Solving Linear Equations

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Matrix Operations

$$\begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} \underline{c_1} & \underline{c_2} & \underline{c_3} \end{bmatrix}$$

$$\underline{c_2} = x_2 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + z_2 \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} = a \begin{bmatrix} 1 & 2 & -4 \end{bmatrix} + b \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} + c \begin{bmatrix} 4 & 1 & 2 \end{bmatrix}$$

Properties of Matrices

$A(BC) = (AB)C$ (Associative law holds)

$AB \neq BA$ (Commutative law does not hold)

$C(A+B) = CA + CB$ or $(A+B)C = AC + BC$ (Distributive laws hold)

Remark. *We can change Guassian Elimination to Matrix multiplication*

Identity Matrix

The identity matrix is a square matrix with ones on the main diagonal and zeros elsewhere. It is denoted by I or I_n for an $n \times n$ matrix. $AI = IA = A$, for any $n \times n$ matrix A.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Inverse Matrix

The inverse of a square matrix A, denoted as A^{-1}, is a matrix such that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix. The inverse matrix can be found using the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

where $\det(A)$ is the determinant of matrix A and $\operatorname{adj}(A)$ is the adjugate of matrix A.

For example, to find the inverse of a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse matrix is given by:

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

To find the inverse of a 3×3 matrix, you can use the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

where $\operatorname{adj}(A)$ is the adjugate of matrix A. The adjugate of a 3×3 matrix is given by:

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where A_{ij} is the cofactor of element a_{ij} in matrix A.

Note that not all matrices have an inverse. A matrix is invertible if and only if its determinant is non-zero.

Attributes

- It is unique.
- The inverse of A^{-1} is A itself.

Claim. Suppose A is invertible. Then its inverse is unique.

Proof. Suppose B and C are both inverses of A. Then $B = BI = B(AC) = (BA)C = IC = C$.

Remark. *left inverse = right inverse = inverse*

Claim. The inverse of A^{-1} is A itself.

Proof. $AA^{-1} = I$ and $A^{-1}A = I$.

Claim. If A is invertible, then the one and only solution to $A\underline{x} = \underline{b}$ is $\underline{x} = A^{-1}\underline{b}$.

Proof. $A\underline{x} = \underline{b} \Rightarrow A^{-1}A\underline{x} = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$.

Claim. Suppose there is a nonzero solution $\underline{x}$ to $A\underline{x} = \underline{0}$ (homogeneous equation). Then A is not invertible.

Proof. If A is invertible, then A^{-1} exists. Then $A^{-1}A\underline{x} = A^{-1}\underline{0} \Rightarrow \underline{x} = \underline{0}$.

Claim. A diagonal matrix has an inverse provided no diagonal entries are zero.

Proof.
If

$$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

then

$$A^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Claim. If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof.

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I$
$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$

Remark. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Gauss-Jordan Elimination

Given A, we want to find its inverse A^{-1}. $AA^{-1} = I$

$$A \begin{bmatrix} \underline{col_1} & \underline{col_3} & \underline{col_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{e_1} & \underline{e_3} & \underline{e_3} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc:ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{\dfrac{3}{2}} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \boxed{\dfrac{4}{3}} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

Until here is Gauss

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \boxed{\dfrac{4}{3}} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \boxed{\dfrac{3}{2}} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \boxed{\dfrac{4}{3}} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

Until here is Jordan

$$col_1 = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, col_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}, col_3 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix}, A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Claim. A matrix is invertible if and only if (iff.) it is nonsingular.

Elimination = Factorization: $A = LU$

TODO

Claim. If $A = L_1 D_1 U_1$ and $A = L_2 D_2 U_2$, where the L's are lower-triangular with unit diagonal, the U's are upper-triangular with unit diagonal, and the D's are diagonal matrices with no zeros on the diagonal, then $L_1 = L_2$, $D_1 = D_2$, and $U_1 = U_2$.

One Square System = Two Triangular Systems

Benifit: ????

$A\underline{x} = \underline{b}$. Suppose elimination requires no row exchanges.

$A = LU \Rightarrow LU\underline{x} = \underline{b} \Rightarrow U\underline{x} = L^{-1}\underline{b} = \underline{c}$

We have $U\underline{x}$ where $L\underline{c} = \underline{b}$.

1. Factor $A = LU$ by Gaussian eliminatio $\underline{c} = L^{-1}\underline{b}$.
2. Solve $\underline{c}$ from $L\underline{c} = \underline{b}$ (forward elimination) and then solve $U\underline{x} = \underline{c}$ (backward elimination)

E.g.

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$
$$A\underline{x} = \underline{b}$$

$\Rightarrow$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A = LU$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$
$$L\underline{c} = \underline{b}$$

$$\therefore \underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$
$$U\underline{x} = \underline{c}$$

$$\therefore \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Complexity of Elimination

1.

Solve $A\underline{x} = \underline{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = LU$$

Gaussian Elimination:

1st stage: $n(n-1) \approx n^2$ (# multiplications and additions)
2nd stage: $(n-1)^2$
3rd stage: $(n-2)^2$
$\vdots$
$n^2 + (n-1)^2 + \dots + 1^2 = \frac{1}{3}n(n+\frac{1}{2})(n+1) \approx \frac{n^3}{3}$

Solve $L\underline{c} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ val & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ val & val & \dots & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$(n-1) + (n-2) + \dots + 2 + 1 = \frac{n(n-1)}{2} \approx \frac{n^2}{2}$

Remark. *b_1 needs to be multiplied $(n-1)$ times, b_2 needs to be multiplied $(n-2)$ times*

Solve $U\underline{x} = \underline{c}$

$$\begin{bmatrix} pivot_1 & val & \cdots & val \\ 0 & p_2 & \cdots & val \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \cdots & p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$1 + 2 + \ldots + n = \frac{n(n-1)}{2} \approx \frac{n^2}{2}$

Total: $\# = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n^2}{2} \approx \frac{n^3}{3}$

2.

Compute A^{-1}

Solve $A \begin{bmatrix} \underline{x_1} & \underline{x_2} & \cdots & \underline{x_n} \end{bmatrix} = \begin{bmatrix} \underline{e_1} & \underline{e_2} & \cdots & \underline{e_n} \end{bmatrix}$

where $\underline{e_i}$ is the ith column of the identity matrix.

According Elimination $A = LU$, solve

$$L\underline{c_i} = \underline{e_i} = \begin{bmatrix} 0 \\ \vdots \\ 1(ith) \\ \vdots \\ 0 \end{bmatrix}$$

$\because$ The non-zero variables are only from c_i to c_n.
$\therefore$ Need to solve (n - i + 1) variables $\Rightarrow$ complexity # is $\frac{(n-i+1)^2}{2}$

Since $1 \leq i \leq n$, $\# = \frac{n^2}{2} + \frac{(n-1)^2}{2} + \cdots + \frac{1^2}{2} \approx \frac{n^3}{6}$

And solve $U\underline{x_i} = \underline{c_i} \# = \frac{n^2}{2}$

Since $1 \leq i \leq n$, $\# = n \times \frac{n^2}{2} = \frac{n^3}{2}$

Total: $\# = \frac{n^3}{3} + \frac{n^3}{6} + \frac{n^3}{2} = n^3$

Compared with e.g. # for A^2: $n^2 \cdot n = n^3$

Conclusion: Use Gaussian Elimination is as good as computing A^2 (not complicated).