Solving Linear Equations

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Matrix Operations

$$\begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} \underline{c_1} & \underline{c_2} & \underline{c_3} \end{bmatrix}$$

$$\underline{c_2} = x_2 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + z_2 \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 \\ -2 & 3 & 1 \\ 4 & 1 & 2 \end{bmatrix} = a \begin{bmatrix} 1 & 2 & -4 \end{bmatrix} + b \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} + c \begin{bmatrix} 4 & 1 & 2 \end{bmatrix}$$

Properties of Matrices

$A(BC) = (AB)C$ (Associative law holds)

$AB \neq BA$ (Commutative law does not hold)

$C(A+B) = CA + CB$ or $(A+B)C = AC + BC$ (Distributive laws hold)

Remark. *We can change Guassian Elimination to Matrix multiplication*

Identity Matrix

The identity matrix is a square matrix with ones on the main diagonal and zeros elsewhere. It is denoted by I or I_n for an $n \times n$ matrix. $AI = IA = A$, for any $n \times n$ matrix A.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Inverse Matrix

The inverse of a square matrix A, denoted as A^{-1}, is a matrix such that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix. The inverse matrix can be found using the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

where $\det(A)$ is the determinant of matrix A and $\operatorname{adj}(A)$ is the adjugate of matrix A.

For example, to find the inverse of a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse matrix is given by:

$$A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

To find the inverse of a 3×3 matrix, you can use the formula:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

where $\operatorname{adj}(A)$ is the adjugate of matrix A. The adjugate of a 3×3 matrix is given by:

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where A_{ij} is the cofactor of element a_{ij} in matrix A.

Note that not all matrices have an inverse. A matrix is invertible if and only if its determinant is non-zero.

Attributes

- It is unique.
- The inverse of A^{-1} is A itself.

Claim. Suppose A is invertible. Then its inverse is unique.

Proof. Suppose B and C are both inverses of A. Then $B = BI = B(AC) = (BA)C = IC = C$.

Remark. *left inverse = right inverse = inverse*

Claim. The inverse of A^{-1} is A itself.

Proof. $AA^{-1} = I$ and $A^{-1}A = I$.

Claim. If A is invertible, then the one and only solution to $A\underline{x} = \underline{b}$ is $\underline{x} = A^{-1}\underline{b}$.

Proof. $A\underline{x} = \underline{b} \Rightarrow A^{-1}A\underline{x} = A^{-1}\underline{b} \Rightarrow \underline{x} = A^{-1}\underline{b}$.

Claim. Suppose there is a nonzero solution $\underline{x}$ to $A\underline{x} = \underline{0}$ (homogeneous equation). Then A is not invertible.

Proof. If A is invertible, then A^{-1} exists. Then $A^{-1}A\underline{x} = A^{-1}\underline{0} \Rightarrow \underline{x} = \underline{0}$.

Claim. A diagonal matrix has an inverse provided no diagonal entries are zero.

Proof.
If

$$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

then

$$A^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Claim. If A and B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof.

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = I$
$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$

Remark. $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Gauss-Jordan Elimination

Given A, we want to find its inverse A^{-1}. $AA^{-1} = I$

$$A \begin{bmatrix} \underline{col_1} & \underline{col_3} & \underline{col_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{e_1} & \underline{e_3} & \underline{e_3} \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc:ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{\dfrac{3}{2}} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \boxed{\dfrac{4}{3}} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

Until here is Gauss

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \boxed{\dfrac{4}{3}} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \boxed{\dfrac{3}{2}} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \boxed{\dfrac{4}{3}} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

$\Rightarrow$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

Until here is Jordan

$$col_1 = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}, col_2 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}, col_3 = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{3}{4} \end{bmatrix}, A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Claim. A matrix is invertible if and only if (iff.) it is nonsingular.

Elimination = Factorization: $A = LU$

TODO

Claim. If $A = L_1 D_1 U_1$ and $A = L_2 D_2 U_2$, where the L's are lower-triangular with unit diagonal, the U's are upper-triangular with unit diagonal, and the D's are diagonal matrices with no zeros on the diagonal, then $L_1 = L_2$, $D_1 = D_2$, and $U_1 = U_2$.

One Square System = Two Triangular Systems

Benifit: ????

$A\underline{x} = \underline{b}$. Suppose elimination requires no row exchanges.

$A = LU \Rightarrow LU\underline{x} = \underline{b} \Rightarrow U\underline{x} = L^{-1}\underline{b} = \underline{c}$

We have $U\underline{x}$ where $L\underline{c} = \underline{b}$.

1. Factor $A = LU$ by Gaussian eliminatio $\underline{c} = L^{-1}\underline{b}$.
2. Solve $\underline{c}$ from $L\underline{c} = \underline{b}$ (forward elimination) and then solve $U\underline{x} = \underline{c}$ (backward elimination)

E.g.

$$\begin{bmatrix} 2 & -1 & 0 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$
$$A\underline{x} = \underline{b}$$

$\Rightarrow$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$A = LU$$

$\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$
$$L\underline{c} = \underline{b}$$

$$\therefore \underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$
$$U\underline{x} = \underline{c}$$

$$\therefore \underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Complexity of Elimination

1.

Solve $A\underline{x} = \underline{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = LU$$

Gaussian Elimination:

1st stage: $n(n-1) \approx n^2$ (# multiplications and additions)
2nd stage: $(n-1)^2$
3rd stage: $(n-2)^2$
$\vdots$
$n^2 + (n-1)^2 + \dots + 1^2 = \frac{1}{3}n(n+\frac{1}{2})(n+1) \approx \frac{n^3}{3}$

Solve $L\underline{c} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ val & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ val & val & \dots & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$(n-1) + (n-2) + \dots + 2 + 1 = \frac{n(n-1)}{2} \approx \frac{n^2}{2}$

Remark. *b_1 needs to be multiplied $(n-1)$ times, b_2 needs to be multiplied $(n-2)$ times*

Solve $U\underline{x} = \underline{c}$

$$\begin{bmatrix} pivot_1 & val & \cdots & val \\ 0 & p_2 & \cdots & val \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \cdots & p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$1 + 2 + \dots + n = \frac{n(n-1)}{2} \approx \frac{n^2}{2}$

Total: $\# = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n^2}{2} \approx \frac{n^3}{3}$

2.

Compute A^{-1}

Solve $A \begin{bmatrix} \underline{x_1} & \underline{x_2} & \cdots & \underline{x_n} \end{bmatrix} = \begin{bmatrix} \underline{e_1} & \underline{e_2} & \cdots & \underline{e_n} \end{bmatrix}$

where $\underline{e_i}$ is the ith column of the identity matrix.

According Elimination $A = LU$, solve

$$L\underline{c_i} = \underline{e_i} = \begin{bmatrix} 0 \\ \vdots \\ 1(ith) \\ \vdots \\ 0 \end{bmatrix}$$

$\because$ The non-zero variables are only from c_i to c_n.
$\therefore$ Need to solve (n - i + 1) variables $\Rightarrow$ complexity # is $\frac{(n-i+1)^2}{2}$

Since $1 \leq i \leq n$, $\# = \frac{n^2}{2} + \frac{(n-1)^2}{2} + \cdots + \frac{1^2}{2} \approx \frac{n^3}{6}$

And solve $U\underline{x_i} = \underline{c_i}\# = \frac{n^2}{2}$

Since $1 \leq i \leq n$, $\# = n \times \frac{n^2}{2} = \frac{n^3}{2}$

Total: $\# = \frac{n^3}{3} + \frac{n^3}{6} + \frac{n^3}{2} = n^3$

Compared with e.g. # for A^2: $n^2 \cdot n = n^3$

Conclusion: Use Gaussian Elimination is as good as computing A^2 (not complicated).

Transpose and Permutation Matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \Rightarrow A^T_{ij} = A_{ji}$$

Claim. $(A + B)^T = A^T + B^T$

Claim. $(AB)^T = B^T A^T$

Proof. TODO

Remark. $(ABC)^T = C^T B^T A^T$

Claim. $(A^-1)^T = (A^T)^{-1}$

Proof. TODO

Claim. $(A^-1)^T = (A^T)^{-1}$

Proof. TODO

Def. An $n \times n$ matrix A is *symmetric* if $A^T = A$.

Remark. $A_{ij} = A_{ji}$*, if* A *is symmetric*

$A = A^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ $D = D^T = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

Claim. Given any matrix $R(n \times m)$, $R^T R$ and RR^T are symmetric.

Proof. $(R^T R)^T = R^T (R^T)^T = R^T R$

Claim. If a symmetric matrix if factored into LDU with no row exchanges, then $U = L^T$.

Proof. $A = LDU \Rightarrow A^T = U^T D^T L^T = U^T D L^T$. Since A is symmetric $A = LDU = A^T = U^T D L^T$. Recall that this Factorization is unique $\Rightarrow U = L^T$

Permutation Matrices

A Permutation matrix has the rows of the identity I in any order. There are *6* 3×3 Permutation matrices.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{32}P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} P_{21}P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\therefore$ There are $n!$ Permutation matrices of order n.

Claim. If P is a Permutation matrix, then $P^T = P^{-1}$

Proof. TODO

Recall if no row exchanges are required, then $A = LU$. If row exchanges are needed, we then have $(\cdots P \cdots E)A = U \Rightarrow A = (\cdots E^{-1} \cdots P^{-1})U$

If row exchanges are needed during elimination, we can do them in advance. The product PA will put the rows in the right order so that no row exchanges are needed for PA. Hence

$$PA = LU$$

We can also hold row exchanges until after elemination, we then have $A = LPU$