HW2 for GPGN605: Minimum-norm model construction with accurate data

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1 PROBLEM

Let us consider a simple problem with the following data equation,

$$d_i = \int_0^1 e^{-(i-1)x} m(x) dx, \; i = 1, \cdots, N, \tag{1}$$

where the kernel functions are exponentials with different decay constants. Assume a true model given by

$$m(x) = 1 - \frac{1}{2} cos(2\pi x). \tag{2}$$

1.1 Expressions and plots for data

Substituting equation 2 into equation 1, we can get the expression of the data as

$$d_i = \int_0^1 [1 - \frac{1}{2} cos(2\pi x)] e^{-(i-1)x} dx = \int_0^1 e^{-(i-1)x} dx - \frac{1}{2} \int_0^1 [cos(2\pi x)] e^{-(i-1)x} dx. \tag{3}$$

For this integral, I first compute its second part which is

$$\frac{1}{2} \int_0^1 [cos(2\pi x)] e^{-(i-1)x} dx = \frac{\pi e^{(1-i)x} sin(2\pi x)|_0^1 + \frac{1-i}{2} e^{(1-i)x} cos(2\pi x)|_0^1}{(1-i)^2 + 4\pi^2} = \frac{(1-i)(e^{(1-i)} - 1)}{2(1-i)^2 + 8\pi^2} \tag{4}$$

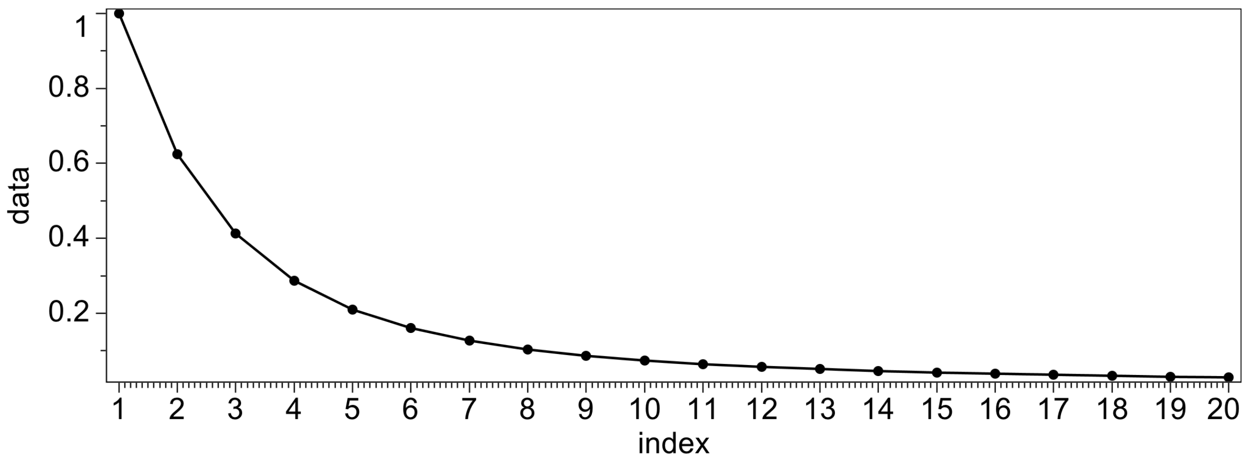

Figure 1. Computed data using the expression shown in equation 5 when $N = 20$.

Then the data d_i expressed in equation 3 can be computed as

$$d_i = \frac{e^{(1-i)} - 1}{1 - i} - \frac{(1-i)(e^{(1-i)} - 1)}{2(1-i)^2 + 8\pi^2}, \quad i = 2, 3, \cdots, N, \tag{5}$$

and $d_i = 1$ when $i = 1$. Figure 1 shows the data with the index when $N = 20$.

1.2 True and constructed models when $N = 6$

The inner product of kernel functions can be expressed as:

$$\Gamma_{ij} = \int_0^1 e^{(2-i-j)x} dx = \frac{e^{(2-i-j)} - 1}{2 - i - j}, \; i, j = 1, 2, \cdots, N, \tag{6}$$

where i and j cannot all be 1, when $i = j = 1$, we have $\Gamma_{11} = 1.0$.

Figure 2a shows the constructed (red) and true (blue) models when $N = 6$, the constructed model is very close to the true model according to Figure 2b, which shows that the difference $(\mathbf{m}_c - \mathbf{m})$ between the two is very small. From Figure 2b, we can also observe that the big differences appear at the beginning and ends of the constructed model.

1.3 Eigenvalues and eigenvectors when $N = 6$

Figure 3 shows the six eigenvalues as a function of the index, and Figure 4 shows the corresponding eigenvectors as a function of the vector index. From Figure 4, we can observe that

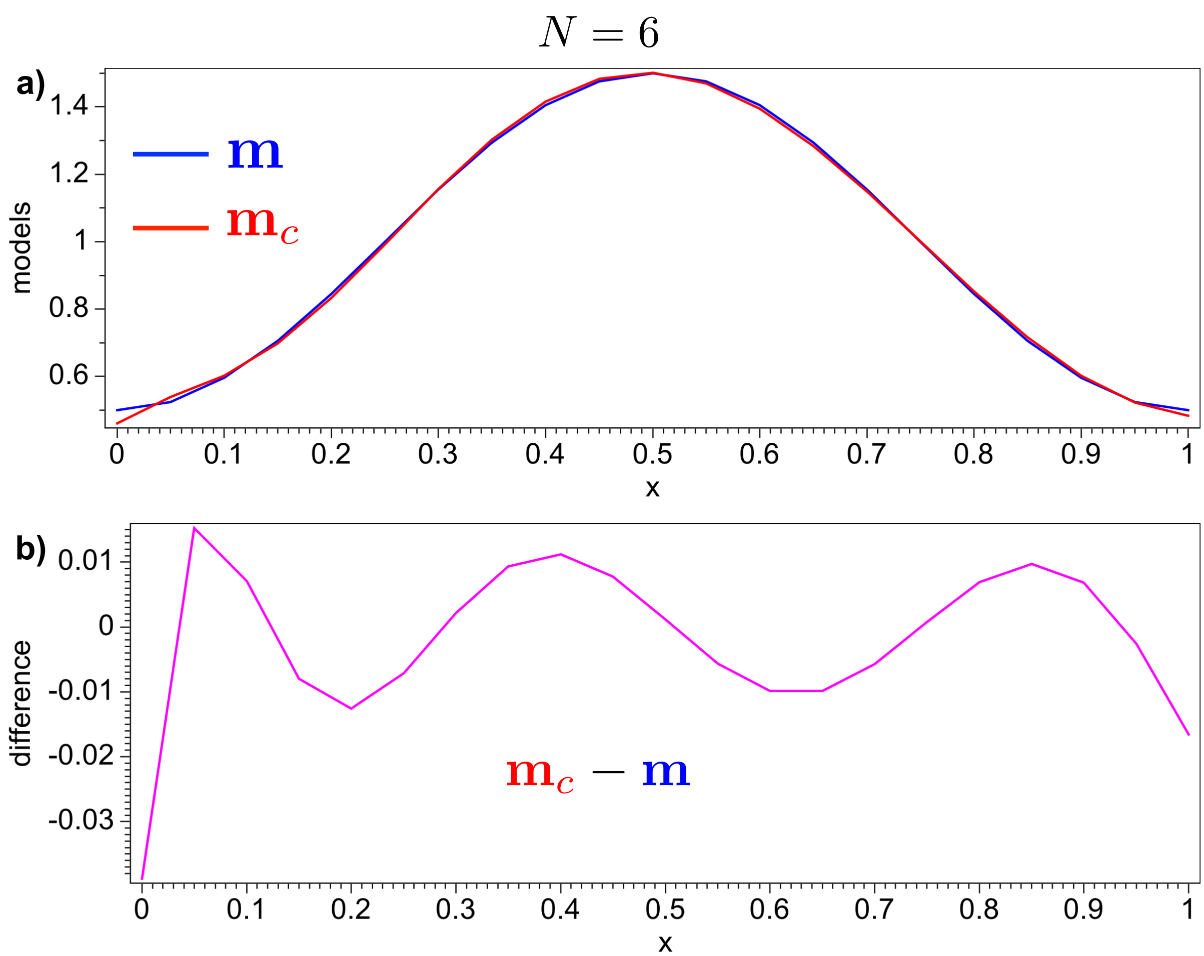

Figure 2. True (blue), constructed (red) models (a) when $N = 6$, and their differences (b).

the complicity of the eigenvectors increases as the corresponding eigenvalues decrease. Eigenvectors corresponding to smaller eigenvalues are more complicated than those corresponding to bigger eigenvalues.

1.4 True and constructed models when $N = 10,\ 20$

Figures 5a and 6a shows the true and constructed models when $N = 10$ and $N = 20$, respectively. Comparing to the constructed model as shown in the Figure 2a, bigger N produces more accurate constructed model. Because the differences (Figures 2b, 5b and 6b) between the constructed model and true model decreases with N increases.

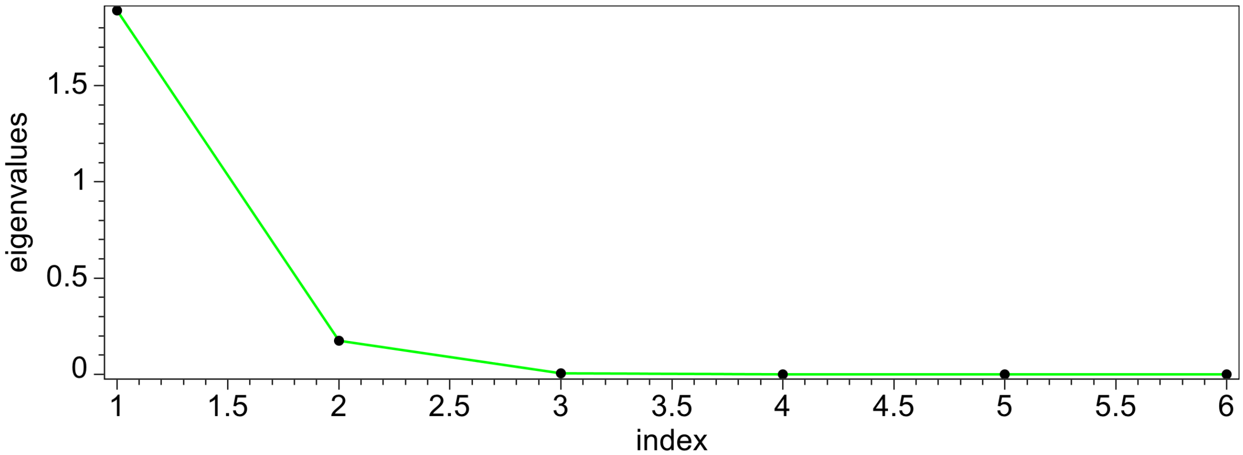

Figure 3. Six eigenvalues from bigger to smaller.

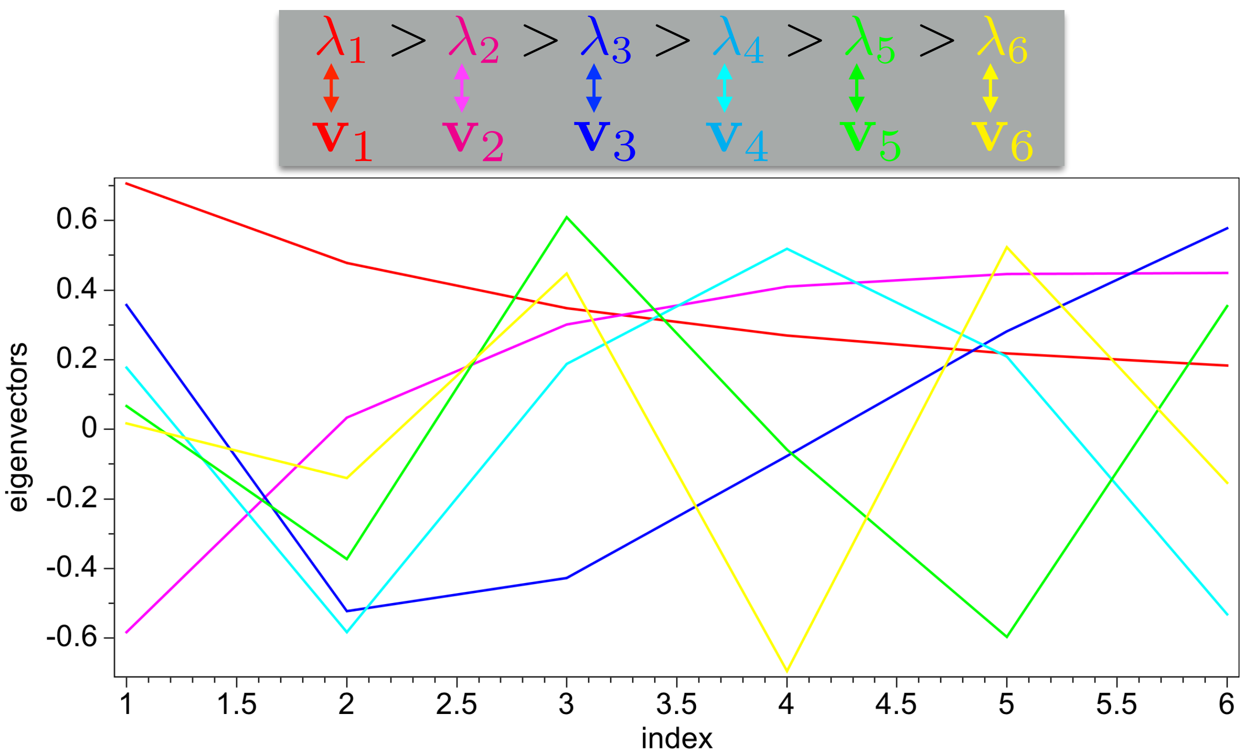

Figure 4. Plots of the six eigenvectors as functions of indexes.

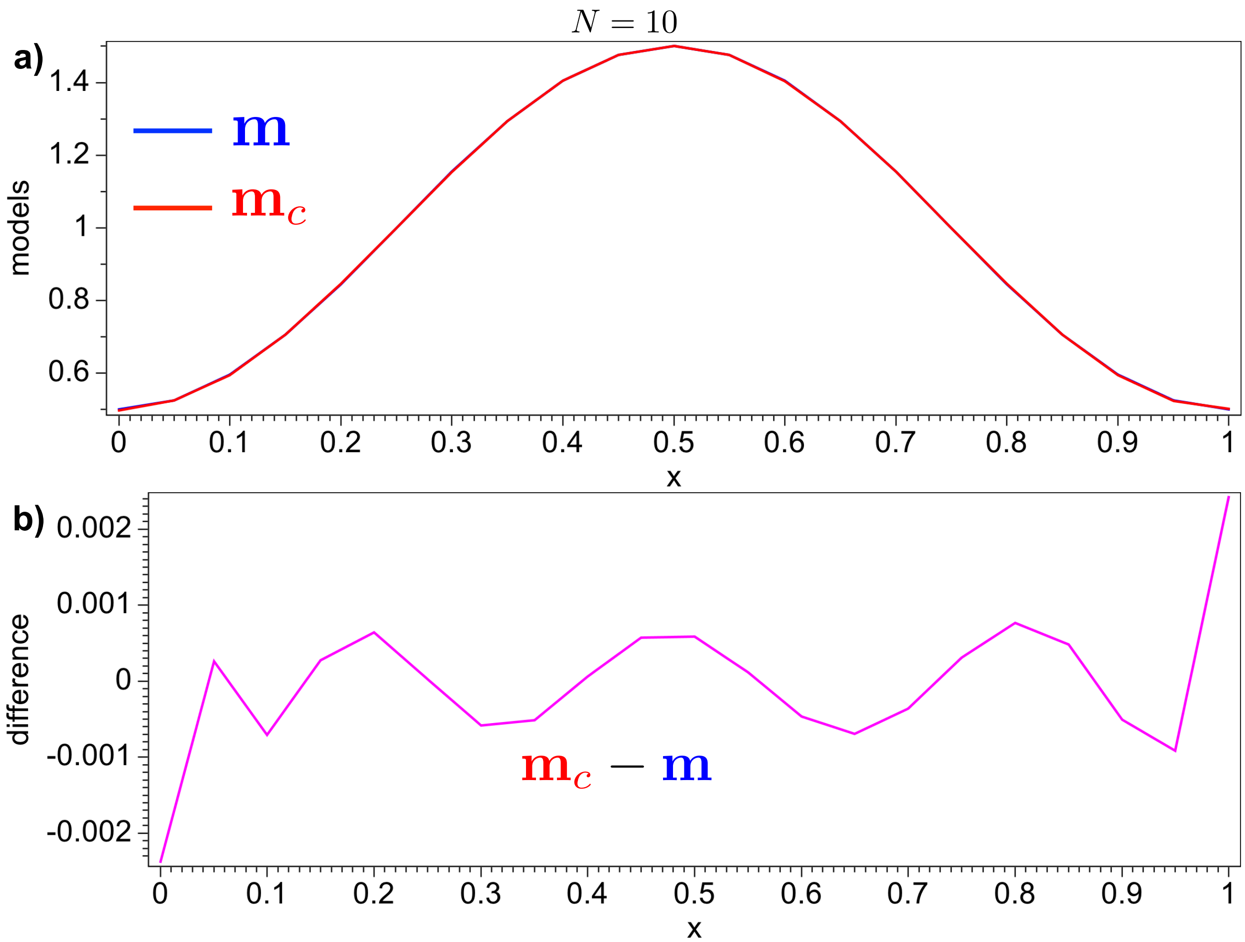

Figure 5. True (blue), constructed (red) models (a) when $N = 10$, and their differences (b).

From the definition of the condition number κ of a matrix A, we have

$$\kappa(A) = \left| \frac{\lambda_{max}(A)}{\lambda_{min}(A)} \right|. \tag{7}$$

Then:

1) $N = 6$: $\kappa = 5511895816.74$

2) $N = 10$: $\kappa = 1.32799783572e + 17$

3) $N = 20$: $\kappa = 1.82775898882e + 16$

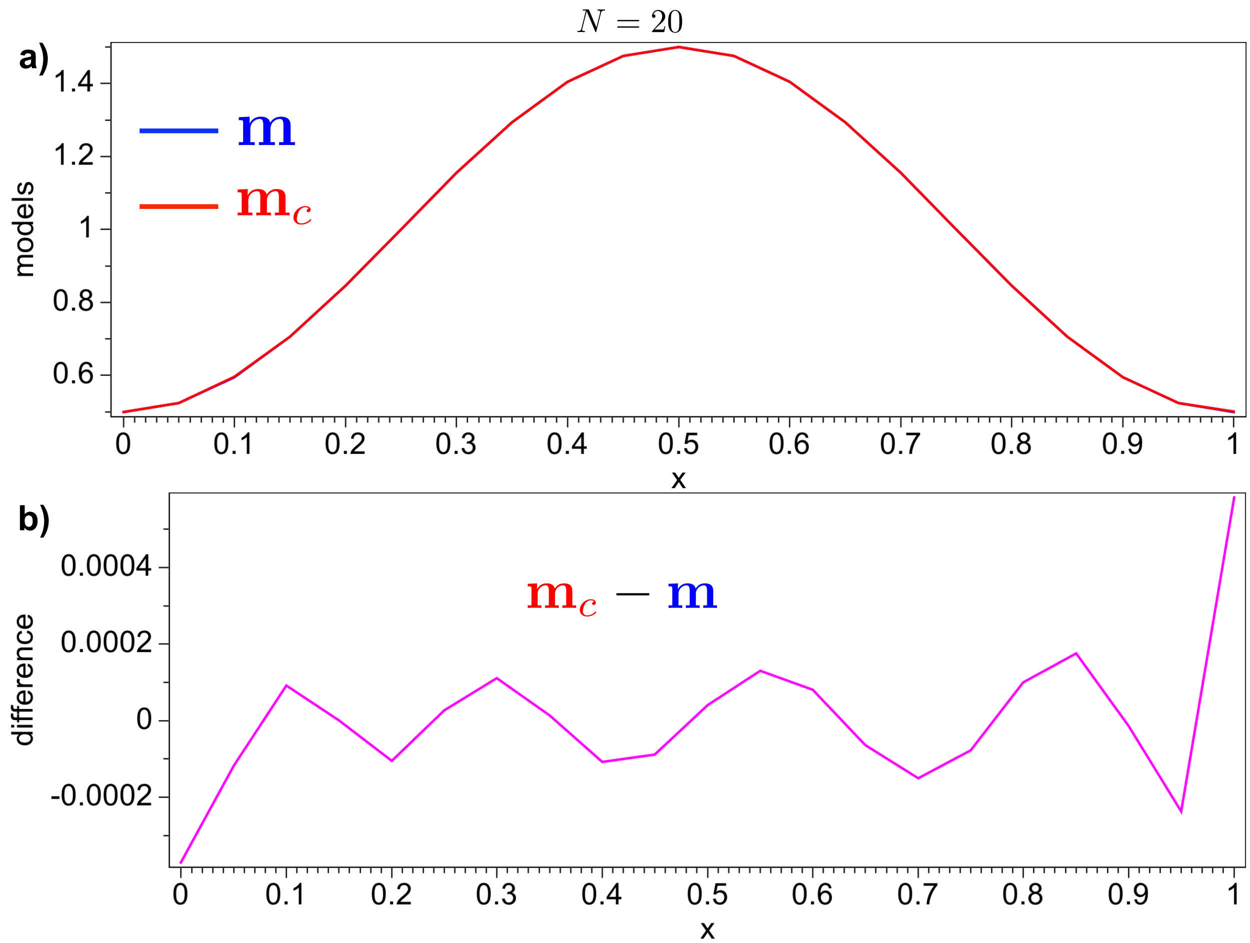

Figure 6. True (blue), constructed (red) models (a) when $N = 20$, and their differences (b).

1.5 Double and float precisions

Using double precisions, I can get pretty good results as shown in Figure 6, but I cannot get good results when using float precisions. Because some eigenvalues are very close to zeros, the running errors will produce big effects on the results.