alpha^+]$, $H_\alpha = [h_\alpha^-, h_\alpha^+]$ be their α-cuts, $\alpha \in [0,1]$. The *addition, subtraction, multiplication*, and *division* of G and H are given by, respectively

$$(G+H)_\alpha = [g_\alpha^- + h_\alpha^-, g_\alpha^+ + h_\alpha^+], \tag{4.1}$$

$$(G-H)_\alpha = [\min(g_\alpha^- - h_\alpha^-, g_\alpha^+ - h_\alpha^+), \max(g_\alpha^- - h_\alpha^-, g_\alpha^+ - h_\alpha^+)], \tag{4.2}$$

$$\begin{aligned}(G\cdot H)_\alpha =& [\min(g_\alpha^- h_\alpha^-, g_\alpha^- h_\alpha^+, g_\alpha^+ h_\alpha^-, g_\alpha^+ h_\alpha^+),\\ & \max(g_\alpha^- h_\alpha^-, g_\alpha^- h_\alpha^+, g_\alpha^+ h_\alpha^-, g_\alpha^+ h_\alpha^+)],\end{aligned} \tag{4.3}$$

$$\begin{aligned}(G/H)_\alpha =& [\min(g_\alpha^-/h_\alpha^-, g_\alpha^-/h_\alpha^+, g_\alpha^+/h_\alpha^-, g_\alpha^+/h_\alpha^+),\\ & \max(g_\alpha^-/h_\alpha^-, g_\alpha^-/h_\alpha^+, g_\alpha^+/h_\alpha^-, g_\alpha^+/h_\alpha^+)].\end{aligned} \tag{4.4}$$

3. Decomposition theorem: Define a fuzzy set $\tilde{D}_\alpha$ in U with the membership function $\mu_{\tilde{D}_\alpha} = \alpha I_{\tilde{D}_\alpha}(x)$ where $I_{\tilde{D}_\alpha}(x) = 1$ if $x \in \tilde{D}_\alpha$ and $I_{\tilde{D}_\alpha}(x) = 0$ if $x \in U - \tilde{D}_\alpha$. Then the fuzzy set D is obtained as

$$D = \bigcup_{\alpha\in[0,1]} \tilde{D}_\alpha, \tag{4.5}$$

where $\cup$ is the union of the fuzzy sets (that is, *sup* over $\alpha \in [0,1]$). Based on these, after the operation of two fuzzy numbers via their α-cuts, one may apply the decomposition theorem to build the membership function of the resulting fuzzy number.

4.3 Fuzzy dynamical model of flexible joint manipulator

In engineering applications, the modeling errors and unpredictable extern noise prevent us from using the explicit knowledge of the system. Therefore, we introduce time-varying uncertainties $\sigma_1(t)$ and $\sigma_2(t)$ which are uncertain parameter vectors into the system. We consider a FJM described as follows

$$\begin{aligned}\begin{bmatrix} M(q_1,\sigma_1) & 0 \\ 0 & J(\sigma_2) \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}+\begin{bmatrix} C(q_1,\dot{q}_1,\sigma_1)\dot{q}_1 \\ F(q_2,\dot{q}_2,\sigma_2)\dot{q}_2 \end{bmatrix}+\begin{bmatrix} G(q_1,\sigma_1) \\ 0 \end{bmatrix} \\ +\begin{bmatrix} K(\sigma_1)(q_1-q_2) \\ -K(\sigma_2)(q_1-q_2) \end{bmatrix}=\begin{bmatrix} 0 \\ \tau \end{bmatrix},\end{aligned} \tag{4.6}$$

where $q_1 = [q_{1,1}\ q_{1,2}\ \cdots\ q_{1,n}]^T$ is link position vector and $q_2 = [q_{2,1}\ q_{2,2}\ \cdots\ q_{2,n}]^T$ is joint position vector. Let $q = [q_1^T\ \ q_2^T]^T$ be a $2n-$ dimensional vector and represent the system's generalized coordinate. The joint flexibility is regarded as a linear torsional spring and elasticity coefficient of the spring is represented by a diagonal positive matrix $K(\sigma_{1,2})$. The link inertia matrix is denoted by $M(q_1,\sigma_1)$. The inertia of actuator is denoted as $J(\sigma_2)$ which is a diagonal matrix. Vectors $C(q_1,\dot{q}_1,\sigma_1)\dot{q}_1$ and $F(q_2,\dot{q}_2,\sigma_2)\dot{q}_2$ represent the Coriolis and centrifugal forces of links and actuators, respectively. $G(q_1,\sigma_1)$ is the gravitation force vector, and τ denotes the input control force from the actuators.

Remark 4.1. In comparing with the system model given in [9], we add an additional Coriolis and centrifugal force term $F(q_2,\dot{q}_2,\sigma_2)\dot{q}_2$ which makes this model become more general. For a specific application, there might not exist this term, but this does not influence our control design procedure by setting this term as 0.

Assumption 4.1. The inertia matrix $M(q_1,\sigma_1)$ is uniformly positive definite. That is, there exists a constant $\underline{\lambda} > 0$, such that

$$M(q_1,\sigma_1) \geq \underline{\lambda} I, \quad \forall q_1 \in \mathbf{R}^n, \tag{4.7}$$

where I denotes the identity matrix.

Since the inertia matrix may be positive semi-definite, the positive definite property is stated as an assumption rather than a fact.

For the upper bound condition, Chen explored and proved a more generic characteristic of the inertia matrix in any serial manipulator[129]. A quadratic polynomial upper bound condition respecting to generalized coordinate is proposed. We take this assertion as a nature property of inertia matrix of FJM.

Property 4.1. *There exist scalar constants* $\bar{\lambda}_1 > 0$, $\bar{\lambda}_2 \geq 0$, $\bar{\lambda}_3 \geq 0$, *such that*

$$\|M(q_1, \sigma_1)\| \leq \bar{\lambda}_1 + \bar{\lambda}_2\|q_1\| + \bar{\lambda}_3\|q_1\|^2, \quad \forall q_1 \in \mathbf{R}^n. \tag{4.8}$$

This upper bound property is generic. For a manipulator with its joints are all revolute, the property is reduced to $\bar{\lambda}_2 = \bar{\lambda}_3 = 0$, such that

$$\|M(q_1, \sigma_1)\| \leq \bar{\lambda}_1, \quad \forall q_1 \in \mathbf{R}^n. \tag{4.9}$$

We propose two steps to design the robust control. Firstly, let us rewrite the first part of (4.6) as

$$\begin{aligned} &M(q_1, \sigma_1)\ddot{q}_1 + C(q_1, \dot{q}_1, \sigma_1)\dot{q}_1 + G(q_1, \sigma_1) \\ &\quad + K(\sigma_1)q_1 - K(\sigma_1)(q_2 - \tilde{\tau}) = K(\sigma_1)\tilde{\tau}, \end{aligned} \tag{4.10}$$

where $\tilde{\tau}$ is a fictitious control implanted into the system, without changing the dynamics of original system. In the later sections, we show that the real control τ can be obtained with the help of $\tilde{\tau}$. With $\tilde{\tau}$ introduced, the system could be divided into two parts, which are: (i) link angles subsystem; (ii) joint angles subsystem. The link position subsystem is controlled by $\tilde{\tau}$, which is fictitious. The joint position subsystem is controlled by the real control τ.

Next, we rewrite the overall system with equivalent state variables transformation. Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2 - \tilde{\tau}$, $x_4 = \dot{q}_2 - \dot{\tilde{\tau}}$, then

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, X_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{4.11}$$

Thus, by multiplying $K^{-1}(\sigma_1)$ on both side of (4.10) the dynamics of FJMs can be expressed as follows by using new state variables:

$$\hat{S}_1 : \hat{M}(x_1, \sigma_1)\dot{x}_2 = -\hat{C}(x_1, x_2, \sigma_1)x_2 - \hat{G}(x_1, \sigma_1) - x_1 + x_3 + \tilde{\tau}, \tag{4.12}$$

$$\begin{aligned}\hat{S}_2: J(\sigma_2)\dot{x}_4 =& -J(\sigma_2)\ddot{\bar{\tau}} - F(x_1,x_2,x_3,x_4,\sigma_2)x_4 - F(x_1,x_2,x_3,x_4,\sigma_2)\dot{\bar{\tau}} \\ & - K(\sigma_2)x_3 + K(\sigma_2)x_1 - K(\sigma_2)\bar{\tau} + \tau,\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}\hat{M}(x_1,\sigma_1) &= K^{-1}(\sigma_1)M(x_1,\sigma_1),\\ \hat{C}(x_1,x_2,\sigma_1) &= K^{-1}(\sigma_1)C(x_1,x_2,\sigma_1),\\ \hat{G}(x_1,\sigma_1) &= K^{-1}(\sigma_1)G(x_1,\sigma_1).\end{aligned} \tag{4.14}$$

Assumption 4.2. [119] (i) Suppose the initial state is $X(t_0) = X_0$, where t_0 is the initial time. For each entry of X_0, namely $X_{0i}, i = 1,2,\cdots,4n$, there exists a fuzzy set U_{0i} in a universe of discourse $\Xi_i \subset \mathbf{R}$, characterized by a membership function $\mu_{\Xi_i}: \Xi_i \to [0,1]$. That is,

$$U_{0i} = \{(X_{0i}, \mu_{\Xi_i}(X_{0i}))|X_{0i} \in \Xi_i\}. \tag{4.15}$$

(ii) For each entry of σ_1,σ_2, namely $\sigma_{1i},\sigma_{2i}, i = 1,2,\cdots,n$, the function $\sigma_{1i}(\cdot),\sigma_{2i}(\cdot)$ are Lebesgue measurable. (iii) For each σ_{1i},σ_{2i}, there exist fuzzy sets N_{1i}, N_{2i} in universe of discourses $\Sigma_{1i} \subset \mathbf{R}, \Sigma_{2i} \subset \mathbf{R}$, characterized by membership functions $\mu_{1i}: \Sigma_{1i} \to [0,1]$, $\mu_{2i}: \Sigma_{2i} \to [0,1]$. That is,

$$\begin{aligned}N_{1i} =& \{(\sigma_{1i}, \mu_{1i}(\sigma_{1i}))|\sigma_{1i} \in \Sigma_{1i}\},\\ N_{2i} =& \{(\sigma_{2i}, \mu_{2i}(\sigma_{2i}))|\sigma_{2i} \in \Sigma_{2i}\}.\end{aligned} \tag{4.16}$$

Remark 4.2. The Assumption 4.2 imposes fuzzy restriction on the uncertainty. We employ the fuzzy description on the uncertainties in the overall system. This description earns much more advantage than the probability avenue which often requires a large number of repetitions to acquire the observed data (always limited by nature).

Assumption 4.3. [138]
(i) There exists an unknown j-dimensional constant vector $\psi_1 \in (0,\infty)^j$ and a known function $\Pi_1: \mathbf{R}^n \times \mathbf{R}^n \times (0,\infty)^j \to \mathbf{R}_+$ such that for all $x_1 \in \mathbf{R}^n, x_2 \in \mathbf{R}^n$,

$$\Pi_1(x_1,x_2,\psi_1) \geq \|\Phi_1(x_1,x_2,\sigma_1)\|, \tag{4.17}$$

where

$$\begin{aligned}\Phi_1(x_1,x_2,\sigma_1) =& \frac{1}{2}\dot{\hat{M}}(x_1,x_2,\sigma_1)(x_2 + S_1x_1) - \hat{C}(x_1,x_2,\sigma_1)x_2\\ & - \hat{G}(x_1,\sigma_1) - x_1 + \hat{M}(x_1,\sigma_1)S_1x_2,\end{aligned} \tag{4.18}$$

$$S_1 = diag[S_{1i}]_{n\times n},\ S_{1i} > 0,\ i = 1,2,\cdots,n. \tag{4.19}$$

(ii) For each entry of ψ_1, namely $\psi_{1i}, i = 1, 2, \cdots, j$, there exists a fuzzy set Q_{1i} in a universe of discourse $\Xi_{1i} \subset \mathbf{R}$, characterized by a membership function $\mu_{\Xi_{1i}} : \Xi_{1i} \to [0, 1]$. That is,

$$Q_{1i} = \{(\psi_{1i}, \mu_{\Xi_{1i}}(\psi_{1i}))|\psi_{1i} \in \Xi_{1i}\}. \tag{4.20}$$

(iii) The function $\Pi_1(x_1, x_2, \cdot) : (0, \infty)^j \to \mathbf{R}_+$ is C^2 (i.e., 2-times continuously differentiable) and concave (i.e., $-\Pi_1(x_1, x_2, \cdot)$ is convex), and it is nondecreasing with respect to each ψ_1.
(iv) The functions $\Pi_1(x_1, x_2, \cdot)$ and $\frac{\partial \Pi_1}{\partial \psi_1}(\cdot)$ are both continuous.

Assumption 4.4. [138]
(i) There exists an unknown s-dimensional constant vector $\psi_2 \in (0, \infty)^s$ and a known function $\Pi_2 : \mathbf{R}^{2n} \times \mathbf{R}^{2n} \times (0, \infty)^s \to \mathbf{R}_+$ such that for all $X_1 \in \mathbf{R}^{2n}, X_2 \in \mathbf{R}^{2n}$,

$$\Pi_2(X_1, X_2, \psi_2) \geq \|\Phi_2(X, \sigma_1, \sigma_2)\|, \tag{4.21}$$

where $S_2 = diag[S_{2i}]_{n\times n}$, $S_{2i} > 0,\ i = 1, 2, \cdots n$,

$$\begin{aligned}\Phi_2(X, \sigma_1, \sigma_2) =& - J(\sigma_2)\ddot{\bar{\tau}}(X, \sigma_1, \sigma_2) - F(X, \sigma_2)(x_4 + \dot{\bar{\tau}}(X_1, \sigma_1)) \\ & - K(\sigma_2)x_3 + K(\sigma_2)x_1 K(\sigma_2)\bar{\tau}(X_1) \\ & + J(\sigma_2)S_2 x_4 + \frac{1}{2}\dot{J}(\sigma_2)(x_4 + S_2 x_3).\end{aligned} \tag{4.22}$$

(ii) For each entry of ψ_2, namely $\psi_{2i}, i = 1, 2, \cdots, s$, there exists a fuzzy set Q_{2i} in a universe of discourse $\Xi_{2i} \in \mathbf{R}$, characterized by a membership function $\mu_{\Xi_{2i}} : \Xi_{2i} \to [0, 1]$. That is,

$$Q_{2i} = \{(\psi_{2i}, \mu_{\Xi_{2i}}(\psi_{2i}))|\psi_{2i} \in \Xi_{2i}\}. \tag{4.23}$$

(iii) The function $\Pi_2(X_1, X_2, \cdot) : (0, \infty)^j \to \mathbf{R}_+$ is C^2 (i.e., 2-times continuously differentiable) and concave (i.e., $-\Pi_2(X_1, X_2, \cdot)$ is convex), and it is nondecreasing with respect to each ψ_2.
(iv) The functions $\Pi_2(X_1, X_2, \cdot)$ and $\frac{\partial \Pi_2}{\partial \psi_2}(\cdot)$ are both continuous.

Remark 4.3. Parts (i) and (ii) of Assumptions 4.3 and 4.4, originally developed in [138], refer to properties on the possible bound of uncertainty. Since the bound of uncertainty is non-unique, an experienced engineer may take the advantage of this to choose an appropriate $\Pi_{1,2}$ to meet the assumptions. In the special case that $\Pi_{1,2}$ is linear with respect to $\psi_{1,2}$, then they are always met. In Section 4.7, we will illustrate this by choosing a linear $\Pi_{1,2}$.

4.4 Deterministic adaptive robust control design

Since uncertainty is introduced, the overall system dose not meet the matching condition [120]. We divide system into the cascades of two subsystems as shown in (4.12) and (4.13). Therefore, both of the two systems have "inputs". The mission is to develop a control τ so that $\hat{S}_1$ and $\hat{S}_2$ can accomplish a good performance. For given constant positive scalars $\gamma_1, \varepsilon_1, k_1$, we construct the fictitious control $\tilde{\tau}$ for the subsystem $\hat{S}_1$ as follows:

$$\tilde{\tau}(t) = -\gamma_1(x_2(t) + S_1x_1(t))\Pi_1{}^2(x_1, x_2, \hat{\psi}_1) - \varepsilon_1(x_2(t) + S_1x_1(t)), \tag{4.24}$$

where $\hat{\psi}_1$ is the adaptation law with the following form:

$$\begin{aligned}\dot{\hat{\psi}}_1(t) &= k_1^{-1}\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1(t), x_2(t), \hat{\psi}_1(t))\|x_2(t) + S_1x_1(t)\| - \hat{\psi}_1(t),\\ \hat{\psi}_1(t_0) &\in (0, \infty)^k.\end{aligned} \tag{4.25}$$

Then, the input torque τ can be constructed as

$$\tau(t) = -\gamma_2(x_4(t) + S_2x_3(t))\Pi_2{}^2(X, \hat{\psi}_2(t)) - \varepsilon_2(x_4(t) + S_2x_3(t)). \tag{4.26}$$

where $\hat{\psi}_2$ is the adaptation law with the following form:

$$\begin{aligned}\dot{\hat{\psi}}_2(t) &= k_2^{-1}\frac{\partial \Pi_2^T}{\partial \psi_2}(X, \hat{\psi}_2(t))\|x_4(t) + S_2x_3(t)\| - \hat{\psi}_2(t),\\ \hat{\psi}_2(t_0) &\in (0, \infty)^j, \quad k_2 > 0.\end{aligned} \tag{4.27}$$

Here, k_1, k_2 determine the "rate" of adaptation, which is vital to the system performance. The adaptive parameter will increase quickly so as to effectively compensate the uncertainty as k_1 is large. This fast response will be helpful. The γ_1 part in (4.24) is designed against the uncertainty, the proper choice of γ_1 and ε_1 will be given later.

Theorem 4.1. For the adaptation laws (4.25) and (4.27), if the initial condition $\hat{\psi}_1(t_0), \hat{\psi}_2(t_0) > 0$, then $\hat{\psi}_1(t), \hat{\psi}_2(t) > 0$, for all $t \geq t_0$, respectively.

Proof: Firstly, let us prove the $\hat{\psi}_1$ part of the theorem, the $\hat{\psi}_2$ part can be got in a similar way. Let U_1 be an open set in $\mathbf{R}^{2n+j}$, an element $(X_1, \hat{\psi}_1) \in U_1$. We rewrite (4.25) as

$$\dot{\hat{\psi}}_1(t) = L_1(X_1(t), \hat{\psi}_1(t)) - \hat{\psi}_1(t), \tag{4.28}$$

where

$$L_1(X_1(t), \hat{\psi}_1(t))) = k_1\frac{\partial \Pi_1^T}{\partial \psi_1}(x_1(t), x_2(t), \hat{\psi}_1(t))\|x_2(t) + S_1x_1(t)\|. \tag{4.29}$$

According to the continuity property of $\frac{\partial \Pi_1}{\partial \psi_1}(\cdot)$ in Assumption 4.3, $L_1(\cdot)$ is continuous on U_1, and there exists at least one solution of (4.28) passing through $(X_1(t_0), \hat{\psi}_1(t_0))$ with the given initial condition. Then, the solution of (4.28) can be given as

$$\begin{aligned}\hat{\psi}_1(t) &= e^{-t}e^{t_0}\hat{\psi}_1(t_0) + e^{-t}\int_{t_0}^{t} e^{s}L_1(X_1(s), \hat{\psi}_1(s))ds \\ &\geq e^{-(t-t_0)}\hat{\psi}_1(t_0) + e^{-t}\bar{L}_1\int_{t_0}^{t} e^{s}ds,\end{aligned} \tag{4.30}$$

where

$$\bar{L}_1 = \inf_{(X_1,\hat{\psi}_1)\in U_1} L_1(X_1, \hat{\psi}_1) \geq 0. \tag{4.31}$$

Therefore, if $\hat{\psi}_1(t_0) > 0$, we have $\hat{\psi}_1(t) > 0$ for all $t \geq t_0$. From the same point of view, it can be concluded that $\hat{\psi}_2(t) > 0$ for all $t \geq t_0$, if $\hat{\psi}_2(t_0) > 0$. Q.E.D.

Theorem 4.2. Let $\tilde{\psi}_1 = \hat{\psi}_1 - \psi_1$, $\tilde{\psi}_2 = \hat{\psi}_2 - \psi_2$, $Z = [X_1^T \quad \tilde{\psi}_1^T \quad X_2^T \quad \tilde{\psi}_2^T]^T$, subject to Assumptions 4.1 to 4.4, the control (4.26) renders the combined mechanical/adaptive system (4.12), (4.13), (4.25) and (4.27) the following performance:

1. **Uniform boundedness**: Given any constant $r_z > 0$ with $\|Z(t_0)\| \leq r_z$,there exists $d_z(r_z) > 0$ such that $\|Z(t)\| \leq d_z(r_z)$ for all $t \in [t_0, \infty)$.

2. **Uniform ultimate boundedness**: Given any $\bar{d}_z > \underline{d}_z$ and any $r_z > 0$, there exists a finite time $T_z(\bar{d}_z, r_z)$ such that $\|Z(t_0)\| \leq r_z$ implies $\|Z(t)\| \leq \bar{d}_z$ for all $t \geq t_0 + T_z(\bar{d}_z, r_z)$.

3. **Uniform stability**: Given any $\bar{d}_z > \underline{d}_z$, there exists a $\delta_z(\bar{d}_z) > 0$ such that $\|Z(t_0)\| \leq \delta_z(\bar{d}_z)$ implies $\|Z(t)\| \leq \bar{d}_z$ for all $t \geq t_0$.

Proof: From now on, we omit the argument on uncertainty in $\hat{M}(x_1, \sigma_1), \hat{C}(x_1, x_2, \sigma_1)$ and $K(\sigma_1)$ etc. if no confusion arises. Otherwise it will be denoted.

Let $\psi = [\tilde{\psi}_1^T \quad \tilde{\psi}_2^T]^T$, choose the Lyapunov function candidates for the system as following:

$$V(X, \psi) = V_1(X_1, \tilde{\psi}_1) + V_2(X_2, \tilde{\psi}_2), \tag{4.32}$$

where

$$
\begin{aligned}
&V_1(X_1,\tilde{\psi}_1) = V_{X_1}(X_1) + V_{\psi_1}(\tilde{\psi}_1),\\
&V_{X_1}(X_1) = \frac{1}{2}(x_2 + S_1 x_1)^T \hat{M}(x_2 + S_1 x_1) + x_1^T \varepsilon_1 S_1 x_1,\\
&V_{\psi_1}(\tilde{\psi}_1) = \frac{1}{2}\tilde{\psi}_1^T k_1 \tilde{\psi}_1,\\
&V_2(X_2,\tilde{\psi}_2) = V_{X_2}(X_2) + V_{\psi_2}(\tilde{\psi}_2),\\
&V_{X_2}(X_2) = \frac{1}{2}(x_4 + S_2 x_3)^T J(x_4 + S_2 x_3) + x_3^T \varepsilon_2 S_2 x_3,\\
&V_{\psi_2}(\tilde{\psi}_2) = \frac{1}{2}\tilde{\psi}_2^T k_2 \tilde{\psi}_2.
\end{aligned}
\tag{4.33}
$$

To show $V(X)$ is a legitimate Lyapunov function candidate for any FJM system, we need to prove that $V(X)$ is positive definite and decrescent. Based on Assumption 4.1,

$$
\begin{aligned}
V_{X_1} &\geq \frac{1}{2}\underline{\lambda}_0\|x_2 + S_1 x_1\|^2 + x_1^T \varepsilon_1 S_1 x_1\\
&= \frac{1}{2}\underline{\lambda}_0 \sum_{i=1}^{n}(\dot{x}_{1i}^2 + 2\dot{x}_{1i}S_{1i}x_{1i} + S_{1i}^2 x_{1i}^2) + \sum_{i=1}^{n}\varepsilon_1 S_{1i} x_{1i}^2\\
&=: \frac{1}{2}\sum_{i=1}^{n}[x_{1i} \quad \dot{x}_{1i}]\Omega_{1i}\begin{bmatrix} x_{1i}\\ \dot{x}_{1i}\end{bmatrix} \geq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\min}(\Omega_{1i})(x_{1i}^2 + x_{2i}^2).
\end{aligned}
\tag{4.34}
$$

here,

$$
\Omega_{1i} = \begin{bmatrix} \underline{\lambda}_0 S_{1i}^2 + 2\varepsilon_1 S_{1i} & \underline{\lambda}_0 S_{1i}\\ \underline{\lambda}_0 S_{1i} & \underline{\lambda}_0 \end{bmatrix}. \tag{4.35}
$$

Since Ω_{1i} is positive definite, V_1 is positive definite. Thus, we have

$$
V_{X_1} \geq \frac{1}{2}\sum_{i=1}^{n}\lambda_{\min}(\Omega_{1i})(x_{1i}^2 + x_{2i}^2). \tag{4.36}
$$

Moreover,

$$
V_{\psi_1} = \frac{1}{2}k_1\|\tilde{\psi}_1\|^2. \tag{4.37}
$$

By combining (4.36) and (4.37), we have

$$
\begin{aligned}
V_1 &= V_{X_1} + V_{\psi_1}\\
&\geq \lambda_0^1(\|X_1\|^2 + \|\tilde{\psi}_1\|^2) =: \lambda_0^1\|Z_1\|^2,
\end{aligned}
\tag{4.38}
$$

where $\lambda_0^1 = \min\{\min_i\{\frac{1}{2}\lambda_{\min}(\Omega_{1i})\}, \frac{1}{2}k_1\}, i = 1,2,\cdots,n$, $Z_1 = [X_1^T \quad \tilde{\psi}_1^T]^T$. For the upper bound condition (4.8) of inertia matrix,

$$
V_{x_1} \leq (\bar{\lambda}_1^1 + \bar{\lambda}_2^1\|x_1\| + \bar{\lambda}_3^1\|x_1\|^2)\|x_2 + S_1 x_1\|^2 + x_1^T \varepsilon_1 S_1 x_1. \tag{4.39}
$$

Since

$$\|x_1\| \leq \|X_1\|, \tag{4.40}$$

$$\|x_1\|^2 \leq \|x_1\|^2 + \|x_2\|^2 = \|X_1\|^2, \tag{4.41}$$

let

$$\Psi_1 = \begin{bmatrix} S_1^2 & S_1 \\ S_1 & I \end{bmatrix}, \tag{4.42}$$

one has

$$\begin{aligned} \|x_2 + S_1 x_1\|^2 &= \begin{bmatrix} x_1^T & x_2^T \end{bmatrix} \begin{bmatrix} S_1^2 & S_1 \\ S_1 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &\leq \lambda_{\max}(\Psi_1)(\|x_1\|^2 + \|x_2\|^2) \\ &=: \bar{S}_1 \|X_1\|^2, \end{aligned} \tag{4.43}$$

where $\bar{S}_1 = \lambda_{\max}(\Psi_1)$. Therefore, we have

$$V_{X_1} \leq (\bar{\lambda}_1^1 \bar{S}_1 + \varepsilon_1 \lambda_{\max}(S_1))\|X_1\|^2 + \bar{\lambda}_2^1 \bar{S}_1 \|X_1\|^3 + \bar{\lambda}_3^1 \bar{S}_1 \|X_1\|^4. \tag{4.44}$$

$$\begin{aligned} V_1 &= V_{X_1} + V_{\psi_1} \\ &\leq (\bar{\lambda}_1^1 \bar{S}_1 + \varepsilon_1 \lambda_{\max}(S_1))\|X_1\|^2 + \bar{\lambda}_2^1 \bar{S}_1 \|X_1\|^3 + \bar{\lambda}_3^1 \bar{S}_1 \|X_1\|^4 + \frac{1}{2} k_1 \|\tilde{\psi}_1\|^2 \\ &\leq \lambda_1^1 \|Z_1\|^2 + \lambda_2^1 \|Z_1\|^3 + \lambda_3^1 \|Z_1\|^4, \end{aligned} \tag{4.45}$$

where $\lambda_1^1 = \max\{(\bar{\lambda}_1^1 \bar{S}_1 + \varepsilon_1 \lambda_{\max}(S_1)), \frac{1}{2} k_1\}, \lambda_2^1 = \bar{\lambda}_2^1 \bar{S}_1, \lambda_3^1 = \bar{\lambda}_3^1 \bar{S}_1$. Similar to V_1, we can prove that V_2 is also positive definite and decrescent, and it follows

$$V_{X_2} \geq \frac{1}{2} \sum_{i=1}^{n} \lambda_{\min}(\underline{\Omega}_{2i})(x_{3i}^2 + x_{4i}^2), \tag{4.46}$$

$$\begin{aligned} \underline{\Omega}_{2i} &= \begin{bmatrix} \underline{\theta} S_{2i}^2 + 2\varepsilon_2 S_{2i} & \underline{\theta} S_{2i} \\ \underline{\theta} S_{2i} & \underline{\theta} \end{bmatrix}, i = 1, 2, \cdots, n, \\ \underline{\theta} &= \lambda_{\min}(J), \end{aligned} \tag{4.47}$$

$$V_{\psi_2} = \frac{1}{2} k_2 \|\tilde{\psi}_2\|^2. \tag{4.48}$$

By combining (4.46) and (4.48), we have

$$\begin{aligned} V_2 &= V_{X_2} + V_{\psi_2} \\ &\geq \lambda_0^2 (\|X_2\|^2 + \|\tilde{\psi}_2\|^2) =: \lambda_0^2 \|Z_2\|^2, \end{aligned} \tag{4.49}$$

where $\lambda_0^2 = \min\{\min_i\{\frac{1}{2}\lambda_{\min}(\Omega_{2i})\}, \frac{1}{2}k_2\}, i = 1, 2, \cdots, n$, $Z_2 = [X_2^T \quad \tilde{\psi}_2^T]^T$.

$$V_{X_2} \leq \frac{1}{2}\sum_{i=1}^{n} \lambda_{\max}(\bar{\Omega}_{2i})(x_{3i}^2 + x_{4i}^2), \tag{4.50}$$

$$\begin{aligned} \bar{\Omega}_{2i} &= \begin{bmatrix} \bar{\theta}S_{2i}^2 + 2\varepsilon_2 S_{2i} & \bar{\theta}S_{2i} \\ \bar{\theta}S_{2i} & \bar{\theta} \end{bmatrix}, i = 1, 2, \cdots, n, \\ \bar{\theta} &= \lambda_{\max}(J), \end{aligned} \tag{4.51}$$

By combining (4.48) and (4.50), we have

$$\begin{aligned} V_2 &= V_{X_2} + V_{\psi_2} \\ &\leq \lambda_1^2(\|X_2\|^2 + \|\tilde{\psi}_2\|^2) =: \lambda_1^2\|Z_2\|^2, \end{aligned} \tag{4.52}$$

where $\lambda_1^2 = \max\{\max_i\{\frac{1}{2}\lambda_{\max}(\bar{\Omega}_{2i}), \frac{1}{2}k_2\}, i = 1, 2, \cdots, n$. Therefore, we have

$$\xi_1(\|Z\|) = \lambda_0\|Z\|^2 \leq V \leq \lambda_1\|Z\|^2 + \lambda_2\|Z\|^3 + \lambda_3\|Z\|^4 = \xi_2(\|Z\|), \tag{4.53}$$

where $\lambda_0 = \min\{\lambda_0^1, \lambda_0^2\}, \lambda_1 = \max\{\lambda_1^1, \lambda_1^2\}, \lambda_2 = \lambda_2^1, \lambda_3 = \lambda_3^1, Z = [Z_1^T \quad Z_2^T]^T$. That is, V is positive definite and decrescent for all $Z \in \mathrm{R}^{2n+j+s}$. Therefore, we have proved that V is a legitimate Lyapunov function candidate. Taking first derivative of $V_1(X_1)$ along the trajectory of the controlled system yields

$$\begin{aligned} \dot{V}_{X_1} =&(x_2 + S_1x_1)^T\hat{M}(\dot{x}_2 + S_1x_2) + \frac{1}{2}(x_2 + S_1x_1)^T\dot{\hat{M}}(x_2 + S_1x_1) + 2x_1^T\varepsilon_1S_1x_2 \\ =&(x_2 + S_1x_1)^T(\frac{1}{2}\dot{\hat{M}}x_2 + \frac{1}{2}\dot{\hat{M}}S_1x_1 - \hat{C}x_2 - \hat{G} - x_1 + \hat{M}S_1\dot{x} + \tilde{\tau} + x_3) \\ &+ 2x_1^T\varepsilon_1S_1x_2. \end{aligned} \tag{4.54}$$

From (4.18), it can be seen that

$$\dot{V}_{X_1} = (x_2 + S_1x_1)^T\Phi_1 + (x_2 + S_1x_1)^T\tilde{\tau} + (x_2 + S_1x_1)^Tx_3 + 2x_1^T\varepsilon_1S_1x_2. \tag{4.55}$$

According to Assumptions 4.3 and (4.17), we substitute the control (4.24), then

$$
\begin{aligned}
\dot{V}_{X_1} =&(x_2+S_1x_1)^T\Phi_1+(x_2+S_1x_1)^T[-\gamma_1(x_2+S_1x_1)\Pi_1^2(X_1,\hat{\psi}_1)-\varepsilon_1(x_2+S_1x_1)]\\
&+(x_2+S_1x_1)^Tx_3+2x_1^T\varepsilon_1S_1x_2\\
\leq&\|x_2+S_1x_1\|\Pi_1(X_1,\psi_1)+(x_2+S_1x_1)^T[-\gamma_1(x_2+S_1x_1)\Pi_1(X_1,\hat{\psi}_1)^2\\
&-\varepsilon_1(x_2+S_1x_1)]+(x_2+S_1x_1)^Tx_3+2x_1^T\varepsilon_1S_1x_2\\
=&\|x_2+S_1x_1\|\Pi_1(X_1,\psi_1)-\gamma_1\|x_2+S_1x_1\|^2\Pi_1^2(X_1,\hat{\psi}_1)\\
&-x_2^T\varepsilon_1x_2-x_1^T\varepsilon_1S_1^2x_1+(x_2+S_1x_1)^Tx_3\\
\leq&\|x_2+S_1x_1\|\Pi_1(X_1,\psi_1)-\gamma_1\|x_2+S_1x_1\|^2\Pi_1^2(X_1,\hat{\psi}_1)\\
&-\lambda_{x1}\|X_1\|^2+(x_2+S_1x_1)^Tx_3,
\end{aligned}
\tag{4.56}
$$

where $\lambda_{x1}=\min\{\varepsilon_1,\lambda_{\min}(\varepsilon_1S_1^2)\}$. Based on (4.25), the derivative of V_{ψ_1} follows

$$
\begin{aligned}
\dot{V}_{\psi_1}&=(\hat{\psi}_1-\psi_1)^Tk_1\dot{\hat{\psi}}_1\\
&=(\hat{\psi}_1-\psi_1)^T\frac{\partial\Pi_1^T}{\partial\psi_1}(X_1,\hat{\psi}_1)\|x_2+S_1x_1\|-(\hat{\psi}_1-\psi_1)^Tk_1\hat{\psi}_1.
\end{aligned}
\tag{4.57}
$$

From Assumption 4.3, we know that $-\Pi_1(X_1,\cdot)$ is convex for all $X_1\in\mathbf{R}^{2n}$, this leads to

$$
\frac{\partial\Pi_1}{\partial\psi_1}(X_1,\hat{\psi}_1)(\hat{\psi}_1-\psi_1)\leq\Pi_1(X_1,\hat{\psi}_1)-\Pi_1(X_1,\psi_1),
\tag{4.58}
$$

therefore, we have

$$
\dot{V}_{\psi_1}\leq(\Pi_1(X_1,\hat{\psi}_1)-\Pi_1(X_1,\psi_1))\|x_2+S_1x_1\|-(\hat{\psi}_1-\psi_1)^Tk_1\hat{\psi}_1.
\tag{4.59}
$$

By combining (4.56) and (4.59), we have

$$
\begin{aligned}
\dot{V}_1=&\|x_2+S_1x_1\|\Pi_1(X_1,\psi_1)-\gamma_1\|x_2+S_1x_1\|^2\Pi_1^2(X_1,\hat{\psi}_1)-\lambda_{x1}\|X_1\|^2\\
&+(x_2+S_1x_1)^Tx_3+(\Pi_1(X_1,\hat{\psi}_1)-\Pi_1(X_1,\psi_1))\|x_2+S_1x_1\|-(\hat{\psi}_1-\psi_1)^Tk_1\hat{\psi}_1\\
=&\|x_2+S_1x_1\|\Pi_1(X_1,\hat{\psi}_1)-\gamma_1\|x_2+S_1x_1\|^2\Pi_1^2(X_1,\hat{\psi}_1)-\lambda_{x1}\|X_1\|^2\\
&+(x_2+S_1x_1)^Tx_3-(\hat{\psi}_1-\psi_1)^Tk_1\hat{\psi}_1\\
\leq&\frac{1}{4\gamma_1}-\lambda_{x1}\|X_1\|^2+(x_2+S_1x_1)^Tx_3-(\hat{\psi}_1-\psi_1)^Tk_1\hat{\psi}_1.
\end{aligned}
\tag{4.60}
$$

According to the inequality $ab\leq\frac{1}{2}(a^2+b^2),\ a,b\in\mathbf{R}$, we have

$$
\begin{aligned}
(x_2+S_1x_1)^Tx_3&\leq\|x_2+S_1x_1\|\|x_3\|\\
&\leq\frac{1}{2}\omega_1\|x_2+S_1x_1\|^2+\frac{1}{2}\omega_1^{-1}\|x_3\|^2,
\end{aligned}
\tag{4.61}
$$

where $\omega_1 > 0$ is a constant. Therefore,

$$
\begin{aligned}
\dot{V}_1 \leq & \frac{1}{4\gamma_1} - \lambda_{x1}\|X_1\|^2 + \frac{1}{2}\omega_1\|x_2 + S_1x_1\|^2 + \frac{1}{2}\omega_1^{-1}\|x_3\|^2 - (\hat{\psi}_1 - \psi_1)^T k_1 \hat{\psi}_1 \\
= & \frac{1}{4\gamma_1} - \lambda_{x1}\|X_1\|^2 + \frac{1}{2}\omega_1\|x_2 + S_1x_1\|^2 + \frac{1}{2}\omega_1^{-1}\|x_3\|^2 - (\hat{\psi}_1 - \psi_1)^T k_1(\hat{\psi}_1 - \psi_1 + \psi_1) \\
\leq & \frac{1}{4\gamma_1} - \lambda_{x1}\|X_1\|^2 + \frac{1}{2}\omega_1\bar{S}_1\|X_1\|^2 + \frac{1}{2}\omega_1^{-1}\|x_3\|^2 - k_1\|\tilde{\psi}_1\|^2 + k_1\|\psi_1\|\|\tilde{\psi}_1\| \\
\leq & \frac{1}{4\gamma_1} - (\lambda_{x1} - \frac{1}{2}\omega_1\bar{S}_1)\|X_1\|^2 + \frac{1}{2}\omega_1^{-1}\|x_3\|^2 - \frac{1}{2}k_1\|\tilde{\psi}_1\|^2 + \frac{1}{2}k_1\|\psi_1\|^2 \\
\leq & -\underline{\lambda}_1\|Z_1\|^2 + \frac{1}{4\gamma_1} + \frac{1}{2}k_1\|\psi_1\|^2 + \frac{1}{2}\omega_1^{-1}\|x_3\|^2,
\end{aligned}
\tag{4.62}
$$

where $\underline{\lambda}_1 = \min\{\lambda_{x1} - \frac{1}{2}\omega_1\bar{S}_1, \frac{1}{2}k_1\}$.

Next, the derivative of V_2 is given by

$$
\dot{V}_2 = \dot{V}_{X_2} + \dot{V}_{\psi_2}. \tag{4.63}
$$

According to (4.13) and (4.22),

$$
\begin{aligned}
\dot{V}_{X_2} = & (x_4 + S_2x_3)^T J(\dot{x}_4 + S_2x_4) + \frac{1}{2}(x_4 + S_2x_3)^T \dot{J}(x_4 + S_2x_3) + 2x_3^T\varepsilon_2 S_2 x_4 \\
= & (x_4 + S_2x_3)^T(-J\ddot{\tilde{\tau}} - Kx_3 - Fx_4 - F\dot{\tilde{\tau}} + Kx_1 - K\tilde{\tau} \\
& + JS_2x_4 + \frac{1}{2}\dot{J}x_4 + \frac{1}{2}\dot{J}S_2x_3 + \tau) + 2x_3^T\varepsilon_2 S_2 x_4 \\
= & (x_4 + S_2x_3)^T\Phi_2 + (x_4 + S_2x_3)^T\tau + 2x_3^T\varepsilon_2 S_2 x_4.
\end{aligned}
\tag{4.64}
$$

With the use of Assumption 4.4 and substituting the control (4.26), we have

$$
\begin{aligned}
\dot{V}_{X_2} \leq & \|x_4 + S_2x_3\|\rho_2 + (x_4 + S_2x_3)^T\tau + 2x_3^T\varepsilon_2 S_2 x_4 \\
= & \|x_4 + S_2x_3\|\Pi_2(X, \psi_2) + 2x_3^T\varepsilon_2 S_2 x_4 \\
& + (x_4 + S_2x_3)^T[-\gamma_2(x_4(t) + S_2x_3(t))\Pi_2{}^2(X, \hat{\psi}_2(t)) - \varepsilon_2(x_4(t) + S_2x_3(t))].
\end{aligned}
\tag{4.65}
$$

Concerning $\dot{V}_{\psi_2}$, it follows from (4.27)

$$
\begin{aligned}
\dot{V}_{\psi_2} = & (\hat{\psi}_2 - \psi_2)^T k_2 \dot{\hat{\psi}}_2 \\
= & (\hat{\psi}_2 - \psi_2)^T \frac{\partial \Pi_2^T}{\partial \psi_2}(X, \hat{\psi}_2)\|x_4 + S_1x_3\| - (\hat{\psi}_2 - \psi_2)^T k_2 \hat{\psi}_2.
\end{aligned}
\tag{4.66}
$$

For $-\Pi_2(X, \cdot)$ is convex with all $X \in \mathbf{R}^{4n}$, this leads to

$$
\frac{\partial \Pi_2}{\partial \psi_2}(X, \hat{\psi}_2)(\hat{\psi}_2 - \psi_2) \leq \Pi_2(X, \hat{\psi}_2) - \Pi_2(X, \psi_2), \tag{4.67}
$$

therefore, we have

$$\dot{V}_{\psi_2} \leq (\Pi_2(X, \hat{\psi}_2 - \Pi_2(X, \psi_2))\|x_4 + S_2 x_3\| - (\hat{\psi}_2 - \psi_2)^T k_2 \hat{\psi}_2. \tag{4.68}$$

By combining (4.65) and (4.68), we obtain

$$\begin{aligned}\dot{V}_2 =&\|x_4 + S_2x_3\|\Pi_2(X,\psi_2) - \gamma_2\|x_4 + S_2x_3\|^2\Pi_2^2(X,\hat{\psi}_2) - x_4^T\varepsilon_2 x_4 - x_3^T\varepsilon_2 S_2^2 x_3 \\ &+ (\Pi_2(X,\hat{\psi}_2) - \Pi_2(X,\psi_2))\|x_4 + S_2x_3\| - (\hat{\psi}_2 - \psi_2)^T k_2\hat{\psi}_2 \\ \leq&\|x_4+S_2x_3\|\Pi_2(X,\hat{\psi}_2) - \gamma_2\|x_4+S_2x_3\|^2\Pi_2^2(X,\hat{\psi}_2) - \lambda_{x2}\|X_2\|^2 - (\hat{\psi}_2-\psi_2)^T k_2\hat{\psi}_2 \\ \leq&\frac{1}{4\gamma_2} - \lambda_{x2}\|X_2\|^2 - (\hat{\psi}_2 - \psi_2)^T k_2\hat{\psi}_2 \\ \leq&\frac{1}{4\gamma_2} - \lambda_{x2}\|X_2\|^2 - \frac{1}{2}k_2\|\tilde{\psi}_2\|^2 + \frac{1}{2}k_2\|\psi_2\|^2.\end{aligned} \tag{4.69}$$

where $\lambda_{x2} = \min\{\varepsilon_2, \lambda_{\min}(\varepsilon_2 S_2^2)\}$. Therefore, with $\|x_3\|^2 \leq \|X_2\|^2$, the total derivative of V is given by

$$\begin{aligned}\dot{V} = \dot{V}_1 + \dot{V}_2 \leq& -\underline{\lambda}_1\|Z_1\|^2 - (\lambda_{x2} - \frac{1}{2}\omega_1^{-1})\|X_2\|^2 - \frac{1}{2}k_2\|\tilde{\psi}_2\|^2 \\ &+ \frac{1}{4\gamma_1} + \frac{1}{4\gamma_2} + \frac{1}{2}k_1\|\psi_1\|^2 + \frac{1}{2}k_2\|\psi_2\|^2 \\ \leq& -\underline{\lambda}\|Z\|^2 + h,\end{aligned} \tag{4.70}$$

where $\underline{\lambda} = \min\{\underline{\lambda}_1, \underline{\lambda}_2\}$, $\underline{\lambda}_2 = \min\{\lambda_{x2} - \frac{1}{2}\omega_1^{-1}, \frac{1}{2}k_2\}$, $h = \frac{1}{4\gamma_1} + \frac{1}{4\gamma_2} + \frac{1}{2}k_1\|\psi_1\|^2 + \frac{1}{2}k_2\|\psi_2\|^2$.

If we choose suitable $\varepsilon_1, \varepsilon_2$ such that

$$\lambda_{x1} - \frac{1}{2}\omega_1\bar{S}_1 > 0, \tag{4.71}$$

$$\lambda_{x2} - \frac{1}{2}\omega_1^{-1} > 0, \tag{4.72}$$

then

$$\dot{V} < 0, \tag{4.73}$$

for all $\|Z\| \geq \sqrt{h/\underline{\lambda}}$. With the standard form shown in [130], we have the uniform boundedness performance. That is given any $r > 0$ with $\|Z(t_0)\| \leq r$, where t_0 is the initial time, there exists a $d(r)$ given by

$$\begin{aligned}d(r) &= \begin{cases} r\sqrt{\frac{\lambda_1+\lambda_2 r+\lambda_3 r^2}{\lambda_0}} & \text{if} \quad r > R \\ R\sqrt{\frac{\lambda_1+\lambda_2 R+\lambda_3 R^2}{\lambda_0}} & \text{if} \quad r \leq R \end{cases}, \\ R &= \sqrt{h/\underline{\lambda}},\end{aligned} \tag{4.74}$$

such that $\|Z(t)\| \leq d(r)$ for all $t \geq t_0$. Uniform ultimate boundedness also follows. That is, given any $\bar{d}$ with

$$\bar{d} > R\sqrt{\frac{\lambda_1 + \lambda_2 R + \lambda_3 R^2}{\lambda_0}}, \tag{4.75}$$

we have $\|Z(t)\| \leq \bar{d}$ for all $t \geq t_0 + T(\bar{d}, r)$, with

$$T(\bar{d}, r) = \begin{cases} 0 & \text{if} \quad r \leq \bar{R} \\ \frac{\lambda_1 r^2 + \lambda_2 r^3 + \lambda_3 r^4 - \lambda_0 \bar{R}^2}{\underline{\lambda} \bar{R}^2 - h} & \text{if} \quad r > \bar{R} \end{cases} \tag{4.76}$$

$$\bar{R} = \xi_2^{-1}(\lambda_0 \bar{d}^2),$$

Q.E.D.

Remark 4.4. There are no restrictions about (positive) gain parameters S_1 and S_2 which we used in the control design procedure. They can be arbitrary. In a practical case, one may choose these according to the specific situation such as the actuator saturation limits. The choice of ε_1 is to guarantee that the coefficient of $\|X\|^2$ is positive. We choose a ω_1, then, ε_1 can be chosen based on (4.71) and (4.72).

Remark 4.5. In many other control designs, which are Lyapunov-based, the controllers are limited to address the revolute joint case due to the upper bound assumption of inertia matrix. The current work, judged from this, can also be extended to prismatic revolute cases by introducing the new upper bound condition.

Remark 4.6. The resulting performance of the controlled mechanical system is deterministic. Given the control parameters and the adaptive scheme, both uniform boundedness and uniform ultimate boundedness are guaranteed. This performance assures the bottom line. In addition, since there is fuzzy information in the uncertain parameter, it allows us to further address performance considerations. This will be explored in Section 4.6.

4.5 Performance analysis of original system

So far we have proved the uniform boundedness and uniform ultimate boundedness properties of the transformed system. However, the original system is described by $q, \dot{q}_1, q_2, \dot{q}_2$ rather than Z. It is necessary to analyze the performance of the original system. For convenience, let $\tilde{q}_1 = [q_1^T \ \dot{q}_1^T]^T$, $\tilde{q}_2 = [q_2^T \ \dot{q}_2^T]^T$, $\tilde{q} = [\tilde{q}_1^T \ \tilde{q}_2^T]^T$.

Suppose Z is bounded by a constant δ,

$$\|Z\| \leq \delta. \tag{4.77}$$

Then we have

$$\|Z\|^2 = \|X_1\|^2 + \|\tilde{\psi}_1\|^2 + \|X_2\|^2 + \|\tilde{\psi}_2\|^2 \leq \delta^2. \tag{4.78}$$

This implies

$$\|X_1\| \leq \delta,\ \|\tilde{\psi}_1\| \leq \delta,\ \|X_2\| \leq \delta,\ \|\tilde{\psi}_2\| \leq \delta. \tag{4.79}$$

By (4.11),

$$\|\tilde{q}_1\| = \|X_1\| \leq \delta, \quad \|\tilde{q}_2 - \begin{bmatrix}\tilde{\tau}^T & \dot{\tilde{\tau}}^T\end{bmatrix}^T\| \leq \delta, \tag{4.80}$$

$$\tilde{q}_2 = X_2 + \begin{bmatrix} \tilde{\tau} \\ \dot{\tilde{\tau}} \end{bmatrix}. \tag{4.81}$$

According to (4.24), we have

$$\|\tilde{\tau}\| \leq \gamma_1\|x_2 + S_1x_1\|\Pi_1(x_1, x_2, \hat{\psi}_1)^2 + \varepsilon_1\|x_2 + S_1x_1\|. \tag{4.82}$$

By the continuity property of $\Pi_1(x_1, x_2, \hat{\psi}_1)$, if $\|X_1\| \leq \delta$, $\|\tilde{\psi}_1\| \leq \delta$, there exists a constant $\hat{\rho}_1(\delta)$ such that

$$\Pi_1(x_1, x_2, \hat{\psi}_1) \leq \hat{\rho}_1(\delta). \tag{4.83}$$

Therefore,

$$\begin{aligned} \|\tilde{\tau}\| \leq& \gamma_1\|x_2 + S_1x_1\|\hat{\rho}_1^2(\delta) + \varepsilon_1\|x_2 + S_1x_1\| \\ \leq& \gamma_1\hat{\rho}_1^2(\delta)(1 + \|S_1\|)\delta + \varepsilon_1(1 + \|S_1\|)\delta \\ =:& p_1(\delta) + h_1\delta, \end{aligned} \tag{4.84}$$

where

$$\begin{aligned} p_1(\delta) =& \gamma_1\hat{\rho}_1^2(\delta)(1 + \|S_1\|)\delta, \\ h_1 =& \varepsilon_1(1 + \|S_1\|). \end{aligned} \tag{4.85}$$

From (4.12) we obtain

$$\begin{aligned} \|\dot{x}_2\| &= \|\hat{M}^{-1}(-\hat{C}x_2 - \hat{G} - x_1 + x_3 + \tilde{\tau})\| \\ &\leq \|\hat{M}^{-1}\hat{C}\|\|x_2\| + \|\hat{M}^{-1}\hat{G}\| + \|\hat{M}^{-1}\|\|x_1 + x_3 + \tilde{\tau}\| \\ &\leq \sup_{\|X_1\|\leq\delta}\|\hat{M}^{-1}\hat{C}\|\delta + \sup_{\|x_1\|\leq\delta}\|\hat{M}^{-1}\hat{G}\| + \sup_{\|x_1\|\leq\delta}\|\hat{M}^{-1}\|(\delta + \delta + p_1(\delta) + h_1\delta) \\ &=: h_2\delta + h_3 + h_4(2\delta + p_1(\delta) + h_1\delta) \\ &= h_4p_1(\delta) + (h_1h_4 + h_2 + 2h_4)\delta + h_3 \\ &=: p_2(\delta) + h_5\delta + h_3, \end{aligned} \tag{4.86}$$

here

$$
\begin{aligned}
h_2 &= \sup_{\|X_1\|\le\delta} \|\hat{M}^{-1}\hat{C}\| < \infty, \\
h_3 &= \sup_{\|x_1\|\le\delta} \|\hat{M}^{-1}\hat{G}\| < \infty, \\
h_4 &= \sup_{\|x_1\|\le\delta} \|\hat{M}^{-1}\| < \infty, \\
h_5 &= h_1 h_4 + h_2 + 2h_4, \\
p_2(\delta) &= h_4 p_1(\delta).
\end{aligned}
\tag{4.87}
$$

The derivative of $\Pi_1(x_1, x_2, \hat{\psi}_1)$ is bounded by

$$
\begin{aligned}
\dot{\Pi}_1(x_1, x_2, \hat{\psi}_1) \le & \|\frac{\partial \Pi_1^T}{\partial x_1} x_2\| + \|\frac{\partial \Pi_1^T}{\partial x_2} \dot{x}_2\| + \|\frac{\partial \Pi_1^T}{\partial \psi_1} \dot{\hat{\psi}}_1\| \\
\le & \sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial x_1}\| \|x_2\| + \sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial x_2}\| \|\dot{x}_2\| + \sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial \psi_1}\| \|\dot{\hat{\psi}}_1\|.
\end{aligned}
\tag{4.88}
$$

According to the adaptive law given in (4.25),

$$
\begin{aligned}
& \sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial \psi_1}\| \|\dot{\hat{\psi}}_1\| \\
& \quad \le \sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial \psi_1}\| (k_1^{-1} \sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial \psi_1}\| \|x_2 + S_1 x_1\| + \hat{\psi}_1\|) \\
& \quad \le h_6 \delta + h_7,
\end{aligned}
\tag{4.89}
$$

with

$$
\begin{aligned}
h_6 &= (\sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial \psi_1}\|)^2 k_1^{-1} (1 + \|S_1\|) < \infty, \\
h_7 &= \sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial \psi_1}\| \|\hat{\psi}_1\| < \infty.
\end{aligned}
\tag{4.90}
$$

Then we have

$$
\begin{aligned}
\dot{\Pi}_1(x_1, x_2, \hat{\psi}_1) \le & h_8 \delta + h_9 (p_2(\delta) + h_5 \delta + h_3) + h_6 \delta + h_7 \\
= & h_9 p_2(\delta) + (h_6 + h_8 + h_5 h_9)\delta + (h_3 h_9 + h_7),
\end{aligned}
\tag{4.91}
$$

$$
\begin{aligned}
h_8 &= \sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial x_1}\| < \infty, \\
h_9 &= \sup_{\substack{\|X_1\|\le\delta \\ \|\tilde{\psi}_1\|\le\delta}} \|\frac{\partial \Pi_1^T}{\partial x_2}\| < \infty.
\end{aligned}
\tag{4.92}
$$

Furthermore, derivative of $\tilde{\tau}$ is bounded by

$$
\begin{aligned}
\|\dot{\tilde{\tau}}\| \leq& \gamma_1\|\dot{x}_2 + S_1x_2\|\Pi_1(x_1,x_2,\hat{\psi}_1)^2 2\Pi_1(x_1,x_2,\hat{\psi}_1)\dot{\Pi}_1(x_1,x_2,\hat{\psi}_1) \\
&+ \gamma_1\|x_2 + S_1x_1\| + \varepsilon_1\|\dot{x}_2 + S_1x_2\| \\
\leq& \gamma_1(p_2(\delta) + h_5\delta + h_3 + \|S_1\|\delta)\hat{\rho}_1^2(\delta) \\
&+ \gamma_1(\delta + \|S_1\|\delta)2\hat{\rho}_1(\delta)\left(h_9p_2(\delta) + (h_6 + h_8 + h_5h_9)\delta + (h_3h_9 + h_7)\right) \\
&+ \varepsilon_1(p_2(\delta) + h_5\delta + h_3 + \|S_1\|\delta) \\
=:& p_3(\delta) + h_{10}\delta + h_{11},
\end{aligned}
\tag{4.93}
$$

where

$$
\begin{aligned}
p_3(\delta) =& \gamma_1(p_2(\delta) + h_5\delta + h_3 + \|S_1\|\delta)\hat{\rho}_1^2(\delta) \\
&+ \gamma_1(\delta + \|S_1\|\delta)2\hat{\rho}_1(\delta)(h_9p_2(\delta) + (h_6 + h_8 + h_5h_9)\delta + (h_3h_9 + h_7)) \\
&+ \varepsilon_1p_2(\delta), \\
h_{10} =& \varepsilon_1(h_5 + \|S_1\|), \\
h_{11} =& \varepsilon_1h_3.
\end{aligned}
\tag{4.94}
$$

Combining (4.81), (4.84) and (4.93) we have

$$
\begin{aligned}
\|\tilde{q}_2\| &\leq \|X_2\| + \|\tilde{\tau}\| + \|\dot{\tilde{\tau}}\| \\
&\leq \delta + p_1(\delta) + h_1\delta + p_3(\delta) + h_{10}\delta + h_{11} \\
&=: H_1(\delta) + H_2\delta + h_{11},
\end{aligned}
\tag{4.95}
$$

where

$$
\begin{aligned}
H_1(\delta) =& p_1(\delta) + p_3(\delta), \\
H_2 =& 1 + h_1 + h_{10}.
\end{aligned}
\tag{4.96}
$$

This implies that

$$
\begin{aligned}
\|\tilde{q}\| &= \sqrt{\|\tilde{q}_1\|^2 + \|\tilde{q}_2\|^2} \\
&\leq \sqrt{\delta^2 + (H_1(\delta) + H_2\delta + h_{11})^2} \\
&=: \zeta(\delta) < \infty.
\end{aligned}
\tag{4.97}
$$

The uniform boundedness and uniform ultimate boundedness performance of original system follows the following statements:

Uniform boundedness: Given any $r > 0$ with $\|Z(t_0)\| \leq r$, there exists a $r_q = d_q^{-1}(\zeta(\delta))$, if $\|\tilde{q}(t_0)\| \leq r_q$, then $\|\tilde{q}(t)\| \leq d_q(r_q)$ with

$$d_q(r_q) = \begin{cases} r_q \left[\frac{\lambda_1+\lambda_2 r_q+\lambda_3 r_q^2}{\lambda_0}\right]^{\frac{1}{2}} & \text{if } r_q > R_q \\ R_q \left[\frac{\lambda_1+\lambda_2 R_q+\lambda_3 R_q^2}{\lambda_0}\right]^{\frac{1}{2}} & \text{if } r_q \leq R_q \end{cases}, \tag{4.98}$$

$$R_q = \zeta(R).$$

Uniform ultimate boundedness: Let

$$\underline{d}_q = R_q \left[\frac{\lambda_1 + \lambda_2 R_q + \lambda_3 R_q^2}{\lambda_0}\right]^{\frac{1}{2}}, \tag{4.99}$$

given any $\bar{d}_q \geq \underline{d}_q$, $\|\tilde{q}(t)\| \leq \bar{d}_q$ for all $t \geq t_0 + T_q(\bar{d}_q, r_q)$ with $T_q(\bar{d}_q, r_q)$ given by

$$T_q(\bar{d}_q, r_q) = \begin{cases} 0 & \text{if } r_q \leq \bar{R}_q \\ \frac{\lambda_1 r_q^2+\lambda_2 r_q^3+\lambda_3 r_q^4-\lambda_0 \bar{R}_q^2}{\underline{\lambda} \bar{R}_q^2-h} & \text{otherwise} \end{cases}, \tag{4.100}$$

$$\bar{R}_q = \xi_2^{-1}(\lambda_0 \bar{d}_q^2).$$

Theorem 4.3. Subject to Assumptions 4.1-4.4, the control (4.26) renders the system (4.6) uniform boundedness and uniform ultimate boundedness properties.

Proof: The system performance analysis has been shown above. Q.E.D.

Remark 4.7. For a long time of using backstepping method in control design area, the original system is necessarily transformed into another system described by new state variables. The other authors only can prove the stability of the transformed system. Then, they infer the original system is also stable based on the equivalent transformation without giving specific proof. In this work, we prove the stability of the original system theoretically.

4.6 Optimal gain design of control parameters

Sections 4.4 and 4.5 show that the overall system performance can be guaranteed by a deterministic control scheme. By the analysis, the size of the uniform ultimate boundedness region decreases as $\gamma_{1,2}$ increase. This rather strong performance is accomplished by a (possibly) large control effort, which is reflected by $\gamma_{1,2}$. From the practical design point of view, the designer may be interested in seeking an optimal choice of $\gamma_{1,2}$ for a compromise among various conflicting criteria. This is associated with the minimization of a performance index.

Lemma 4.1. For any $Z(t) \in \mathrm{R}^{4n+j+s}$, $t \geq t_0$, there exists a known continuous function $g(\cdot)$ such that

$$g(\|Z\|) = \rho_1\|Z\|^4 + \rho_2\|Z\|^2 \geq \xi_2(\|Z\|) \geq V. \tag{4.101}$$

Proof: Let

$$\begin{aligned} f(\|Z\|) =& g(\|Z\|) - \xi_2(\|Z\|) \\ =& \rho_1\|Z\|^4 + \rho_2\|Z\|^2 - \lambda_1\|Z\|^2 - \lambda_2\|Z\|^3 - \lambda_3\|Z\|^4 \\ =& [(\rho_1 - \lambda_3)\|Z\|^2 - \lambda_2\|Z\| + (\rho_2 - \lambda_1)]\|Z\|^2. \end{aligned} \tag{4.102}$$

If we choose ρ_1, ρ_2 to make

$$\begin{aligned} &\rho_1 - \lambda_3 > 0, \\ &4(\rho_1 - \lambda_3)(\rho_2 - \lambda_1) - \lambda_2 \geq 0, \end{aligned} \tag{4.103}$$

hold, then $f(\|Z\|) \geq 0$ for all $\|Z\| \geq 0$. This implies that $g(\|Z\|) \geq \xi_2(\|Z\|)$. Q.E.D.

Based on the uniform boundedness and uniform ultimate boundedness performance proved above, we know that $(\xi_1^{-1} \circ g)(Z(t_0))$ serves as the upper bound of the transient state performance and $(\xi_1^{-1} \circ g \circ \xi_3^{-1})(h)$ serves as the upper bound of the steady state performance. Let

$$\eta^2(t_0) := (\xi_1^{-1} \circ g)(Z(t_0)) = \frac{1}{\lambda_0}(\rho_1\|Z(t_0)\|^4 + \rho_2\|Z(t_0)\|^2), \tag{4.104}$$

$$\begin{aligned} &\eta_\infty^2(\gamma_1, \gamma_2, \psi_1, \psi_2) \\ =&(\xi_1^{-1} \circ g \circ \xi_3^{-1})(h) \\ =&\frac{1}{\lambda_0}(\frac{\rho_2}{\underline{\lambda}}(\frac{1}{4\gamma_1} + \frac{1}{4\gamma_2} + \frac{1}{2}k_1\|\psi_1\|^2 + \frac{1}{2}k_2\|\psi_2\|^2) \\ &+ \frac{\rho_1}{\underline{\lambda}^2}(\frac{1}{4\gamma_1} + \frac{1}{4\gamma_2} + \frac{1}{2}k_1\|\psi_1\|^2 + \frac{1}{2}k_2\|\psi_2\|^2)^2) \\ =&\frac{1}{\lambda_0}(\frac{1}{\gamma_1}(\frac{\rho_2}{4\underline{\lambda}} + k_1\|\psi_1\|^2 + k_2\|\psi_2\|^2) + \frac{1}{\gamma_2}(\frac{\rho_2}{4\underline{\lambda}} + k_1\|\psi_1\|^2 + k_2\|\psi_2\|^2) \\ &+ \frac{1}{\gamma_1^2}\frac{\rho_1}{16\underline{\lambda}^2} + \frac{1}{\gamma_2^2}\frac{\rho_1}{16\underline{\lambda}^2} + \frac{1}{\gamma_1\gamma_2}\frac{\rho_1}{8\underline{\lambda}} + \frac{\rho_2}{\underline{\lambda}}(\frac{1}{2}k_1\|\psi_1\|^2 + \frac{1}{2}k_2\|\psi_2\|^2) \\ &+ \frac{\rho_1}{4\underline{\lambda}^2}(k_1^2\|\psi_1\|^4 + 2k_1k_2\|\psi_1\|^2\|\psi_2\|^2 + k_2^2\|\psi_2\|^4)) \\ =&\frac{1}{\gamma_1}\frac{\rho_2}{4\lambda_0\underline{\lambda}} + \frac{1}{\gamma_1}\frac{k_1}{\lambda_0}\|\psi_1\|^2 + \frac{1}{\gamma_1}\frac{k_2}{\lambda_0}\|\psi_2\|^2 + \frac{1}{\gamma_2}\frac{\rho_2}{4\lambda_0\underline{\lambda}} + \frac{1}{\gamma_2}\frac{k_1}{\lambda_0}\|\psi_1\|^2 + \frac{1}{\gamma_2}\frac{k_2}{\lambda_0}\|\psi_2\|^2 \\ &+ \frac{1}{\gamma_1^2}\frac{\rho_1}{16\underline{\lambda}^2\lambda_0} + \frac{1}{\gamma_2^2}\frac{\rho_1}{16\underline{\lambda}^2\lambda_0} + \frac{1}{\gamma_1\gamma_2}\frac{\rho_1}{8\underline{\lambda}^2\lambda_0} + \frac{\rho_1}{4\underline{\lambda}^2\lambda_0}k_1^2\|\psi_1\|^4 + \frac{\rho_2}{2\underline{\lambda}\lambda_0}k_1\|\psi_1\|^2 \\ &+ \frac{\rho_1}{4\underline{\lambda}^2\lambda_0}k_2^2\|\psi_2\|^4 + \frac{\rho_2}{2\underline{\lambda}\lambda_0}k_2\|\psi_2\|^2 + \frac{k_1k_2}{2\underline{\lambda}^2\lambda_0}\|\psi_1\|^2\|\psi_2\|^2. \end{aligned} \tag{4.105}$$

Since there is no exact knowledge of the uncertainty, it is only realistic to refer to $\eta(t_0)$ and $\eta_\infty(\bar{\delta}_1, \bar{\delta}_2, \gamma_1, \gamma_2)$ while analyzing the system performance.

Definition 4.1. Consider a fuzzy set

$$\mathcal{N} = \{(\nu, \mu_N(\nu))|\nu \in N\}, \tag{4.106}$$

for any function $f : N \to \mathbf{R}$, the D-operation $D[f(\nu)]$ is given by

$$D[f(\nu)] = \frac{\int_N f(\nu)\mu_N(\nu)d\nu}{\int_N \mu_N(\nu)d\nu}. \tag{4.107}$$

Remark 4.8. In a sense, the D-operation $D[f(\nu)]$ takes an average value of $f(\nu)$ over $\mu_N(\nu)$. In the special case that $f(\nu) = \nu$, this is reduced to the well-known center-of-gravity defuzzification method. Particularly, if N is crisp (i.e., $\mu_N(\nu) = 1$ for all $\nu \in N$), $D[f(\nu)] = f(\nu)$.

Lemma 4.2. For any crisp constant $a \in \mathbf{R}$,

$$\begin{aligned} D[af(\nu)] =& \frac{\int_N af(\nu)\mu_N(\nu)d\nu}{\int_N \mu_N(\nu)d\nu} \\ =& a\frac{\int_N f(\nu)\mu_N(\nu)d\nu}{\int_N \mu_N(\nu)d\nu} \\ =& aD[f(\nu)]. \end{aligned} \tag{4.108}$$

We now propose an index for the system performance: For any $Z(t_0)$, let

$$J(\gamma_1, \gamma_2, t_0) = J_1(\gamma_1, \gamma_2, t_0) + \alpha J_2(\psi_1, \psi_2, \gamma_1, \gamma_2) + \beta J_3(\gamma_1, \gamma_2), \tag{4.109}$$

where

$$J_1(\gamma_1, \gamma_2, t_0) = D[\eta^2(t_0)] \tag{4.110}$$

$$\begin{aligned} &J_2(\psi_1, \psi_2, \gamma_1, \gamma_2) \\ =& D\left[\frac{1}{\gamma_1}\frac{\rho_2}{4\lambda_0\underline{\lambda}}\right] + D\left[\frac{1}{\gamma_1}\frac{k_1}{\lambda_0}\|\psi_1\|^2\right] + D\left[\frac{1}{\gamma_1}\frac{k_2}{\lambda_0}\|\psi_2\|^2\right] + D\left[\frac{1}{\gamma_2}\frac{\rho_2}{4\lambda_0\underline{\lambda}}\right] \\ &+ D\left[\frac{1}{\gamma_2}\frac{k_1}{\lambda_0}\|\psi_1\|^2\right] + D\left[\frac{1}{\gamma_2}\frac{k_2}{\lambda_0}\|\psi_2\|^2\right] + D\left[\frac{1}{\gamma_1^2}\frac{\rho_1}{16\underline{\lambda}^2\lambda_0}\right] + D\left[\frac{1}{\gamma_2^2}\frac{\rho_1}{16\underline{\lambda}^2\lambda_0}\right] \\ &+ D\left[\frac{1}{\gamma_1\gamma_2}\frac{\rho_1}{8\underline{\lambda}^2\lambda_0}\right] + D\left[\frac{\rho_1}{4\underline{\lambda}^2\lambda_0}k_1^2\|\psi_1\|^4 + \frac{\rho_2}{2\underline{\lambda}\lambda_0}k_1\|\psi_1\|^2\right] \\ &+ D\left[\frac{\rho_1}{4\underline{\lambda}^2\lambda_0}k_2^2\|\psi_2\|^4 + \frac{\rho_2}{2\underline{\lambda}\lambda_0}k_2\|\psi_2\|^2\right] + D\left[\frac{k_1k_2}{2\underline{\lambda}^2\lambda_0}\|\psi_1\|^2\|\psi_2\|^2\right], \end{aligned} \tag{4.111}$$

$$J_3(\gamma_1,\gamma_2) = \gamma_1^2 + \gamma_2^2. \tag{4.112}$$

Notice that, $\alpha > 0$ and $\beta > 0$ are weighting factors. Since $\eta(t_0)$ does not depend on the design parameters $\gamma_{1,2}$, J_1 is a constant

$$J_1(\gamma_1,\gamma_2,t_0) = D[\eta^2(t_0)] =: \kappa_0. \tag{4.113}$$

Next, we analyze the J_2 term by Lemma 4.2:

$$\begin{aligned} J_2(\psi_1,\psi_2,\gamma_1,\gamma_2) =& \frac{1}{\gamma_1}\left(\frac{\rho_2}{4\lambda_0\underline{\lambda}} + \frac{k_1}{\lambda_0}D\left[\|\psi_1\|^2\right] + \frac{k_2}{\lambda_0}D\left[\|\psi_2\|^2\right]\right) + \frac{1}{\gamma_2}\left(\frac{\rho_2}{4\lambda_0\underline{\lambda}}\right. \\ &\left. + \frac{k_1}{\lambda_0}D\left[\|\psi_1\|^2\right] + \frac{k_2}{\lambda_0}D\left[\|\psi_2\|^2\right]\right) + \frac{1}{\gamma_1^2}\frac{\rho_1}{16\underline{\lambda}^2\lambda_0} + \frac{1}{\gamma_2^2}\frac{\rho_1}{16\underline{\lambda}^2\lambda_0} \\ &+ \frac{1}{\gamma_1\gamma_2}\frac{\rho_1}{8\underline{\lambda}^2\lambda_0} + D\left[\frac{\rho_1}{4\underline{\lambda}^2\lambda_0}k_1^2\|\psi_1\|^4 + \frac{\rho_2}{2\underline{\lambda}\lambda_0}k_1\|\psi_1\|^2\right] \\ &+ D\left[\frac{\rho_1}{4\underline{\lambda}^2\lambda_0}k_2^2\|\psi_2\|^4 + \frac{\rho_2}{2\underline{\lambda}\lambda_0}k_2\|\psi_2\|^2\right] + D\left[\frac{k_1k_2}{2\underline{\lambda}^2\lambda_0}\|\psi_1\|^2\|\psi_2\|^2\right]. \end{aligned} \tag{4.114}$$

Let

$$\begin{aligned} \kappa_1 =& D\left[\frac{\rho_1}{4\underline{\lambda}^2\lambda_0}k_1^2\|\psi_1\|^4 + \frac{\rho_2}{2\underline{\lambda}\lambda_0}k_1\|\psi_1\|^2\right] \\ &+ D\left[\frac{\rho_1}{4\underline{\lambda}^2\lambda_0}k_2^2\|\psi_2\|^4 + \frac{\rho_2}{2\underline{\lambda}\lambda_0}k_2\|\psi_2\|^2\right] + D\left[\frac{k_1k_2}{2\underline{\lambda}^2\lambda_0}\|\psi_1\|^2\|\psi_2\|^2\right] \\ \kappa_2 =& \frac{\rho_2}{4\lambda_0\underline{\lambda}} + \frac{k_1}{\lambda_0}D\left[\|\psi_1\|^2\right] + \frac{k_2}{\lambda_0}D\left[\|\psi_2\|^2\right], \\ \kappa_3 =& \frac{\rho_1}{16\underline{\lambda}^2\lambda_0}, \\ \kappa_4 =& \frac{\rho_1}{8\underline{\lambda}^2\lambda_0}, \end{aligned} \tag{4.115}$$

with (4.112), (4.113) and (4.114) into (4.109), we have

$$J(\gamma_1,\gamma_2,t_0) = \kappa_0 + \alpha\left(\kappa_1 + \frac{\kappa_2}{\gamma_1} + \frac{\kappa_2}{\gamma_2} + \frac{\kappa_3}{\gamma_1^2} + \frac{\kappa_3}{\gamma_2^2} + \frac{\kappa_4}{\gamma_1\gamma_2}\right) + \beta(\gamma_1^2+\gamma_2^2). \tag{4.116}$$

The optimal design problem is then equivalent to the following constrained optimization problem: For any t_0

$$\min_{\gamma_1,\gamma_2} J(\gamma_1,\gamma_2,t_0) \qquad \text{subject} \quad \text{to} \gamma_1,\gamma_2 > 0. \tag{4.117}$$

For any t_0, taking the first order derivative of J respecting to γ_1,γ_2,

$$\begin{aligned} \frac{\partial J}{\partial \gamma_1} &= -\frac{\alpha\kappa_2}{\gamma_1^2} - \frac{\alpha\kappa_3}{\gamma_1^3} - \frac{\alpha\kappa_4}{\gamma_1^2\gamma_2} + 2\beta\gamma_1, \\ \frac{\partial J}{\partial \gamma_2} &= -\frac{\alpha\kappa_2}{\gamma_2^2} - \frac{\alpha\kappa_3}{\gamma_2^3} - \frac{\alpha\kappa_4}{\gamma_1\gamma_2^2} + 2\beta\gamma_2. \end{aligned} \tag{4.118}$$

That

$$\frac{\partial J}{\partial \gamma_1} = 0, \qquad \frac{\partial J}{\partial \gamma_2} = 0 \tag{4.119}$$

leads to

$$\begin{cases} 2\beta\gamma_1^4 = \alpha\kappa_3 + \alpha(\kappa_2 + \frac{\kappa_4}{\gamma_2})\gamma_1 \\ 2\beta\gamma_2^4 = \alpha\kappa_3 + \alpha(\kappa_2 + \frac{\kappa_4}{\gamma_1})\gamma_2 \end{cases} \tag{4.120}$$

with $\gamma_1, \gamma_2 > 0$. The solution of (4.120) can be obtained numerically. Suppose the solution are γ_1^*, γ_2^*, taking the second derivative of J, one gets

$$\begin{aligned} A =& \frac{\partial^2 J}{\partial \gamma_1^2}|_{\gamma_1^*,\gamma_2^*} = \frac{2\alpha\kappa_2}{\gamma_1^{*3}} + \frac{3\alpha\kappa_3}{\gamma_1^{*4}} + \frac{2\alpha\kappa_4}{\gamma_1^{*3}\gamma_2^*} + 2\beta =: h_1(\gamma_1^*) + \frac{2\alpha\kappa_4}{\gamma_1^{*3}\gamma_2^*}, \\ B =& \frac{\partial}{\partial \gamma_2}\left(\frac{\partial J}{\partial \gamma_1}\right)|_{\gamma_1^*,\gamma_2^*} = \frac{\partial}{\partial \gamma_1}\left(\frac{\partial J}{\partial \gamma_2}\right)|_{\gamma_1^*,\gamma_2^*} = \frac{\alpha\kappa_4}{\gamma_1^{*2}\gamma_2^{*2}}, \\ C =& \frac{\partial^2 J}{\partial \gamma_2^2}|_{\gamma_1^*,\gamma_2^*} = \frac{2\alpha\kappa_2}{\gamma_2^{*3}} + \frac{3\alpha\kappa_3}{\gamma_2^{*4}} + \frac{2\alpha\kappa_4}{\gamma_1^*\gamma_2^{*3}} + 2\beta =: h_2(\gamma_2^*) + \frac{2\alpha\kappa_4}{\gamma_1^*\gamma_2^{*3}}, \end{aligned} \tag{4.121}$$

here, $h_1(\gamma_1^*) = \frac{2\alpha\kappa_2}{\gamma_1^{*3}} + \frac{3\alpha\kappa_3}{\gamma_1^{*4}} + 2\beta > 0$, $h_2(\gamma_2^*) = \frac{2\alpha\kappa_2}{\gamma_2^{*3}} + \frac{3\alpha\kappa_3}{\gamma_2^{*4}} + 2\beta > 0$.

$$\begin{aligned} B^2 - AC =& (\frac{\alpha\kappa_4}{\gamma_1^{*2}\gamma_2^{*2}})^2 - h_1(\gamma_1^*)\frac{2\alpha\kappa_4}{\gamma_1^*\gamma_2^{*3}} - h_2(\gamma_2^*)\frac{2\alpha\kappa_4}{\gamma_1^{*3}\gamma_2^*} - h_1(\gamma_1^*)h_2(\gamma_2^*) - \frac{4\alpha^2\kappa_4^2}{\gamma_1^{*4}\gamma_2^{*4}} \\ =:& -\frac{3\alpha^2\kappa_4^2}{\gamma_1^{*4}\gamma_2^{*4}} - h_3(\gamma_1^*, \gamma_2^*), \end{aligned} \tag{4.122}$$

where

$$h_3(\gamma_1^*, \gamma_2^*) = h_1(\gamma_1^*)\frac{2\alpha\kappa_4}{\gamma_1^*\gamma_2^{*3}} + h_2(\gamma_2^*)\frac{2\alpha\kappa_4}{\gamma_1^{*3}\gamma_2^*} + h_1(\gamma_1^*)h_2(\gamma_2^*) > 0. \tag{4.123}$$

Therefore, $A > 0$ and $B^2 - AC < 0$. Thus, γ_1^*, γ_2^* solve the constrained minimization problem.

4.7 Illustrative example

A two-link FJM, see Figure 2.5, is considered to show the validity of the control proposed in this chapter. Let link angle vector $q_1 = [q_{1,1} \quad q_{1,2}]^T$, joint angle vector $q_2 = [q_{2,1} \quad q_{2,2}]^T$. The system model of the two-link FJM is given by

$$
\begin{aligned}
M(q_1) &= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \\
C(q_1, \dot{q}_1) &= \begin{bmatrix} -m_1 l_1 l_{c2} \sin q_{1,2} \dot{q}_{1,2} & -m_2 l_1 l_{c2} \sin q_{1,2} (\dot{q}_{1,1} + \dot{q}_{1,2}) \\ m_2 l_1 l_{c2} \sin q_{1,2} \dot{q}_{1,1} & 0 \end{bmatrix}, \\
G(q_1) &= \begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \sin q_{1,1} + m_2 l_{c2} g \sin(q_{1,1} + q_{1,2}) \\ m_2 l_{c2} g \sin(q_{1,1} + q_{1,2}) \end{bmatrix}, \\
J &= \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}, S_1 = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{12} \end{bmatrix}, S_2 = \begin{bmatrix} S_{21} & 0 \\ 0 & S_{22} \end{bmatrix},
\end{aligned}
\tag{4.124}
$$

where

$$
\begin{aligned}
M_{11} &= m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos q_{1,2}) + m_1 l_{c1}^2 + I_1 + I_2, \\
M_{12} &= m_2(l_{c2}^2 + l_1 l_{c2} \cos q_{1,2}) + I_2, \\
M_{21} &= M_{12}, \\
M_{22} &= m_2 l_{c2}^2 + I_2.
\end{aligned}
\tag{4.125}
$$

All elements of inertia matrix are bounded by following

$$
\begin{aligned}
|M_{11}| &\le m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}) + m_1 l_{c1}^2 + I_1 + I_2, \\
|M_{12}| &\le m_2(l_{c2}^2 + l_1 l_{c2}) + I_2, \\
|M_{22}| &\le m_2 l_{c2}^2 + I_2.
\end{aligned}
\tag{4.126}
$$

To evaluate $||M||$, recall again that here the norms are Euclidean, we invoke the norm inequality that for a matrix $\Lambda \in \mathbf{R}^{p \times q}$, $\frac{1}{\sqrt{p}}||\Lambda||_1 \le ||\Lambda||_2 \le \sqrt{p}||\Lambda||_1$. This means, $\frac{1}{\sqrt{2}}||M||_1 \le ||M||_2 \le \sqrt{2}||M||_1$. Here, to avoid confusions we use subscripts 1 and 2 to denote two classes of induced matrix norms. Then we have

$$
\begin{aligned}
||M||_1 &= \max_j \sum_i |d_{ij}| \\
&= \max\{|d_{11}| + |d_{21}|, |d_{12}| + |d_{22}|\} \\
&= m_1 l_{c1}^2 + m_2(l_1^2 + 2 l_{c2}^2 + 3 l_1 l_{c2}) + I_1 + 2 I_2.
\end{aligned}
\tag{4.127}
$$

We choose bounding function as

$$
\begin{aligned}
\Pi_1(q_1, \dot{q}_1, \hat{\psi}_1) &= \hat{\psi}_{11}(q_{1,1}^2 + q_{1,2}^2) + \hat{\psi}_{12}(\dot{q}_{1,2}^2 + \dot{q}_{1,2}^2), \\
\Pi_2(q_1, \dot{q}_1, q_2, \dot{q}_2, \hat{\psi}_2) &= \hat{\psi}_{21}(q_{1,1}^2 + q_{1,2}^2) + \hat{\psi}_{22}(q_{2,1}^2 + q_{2,2}^2) + \hat{\psi}_{23}(\dot{q}_{1,1}^2 + \dot{q}_{1,2}^2) \\
&\quad + \hat{\psi}_{24}(\dot{q}_{2,1}^2 + \dot{q}_{2,2}^2).
\end{aligned}
\tag{4.128}
$$

The adaptive laws $\hat{\psi}_1$ and $\hat{\psi}_2$ are given by

$$
\begin{aligned}
\dot{\hat{\psi}}_1 &= \begin{bmatrix} \dot{\hat{\psi}}_{11} & \dot{\hat{\psi}}_{12} \end{bmatrix}^T \\
&= \frac{1}{k_1}\begin{bmatrix} \|x_2 + S_1x_1\|(q_{1,1}^2 + q_{1,2}^2) \\ \|x_2 + S_1x_1\|(\dot{q}_{1,1}^2 + \dot{q}_{1,2}^2) \end{bmatrix} - \begin{bmatrix} \hat{\psi}_{11} \\ \hat{\psi}_{12} \end{bmatrix}, \\
\dot{\hat{\psi}}_2 &= \begin{bmatrix} \dot{\hat{\psi}}_{21} & \dot{\hat{\psi}}_{22} & \dot{\hat{\psi}}_{23} & \dot{\hat{\psi}}_{24} \end{bmatrix}^T \\
&= \frac{1}{k_2}\begin{bmatrix} \|x_4 + S_2x_3\|(q_{1,1}^2 + q_{1,2}^2) \\ \|x_4 + S_2x_3\|(q_{2,1}^2 + q_{2,2}^2) \\ \|x_4 + S_2x_3\|(\dot{q}_{1,1}^2 + \dot{q}_{1,2}^2) \\ \|x_4 + S_2x_3\|(\dot{q}_{2,1}^2 + \dot{q}_{2,2}^2) \end{bmatrix} - \begin{bmatrix} \hat{\psi}_{21} \\ \hat{\psi}_{22} \\ \hat{\psi}_{23} \\ \hat{\psi}_{24} \end{bmatrix}.
\end{aligned} \tag{4.129}
$$

Then we have the controllers as

$$
\begin{aligned}
\tilde{\tau} &= -\gamma_1(x_2 + S_1x_1){\Pi_1}^2 - \varepsilon_1(x_2 + S_1x_1), \\
\tau &= -\gamma_2(x_4 + S_2x_3){\Pi_2}^2 - \varepsilon_2(x_4 + S_2x_3),
\end{aligned} \tag{4.130}
$$

where $x_1 = q_1, x_3 = q_2 - \tilde{\tau}$. Note that, $\tau = [\tau_1 \quad \tau_2]^T$. For numerical demonstration, we choose $l_1 = 1$, $l_{c1} = l_{c2} = 0.5$, $I_1 = I_2 = 1$, $J_{11} = J_{22} = 1$, $g = 9.81$, $S_{11} = S_{12} = 1$, $S_{21} = S_{22} = 2$, $\omega_1 = 1, \varepsilon_1 = \varepsilon_2 = 1$. Assume that $m_1 = \bar{m}_1 + \Delta m_1$ and $m_2 = \bar{m}_2 + \Delta m_2$, $K_1 = \bar{K}_1 + \Delta K_1$ and $K_2 = \bar{K}_2 + \Delta K_2$, where $\bar{m}_{1,2}, \bar{K}_{1,2}$ are the nominal portions of the mass and elasticity coefficient while $\Delta m_{1,2}$ and $\Delta K_{1,2}$ are uncertain portions. Let $\bar{m}_1 = \bar{m}_2 = 1, \bar{K}_1 = \bar{K}_2 = 1$. For the uncertainty, we choose the following: the uncertainties in m_1, m_2 are the same and denoted as $\Delta m_1 = \Delta m_2 = \Delta m$ which is "close to 0.3". The associated membership function (triangular) is given by

$$
\mu_{\Delta m}(\nu) = \begin{cases} \frac{10}{3}\nu, & 0 \le \nu \le 0.3, \\ -\frac{10}{3}\nu + 2, & 0.3 \le \nu \le 0.6. \end{cases} \tag{4.131}
$$

The uncertainty is elasticity coefficient is "close to 0.2" and denoted by $\Delta K_1 = \Delta K_2 = \Delta K$ with the membership function (triangular) given by

$$
\mu_{\Delta K}(\nu) = \begin{cases} \frac{10}{2}\nu, & 0 \le \nu \le 0.2, \\ -\frac{10}{2}\nu + 2, & 0.2 \le \nu \le 0.4. \end{cases} \tag{4.132}
$$

The uncertain portions of $\psi_{1,2}$ in adaptation laws are "close to 0.25", the associated membership function (triangular) is given by

$$
\mu_{\psi_{1,2}}(\nu) = \begin{cases} \frac{10}{2.5}\nu, & 0 \le \nu \le 0.25, \\ -\frac{10}{2.5}\nu + 2, & 0.25 \le \nu \le 0.5. \end{cases} \tag{4.133}
$$

With these, we have $\lambda_0 = 0.5, \lambda_1 = 8.8388, \lambda_2 = \lambda_3 = 0, \underline{\lambda} = 0.5, \rho_1 = 0, \rho_2 = \lambda_1$. Let us choose crisp initial condition $\|Z(t_0)\| = \|Z(0)\| = 2.1012$. This means $J_1 = D[\eta^2(t_0)] = \kappa_0 = 78.0473$. By utilizing the fuzzy number arithmetic and the decomposition theorem mentioned before, we have $\kappa_1 = 8.8597, \kappa_2 = 9.0938, \kappa_3 = \kappa_4 = 0$. Then the quartic equation (4.120) is given by

$$\begin{cases} 2\beta\gamma_1^4 = 9.0938\alpha\gamma_1 \\ 2\beta\gamma_2^4 = 9.0938\alpha\gamma_2. \end{cases} \tag{4.134}$$

For the weighting factors, we choose five (α, β) combinations. Their values and the corresponding $\gamma_{1,2opt}$ and J_{min} are summarized in Table 4.1.

Table 4.1: Weighting / optimal gain / minimum cost

(α,β)	α/β	γ_{1opt}	γ_{2opt}	J_{min}
(1,1)	1	1.6567	1.6567	103.3745
(1,10)	0.1	0.7690	0.7690	122.3851
(1,100)	0.01	0.3569	0.3569	163.3423
(10,1)	10	3.5692	3.5692	243.0792
(100,1)	100	7.6896	7.6896	1.318×10^3

The simulations are performed by using $\Delta m = 0.4|\sin(10t)|$, $\Delta K = 0.4|\cos(10t)|$. Figure 4.1 shows the state norm (i.e., $\|q(t)\|$) trajectory with adaptive robust control (under $\gamma_1 = \gamma_2 = 3.5692, \alpha = 10, \beta = 1$) and PD control. The controlled trajectory $\|q(t)\|$ enters into a small area near 0 after a short time, hence, the system is ultimately bounded. The PD controlled system has a large fluctuation at the first several seconds. The velocity in controlled system (i.e., $\|\dot{q}\|$, see Figure 4.2) also has a smaller scale and less fluctuation. The corresponding histories of the control inputs are shown in Figure 4.3, where $\|\tau\| = \sqrt{\tau_1^2 + \tau_2^2}$. The adaptive parameters $\hat{\psi}_1$ and $\hat{\psi}_2$ are used to approximate the parameters ψ_1 and ψ_2 in functions Π_1 and Π_2 which are related to the bounds of uncertain terms in the control. Since the leakage parts exist in the adaptive laws, $\hat{\psi}_1$ and $\hat{\psi}_2$ may decrease by the reduction of the following errors. The trajectory of the norms of adaptive parameters $\|\hat{\psi}_1\|, \|\hat{\psi}_2\|$ are shown in Figure 4.4.

Figure 4.5 shows the trajectories of controlled system by using different weighting factors (α, β) combinations. Figure 4.6 shows the corresponding histories of control efforts $\|\tau(t)\|$.

In order to describe effect of the magnitude of uncertainty bound in this example, we define two parameters as $\hat{m} = \max_t \Delta m(t), \quad \hat{K} = \max_t \Delta K(t)$ which stand for the bounds of uncertainty in m, K, respectively. For simplicity, we take the optimal $\gamma_1 = \gamma_2 = 3.5692$ (under $\alpha = 10, \beta = 1$)

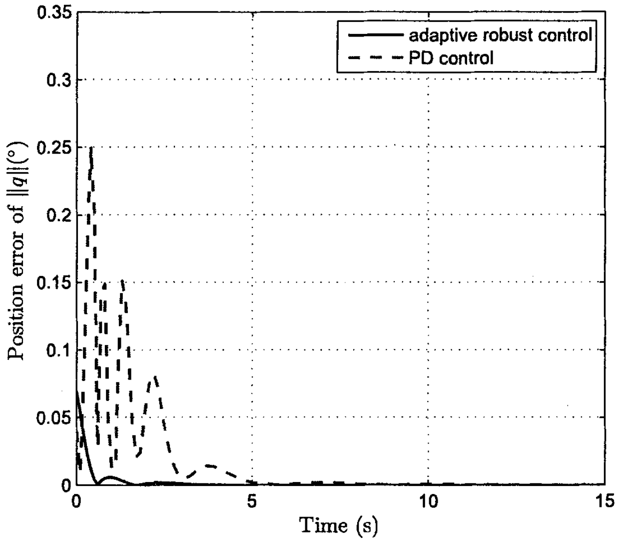

Figure 4.1: Comparison of controlled system performance.

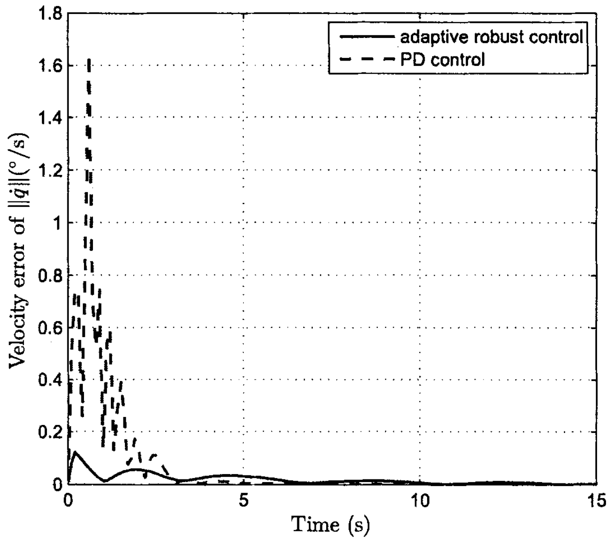

Figure 4.2: Velocities Comparison of controlled system performance.

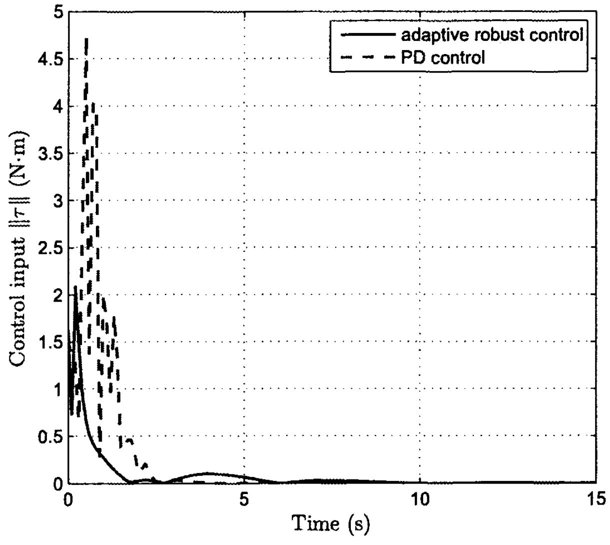

Figure 4.3: Comparison of control efforts.

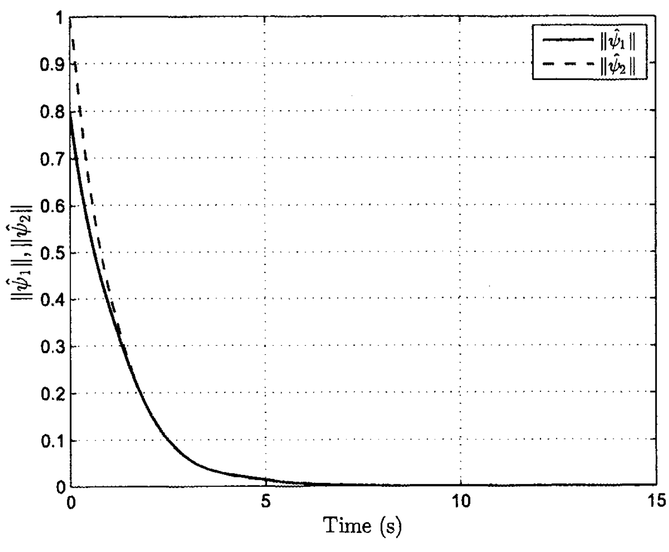

Figure 4.4: Trajectories of adaptive parameters.

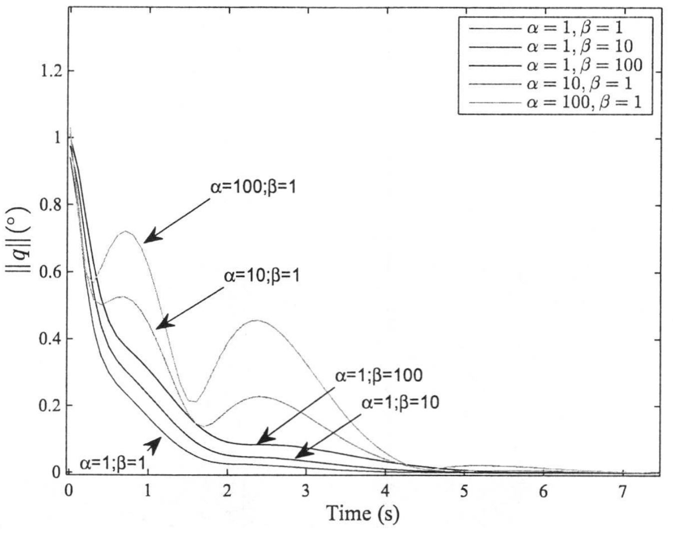

Figure 4.5: Comparison of system performance under different (α, β) sets.

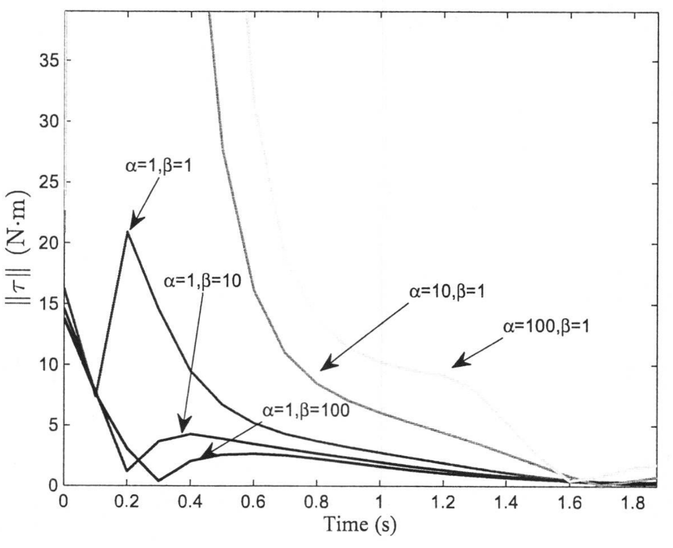

Figure 4.6: Comparison of control magnitude under different (α, β) sets.

for example, the relation curve of the average control effort τ_{ave} and the uncertainty bound is shown in Figure 4.7, with τ_{ave} given by

$$\tau_{ave} = \frac{\int_0^T \tau(t)dt}{T}|_{(\hat{m},\hat{K})}. \tag{4.135}$$

The relation between the maximum of the control $||\tau||_{\max}$ and the bounds of uncertainty is shown

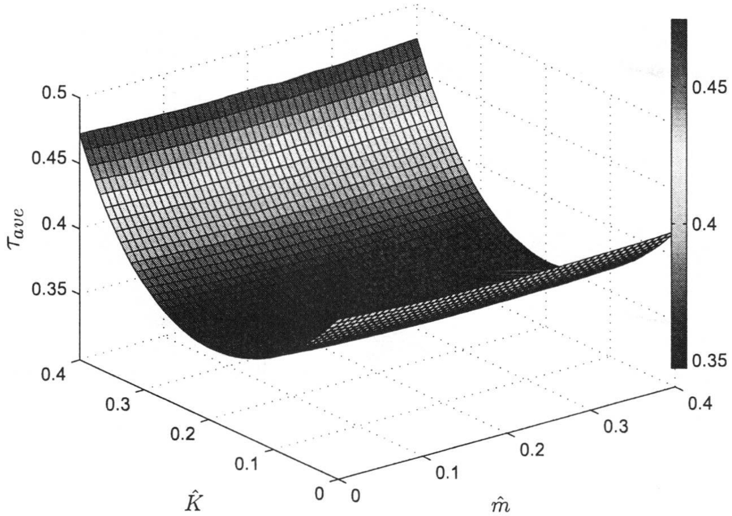

Figure 4.7: Average control effort τ_{ave} under optimal gains.

in Figure 4.8. The $||\tau||_{\max}$ is given by

$$||\tau||_{\max} = \max_t ||\tau||(t)|_{(\hat{m},\hat{K})}. \tag{4.136}$$

4.8 Conclusions

The control for mismatched FJM system whose uncertainty and states are fuzzily described is considered. By implanting a fictitious control, the original mismatched system is divided into two subsystems so that the matching condition is satisfied. An adaptive robust control, which is deterministic and is not IF-THEN rules-based, is proposed to guarantee the system uniformly bounded and uniformly ultimately bounded. This is proved via Lyapunov minimax approach. Meanwhile, not only the stability (uniform boundedness and uniform ultimate boundedness) of the system is

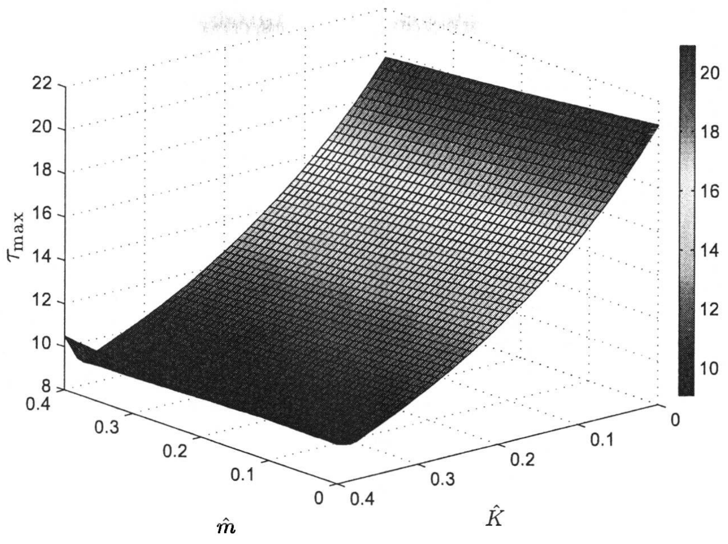

Figure 4.8: $\|\tau\|_{\max}$ with different bound of uncertainty.

guaranteed, but also the fuzzy performance is optimized in the consideration of control cost. The optimal control problem is equivalently transformed to a performance index minimization problem. We show that the extreme solution to this minimization problem is indeed the optimal parameter to the control. This assertion is verified by the second order sufficient condition.

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Chapter 5 Dynamical model and control of constrained mechanical system

5.1 Introduction

As we know, the principles of mechanics are so sufficiently investigated that it is hard to establish a new fundamental principal for the theory of motion of dynamical system. However, a rather different angle of view has been proposed by Udwadia and Kalaba which are useful to help us understand Nature's law.

One of the central problem in analytical dynamics is the determination of equations of motion for constrained mechanical systems, which was firstly formulated 200 years ago and has been energetically worked on by many engineers, mathematicians and scientists since then. Lagrange established the initial description of constrained motion and invented Lagrange multiplier method to handle constrained motion. Gauss introduced a general, new principle of mechanics (commonly called as Gauss's Principle) for handling constrained motion. Gibbs [89] and Appell [140] independently formulated the equations of motion by virtue of the concept of quasi-coordinates. Dirac used Poisson bracket and developed a recursive scheme for determining the Lagrange multipliers for singular, Hamiltonian systems. Udwadia and Kalaba [94] came up with a simple, explicit set of equations of motion for general constrained mechanical system, which can deal with holonomic and/or non-holonomic constraints. Above principles are all governed by the usual d'Alembert's principle which indicates the forces of constraints are considered to be ideal and do zero work under virtual displacements. However, the constraints could also be nonideal. Thus, Udwadia and Kalaba [95, 141] generalized their equations to constrained mechanical systems that may not satisfy d'Alembert's Principle.

5.2 Moore-Penrose generalized inverse

In 1903, Fredholm introduced the concept of pseudoinverse for the integral operators. In linear algebra, the pseudoinverse A^+ of a matrix A is a generalization of the inverse matrix A^{-1}. The term generalized inverse is also regarded as a synonym for pseudoinverse if there is no further specification. The most famous type of the pseudoinverse is the Moore-Penrose generalized inverse, which was reported by Eliakim Hastings Moore [142] in 1920, Arne Bjerhammar [143] in 1951 and Roger Penrose [144] in 1955, independently. The common application of using Moore-Penrose generalized inverse is to find a "best fit" (least squares) solution to linear equations which lack unique solution. Besides, it can be used to compute the minimum norm solution to linear equations which have multiple solutions.

For a given matrix $A_{m\times n}$, the Moore-Penrose generalized inverse A^+ satisfies the following four equations

$$\begin{aligned} AA^+A &= A, \\ A^+AA^+ &= A^+, \\ (A^+A)^T &= AA^+, \\ (AA^+)^T &= A^+A. \end{aligned} \tag{5.1}$$

A^+ always exists for any given A. When A is with full rank, in particular, A is with linearly independent columns (i.e., A^TA is invertible), then A^+ can be find by

$$A^+ = (A^TA)^{-1}A^T. \tag{5.2}$$

This special case is so-called *left inverse*, since, here $A^+A = I$. In contrast, when the rows of A are linearly independent and AA^T is invertible, then we have

$$A^+ = A^T(AA^T)^{-1}. \tag{5.3}$$

This is called *right inverse* for $AA^+ = I$.

We have several approaches to find the solution to (5.1):

(i) For a given $A \in \mathbf{R}^{m\times n}$, the rank of A is $r_A \geq 1$, the eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. Assume the singular value decomposition (SVD) of A is

$$A = U\Sigma V^T = \sum_{i=1}^{r} \lambda_i u_i v_i^T, \tag{5.4}$$

where $\Sigma = diag[\lambda_i]_{r\times r}$, $U = [u_1, u_2, \cdots, u_r] \in \mathbf{R}^{m\times r}$ and $V = [v_1, v_2, \cdots, v_r] \in \mathbf{R}^{n\times r}$ are unitary matrixes. Furthermore, the $\{u_1, u_2, \cdots, u_r\} \in \mathbf{R}^{m\times r}$ and $\{v_1, v_2, \cdots, v_r\} \in \mathbf{R}^{n\times r}$ are orthonormal sets over $\mathbf{R}^m$ and $\mathbf{R}^n$. Then the Moore-Penrose generalized inverse of A is

$$A^+ = V\Sigma^{-1}U^T. \tag{5.5}$$

(ii) Assume that the decomposition of A is $A = B_1B_2$, $B_1 \in \mathbf{R}^{m\times r}$, $B_2 \in \mathbf{R}^{r\times n}$, and $rank(A) = rank(B_1) = rank(B_2) = r$, then the Moore-Penrose generalized inverse of A can be obtained by

$$A^+ = B_2^T(B_2B_2^T)^{-1}(B_1^TB_1)^{-1}B_1^T. \tag{5.6}$$

(iii) Let A_k be the submatrix of A including the first k columns of A. Then A_k can be decomposed as

$$A_k = [A_{k-1} \quad a_k], \tag{5.7}$$

where a_k is the k-th column of A_k. Defining the following vectors:

$$\begin{aligned} d_k :=& A_{k-1}^+ a_k, \\ c_k :=& a_k - A_{k-1}d_k \\ =& a_k - A_{k-1}A_{k-1}^+ a_k \\ =& (I - A_{k-1}A_{k-1}^+)a_k, \end{aligned} \tag{5.8}$$

$$b_k^T := \begin{cases} c_k^+, & \text{if } c_k \neq 0 \\ (1 + d_k^Td_k)^{-1}d_k^TA_{k-1}^+, & \text{if } c_k = 0 \end{cases}, \tag{5.9}$$

The Moore-Penrose generalized inverse of A can then be computed step-by-step $A^+ (A^+ = A_n^+)$

$$A_k^+ = \begin{bmatrix} A_{k-1}^+ - d_kb_k^T \\ b_k^T \end{bmatrix}. \tag{5.10}$$

Theorem 5.1. [145] For any given matrix $A \in \mathbf{R}^{m\times n}$, there always exists a Moore-Penrose generalized inverse $A^+ \in \mathbf{R}^{n\times m}$, and the Moore-Penrose generalized inverse is unique.

Theorem 5.2. Consider a matrix $A \in \mathbf{R}^{m\times n}$, with $rank(A) = r \geq 1$, the following properties hold:

$$\mathcal{R}(A^T) = \mathcal{R}(A^+) = \mathcal{R}(A^+A), \tag{5.11}$$

$$\mathcal{N}(A) = \mathcal{R}(I - A^+A), \tag{5.12}$$

where $\mathcal{R}(\cdot)$ denotes the range space of matrix, $\mathcal{N}(\cdot)$ denotes the null space of matrix.

Proof: With the singular-value decomposition (5.4), then A^T can be decomposed as

$$\begin{aligned}
A^T =& V\Sigma U^T = \sum_{i=1}^{r} \lambda_i v_i u_i^T \\
=& \lambda_1 v_1 u_1^T + \lambda_2 v_2 u_2^T + \cdots + \lambda_r v_r u_r^T \\
=& \lambda_1 v_1 [u_{11} \quad u_{12} \quad \cdots \quad u_{1m}] \\
& + \lambda_2 v_2 [u_{21} \quad u_{22} \quad \cdots \quad u_{2m}] \\
& + \cdots \\
& + \lambda_r v_r [u_{r1} \quad u_{r2} \quad \cdots \quad u_{rm}] \\
=& \left[\sum_{i=1}^{r} \lambda_i u_{i1} v_i \quad \sum_{i=1}^{r} \lambda_i u_{i2} v_i \quad \cdots \sum_{i=1}^{r} \lambda_i u_{im} v_i \right],
\end{aligned} \tag{5.13}$$

here u_{ij} is the j-th component of $u_i \in \mathbf{R}^m$. This in turn means each column of the matrix A^T is consisted of a combination of $v_1, v_2, \cdots, v_r$. In other words, the range space of A^T (i.e., $\mathcal{R}(A^T)$) is spanned by $v_1, v_2, \cdots, v_r$. Similarly, with (5.5), we have

$$A^+ = \left[\sum_{i=1}^{r} \frac{1}{\lambda_i} u_{i1} v_i \quad \sum_{i=1}^{r} \frac{1}{\lambda_i} u_{i2} v_i \quad \cdots \sum_{i=1}^{r} \frac{1}{\lambda_i} u_{im} v_i \right] \tag{5.14}$$

This means the range space of A^T is also spanned by $v_1, v_2, \cdots, v_r$. According to (5.4) and (5.6), we have

$$\begin{aligned}
A^+ A =& (V\Sigma^{-1}U^T)(U\Sigma V^T) \\
=& V\Sigma^{-1}(U^T U)\Sigma V^T) \\
=& VV^T = \sum_{i=1}^{r} v_i v_i^T.
\end{aligned} \tag{5.15}$$

Obviously, each column of A^+A is consisted of linear combination of $v_1, v_2, \cdots, v_r$. That is to say, the range space of matrix A^+A is spanned by $v_1, v_2, \cdots, v_r$.

Next, let us prove the second part of Theorem 5.2. Consider a vector $x \in \mathcal{R}(I - A^+A)$, then there exists a vector $y \in \mathbf{R}^n$ such that

$$x = (I - A^+A)y, \tag{5.16}$$

then we have

$$Ax = A(I - A^+A)y = (A - AA^+A)y. \tag{5.17}$$

Accordoing to (5.1), we have

$$\begin{aligned}
Ax =& (A - AA^+A)y \\
=& (A - A)y \\
=& 0,
\end{aligned} \tag{5.18}$$

which in turn means if $x \in \mathcal{R}(I - A^+A)$ then $x \in \mathcal{N}(A)$. Next, consider a vector $x \in \mathcal{N}(A)$, i.e., $Ax = 0$ or $x^TA^T = 0$. For any given matrix A, $\mathcal{R}(A^T) \perp \mathcal{N}(A)$, that is to say, $x \perp \mathcal{R}(A^T)$, or, in other words, x is perpendicular to the space which is spanned by $v_1, v_2, \cdots, v_r$. Let us choose vectors $v_{r+1}, \cdots, v_n$, then in the space $\mathbf{R}^n$, by combining with $v_1, v_2, \cdots, v_r$, we form an orthonormal basis: $I = \sum_{i=1}^{n} v_i v_i^T$. Then we have

$$\begin{aligned} I - A^+A &= \sum_{i=1}^{n} v_i v_i^T - \sum_{i=1}^{r} v_i v_i^T \\ &= \sum_{i=1}^{r} v_i v_i^T + \sum_{i=r+1}^{n} v_i v_i^T - \sum_{i=1}^{r} v_i v_i^T \\ &= \sum_{i=r+1}^{n} v_i v_i^T. \end{aligned} \tag{5.19}$$

Similar to the proof of the first part of Theorem 5.2, it can be easily concluded that the range space of $I - A^+A$ is spanned by vectors $v_{r+1}, \cdots, v_n$. This in turn leads to $\mathcal{R}(I - A^+A) \perp \mathcal{R}(A^T)$. Recall that $x \perp \mathcal{R}(A^T)$, therefore, we have $x \in \mathcal{R}(I - A^+A)$.

Q.E.D.

5.3 Mechanical system subject to the constraint

Firstly, we consider a dynamical system described by n-dimensional generalized coordinate $q = [q_1, q_2, \cdots, q_n]^T \in \mathbf{R}^n$, the kinetic energy of the system is given by

$$T(q, \dot{q}, t) = \frac{1}{2}\dot{q}^T M(q,t)\dot{q} + N(q,t)\dot{q} + P(q,t), \tag{5.20}$$

where $M(q,t) = M^T(q,t) \in \mathbf{R}^{n\times n}$ is the inertia matrix, $N(q,t) \in \mathbf{R}^{1\times n}$, $P(q,t) \in \mathbf{R}$ [146].

Assume that there is a generalized force, which is denoted as $Q \in \mathbf{R}^n$, to the system. It is determined by the time t, the generalized position q and the generalized velocity $\dot{q} = dq/dt$, i.e., $Q = Q(q, \dot{q}, t)$. It should be noted that the effect of generalized acceleration $\ddot{q} = d^2q/dt^2$ is ignored here for simplicity, readers can refer to [147] for further information. The equation of motion is described as following

$$\frac{d}{dt}\frac{\partial T(q,\dot{q},t)}{\partial \dot{q}} - \frac{\partial T(q,\dot{q},t)}{\partial q} = Q(q,\dot{q},t) + Q^c, \tag{5.21}$$

where $Q^c = [Q_1^c \quad Q_2^c \quad \cdots \quad Q_n^c] \in \mathbf{R}^n$, arising by the constraints, is usually treated as the constraint force of the system. In literatures, how to obtain $Q^c(q, \dot{q}, t)$ becomes a vital issue in the modeling of constrained mechanical system.

With (5.20) into (5.21) yields

$$M(q,t)\ddot{q} + \left(\frac{d}{dt}M(q,t)\right)\dot{q} + \frac{dN(q,t)}{dt} - \frac{\partial T(q,\dot{q},t)}{\partial q} = Q(q,\dot{q},q) + Q^c. \tag{5.22}$$

Let

$$F(q,\dot{q},t) = \left(\frac{d}{dt}M(q,t)\right)\dot{q} + \frac{dN(q,t)}{dt} - \frac{\partial T(q,\dot{q},t)}{\partial q}, \tag{5.23}$$

then we have

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) - F(q,\dot{q},t) + Q^c(q,\dot{q},t). \tag{5.24}$$

Assume the system is subject to the following m constraints ($m < n$):

$$\sum_{i=1}^{n} A_{li}(q,t)\dot{q}_i + A_l(q,t) = 0, \quad l = 1,2,\cdots,m, \tag{5.25}$$

where $A_{li}(\cdot) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ and $A_l(\cdot) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ is C^1 continuous. The constraint above can be put into matrix form

$$A(q,t)\dot{q} = c(q,t), \tag{5.26}$$

here, $A = [A_{li}]_{m\times n}$, $c = [c_1, c_2, \cdots, c_m]^T$. The constraints may be holonomic or nonholonomic. Then the Pfaffican form of (5.25) is

$$\sum_{i=1}^{n} A_{li}(q,t)dq_i + A_l(q,t)dt = 0, \quad l = 1,2,\cdots,m, \tag{5.27}$$

where q_i is the i-th element of q. In Lagrangian mechanics, according to d'Alember's principle, the total work arising by virtue of constraint force is zero under virtual displacements, which means

$$\sum_{i=1}^{n} A_{li}(q,t)\delta q_i = 0, \quad l = 1,2,\cdots,m, \tag{5.28}$$

where δq_i is the i-th element of δq. Let $A(q,t) = [A_{li}(q,t)]_{m\times n}$, then (5.26) can be reexpressed as

$$A(q,t)\delta q = 0. \tag{5.29}$$

This equation means δq is in the Null Space of $A(q,t)$: $\delta q \in \mathcal{N}(A(q,t))$. Furthermore, the constraint force is in the Column Space of $A^T(q,t)$, i.e., $Q^c(q,\dot{q},t) \in \mathcal{R}(A^T(q,t))$. In Lagrangian mechanics, Lagrange Multiplier is the most common method to find the model of system. According to this approach, the constraint force is

$$Q^c = A^T(q,t)\lambda, \tag{5.30}$$

where $\lambda \in \mathbf{R}^m$ is the Lagrange Multiplier. Then Eq. (5.24) can be rewritten as

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) - F(q,\dot{q},t) + A^T(q,t)\lambda. \tag{5.31}$$

Remark 5.1. Generally, for given initial conditions, Lagrange Multiplier $\lambda \in \mathbf{R}^m$ and the generalized acceleration $\ddot{q}$ could be computed by (5.26) and (5.31). However, the solution to these equations is usually not in closed-form and depends on the initial condition. Thus, this approach can not be adopted as a common method for the modeling of constrained mechanical system. As there is not a explicit expression of the constraint force, this approach can not be used in the control design either. In the next section, the latest development of the mechanical system dynamics will be introduced.

5.4 Udwadia-Kalaba equation

Taking the time derivative of (5.26) yields

$$A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t), \tag{5.32}$$

where $A(q, \dot{q}, t) \in \mathbf{R}^{m \times n}$, $m < n$, is constraint matrix and $b(q, \dot{q}, t)$ is a m-dimensional vector.

Assumption 5.1. For each $(q, t) \in \mathbf{R}^n \times \mathbf{R}$, $M(q, t) > 0$.

Remark 5.2. In many past studies, the inertia matrix was always believed to be positive definite. However, this property is not always true because it is semi-definite in some cases. The detailed analysis can be found in Chen et al. [129]. Therefore, we state it as an assumption rather than a fact.

Definition 5.1. For given matrix $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$, the constraint (5.32) is said to be consistent if there exists at least one solution $\ddot{q}$.

Remark 5.3. For a constraint force servo control problem, this consistency is a necessary condition. That is to say, if the constraints cannot render equation (5.32) consistent, then we cannot find a solution τ to the constraint force servo control problem.

Assumption 5.2. Consider the constraint (5.32), the constraint is consistent and $rank(A) \geq 1$, which means the Moore-Penrose generalized inverse of A exists.

Let $\ddot{x} = M^{-1/2}\ddot{q}, a = M^{-1/2}(Q - F), F^c = M^{-1/2}Q^c$, then we rewrite (5.24) as

$$\ddot{x} = a + F^c = a + M^{-\frac{1}{2}}B\tau =: a + \bar{B}\tau, \tag{5.33}$$

where $B = [B_{il}]_{n\times p}$ is the input matrix, $\tau = [\tau_1 \quad \tau_2 \quad \cdots \quad \tau_p]$ is the control to be designed. Then the constraint force servo control problem is equivalent to find τ such that the system (5.33) follows the constraint

$$C\ddot{x} = b, \tag{5.34}$$

where $C = AM^{-\frac{1}{2}}$.

Lemma 5.1. [148] Subject to Assumption 5.2, the constraint equation (5.32) is consistent if and only if

$$AA^{+}b = b. \tag{5.35}$$

Here, it should be noted that the mapping from $\ddot{q}$ to $\ddot{x}$ is one-to-one correspondence, thus, the statement "for given C and b, the constraint equation (5.34) is consistent" is equivalent to "for given A and b, the constraint equation (5.32) is consistent". Therefore, the constraint equation (5.34) is consistent if and only if

$$CC^{+}b = b. \tag{5.36}$$

Lemma 5.2. Subject to Assumptions 5.1 and 5.2, the solution to the constraint equation (5.34) is given by

$$\ddot{x} = C^{+}b + (I - C^{+}C)r, \tag{5.37}$$

where $r \in \mathbf{R}^n$ is an arbitrary vector.

Proof: it can be easily shown that

$$rank(C) = rank(AM^{-\frac{1}{2}}) = rank(A) \geq 1, \tag{5.38}$$

then we have $rank(C) \geq 1$, which means C^{+} exists. By using (5.1) and multiplying both sides of (5.37), one gets

$$C\ddot{x} = CC^{+}b + (C - CC^{+}C)r = b. \tag{5.39}$$

Theorem 5.3. Subject to Assumptions 5.1 and 5.2, consider the F^c which has the following form

$$F^c = -C^{+}Ca + C^{+}b + (I - CC^{+})r, \tag{5.40}$$

where $r \in \mathbf{R}^n$ is an arbitrary vector and may be independent on the time and system state. The system (5.33) satisfies the constraint equation (5.34) when F^c is applied.

Proof: Since C^+, which is the Moore-Penrose generalized inverse of C, exists. We can decompose a into two portions which are in $\mathcal{R}(C^T)$ and $\mathcal{N}(C)$, respectively:

$$a = C^+Ca + (I - C^+C)a. \tag{5.41}$$

By substituting F^c and (5.41) into the system, we have

$$\begin{aligned} \ddot{x} =& \underbrace{C^+Ca + (I - C^+C)a}_{a} - \underbrace{C^+Ca + C^+b + (I - C^+C)r}_{F^c} \\ =& C^+b + (I - C^+C)(a + r). \end{aligned} \tag{5.42}$$

Multiplying C on both sides of (5.42) yields

$$\begin{aligned} C\ddot{x} =& CC^+b + C(I - C^+C)(a + r) \\ =& CC^+b + (C - CC^+C)(a + r). \end{aligned} \tag{5.43}$$

Recall that $CC^+b = b$, using (5.1), we obtain

$$C\ddot{x} = b + (C - C)(a + r) = b. \tag{5.44}$$

Q.E.D.

Let us rewrite (5.40) as

$$F^c = C^+(b - Ca) + (I - C^+C)r. \tag{5.45}$$

It can be seen that what F^c does is to adjust the projection of $\ddot{x}$ in $\mathcal{R}(C^T)$ to be C^+b. Since for any $r \in \mathbf{R}^n$, F^c is acceptable, the projection of $\ddot{x}$ in $\mathcal{N}(C)$ seems to be unimportant. Then we have

$$\min_{r \in \mathbf{R}^n} \|F^c\| = C^+(b - Ca), \tag{5.46}$$

the norm here is Euclidean. This means, F^c obtains its minimum norm when $r = 0$. Udwadia and Kalaba studied this problem and gave the formula of the system under the constraint force

$$\ddot{x} = a + C^+(b - Ca), \tag{5.47}$$

or

$$M\ddot{q} = (Q - F) + \underbrace{M^{\frac{1}{2}}(AM^{-\frac{1}{2}})^+[b - AM^{-1}(Q - F)]}_{constranintforce}. \tag{5.48}$$

This is the Udwadia-Kalaba equation. The closed-form of the constraint force is given by

$$Q^c = M^{\frac{1}{2}}(AM^{-\frac{1}{2}})^+[b - AM^{-1}(Q - F)]. \tag{5.49}$$

Remark 5.4. The constraint force Q^c which is introduced into the system in (5.21) is a joint space expression. In most of practical cases, the constraint (may be the trajectory constraint, velocity constraint, force constraint, etc.) is imposed on the end-effector, which is in the operation space. Conventional method to obtain Q^c requires a constraint force transformation from operation space to joint space. In this work, by using Udawadia-Kalaba equation, we can have explicit joint space constraint force directly. The procedure of calculating Q^c is shown above. More applications of using Udawadia-Kalaba equation in solving constrained manipulator problems can be found in Chen [149], Zhang [150] and Zhao [151].

Based on the analysis above, the modeling of the dynamics of constrained system via Udwadia-Kalaba approach can be summarized into three steps.

1. We consider the equation of motion of the system without constraints. As it is shown in (5.22), it can be obtained by Newtonian or Lagrangian mechanics with respect to the generalized coordinate.

2. The constraints should be taken into account. Here, the constraints could be holonomic constraints, nonholonomic constraints, scleronomic constraints, rheonomic constraints, catastatic constraints or even their combinations. Rewrite the constraints into matrix form, we have (5.26).

3. Taking the second order of the constraints yields (5.32). Then we can obtain the constraint force based on (5.49).

Remark 5.5. The advantage of finding the equations of motion for constrained system via this new theory is obvious. In comparing with the Lagrange multiplier method, there is no need to compute the multiplier which is difficult to obtain, especially for systems with a large number of degrees of freedom and/or under many non-integrable constraints.

5.5 Discussion of the constraint

The second-order form of constraint (5.32), which includes some typical constraints as shown in, for example, Rosenberg [146] and Papastavridis [92], is a very common form. A number of standard constrained problems in our control design can be cast into (5.32).

5.5.1 Stabilization problem

Suppose designer is to develop a control such that the partial dynamics of the system is stable. Let $\bar{q} \in \mathbf{R}^m$ represent a portion of the system state described by generalized coordinate $q \in \mathbf{R}^n, m \le n$. Let $A \in \mathbf{R}^{m\times n}$ be a constant matrix whose elements are 0's and 1's, then the relation between $\bar{q}$ and q can be denoted as $\bar{q} = Aq$. For example, if $q = [q_1, q_2, q_3, q_4, q_5]$ and $\bar{q} = [q_1, q_2, q_4]$, then we have

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5.50}$$

Consider the dynamics of $\bar{q}$ described by the following system:

$$\dot{\bar{q}} = h(\bar{q}, t) \tag{5.51}$$

where $h(\cdot) : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^m, h(0,t) = 0, \forall t \in \mathbf{R}$, and $\bar{q} = 0$ of (5.51) is asymptotically stable. That is to say, a control is required to stabilize $\bar{q}$. Let us take the time derivative of both sides of (5.51), note that $\bar{q} = Aq, \dot{\bar{q}} = A\dot{q}$, we have

$$\ddot{\bar{q}} = \frac{\partial h}{\partial \bar{q}} + \frac{\partial h}{\partial t} =: \bar{b}(\dot{\bar{q}}, \bar{q}, t) = \bar{b}(A\dot{q}, Aq, t) =: b(\dot{q}, q, t). \tag{5.52}$$

Since $\ddot{\bar{q}} = A\ddot{q}$, equation (5.52) can be reexpressed as $A\ddot{q} = b$, this is in the same form of equation (5.32). In special cases, if the dynamics of (5.51) is chosen to be linear and time-invariant:

$$\dot{\bar{q}} = \Lambda \bar{q}, \tag{5.53}$$

where Λ is Hurwitz and $\Lambda \in \mathbf{R}^{m\times m}$ with all elements are constant scalars.

5.5.2 Trajectory following

Suppose that $\bar{q}$ is required to follow a prescribed desired trajectory $\bar{q}_d(t) \in \mathbf{R}^m$, where the function $\bar{q}_d(\cdot) : \mathbf{R} \to \mathbf{R}^m$ is C^2. Define the follow error $\bar{e} := \bar{q} - \bar{q}_d$. Then the dynamics of $\bar{e}$ can be described as

$$\dot{\bar{e}} = \bar{h}(\bar{e}, t), \tag{5.54}$$

where $\bar{h}(\cdot) : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^m$, $\bar{h}(0,t) = 0, \forall t \in \mathbf{R}$, and $\bar{e} = 0$ of the equation (5.54) is asymptotically stable. After taking the time derivative of both side of (5.54), we have

$$\ddot{\bar{e}} = \frac{\partial \bar{h}}{\partial \bar{e}}\dot{\bar{e}} + \frac{\partial \bar{h}}{\partial t} =: \bar{b}_e(\dot{\bar{e}}, \bar{e}, t). \tag{5.55}$$

Since $\ddot{\bar{e}} = \ddot{\bar{q}} - \ddot{\bar{q}}_d(t), \ddot{\bar{q}} = \ddot{\bar{e}} + \ddot{\bar{q}}_d(t)$, one gets

$$\begin{aligned}\ddot{q} =& A\ddot{\bar{q}} \\ =& A\ddot{\bar{e}} + A\ddot{\bar{q}}_d(t) \\ =& A\bar{b}_e(\dot{\bar{e}}, \bar{e}, t) + A\ddot{\bar{q}}_d(t) \\ =& A\bar{b}_e(\dot{\bar{q}} - \dot{\bar{q}}_d(t), \bar{q} - \bar{q}_d(t), t) + A\ddot{\bar{q}}_d(t) \\ =:& b_e(\dot{q}, q, t).\end{aligned} \tag{5.56}$$

This is of the same form as constraint equation (5.32). In special cases, if the dynamics of (5.54) is chosen to be linear and time-invariant:

$$\dot{\bar{e}} = \Lambda_e \bar{e}. \tag{5.57}$$

where Λ_e is Hurwitz and $\Lambda_e \in \mathbf{R}^{m\times m}$ with all elements are constant scalars. Then we have

$$\begin{aligned}\ddot{q} =& A\Lambda_e\dot{\bar{e}} + A\ddot{\bar{q}}_d(t) \\ =& A\Lambda_e(\dot{\bar{q}} - \dot{\bar{q}}_d(t)) + A\ddot{\bar{q}}_d(t) \\ =& A\Lambda_e A\dot{q} - A\Lambda_e\dot{\bar{q}}_d(t) + A\ddot{\bar{q}}_d(t) \\ =:& b_e(\dot{q}, q, t).\end{aligned} \tag{5.58}$$

5.5.3 Optimization problem

Suppose that the partial state $\bar{q}$ is governed by the following equation

$$\dot{\bar{q}} = h_0(\bar{q}, t) \tag{5.59}$$

where $h_0(\cdot) : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^m$. Meanwhile, function $h(\cdot)$ is chosen to minimize one of the following performance indices:

$$\int_{t_0}^{T} \|W\bar{q}\| dt, \tag{5.60}$$

$$\int_{t_0}^{T} \|W\bar{q}\|^2 dt, \tag{5.61}$$

$$\sup_{t\in[t_0,T]} \|W\bar{q}\|, \tag{5.62}$$

where the matrix $W \in \mathbf{R}^{m\times m}$ is weighting. The performance indices here, representing the transient performance of (5.59), are indeed the weighted $\mathcal{L}_1$-norm, $\mathcal{L}_2$-norm and $\mathcal{L}_\infty$-norm, respectively. When the time T approaches to infinity, this problem turns into an infinite-time horizon optimality problem. If the system (5.59) is autonomous, which means $h_0(\bar{q}, t) = h_0(\bar{q})$, then the initial time can be chosen as $t_0 = 0$ such that the indices are more generic. After taking a similar procedure shown in Section 5.5.1, the constrained problem (5.59) can then be transformed into the form of constrained problem (5.32).

In practical cases, a dynamic system also needs to meet numerical constraint which has been fully discussed by, e.g., Papastavridis [92]. For example, as it is sometimes reported, the numerical simulation of the system which is under prescribed constraint generates numerical drift of constraints and integrals. Therefore, addition (numerical) constraint is required to counteract such numerical drift. A standard method can be found in [152]. By using this technique, numerical constraint and practical constraint can be combined and then turn to be in the form of (5.32).

5.6 Constraint force servo control design

After obtaining the dynamical model of constrained system via Udwaida-Kalaba equation, then we can consider the design of τ.

Definition 5.2. For a system described by (5.33) with respect to a set $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, if there is a control τ such that by applying this control, the system meets the given constraints (5.34), then the system is said to be servo constraint controllable for all $(\dot{q}, q, t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$.

Let $\bar{b} = b - Ca$. For given $C, \bar{B}, \bar{b}$, our task is to find a $\tau \in \mathbf{R}^m$ such that

$$(C\bar{B})\tau = \bar{b}. \tag{5.63}$$

Assumption 5.3. For the given $C(q,t), \bar{B}(\dot{q}, q, t)$ and $\bar{b}(\dot{q}, q, t)$, the equation (5.63) is consistent such that

$$(C\bar{B})(C\bar{B})^{+}\bar{b} = \bar{b}. \tag{5.64}$$

Theorem 5.4. Subject to Assumptions 5.1 to 5.3, for all $(\dot{q}, q, t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, the control τ which is in the following form

$$\tau = (C\bar{B})^{+}\bar{b} + [I - (C\bar{B})^{+}(C\bar{B})]s, \tag{5.65}$$

where $s \in \mathbf{R}^m$ is an arbitrary vector, renders the system (5.33) servo constraint controllable.

Proof. Since the Moore-Penrose generalize inverse of $(C\bar{B})$ exists, by applying the control τ

in (5.65) and multiplying C on both sides of (5.33), we have

$$
\begin{aligned}
C\ddot{x} =& Ca + C\bar{B}\{(C\bar{B})^{+}\bar{b} + [I - (C\bar{B})^{+}(C\bar{B})]s\} \\
=& Ca + \underbrace{C\bar{B}(C\bar{B})^{+}\bar{b}}_{\bar{b}} + [C\bar{B} - \underbrace{C\bar{B}(C\bar{B})^{+}(C\bar{B})}_{C\bar{B}}]s \\
=& Ca + \underbrace{\bar{b}}_{b-Ca} \\
=& b
\end{aligned}
\tag{5.66}
$$

It can be seen that servo control τ in (5.65) is a model-based state feedback control and is continuous in the state.

5.7 Conclusions

Udwadia-Kalaba approach is a new understanding of the constrained motion. It provides an explicit equation to treat the holonomic and/or non-holonomic constraints, even the non-ideal constraint force. By using this approach, the Lagrangian multipliers are not required, and the difficulty in computing the constraint force is overcome. Moreover, the constraint force is presented in the closed-form so that can be applied in control easily. Moore-Penrose generalized inverse plays as a momentous role in Udwadia-Kalaba equation. It reveals the geometrical structure of the constraint force. There is no transformation or elimination of coordinates when constraints are imposed. The coordinates used to describe the unconstrained system are the same as those in describing the constrained system.

The significance of Udwaida-Kalaba equation is twofold. First, constraints are derived into the second-order differentiated form, other than integrated form. Thus, not only nonholonomic constraints can be included, but also many other control problems, including stabilization problem, trajectory tracking problem, optimization problem, can be cast into this form. Second, it utilizes a fixed manner to treat the constraints no matter what the constraints are. Designers do not need to select the quasi-variables nor compute the Lagrange multiplier. They can obtain the closed-form of constraint force other than numerical solutions. It is believed that, Udwadia-Kalaba equation not only benefits the control for FJM system, but also can be applied to other fields of modeling and control, including biomechanics, multibody system, astronomy, etc.

Chapter 6 Constraint force servo control for uncertain flexible joint manipulator

6.1 Introduction

In practical situation, the movement of flexible joint manipulator (FJM) is usually required to follow certain constraints. To achieve this performance, the ideal case is to design a controller which can provide the desired constraint force. However, the research was blocked in obtaining this constraint force. As we know, the constraint has different forms and may be nonholonomic which makes the constraint equations unintegrable. Moreover, control inputs of the FJM system are less than its degrees of freedom (DOFs) , in other words, it is under-actuated. Therefore, it is hard to have precise description of constraint force. As an alternative way, the problem can be treated as a passive constraint problem, which means the environment (including the structure requirements and physical limitations) can generate the required constraint force to restrain the motion [92]. For example, the planar robot can only move in a surface under the constraint force from the structure. The designer is only responsible for providing force to follow the given trajectory. Based on this precondition, many outstanding investigations have been made to assure that the constraints are obeyed (see [93, 153, 154]).

The situation has been turned recently. Under the frame work of Udwadia and Kalaba [94, 95], the required constraint force can be obtained by adopting the Moore-Penrose generalized inverse to explore the geometric structure of the constraint [96]. The significant progress is that the constraint force is presented in the closed form so that it can be applied in control design easily. This makes *engineer* imitate what the *Nature* does so that the constraints are followed [92]. Furthermore, several classes of control problems can be formulated as constraint servo control [97].

Fuzzy theory was originally introduced to describe information (for example, linguistic information) that is in lack of a sharp boundary with its environment [80]. Most interest in fuzzy theory is attracted to fuzzy reasoning for control, estimation, decision–making, etc. This study, on the other hand, propose a rather different angle. With the fuzzy description of the system state and uncertainty (so called fuzzy dynamical system), it endeavors to design a fuzzy approach for the control of FJM system.

6.2 Preliminaries

Consider the following mechanical system [146]:

$$M(q(t),\sigma(t),t)\ddot{q}(t)+C(q(t),\dot{q}(t),\sigma(t),t)\dot{q}(t)+G(q(t),\sigma(t),t)=\tau(t), \tag{6.1}$$

here $t \in \mathbf{R}$ is the independent variable, $q \in \mathbf{R}^n$ is the vector of coordinates, $\dot{q} \in \mathbf{R}^n$ is the velocity, $\ddot{q} \in \mathbf{R}^n$ is the acceleration, $\sigma \in \Sigma \subset \mathbf{R}^p$ is the uncertain parameter, and $\tau \in \mathbf{R}^n$ is the control input. Σ is compact and bounded, which stands for the possible bound of σ. Furthermore, $M(q,\sigma,t)$ is the inertia matrix, $C(q,\dot{q},\sigma,t)\dot{q}$ is the Coriolis/centrifugal force, and $G(q,\sigma,t)$ is the gravitational force. The matrices/vector $M(q,\sigma,t), C(q,\dot{q},\sigma,t)$ and $G(q,\sigma,t)$ are of appropriate dimensions. We assume that the functions $M(\cdot), C(\cdot)$ and $G(\cdot)$ are continuous (this can be generalized to be Lebesgue measurable in t).

Remark 6.1. The vector of coordinates can be selected based on the specifics of the problem and does not need to be the *generalized coordinate*.

The following constraints are proposed:

$$\sum_{i=1}^{n} A_{li}(q,t)\dot{q}_i = c_l(q,t), \qquad l=1,2,\cdots,m, \tag{6.2}$$

where the $\dot{q}_i$ is the i-th component of $\dot{q}$. $A_{li}(\cdot)$ and $c_l(\cdot)$ are both C^1, $m \leq n$. They are the *first order* form of the constraints. The constraints may not be integrable and may be nonholonomic in general. The constraints can be put in matrix form:

$$A(q,t)\dot{q} = c(q,t), \tag{6.3}$$

where $A=[A_{li}]_{m\times n}$, $c=[c_1 \quad c_2 \quad \cdots \quad c_m]^T$.

There are two ways of interpreting the constraints. First, they may be passive. That is, the environment (or the structure) is to supply the constraint force for the system in order to comply with the constraint. Second, they may be active. That is, the system's control input supplies the required force so that the constraints are met. In this study, we shall adopt the second view.

We now convert the first order form into *second order* form [155]. Taking the derivative of constraint (6.2) with respect to t yields

$$\sum_{i=1}^{n}\left(\frac{d}{dt}A_{li}(q,t)\right)\dot{q}_i+\sum_{i=1}^{n}A_{li}(q,t)\ddot{q}_i=\frac{d}{dt}c_l(q,t), \tag{6.4}$$

where

$$\frac{d}{dt}A_{li}(q,t)=\sum_{k=1}^{n}\frac{\partial A_{li}(q,t)}{\partial q_k}\dot{q}_k+\frac{\partial A_{li}(q,t)}{\partial t}, \tag{6.5}$$

$$\frac{d}{dt}c_l(q,t)=\sum_{k=1}^{n}\frac{\partial c_l(q,t)}{\partial q_k}\dot{q}_k+\frac{\partial c_l(q,t)}{\partial t}. \tag{6.6}$$

Equation (6.4), the second order form of the constraints, can be rewritten as

$$\begin{aligned}\sum_{i=1}^{n}A_{li}(q,t)\ddot{q}_i&=-\sum_{i=1}^{n}\left(\frac{d}{dt}A_{li}(q,t)\right)\dot{q}_i+\frac{d}{dt}c_l(q,t)\\&=:b_l(q,\dot{q},t),\end{aligned} \tag{6.7}$$

$l=1,2,\cdots,m$, or in matrix form

$$A(q,t)\ddot{q}=b(q,\dot{q},t), \tag{6.8}$$

where $b=[b_1 \quad b_2 \quad \cdots \quad b_m]^T$.

Remark 6.2. In Section 5.5 of Chapter 5, it has been demonstrated that various control problems, including stabilization, trajectory following and optimality, can be cast into the form (6.8).

Assumption 6.1. For each $(q,t)\in \mathbf{R}^n\times\mathbf{R}, \sigma\in\Sigma, M(q,\sigma,t)>0$.

Remark 6.3. The assumption on the positive definiteness of the inertia matrix will be vital in later development. In the past, it was often believed that this was always true, and therefore a fact rather than an assumption. However, there are counter examples, as listed in [129], when q is not selected to be the generalized coordinate.

Definition 6.1. For given A and b, the constraint (6.8) is called *consistent* if there exists at least one solution $\ddot{q}$.

Assumption 6.2. The constraint (6.8) is consistent.

Theorem 6.1. (Udwadia and Kalaba [94]). Consider the system (6.1) and the constraint (6.8), subject to Assumptions 6.1 and 6.2, the constraint force

$$\begin{aligned} Q^c =& M^{1/2}(q,\sigma,t)(A(q,t)M^{-1/2}(q,\sigma,t))^+ \\ & \times [b(q,\dot{q},t) + A(q,t)M^{-1}(q,\sigma,t)(C(q,\dot{q},\sigma,t)\dot{q} + G(q,\sigma,t))] \end{aligned} \tag{6.9}$$

obeys the Lagrange's form of d'Alembert's principle [92] and renders the system to meet the constraint. Here "+" stands for the Moore-Penrose generalized inverse [148].

Remark 6.4. The Lagrange's form of d'Alembert's principle renders the constraint force (6.9) to be the one with minimum norm, out of all possible alternative forces which can also meet Eq. (6.8) [94]. Note that $Q^c \in \mathcal{R}(A^T)$ (with the understanding that in this d'Alembert's principle, the virtual displacement $\delta q \in \mathcal{N}(A)$; in addition, $\mathcal{R}(A^T) \perp \mathcal{N}(A)$).

Remark 6.5. Theorem 6.1 shows the strategy the Nature will undertake to meet the constraint. The constraint force is model-based. That is, it is based on the exact model information. Based on the theorem, one could apply the control input $\tau = Q^c$ to drive the system to meet equation (6.8), if the uncertainty were known. A more realistic design, from the engineer's point of view, when the uncertainty is unknown, is investigated in the next section.

6.3 Fuzzy described system model of constrained flexible joint manipulator

We consider a flexible manipulator system which contains uncertainties described by the following equation ([9]):

$$\begin{cases} M(q_1,\sigma_1)\ddot{q}_1 + C(q_1,\dot{q}_1,\sigma_1)\dot{q}_1 + G(q_1,\sigma_1) + K(\sigma_1)(q_1 - q_2) = 0 \\ J(\sigma_2)\ddot{q}_2 - K(\sigma_2)(q_1 - q_2) = \tau, \end{cases} \tag{6.10}$$

where $q_1 = [q_{1,1}\ q_{1,2}\ \cdots\ q_{1,n}]^T$ is link position vector and $q_2 = [q_{2,1}\ q_{2,2}\ \cdots\ q_{2,n}]^T$ is joint position vector. Let $q = [q_1^T \quad q_2^T]^T$ be a $2n-$ dimensional vector and represent the system's generalized coordinate. The joint flexibility is regarded as a linear torsional spring and elasticity coefficient of the spring is represented by a diagonal positive matrix $K(\sigma_{1,2})$. The link inertia matrix is denoted by $M(q_1,\sigma_1)$; The inertia of actuator is denoted as $J(\sigma_2)$ which is a diagonal matrix; $C(q_1,\dot{q}_1,\sigma_1)\dot{q}_1$ represents the Coriolis and centrifugal forces of links; $G(q_1,\sigma_1)$ is the gravitation force vector, and τ denotes the input control force from the actuators. σ_1 and σ_2 which are uncertain parameter vectors in the system may vary fast. Let $\tilde{q} = \left[q_1^T\ \dot{q}_1^T\ q_2^T\ \dot{q}_2^T\right]^T$.

Assumption 6.3. (i) Suppose the initial state is $\tilde{q}(t_0) = \tilde{q}_0$, where t_0 is the initial time. For each entry of $\tilde{q}_0$, namely $\tilde{q}_{0i}, i = 1, 2, \cdots, 4n$, there exists a fuzzy set U_{0i} in a universe of discourse $\Xi_i \subset \mathbf{R}$, characterized by a membership function $\mu_{\Xi_i} : \Xi_i \to [0, 1]$. That is,

$$U_{0i} = \{(\tilde{q}_{0i}, \mu_{\Xi_i}(\tilde{q}_{0i}))|\tilde{q}_{0i} \in \Xi_i\}. \tag{6.11}$$

(ii) For each entry of σ_1, σ_2, namely $\sigma_{1i}, \sigma_{2i}, i = 1, 2, \cdots, n$, the function $\sigma_{1i}(\cdot), \sigma_{2i}(\cdot)$ are Lebesgue measurable.
(iii) For each σ_{1i}, σ_{2i}, there exist fuzzy sets N_{1i}, N_{2i} in universe of discourses $\Sigma_{1i} \subset \mathbf{R}, \Sigma_{2i} \subset \mathbf{R}$, characterized by membership functions $\mu_{1i} : \Sigma_{1i} \to [0, 1]$, $\mu_{2i} : \Sigma_{2i} \to [0, 1]$. That is,

$$\begin{aligned} N_{1i} &= \{(\sigma_{1i}, \mu_{1i}(\sigma_{1i}))|\sigma_{1i} \in \Sigma_{1i}\}, \\ N_{2i} &= \{(\sigma_{2i}, \mu_{2i}(\sigma_{2i}))|\sigma_{2i} \in \Sigma_{2i}\}. \end{aligned} \tag{6.12}$$

The system with its state fuzzy described is called fuzzy dynamical system.

Assumption 6.4. The inertia matrix $M(q_1, \sigma_1)$ is uniformly positive definite and uniformly ultimately upper bounded. That is, there exist constants $\bar{\rho}, \underline{\rho} > 0$, such that

$$\underline{\rho} \leq \|M(q_1, \sigma_1)\| \leq \bar{\rho}, \quad \forall q_1 \in \mathbf{R}^n. \tag{6.13}$$

We propose two steps to design the robust control. Firstly, the first part of (6.10) can be rewritten as follows

$$\begin{aligned} M(q_1,\sigma_1)\ddot{q}_1 &+ C(q_1, \dot{q}_1, \sigma_1)\dot{q}_1 + G(q_1, \sigma_1) \\ &+ K(\sigma_1)q_1 = K(\sigma_1)(q_2 - \tilde{\tau}) + K(\sigma_1)\tilde{\tau}, \end{aligned} \tag{6.14}$$

where $\tilde{\tau}$ is a *fictitious* control implanted into the system and is only used to design the real control τ. With $\tilde{\tau}$ introduced, the overall system then can be divided into two second-order systems, which are: (i) link angles subsystem, (ii) joint angles subsystem. The link position subsystem is controlled by $\tilde{\tau}$, which is fictitious. The joint position subsystem is controlled by the real control τ.

Secondly, by multiplying $K(\sigma_1)$ to both sides of (6.14), let $x_2 = q_2 - \tilde{\tau}$, $x_3 = \dot{q}_2 - \dot{\tilde{\tau}}$, then the system can be transformed into the following equations

$$\hat{M}(q_1, \sigma_1)\ddot{q}_1 + \hat{C}(q_1, \dot{q}_1, \sigma_1)\dot{q}_1 + \hat{G}(q_1, \sigma_1) + q_1 = x_2 + \tilde{\tau}, \tag{6.15}$$

$$J(\sigma_2)x_4 + J(\sigma_2)\ddot{\tilde{\tau}} + K(\sigma_2)x_2 - K(\sigma_2)q_1 + K(\sigma_2)\tilde{\tau} = \tau. \tag{6.16}$$

Here

$$\begin{aligned}\hat{M}(q_1,\sigma_1) :=&K(\sigma_1)^{-1}M(q_1,\sigma_1),\\ \hat{C}(q_1,\dot{q}_1,\sigma_1) :=&K(\sigma_1)^{-1}C(q_1,\dot{q}_1,\sigma_1),\\ \hat{G}(q_1,\sigma_1) :=&K(\sigma_1)^{-1}G(q_1,\sigma_1).\end{aligned}\tag{6.17}$$

Since we take the uncertainty into account while designing the control $\tilde{\tau}$ and τ, this prevents us from using the precise information of M, C, K and G. Thus, we decompose the $\hat{M},\hat{C},\hat{G},J$ and K as follows:

$$\begin{aligned}\hat{M}(q_1,\sigma_1) =:&\bar{M}(q_1)+\Delta M(q_1,\sigma_1),\\ \hat{C}(q_1,\dot{q}_1,\sigma_1) =:&\bar{C}(q_1,\dot{q}_1)+\Delta C(q_1,\dot{q}_1,\sigma_1),\\ \hat{G}(q_1,\sigma_1) =:&\bar{G}(q_1)+\Delta G(q_1,\sigma_1),\\ J(\sigma_2) =:&\bar{J}+\Delta J(\sigma_2),\\ K(\sigma_2) =:&\bar{K}+\Delta K(\sigma_2).\end{aligned}\tag{6.18}$$

Here, $\bar{M},\bar{C},\bar{G},\bar{J}$ and $\bar{K}$ stand for the nominal portions, while $\Delta M,\Delta C,\Delta G,\Delta J$ and ΔK are uncertain portions. Let

$$\begin{aligned}D(q_1) :=&\bar{M}^{-1}(q_1),\\ \Delta D(q_1,\sigma_1) :=&\hat{M}^{-1}(q_1,\sigma_1)-\bar{M}^{-1}(q_1),\\ E(q_1,\sigma_1) :=&\bar{M}(q_1)M^{-1}(q_1,\sigma_1)-I,\end{aligned}\tag{6.19}$$

hence $\Delta D(q_1,\sigma_1)=D(q_1)E(q_1,\sigma_1)$.

In engineering applications, manipulator is used to position a target or follow a given trajectory. Actuator is usually fixed on the link, thus, link angles performance is our control purpose. We drive this subsystem to follow pre-specified constraints (in the form of $A(q_1,t)\dot{q}_1=c(q_1,t)$) approximately.

Assumption 6.5. (i) For each $q_1\in\mathbf{R}^n$, $A(q_1,t)$ is of full rank, which means $A(q_1,t)A^T(q_1,t)$ is invertible.

(ii) Under the provision of Assumption 6.5.1, for given $P\in\mathbf{R}^{m\times m}$, $P>0$, let

$$\begin{aligned}W(q_1,\sigma_1,t) :=&PA(q_1,t)D(q_1)E(q_1,\sigma_1)\bar{M}(q_1)\\ &\times A^T(q_1,t)(A(q_1,t)A^T(q_1,t))^{-1}P^{-1}.\end{aligned}\tag{6.20}$$

Furthermore, there exists a (possibly unknown) constant $\rho_E>-1$ (ρ_E belongs to a known fuzzy number) such that for all $(q_1,t)\in\mathbf{R}^n\times\mathbf{R}$,

$$\frac{1}{2}\min_{\sigma_1\in\Sigma_1}\lambda_m(W(q_1,\sigma_1,t)+W^T(q_1,\sigma_1,t))\geq\rho_E.\tag{6.21}$$

Remark 6.6. The constant ρ_E is in general unknown since the uncertainty bound is unknown. In the special case that $M = \bar{M}$ (i.e., no uncertainty),$E = 0, W = 0$ and hence one can choose $\rho_E = 0$. Thus by continuity this assumption imposes the effect of uncertainty on the possible deviation of M from $\bar{M}$ to be within a certain threshold. We also stress that this threshold is unidirectional (that is, it is not bounded in one direction and not bounded in the other direction).

Assumption 6.6. (i) There exists a (possibly unknown) constant vector $\psi_1 \in \mathbf{R}^j$ and a known function $\Pi(\cdot) : \mathbf{R}^j \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}_+$ such that for all $(q, \dot{q}, t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, $\sigma \in \Sigma$,

$$\begin{aligned}\Pi_1(\psi_1, q_1, \dot{q}_1, t) \geq &(1+\rho_E)^{-1} \max_{\sigma_1 \in \Sigma_1} \| PA(q_1, t)[D(q_1)(-\Delta C(q_1, \dot{q}_1, \sigma_1)\dot{q}_1 \\ &- \Delta G(q_1, \sigma_1, t)) + \Delta D(q_1, \sigma_1)(-\hat{C}(q_1, \dot{q}_1, \sigma_1)\dot{q}_1 - \hat{G}(q_1, \sigma_1) \\ &- q_1 + p_{11}(q_1, \dot{q}_1, t) + p_{12}(q_1, \dot{q}_1, t))] \|.\end{aligned} \tag{6.22}$$

(ii) For each $(q_1, \dot{q}_1, t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, the function $\Pi_1(\cdot, q_1, \dot{q}_1, t)$ is C^1 and concave; that is, for any ψ_{11}, ψ_{12},

$$\Pi_1(\psi_{11}, q_1, \dot{q}_1, t) - \Pi_1(\psi_{12}, q_1, \dot{q}_1, t) \leq \frac{\partial \Pi_1}{\partial \psi_1}(\psi_{12}, q_1, \dot{q}_1, t)(\psi_{11} - \psi_{12}). \tag{6.23}$$

Besides, it is non-decreasing with respect to each component of its argument ψ_1.

(iii) Each entry of ψ_1, namely ψ_{1i}, $i = 1, 2, \cdots, j$, belongs to a known fuzzy number.

Assumption 6.7. (i) There exists a (possibly unknown) constant vector $\psi_2 \in \mathbf{R}^s$ and a known function $\Pi_2(\cdot) : \mathbf{R}^s \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}_+$ such that for all $(q, \dot{q}, t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, $\sigma \in \Sigma$,

$$\begin{aligned}\Pi_2(\psi_2, q_1, \dot{q}_1, x_2, x_3, t) \geq \max_{\sigma_2 \in \Sigma_2} \| &-J(\sigma_2)\ddot{\tilde{\tau}}(q_1, \dot{q}_1, t) \\ &- \Delta K(\sigma_2)x_2 + \Delta K(\sigma_2)q_1 + \Delta J(\sigma_2)Sx_2 \\ &- K(\sigma_2)\tilde{\tau}(q_1, \dot{q}_1, t) + \frac{1}{2}\dot{J}(\sigma_2)(x_3 + Sx_2) \|.\end{aligned} \tag{6.24}$$

(ii) For each $(q_1, \dot{q}_1, x_2, x_3, t) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, the function $\Pi_2(\cdot, q_1, \dot{q}_1, x_2, x_3, t)$ is C^1 and concave; that is, for any ψ_{21}, ψ_{22},

$$\Pi_2(\psi_{21}, q_1, \dot{q}_1, x_2, x_3, t) - \Pi_2(\psi_{22}, q_1, \dot{q}_1, x_2, x_3, t) \leq \frac{\partial \Pi_2}{\partial \psi_2}(\psi_{22}, q_1, \dot{q}_1, x_2, x_3, t)(\psi_{21} - \psi_{22}). \tag{6.25}$$

Besides, it is non-decreasing with respect to each component of its argument ψ_2.

(iii) Each entry of ψ_2, namely ψ_{2i}, $i = 1, 2, \cdots, s$, belongs to a known fuzzy number.

6.4 Adaptive robust constraint force servo control design

Let

$$
\begin{aligned}
p_{11}(q_1,\dot{q}_1,t) =&\bar{M}^{\frac{1}{2}}(q_1)(A(q_1,t)\bar{M}^{-\frac{1}{2}}(q_1))^{+}[b_1(q_1,\dot{q}_1,t) \\
&+ A(q_1,t)\bar{M}^{-1}(q_1)(\bar{C}(q_1,\dot{q}_1)\dot{q}_1+\bar{G}(q_1)+q_1)],
\end{aligned} \tag{6.26}
$$

$$
\begin{aligned}
p_{12}(q_1,\dot{q}_1,t) =&-\gamma_1\bar{M}(q_1)A^T(q_1,t)(A(q_1,t)A^T(q_1,t))^{-1} \\
&\times P^{-1}(A(q_1,t)\dot{q}_1-c(q_1,t)).
\end{aligned} \tag{6.27}
$$

Since the overall system is regarded as the cascades of two subsystems, then we can design the control for each subsystem respectively. Firstly, we design the virtual control $\tilde{\tau}$ of the link position subsystem (6.15). For a given positive gain $\varepsilon_1 > 0$, the implanted virtual control $\tilde{\tau}$ could be given as

$$
\tilde{\tau}(t) = p_{11}(q_1(t),\dot{q}_1(t),t) + p_{12}(q_1(t),\dot{q}_1(t),t) + p_{13}(q_1(t),\dot{q}_1(t),t), \tag{6.28}
$$

with

$$
\begin{aligned}
p_{13}(q_1,\dot{q}_1,t) =&-[\bar{M}(q_1)A^T(q_1,t)(A(q_1,t)A^T(q_1,t))^{-1}P^{-1}] \\
&\times \varphi_1(q_1,\dot{q}_1,t)\rho_1(q_1,\dot{q}_1,t)\Pi_1(\hat{\psi}_1,q_1,\dot{q}_1,t),
\end{aligned} \tag{6.29}
$$

where

$$
\varphi_1(q_1,\dot{q}_1,t) = \begin{cases} \frac{1}{\|\rho_1(q_1,\dot{q}_1,t)\|}, & \text{if} \quad \|\rho_1(q_1,\dot{q}_1,t)\| > \varepsilon_1, \\ \sin(\frac{\pi}{2\varepsilon_1}), & \text{if} \quad \|\rho_1(q_1,\dot{q}_1,t)\| \le \varepsilon_1, \end{cases} \tag{6.30}
$$

$$
\rho_1(q_1,\dot{q}_1,t) = (A(q_1,t)\dot{q}_1 - c(q_1,t))\Pi_1(\hat{\psi}_1,q_1,\dot{q}_1,t). \tag{6.31}
$$

The parameter $\hat{\psi}_1$ is given by following adaptive law

$$
\dot{\hat{\psi}}_1 = k_1\frac{\partial \Pi_1^T}{\partial \psi_1}(\hat{\psi}_1,q_1,\dot{q}_1,t)\|A(q_1,t)\dot{q}_1 - c(q_1,t)\| - \hat{\psi}_1, \tag{6.32}
$$

where $k_1 > 0$ is a scalar.

Remark 6.7. The control task is the approximate constraint following control for the link subsystem problem, which means, it is possible that $\|A(q_1,t)\dot{q}_1 - c(q_1,t)\| \neq 0$. This may be caused by the unmodeled uncertainty or external disturbances. Engineers can not have precise knowledge to have Q^c in (6.9). Meanwhile, the initial state of the system may be far away from the given constraints.

To give the control torque τ with a given $S = diag[S_i]_{n\times n}$, $S_i > 0$, $i = 1, 2, \cdots n$, and a constant $\varepsilon_2 > 0$, we propose the real input torque τ as follows

$$\tau(t) = -K_p x_2(t) - K_d x_3(t) + \bar{K} x_2(t) - \bar{K} q_1(t) - \bar{J} S x_3(t) + p_2(t), \tag{6.33}$$

with

$$p_2 = \begin{cases} -\frac{\rho_2}{\|\rho_2\|}\Pi_2(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3, t), & \text{if } \|\rho_2\| > \varepsilon_2, \\ -\frac{\rho_2}{\varepsilon_2}\Pi_2(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3, t), & \text{if } \|\rho_2\| \le \varepsilon_2, \end{cases} \tag{6.34}$$

here, $\rho_2 = (x_3 + Sx_2)\Pi_2(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3, t)$, K_p, K_d are diagonal positive gain matrices with corresponding dimension.

The parameter $\hat{\psi}_2$ is given by following adaptive law

$$\dot{\hat{\psi}}_2 = k_2 \frac{\partial \Pi_2^T}{\partial \psi_2}(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3, t)\|x_3 + Sx_2\| - \hat{\psi}_2, \tag{6.35}$$

where $k_2 > 0$ is a scalar.

Remark 6.8. The adaptive laws (6.32) and (6.35) are of leakage type. Thus, $\hat{\psi}_1$ and $\hat{\psi}_2$ may decrease by the reduction of the following error. k_1 and k_2 determine the rate of convergence, the optimal choice of these two parameters will be given later.

Theorem 6.2. Let $x_1(q_1(t), \dot{q}_1(t), t) = A(q_1(t), t)\dot{q}_1(t) - c(q_1(t), t)$, $\tilde{\psi}_1(t) = \hat{\psi}_1(t) - \psi_1(t)$, $\delta_1(t) = [x_1^T(t) \quad \tilde{\psi}_1^T(t)]^T$; $\tilde{\psi}_2(t) = \hat{\psi}_2(t) - \psi_2(t)$, $\delta_2(t) = [x_2^T(t) \quad x_3^T(t) \quad \tilde{\psi}_2^T(t)]^T$ and $\delta(t) = [\delta_1^T(t) \quad \delta_2^T(t)]^T$. Subject to Assumptions 6.3- 6.7, the control (6.28) renders the combined systems (6.15), (6.16), (6.32) and (6.35) the following performances:

(i) Uniform boundedness: For any $r > 0$, there is a $d(r) < \infty$ such that if $\delta(\cdot)$ is any solution with $\|\delta(t_0)\| \le r$, then $\|\delta(t)\| \le d(r)$ for all $t \ge t_0$.

(ii) Uniformly ultimately bounded: For any $r > 0$ with $\|\delta(t_0)\| \le r$, there exists a $\underline{d} > 0$ such that $\|\delta(t)\| \le \bar{d}$ for any $\bar{d} > \underline{d}$ as $t \ge t_0 + T(\bar{d}, r)$, where $T(\bar{d}, r) < \infty$.

(iii) Convergence to zero: For any given trajectory $\delta(\cdot)$,

$$\lim_{t\to\infty} x_1(t) = 0. \tag{6.36}$$

Proof: In the proof, for simplicity, arguments of functions are omitted when no confusions are likely to arise, otherwise it will be stated. Choosing Lyapunov candidate as

$$V(\delta) = V_1(\delta_1) + V_2(\delta_2), \tag{6.37}$$

with

$$V_1(\delta_1) = \frac{1}{2}x_1^T P x_1 + \frac{1}{2}(1+\rho_E)\tilde{\psi}_1^T k_1^{-1}\tilde{\psi}_1, \tag{6.38}$$

$$V_2(\delta_2) =\frac{1}{2}(x_3 + Sx_2)^T J(x_3 + Sx_2) + \frac{1}{2}x_2^T(K_p + SK_d)x_2 + \frac{1}{2}\tilde{\psi}_2^T k_2^{-1}\tilde{\psi}_2, \tag{6.39}$$

where P is a positive definite and symmetric matrix. First of all, we should prove that V is a legitimate Lyapunov function candidate ([156]). From (6.38), it yields

$$\begin{aligned} V_1 \geq& \frac{1}{2}\lambda_{\min}(P)\|x_1\|^2 + \frac{1}{2}(1+\rho_E)k_1^{-1}\|\tilde{\psi}_1\|^2 \\ \geq& h_1\|\delta_1\|^2, \end{aligned} \tag{6.40}$$

where $h_1 = \min\{\frac{1}{2}\lambda_{\min}(P), \frac{1}{2}(1+\rho_E)k_1^{-1}\}$.

$$\begin{aligned} V_1 \leq& \frac{1}{2}\lambda_{\max}(P)\|x_1\|^2 + \frac{1}{2}(1+\rho_E)k_1^{-1}\|\tilde{\psi}_1\|^2 \\ \leq& h_2\|\delta_1\|^2, \end{aligned} \tag{6.41}$$

where $h_2 = \max\{\frac{1}{2}\lambda_{\max}(P), \frac{1}{2}(1+\rho_E)k_1^{-1}\}$.

Based on (6.39),

$$\begin{aligned} V_2 \geq& \frac{1}{2}\underline{\theta}\|x_3 + Sx_2\|^2 + \frac{1}{2}x_2^T(K_p + SK_d)x_2 + \frac{1}{2}k_2^{-1}\|\tilde{\psi}_2\|^2 \\ =& \frac{1}{2}\underline{\theta}\begin{bmatrix} x_2^T & x_3^T \end{bmatrix}\begin{bmatrix} S^2 & S \\ S & I \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_2^T & x_3^T \end{bmatrix}\begin{bmatrix} K_p + SK_d & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \\ &+ \frac{1}{2}k_2^{-1}\|\tilde{\psi}_2\|^2 \\ \geq& \frac{1}{2}\lambda_{\min}(\Omega_1)(\|x_2\|^2 + \|x_3\|^2) + \frac{1}{2}k_2^{-1}\|\tilde{\psi}_2\|^2 \\ \geq& h_3\|\delta_2\|^2, \end{aligned} \tag{6.42}$$

where

$$\underline{\theta} = \lambda_{\min}(J), \tag{6.43}$$

$$\Omega_1 = \begin{bmatrix} \underline{\theta}S^2 + K_p + SK_d & \underline{\theta}S \\ \underline{\theta}S & \underline{\theta} \end{bmatrix}, \tag{6.44}$$

$$h_3 = \min\{\frac{1}{2}\lambda_{\min}(\Omega_1), \frac{1}{2}k_2^{-1}\}. \tag{6.45}$$

Since Ω_1 is positive definite, V_2 is positive definite. Similarly, for the upper bound of V_2 we have

$$\begin{aligned} V_2 \leq& \frac{1}{2}\lambda_{\max}(\Omega_2)(\|x_2\|^2 + \|x_3\|^2) + \frac{1}{2}k_2^{-1}\|\tilde{\psi}_2\|^2 \\ \leq& h_4\|\delta_2\|^2. \end{aligned} \tag{6.46}$$

with

$$\Omega_2 = \begin{bmatrix} \bar{\theta}S^2 + K_p + SK_d & \bar{\theta}S \\ \bar{\theta}S & \bar{\theta} \end{bmatrix}, \tag{6.47}$$

$$\bar{\theta} = \lambda_{\max}(J), \tag{6.48}$$

$$h_4 = \max\{\frac{1}{2}\lambda_{\max}(\Omega_2), \frac{1}{2}k_2^{-1}\}. \tag{6.49}$$

Therefore, according to (6.40), (6.41), (6.42) and (6.46),

$$a_1\|\delta\|^2 \le V \le a_2\|\delta\|^2, \tag{6.50}$$

here, $a_1 = \min\{h_1, h_3\}, a_2 = \max\{h_2, h_4\}$. This shows that V is positive definite and decrescent.

For given uncertainties σ_1, σ_2 and the corresponding trajectory $\delta(t)$ of the controlled system, the derivative of V_1 is given by

$$\begin{aligned} \dot{V}_1 =& x_1^T P x_2 + (1+\rho_E)\tilde{\psi}_1^T k_1^{-1}\dot{\hat{\psi}}_1 \\ =& x_1^T P(A\ddot{q}_1 - b) + (1+\rho_E)\tilde{\psi}_1^T k_1^{-1}\dot{\hat{\psi}}_1 \\ =& x_1^T P\{A[\hat{M}^{-1}(-\hat{C}\dot{q}_1 - \hat{G} - q_1 + x_2) + \hat{M}^{-1}(p_{11} + p_{12} + p_{13})] - b\} \\ &+ (1+\rho_E)\tilde{\psi}_1^T k_1^{-1}\dot{\hat{\psi}}_1. \end{aligned} \tag{6.51}$$

We analyze these terms separately. With the decomposition of $\hat{M}^{-1}, \hat{C}, \hat{G}$ in (6.18), we have

$$\begin{aligned} & A[\hat{M}^{-1}(-\hat{C}\dot{q}_1 - \hat{G} - q_1 + x_2) + \hat{M}^{-1}(p_{11} + p_{12} + p_{13})] - b \\ =& A[(D+\Delta D)(-\bar{C}\dot{q}_1 - \bar{G} - x_1 - \Delta C x_2 - \Delta G) + (D+\Delta D)(p_{11}+p_{12}+p_{13}) + \hat{M}^{-1}x_2] - b \\ =& A[D(-\bar{C}\dot{q}_1 - \bar{G} - q_1) + D(p_{11} + p_{12}) + D(-\Delta C\dot{q}_1 - \Delta G) \\ &+ \Delta D(-\hat{C}\dot{q}_1 - \hat{G} - q_1 + p_{11} + p_{12}) + (D+\Delta D)p_{13} + \hat{M}^{-1}x_2] - b \\ =& A[D(-\bar{C}x_2 - \bar{G} - x_1) + Dp_{11}] - b + ADp_{12} + A(D+\Delta D)p_{13} \\ &+ A[D(-\Delta C\dot{q}_1 - \Delta G) + \Delta D(-\hat{C}\dot{q}_1 - \hat{G} - q_1 + p_{11} + p_{12})] + A\hat{M}^{-1}x_2. \end{aligned} \tag{6.52}$$

By (6.9) and (6.26), under the ideal condition that $\sigma_1 \equiv 0$ (i.e., $\hat{M}^{-1} = \bar{M}^{-1}, \hat{C} = \bar{C}, \hat{G} = \bar{G}, Q^c = p_{11}$),

$$A[D(-\bar{C}x_2 - \bar{G} - x_1) + Dp_{11}] - b = 0. \tag{6.53}$$

According to the given p_{12} in (6.27) and after performing some algebra, we have

$$\begin{aligned} x_1^T PADp_{12} =& x_1^T PAD[-\gamma_1 \bar{M} A^T (AA^T)^{-1} P^{-1}(A\dot{q}_1 - c) \\ =& -\gamma_1 x_1^T (A\dot{q}_1 - c) \\ =& -\gamma_1 \|x_1\|^2. \end{aligned} \tag{6.54}$$

By (6.29) and with $\Delta D = DE$, one has

$$
\begin{aligned}
x_1^T PA(D+\Delta D)p_{13} =& x_1^T PAD\{-[\bar{M}A^T(AA^T)^{-1}P^{-1}]\varphi_1\rho_1\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t)\} \\
&+ x_1^T PADE\{-[\bar{M}A^T(AA^T)^{-1}P^{-1}]\varphi_1\rho_1\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t)\}.
\end{aligned}
\tag{6.55}
$$

After performing matrix cancellation, we have (notice that $\rho_1 = x_1\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t)$)

$$
\begin{aligned}
& x_1^T PAD\{-[\bar{M}A^T(AA^T)^{-1}P^{-1}]\varphi_1\rho_1\Pi_1(\psi_1, q_1, \dot{q}_1, t)\} \\
=& -(x_1\Pi_1(\psi_1, q_1, \dot{q}_1, t))^T\varphi_1\rho_1 \\
=& -\varphi_1\|\rho_1\|^2.
\end{aligned}
\tag{6.56}
$$

Adopting the Rayleigh's principle, one gets

$$
\begin{aligned}
& x_1^T PADE\{-[\bar{M}A^T(AA^T)^{-1}P^{-1}]\varphi_1\rho_1\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t)\} \\
=& -\rho_1^T[PADE\bar{M}A^T(AA^T)^{-1}P^{-1}]\varphi_1\rho_1] \\
=& -\varphi_1\frac{1}{2}\rho_1[PADE\bar{M}A^T(AA^T)^{-1}P^{-1} + P^{-1}(AA^T)^{-T}A\bar{M}E^TDA^TP]\rho_1 \\
\le& -\varphi_1\frac{1}{2}\lambda_m(W+W^T)\|\rho_1\|^2 \\
\le& -\varphi_1\rho_E\|\rho_1\|^2.
\end{aligned}
\tag{6.57}
$$

Therefore, we can show that

$$
x_1^T PA(D+\Delta D)p_{13} \le -\varphi_1(1+\rho_E)\|\rho_1\|^2.
\tag{6.58}
$$

With the use of Assumption 6.5,

$$
\begin{aligned}
& x_1^T PA[D(-\Delta C\dot{q}_1 - \Delta G) + \Delta D(-\hat{C}\dot{q}_1 - \hat{G} - q_1 + p_{11} + p_{12})] \\
\le& \|x_1\|\|PA[D(-\Delta C\dot{q}_1 - \Delta G) + \Delta D(-\hat{C}\dot{q}_1 - \hat{G} - q_1 + p_{11} + p_{12})]\| \\
\le& \|x_1\|(1+\rho_E)\Pi_1(\psi_1, q_1, \dot{q}_1, t).
\end{aligned}
\tag{6.59}
$$

According to the inequality $ab \le \frac{1}{2}(a^2+b^2), a, b \in \mathbf{R}$, we have that for any $\omega > 0$,

$$
\begin{aligned}
x_1^T PA\hat{M}^{-1}x_2 \le& \|x_1\|\|PA\hat{M}^{-1}\|\|x_2\| \\
\le& \frac{1}{2}\|PA\hat{M}^{-1}\|(\omega\|x_1\|^2 + \omega^{-1}\|x_2\|^2) \\
\le& \frac{1}{2}\lambda_A\omega\|x_1\|^2 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2,
\end{aligned}
\tag{6.60}
$$

where $\lambda_A \ge \|PA\hat{M}^{-1}\|$ is a constant. Combining (6.54), (6.58), (6.59)and (6.60), if $\|\rho_1\| > \varepsilon_1$,

$$\begin{aligned} x_1^T P x_2 =& -\gamma_1\|x_1\|^2 - \varphi_1(1+\rho_E)\|\rho_1\|^2 + (1+\rho_E)\|x_1\|\Pi_1(\psi_1, q_1, \dot{q}_1, t) \\ &+ \frac{1}{2}\lambda_A\omega\|x_1\|^2 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2 \\ =& -\gamma_1\|x_1\|^2 - (1+\rho_E)\frac{1}{\|\rho_1\|}\|\rho_1\|^2 + (1+\rho_E)\|x_1\|\Pi_1(\psi_1, q_1, \dot{q}_1, t) \\ &+ \frac{1}{2}\lambda_A\omega\|x_1\|^2 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2 \\ =& -\gamma_1\|x_1\|^2 + (1+\rho_E)\{-\|x_1\|\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t) + \|x_1\|\Pi_1(\psi_1, q_1, \dot{q}_1, t)\} \\ &+ \frac{1}{2}\lambda_A\omega\|x_1\|^2 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2. \end{aligned} \tag{6.61}$$

If $\|\rho_1\| \leq \varepsilon_1$,

$$\begin{aligned} x_1^T P x_2 =& -\gamma_1\|x_1\|^2 - \varphi_1(1+\rho_E)\|\rho_1\|^2 + \|x_1\|(1+\rho_E)\Pi_1(\psi_1, q_1, \dot{q}_1, t) \\ &+ \frac{1}{2}\lambda_A\omega\|x_1\|^2 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2 \\ \leq& -\gamma_1\|x_1\|^2 + (1+\rho_E)\left(-\frac{1}{\varepsilon_1}\|x_1\|^2\Pi_1^2(\hat{\psi}_1, q_1, \dot{q}_1, t) + \underbrace{\|x_1\|\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t)}\right) \\ &+ (1+\rho_E)\left(\underbrace{-\|x_1\|\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t)} + \|x_1\|\Pi_1(\psi_1, q_1, \dot{q}_1, t)\right) \\ &+ \frac{1}{2}\lambda_A\omega\|x_1\|^2 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2 \\ \leq& -(\gamma_1 - \frac{1}{2}\lambda_A\omega)\|x_1\|^2 + \frac{1+\rho_E}{4}\varepsilon_1 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2 \\ &+ (1+\rho_E)\{-\|x_1\|\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t) + \|x_1\|\Pi_1(\psi_1, q_1, \dot{q}_1, t)\}. \end{aligned} \tag{6.62}$$

Therefore, for all $\varepsilon_1 > 0$,

$$\begin{aligned} x_1^T P x_2 \leq& -(\gamma_1 - \frac{1}{2}\lambda_A\omega)\|x_1\|^2 + \frac{1+\rho_E}{4}\varepsilon_1 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2 \\ &+ (1+\rho_E)\{-\|x_1\|\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t) + \|x_1\|\Pi_1(\psi_1, q_1, \dot{q}_1, t)\}. \end{aligned} \tag{6.63}$$

By Assumption 6.6, we have

$$-\Pi_1(\hat{\psi}_1, q_1, \dot{q}_1, t) + \Pi_1(\psi_1, q_1, \dot{q}_1, t) \leq \frac{\partial \Pi_1^T}{\partial \psi_1}(\hat{\psi}_1, q_1, \dot{q}_1, t)(\psi_1 - \hat{\psi}_1). \tag{6.64}$$

This leads to

$$\begin{aligned} x_1^T P x_2 \leq& -(\gamma_1 - \frac{1}{2}\lambda_A\omega)\|x_1\|^2 + \frac{1+\rho_E}{4}\varepsilon_1 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2 \\ &+ (1+\rho_E)\|x_1\|\frac{\partial \Pi_1}{\partial \psi_1}(\hat{\psi}_1, q_1, \dot{q}_1, t)(\psi_1 - \hat{\psi}_1). \end{aligned} \tag{6.65}$$

For the second term on the right hand side (RHS) of (6.51), by applying the adaptive law (6.32), one has

$$
\begin{aligned}
(1+\rho_E)\tilde{\psi}_1^T k_1^{-1}\dot{\hat{\psi}}_1 =&(1+\rho_E)\tilde{\psi}_1^T k_1^{-1}(k_1\frac{\partial \Pi_1^T}{\partial \psi_1}(\hat{\psi}_1,q_1,\dot{q}_1,t)\|x_1\|-\hat{\psi}_1)\\
=&(1+\rho_E)\tilde{\psi}_1^T k_1^{-1}[k_1\frac{\partial \Pi_1^T}{\partial \psi_1}(\hat{\psi}_1,q_1,\dot{q}_1,t)\|x_1\|-(\hat{\psi}_1-\psi_1)-\psi_1]\\
\leq&(1+\rho_E)\|x_1\|\frac{\partial \Pi_1}{\partial \psi_1}(\hat{\psi}_1,q_1,\dot{q}_1,t)\tilde{\psi}_1\\
&-k_1^{-1}(1+\rho_E)\|\tilde{\psi}_1\|^2+k_1^{-1}(1+\rho_E)\|\tilde{\psi}_1\|\|\psi_1\|\\
\leq&(1+\rho_E)\|x_1\|\frac{\partial \Pi_1}{\partial \psi_1}(\hat{\psi}_1,q_1,\dot{q}_1,t)(\hat{\psi}_1-\psi_1)\\
&-\frac{1}{2}k_1^{-1}(1+\rho_E)\|\tilde{\psi}_1\|^2+\frac{1}{2}k_1^{-1}(1+\rho_E)\|\psi_1\|^2.
\end{aligned}
\tag{6.66}
$$

According to (6.65) and (6.66), we have

$$
\begin{aligned}
\dot{V}_1 =&-(\gamma_1-\frac{1}{2}\lambda_A\omega)\|x_1\|^2+\frac{1+\rho_E}{4}\varepsilon_1+\frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2\\
&+(1+\rho_E)\|x_1\|\frac{\partial \Pi_1}{\partial \psi_1}(\hat{\psi}_1,q_1,\dot{q}_1,t)(\psi_1-\hat{\psi}_1)\\
&+(1+\rho_E)\|x_1\|\frac{\partial \Pi_1}{\partial \psi_1}(\hat{\psi}_1,q_1,\dot{q}_1,t)(\hat{\psi}_1-\psi_1)\\
&-\frac{1}{2}k_1^{-1}(1+\rho_E)\|\tilde{\psi}_1\|^2+\frac{1}{2}k_1^{-1}(1+\rho_E)\|\psi_1\|^2\\
=&-(\gamma_1-\frac{1}{2}\lambda_A\omega)\|x_1\|^2-\frac{1}{2}k_1^{-1}(1+\rho_E)\|\tilde{\psi}_1\|^2\\
&+\frac{1+\rho_E}{4}\varepsilon_1+\frac{1}{2}k_1^{-1}(1+\rho_E)\|\psi_1\|^2+\frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2\\
\leq&-\lambda_1\|\delta_1\|^2+\frac{1+\rho_E}{4}\varepsilon_1+\frac{1}{2}k_1^{-1}(1+\rho_E)\|\psi_1\|^2+\frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2,
\end{aligned}
\tag{6.67}
$$

here, $\lambda_1=\min\{\gamma_1-\frac{1}{2}\lambda_A\omega, \frac{1}{2}k_1^{-1}(1+\rho_E)\}$.

Next, we analyze the derivative of V_2, then it can be evaluated as

$$
\begin{aligned}
\dot{V}_2 =&(x_3+Sx_2)^T J(x_4+S\dot{x}_2)+\frac{1}{2}(x_3+Sx_2)^T\dot{J}(x_3+Sx_2)+x_2^T(K_p+SK_d)x_3\\
&+\tilde{\psi}_2^T k_2^{-1}\dot{\hat{\psi}}_2\\
=&(x_3+Sx_2)^T(-J\ddot{\tilde{\tau}}-Kx_2+Kq_1-K\tilde{\tau}+\tau+JSx_3+\frac{1}{2}\dot{J}x_3+\frac{1}{2}\dot{J}Sx_2)\\
&+x_2^T(K_p+SK_d)x_3+\tilde{\psi}_2^T k_2^{-1}\dot{\hat{\psi}}_2.
\end{aligned}
\tag{6.68}
$$

With the decomposition of J^{-1}, K in (6.18), we have

$$
\begin{aligned}
\dot{V}_2 =&(x_3+Sx_2)^T(-J\ddot{\tilde{\tau}}-\bar{K}x_2+\bar{K}q_1-K\tilde{\tau}+\tau+\bar{J}Sx_3-\Delta Kx_2+\Delta Kq_1\\
&+\Delta JSx_2+\frac{1}{2}\dot{J}x_3+\frac{1}{2}\dot{J}Sx_2)+x_2^T(K_p+SK_d)x_3+\tilde{\psi}_2^T k_2^{-1}\dot{\hat{\psi}}_2.
\end{aligned}
\tag{6.69}
$$

Based on Assumption 6.6 and the control (6.33), it can be seen that

$$\begin{aligned}\dot{V}_2 =&(x_3+Sx_2)^T(-J\ddot{\tilde{\tau}}-K\tilde{\tau}-\Delta Kx_2+\Delta Kq_1+\Delta JSx_2+\frac{1}{2}\dot{J}x_3+\frac{1}{2}\dot{J}Sx_2\\&-K_px_2-K_dx_3+p_2)+x_2^T(K_p+SK_d)x_3+\tilde{\psi}_2^Tk_2^{-1}\dot{\hat{\psi}}_2\\\leq&\|x_3+Sx_2\|\Pi_2(\psi_2,q_1,\dot{q}_1,x_2,x_3)+(x_3+Sx_2)^Tp_2+(x_3+Sx_2)^T(-K_px_2-K_dx_3)\\&+x_2^T(K_p+SK_d)x_3+\tilde{\psi}_2^Tk_2^{-1}\dot{\hat{\psi}}_2.\end{aligned}\tag{6.70}$$

If $\|\rho_2\|>\varepsilon_2$,

$$\begin{aligned}&\|x_3+Sx_2\|\Pi_2(\psi_2,q_1,\dot{q}_1,x_2,x_3)+(x_3+Sx_2)^Tp_2\\&\quad\leq\|x_3+Sx_2\|\Pi_2(\psi_2,q_1,\dot{q}_1,x_2,x_3)-\|x_3+Sx_2\|\frac{\|\rho_2\|}{\|\rho_2\|}\Pi_2(\hat{\psi}_2,q_1,\dot{q}_1,x_2,x_3)\\&\quad=\|x_3+Sx_2\|\Pi_2(\psi_2,q_1,\dot{q}_1,x_2,x_3)-\|x_3+Sx_2\|\Pi_2(\hat{\psi}_2,q_1,\dot{q}_1,x_2,x_3)\\&\quad\leq\|x_3+Sx_2\|\frac{\partial\Pi_2}{\partial\psi_2}(\hat{\psi}_2,q_1,\dot{q}_1,x_2,x_3)(\psi_2-\hat{\psi}_2).\end{aligned}\tag{6.71}$$

If $\|\rho_2\|\leq\varepsilon_2$,

$$\begin{aligned}&\|x_3+Sx_2\|\Pi_2(\psi_2,q_1,\dot{q}_1,x_2,x_3)+(x_3+Sx_2)^Tp_2\\&\quad\leq\|x_3+Sx_2\|\Pi_2(\psi_2,q_1,\dot{q}_1,x_2,x_3)-\|x_3+Sx_2\|\frac{\|\rho_2\|}{\varepsilon_2}\Pi_2(\hat{\psi}_2,q_1,\dot{q}_1,x_2,x_3)\\&\quad=\|x_3+Sx_2\|\Pi_2(\psi_2,q_1,\dot{q}_1,x_2,x_3)-\underbrace{\|x_3+Sx_2\|\Pi_2(\hat{\psi}_2,q_1,\dot{q}_1,x_2,x_3)}\\&\qquad+\underbrace{\|x_3+Sx_2\|\Pi_2(\hat{\psi}_2,q_1,\dot{q}_1,x_2,x_3)}-\frac{1}{\varepsilon_2}\|x_3+Sx_2\|^2\Pi_2^2(\hat{\psi}_2,q_1,\dot{q}_1,x_2,x_3)\\&\quad\leq\|x_3+Sx_2\|\frac{\partial\Pi_2}{\partial\psi_2}(\hat{\psi}_2,q_1,\dot{q}_1,x_2,x_3)(\psi_2-\hat{\psi}_2)+\frac{\varepsilon_2}{4}.\end{aligned}\tag{6.72}$$

Thus, for all $\varepsilon_2>0$,

$$\begin{aligned}&\|x_3+Sx_2\|\Pi_2(\psi_2,q_1,\dot{q}_1,x_2,x_3)+(x_3+Sx_2)^Tp_2\\&\quad\leq\|x_3+Sx_2\|\frac{\partial\Pi_2}{\partial\psi_2}(\hat{\psi}_2,q_1,\dot{q}_1,x_2,x_3)(\psi_2-\hat{\psi}_2)+\frac{\varepsilon_2}{4}.\end{aligned}\tag{6.73}$$

Next, we analyze the third and forth terms on the right-hand side (RHS) of (6.70) as

$$\begin{aligned}&(x_3+Sx_2)^T(-K_px_2-K_dx_3)+x_2^T(K_p+SK_d)x_3\\&\quad=-x_3^TK_px_2-x_3^TK_dx_3-x_2^TSK_px_2-x_2^TSK_dx_3+x_2^T(K_p+SK_d)x_3\\&\quad=-x_2^TSK_px_2-x_3^TK_dx_3\\&\quad\leq-\lambda_{K_p}\|x_2\|^2-\lambda_{K_d}\|x_3\|^2,\end{aligned}\tag{6.74}$$

where $\lambda_{K_p} = \min\{\lambda(SK_p)\}, \lambda_{K_d} = \min\{\lambda(K_d)\}$. After substituting the adaptive law (6.35) for the last term on the RHS of (6.70), we have

$$\begin{aligned}\tilde{\psi}_2^T k_1^{-1}\dot{\hat{\psi}}_2 =&\tilde{\psi}_2^T k_2^{-1}(k_2\frac{\partial\Pi_2^T}{\partial\psi_2}(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3)\|x_3 + Sx_2\| - \hat{\psi}_2)\\ =&\tilde{\psi}_2^T k_2^{-1}[k_2\frac{\partial\Pi_2^T}{\partial\psi_2}(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3)\|x_3 + Sx_2\| - (\hat{\psi}_2 - \psi_2) - \psi_2]\\ \le&\|x_3 + Sx_2\|\frac{\partial\Pi_2}{\partial\psi_2}(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3)\tilde{\psi}_2 - k_2^{-1}\|\tilde{\psi}_2\|^2 + k_2^{-1}\|\tilde{\psi}_2\|\|\psi_2\|\\ \le&\|x_3 + Sx_2\|\frac{\partial\Pi_2}{\partial\psi_2}(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3)(\hat{\psi}_2 - \psi_2) - \frac{1}{2}k_2^{-1}\|\tilde{\psi}_2\|^2 + \frac{1}{2}k_2^{-1}\|\psi_2\|^2.\end{aligned} \tag{6.75}$$

Combining (6.73), (6.74) and (6.75), we have

$$\begin{aligned}\dot{V}_2 \le&\|x_3 + Sx_2\|\frac{\partial\Pi_2}{\partial\psi_2}(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3)(\psi_2 - \hat{\psi}_2) + \frac{\varepsilon_2}{4} - \lambda_{K_p}\|x_2\|^2 - \lambda_{K_d}\|x_3\|^2\\ &+ \|x_3 + Sx_2\|\frac{\partial\Pi_2}{\partial\psi_2}(\hat{\psi}_2, q_1, \dot{q}_1, x_2, x_3)(\hat{\psi}_2 - \psi_2) - \frac{1}{2}k_2^{-1}\|\tilde{\psi}_2\|^2 + \frac{1}{2}k_2^{-1}\|\psi_2\|^2 + \frac{\varepsilon_2}{4}\\ =& - \lambda_{K_p}\|x_2\|^2 - \lambda_{K_d}\|x_3\|^2 - \frac{1}{2}k_2^{-1}\|\tilde{\psi}_2\|^2 + \frac{1}{2}k_2^{-1}\|\psi_2\|^2 + \frac{\varepsilon_2}{4}\end{aligned} \tag{6.76}$$

With (6.67), (6.76), the derivative of V is given by

$$\begin{aligned}\dot{V} =&\dot{V}_1 + \dot{V}_2\\ \le& - \lambda_1\|\delta_1\|^2 + \frac{1+\rho_E}{4}\varepsilon_1 + \frac{1}{2}k_1^{-1}(1+\rho_E)\|\psi_1\|^2 + \frac{1}{2}\lambda_A\omega^{-1}\|x_2\|^2\\ &- \lambda_{K_p}\|x_2\|^2 - \lambda_{K_d}\|x_3\|^2 - \frac{1}{2}k_2^{-1}\|\tilde{\psi}_2\|^2 + \frac{1}{2}k_2^{-1}\|\psi_2\|^2 + \frac{\varepsilon_2}{4}\\ =& - \lambda_1\|\delta_1\|^2 - (\lambda_{K_p} - \frac{1}{2}\lambda_A\omega^{-1})\|x_2\|^2 - \lambda_{K_d}\|x_3\|^2 - \frac{1}{2}k_2^{-1}\|\tilde{\psi}_2\|^2 + \frac{1+\rho_E}{4}\varepsilon_1 + \frac{\varepsilon_2}{4}\\ &+ \frac{1}{2}k_1^{-1}(1+\rho_E)\|\psi_1\|^2 + \frac{1}{2}k_2^{-1}\|\psi_2\|^2\\ =:& - \lambda_1\|\delta_1\|^2 - \lambda_2\|\delta_2\|^2 + \frac{1+\rho_E}{4}\varepsilon_1 + \frac{\varepsilon_2}{4} + \frac{1}{2}k_1^{-1}(1+\rho_E)\|\psi_1\|^2 + \frac{1}{2}k_2^{-1}\|\psi_2\|^2\\ \le& - \lambda\|\delta\|^2 + h,\end{aligned} \tag{6.77}$$

where $\lambda_2 = \min\{\lambda_{K_p} - \frac{1}{2}\lambda_A\omega^{-1}, \lambda_{K_d}, \frac{1}{2}k_2^{-1}\}$, $\lambda = \min\{\lambda_1, \lambda_2\}$,

$$h = \frac{1+\rho_E}{4}\varepsilon_1 + \frac{\varepsilon_2}{4} + \frac{1}{2}k_1^{-1}(1+\rho_E)\|\psi_1\|^2 + \frac{1}{2}k_2^{-1}\|\psi_2\|^2. \tag{6.78}$$

If we choose suitable γ_1, S and K_p to satisfy $(\gamma_1 - \frac{1}{2}\lambda_A\omega) > 0$ and $(\lambda_{K_p} - \frac{1}{2}\lambda_A\omega^{-1}) > 0$, then $\dot{V}$ is negative definite for all

$$\|\delta\| \ge \sqrt{h/\lambda} \tag{6.79}$$

The uniform boundedness performance follows [130], and $d(r)$ is given by

$$d(r) = \begin{cases} R\sqrt{\frac{a_2}{a_1}}, & \text{if } r \le R,\\ r\sqrt{\frac{a_2}{a_1}}, & \text{if } r > R,\end{cases} \tag{6.80}$$

$$R = \sqrt{h/\lambda}. \tag{6.81}$$

Furthermore, uniform ultimate boundedness performance also follows with

$$\underline{d} = R\sqrt{a_2/a_1}, \tag{6.82}$$

for a given $\bar{d} > \underline{d}$,

$$T(\bar{d}, r) = \begin{cases} 0, & \text{if } r \leq \bar{R}, \\ \frac{a_2 r^2 - a_1 \bar{R}^2}{\lambda \bar{R}^2 - h}, & \text{otherwise.} \end{cases} \tag{6.83}$$

$$\bar{R} = \bar{d}\sqrt{a_1/a_2}. \tag{6.84}$$

The radius of uniform ultimate boundedness ball is determined by $\underline{d}$. As it is shown $\underline{d}$ approaches to 0 when R approaches to 0, which means both ε_1 and ε_2 are close to 0, and k_1, k_2 approach to infinity. Thus, if $\varepsilon_{1,2} \to 0$, $k_{1,2} \to \infty$, then $\underline{d} \to 0$.

Q.E.D.

6.5 Optimal gains design

Section 6.4 shows a system performance, which can be guaranteed by a deterministic control design. By the analysis, the size of the uniform ultimate boundedness region decreases as $k_{1,2}$ increase. This means strong performance is accompanied by a (possibly) large control effort, which is reflected by $k_{1,2}$. From the practical design point of view, the designer may be also interested in seeking an optimal choice of $k_{1,2}$ for a compromise among various conflicting criteria. This is associated with the minimization of a performance index.

We first explore more on the deterministic performance of the uncertain system. According to (6.50),

$$-\|\delta\|^2 \leq -\frac{1}{a_2}V. \tag{6.85}$$

With (6.85) in (6.77), we have

$$\dot{V}(t) \leq -\frac{\lambda}{a_2}V(t) + h. \tag{6.86}$$

This is a differential inequality[157] with $V_0 = V(t_0)$ as its initial condition. The analysis of this inequality can be made according to Chen[158] as follows. The following is needed for our analysis of (6.86).

Definition 6.2. If $\xi(e,t)$ is a scalar function of the scalars e, t is in some open connected set D, we say a function $e(t)$, $t_0 \le t \le t_1$, $t_1 \ge t_0$ is a solution of the differential inequality

$$\dot{e}(t) \le \xi(e(t), t) \tag{6.87}$$

on $[t_0, t_1]$ if $e(t)$ is continuous on $[t_0, t_1]$ and its derivative on $[t_0, t_1]$ satisfies (6.87).

Theorem 6.3. [119] Let $\xi(\phi, t)$ be continuous on an open connected set $\mathcal{D} \in \mathbf{R}^2$ and such that the initial value problem for the scalar equation

$$\dot{\phi}(t) = \xi(\phi(t), t), \quad \phi(t_0) = \phi_0, \tag{6.88}$$

has a unique solution. If $\phi(t)$ is a solution of (6.88) on $t_0 \le t \le t_1$ and $e(t)$ is a solution of (6.87) on $t_0 \le t \le t_1$ with $e(t_0) \le \phi(t_0)$, then $e(t) \le \phi(t)$ for $t_0 \le t \le t_1$.

This theorem suggests that it may be possible to study the upper bound of the solution instead of directly exploring the solution to the differential inequality. This is because the solution to (6.86) is often non-unique and not available while the solution to (6.88) is unique.

Theorem 6.4. [119] Consider the differential inequality (6.87) and the differential equation (6.88). Suppose that for some constant $L > 0$, the function $\xi(\cdot)$ satisfies the Lipschitz condition

$$|\xi(v_1, t) - \xi(v_2, t)| \le |v_1 - v_2|, \tag{6.89}$$

for all points $(v_1, t), (v_2, t) \in \mathcal{D}$. Then any function $e(t)$ that satisfies the differential inequality (6.87) for $t_0 \le t < t_1$ also satisfies the inequality

$$e(t) \le \phi(t), \tag{6.90}$$

for $t_0 \le t \le t_1$.

We consider the following differential inequality

$$\dot{r}(t) = -\frac{\lambda}{a_2} r(t) + h, r(t_0) = V_0 = V(t_0). \tag{6.91}$$

The RHS satisfies the global Lipschitz condition with $L = \lambda/a_2$. By solving the differential equation (6.91), we have

$$r(t) = (V_0 - \frac{\lambda}{a_2} h) \exp[-\frac{\lambda}{a_2}(t - t_0)] + \frac{\lambda}{a_2} h. \tag{6.92}$$

This results in

$$V(t) \le r(t), \tag{6.93}$$

or

$$V(t) \le (V_0 - \frac{\lambda}{a_2}h)\exp[-\frac{\lambda}{a_2}(t-t_0)] + \frac{\lambda}{a_2}h \tag{6.94}$$

for all $t \ge t_0$. By the same argument, we also have, for any t_s and any $\nu \ge t_s$,

$$V(\nu) \le (V_s - \frac{\lambda}{a_2}h)\exp[-\frac{\lambda}{a_2}(\nu-t_s)] + \frac{\lambda}{a_2}h, \tag{6.95}$$

where $V_s = V(t_s)$. The time t_s is when the control law (6.33) starts to be executed. It does not necessarily need to be t_0.

For given $\varepsilon_1, \varepsilon_2$, let

$$\eta(k_1,k_2,\psi_1,\psi_2,\nu,t_s) = (V_s - \frac{\lambda}{a_2}h)\exp[-\frac{\lambda}{a_2}(\nu-t_s)], \tag{6.96}$$

$$\eta_\infty(k_1,k_2,\psi_1,\psi_2) = \frac{\lambda}{a_2}h. \tag{6.97}$$

Let $\kappa = \frac{\lambda}{a_2}$, it can be seen that

$$\begin{aligned}\int_{t_s}^{\infty}\eta^2(k_1,k_2,\psi_1,\psi_2,\nu,t_s)d\nu =&(V_s - \frac{\lambda}{a_2}h)^2\int_{t_s}^{\infty}\exp[-2\frac{\lambda}{a_2}(\nu-t_s)]d\nu\\ =&(V_s - \frac{\lambda}{a_2}h)^2(\frac{1}{-2\frac{\lambda}{a_2}})\exp[-2\frac{\lambda}{a_2}(\nu-t_s)]|_{t_s}^{\infty}\\ =&(V_s-\kappa h)^2\frac{\kappa}{2}.\end{aligned} \tag{6.98}$$

With (6.78), let $\varepsilon = \frac{(1+\rho_E)\varepsilon_1}{4} + \frac{\varepsilon_2}{4}$, $b_1 = \frac{1}{2}(1+\rho_E)\|\psi_1\|^2$, $b_2 = \frac{1}{2}\|\psi_2\|^2$, one has

$$\begin{aligned}&\int_{t_s}^{\infty}\eta^2(k_1,k_2,\psi_1,\psi_2,\nu,t_s)d\nu\\ &=\frac{\kappa}{2}(V_s^2 - 2V_s\kappa h + \kappa^2h^2)\\ &=\frac{\kappa}{2}[V_s^2 - 2V_s\kappa(\varepsilon + \frac{b_1}{k_1} + \frac{b_2}{k_2}) + \kappa^2(\varepsilon + \frac{b_1}{k_1} + \frac{b_2}{k_2})^2]\\ &=\frac{\kappa}{2}[V_s^2 - 2V_s\kappa(\varepsilon + \frac{b_1}{k_1} + \frac{b_2}{k_2}) + \kappa^2(\varepsilon^2 + \frac{b_1^2}{k_1^2} + \frac{b_2^2}{k_2^2} + 2\frac{\varepsilon b_1}{k_1} + 2\frac{\varepsilon b_2}{k_2} + 2\frac{b_1b_2}{k_1k_2})]\\ &=\frac{\kappa}{2}(V_s^2 - 2V_s\kappa\varepsilon + \kappa^2\varepsilon^2) - \frac{V_s\kappa^2\varepsilon b_1}{k_1} + \frac{\kappa^3\varepsilon b_1}{k_1}\\ &\quad - \frac{V_s\kappa^2\varepsilon b_2}{k_2} + \frac{\kappa^3\varepsilon b_2}{k_2} + \frac{\kappa^3 b_1^2}{2k_1^2} + \frac{\kappa^3 b_1^2}{2k_1^2} + \frac{\kappa^3 b_1 b_2}{k_1k_2}.\end{aligned} \tag{6.99}$$

$$\begin{aligned}\eta_\infty^2(k_1,k_2,\psi_1,\psi_2) =&\kappa^2(\varepsilon + \frac{b_1}{k_1} + \frac{b_2}{k_2})^2.\\ =&\kappa^2\varepsilon^2 + \frac{\kappa^2b_1^2}{k_1^2} + \frac{\kappa^2b_2^2}{k_2^2} + 2\frac{\kappa^2\varepsilon b_1}{k_1} + 2\frac{\kappa^2\varepsilon b_2}{k_2} + 2\frac{\kappa^2 b_1b_2}{k_1k_2}\end{aligned} \tag{6.100}$$

Definition 6.3. Consider a fuzzy set

$$\mathcal{N} = \{(\nu, \mu_N(\nu)) | \nu \in N\}, \tag{6.101}$$

for any function $f : N \to \mathbf{R}$, the D-operation $D[f(\nu)]$ is given by

$$D[f(\nu)] = \frac{\int_N f(\nu)\mu_N(\nu)d\nu}{\int_N \mu_N(\nu)d\nu}. \tag{6.102}$$

Remark 6.9. In a sense, the D-operation $D[f(\nu)]$ takes an average value of $f(\nu)$ over $\mu_N(\nu)$. In the special case that $f(\nu) = \nu$, this is reduced to the well-known center-of-gravity defuzzification method. Particularly, if N is crisp (i.e., $\mu_N(\nu) = 1$ for all $\nu \in N$), $D[f(\nu)] = f(\nu)$.

Lemma 6.1. For any crisp constant $a \in \mathbf{R}$,

$$D[af(\nu)] = aD[f(\nu)]. \tag{6.103}$$

Proof: By definition 6.3,

$$\begin{aligned} D[af(\nu)] =& \frac{\int_N af(\nu)\mu_N(\nu)d\nu}{\int_N \mu_N(\nu)d\nu} \\ =& a\frac{\int_N f(\nu)\mu_N(\nu)d\nu}{\int_N \mu_N(\nu)d\nu} \\ =& aD[f(\nu)] \end{aligned} \tag{6.104}$$

Q.E.D.

We now propose the following performance index: For any t_s, and weighting factors α, β, let

$$J(k_1, k_2, t_s) = J_1(k_1, k_2, t_s) + \alpha J_2(k_1, k_2) + \beta J_3(k_1, k_2), \tag{6.105}$$

where

$$\begin{aligned} J_1(k_1, k_2, t_s) =& D[\frac{\kappa}{2}(V_s^2 - 2V_s\kappa\varepsilon + \kappa^2\varepsilon^2)] - D[\frac{V_s\kappa^2\varepsilon b_1}{k_1}] + D[\frac{\kappa^3\varepsilon b_1}{k_1}] \\ & - D[\frac{V_s\kappa^2\varepsilon b_2}{k_2}] + D[\frac{\kappa^3\varepsilon b_2}{k_2}] + D[\frac{\kappa^3 b_1^2}{2k_1^2}] + D[\frac{\kappa^3 b_2^2}{2k_2^2}] + D[\frac{\kappa^3 b_1 b_2}{k_1 k_2}], \\ J_2(k_1, k_2) =& D[\kappa^2\varepsilon^2] + D[\frac{\kappa^2 b_1^2}{k_1^2}] + D[\frac{\kappa^2 b_2^2}{k_2^2}] + D[2\frac{\kappa^2\varepsilon b_1}{k_1}] + D[2\frac{\kappa^2\varepsilon b_2}{k_2}] + D[2\frac{\kappa^2 b_1 b_2}{k_1 k_2}], \\ J_3(k_1, k_2) =& (k_1^2 + k_2^2). \end{aligned} \tag{6.106}$$

With (6.106) into (6.105), applying Lemma 6.1, we have

$$\begin{aligned}
&J(k_1,k_2,t_s)\\
=&D[\frac{\kappa}{2}(V_s^2-2V_s\kappa\varepsilon+\kappa^2\varepsilon^2)]-\frac{1}{k_1}D[V_s\kappa^2\varepsilon b_1]+\frac{1}{k_1}D[\kappa^3\varepsilon b_1]\\
&-\frac{1}{k_2}D[V_s\kappa^2\varepsilon b_2]+\frac{1}{k_2}D[\kappa^3\varepsilon b_2]+\frac{1}{k_1^2}D[\frac{\kappa^3b_1^2}{2}]+\frac{1}{k_2^2}D[\frac{\kappa^3b_2^2}{2}]+\frac{1}{k_1k_2}D[\kappa^3b_1b_2]\\
&+\alpha(D[\kappa^2\varepsilon^2]+\frac{1}{k_1^2}D[\kappa^2b_1^2]+\frac{1}{k_2^2}D[\kappa^2b_2^2]+\frac{1}{k_1}D[2\kappa^2\varepsilon b_1]+\frac{1}{k_2}D[2\kappa^2\varepsilon b_2]\\
&+\frac{1}{k_1k_2}D[2\kappa^2b_1b_2])+\beta(k_1^2+k_2^2)\\
=&\kappa_1-\frac{\kappa_2}{k_1}+\frac{\kappa_3}{k_1}-\frac{\kappa_4}{k_2}+\frac{\kappa_5}{k_2}+\frac{\kappa_6}{k_1^2}+\frac{\kappa_7}{k_2^2}+\frac{\kappa_8}{k_1k_2}\\
&+\alpha(\frac{\kappa_9}{k_1}+\frac{\kappa_{10}}{k_2}+\frac{\kappa_{11}}{k_1^2}+\frac{\kappa_{12}}{k_2^2}+\frac{\kappa_{13}}{k_1k_2})+\beta(k_1^2+k_2^2),
\end{aligned}\tag{6.107}$$

where

$$\begin{aligned}
\kappa_1=&D[\frac{\kappa}{2}(V_s^2-2V_s\kappa\varepsilon+\kappa^2\varepsilon^2)]+\alpha D[\kappa^2\varepsilon^2],\\
\kappa_2=&D[V_s\kappa^2\varepsilon b_1],\quad \kappa_3=D[\kappa^3\varepsilon b_1],\quad \kappa_4=D[V_s\kappa^2\varepsilon b_2],\\
\kappa_5=&D[\kappa^3\varepsilon b_2],\quad \kappa_6=D[\frac{\kappa^3b_1^2}{2}],\quad \kappa_7=D[\frac{\kappa^3b_2^2}{2}],\\
\kappa_8=&D[\kappa^3b_1b_2],\quad \kappa_9=D[\kappa^2b_1^2],\quad \kappa_{10}=D[\kappa^2b_2^2],\\
\kappa_{11}=&D[2\kappa^2\varepsilon b_1],\quad \kappa_{12}=D[2\kappa^2\varepsilon b_2],\quad \kappa_{13}=D[2\kappa^2b_1b_2].
\end{aligned}\tag{6.108}$$

The optimal design problem is then equivalent to the following constrained optimization problem: For any t_s

$$\min_{k_1,k_2} J(k_1,k_2,t_s)\qquad \text{subject to}\qquad k_1,k_2>0.\tag{6.109}$$

For any t_s, taking the first derivative of J with respect to k_1,k_2,

$$\begin{cases}\frac{\partial J}{\partial k_1}= & \frac{\kappa_2}{k_1^2}-\frac{\kappa_3}{k_1^2}-\frac{2\kappa_6}{k_1^3}-\frac{\kappa_8}{k_1^2k_2}-\alpha(\frac{\kappa_9}{k_1^2}+\frac{2\kappa_{11}}{k_1^3}+\frac{\kappa_{13}}{k_1^2k_2})+2\beta k_1,\\ \frac{\partial J}{\partial k_2}= & \frac{\kappa_4}{k_2^2}-\frac{\kappa_5}{k_2^2}-\frac{2\kappa_7}{k_2^3}-\frac{\kappa_8}{k_1k_2^2}-\alpha(\frac{\kappa_{10}}{k_2^2}+\frac{2\kappa_{12}}{k_2^3}+\frac{\kappa_{13}}{k_1k_2^2})+2\beta k_2.\end{cases}\tag{6.110}$$

That

$$\frac{\partial J}{\partial k_1}=0,\frac{\partial J}{\partial k_2}=0\tag{6.111}$$

leads to

$$\begin{cases}(\kappa_2-\kappa_3-\alpha\kappa_9-\frac{\kappa_8}{k_2}-\alpha\frac{\kappa_{13}}{k_2})k_1+2\beta k_1^4 & =2(\kappa_6+\alpha\kappa_{11}),\\ (\kappa_4-\kappa_5-\alpha\kappa_{10}-\frac{\kappa_8}{k_1}-\alpha\frac{\kappa_{13}}{k_1})k_2+2\beta k_2^4 & =2(\kappa_7+\alpha\kappa_{12}).\end{cases}\tag{6.112}$$

The solution of (6.112) can be obtained numerically. Suppose the solutions are k_1^*, k_2^*, we analyze the second order derivative of J. Let

$$
\begin{aligned}
A =& \frac{\partial^2 J}{\partial k_1^2}\Big|_{k_1=k_1^*,k_2=k_2^*} \\
=& (-\kappa_2+\kappa_3+\frac{\kappa_8}{k_2^*}+\alpha\kappa_9+\frac{\alpha\kappa_{13}}{k_2^*})\frac{2}{k_1^{*3}}+\frac{6(\kappa_6+\alpha\kappa_{11})}{k_1^{*4}}+2\beta \\
=:& \zeta_1(k_1^*)+\frac{2(\kappa_8+\alpha\kappa_{13})}{k_1^{*3}k_2^*}, \\
B =& \frac{\partial}{\partial k_2}(\frac{\partial J}{k_1})\Big|_{k_1=k_1^*,k_2=k_2^*}=\frac{\partial}{\partial k_1}(\frac{\partial J}{k_2})\Big|_{k_1=k_1^*,k_2=k_2^*}=\frac{\kappa_8+\alpha\kappa_{13}}{k_1^{*2}k_2^{*2}} \\
C =& \frac{\partial^2 J}{\partial k_2^2}\Big|_{k_1=k_1^*,k_2=k_2^*} \\
=& (-\kappa_4+\kappa_5+\frac{\kappa_8}{k_1^*}+\alpha\kappa_{10}+\frac{\alpha\kappa_{13}}{k_1^*})\frac{2}{k_2^{*3}}+\frac{6(\kappa_7+\alpha\kappa_{12})}{k_2^{*4}}+2\beta \\
=:& \zeta_2(k_2^*)+\frac{2(\kappa_8+\alpha\kappa_{13})}{k_1^*k_2^{*3}},
\end{aligned}
\tag{6.113}
$$

here

$$
\begin{aligned}
\zeta_1(k_1^*) =& \frac{2(-\kappa_2+\kappa_3+\alpha\kappa_9)}{k_1^{*3}}+\frac{6(\kappa_6+\alpha\kappa_{11})}{k_1^{*4}}+2\beta, \\
\zeta_2(k_2^*) =& \frac{2(-\kappa_4+\kappa_5+\alpha\kappa_{10})}{k_2^{*3}}+\frac{6(\kappa_7+\alpha\kappa_{12})}{k_2^{*4}}+2\beta.
\end{aligned}
\tag{6.114}
$$

As we know, whether the solution of (6.112) minimizes the index J or not, it depends on the signs of A and B^2-AC. After performing some algebra we have

$$
\begin{aligned}
& B^2-AC \\
=& \frac{(\kappa_8+\alpha\kappa_{13})^2}{k_1^{*4}k_2^{*4}}-(\zeta_1(k_1^*)+\frac{2(\kappa_8+\alpha\kappa_{13})}{k_1^{*3}k_2^*})(\zeta_2(k_2^*)+\frac{2(\kappa_8+\alpha\kappa_{13})}{k_1^*k_2^{*3}}) \\
=& -\zeta_1(k_1^*)\frac{2(\kappa_8+\alpha\kappa_{13})}{k_1^*k_2^{*3}}-\zeta_2(k_2^*)\frac{2(\kappa_8+\alpha\kappa_{13})}{k_1^{*3}k_2^*}-\zeta_1(k_1^*)\zeta_2(k_2^*)-\frac{3(\kappa_8+\alpha\kappa_{13})^2}{k_1^{*4}k_2^{*4}} \\
& <0
\end{aligned}
\tag{6.115}
$$

Therefore, the solution of (6.112) solves the minimization problem.

Remark 6.10. The choice of $\varepsilon_1, \varepsilon_2$ will affect $J_{\min}$. Thus, it is interesting for the designer to consider the optimal choice of $\varepsilon_1, \varepsilon_2$ for the minimization of $J_{\min}$. This will be pursued in a future study.

6.6 Illustrative Example

A two-link FJM, see Figure 2.5, is considered to show the validity of the control proposed in this Chapter. Let link angle vector $q_1 = [q_{1,1} \quad q_{1,2}]^T$, joint angle vector $q_2 = [q_{2,1} \quad q_{2,2}]^T$. $m_1(m_2)$ is the mass of link, l_1 is the length of the first link, $l_{c1}(l_{c2})$ is the center position of the link (suppose that mass is uniform on the link), g is gravitational constant. The system model of the two-link FJM is given by

$$M(q_1) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{6.116}$$

$$C(q_1, \dot{q}_1) = \begin{bmatrix} -m_1 l_1 l_{c2} \sin q_{1,2} \dot{q}_{1,2} & -m_2 l_1 l_{c2} \sin q_{1,2}(\dot{q}_{1,1} + \dot{q}_{1,2}) \\ m_2 l_1 l_{c2} \sin q_{1,2} \dot{q}_{1,1} & 0 \end{bmatrix}, \tag{6.117}$$

$$\begin{aligned} G(q_1) &= \begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \sin q_{1,1} + m_2 l_{c2} g \sin(q_{1,1} + q_{1,2}) \\ m_2 l_{c2} g \sin(q_{1,1} + q_{1,2}) \end{bmatrix}, \\ J &= \begin{bmatrix} J_{11} & 0 \\ 0 & J_{22} \end{bmatrix}, S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}, \end{aligned} \tag{6.118}$$

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, K_p = \begin{bmatrix} k_{p1} & 0 \\ 0 & k_{p2} \end{bmatrix}, K_d = \begin{bmatrix} k_{d1} & 0 \\ 0 & k_{d2} \end{bmatrix}, \tag{6.119}$$

where

$$\begin{aligned} M_{11} &= m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2} \cos q_{1,2}) + m_1 l_{c1}^2 + I_1 + I_2, \\ M_{12} &= m_2(l_{c2}^2 + l_1 l_{c2} \cos q_{1,2}) + I_2, \\ M_{21} &= M_{12}, \\ M_{22} &= m_2 l_{c2}^2 + I_2. \end{aligned} \tag{6.120}$$

All elements of inertia matrix are bounded by following

$$\begin{aligned} |M_{11}| &\le m_2(l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}) + m_1 l_{c1}^2 + I_1 + I_2, \\ |M_{12}| &\le m_2(l_{c2}^2 + l_1 l_{c2}) + I_2, \\ |M_{22}| &\le m_2 l_{c2}^2 + I_2. \end{aligned} \tag{6.121}$$

For simplicity, we require that the velocities of the link angles meet the following constraint:

$$\dot{q}_{1,1} + \dot{q}_{1,2} = 0, \tag{6.122}$$

which means

$$A = [1 \quad 1], \qquad c = 0, \qquad b = 0. \tag{6.123}$$

Notice that, the desired constraint (6.122) is linear in velocities. The Assumptions 6.6 and 6.7 are met by choosing

$$\begin{aligned} \Pi_1(\psi_1, q_1, \dot{q}_1) =& \psi_{11}\|\dot{q}_1\|^2 + \psi_{12}\|\dot{q}_1\| + \psi_{13} \\ =& [\psi_{11}\ \psi_{12}\ \psi_{13}] \begin{bmatrix} \|\dot{q}_1\|^2 \\ \|\dot{q}_1\| \\ 1 \end{bmatrix} \\ =:& \psi_1^T \tilde{\Pi}_1(\psi_1, q_1, \dot{q}_1), \end{aligned} \tag{6.124}$$

$$\begin{aligned} \Pi_2(\psi_2, q_1, \dot{q}_1, x_2, x_3) =& \psi_{21}(\|x_3\| + \|x_2\|)^2 + \psi_{22}(\|x_3\| + \|x_2\|) + \psi_{23} \\ =& [\psi_{21}\ \psi_{22}\ \psi_{23}] \begin{bmatrix} (\|x_3\| + \|x_2\|)^2 \\ (\|x_3\| + \|x_2\|) \\ 1 \end{bmatrix} \\ =:& \psi_2^T \tilde{\Pi}_2(\psi_2, q_1, \dot{q}_1, x_2, x_3). \end{aligned} \tag{6.125}$$

Then the adaptive laws are given by

$$\begin{aligned} \dot{\hat{\psi}}_1 =& [\hat{\psi}_{11}\ \hat{\psi}_{12}\ \hat{\psi}_{13}]^T \\ =& k_1 \begin{bmatrix} \|\dot{q}_1\|^2 \\ \|\dot{q}_1\| \\ 1 \end{bmatrix} \|A\dot{q}_1 - c\| - \begin{bmatrix} \hat{\psi}_{11} \\ \hat{\psi}_{12} \\ \hat{\psi}_{13} \end{bmatrix} \end{aligned} \tag{6.126}$$

$$\begin{aligned} \dot{\hat{\psi}}_2 =& [\hat{\psi}_{21}\ \hat{\psi}_{22}\ \hat{\psi}_{23}]^T \\ =& k_2 \begin{bmatrix} (\|x_3\| + \|x_2\|)^2 \\ (\|x_3\| + \|x_2\|) \\ 1 \end{bmatrix} \|x_3 + Sx_2\| - \begin{bmatrix} \hat{\psi}_{21} \\ \hat{\psi}_{22} \\ \hat{\psi}_{23} \end{bmatrix} \end{aligned} \tag{6.127}$$

We consider there are uncertainties in masses and K, i.e., $m_1 = \bar{m}_1 + \Delta m_1(t), m_2 = \bar{m}_2 + \Delta m_2(t), K_1 = \bar{K}_1 + \Delta K_1(t), K_2 = \bar{K}_2 + \Delta K_2(t)$. Here, we suppose that $\Delta m_1(t) = \Delta m_2(t) = \Delta m(t)$ which is "close to 0.3",and $\Delta K_1(t) = \Delta K_2(t) = \Delta K(t)$ which is "close to 0.2". Their associated membership functions (triangular type) are given by

$$\mu_{\Delta m}(\nu) = \begin{cases} \frac{10}{3}\nu, & 0 \le \nu \le 0.3, \\ -\frac{10}{3}\nu + 2, & 0.3 \le \nu \le 0.6, \end{cases} \tag{6.128}$$

$$\mu_{\Delta K}(\nu)=\begin{cases}\frac{10}{2}\nu, & 0\le\nu\le 0.2,\\ -\frac{10}{2}\nu+2, & 0.2\le\nu\le 0.4.\end{cases} \tag{6.129}$$

The uncertain parameters $\psi_{1,2}$ in adaptation laws are "close to 0.25", the associated membership function (triangular type) is given by

$$\mu_{\psi_{1,2}}(\nu)=\begin{cases}\frac{10}{2.5}\nu, & 0\le\nu\le 0.25,\\ -\frac{10}{2.5}\nu+2, & 0.25\le\nu\le 0.5.\end{cases} \tag{6.130}$$

The simulation is performed with $\gamma_1=4$ to meet $(\gamma_1-\frac{1}{2}\lambda_A)>0$. Similarly, we set $k_{p1}=k_{p2}=4$. Then, let $m_1=m_2=1, l_1=1, l_{c1}=l_{c2}=0.5, \bar{K}_1=\bar{K}_2=1, I_1=I_2=1, J_{11}=J_{22}=1, g=9.81, s_1=s_2=1, \omega=1, P=6, \varepsilon_1=\varepsilon_2=0.1, k_{d1}=k_{d2}=2$. With these, we have

$$\|PA\hat{M}^{-1}\|\le\|PA\|\|\|M^{-1}\|\|\|K\|\le 20.2330=\lambda_A. \tag{6.131}$$

Let us choose a crisp initial condition that $\delta(t_s)=\delta(t_0)=\delta(0)$ with $q_{2,1}(0)=0.8$, $q_{1,1}(0)=0.5$, $q_{2,2}(0)=0.4$, $q_{1,2}(0)=0.5$, $\dot{q}_{2,1}(0)=0.1$, $\dot{q}_{1,1}(0)=1$, $\dot{q}_{2,2}(0)=-0.1$, $\dot{q}_{1,2}(0)=-0.5$. Here $\dot{q}_{1,1}(0)+\dot{q}_{1,2}(0)=0.5\neq 0$, which means the initial status dose not meet the constraint. Let $\psi_1(0)=\psi_2(0)=[0.5,0.5,0.5]^T$, and choose the weighting factors $\alpha=\beta=1$ then we have $V_s=47.9547$. By using the D-operation and fuzzy arithmetic, we obtain $\kappa_1=1.125\times10^3$, $\kappa_2=1.0441$, $\kappa_3=0.0218$, $\kappa_4=0.9228$, $\kappa_5=0.0192$, $\kappa_6=6.5172\times10^{-4}$, $\kappa_7=6.5172\times10^{-4}$, $\kappa_8=3.6871\times10^{-4}$, $\kappa_9=0.0013$, $\kappa_{10}=0.0013$, $\kappa_{11}=0.0435$, $\kappa_{12}=0.0385$, $\kappa_{13}=7.3742\times10^{-4}$. By solving (6.112), we have $k_1=0.1203$, $k_2=0.1304$.

Figure 6.1 shows the constrained system performance (i.e., $\|x_1\|=\|\dot{q}_{1,1}+\dot{q}_{1,2}\|$) by adopting the control (6.33) under the weighting factors $\alpha=\beta=1$. The performance under nominal control (i.e., $p_{13}\equiv 0$ in (6.33)) is also shown for comparison. This corresponds to Udwadia-Kalaba-like control for asymptotic convergence. By using the proposed control scheme, system performance meets the constraints after a certain time, while the nominal controlled one can not converge to a region around 0. Figure 6.2 gives the trajectories of the link angles velocities $\dot{q}_{1,1}$ and $\dot{q}_{1,2}$, respectively.

The corresponding norms of control inputs ($\|\tau\|$) are shown in Figure 6.3. Obviously, not only the performance of adaptive robust controlled system is much better than that of nominal controlled, but also the control costs of adaptive robust control is much less than the other one. Furthermore, at the end of the simulation, cost of nominal control seems to be diverging. Figure 6.4 shows the histories of the adaptive parameters. The proposed control not only drives the system

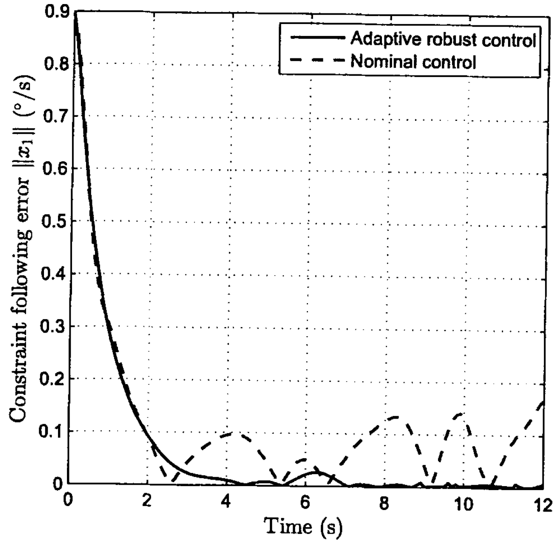

Figure 6.1: The comparison of desired constraint performance.

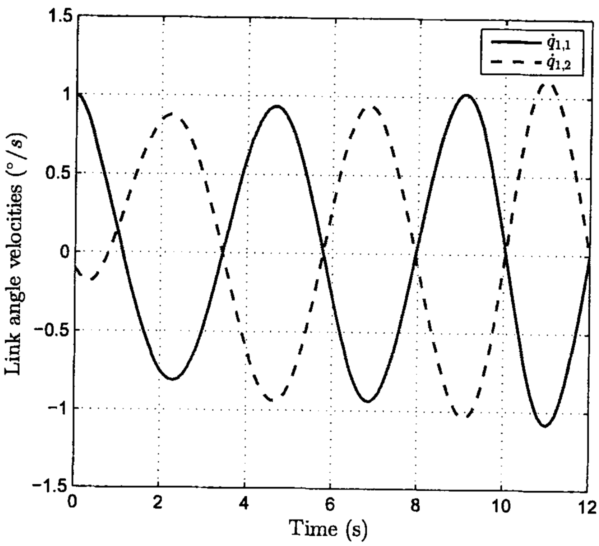

Figure 6.2: The histories of constrained link angles velocities.

to meet the constraint, but also renders the other system state (e.g., trajectory of x_2 in Figure 6.5) uniform boundedness and uniform ultimate boundedness.

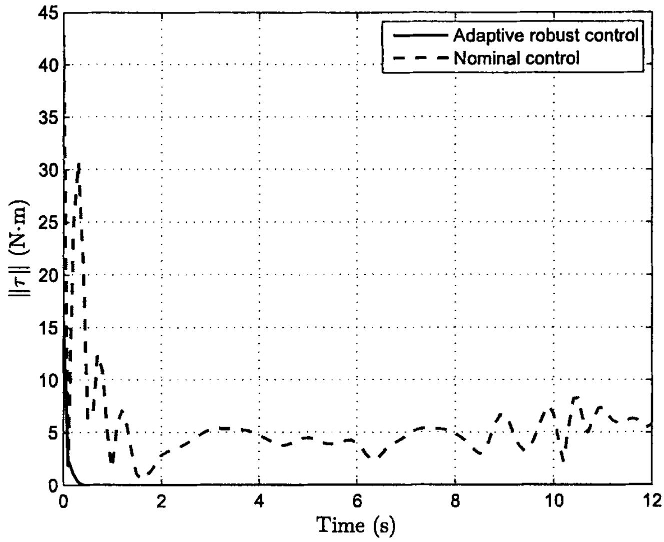

Figure 6.3: The histories of the corresponding control efforts.

We choose five sets of weighting α and β, their values and the corresponding k_1, k_2 and $J_{\min}$ are summarized in Table 6.6. The control costs with optimal k_1, k_2 under different weighting factors are shown in Figure 6.6.

Table 6.1: Weighting / optimal gain / minimum cost

(α,β)	α/β	k_1	k_2	$J_{\min}$
(1,1)	1	0.1203	0.1340	1125.3
(1,10)	0.1	0.1180	0.1274	1125.4
(1,100)	0.01	0.1024	0.1044	1126.6
(10,1)	10	0.7191	0.7124	1136.0
(100,1)	100	1.5408	1.5063	1151.7

In our control design of the adaptive laws, k_1 and k_2 determine the rate of how fast $\hat{\psi}_{1,2}$ approach to $\psi_{1,2}$. Different combinations of $(k_1, k2)$ surely affect the control efforts and the performances of $\hat{\psi}_1$ and $\hat{\psi}_2$. The influence, caused by different (k_1, k_2), on τ_{ave}, $\|\tau\|_{\max}$ are shown in Figure 6.7 and Figure 6.8.

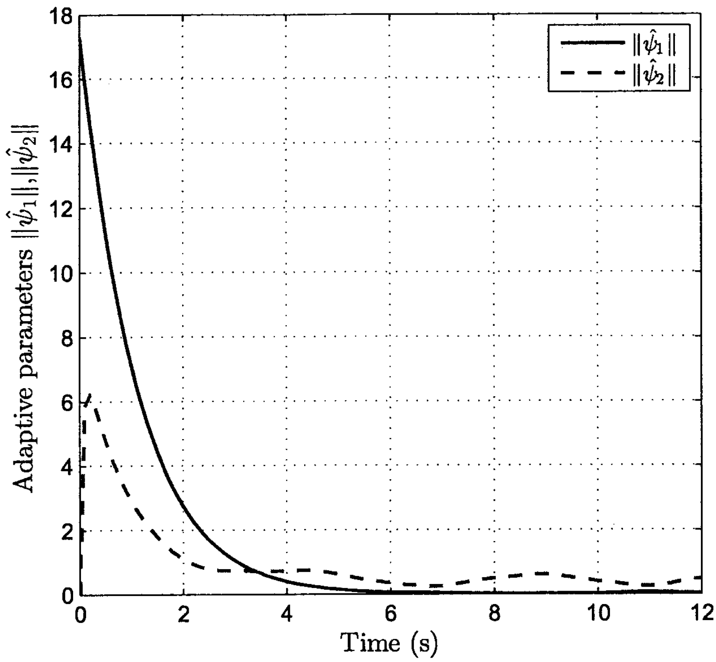

Figure 6.4: The histories of the norm of adaptive parameters.

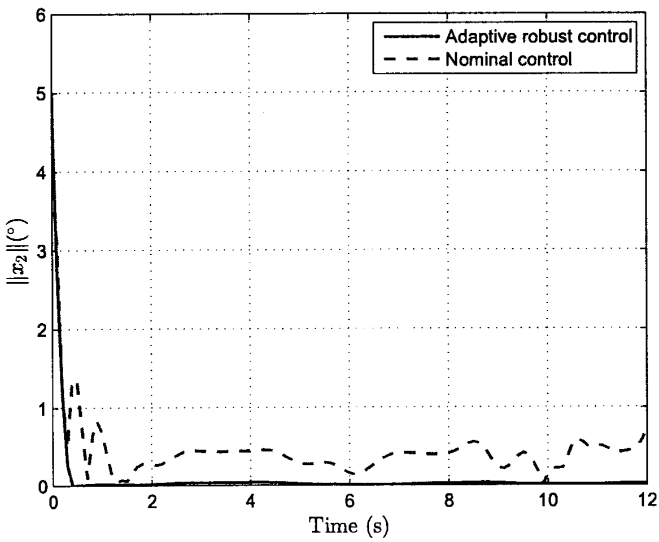

Figure 6.5: The comparison of the system performance x_2.

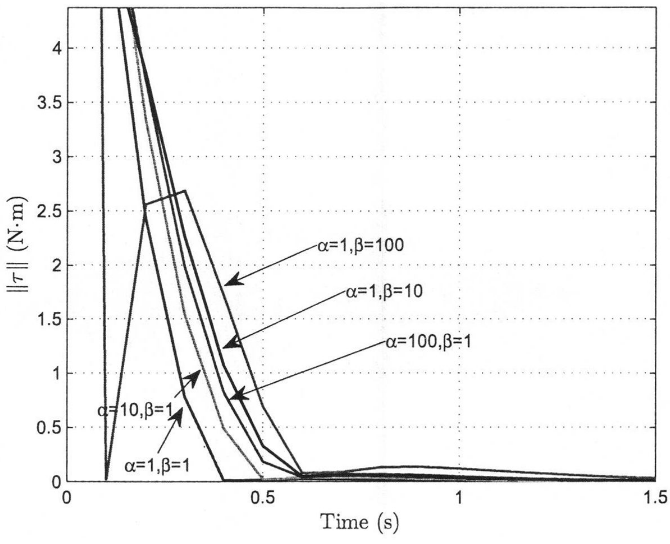

Figure 6.6: Comparison of control costs under different weighting factors.

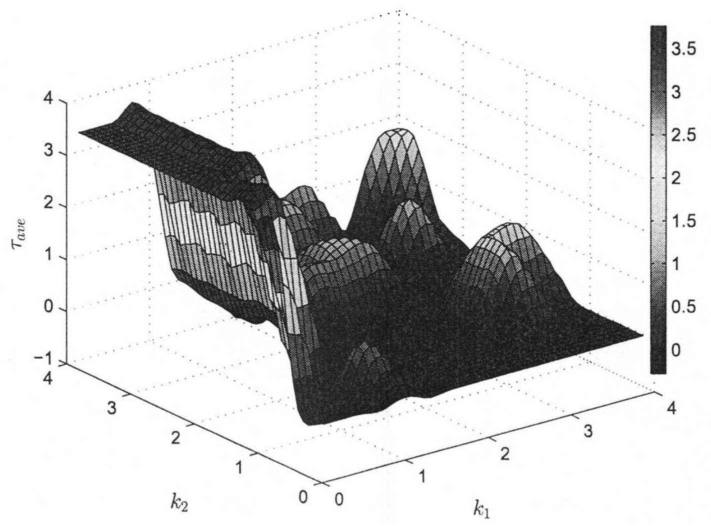

Figure 6.7: The average control efforts under different (k_1, k_2).

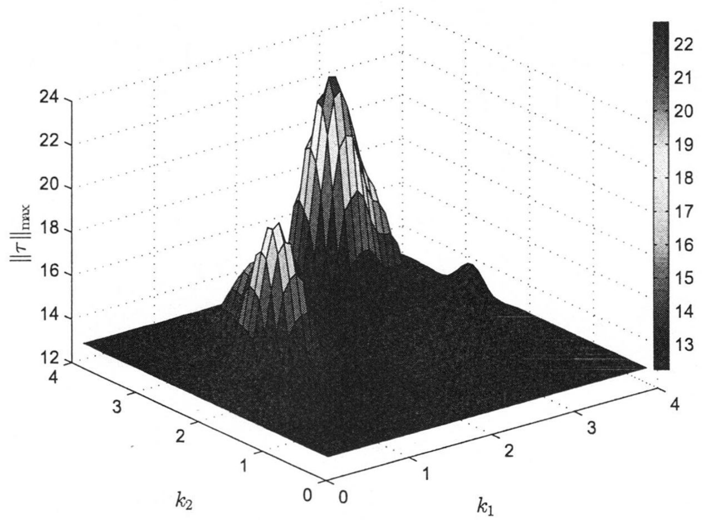

Figure 6.8: The maximum control efforts under different (k_1, k_2).

Similar to the definitions of τ_{ave}, $\|\tau\|_{\max}$, let

$$\bar{\psi} = \frac{\int_0^T \hat{\psi}(t)dt}{T}|_{(k_1,k_2)} \tag{6.132}$$

as the average of the norm of the adaptive parameters, and

$$\|\hat{\psi}\|_{\max} = \max_t \|\hat{\psi}(t)\||_{(k_1,k_2)} \tag{6.133}$$

as the maximum norm of the adaptive parameters. The relations among $\bar{\psi}$, $\|\hat{\psi}\|_{\max}$ and (k_1, k_2) are shown in Figure 6.9 and Figure 6.10.

6.7 Conclusions

The constraint force servo control problem for under-actuated flexible joint manipulator is addressed. By assuming the system state and uncertainty lie in some certain fuzzy sets, the system becomes so-called *fuzzy dynamical system.* After implanting a virtual control, the whole system can be treated as cascades of two subsystems. The constraint force, which is used in proposed control, is obtained via Udwadia-Kalaba equation. The optimal control effort also can be achieved by minimizing a performance which includes both the deterministic performance and average fuzzy control efforts. The simulation shows that the constraint is approximately followed while the uniform boundedness and uniform ultimate boundedness performances of the whole system are guaranteed.

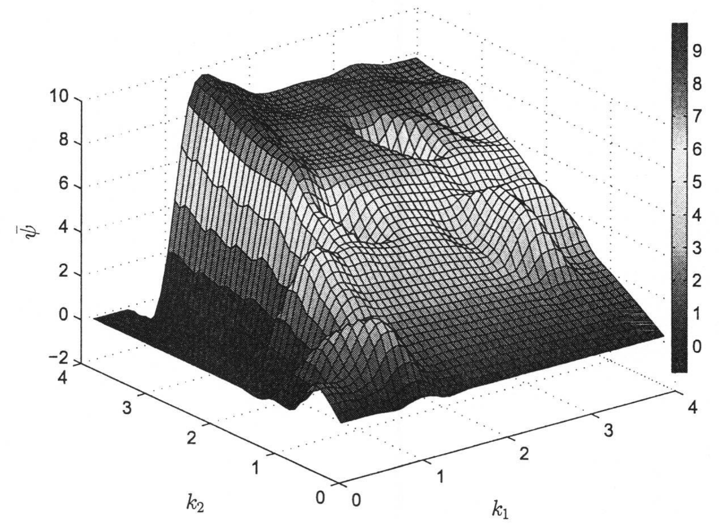

Figure 6.9: The average of adaptive parameters under different (k_1, k_2).

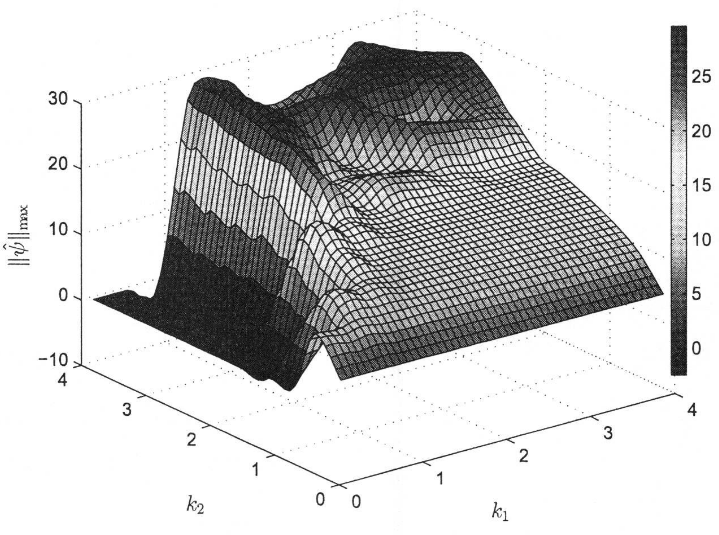

Figure 6.10: The maximum norm of adaptive parameters under different (k_1, k_2).

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Chapter 7 Conclusions and future work

The studies presented in this dissertation are motivated by the demands of the processing uncertainty in flexible joint manipulator (FJM) system. The contributions are in the areas of FJM systems with uncertainty, robust control, adaptive control, optimal control and constrained system. The proposed control schemes are general enough in nature to be applicable to both revolute joint and prismatic joint FJM systems.

7.1 Dissertation summary

Motivated by the practical success of the applications of FJMs, the studies on the control of FJM with uncertainty increases rapidly in the past three decades. A systematic approach to the system description, control design, parameter optimization of the uncertain FJM system is proposed. The work of the dissertation is presented as follows.

As a drawback of many other Lyapunov-based robust control schemes, the inertia matrix is assumed to be upper bounded so that the stability proof can be guaranteed. This limits such controls to only be applied on the revolute type FJM. In Chapter 2, we propose a more generic upper bound condition which extends the control to prismatic type FJM. The presented control does not require any other information of the uncertainty other than its possible bound. Since the FJM system is under-actuated, the backstepping-like technique is adopted for the control design. By creatively implanting a fictitious control, the system is divided into two connected subsystems, then the actual control input can be constructed step by step.

In engineering area, the information of uncertainty may be poorly known or totally unknown. The improper choice of uncertainty bound in robust control may lead to extra control cost. In order to deal with this issue, an adaptive robust control is presented in Chapter 3. The basic assumption of this method is that the uncertainty bound can be described by a function with known form although some parameter of this function is unknown. Thus, we can construct an adaptive law to obtain the estimation of the unknown parameter and then obtain the estimation of uncertainty bound. The procedure of the control design is the same as that in Chapter 2. The adaptive law here is of leakage type which means that the adaptive parameters do not always grow with time, since once the system performance is satisfactory and the gradients of the boundary function are small. During the control design, the system is reexpressed with new state variables. This representation introduces a problem which was neglected before while giving designer convenience. We should notice that the system is not the original system any more. It is not suitable to say the original system is stable because the transformed system is stable, directly. Form this point of view, we prove the stability of the original system theoretically.

The fuzzy description of uncertainties earns more advantages than that of probability method. It is both significantly and theoretically valuable in engineering to explore the incorporating of fuzzy theory into system theory. In Chapter 4, a new fuzzy approach to the control design for uncertain FJM system is proposed. The fuzzy set theory is employed to describe the uncertainty and the system states. By applying a deterministic adaptive robust controller, the uniform boundedness and uniform ultimate boundedness properties are guaranteed. Since the designer may be interested in seeking a control scheme which renders high-performance at lower cost, a fuzzy based optimal method is presented. With the help of D-operation, the optimization problem is equivalent to a performance index minimization problem. The index combines both the deterministic performance and fuzzy performance. We show this problem can be solved by solving two quartic algebraic equations.

For the control of constrained uncertain FJM system, we firstly introduce a new modeling approach based on Udwadia-Kalaba theory in Chapter 5. The Moore-Penrose generalized inverse is employed to investigate the geometry structure of constraint force. By the proof, it can be seen that the constraint force is formulated in a fixed manner and has a clear relation with the impressed force (Udwadia-Kalaba equation). The constraints, which can be holonomic or nonholonomic, are given in second order derivative form. We show that various control problems, including stabilization, trajectory following, optimization problem, etc., can be cast into this form.

In Chapter 6, we consider the constraint fore servo control problem of FJM system. Under the frame work of Chapter 5, by utilizing the Udwadia-Kalaba equation, the closed form of the constraint force is obtained so that it can be used in our control design. The control here is consisted of three parts: a nominal part which is used to control "nominal" part of the system, a Udwadia-Kalaba-like part which is used to present the constraint force, an adaptive robust part which is used to deal with the effects of the uncertainty. The proposed adaptive robust control makes the controlled system approximately follow the given constraints. To be specific, a part of the overall system (i.e., the link position subsystem) is desired to follow the given constraints. After applying the control, although the initial condition is far away from the constraints, the link position subsystem approximately follows the constraints. Meanwhile, the rest part of the system (i.e., the joint position subsystem) is uniformly bounded and uniformly ultimately bounded.

7.2 Innovations

The main contributions of this paper are fourfold:

First, for the control of uncertain FJM system whose uncertainty bound is known a priori, we propose a more general Lyapunov-based robust controller. In the past literatures, the inertia matrix should be uniformly upper bounded which limits the applications to only revolute joint type FJM. Our work proposes a more generic upper bound condition of inertia matrix so that the control objective can also include prismatic joint type FJM. Then, a robust controller is proposed which can deal with the uncertainty in FJM system. The control design and stability analysis are only based on the possible upper bound of the uncertainty, and no other information is required. Furthermore, the sizes of the uniform ultimate boundedness ball and uniform stability ball can be made arbitrary small by choosing suitable parameters.

Second, for the control of uncertain FJM system whose uncertainty bound is unknown, we propose an adaptive robust controller to guarantee the stability of the system. Specifically speaking, although the bound of uncertainty is unknown, we know it does exist. We assume the uncertainty bound is in a known manner on an unknown parameter. Then an adaptive law is constructed to approximately estimate this unknown parameter. The stability of the adaptive/mechanical system is proven theoretically and numerically. Besides, by using backstepping-like technique, the difficulty in control design for mismatched uncertain system is overcome. Furthermore, not only the stability of transformed system is guaranteed, but also the stability of original system is proven theoretically.

Third, we propose a novel fuzzy dynamical system approach to the optimal control design for uncertain FJM system. The control, which is deterministic and is not IF-THEN rules-based, is proposed to guarantee the system uniformly bounded and uniformly ultimately bounded. We not only guarantee the deterministic performance (including uniform boundedness and uniform ultimate boundedness), but also explore the control cost under the fuzzy description of the uncertainty. This is achieved by minimizing a comprehensive performance index. It is proven that the extreme solution to this optimization problem, which is solved by a first-order necessary condition, is indeed the minimum solution, which is verified by a second-order sufficient condition.

Forth, following the frame work of Udwadia and Kalaba, the constraint force servo control problem of FJM system is addressed. The uncertainty can be (possibly fast) time-varying. No Lagrange multiplier is needed for control formulation, hence no force feedback. No initial condition restrictions are imposed and the starting configuration of the mechanical system can be far away from the desired constraint. We obtain the closed form of constraint force which can be applied in the control design. This is feasible for both passive constraint and servo constraint applications. The proposed control, including Udwadia-Kalaba-like control part, makes the system approximately follow the given constraints, while guaranteeing the uniform boundedness and uniform ultimate boundedness performance of the rest of system.

7.3 Prospect

The control problem of FJM is to design an appropriate controller which guarantees some desired system performances. Such performance should be always maintained regardless of what the actual value of the uncertainty may be. The uncertainty in practical application is often obtained via observed data while the data are, by nature, always limited for their non-repeatability (e.g., the earthquake data). Hence, the novel ideas on uncertainty description from this dissertation will soon benefit other system control, such as sliding mode control, neural network control, etc. The frame work of adaptive robust control for fuzzy FJM system proposed in this paper can also be updated to more general system conditions, such as, nonlinear interconnections with uncertainty, unknown initial condition, etc.

In this work, we have proposed various control schemes by the way of backstepping-like method. In utilizing this technique, skills and experiences are required in choosing of bounding functions. In the stability analysis of robust control, the knowledge of derivative of the implanted fictitious control is required. If we can find another different procedure which does not depend on the derivative of virtual control, less computation may be needed and efficiency would be improved.

The control schemes developed in this work depend on the full state feedback which means the positions and velocities of links and actuators are required. Therefore, designers need to consider how these states are acquired so that they can be applied in the controls. Usually, the sensors of links and actuators can provide these state variables. However, additional cost on the sensors may be high. This has motivated the design of state observer to reduce the use of sensors and such works can be found in [49–51, 159]. However, robustness of these observers to mismatched time-varying uncertainty is not clearly known. Thus, it is worth finding a new way of incorporating state observers into controls proposed in this thesis.

In the end, at the implementing technique level, electrical dynamics of servo actuators are required to construct an integrated dynamic equation. Further work on modifying the control design procedure is needed by combining electrical and mechanical dynamics.

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附录：总结和展望（中文）

本文的研究是基于不确定柔性关节机器人控制的实际需要而展开的，所取得的成果集中在含有不确定性的柔性关节机器人的鲁棒控制、自适应控制、最优控制和受约束控制等方面。所提出的控制方案对于旋转关节或者平移关节的机器人系统是通用且有效的。

1. 论文总结

由于对柔性关节机器人的应用需求旺盛，而它的性能又总是很难提升，因此对柔性关节机器人控制方法的研究在过去三十年中一直是一个研究热点。本文系统地给出了含有不确定性的柔性关节机器人的建模、不确定性描述、控制设计、参数优化方法。主要工作总结如下：

其他许多基于李雅普诺夫稳定性理论的鲁棒控制中，需要假设惯性矩阵具有一致有界性才能证明系统的稳定性。这使得这一类控制只能应用到旋转关节的柔性机器人中。在第二章，我们提出了一个更加通用的上界条件，使得所设计的控制能同时应用在旋转关节和平移关节的情况中。所设计的控制仅仅需要不确定性的一个可能的边界而不需要其他任何关于不确定性的信息。由于柔性关节机器人是一个欠驱动系统，系统中的不确定性便不满足匹配性条件。为了解决这个问题，在控制设计过程中采用了类反步法的技巧。通过创造性地嵌入一个虚拟控制，将原先的系统分解成两个子系统，然后递推得到真实的控制。

在实际应用中，不确定性的信息有时很难得到或者完全不能获得。这就给鲁棒控制中不确定性边界的选取带来了困难。如果不确定性的边界函数选取的不恰当会导致额外的控制消耗。为了解决这个问题，在第三章中提出了自适应鲁棒控制。这种方法的基本假设是存在一个可以描述不确定性边界的函数，同时这个函数中有些参数是未知的。换句话说，我们仅假设不确定性边界的存在性。因而，可以设计一个自适应律来得到这个不确定性边界的估计值。控制设计的流程和第二章类似。所设计的自适应律是渐亏类型，即当系统性能满足要求或者边界函数的梯度值很小的时候，自适应参数就不随着时间增长。在控制设计过程中，由于使用了类反步法，系统用新的状态变量重新表达了。这在给设计带来便利性的同时也引入了以前被忽视新问题：系统不再是原先的系统了。直接依据转化后系统的稳定性就推断原系统的稳定性是不恰当的。从这个问题出发，我们从理论上证明了原系统的稳定性。

用模糊集来描述不确定性比用概率法有更多的优势。不论是工程上还是理论上，将模糊理论和系统理论结合都具有重要价值。在第四章中，针对不确定柔性关节机器人的控制我们提出了一个新的模糊动态系统方法。模糊集理论被用来描述系统的状态和不确定性。然后提出了一个确定性的自适应鲁棒控制器来保证系统的一致有界性和一致最终有界性。由于设计者也需要一组合适的控制器参数使得在系统性能提高的同时降低所需要的控制消耗，提出了一个基于模糊理论的优化方法。通过D运算，控制的最优化问题转化为一个性能指标的最小化问题。这个性能指标包含了系统的确定性性能和模糊性能。我们证明了这个问题可以通过解一个四次代数方程组来解决。

对于受约束的不确定柔性关节机器人系统控制，首先在第五章介绍了一种基于Udwadia-Kalaba方法的新的约束力建模方法。Moore-Penrose广义逆矩阵被用来分析约束力的几何结构。通过理论分析可以发现，约束力可以通过一个固定的方法确定并且与系统所受的外力有清晰的关系（Udwadia-Kalaba方程）。系统所受的约束条件可以是完整的也可以是非完整的，并以二阶微分的形式给出。同时可以证明，包括镇定问题、轨迹跟踪问题、优化问题在内的许多控制问题都可以归结于这种二阶微分形式的约束问题。

在第六章，我们考虑了不确定柔性关节机器人的约束力伺服控制问题。在第五章的基础上，通过应用Udwadia-Kalaba方程，可以得到约束力的解析解形式并可以应用在控制中。提出的控制分为三个部分：理想控制部分，用来控制系统的理想值部分；类Udwadia-Kalaba控制部分，用来产生满足约束条件的约束力；自适应鲁棒控制部分，用来抵消不确定性的影响。提出的控制器能够保证系统跟随给定的约束条件。具体来说，系统的受约束部分满足给定的约束条件，即使初始状态并不满足约束，在应用提出的控制之后，约束逐渐被满足；同时系统的其他部分具有一致有界性和一致最终有界性。

2. 本文主要创新点

本文的主要创新点有四个：

第一，对于不确定性边界已知的柔性关节机器人，提出了一个更具有一般性的基于李雅普诺夫方法的鲁棒控制器。在过去的研究中，总是需要假设惯性矩阵具有一致性上界，这使的控制只能应用在柔性旋转关节机器人上。本文提出了一个更具有一般性的上界条件，使平移关节的柔性机器人也能包含到被控对象中。然后设计了一个鲁棒控制器来处理系统中的不确定性。控制设计和稳定性分析仅仅是基于不确定性的一个可能的边界而并不需要其他信息。并且，一致最终有界性区域的大小可以通过选择合适的控制参数来达到无限小。

第二，对于不确定性边界未知的柔性关节机器人，提出了一个自适应鲁棒控制器来保证系统的稳定性。具体来说，尽管不确定性的边界是未知的，但是它是确定存在的。我们假设它是由一个受未知参数影响的确定函数决定。然后构造一个自适应规律来估计这个

未知的参数。这个机械/自适应系统的稳定性可以通过李雅普诺夫第二方法来证明。同时，通过采用一个类似反步法的控制设计方法，含有不匹配不确定性系统的控制设计难点被克服。此外，不仅转化后的机械/自适应系统稳定性得到了证明，我们还从理论上证明了原系统的稳定性。这一点一直被其他学者所忽视。

第三，提出了一个新的基于模糊动态系统法的控制设计和优化方法。系统的状态变量和不确定性由模糊集理论来描述，系统也被转化为模糊动态系统。尽管系统是模糊描述的，但是提出的控制是确定的，并且不同于其他基于IF-THEN规则的模糊控制。理论分析表明，所提出的设计方法不仅保证了系统的确定性性能（一致有界性和一致最终有界性），在系统的模糊描述基础上，同时解决了系统的控制消耗优化问题。其中优化问题可以通过最小化一个综合的性能指标来实现。证明了由一阶必要条件得到的极值解就是这个最优化问题的最优解，并用二阶充分条件验证了这一结论。

第四，在Udwadia和Kalaba工作的基础上，解决了受约束不确定柔性关节机器人的约束力伺服控制问题。其中的不确定性可以是（快速）时变的。控制中不需要使用拉格朗日乘子，也不需要力反馈控制。对系统初始条件也没有限制，初始点的位置可以远离给定的约束。通过分析约束力的结构，获得了约束力的解析解并在控制中加以应用。这种分析方法对被动约束和主动约束同时有效。提出的包含类Udwadia-Kalaba控制部分的控制器能够使系统满足给定约束条件，同时保证系统其余状态的一致有界性和一致最终有界性。

3. 研究展望

不确定柔性关节机器人的控制任务是设计一个满足期望性能的控制器。这样的性能总是可以保持而不论不确定性的真实值是什么。现实中的不确定性一般是通过观测数据得到的，但观测数据又总是受限于现象的不可重复性。因而本文关于不确定性描述的新方法也可为其他诸如滑膜控制、神经网络控制等提供借鉴。本文关于模糊描述的柔性关节机器人的自适应鲁棒控制也可能应用于更多的系统条件下，例如不确定性的非线性交互、初始条件未知等。

本文运用类似反步法的设计方法提出了多种控制策略。在使用这一方法的时候，选择不确定性的边界函数需要一定的经验，鲁棒控制的稳定性分析中需要虚拟控制的微分。如果能找到另外一种不需要虚拟控制二阶微分的控制方法的话，分析就能得到简化，同时不确定性的边界函数的选取也会更加方便。

本文是基于状态全反馈来设计控制的，也就是说连杆和关节电机的速度及位置都需要测量。因此设计者需要考虑如何得到这些数据从而在控制中加以应用。通常安装在连杆和电机上的传感器可以提供这些数据，但是由传感器产生的额外花费可能很大。这促使了状态观测器的开发来减少传感器的使用。这一类的研究可以在文献[49–51, 159]中找到。然而，对于这些观测器在含有不匹配、时变不确定性时的鲁棒性的研究目前还不充分。因

此，有必要研究在本文提出的控制中加入状态观测器后的性能。

最后，在算法实现的层面上，也应该一并考虑伺服电机的性能从而建立一个集成的动力学方程。未来在改进控制方法的过程中，需要合并电机的动力学模型和机械系统的动力学模型。

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Academic activities and achievements

1. Publications

1. Fangfang Dong, Jiang Han, and Ye-Hwa Chen. Adaptive robust control for flexible joint manipulators with mismatched uncertainty: Dead-zone adaptation. *Control and Intelligent Systems*, 2016, 44(1), pp. 27–35. (ESCI,EI)

2. Fangfang Dong, Jiang Han, and Ye-Hwa Chen. Improved robust control for multi-link flexible manipulator with mismatched uncertainties. In *Fluid Power and Mechatronics (FPM), 2015 International Conference on*, pp. 1102-1107, IEEE 2015.(EI)

3. Fangfang Dong, Jiang Han, and Lian Xia. Adaptive robust control and fuzzy-based optimization for flexible serial robot. *Mechatronics and Robotics Engineering for Advanced and Intelligent Manufacturing* in *Lecture Notes in Mechanical Engineering*. Springer International Publishing, 2017: 151-165.(Scopus)

4. Jiang Han, Ye-Hwa Chen, Lian Xia, Fangfang Dong. Constraint-Following Servo Control for Uncertain Elastic Joint Robot via Udwadia-Kalaba Approach. In *The 20th International Conference on Mechatronics Technology*, 2016.

5. 韩江，董方方，夏链，段少磊.开放式软PLC集成开发系统V1.0，中国软件著作权：2013SR111707

6. Jiang Han, Fangfang Dong, Ye-Hwa Chen. Optimal design for robust control of uncertain flexible joint manipulators: A fuzzy dynamical system approach, *International Journal of Robust and Nonlinear Control*. (Under review)

7. Fangfang Dong, Jiang Han, Ye-Hwa Chen and Lian Xia. A new fuzzy approach to optimal adaptive control design for uncertain flexible joint manipulator, *Control Theory and Technology (English Edition)*. (Under review)

8. Fangfang Dong, Jiang Han, and Ye-Hwa Chen. A novel robust constraint force servo control for under-actuated manipulator system: Fuzzy and optimal, *IET Control Theory & Applications*. (Under review)

2. Academic activities

1. 标准型数控系统的产业化及专用型齿轮机床数控系统的研究开发（编号：2012ZX04001-21），国家科技重大专项，2012-2015；
2. 安徽省机械产品数控化创新研发及应用示范（编号：2015BAF26B01），国家科技支撑计划，2015-2017；
3. 高性能智能化数控系统开发（编号：2013AKKG0394），安徽省自主创新校内项目（秋实计划），2013-2015；
4. 国际交流：美国Georgia Institute of Technology，联合培养，2014-2015.