HW3 for GPGN605: SVD solution of a discretized linear inverse problem

Xinming Wu

CWID: 10622240

x Center for Wave Phenomena, Colorado School of Mines, Golden, CO 80401, USA

1 SENSITIVITY MATRIX G

G is an $N \times M$ matrix, where $N = 20$ is the number of kernel functions, and $M = 100$ is the number of samples in a discretized model. Each element of **G** can be computed as

$$\begin{aligned} G_{i,j} &= \int_{x_{j-1}}^{x_j} g_i(x)dx \approx g_i(x_{j-0.5})dx \\ &= cos(\frac{i-1}{2}\pi x_{j-0.5})e^{-(i-1)x_{j-0.5}/4}, \end{aligned} \tag{1}$$

where $i = 1, \cdots, N$ and $j = 1, \cdots, M$. The matrix **G** is shown in Figure 1.

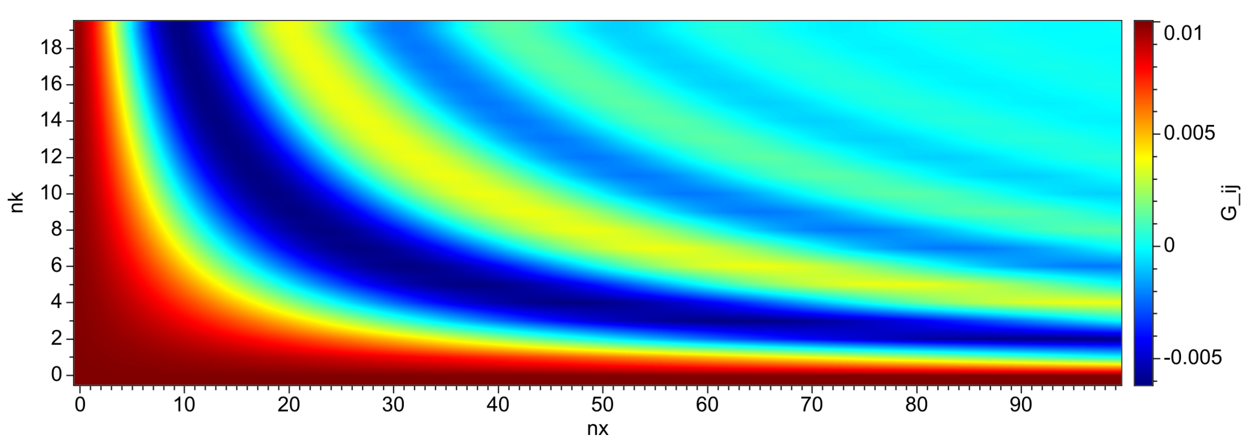

Figure 1. The $N \times M$ matrix **G**.

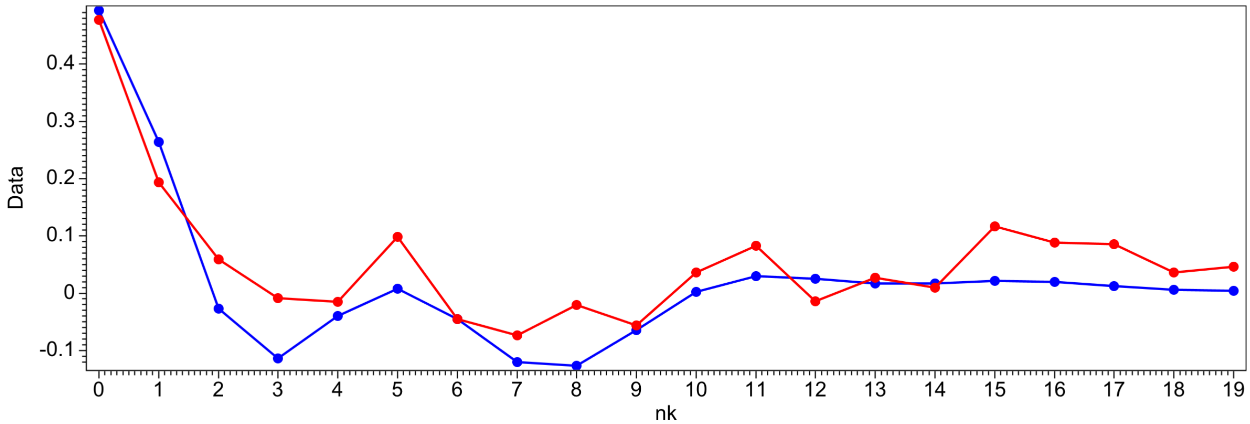

Figure 2. The true data $\mathbf{d}^t$ (blue curve) and observe data $\mathbf{d}^o$ (red curve) simulated by adding Gaussian noise to the true data.

2 TRUE AND NOISY DATA

Knowing the model $\mathbf{m}$ which is a $M-$dimensional vector, I compute the accurate data $\mathbf{d^t}$ as

$$\mathbf{d}^t = \mathbf{Gm}, \tag{2}$$

and I add a Gaussian noise with a zero mean and standard deviation of 0.05 to $\mathbf{d}^t$ to simulate noisy observation data $\mathbf{d}^o$. The true and noisy data are shown in Figure 2.

3 SVD OF MATRIX G

Perform SVD of the sensitivity matrix $\mathbf{G}$, we obtain:

$$\mathbf{G} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^\top, \tag{3}$$

where $\mathbf{U} = [\mathbf{u_1}|\cdots|\mathbf{u}_N]$ is an $N \times N$ matrix, $\boldsymbol{\Lambda}$ is a diagonal matrix $\boldsymbol{\Lambda} = diag\{\lambda_1, \cdots, \lambda_N\}$ and $\mathbf{V} = [\mathbf{v_1}|\cdots|\mathbf{v}_N]$ is a $M \times N$ matrix. The singular values $\{\lambda_1, \cdots, \lambda_N\}$ are shown in Figure 3a, the left singular vectors $\mathbf{u}_i$ are shown in Figure 3b and the right singular vectors $\mathbf{v}_i$ are shown in Figure 3c.

From Figure 3, we can observe that both the right and left singular vectors, corresponding

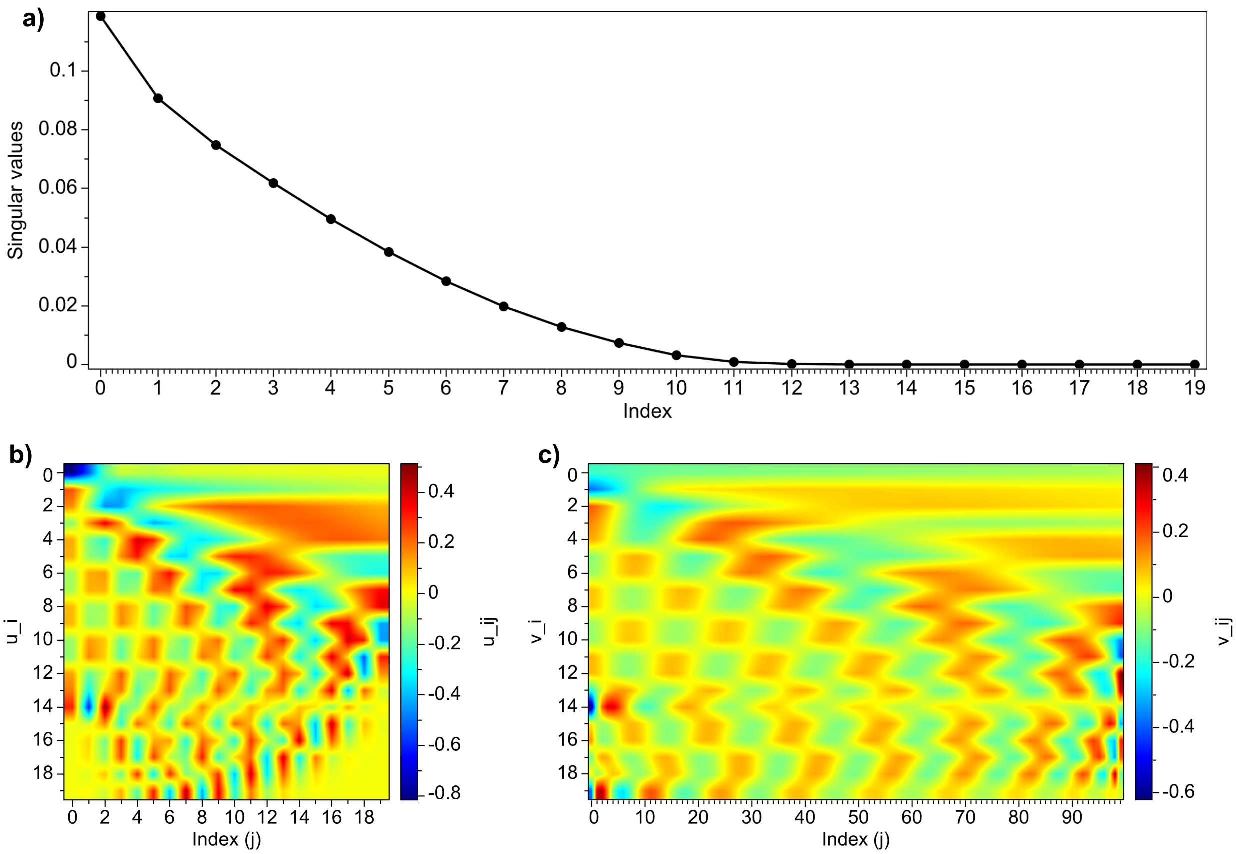

Figure 3. Singular values $\lambda_1, \cdots, \lambda_N$ (a), left singular vectors $\mathbf{u}_i$ (b) and right singular vectors $\mathbf{v}_i$ (c).

to smaller singular values, are more complex. The range of the singular values are from 0 to 0.12, and they are close to zero for indexes $i = 11, \cdots, 19$.

4 ROTATED DATA, SINGULAR VALUES AND RATIO

As shown in Figure 4, rotated data $\mathbf{u}^\top{}_i \mathbf{d}^t$, singular values λ_i and the ratio $\mathbf{u}^\top{}_i \mathbf{d}^t / \lambda_i$ are the red, green and blue curves , respectively. Both the rotated data (red) and singular values (green) have the same decreasing trend as the index increases. Therefore, the ratio of the rotated data and singular value is a flat curve (blue) as shown in Figure 4.

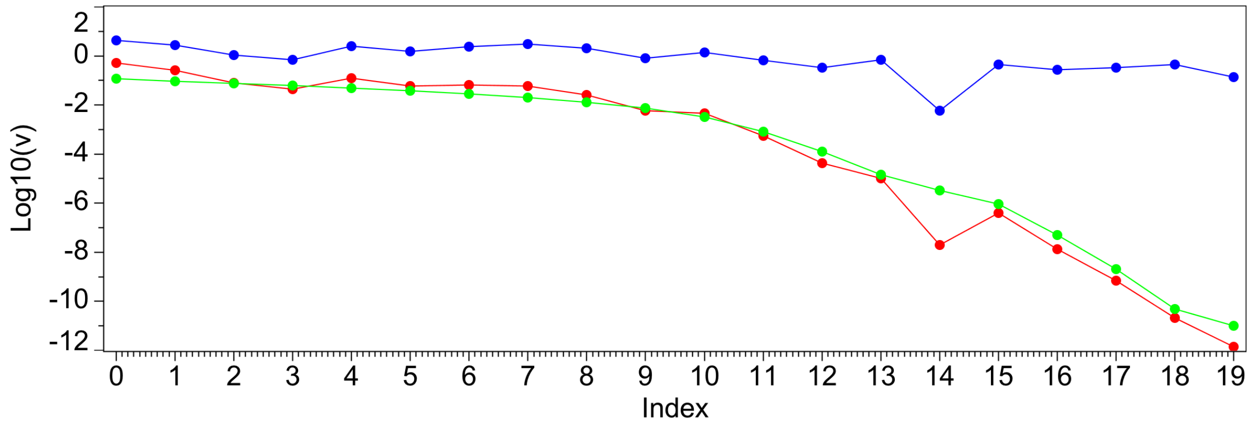

Figure 4. Rotated data $\mathbf{u}^{\top}{}_i\mathbf{d}^t$ (red), singular values λ_i (green) and the ratio $\mathbf{u}^{\top}{}_i\mathbf{d}^t/\lambda_i$ (blue).

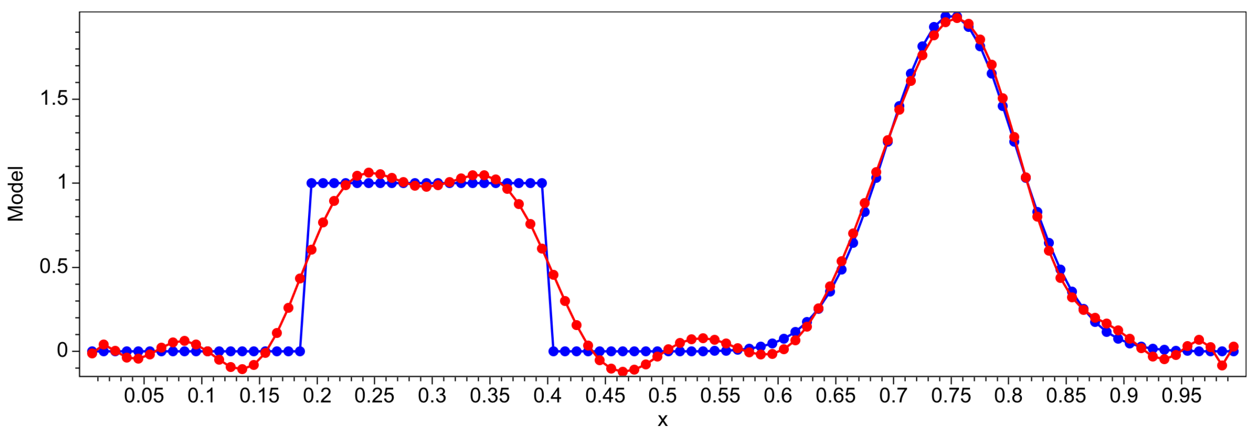

Figure 5. True (blue) and constructed (red) model with all terms in the spectral expansion form.

5 CONSTRUCTED MODEL

The minimum-norm model constructed through SVD can be expressed as

$$\mathbf{m}_c = \sum_{i=1}^{N} \frac{\mathbf{u}^{\top}{}_i\mathbf{d}^t}{\lambda_i}\mathbf{v}_i. \tag{4}$$

Figure 5 shows the true (blue) and constructed (red) models. The constructed model is computed with all terms in the spectral expansion form as shown in equation 4.

Using the first seven terms of the spectral expansion form, the constructed model is the red curve shown in Figure 6.

Using the second seven terms of the spectral expansion form, the constructed model is

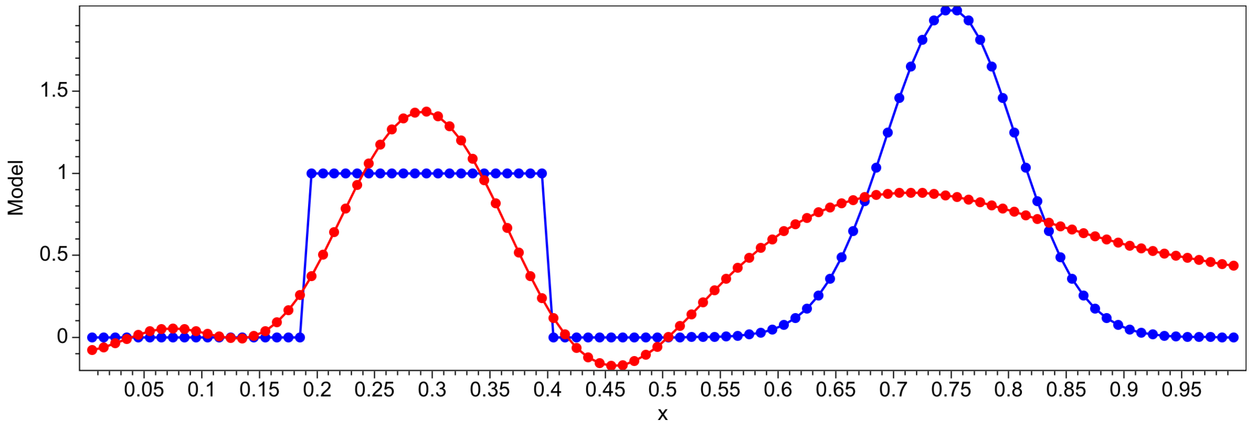

Figure 6. True (blue) and constructed (red) model with the first seven terms in the spectral expansion form.

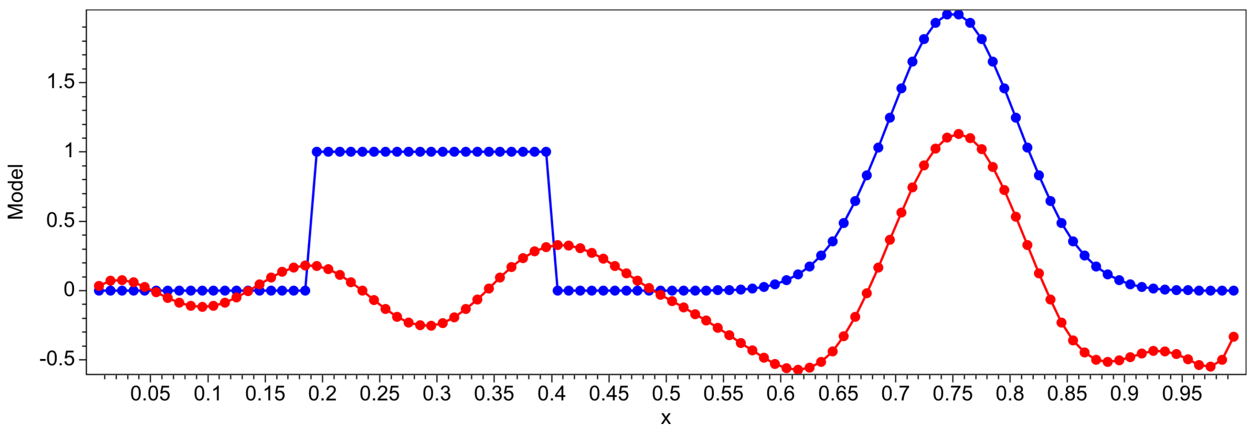

Figure 7. True (blue) and constructed (red) model with the second seven terms in the spectral expansion form.

the red curve shown in Figure 7.

Using the last six terms of the spectral expansion form, the constructed model is the red curve shown in Figure 8.

From the Figure 6, 7 and 8, we can observe that the lower index terms construct the higher energy and lower frequency components of the model, while higher index terms construct the lower energy but higher frequency (or more complex) components of the model.

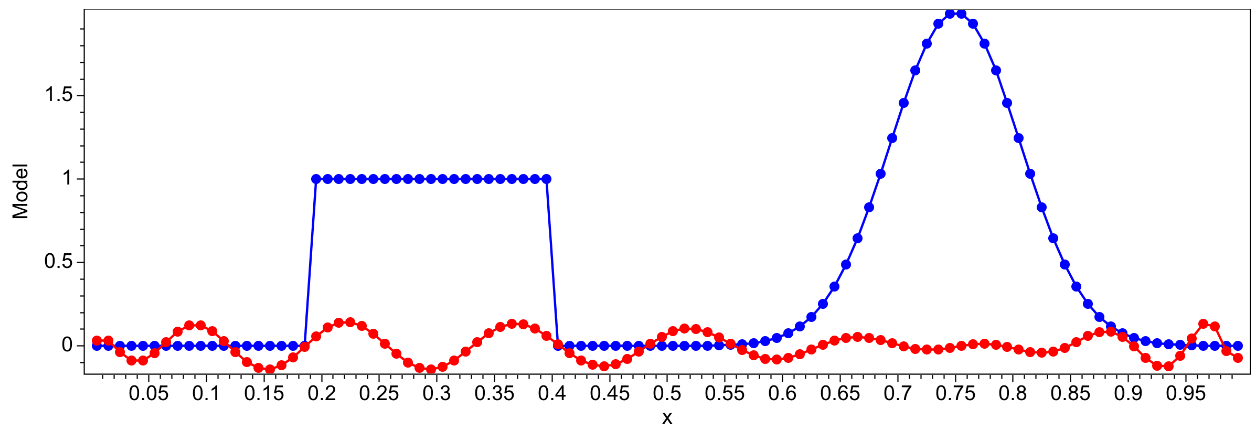

Figure 8. True (blue) and constructed (red) model with the last six terms in the spectral expansion form.