

## HW4 for GPGN658: Finite-difference kernels

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### 1 SECOND ORDER

Prove the stencil of the second derivative below has second-order accuracy.

$$\frac{\partial^2 W}{\partial h^2} \approx \frac{W_{h+(\Delta h)} - 2W_h + W_{h-(\Delta h)}}{(\Delta h)^2} \quad (1)$$

**Proof:**

From the Taylor's theorem, we have

$$W_{h+\Delta h} \approx W_h + \frac{1}{1!} \frac{\partial W}{\partial h} (\Delta h) + \frac{1}{2!} \frac{\partial^2 W}{\partial h^2} (\Delta h)^2 + \frac{1}{3!} \frac{\partial^3 W}{\partial h^3} (\Delta h)^3 + \frac{1}{4!} \frac{\partial^4 W}{\partial h^4} (\Delta h)^4 + \dots \quad (2)$$

$$W_{h-\Delta h} \approx W_h - \frac{1}{1!} \frac{\partial W}{\partial h} (\Delta h) + \frac{1}{2!} \frac{\partial^2 W}{\partial h^2} (\Delta h)^2 - \frac{1}{3!} \frac{\partial^3 W}{\partial h^3} (\Delta h)^3 + \frac{1}{4!} \frac{\partial^4 W}{\partial h^4} (\Delta h)^4 - \dots \quad (3)$$

From equations 2 and 3, we have

$$W_{h+\Delta h} + W_{h-\Delta h} \approx 2W_h + \frac{\partial^2 W}{\partial h^2} (\Delta h)^2 + \frac{1}{12} \frac{\partial^4 W}{\partial h^4} (\Delta h)^4 + \dots \quad (4)$$

Then we have

$$\frac{W_{h+\Delta h} - 2W_h + W_{h-\Delta h}}{(\Delta h)^2} \approx \frac{\partial^2 W}{\partial h^2} + \frac{1}{12} \frac{\partial^4 W}{\partial h^4} (\Delta h)^2 + \dots \quad (5)$$

From equation 5, we have

$$\frac{\partial^2 W}{\partial h^2} \approx \frac{W_{h+\Delta h} - 2W_h + W_{h-\Delta h}}{(\Delta h)^2} - \frac{1}{12} \frac{\partial^4 W}{\partial h^4} (\Delta h)^2 - \dots \quad (6)$$

Therefore, the finite-difference stencil in equation 1 has the second-order accuracy.

**2 FOURTH ORDER**

Prove the stencil of the second derivative below has fourth-order accuracy.

$$\frac{\partial^2 W}{\partial h^2} \approx \frac{-W_{h+2\Delta h} + 16W_{h+\Delta h} - 30W_h + 16W_{h-\Delta h} - W_{h-2\Delta h}}{12(\Delta h)^2} \quad (7)$$

According to the Taylor's theorem, we have

$$\begin{aligned} W_{h+2\Delta h} \approx W_h &+ \frac{1}{1!} \frac{\partial W}{\partial h} (2\Delta h) + \frac{1}{2!} \frac{\partial^2 W}{\partial h^2} (2\Delta h)^2 + \frac{1}{3!} \frac{\partial^3 W}{\partial h^3} (2\Delta h)^3 \\ &+ \frac{1}{4!} \frac{\partial^4 W}{\partial h^4} (2\Delta h)^4 + \frac{1}{5!} \frac{\partial^5 W}{\partial h^5} (2\Delta h)^5 + \frac{1}{6!} \frac{\partial^6 W}{\partial h^6} (2\Delta h)^6 + \dots \end{aligned} \quad (8)$$

$$\begin{aligned} W_{h-2\Delta h} \approx W_h &- \frac{1}{1!} \frac{\partial W}{\partial h} (2\Delta h) + \frac{1}{2!} \frac{\partial^2 W}{\partial h^2} (2\Delta h)^2 - \frac{1}{3!} \frac{\partial^3 W}{\partial h^3} (2\Delta h)^3 \\ &+ \frac{1}{4!} \frac{\partial^4 W}{\partial h^4} (2\Delta h)^4 - \frac{1}{5!} \frac{\partial^5 W}{\partial h^5} (2\Delta h)^5 + \frac{1}{6!} \frac{\partial^6 W}{\partial h^6} (2\Delta h)^6 - \dots \end{aligned} \quad (9)$$

From equations 2, 3, 8 and 9, we have

$$-W_{h+2\Delta h} + 16W_{h+\Delta h} - 30W_h + 16W_{h-\Delta h} - W_{h-2\Delta h} \approx 12 \frac{\partial^2 W}{\partial h^2} (\Delta h)^2 + \frac{8}{3} \frac{\partial^6 W}{\partial h^6} (\Delta h)^6 + \dots \quad (10)$$

Therefore, we have

$$\frac{\partial^2 W}{\partial h^2} \approx \frac{-W_{h+2\Delta h} + 16W_{h+\Delta h} - 30W_h + 16W_{h-\Delta h} - W_{h-2\Delta h}}{12(\Delta h)^2} - \frac{2}{9} \frac{\partial^6 W}{\partial h^6} (\Delta h)^4 - \dots, \quad (11)$$

which means that the finite-difference stencil in equation 7 has the fourth-order accuracy.