Vector Spaces and Subspaces

R4 Cheng

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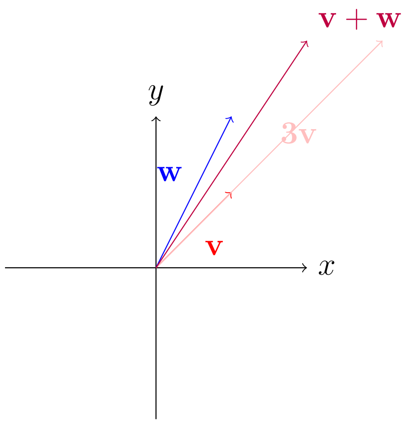
$\mathbb{R}^n$ = all (column) vectors with n (real) components.
$= \{(v_1, v_2, \cdots, v_n) : v_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \in \mathbb{R}^2, \quad (1,1,0,1,1) \in \mathbb{R}^5$$



Vector Space V

V: a set of vectors

1. Two operations:
   - vector addition: $\underline{v}, \underline{w} \in V \Rightarrow \underline{v} + \underline{w} \in V$
   - scalar multiplication: $c\underline{v} \in V$
2. Eight rules:
   (a) $\underline{v} + \underline{w} = \underline{w} + \underline{v}$ (commutative)
   (b) $(\underline{v} + \underline{w}) + \underline{z} = \underline{v} + (\underline{w} + \underline{z})$ (associative)
   (c) There is a unique "zero vector" $\underline{0}$ such that $\underline{v} + \underline{0} = \underline{v}$ for all $\underline{v} \in V$
   (d) For each $\underline{v}$, there is a unique vector $-\underline{v}$ such that $\underline{v} + (-\underline{v}) = \underline{0}$
   (e) $1 \times \underline{v} = \underline{v}$

(f) $(c_1c_2)\underline{v} = c_1(c_2\underline{v})$

(g) $c(\underline{v} + \underline{w}) = c\underline{v} + c\underline{w}$

(h) $(c_1 + c_2)\underline{v} = c_1\underline{v} + c_2\underline{v}$

$\Rightarrow 0 \times \underline{v} = \underline{0}$ (not 0)
$\Rightarrow (-1)\underline{v} = -\underline{v}$

Examples:

- $\mathbb{R}^n$ is a vector space
- $M = \{$all real 2×2 matrices$\}$ is a vector space
- $F = \{$all real functions $f(x)$ $\}$ is a vector space
- $z = \{\underline{0}\}$ is a vector space

Subspaces

Def. A subset W of a vector space V is a <u>subspaces</u> if W itself is a vector space.

Claim Every subspace contains the zero vector.

Proof $0 \times \underline{v} = \underline{0} \in W$