HW4 for GPGN658: Finite-difference kernels

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1 SECOND ORDER

Prove the stencil of the second derivative below has second-order accuracy.

$$\frac{\partial^2 W}{\partial h^2} \approx \frac{W_{h+(\triangle h)} - 2W_h + W_{h-(\triangle h)}}{(\triangle h)^2} \tag{1}$$

Proof:

From the Taylor's theorem, we have

$$W_{h+\triangle h} \approx W_h + \frac{1}{1!}\frac{\partial W}{\partial h}(\triangle h) + \frac{1}{2!}\frac{\partial^2 W}{\partial h^2}(\triangle h)^2 + \frac{1}{3!}\frac{\partial^3 W}{\partial h^3}(\triangle h)^3 + \frac{1}{4!}\frac{\partial^4 W}{\partial h^4}(\triangle h)^4 + \cdots \tag{2}$$

$$W_{h-\triangle h} \approx W_h - \frac{1}{1!}\frac{\partial W}{\partial h}(\triangle h) + \frac{1}{2!}\frac{\partial^2 W}{\partial h^2}(\triangle h)^2 - \frac{1}{3!}\frac{\partial^3 W}{\partial h^3}(\triangle h)^3 + \frac{1}{4!}\frac{\partial^4 W}{\partial h^4}(\triangle h)^4 - \cdots \tag{3}$$

From equations 2 and 3, we have

$$W_{h+\triangle h} + W_{h-\triangle h} \approx 2W_h + \frac{\partial^2 W}{\partial h^2}(\triangle h)^2 + \frac{1}{12}\frac{\partial^4 W}{\partial h^4}(\triangle h)^4 + \cdots . \tag{4}$$

Then we have

$$\frac{W_{h+\triangle h} - 2W_h + W_{h-\triangle h}}{(\triangle h)^2} \approx \frac{\partial^2 W}{\partial h^2} + \frac{1}{12}\frac{\partial^4 W}{\partial h^4}(\triangle h)^2 + \cdots . \tag{5}$$

From equation 5, we have

$$\frac{\partial^2 W}{\partial h^2} \approx \frac{W_{h+\triangle h} - 2W_h + W_{h-\triangle h}}{(\triangle h)^2} - \frac{1}{12}\frac{\partial^4 W}{\partial h^4}(\triangle h)^2 - \cdots . \tag{6}$$

Therefore, the finite-difference stencil in equation 1 has the second-order accuracy.

2 FOURTH ORDER

Prove the stencil of the second derivative below has fourth-order accuracy.

$$\frac{\partial^2 W}{\partial h^2} \approx \frac{-W_{h+2\triangle h} + 16W_{h+\triangle h} - 30W_h + 16W_{h-\triangle h} - W_{h-2\triangle h}}{12(\triangle h)^2} \tag{7}$$

According to the Taylor's theorem, we have

$$\begin{aligned} W_{h+2\triangle h} \approx W_h &+ \frac{1}{1!}\frac{\partial W}{\partial h}(2\triangle h) + \frac{1}{2!}\frac{\partial^2 W}{\partial h^2}(2\triangle h)^2 + \frac{1}{3!}\frac{\partial^3 W}{\partial h^3}(2\triangle h)^3 \\ &+ \frac{1}{4!}\frac{\partial^4 W}{\partial h^4}(2\triangle h)^4 + \frac{1}{5!}\frac{\partial^5 W}{\partial h^5}(2\triangle h)^5 + \frac{1}{6!}\frac{\partial^6 W}{\partial h^6}(2\triangle h)^6 + \cdots \end{aligned} \tag{8}$$

$$\begin{aligned} W_{h-2\triangle h} \approx W_h &- \frac{1}{1!}\frac{\partial W}{\partial h}(2\triangle h) + \frac{1}{2!}\frac{\partial^2 W}{\partial h^2}(2\triangle h)^2 - \frac{1}{3!}\frac{\partial^3 W}{\partial h^3}(2\triangle h)^3 \\ &+ \frac{1}{4!}\frac{\partial^4 W}{\partial h^4}(2\triangle h)^4 - \frac{1}{5!}\frac{\partial^5 W}{\partial h^5}(2\triangle h)^5 + \frac{1}{6!}\frac{\partial^6 W}{\partial h^6}(2\triangle h)^6 - \cdots \end{aligned} \tag{9}$$

From equations 2, 3, 8 and 9, we have

$$-W_{h+2\triangle h} + 16W_{h+\triangle h} - 30W_h + 16W_{h-\triangle h} - W_{h-2\triangle h} \approx 12\frac{\partial^2 W}{\partial h^2}(\triangle h)^2 + \frac{8}{3}\frac{\partial^6 W}{\partial h^6}(\triangle h)^6 + \cdots \tag{10}$$

Therefore, we have

$$\frac{\partial^2 W}{\partial h^2} \approx \frac{-W_{h+2\triangle h} + 16W_{h+\triangle h} - 30W_h + 16W_{h-\triangle h} - W_{h-2\triangle h}}{12(\triangle h)^2} - \frac{2}{9}\frac{\partial^6 W}{\partial h^6}(\triangle h)^4 - \cdots, \tag{11}$$

which means that the finite-difference stencil in equation 7 has the fourth-order accuracy.