HW2 for GPGN658: Displacement from potentials

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1 POTENTIALS FROM A DISPLACEMENT FIELD

A displacement field $\mathbf{u}(\mathbf{x}, t)$ can be expressed as a sum of curl-free and divergence-free forms by using the Helmholtz decomposition

$$\mathbf{u} = \nabla\theta + \nabla \times \boldsymbol{\psi}. \tag{1}$$

Given a displacement field $\mathbf{u}$, we can easily compute its corresponding scalar potential

$$\Theta = \nabla \cdot \mathbf{u} = \nabla \cdot (\nabla\theta), \tag{2}$$

and vector potential

$$\boldsymbol{\Psi} = \nabla \times \mathbf{u} = \nabla \times (\nabla \times \boldsymbol{\psi}). \tag{3}$$

2 DISPLACEMENT FROM POTENTIALS

Given the scalar and vector potentials Θ and $\boldsymbol{\Psi}$, we can also reconstruct its corresponding displacement filed $\mathbf{u}$.

We know that the vector Laplacian of the displacement field $\mathbf{u}$ can be expressed as

$$\begin{aligned} \nabla^2\mathbf{u} &= \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) \\ &= \nabla\Theta - \nabla \times \boldsymbol{\Psi}. \end{aligned} \tag{4}$$

Both the left-hand and right-hand sides of the above equations are vectors. Given the scalar and vector potentials Θ and $\mathbf{\Psi}$, we can first compute the vector on the right-hand side, then we can compute all the three components of the displacement field $\mathbf{u}$ by solving three Laplacian equations with a specific boundary condition.

Generally, such a Laplacian equation can be solved by using the Conjugate Gradient method because a negative Laplacian operator is symmetric positive definite.