HW4 for GPGN605: Truncated SVD

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1 NORMALIZED SENSITIVITY MATRIX A AND DATA B

Normalized sensitivity matrix $\mathbf{A}$ and data $\mathbf{b}$ are computed from matrix $\mathbf{G}$ and noisy data $\mathbf{d}^{obs}$ by

$$\begin{aligned} \mathbf{A} &= \mathbf{W}_d \mathbf{G}, \\ \mathbf{b} &= \mathbf{W}_d \mathbf{d}^{obs}, \end{aligned} \tag{1}$$

where $\mathbf{W}_d$ is a diagonal data weighting matrix whose delements are the inverse of the data standard deviations, which is $\sigma = 0.05$ in this example.

The normalized sensitivity matrix $\mathbf{A}$ Figure 1.

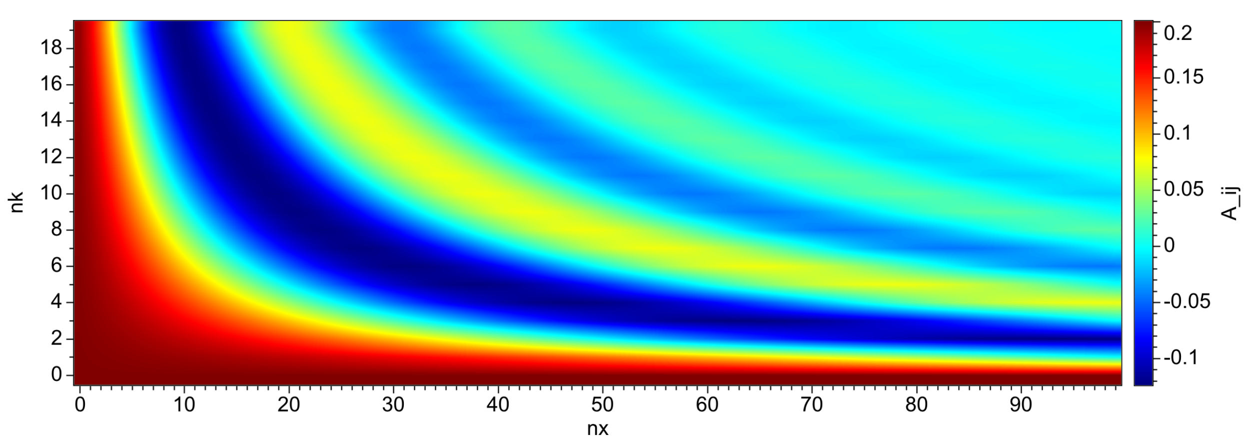

Figure 1. The $N \times M$ normalized sensitivity matrix $\mathbf{A}$.

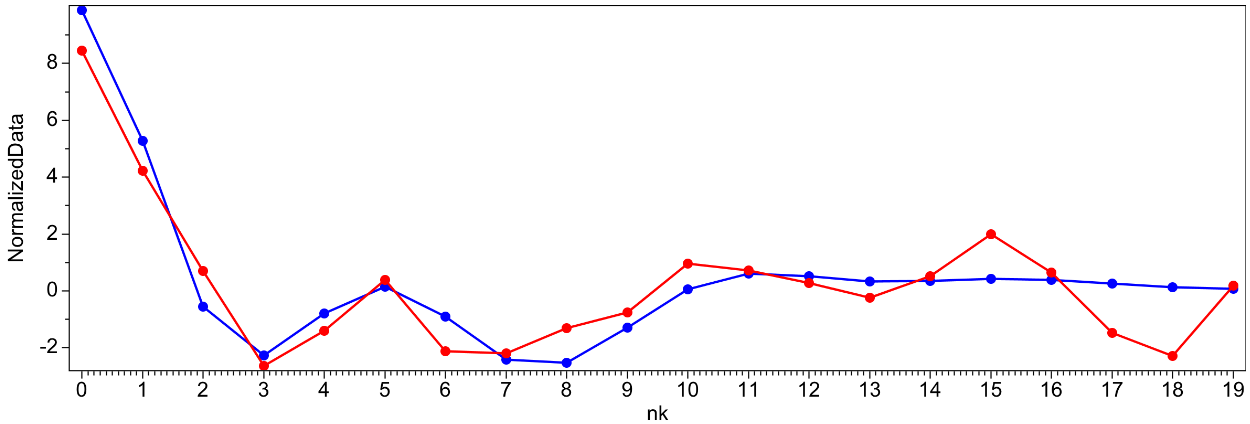

Figure 2. The normalized true data $\mathbf{d}^t$ (blue curve) and observe data $\mathbf{b}$ (red curve).

Knowing the model $\mathbf{m}$ that is a $M-$dimensional vector, I compute the accurate data $\mathbf{d^t}$ as

$$\mathbf{d}^t = \mathbf{G}\mathbf{m}, \tag{2}$$

and I add a Gaussian noise with a zero mean and standard deviation of 0.05 to $\mathbf{d}^t$ to simulate noisy observation data $\mathbf{d}^{obs}$. These true and noisy data are further normalized by the diagonal matrix $\mathbf{W}_d$. These normalized data are shown in Figure 2.

2 ROTATED DATA, SINGULAR VALUES AND RATIO

As shown in Figure 3, rotated data $\mathbf{u}_i^\top \mathbf{b}$, singular values λ_i and the ratio $\mathbf{u}_i^\top \mathbf{b}/\lambda_i$ are the red, green and blue curves, respectively. The rotated data (red) is a flat curve with slight fluctuations, while the singular values (green) decreases with index increasing. Therefore, the ratio of the rotated data and singular value increase with index increasing as shown in Figure 3, which means that the ratio does not converge with index.

Comparing to those for the true data as shown in Figure 4, we can observe that both the rotated data (red) and singular values (green) have the same decreasing trend as the index increases. Therefore, the ratio of the rotated data and singular value is a flat curve (blue) as shown in Figure 4.

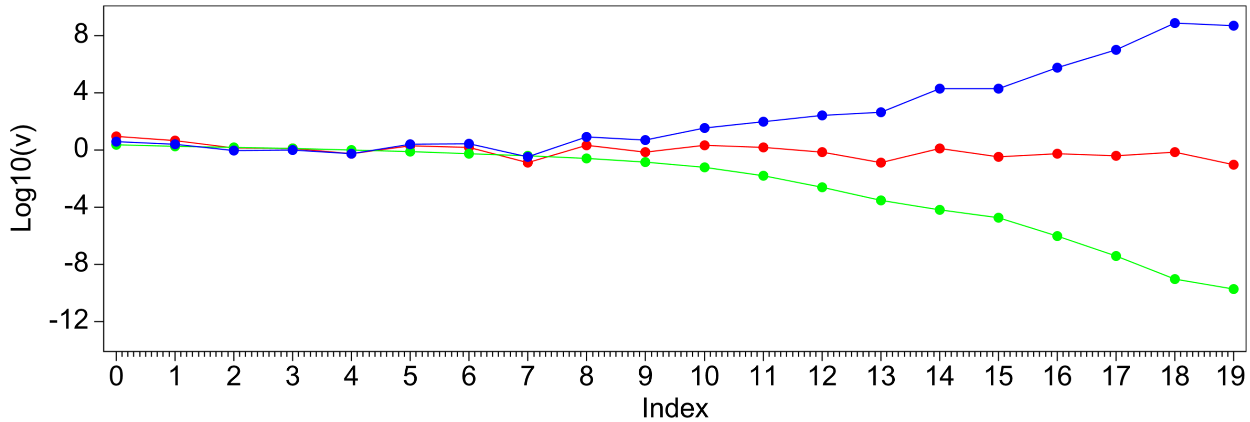

Figure 3. Rotated data $\mathbf{u}_i^\top \mathbf{b}$ (red), singular values λ_i (green) and the ratio $\mathbf{u}_i^\top \mathbf{b}/\lambda_i$ (blue).

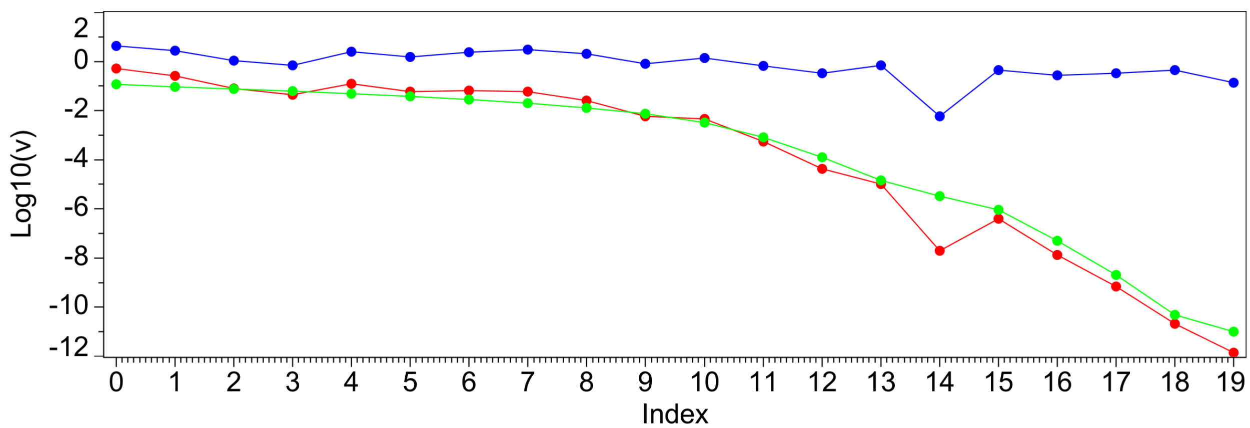

Figure 4. Rotated data $\mathbf{u}_i^\top \mathbf{d}^t$ (red), singular values λ_i (green) and the ratio $\mathbf{u}_i^\top \mathbf{d}^t/\lambda_i$ (blue).

3 DATA MISFIT, MODEL OBJECTIVE FUNCTION AND TIKHONOV CURVES

Data misfit ϕ_d:

$$\phi_d = \sum_{i=q+1}^{N} (\mathbf{u}_i^\top \mathbf{b})^2, \tag{3}$$

and model objective function ϕ_m:

$$\phi_m = \sum_{i=1}^{q} (\frac{\mathbf{u}_i^\top \mathbf{b}}{\lambda_i})^2, \tag{4}$$

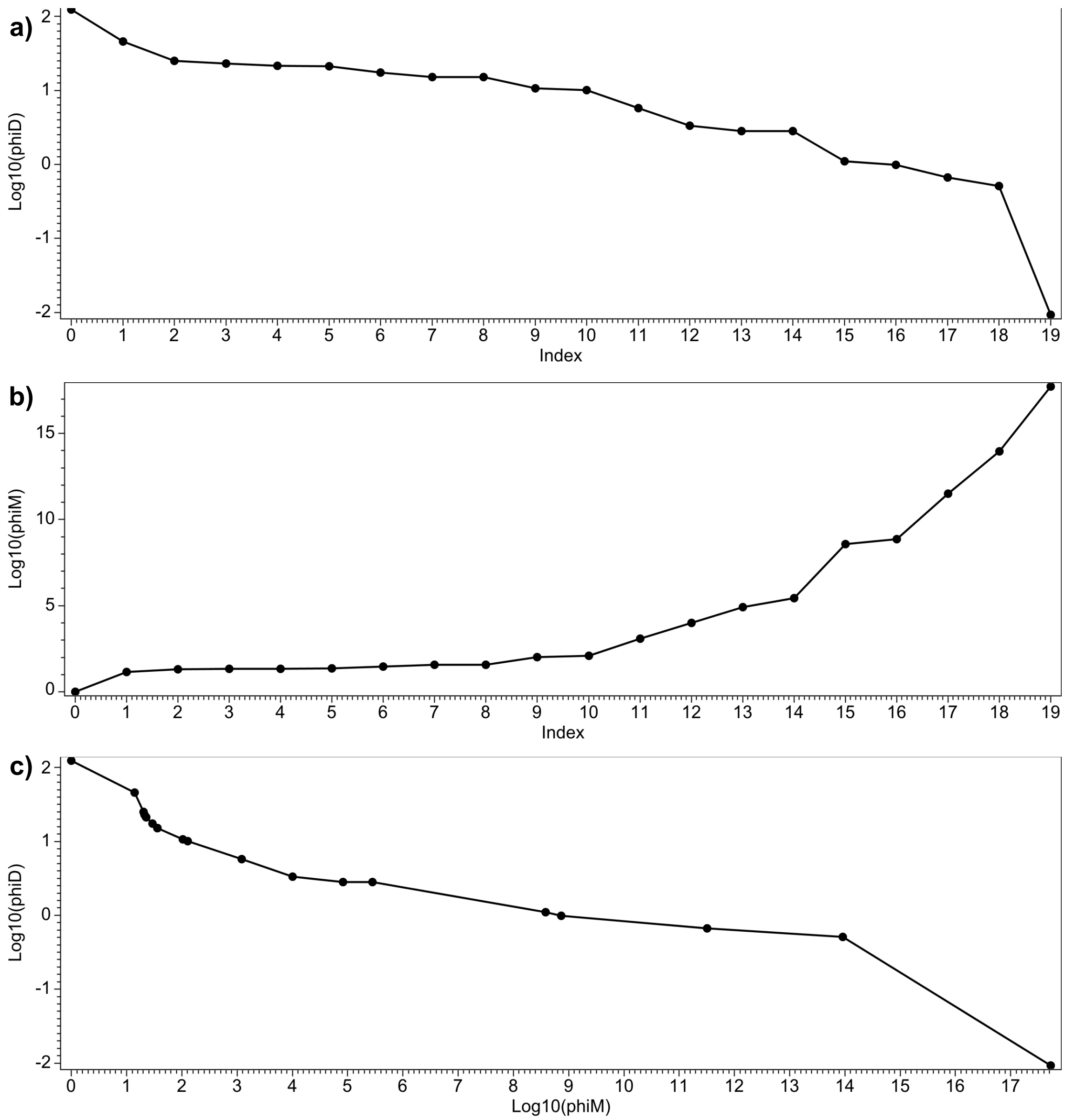

Figure 5. Data misfit (a), model objective function (b) and Tikhonov (c) curves.

where q is the number of terms kept in the truncated SVD solution.

Figure 5 shows the data misfit (Figure 5a), model objective function (Figure 5b) and Tikhonov (Figure 5c) curves.

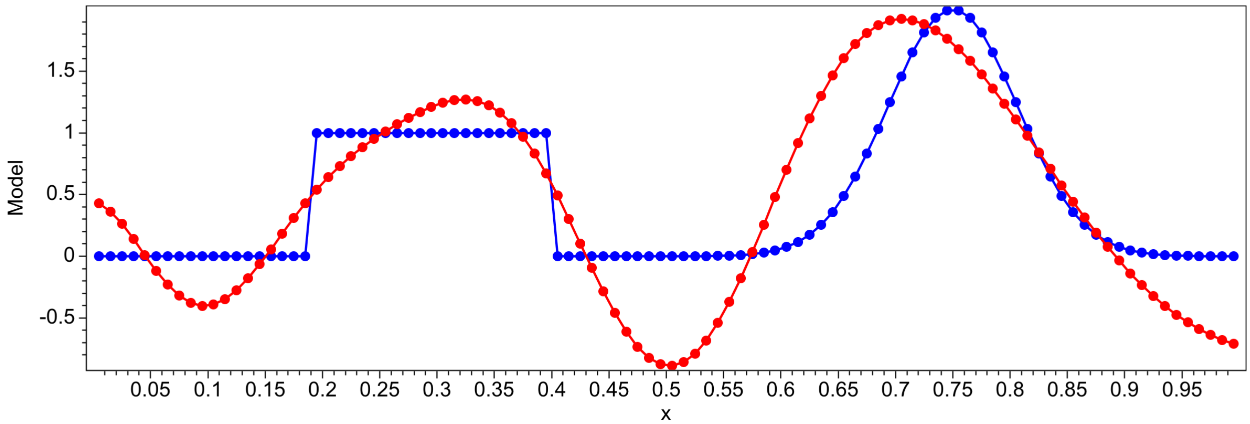

Figure 6. True (blue) and constructed (red) model computed with the optimal truncated SVD.

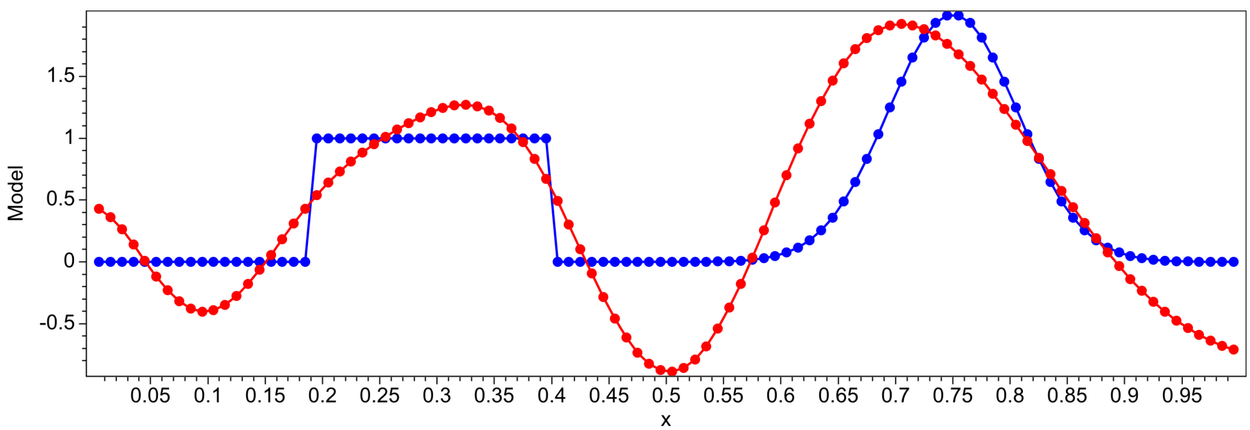

Figure 7. True (blue) and constructed (red) model with the first seven terms in the spectral expansion form.

4 CONSTRUCTED MODEL

Using the Chi-squared definition, I found the number of spectral expansion terms $q^* = 7$. Figure 6 shows the constructed model (red curve) from the optimal truncated SVD and the true model (blue curve).

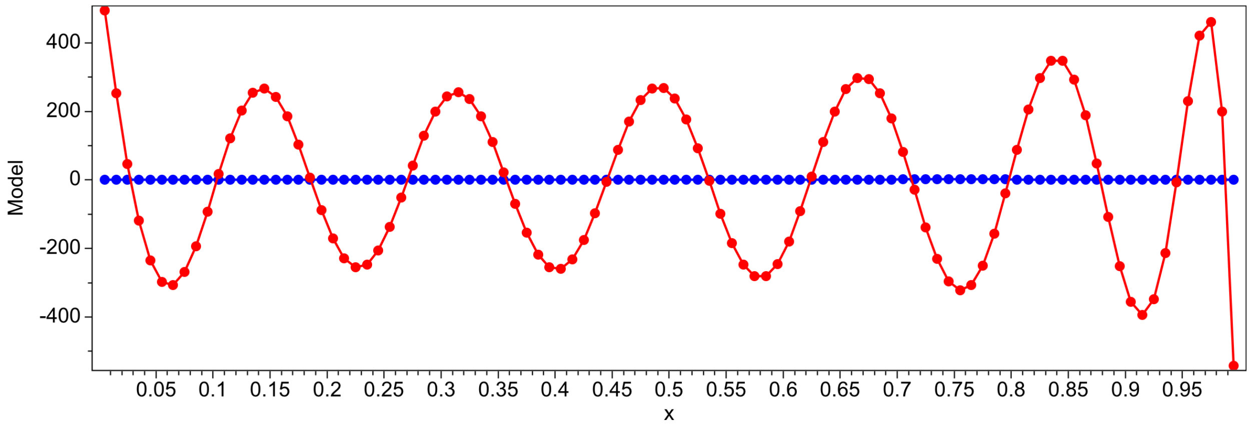

Figure 8. True (blue) and constructed (red) model with the second seven terms in the spectral expansion form.

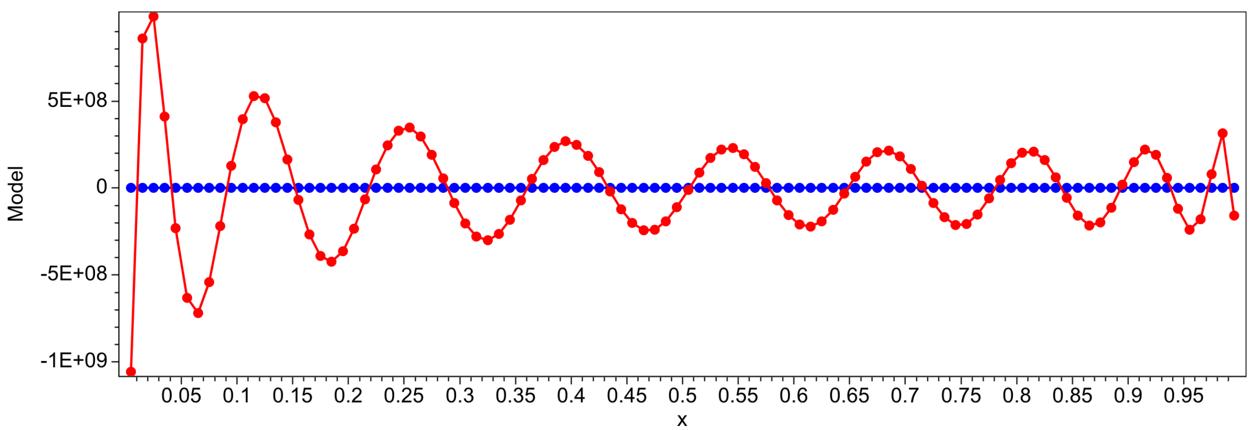

Figure 9. True (blue) and constructed (red) model with the last six terms in the spectral expansion form.

5 CONSTRUCTED MODELS WITH DIFFERENT SPECTRAL EXPANSION TERMS

Using the first seven terms of the spectral expansion form, the constructed model is the red curve shown in Figure 7.

Using the second seven terms of the spectral expansion form, the constructed model is the red curve shown in Figure 8.

Using the last six terms of the spectral expansion form, the constructed model is the red curve shown in Figure 9.

From the Figure 7, 8 and 9, we can observe that the lower index terms construct the effective and lower frequency components of the model. The higher index terms appears to construct the higher frequency (more complex) components of the model, but they are not effective for the model constructions, because the magnitudes of constructed models with higher-index terms are huge (much larger than the magnitude of the true model).