Vector Spaces and Subspaces

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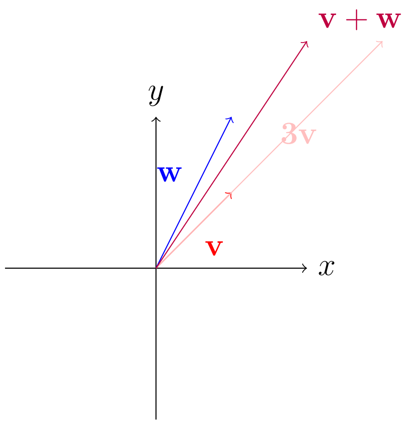
$\mathbb{R}^n$ = all (column) vectors with n (real) components.
$= \{(v_1, v_2, \cdots, v_n) : v_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \in \mathbb{R}^2, \quad (1,1,0,1,1) \in \mathbb{R}^5$$



Vector Space V

V: a set of vectors

1. Two operations:
   - vector addition: $\underline{v}, \underline{w} \in V \Rightarrow \underline{v} + \underline{w} \in V$
   - scalar multiplication: $c\underline{v} \in V$
2. Eight rules:
   (a) $\underline{v} + \underline{w} = \underline{w} + \underline{v}$ (commutative)
   (b) $(\underline{v} + \underline{w}) + \underline{z} = \underline{v} + (\underline{w} + \underline{z})$ (associative)
   (c) There is a unique "zero vector" $\underline{0}$ such that $\underline{v} + \underline{0} = \underline{v}$ for all $\underline{v} \in V$
   (d) For each $\underline{v}$, there is a unique vector $-\underline{v}$ such that $\underline{v} + (-\underline{v}) = \underline{0}$
   (e) $1 \times \underline{v} = \underline{v}$

(f) $(c_1c_2)\underline{v} = c_1(c_2\underline{v})$

(g) $c(\underline{v} + \underline{w}) = c\underline{v} + c\underline{w}$

(h) $(c_1 + c_2)\underline{v} = c_1\underline{v} + c_2\underline{v}$

$\Rightarrow 0 \times \underline{v} = \underline{0}$ (not 0)
$\Rightarrow (-1)\underline{v} = -\underline{v}$

Example:

- $\mathbb{R}^n$ is a vector space
- $M = \{$all real 2×2 matrices$\}$ is a vector space
- $F = \{$all real functions $f(x)$ $\}$ is a vector space
- $z = \{\underline{0}\}$ is a vector space

Subspaces

Def. A subset W of a vector space V is a <u>subspaces</u> if W itself is a vector space.

Check:

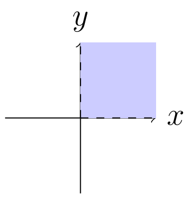
- $\underline{v}, \underline{w} \in W \Rightarrow \underline{v} + \underline{w} \in W$
- $\underline{v} \in W, c\underline{v} \in W$ for any c

Claim Every subspace contains the zero vector.

Proof TODO

Example:

1.



$U = \{(x, y) : x \geq 0, y \geq 0\}$, is U a subspace?

No, since $-1(1, 0) = (-1, 0) \notin U$ even of $(1, 0) \in U$.

2.

$M = \{$all real 2×2 matrices$\}$
$\{U = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a,\ b,\ d \in \mathbb{R}\}$

$A, B \in U,\ A + B \in U$ and $cA \in U$

$\therefore U$ is a subspace of M

Column Space

Def. The column space $C(A)$ of a matrix A consists of all linear combinations of the columns of A.

Remark. *C(A): C of A*

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\begin{aligned} C(A) = \ & \{c_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} : c_1, c_2 \in \mathbb{R}\} \\ & = \{A\underline{c} : \underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2\} (A\underline{x} = \underline{b}) \end{aligned}$$

The set of all $A\underline{x}$ for all x is called the column space.

$$\iff c_1[\underline{a_1}] + c_2[\underline{a_2}] + \cdots + c_n[\underline{a_n}] = \underline{b}$$

$\therefore$ The system $A\underline{x} = \underline{b}$ is solvable iff $\underline{b} \in C(A)$

Example:

What are the column spaces of 1. I, 2. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, 3. $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$?

1.

$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \therefore C(I) = \mathbb{R}^2$

2.

$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2 \Rightarrow x_{real} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore C(A) = \{x \begin{bmatrix} 1 \\ 2 \end{bmatrix} : x \in \mathbb{R}\}$

3.

$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = (x_1 + 2x_2 (= x_4)) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is always solvable for any b_1, b_2

$\because \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$ is upper triangular matrix (b_1, b_2 must be found) $\therefore C(B) = \mathbb{R}^2$

$\Rightarrow$ All of them are subspaces of $\mathbb{R}^2$

Claim If A is an $m \times n$ real matrix, then $C(A)$ is a subspace of $\mathbb{R}^m$

Proof. TODO

S = the set of vectors in a vector space V (probably not a subspace)
SS = the set of all linear combinations of vectors in S

We call SS the "span" of S.

Then SS is a subspace of V, called the subspace "spanned" by S.

E.g.

S = the set of columns of $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

SS = the column space of $A = C(A)$

Null Space of A

$$N(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$$

Remark. *related to the "rank"*

Claim If A is $m \times n$, then $N(A)$ is a subspace of $\mathbb{R}^n$.

Proof TODO

Example: $C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} = [A \quad 2A]$ (Two equations in four unknowns)

TODO

$$N(C) = \{\underline{x} : \underline{x} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_3, x_4 \in \mathbb{R}\}$$

Remark. *Reduced Row Echelon form (RRE form) 1. Produce 0 above/below pivots 2. Produce 1 in pivots*

Example: $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$

TODO

$$N(A) = \{\underline{x} : \underline{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, x_2, x_4 \in \mathbb{R}\}$$

Suppose A is $m \times n$. If there are r pivots ($r \leq m, n$), there are $(n - r)$ free variables. And there are $(n - r)$ special solutions. $N(A)$ consists of all the linear combinations of these $(n - r)$ special solutions.

$$N(A) = \text{the subspace spanned by these } (n-r) \text{ special solutions.}$$

Def. The rank of a matrix A is the number of pivots.

Remark. *All free columns are linear combinations of the pivot columns, and special solutions describe these combinations.*

Complete Solution to $A\underline{x} = \underline{b}$

$$\underline{x} = \underline{x}_{particular} + \underline{x}_{nullspace}$$

Claim If $A\underline{x} = \underline{b}$, then the complete solution is $\underline{x} = \underline{x}_{particular} + \underline{x}_{nullspace}$, where $\underline{x}_{particular}$ is a particular solution to $A\underline{x} = \underline{b}$, and $\underline{x}_{nullspace}$ is a general solution to $A\underline{x} = \underline{0}$.

Proof

1. If $\underline{x} = \underline{x}_{particular} + \underline{x}_{nullspace}$, then $A\underline{x} = A(\underline{x}_{particular} + \underline{x}_{nullspace}) = A\underline{x}_{particular} + A\underline{x}_{nullspace} = \underline{b} + \underline{0} = \underline{b}$

$\therefore$ $\underline{x}$ is a solution to $A\underline{x} = \underline{b}$

2. If $\underline{x}$ is a solution to $A\underline{x} = \underline{b}$, then $A(\underline{x} - \underline{x}_p) = A\underline{x} - A\underline{x}_p = \underline{b} - \underline{b} = \underline{0}$

$\therefore$ $\underline{x} - \underline{x}_p = \underline{x}_n$ is a solution to $A\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{x}_p + \underline{x}_n$

Q. Suppose A is a square invertible matrix ($m = n = pivots$). What are $\underline{x}_p$ and $\underline{x}_n$?
A.
The only particular solution to $A\underline{x} = \underline{b}$ is $\underline{x}_p = A^{-1}\underline{b}$. (because it is invertible)
The only solution to $A\underline{x} = \underline{0}$ is $\underline{x}_n = \underline{0}$ (Inverse matrix's attribute).

Example: Full Column Rank

TODO

In general, if $r = n (m \geq n)$ (A has full column rank),

$$A = [Matrix]_{m\times n} \Rightarrow R = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Therefore, if A has full column rank ($r = n$), then

1. All columns of A are pivot columns
2. There are no free variables or no special solutions
3. $N(a) = \underline{0}$
4. If $A\underline{x} = \underline{b}$ is solvable, then the solution is unique.

Example: Full Row Rank

TODO

In general, if $r = m (m \leq n)$ (A has full row rank),

$$A = [Matrix]_{m \times n} \ \Rightarrow R = [I \quad F]$$

Therefore, if A has full row rank ($r = m$), then

1. All rows of A are pivot columns and R has no zero rows
2. $A\underline{x} = \underline{b}$ is solvable for all $\underline{b}$
3. $C(A) = \mathbb{R}^m$ (C: column space)
4. There are $n - r = n - m$ special solutions in $N(A)$

Summary

Case 1: $r = m = n$ (A is invertible)

$R = I \Rightarrow A\underline{x} = \underline{b}$ has 1 solution: $\underline{x} = A^{-1}\underline{b}$

Case 2: $r = m < n$ (full row rank) (A: short and wide)

$R = [I \quad F] \Rightarrow A\underline{x} = \underline{b}$ is solvable for all $\underline{b}$ and solutions are ∞.

Case 3: $r = n < m$ (full column rank) (A: tall and thin)

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$A\underline{x} = \underline{b}$ has 0 or 1 solution.

Case 4: $r < m, n$ (neither full row rank nor full column rank)

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$A\underline{x} = \underline{b}$ has 0 or ∞ solutions.

Independence, Basis, and Dimension

(Linearly) Independence

Def. If $x_1\underline{v_1} + x_2\underline{v_2} + \cdots + x_n\underline{v_n} = \underline{0}$ and only happens when $x_1 = x_2 = \cdots = x_n = 0$, then $\underline{v_1}, \underline{v_2}, \cdots, \underline{v_n}$ are <u>linearly independent</u>. Otherwise, they are <u>linearly dependent</u>.

Examples

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

TODO

They are linearly independent.

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

TODO

$\Rightarrow x_1 = x_2 = 0$

$\therefore$ They are linearly independent.

Remark. *These special solutions we found in the nullspace of a matrix are independent*

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$$

Suppose

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Calculate the nullspace of the matrix)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = R \therefore \underline{\mathrm{x}} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

We can have

$$-3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ They are linearly dependent.

In general, the columns of A are linearly independent exactly when $rank = n$ (aka. full column rank).

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$ There are n pivots and no free variables. Hence $N(A) = \underline{0}$

Claim Any set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $n > m$.

Proof TODO

Def. A set of vectors 'spans' a vector space if there linear combinations fill the space.

E.g. The column of a matrix span its column space.

E.g. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ span $\mathbb{R}^2 \Rightarrow \begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$

Def. The row space of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

The row space of A is $C(A^T)$

Basis

Def. A basis for a vector space a sequence of vectors satisfying two properties:

1. The basis vectors are linearly independent
2. The basis vectors span the vector space

Examples

$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$ constitute a basis for $\mathbb{R}^2$

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\1\end{bmatrix}$ Let $A = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \Rightarrow$ 3 pivot (no free variables) $\Rightarrow$ linearly independent.

And $\begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix} \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ is always solvable for any $b_1, b_2, b_3 \Rightarrow \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\1\end{bmatrix}$ are a basis for $\mathbb{R}^3$

Claim Any three linearly independent vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

Proof Assume $\underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$ are linearly independent. $\Rightarrow A = [\underline{v}_1 \quad \underline{v}_2 \quad \underline{v}_3] \Rightarrow \text{rank} = 3 \Rightarrow A$ is invertible.

$A\underline{x} = \underline{b}$ is solvable for every $\underline{b}$. $\therefore C(A) = \mathbb{R}^3 \Rightarrow \underline{v}_1$, $\underline{v}_2$, $\underline{v}_3$ are a basis for $\mathbb{R}^3$

Claim Any n linearly independent vectors in $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$.

Claim The vectors $\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_n$ are a basis for $\mathbb{R}^n$ exactly when they are the columns of an invertible matrix.

Remark. *$\mathbb{R}^n$ has infinitely many different bases.*

Claim There is one and only one way to write any vector in a vector space as a linear combination of the basis vectors.

Proof TODO

Example: Search for Basis

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \Rightarrow R = \begin{bmatrix} \boxed{1} & 3 & 0 & 2 & -1 \\ 0 & 0 & \boxed{1} & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Recall free columns of R/A are linear combinations of pivot columns of R/A.

$\because$ pivot columns of R/A are linearly independent ($A\underline{x} = \underline{0} \iff R\underline{x} = \underline{0}$), $\therefore$ Pivot columns of R/A are a basis for $C(R)/C(A)$.

$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are a basis for $C(A)$, and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are a basis for $C(R)$.

Remark. $C(A) \neq C(R)$

Dimension

Claim If $\underline{v_1}, \underline{v_2}, \cdots, \underline{v_n}$ and $\underline{w_1}, \underline{w_2}, \cdots, \underline{w_m}$ are both bases for the same vector vector space, then the number of elements $m = n$.

Proof TODO

Def. The dimension of a vector space is the number of vectors in every basis.

Examples $dim(\mathbb{R}^2) = 2 \Rightarrow dim(\mathbb{R}^n) = n$

Q. What is the dimension of the column space of a matrix A?

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow dim(C(A)) = dim(C(R)) = 2 = rank$$

$M =$ the vector space of all 2×2 real matrices. A basis for M is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ $dim(m) = 4 \Rightarrow$ Dim of the $n \times n$ real matrix space is n^2.

- Dim of the subspace (Why subspace rahter than vector space?) of all $n \times n$ upper triangular matrices $= n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}$
- Dim of the subspace of all $n \times n$ diagonal matrices $= n$
- Dim of the subspace of all $n \times n$ symmetric matrices $= \frac{n(n+1)}{2}$

Symmetric: $\begin{bmatrix} \ddots & a_{ij} \\ a_{ji} & \ddots \end{bmatrix}$ $a_{ij} = a_{ji}$

Dimension of the Four Subspaces

- Dim of $C(A^T)$ (row space) = rank of A, A subspace of $\mathbb{R}^n$
- Dim of $C(A)$ (col space) = rank of A, A subspace of $\mathbb{R}^m$
- Dim of $N(A)$ (null space) = $n - rank$, A subspace of $\mathbb{R}^n$
- Dim of $N(A^T)$ (left nullspace) = $m - rank$, A subspace of $\mathbb{R}^m$

Remark. $N(A^T) = \{\underline{y} : A^T \underline{y} = \underline{0}\} = \{\underline{y} : y^T A = \underline{0}^T\}$