Vector Spaces and Subspaces

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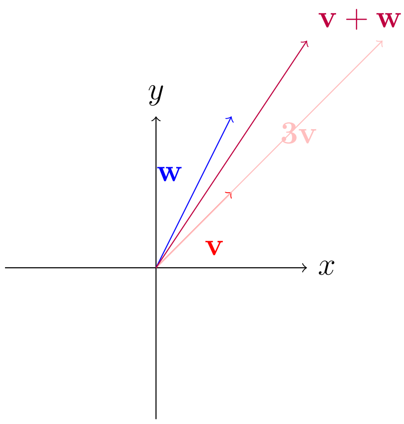
$\mathbb{R}^n$ = all (column) vectors with n (real) components.
$= \{(v_1, v_2, \cdots, v_n) : v_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \in \mathbb{R}^2, \quad (1,1,0,1,1) \in \mathbb{R}^5$$



Vector Space V

V: a set of vectors

1. Two operations:
   - vector addition: $\underline{v}, \underline{w} \in V \Rightarrow \underline{v} + \underline{w} \in V$
   - scalar multiplication: $c\underline{v} \in V$
2. Eight rules:
   (a) $\underline{v} + \underline{w} = \underline{w} + \underline{v}$ (commutative)
   (b) $(\underline{v} + \underline{w}) + \underline{z} = \underline{v} + (\underline{w} + \underline{z})$ (associative)
   (c) There is a unique "zero vector" $\underline{0}$ such that $\underline{v} + \underline{0} = \underline{v}$ for all $\underline{v} \in V$
   (d) For each $\underline{v}$, there is a unique vector $-\underline{v}$ such that $\underline{v} + (-\underline{v}) = \underline{0}$
   (e) $1 \times \underline{v} = \underline{v}$

(f) $(c_1c_2)\underline{v} = c_1(c_2\underline{v})$

(g) $c(\underline{v} + \underline{w}) = c\underline{v} + c\underline{w}$

(h) $(c_1 + c_2)\underline{v} = c_1\underline{v} + c_2\underline{v}$

$\Rightarrow 0 \times \underline{v} = \underline{0}$ (not 0)
$\Rightarrow (-1)\underline{v} = -\underline{v}$

Example:

- $\mathbb{R}^n$ is a vector space
- $M = \{\text{all real } 2 \times 2 \text{ matrices}\}$ is a vector space
- $F = \{\text{all real functions } f(x)\ \}$ is a vector space
- $z = \{\underline{0}\}$ is a vector space

Subspaces

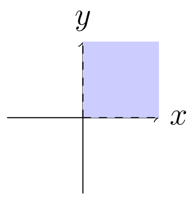
Def. A subset W of a vector space V is a <u>subspaces</u> if W itself is a vector space.

Claim Every subspace contains the zero vector.

Proof TODO

Example:

1.



$U = \{(x, y) : x \geq 0, y \geq 0\}$, is U a subspace?

No, since $-1(1, 0) = (-1, 0) \notin U$ even of $(1, 0) \in U$.

2.

$M = \{\text{all real } 2 \times 2 \text{ matrices}\}$
$\{U = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a,\ b,\ d \in \mathbb{R}\}$

$A, B \in U$, $A + B \in U$ and $cA \in U$

$\therefore U$ is a subspace of M

Column Space

Def. The column space $C(A)$ of a matrix A consists of all linear combinations of the columns of A.

Remark. *C(A): C of A*

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$C(A) = \ \{c_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} : c_1, c_2 \in \mathbb{R}\}$$

$$= \{A\underline{c} : \underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2\}(A\underline{x} = \underline{b})$$

The set of all $A\underline{x}$ for all x is called the column space.

$$\iff c_1[\underline{a_1}] + c_2[\underline{a_2}] + \cdots + c_n[\underline{a_n}] = \underline{b}$$

$\therefore$ The system $A\underline{x} = \underline{b}$ is solvable iff $\underline{b} \in C(A)$

Example:

What are the column spaces of 1. I, 2. $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, 3. $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$?

1.

$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \therefore C(I) = \mathbb{R}^2$

2.

$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2 \Rightarrow x_{real} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore C(A) = \{x \begin{bmatrix} 1 \\ 2 \end{bmatrix} : x \in \mathbb{R}\}$

3.

$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = (x_1 + 2x_2(= x_4)) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is always solvable for any b_1, b_2

$\because \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$ is upper triangular matrix (b_1, b_2 must be found) $\therefore C(B) = \mathbb{R}^2$

$\Rightarrow$ All of them are subspaces of $\mathbb{R}^2$

Claim If A is an $m \times n$ real matrix, then $C(A)$ is a subspace of $\mathbb{R}^m$

Proof. TODO

$S =$ the set of vectors in a vector space V (probably not a subspace)
$SS =$ the set of all linear combinations of vectors in S

We call SS the "span" of S.

Then SS is a subspace of V, called the subspace "spanned" by S.

E.g.

$S =$ the set of columns of $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$SS =$ the column space of $A = C(A)$

Null Space of A

$$N(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$$

Remark. *related to the "rank"*

Claim If A is $m \times n$, then $N(A)$ is a subspace of $\mathbb{R}^n$.

Proof TODO

Example: $C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} = [A \quad 2A]$ (Two equations in four unknowns)

TODO

$$N(C) = \{\underline{x} : \underline{x} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_3, x_4 \in \mathbb{R}\}$$

Remark. *Reduced Row Echelon form (RRE form) 1. Produce 0 above/below pivots 2. Produce 1 in pivots*

Example: $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$

TODO

$$N(A) = \{\underline{x} : \underline{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, x_2, x_4 \in \mathbb{R}\}$$

Suppose A is $m \times n$. If there are r pivots ($r \leq m, n$), there are $(n - r)$ free variables. And there are $(n - r)$ special solutions. $N(A)$ consistss of all the linear combinations of these $(n - r)$ special solutions.

$$N(A) = \text{the subspace spanned by these } (n - r) \text{ special solutions.}$$

Def. The rank of a matrix A is the number of pivots.

Remark. *All free columns are linear combinations of the pivot columns, and special solutions describe these combinations.*

Complete Solution to $A\underline{x} = \underline{b}$

$$\underline{x} = \underline{x}_{particular} + \underline{x}_{nullspace}$$

Claim If $A\underline{x} = \underline{b}$, then the complete solution is $\underline{x} = \underline{x}_{particular} + \underline{x}_{nullspace}$, where $\underline{x}_{particular}$ is a particular solution to $A\underline{x} = \underline{b}$, and $\underline{x}_{nullspace}$ is a general solution to $A\underline{x} = \underline{0}$.

Proof

1. If $\underline{x} = \underline{x}_{particular} + \underline{x}_{nullspace}$, then $A\underline{x} = A(\underline{x}_{particular} + \underline{x}_{nullspace}) = A\underline{x}_{particular} + A\underline{x}_{nullspace} = \underline{b} + \underline{0} = \underline{b}$

$\therefore$ $\underline{x}$ is a solution to $A\underline{x} = \underline{b}$

2. If $\underline{x}$ is a solution to $A\underline{x} = \underline{b}$, then $A(\underline{x} - \underline{x}_p) = A\underline{x} - A\underline{x}_p = \underline{b} - \underline{b} = \underline{0}$

$\therefore$ $\underline{x} - \underline{x}_p = \underline{x}_n$ is a solution to $A\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{x}_p + \underline{x}_n$