

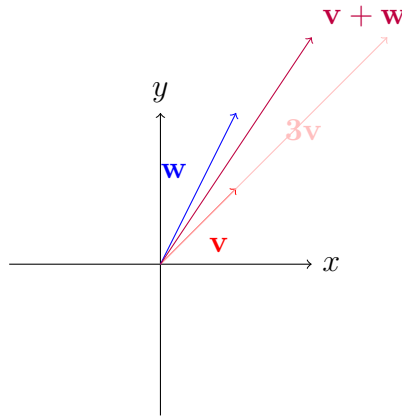
# Vector Spaces and Subspaces

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$\mathbb{R}^n$  = all (column) vectors with n (real) components.  
 $= \{(v_1, v_2, \dots, v_n) : v_i \in \mathbb{R}, i = 1, 2, \dots, n\}$

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \in \mathbb{R}^2, \quad (1, 1, 0, 1, 1) \in \mathbb{R}^5$$



## Vector Space $V$

$V$ : a set of vectors

1. Two operations:

- vector addition:  $\underline{v}, \underline{w} \in V \Rightarrow \underline{v} + \underline{w} \in V$
- scalar multiplication:  $c\underline{v} \in V$

2. Eight rules:

- (a)  $\underline{v} + \underline{w} = \underline{w} + \underline{v}$  (commutative)
- (b)  $(\underline{v} + \underline{w}) + \underline{z} = \underline{v} + (\underline{w} + \underline{z})$  (associative)
- (c) There is a unique "zero vector"  $\underline{0}$  such that  $\underline{v} + \underline{0} = \underline{v}$  for all  $\underline{v} \in V$
- (d) For each  $\underline{v}$ , there is a unique vector  $-\underline{v}$  such that  $\underline{v} + (-\underline{v}) = \underline{0}$
- (e)  $1 \times \underline{v} = \underline{v}$

$$(f) \quad (c_1 c_2) \underline{v} = c_1 (c_2 \underline{v})$$

$$(g) \quad c(\underline{v} + \underline{w}) = c\underline{v} + c\underline{w}$$

$$(h) \quad (c_1 + c_2) \underline{v} = c_1 \underline{v} + c_2 \underline{v}$$

$$\Rightarrow 0 \times \underline{v} = \underline{0} \text{ (not } 0)$$

$$\Rightarrow (-1)\underline{v} = -\underline{v}$$

Example:

- $\mathbb{R}^n$  is a vector space
- $M = \{\text{all real } 2 \times 2 \text{ matrices}\}$  is a vector space
- $F = \{\text{all real functions } f(x) \}$  is a vector space
- $z = \{\underline{0}\}$  is a vector space

## Subspaces

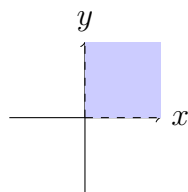
**Def.** A subset  $W$  of a vector space  $V$  is a subspace if  $W$  itself is a vector space.

**Claim** Every subspace contains the zero vector.

**Proof** TODO

Example:

1.



$U = \{(x, y) : x \geq 0, y \geq 0\}$ , is  $U$  a subspace?

No, since  $-1(1, 0) = (-1, 0) \notin U$  even if  $(1, 0) \in U$ .

2.

$M = \{\text{all real } 2 \times 2 \text{ matrices}\}$

$\{U = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in \mathbb{R}\}$

$A, B \in U, A + B \in U$  and  $cA \in U$

$\therefore U$  is a subspace of  $M$

## Column Space

**Def.** The column space  $C(A)$  of a matrix  $A$  consists of all linear combinations of the columns of  $A$ .

**Remark.**  $C(A)$ :  $C$  of  $A$

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$C(A) = \left\{ c_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

$$= \{ A\underline{c} : \underline{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2 \} (A\underline{x} = \underline{b})$$

The set of all  $A\underline{x}$  for all  $x$  is called the column space.

$$\iff c_1[a_1] + c_2[a_2] + \dots + c_n[a_n] = \underline{b}$$

$\therefore$  The system  $A\underline{x} = \underline{b}$  is solvable iff  $\underline{b} \in C(A)$

Example:

What are the column spaces of 1.  $I$ , 2.  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ , 3.  $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ ?

1.

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \therefore C(I) = \mathbb{R}^2$$

2.

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (x_1 + 2x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2 \Rightarrow x_{real} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore C(A) = \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} : x \in \mathbb{R} \right\}$$

3.

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = (x_1 + 2x_2 (= x_4)) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \text{ is always solvable for any } b_1, b_2$$

$$\therefore \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \text{ is upper triangular matrix } (b_1, b_2 \text{ must be found}) \therefore C(B) = \mathbb{R}^2$$

$\Rightarrow$  All of them are subspaces of  $\mathbb{R}^2$

**Claim** If  $A$  is an  $m \times n$  real matrix, then  $C(A)$  is a subspace of  $\mathbb{R}^m$

**Proof.** TODO

$S$  = the set of vectors in a vector space  $V$  (probably not a subspace)

$SS$  = the set of all linear combinations of vectors in  $S$

We call  $SS$  the "span" of  $S$ .

Then  $SS$  is a subspace of  $V$ , called the subspace “spanned” by  $S$ .

E.g.

$S$  = the set of columns of  $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$SS$  = the column space of  $A = C(A)$

## Null Space of $A$

$$N(A) = \{\underline{x} : A\underline{x} = \underline{0}\}$$

**Remark.** *related to the "rank"*

**Claim** If  $A$  is  $m \times n$ , then  $N(A)$  is a subspace of  $\mathbb{R}^n$ .

**Proof** TODO

Example:  $C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} = [A \quad 2A]$  (Two equations in four unknowns)

TODO

$$N(C) = \{\underline{x} : \underline{x} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, x_3, x_4 \in \mathbb{R}\}$$

**Remark.** *Reduced Row Echelon form (RRE form) 1. Produce 0 above/below pivots 2. Produce 1 in pivots*

Example:  $A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$

TODO

$$N(A) = \{\underline{x} : \underline{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, x_2, x_4 \in \mathbb{R}\}$$

Suppose  $A$  is  $m \times n$ . If there are  $r$  pivots ( $r \leq m, n$ ), there are  $(n - r)$  free variables. And there are  $(n - r)$  special solutions.  $N(A)$  consists of all the linear combinations of these  $(n - r)$  special solutions.

$N(A)$  = the subspace spanned by these  $(n - r)$  special solutions.

**Def.** The **rank** of a matrix  $A$  is the number of pivots.

**Remark.** *All free columns are linear combinations of the pivot columns, and special solutions describe these combinations.*

## Complete Solution to $A\underline{x} = \underline{b}$

$$\underline{x} = \underline{x}_{\text{particular}} + \underline{x}_{\text{nullspace}}$$

**Claim** If  $A\underline{x} = \underline{b}$ , then the complete solution is  $\underline{x} = \underline{x}_{\text{particular}} + \underline{x}_{\text{nullspace}}$ , where  $\underline{x}_{\text{particular}}$  is a particular solution to  $A\underline{x} = \underline{b}$ , and  $\underline{x}_{\text{nullspace}}$  is a general solution to  $A\underline{x} = \underline{0}$ .

### Proof

1. If  $\underline{x} = \underline{x}_{\text{particular}} + \underline{x}_{\text{nullspace}}$ , then  $A\underline{x} = A(\underline{x}_{\text{particular}} + \underline{x}_{\text{nullspace}}) = A\underline{x}_{\text{particular}} + A\underline{x}_{\text{nullspace}} = \underline{b} + \underline{0} = \underline{b}$

$\therefore \underline{x}$  is a solution to  $A\underline{x} = \underline{b}$

2. If  $\underline{x}$  is a solution to  $A\underline{x} = \underline{b}$ , then  $A(\underline{x} - \underline{x}_p) = A\underline{x} - A\underline{x}_p = \underline{b} - \underline{b} = \underline{0}$

$\therefore \underline{x} - \underline{x}_p = \underline{x}_n$  is a solution to  $A\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{x}_p + \underline{x}_n$

**Q.** Suppose  $A$  is a square invertible matrix ( $m = n = \text{pivots}$ ). What are  $\underline{x}_p$  and  $\underline{x}_n$ ?  
**A.**

The only particular solution to  $A\underline{x} = \underline{b}$  is  $\underline{x}_p = A^{-1}\underline{b}$ . (because it is invertible)

The only solution to  $A\underline{x} = \underline{0}$  is  $\underline{x}_n = \underline{0}$  (Inverse matrix's attribute).

### Example: Full Column Rank

TODO

In general, if  $r = n$  ( $m \geq n$ ) ( $A$  has full column rank),

$$A = [\text{Matrix}]_{m \times n} \Rightarrow R = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Therefore, if  $A$  has full column rank ( $r = n$ ), then

1. All columns of  $A$  are pivot columns
2. There are no free variables or no special solutions
3.  $N(a) = \underline{0}$
4. If  $A\underline{x} = \underline{b}$  is solvable, then the solution is unique.

### Example: Full Row Rank

TODO

In general, if  $r = m$  ( $m \leq n$ ) ( $A$  has full row rank),

$$A = [\text{Matrix}]_{m \times n} \Rightarrow R = [I \quad F]$$

Therefore, if  $A$  has full row rank ( $r = m$ ), then

1. All rows of  $A$  are pivot columns and  $R$  has no zero rows
2.  $A\underline{x} = \underline{b}$  is solvable for all  $\underline{b}$

3.  $C(A) = \mathbb{R}^m$  ( $C$ : column space)
4. There are  $n - r = n - m$  special solutions in  $N(A)$

## Summary

Case 1:  $r = m = n$  ( $A$  is invertible)

$R = I \Rightarrow A\underline{x} = \underline{b}$  has 1 solution:  $\underline{x} = A^{-1}\underline{b}$

Case 2:  $r = m < n$  (full row rank) ( $A$ : short and wide)

$R = [I \quad F] \Rightarrow A\underline{x} = \underline{b}$  is solvable for all  $\underline{b}$  and solutions are  $\infty$ .

Case 3:  $r = n < m$  (full column rank) ( $A$ : tall and thin)

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

$A\underline{x} = \underline{b}$  has 0 or 1 solution.

Case 4:  $r < m, n$  (neither full row rank nor full column rank)

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$A\underline{x} = \underline{b}$  has 0 or  $\infty$  solutions.

## Independence, Basis, and Dimension

### (Linearly) Independence

**Def.** If  $x_1\underline{v}_1 + x_2\underline{v}_2 + \cdots + x_n\underline{v}_n = \underline{0}$  and only happens when  $x_1 = x_2 = \cdots = x_n = 0$ , then  $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$  are linearly independent. Otherwise, they are linearly dependent.

### Examples

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

TODO

They are linearly independent.

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

TODO

$$\Rightarrow x_1 = x_2 = 0$$

$\therefore$  They are linearly independent.

**Remark.** *These special solutions we found in the nullspace of a matrix are independent*

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$$

Suppose

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Calculate the nullspace of the matrix)

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = R \therefore \underline{x} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

We can have

$$-3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$  They are linearly dependent.

In general, the columns of  $A$  are linearly independent exactly when  $rank = n$  (aka. full column rank).

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$  There are  $n$  pivots and no free variables. Hence  $N(A) = \underline{0}$

**Claim** Any set of  $n$  vectors in  $\mathbb{R}^m$  must be linearly dependent if  $n > m$ .

**Proof** TODO

**Def.** A set of vectors ‘spans’ a vector space if there linear combinations fill the space.

E.g. The column of a matrix span its column space.

$$\text{E.g. } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ span } \mathbb{R}^2 \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Def.** The **row space** of an  $m \times n$  matrix  $A$  is the subspace of  $\mathbb{R}^n$  spanned by the rows of  $A$ .

The row space of  $A$  is  $C(A^T)$

## Basis

**Def.** A **basis** for a vector space is a sequence of vectors satisfying two properties:

1. The basis vectors are linearly independent
2. The basis vectors span the vector space

### Examples

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  constitute a basis for  $\mathbb{R}^2$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  Let  $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow 3$  pivot (no free variables)  $\Rightarrow$  linearly independent.

And  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$  is always solvable for any  $b_1, b_2, b_3 \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  are a basis for  $\mathbb{R}^3$

**Claim** Any three linearly independent vectors in  $\mathbb{R}^3$  form a basis for  $\mathbb{R}^3$ .

**Proof** Assume  $\underline{v}_1, \underline{v}_2, \underline{v}_3$  are linearly independent.  $\Rightarrow A = [\underline{v}_1 \ \underline{v}_2 \ \underline{v}_3] \Rightarrow \text{rank} = 3 \Rightarrow A$  is invertible.

$A\underline{x} = \underline{b}$  is solvable for every  $\underline{b}$ .  $\therefore C(A) = \mathbb{R}^3 \Rightarrow \underline{v}_1, \underline{v}_2, \underline{v}_3$  are a basis for  $\mathbb{R}^3$

**Claim** Any  $n$  linearly independent vectors in  $\mathbb{R}^n$  are a basis for  $\mathbb{R}^n$ .

**Claim** The vectors  $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$  are a basis for  $\mathbb{R}^n$  exactly when they are the columns of an invertible matrix.

**Remark.**  $\mathbb{R}^n$  has infinitely many different bases.

**Claim** There is **one and only one** way to write any vector in a vector space as a linear combination of the basis vectors.

**Proof** TODO

### Example: Search for Basis

$$A = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \Rightarrow R = \begin{bmatrix} \boxed{1} & 3 & 0 & 2 & -1 \\ 0 & 0 & \boxed{1} & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Recall** free columns of  $R/A$  are linear combinations of pivot columns of  $R/A$ .

$\therefore$  pivot columns of  $R/A$  are linearly independent ( $A\underline{x} = \underline{0} \iff R\underline{x} = \underline{0}$ ),  $\therefore$  Pivot columns of  $R/A$  are a basis for  $C(R)/C(A)$ .

$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$  are a basis for  $C(A)$ , and  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  are a basis for  $C(R)$ .



**Remark.**  $C(A) \neq C(R)$