$y = x$ linear $\to$ solution $y = x$

v.s.

$y' \cdot y = x$ non linear $\to$ solution $y = \sqrt{x^2 + C}$

First order de

① $\frac{dy}{dx} = 2x \xrightarrow{\text{anti derivative}} y = x^2$ 原式

不可反推 ② $\frac{dy}{dx} = 2x$ (一) $\underbrace{dy}_{\text{only } y} = \underbrace{2x\,dx}_{\text{only } x}$ (分離 x, y) $\overset{\text{integrate}}{\Rightarrow} y = x^2 + c$

substitution

simplify — Separable equation (variables) — direct

t1: $y' = g(x)h(x)$

$\frac{dy}{h(y)} = g(x)\,dx$

$\boxed{\int \frac{dy}{h(y)} = \int g(x)\,dx + C}$

e.g.

$y' = \frac{y}{1+x}$

$\Rightarrow \frac{y'}{y} = (1+x)^{-1} \Rightarrow \frac{dy}{y} = \frac{1}{1+x}dx \overset{\text{integra}}{\Rightarrow} \int = \int$

$\Rightarrow \ln|y| = \ln|1+x| + C$

$y = e^{C}(1+x)$ (general solution)

[$y = 0$ trivial solution / particular sol

↓

it's not coming from the general solution and also called singular solution;

發散

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t2: $M_1(x)M_2(y)\,dx + N_1(x)N_2(y)\,dy = 0$

$\Rightarrow \frac{M_1(x)}{N_1(x)}dx + \frac{N_2(y)}{M_2(y)}d(y) = 0$

$\Rightarrow \int \bigcirc + \int \bigcirc = C$

$(x-1)(y+3)\,dx = (x+4)(y-2)\,dy$

$\Rightarrow \frac{x-1}{x+4}dx = \frac{y-2}{y+3}dy \overset{\text{integratio}}{\Rightarrow} x - 5\ln|x+4| = y - 5\ln|y+3| + c$

$1 - \frac{5}{x+4}$ dx $\quad 1 - \frac{5}{y+3}$

$\frac{y}{x}$ or $\frac{x}{y}$ ① 齊次 (order) homogeneous

e.g.

$(x+y)\,dx - x\,dy = 0$ 齊一次

not easy to seperate

$\Rightarrow M(\frac{x}{y})dx + N(\frac{y}{x})dy = 0$

$\frac{y}{x} = u \Rightarrow dy = u\,dx + x\,du$ ($y = xu$)

$M(u)dx + N(u)(u\,dx + x\,du) = 0$

E.g.

1 order / ordinary $\quad y' = \frac{x-y}{x+y} = \frac{dy}{dx} \Rightarrow (x-y)dx = (x+y)dy$

同除 x, $\frac{y}{x} = u$ $\quad \Rightarrow (1 - \frac{y}{x})dx = (1 + \frac{y}{x})dy$

$u = \frac{dy}{dx} \Rightarrow dy = u\,dx + x\,du$

$(1-u)dx = (1+u)dy$

$\Rightarrow \frac{1-u}{1+u} = \frac{u\,dx + x\,du}{dx} = u + x\frac{du}{dx}$

$\Rightarrow \frac{1-2u-u^2}{1+u} = x\frac{du}{dx} \Rightarrow \int \frac{dx}{x} = \int \frac{u+1}{u^2+2u-1}\,dx$

$\Rightarrow -\ln|x| + c = \frac{1}{2}\ln|u^2+2u-1|$

$\Rightarrow x^2(u^2+2u-1) = C \Rightarrow$ return $u = \frac{y}{x}$

② not homogeneous equation ~~type2~~ → not homogeneous

$(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$ (M, N) solve remove $\Rightarrow c_1, c_2$

if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow$ has 交點 $\Rightarrow$ $x-\alpha=u$, $y-\beta=v$ $\Rightarrow (a_1u+b_1v)du+(a_2u+b_2v)dv=0$

$\Rightarrow$ 用一階齊次 ODE

e.g. $(2x-5y+3)dx-(2x+4y-6)dy=0$ remove

$\begin{cases} 2x-5y+3=0 \\ 2x+4y-6=0 \end{cases} \Rightarrow x=1, y=1$ ~~x+1=u~~ ~~y+1=v~~ $x-1=u$, $y-1=v$ $du=dx$, $dv=dy$

→ 新坐標軸 (1, 1)

$\begin{cases} 2u-5v=0 \\ 2u+4v=0 \end{cases} \Rightarrow (2u-5v)du-(2u+4v)dv=0$ (homogeneous equation)

if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=m \neq \frac{c_1}{c_2} \Rightarrow a_2x+b_2y=t$, $a_1x+b_1y=mt$ (t 取代 y)

↓ maybe total differential

$dy=\frac{dt-a_2dx}{b_2}$

$\frac{a_1}{a_2}=\frac{b_1}{b_2}$

e.g. $(2x-4y-3)dy+(x-2y+3)dx=0 \Rightarrow$ $t=x-2y \Rightarrow dx=dt+2dy$

$(2t-3)dy+(t+3)(dt+2dy)=0$

$\Rightarrow \frac{2t-3}{t+3}dy=-dt-2dy \Rightarrow \frac{4t+3}{t+3}dy=-dt \Rightarrow \ldots$

③ $y'=f(ax+by+c)$

$\Rightarrow t=ax+by+c \Rightarrow dy=\frac{dt-adx}{b} \Rightarrow \frac{dy}{dx}=\frac{\frac{dt-adx}{b}}{dx}=f(t) \Rightarrow \int\frac{dt}{bf(t)+a}=\int dx+C$

E.g. Find $y'=\tan^2(x+y)$ $\Rightarrow y'=\frac{dt-dx}{dx}=\tan^2 t \Rightarrow \frac{dt}{dx}=1+\tan^2 t = \sec^2 t$

$t=x+y \Rightarrow dy=dt-dx$

$\Rightarrow \frac{1}{\sec^2 t}dt=dx \Rightarrow \cos^2 t\,dt=dx$

$\Rightarrow \frac{1+\cos 2t}{2}=dx \Rightarrow \int\ \int +C \Rightarrow \frac{t}{2}+\frac{1}{4}\sin 2t=x+C \rightarrow \frac{x+y}{2}+\frac{1}{4}\sin 2(x+y)=x+C$

exact equation / modified equation

5 V — $\boxed{\text{Exact equation}}$ 全解 $\phi(x,y)$

on types

$\int d(3x^2y)=\phi\ 3x^2y$

積分因子 integrating factor $I(x,y)$ 3 types

設 1st-order ODE 為 $\boxed{M(x,y)dx+N(x,y)dy=0}$ 不懂

若存在 $\phi(x,y)$ satisfy $M(x,y)dx+N(x,y)dy=d\phi=0$

→ 正合 E.g. $6xydx+3x^2dy=0$

$=d(3x^2y)=0 \Rightarrow 3x^2y=C$ (total differential)