

# Solving Linear Equations

R4 Cheng

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## Matrix Operations

$$\begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} \underline{c_1} & \underline{c_2} & \underline{c_3} \end{bmatrix}$$

$$\underline{c_2} = x_2 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + z_2 \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} = a \begin{bmatrix} 1 & 2 & -4 \end{bmatrix} + b \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} + c \begin{bmatrix} 4 & 1 & 2 \end{bmatrix}$$

## Properties of Matrices

$A(BC) = (AB)C$  (Associative law holds)

$AB \neq BA$  (Commutative law does not hold)

$C(A + B) = CA + CB$  or  $(A + B)C = AC + BC$  (Distributive laws hold)

**Remark.** *We can change Gaussian Elimination to Matrix multiplication*

## Identity Matrix

The identity matrix is a square matrix with ones on the main diagonal and zeros elsewhere. It is denoted by  $I$  or  $I_n$  for an  $n \times n$  matrix.  $AI = IA = A$ , for any  $n \times n$  matrix  $A$ .

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

## Inverse Matrix

The inverse of a square matrix  $A$ , denoted as  $A^{-1}$ , is a matrix such that  $AA^{-1} = A^{-1}A = I$ , where  $I$  is the identity matrix. The inverse matrix can be found using the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

where  $\det(A)$  is the determinant of matrix  $A$  and  $\text{adj}(A)$  is the adjugate of matrix  $A$ .

For example, to find the inverse of a  $2 \times 2$  matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse matrix is given by:

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

To find the inverse of a  $3 \times 3$  matrix, you can use the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

where  $\text{adj}(A)$  is the adjugate of matrix  $A$ . The adjugate of a  $3 \times 3$  matrix is given by:

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where  $A_{ij}$  is the cofactor of element  $a_{ij}$  in matrix  $A$ .

Note that not all matrices have an inverse. A matrix is invertible if and only if its determinant is non-zero.

### Attributes

- It is unique.
- The inverse of  $A^{-1}$  is  $A$  itself.

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**Claim.** Suppose  $A$  is invertible. Then its inverse is unique.

**Proof.** Suppose  $B$  and  $C$  are both inverses of  $A$ . Then  $B = BI = B(AC) = (BA)C = IC = C$ .

**Remark.** *left inverse = right inverse = inverse*

**Claim.** The inverse of  $A^{-1}$  is  $A$  itself.

**Proof.**  $AA^{-1} = I$  and  $A^{-1}A = I$ .

**Claim.** If  $A$  is invertible, then the one and only solution to  $A\underline{x} = \underline{b}$  is  $\underline{x} = A^{-1}\underline{b}$ .

**Proof.**  $A\underline{x} = \underline{b} \Rightarrow A^{-1}A\underline{x} = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$ .

**Claim.** Suppose there is a nonzero solution  $\underline{x}$  to  $A\underline{x} = \underline{0}$  (homogeneous equation). Then  $A$  is not invertible.

**Proof.** If  $A$  is invertible, then  $A^{-1}$  exists. Then  $A^{-1}A\underline{x} = A^{-1}\underline{0} \Rightarrow \underline{x} = \underline{0}$ .

**Claim.** A diagonal matrix has an inverse provided no diagonal entries are zero.

**Proof.**

If

$$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

then

$$A^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

**Claim.** If  $A$  and  $B$  are invertible, then  $AB$  is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$ .

**Proof.**

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$$

**Remark.**  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

## Gauss-Jordan Elimination

Given  $A$ , we want to find its inverse  $A^{-1}$ .  $AA^{-1} = I$

$$A \begin{bmatrix} \underline{col}_1 & \underline{col}_2 & \underline{col}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{2} & -1 & 0 & \vdots & 1 & 0 & 0 \\ -1 & 2 & -1 & \vdots & 0 & 1 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} 2 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & \boxed{\frac{3}{2}} & -1 & \vdots & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} 2 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \vdots & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \boxed{\frac{4}{3}} & \vdots & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

Until here is Gauss

$\Rightarrow$

$$\begin{bmatrix} 2 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \vdots & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \boxed{\frac{4}{3}} & \vdots & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} 2 & 0 & 0 & \vdots & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \boxed{\frac{3}{2}} & 0 & \vdots & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \boxed{\frac{4}{3}} & \vdots & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \vdots & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Until here is Jordan

$$col_1 = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, col_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}, col_3 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix}, A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

**Claim.** A matrix is invertible if and only if (iff.) it is nonsingular.

## Elimination = Factorization: $A = LU$

TODO

**Claim.** If  $A = L_1 D_1 U_1$  and  $A = L_2 D_2 U_2$ , where the  $L$ 's are lower-triangular with unit diagonal, the  $U$ 's are upper-triangular with unit diagonal, and the  $D$ 's are diagonal matrices with no zeros on the diagonal, then  $L_1 = L_2$ ,  $D_1 = D_2$ , and  $U_1 = U_2$ .

## One Square System = Two Triangular Systems

Benefit: ????

$A\underline{x} = \underline{b}$ . Suppose elimination requires no row exchanges.

$$A = LU \Rightarrow LU\underline{x} = \underline{b} \Rightarrow U\underline{x} = L^{-1}\underline{b} = \underline{c}$$

We have  $U\underline{x}$  where  $L\underline{c} = \underline{b}$ .

1. Factor  $A = LU$  by Gaussian elimination  $\underline{c} = L^{-1}\underline{b}$ .
2. Solve  $\underline{c}$  from  $L\underline{c} = \underline{b}$  (forward elimination) and then solve  $U\underline{x} = \underline{c}$  (backward elimination)

E.g.

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$
$$A\underline{x} = \underline{b}$$

$\Rightarrow$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A = LU$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$
$$L\underline{c} = \underline{b}$$

$$\therefore \underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$
$$U\underline{x} = \underline{c}$$

$$\therefore \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

## Complexity of Elimination

1.

Solve  $A\underline{x} = \underline{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = LU$$

Gaussian Elimination:

1st stage:  $n(n-1) \approx n^2$  (# multiplications and additions)

2nd stage:  $(n-1)^2$

3rd stage:  $(n-2)^2$

$\vdots$

$$n^2 + (n-1)^2 + \dots + 1^2 = \frac{1}{3}n(n+\frac{1}{2})(n+1) \approx \frac{n^3}{3}$$

Solve  $L\underline{c} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ val & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ val & val & \dots & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$(n-1) + (n-2) + \dots + 2 + 1 = \frac{n(n-1)}{2} \approx \frac{n^2}{2}$$

**Remark.**  $b_1$  needs to be multiplied  $(n-1)$  times,  $b_2$  needs to be multiplied  $(n-2)$  times

Solve  $U\underline{x} = \underline{c}$

$$\begin{bmatrix} pivot_1 & val & \dots & val \\ 0 & p_2 & \dots & val \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$1 + 2 + \dots + n = \frac{n(n-1)}{2} \approx \frac{n^2}{2}$$

$$\text{Total: } \# = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n^2}{2} \approx \frac{n^3}{3}$$

2.

Compute  $A^{-1}$

$$\text{Solve } A \begin{bmatrix} \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_n \end{bmatrix} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_n \end{bmatrix}$$

where  $\underline{e}_i$  is the  $i$ th column of the identity matrix.

According Elimination  $A = LU$ , solve

$$L\underline{c}_i = \underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1(i\text{th}) \\ \vdots \\ 0 \end{bmatrix}$$

$\therefore$  The non-zero variables are only from  $c_i$  to  $c_n$ .

$\therefore$  Need to solve  $(n - i + 1)$  variables  $\Rightarrow$  complexity  $\#$  is  $\frac{(n-i+1)^2}{2}$

$$\text{Since } 1 \leq i \leq n, \# = \frac{n^2}{2} + \frac{(n-1)^2}{2} + \cdots + \frac{1^2}{2} \approx \frac{n^3}{6}$$

$$\text{And solve } U\underline{x}_i = \underline{c}_i \# = \frac{n^2}{2}$$

$$\text{Since } 1 \leq i \leq n, \# = n \times \frac{n^2}{2} = \frac{n^3}{2}$$

$$\text{Total: } \# = \frac{n^3}{3} + \frac{n^3}{6} + \frac{n^3}{2} = n^3$$

Compared with e.g.  $\#$  for  $A^2$ :  $n^2 \cdot n = n^3$

Conclusion: Use Gaussian Elimination is as good as computing  $A^2$  (not complicated).

## Transpose and Permutation Matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \Rightarrow A_{ij}^T = A_{ji}$$

$$\text{Claim. } (A + B)^T = A^T + B^T$$

$$\text{Claim. } (AB)^T = B^T A^T$$

**Proof.** TODO

$$\text{Remark. } (ABC)^T = C^T B^T A^T$$

$$\text{Claim. } (A^{-1})^T = (A^T)^{-1}$$

**Proof.** TODO

$$\text{Claim. } (A^{-1})^T = (A^T)^{-1}$$

**Proof.** TODO

**Def.** An  $n \times n$  matrix  $A$  is *symmetric* if  $A^T = A$ .

**Remark.**  $A_{ij} = A_{ji}$ , if  $A$  is symmetric

$$A = A^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} D = D^T = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

**Claim.** Given any matrix  $R(n \times m)$ ,  $R^T R$  and  $RR^T$  are symmetric.

**Proof.**  $(R^T R)^T = R^T (R^T)^T = R^T R$

**Claim.** If a symmetric matrix is factored into  $LDU$  with no row exchanges, then  $U = L^T$ .

**Proof.**  $A = LDU \Rightarrow A^T = U^T D^T L^T = U^T D L^T$ . Since  $A$  is symmetric  $A = LDU = A^T = U^T D L^T$ . Recall that this Factorization is unique  $\Rightarrow U = L^T$

## Permutation Matrices

A Permutation matrix has the rows of the identity  $I$  in any order. There are  $6 \cdot 3 \times 3$  Permutation matrices.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{32}P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} P_{21}P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\therefore$  There are  $n!$  Permutation matrices of order  $n$ .

**Claim.** If  $P$  is a Permutation matrix, then  $P^T = P^{-1}$

**Proof.** TODO

Recall if no row exchanges are required, then  $A = LU$ . If row exchanges are needed, we then have  $(\cdots P \cdots E)A = U \Rightarrow A = (\cdots E^{-1} \cdots P^{-1})U$

If row exchanges are needed during elimination, we can do them in advance. The product  $PA$  will put the rows in the right order so that no row exchanges are needed for  $PA$ . Hence

$$PA = LU$$

We can also hold row exchanges until after elimination, we then have  $A = LPU$