HW3 for GPGN658: Dispersion relation

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1 DISPERSION RELATION FOR ISOTROPIC MEDIA

In isotropic media, the constitutive law for stress and strain can be expressed as

$$t_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}. \tag{1}$$

According to the equation of motions

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial t_{ij}}{\partial x_j}, \tag{2}$$

we have

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \lambda e_{kk}\delta_{ij} + 2\mu e_{ij}}{\partial x_j} = \lambda\delta_{ij}\frac{\partial e_{kk}}{\partial x_j} + 2\mu\frac{e_{ij}}{\partial x_j}. \tag{3}$$

We also have the geometric law below

$$e_{kl} = \frac{1}{2}(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}). \tag{4}$$

Substituting equation 4 into equation 3, and assuming $\nabla\lambda \approx 0$, $\nabla\mu \approx 0$, we have

$$\begin{aligned}\rho\frac{\partial^2 u_i}{\partial t^2} &= \lambda\delta_{ij}\frac{\partial^2 u_k}{\partial x_k \partial x_j} + \mu(\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_i \partial x_j}) \\ &= \lambda\frac{\partial^2 u_k}{\partial x_k \partial x_i} + \mu(\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_i \partial x_j}).\end{aligned} \tag{5}$$

Since $\frac{\partial^2 u_k}{\partial x_k \partial x_i}$ and $\frac{\partial^2 u_j}{\partial x_i \partial x_j}$ have the same structure, then we can combine them and obtain

$$\rho\frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i} + \mu\frac{\partial^2 u_i}{\partial^2 x_j}. \tag{6}$$

If we assume a planar wave $u_i = A_i e^{i(k_j x_j - \omega t)}$, and substitute it into the above equation, we obtain

$$\rho(-i\omega)^2 u_i = (\lambda + \mu)[(ik_j)(ik_i)]u_j + \mu[(ik_j)^2]u_i, \tag{7}$$

which can be further simplified as

$$\rho\omega^2 A_i = (\lambda + \mu)k_j k_i A_j + \mu k_j^2 A_i. \tag{8}$$

The equation above is the dispersion relation for isotropic media.

2 DISPERSION RELATION FOR VTI MEDIA

As discussed in class, we the dispersion relation for general anisotropic media as below

$$C_{ijkl} k_j k_l A_k = \rho\omega^2 A_i. \tag{9}$$

If we let $n_j = \frac{k_j}{\omega s}$ and $n_l = \frac{k_l}{\omega s}$, where s is slowness, then we have

$$C_{ijkl} n_j n_l A_k = \rho v^2 A_i. \tag{10}$$

To further simplify the dispersion relation above, we define a 3×3 matrix $\mathbf{G}$ with elements

$$G_{ik} = C_{ijkl} n_j n_l. \tag{11}$$

Then we obtain the below Christoffel equation from equation 10

$$(G_{ik} - \rho v^2 \delta_{ik}) A_k = 0. \tag{12}$$

In the VTI media, the stiffness matrix is given by

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}, \tag{13}$$

where $C_{12} = C_{11} - 2C_{66}$. Then for the 3×3 matrix $\mathbf{G}$, we have

$$\begin{aligned} G_{11} &= C_{11}n_1^2 + C_{66}n_2^2 + C_{55}n_3^2 \\ G_{22} &= C_{66}n_1^2 + C_{22}n_2^2 + C_{55}n_3^2 \\ G_{33} &= C_{55}n_1^2 + C_{55}n_2^2 + C_{33}n_3^2 \\ G_{21} &= G_{12} = C_{12}n_1n_2 + C_{66}n_2n_1 = (C_{11} - C_{66})n_1n_2 \\ G_{31} &= G_{13} = C_{13}n_1n_3 + C_{55}n_3n_1 = (C_{13} + C_{55})n_1n_3 \\ G_{32} &= G_{23} = C_{23}n_2n_3 + C_{55}n_3n_2 = (C_{23} + C_{55})n_2n_3 \end{aligned} \tag{14}$$

Therefore, the dispersion relation for the VTI media can be expressed as

$$\begin{bmatrix} G_{11} - \rho v^2 & G_{12} & G_{13} \\ G_{12} & G_{22} - \rho v^2 & G_{23} \\ G_{13} & G_{23} & G_{33} - \rho v^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

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