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Geometric Control of Multiple Quadrotors Transporting a Rigid-body Load

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***Abstract*— We address the problem of cooperative transportation of a cable-suspended rigid-body payload by multiple quadrotors. We develop a coordinate-free dynamical model of the system by developing equations of motions directly on the unit sphere and the special orthogonal group. This model is used to design a geometric feedback control to track a reference trajectory for the load's pose (position and orientation), as well as the yaw angle of each quadrotor, and the orientation of each cable. Simulation results and formal proofs of the controller are presented to demonstrate the stability properties of the controller.**

I. Introduction

Aerial robotics has a wide range of civil and military applications with use in surveillance, sensor networks, and in education. In recent years, we have seen the technology mature to a point where we have quadrotors ranging in size from centimeters to meters with payloads up to a few kilograms. Moreover, nonlinear control systems for complex maneuvers of quadrotors have been studied, and aggressive maneuvers have also been demonstrated experimentally, [10]. Thus, quadrotors are naturally suitable for achieving aerial manipulation for the purposes of load carrying and transportation.

There is a broad spectrum of approaches in realizing aerial manipulation and transportation. At one end are aerial robots equipped with fixed grippers, where the payload is rigidly attached to the aerial robot through the gripper, and the same control technique for flying without a load is used. These robots are typically characterized by slow, quasi-static motions for hovering and picking up objects [9]. Moreover, carrying a heavy load so close to the body, increases the inertia of the system considerably and thereby makes the system's attitude response very sluggish, significantly degrading performance. Cooperative aerial manipulation using multiple aerial robots equipped with grippers for aerial transportation of loads has also been carried out, [12]. However, once again, the motions are slow, as the load inertia becomes even more significant.

An alternative is to suspend the load through a cable, thereby retaining the agility of the aerial vehicle while still achieving the task of transportation of the suspended load. Although this preserves the fast attitude response of the aerial robot, it introduces additional degrees of underactuation at the cable suspension point. Moreover, cooperative load transportation with multiple quadrotors is useful for manipulating large and heavy loads through constrained urban spaces where additional safety is required through redundancy. This is challenging due to the dynamic coupling between the quadrotors through the load. Contrast this to existing results on formation control of multi-agent systems which are not only dynamically decoupled, but are also not subject to switching dynamics and unilateral constraints.

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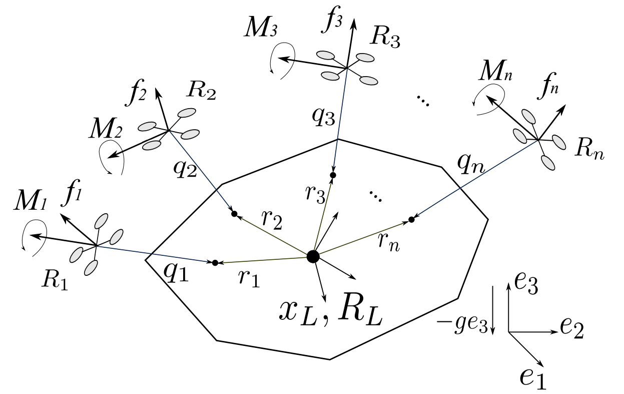


Fig. 1: Schematic diagram for the multiple quadrotor system with rigid-body load.

Feedback control for dynamic multi-agent aerial manipulation of cable-suspended loads presents many challenges. Some of these challenges arise from the dynamics of the system, which evolve on a complex nonlinear manifold, and which is also hybrid due to the unilateral tension forces. Other challenges are due to the many degrees-of-freedom in the model, their associated underactuation, and the many hybrid dynamical modes, each of which scale linearly with the number of aerial robots. Moreover, the dynamic coupling between the multiple aerial robots has to be explicitly addressed.

In our prior work, [16], we developed a coordinate-free dynamic model and a geometric controller for a single quadrotor with a cable suspended point-mass load, that offered almost-global stability for tracking any desired load trajectory. We also established that the single quadrotor with a cable-suspended load was a differentially flat hybrid system. Moreover, in [14], we established that the n multiple quadrotors case with a shared point-mass load is a differentially flat hybrid system for $n \geq 1$. We also established that the multiple quadrotors with a shared rigid-body load is a differentially flat hybrid system for $n > 3$. The problem of feedback control for multiple quadrotors cooperatively transporting a shared point-mass load was presented in [8]. In this paper, we will develop a coordinate-free dynamical model for the multiple quadrotors with a shared cable-suspended rigid-body load and also present a geometric control design for the case of $n \geq 6$ quadrotors for tracking the position, orientation of the rigid-body and the attitudes of the cables.

The rest of the paper is structured as follows. Section II presents the coordinate-free dynamic model for the n quadrotors with a shared rigid-body load, Section III presents a geometric control design for tracking a desired reference pose trajectory for the rigid-body load, and Section V provides some concluding remarks.

II. Dynamic Model of this System with Multiple Quadrotors

We develop a coordinate-free dynamical model for n quadrotors with a shared, cable-suspended, rigid-body load. We utilize rotation matrices to represent the load and quadrotor attitudes, and the two-sphere to represent the cable attitude. The system is depicted in Figure 1, with the various symbols defined in Table I.

In particular, $x_L \in \mathbb{R}^3$ is the position of center of mass of the load, $R_L \in SO(3)$ is the rotation matrix of the load from the body frame to the inertial frame, $R_i \in SO(3)$ is the rotation matrix of the i^{th} quadrotor from the body-fixed frame to the inertial frame, where i lies in the set $\{1, 2, \cdots, n\}$, $q_i \in S^2$ is the unit vector, in the inertial frame, from the i^{th} quadrotor to its attachment point on the load, and r_i is the vector from the center-of-mass (CoM) of the load to the attachment point of the i^{th} cable, expressed in the body-fixed frame of the load. Thus, the configuration space of the system is given by $Q = SE(3) \times (S^2 \times SO(3))^n$, and the position of the i^{th} quadrotor given by the following kinematic relation,

$$x_i = x_L + R_L r_i - L_i q_i.$$

The method of Lagrange is used to develop the dynamical equations of motion. The Lagrangian of the system, $\mathscr{L} : TQ \to \mathbb{R}$, is defined by $\mathscr{L} = \mathscr{T} - \mathscr{U}$, where the kinetic and potential energies are denoted as $\mathscr{T} : TQ \to \mathbb{R}$, and $\mathscr{U} : TQ \to \mathbb{R}$, respectively, and are given by,

$$\begin{aligned}\mathscr{T} &= \frac{1}{2}\{m_L||v_L||^2 + \Omega_L^T J_L \Omega_L + \\ &\quad \sum_{i=1}^{n}(m_i||v_L + R_L\hat{\Omega}_L r_i - L_i\dot{q}_i||^2 + \Omega_i^T J_i \Omega_i)\},\\ \mathscr{U} &= m_L x_L \cdot g e_3 + \sum_{i=1}^{n}(m_i(x_L + R_L r_i - L_i q_i)\cdot g e_3).\end{aligned}$$

The dynamics of the system then satisfy the Lagrange-d'Alembert principle,

$$\delta \int_0^\tau \mathscr{L}\, dt + \int_0^\tau \sum_{i=1}^{n}(\langle W_i^a, \hat{M}_i\rangle + W_i^b \cdot f_i R_i e_3)\, dt = 0, \quad (1)$$

where f_i is the thrust magnitude of the i^{th} quadrotor, M_i is the moment vector of the i^{th} quadrotor, $\langle\cdot,\cdot\rangle : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathbb{R}$ is the inner product on $\mathfrak{so}(3)$, the *hat map* $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is defined such that $\hat{x}y = x \times y, \forall x, y \in \mathbb{R}^3$, and $W_i^a = R_i^T \delta R_i$, $W_i^b = \delta x_i = \delta x_L + \delta R_L r_i - L_i \delta q_i$ are variational vector fields [13], with the infinitesimal variations satisfying [1], [6], [7],

$$\begin{aligned}\delta q_i &= \xi_i \times q_i,\ \xi_i \in \mathbb{R}^3 \text{ s.t. } \xi_i \cdot q_i = 0,\\ \delta \dot{q}_i &= \dot{\xi}_i \times q_i + \xi_i \times \dot{q}_i,\\ \delta R_i &= R_i \hat{\eta}_i,\ \eta_i \in \mathbb{R}^3,\\ \delta \Omega_i &= \hat{\Omega}_i \eta_i + \dot{\eta}_i\end{aligned}$$

with δq_i a variation on S^2, and δR_i a variation on $SO(3)$.

The equations of motion are then obtained by ensuring (1) is satisfied for all possible variations. These equations comprise of the load pose dynamics, cable attitude dynamics, and the attitude dynamics for each quadrotor:

Load pose dynamics:

$$\begin{aligned}\dot{x}_L &= v_L,\\ \dot{R}_L &= R_L \hat{\Omega}_L,\\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \dot{v}_L + g e_3 \\ \dot{\Omega}_L \end{bmatrix} &= \sum_{i=1}^{n}\begin{bmatrix} b_{i1} \\ b_{i2} \end{bmatrix}(q_i \cdot \frac{f_i}{m_i L_i} R_i e_3 - \omega_i \cdot \omega_i)\\ &\quad + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},\end{aligned}$$

TABLE I: NOMENCLATURE.

$n \in \mathbb{Z}^+$	The number of quadrotors involved
$g \in \mathbb{R}$	Acceleration due to gravity
$m_L \in \mathbb{R}$	Mass of the suspended load
$J_L \in \mathbb{R}^{3\times3}$	Body inertia matrix of the load
$m_i \in \mathbb{R}$	Mass of the i^{th} quadrotor
$J_i \in \mathbb{R}^{3\times3}$	Inertia matrix of the i^{th} quadrotor
$L_i \in \mathbb{R}^+$	Length of the cable between the i^{th} quadrotor and the load
$r_i \in \mathbb{R}^3$	Vector from the load's CoM to the i^{th} suspension point in the load's body frame
$x_L \in \mathbb{R}^3$	Position vector of the load's CoM
$R_L \in SO(3)$	Rotation matrix of the load from body-fixed frame to inertial frame
$\Omega_L \in \mathbb{R}^3$	Body angular velocity of the load
$q_i \in \mathbb{S}^2$	Directional vector of the i^{th} cable in inertial frame
$q_{ib} \in \mathbb{S}^2$	Directional vector of the i^{th} cable in body-fixed frame of the load
$\omega_i \in \mathbb{R}^3$	Angular velocity of q_i defined as $q_i \times \dot{q}_i$
$T_i \in \mathbb{R}$	Tension in the i^{th} cable
$R_i \in SO(3)$	Rotation matrix of the i^{th} quadrotor from body-fixed frame to inertial frame
$\Omega_i \in \mathbb{R}^3$	Body angular velocity of the i^{th} quadrotor
$f_i \in \mathbb{R}$	Thrust applied by the i^{th} quadrotor
$M_i \in \mathbb{R}^3$	Moment applied by the i^{th} quadrotor

where each term is defined as,

$$
\begin{aligned}
A_{11} &= m_L I_3 + \sum_{i=1}^{n} m_i q_i q_i^T, \quad A_{12} = -\sum_{i=1}^{n} m_i q_i q_{ib}^T \hat{r}_i, \\
A_{21} &= \sum_{i=1}^{n} m_i \hat{r}_i q_{ib} q_i^T = A_{12}^T, \\
A_{22} &= J_L + \sum_{i=1}^{n} m_i (\hat{r}_i q_{ib})(\hat{r}_i q_{ib})^T, \\
b_{i1} &= m_i L_i q_i, \quad b_{i2} = m_i L_i \hat{r}_i q_{ib}, \\
c_1 &= -\sum_{i=1}^{n} m_i (q_{ib}^T \hat{\Omega}_L^2 r_i) q_i, \\
c_2 &= -(\Omega_L \times J_L \Omega_L + \sum_{i=1}^{n} m_i (q_{ib}^T \hat{\Omega}_L^2 r_i) \hat{r}_i q_{ib}).
\end{aligned}
$$

Cable attitude dynamics for the i^{th} quadrotor:

$$
\begin{aligned}
\dot{q}_i &= \omega_i \times q_i, \\
\dot{\omega}_i &= q_i \times \Bigg(-\frac{f_i}{m_i L_i} R_i e_3 + \frac{1}{L_i}(\dot{v}_L + g e_3) \\
&\quad + \frac{1}{L_i} R_L (\hat{\Omega}_L^2 r_i + \dot{\hat{\Omega}}_L r_i) \Bigg).
\end{aligned}
$$

Attitude dynamics for the i^{th} quadrotor:

$$
\begin{aligned}
\dot{R}_i &= R_i \hat{\Omega}_i, \\
\dot{\Omega}_i &= J_i^{-1}(-\Omega_i \times J_i \Omega_i + M_i).
\end{aligned}
$$

We can convert these equations into the following compact form as

$$
\underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} \begin{bmatrix} \ddot{x}_L + g e_3 \\ \dot{\Omega}_L \end{bmatrix} = \underbrace{\begin{bmatrix} G_1 & G_2 & \cdots & G_n \end{bmatrix}}_{G} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \underbrace{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}}_{d},
$$

$$
\begin{aligned}
\dot{R}_L &= R_L \hat{\Omega}_L, \\
\dot{q}_i &= \omega_i \times q_i, \\
\dot{\omega}_i &= -\hat{q}_i u_i \\
&\quad + q_i \times \frac{1}{L_i}(\ddot{x}_L + g e_3 + R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_L) r_i), \\
\dot{R}_i &= R_i \hat{\Omega}_i, \\
\dot{\Omega}_i &= J_i^{-1}(-\Omega_i \times J_i \Omega_i + M_i)
\end{aligned}
$$

where

$$
G_i = \begin{bmatrix} m_i L_i q_i q_i^T \\ m_i L_i \hat{r}_i q_{ib} q_i^T \end{bmatrix}, \quad u_i = \frac{f_i}{m_i L_i} R_i e_3,
$$

and

$$
\begin{aligned}
d_1 &= -\sum_{i=1}^{n} m_i (q_{ib}^T \hat{\Omega}_L^2 r_i + L_i(\omega_i \cdot \omega_i)) q_i, \\
d_2 &= -\Omega_L \times J_L \Omega_L - \sum_{i=1}^{n} m_i (q_{ib}^T \hat{\Omega}_L^2 r_i + L_i(\omega_i \cdot \omega_i))(\hat{r}_i q_{ib}).
\end{aligned}
$$

Remark 1: The system considered has $5n+6$ degrees-of-freedom (DoFs) and $4n$ actuators, thereby having $n+6$ degrees-of-underactuation.

Remark 2: For future reference, the tension in the the i^{th} cable can be computed as $T_i = \; q_i \cdot [\ddot{x}_L + g e_3 + R_L(\dot{\hat{\Omega}}_L + \hat{\Omega}_L^2) r_i - f_i R_i e_3] + m_i L_i (\omega_i \cdot \omega_i)$, which arises from force analysis for each individual quadrotor.

III. Feedback Controller Design

Having derived the dynamical model of the complete system directly in terms of the rotation matrices representing the load and quadrotor attitudes, and using vectors on the unit-sphere to represent the cable attitudes, we now develop a geometric feedback controller to track a desired load position and attitude. In order to better illustrate how this controller works, we first provide a qualitative analysis of the system dynamics, followed by the control problem formulation and the controller development that uses the method of *singular perturbation.*

A. Qualitative Analysis of System Dynamics

From the dynamics of the load, we see that the matrix G is a *projection operator* that maps the external force being applied on the system by the quadrotors to a *wrench* that's applied to the rigid-body payload. Simplfying further, the load pose dynamics can be written as,

$$
A \begin{bmatrix} \ddot{x}_L + g e_3 \\ \dot{\Omega}_L \end{bmatrix} = G^* \begin{bmatrix} f_1 R_1 e_3 \cdot q_1 \\ f_2 R_2 e_3 \cdot q_2 \\ \vdots \\ f_n R_n e_3 \cdot q_n \end{bmatrix} + d,
$$

where the new matrix

$$
G^* = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \\ \hat{r}_1 q_{1b} & \hat{r}_2 q_{2b} & \cdots & \hat{r}_n q_{nb} \end{bmatrix}.
$$

Moreover, the matrix A on the left hand side can also be represented as,

$$
A = M_L + G^* M_Q G^{*T}
$$

where

$$
M_L = \begin{bmatrix} m_L I_3 & 0_{3\times3} \\ 0_{3\times3} & J_L \end{bmatrix}, \quad M_Q = diag[m_1, m_2, \cdots, m_n].
$$

Remark 3: This concise expression for A clearly illustrates its physical meaning. In particular, A is the *state-dependent generalized inertia* of the system, where the term M_L is the intrinsic inertia of the load and the term $G^* M_Q G^{*T}$ is the induced inertia due to the attitude change of each cable.

Next, similar to the case in [4], we decompose the control input into the following two parts,

$$
\begin{aligned}
u_i^{\parallel} &= (q_i \cdot u_i) q_i = q_i q_i^T u_i, \\
u_i^{\perp} &= (I_3 - q_i q_i^T) u_i = -\hat{q}_i^2 u_i,
\end{aligned}
$$

corresponding to parallel and perpendicular components of u_i with respect to the cable attitude, q_i. Further, noting that

$-\hat{q}_i u_i = -\hat{q}_i u_i^{\perp} = \hat{q}_i^3 u_i$, we have,

$$G\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = G\begin{bmatrix} u_1^{\parallel} \\ u_2^{\parallel} \\ \vdots \\ u_n^{\parallel} \end{bmatrix}.$$

***Remark** 4:* From the above equation, it is evident that the load pose dynamics is *only* affected by the net effect of all the $u_i^{\parallel}$. In particular, $u_i^{\perp}$ does not affect the load pose.

***Remark** 5:* Furthermore, note that both $u_i^{\parallel}$ and $u_i^{\perp}$ affect the attitude of the i^{th} cable, q_i. The difference lies in the fact that $u_i^{\perp}$ appears *explicitly* in the dynamics of q_i, where as $u_i^{\parallel}$ affects q_i only *implicitly* through the load attitude R_L.

B. Inertial Controller - Problem Formulation

Having gained some insight into the structure of the system's dynamics, we now formulate the control problem. Since the system has a high-degree of underactuation (see Remark 1), the controller is formulated as an output tracking controller with the goal being to track a set of outputs.

In particular, the control goal is that given a bounded and sufficiently smooth reference output for the system, defined as $r_d(t) = [x_{Ld}(t), R_{Ld}(t), q_{id}(t), \theta_{id}(t)]$, for $i = 1, 2, \cdots, n$, where $t \in [t_0, \infty]$ and θ_{id} is the desired yaw angle for each quadrotor, design a state feedback law

$$\begin{aligned}\Gamma : [t_0, \infty] \times TQ &\to (\mathbb{R}^4)^n, \\ (t, x) &\mapsto (f_1, M_1, f_2, M_2, \cdots, f_n, M_n),\end{aligned}$$

such that the system trajectory tracks the outputs exponentially. Here, Q is the configuration of the system as defined in Section II, TQ is *the tangent bundle of* Q, and $x \in TQ$ is the system state.

To achieve the control goal just defined, we draw inspiration from the "inertial controller" design in [8]. In particular, we assume the number of quadrotors employed for the load transportation problem is large enough so that the column vectors of G *span* $\mathbb{R}^6$ for all time. This enables the application of an arbitrary wrench to the load, subject to the dynamics of the quadrotors being sufficiently fast.

Our control design can be divided into two parts shown in the following subsections: (a) control design for each individual quadrotor to track a virtual force input and desired yaw angle sufficiently fast, and (b) control design for specifying these virtual forces to enable exponential tracking of the load pose and cable attitudes.

C. Quadrotor force and yaw tracking control

We consider the problem of tracking the force and yaw angle of the i^{th} quadrotor, i.e given a smooth, bounded time-varying force $v_i : [t_0, \infty] \to \mathbb{R}^3$, and quadrotor yaw angle $\theta_{id} : [t_0, \infty] \to \mathbb{S}^1$, design the quadrotor control input $(f_i, M_i) \in \mathbb{R} \times \mathbb{R}^3$ such that the force being applied by the quadrotor $f_i R_i e_3$ tracks the specified force v_i, and the quadrotor body yaw angle θ_i tracks θ_{id} for all time.

To do this, we begin by defining the unit vector $e_3^{ic} = v_i / ||v_i||$, along with another unit vector that represents the yaw attitude to track as $e_1^{id} = [\cos\theta_{id} \;\; \sin\theta_{id} \;\; 0]^T$. Next we define a rotation matrix based on these, as,

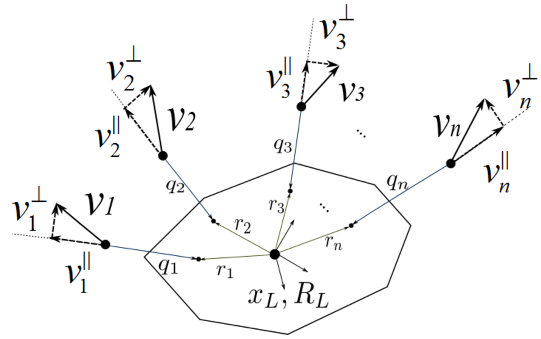


Fig. 2: Diagram showing the reduced system.

$$R_{ic} = \begin{bmatrix} e_1^{ic} & e_3^{ic} \times e_1^{ic} & e_3^{ic} \end{bmatrix}, \text{where } e_1^{ic} = -\frac{e_3^{ic} \times (e_3^{ic} \times e_1^{id})}{||e_3^{ic} \times (e_3^{ic} \times e_1^{id})||}.$$

Then, we define the orientation error for geometric control as,

$$e_{R_i} = \frac{1}{2}(R_{ic}^T R_i - R_i^T R_{ic})^{\vee}, \quad e_{\Omega_i} = \Omega_i - R_i^T R_{ic} \Omega_{ic}$$

where $\Omega_{ic} = (R_{ic}^T \dot{R}_{ic})^{\vee}$, and the *vee map* $^{\vee} : \mathfrak{so}(3) \to \mathbb{R}^3$ is the reverse of the hat map $\hat{\cdot}$.

***Proposition** 1: (Force and Yaw Tracking for a Single Quadrotor)* Consider a desired force v_i to be applied by a quadrotor with a desired quadrotor yaw θ_{id}. Also consider the following quadrotor inputs,

$$\begin{aligned} f_i = v_i \cdot R_i e_3, \quad M_i = &-\frac{k_{R_i}}{\varepsilon^2} e_{R_i} - \frac{k_{\Omega_i}}{\varepsilon} e_{\Omega_i} + \Omega_i \times J_i \Omega_i \\ &- J_i(\hat{\Omega}_i R_i^T R_{ic} \Omega_{ic} - R_i^T R_{ic} \dot{\Omega}_{ic}), \end{aligned}$$

with $R_{ic}, \Omega_{ic}, e_{R_i}, e_{\Omega_i}$ as defined earlier, then there exists parameters k_{R_i}, k_{Ω_i} and $\bar{\varepsilon} > 0$ such that the errors $v_i - f_i R_i e_3$, $\theta_i - \theta_{id}$ tend to zero exponentially for any $i \in \{1, 2, \cdots, n\}$ when $\varepsilon < \bar{\varepsilon}$. Moreover, the convergence rate can be increased by reducing the value of ε.

Proof: See Appendix A. ■

***Remark** 6:* For the extreme case when $\varepsilon = 0$, the force applied by the quadrotor $f_i R_i e_3$ can be controlled directly. In this case, the quadrotor attitude dynamics is entirely omitted, and the original system is reduced to the simplified one shown in Figure 2.

D. Load pose and cable attitude tracking control

We first consider the case when $\varepsilon = 0$, and design a control that tracks a desired load pose and cable attitude for the *reduced system* in Figure 2. Next, we extend this control to the *full system* dynamics that includes the quadrotor dynamics.

We begin by designing a feedback law for the reduced system so as to specify the virtual force v_i to be applied by each quadrotor for tracking a desired load pose and cable attitude. To do this, we make the following assumptions:

- The rank of the matrix G is 6 for all time.
- There's no cable that becomes slack during the control process.

Assumption 1 is equivalent to the *column span* of G being $\mathbb{R}^6$, as discussed in the previous section. From the expression of G it follows that we need at least 6 quadrotors in order to realize this. Moreover, an equivalent condition is that G has Moore-Penrose pseudoinverse, denoted as $G^{\dagger}$. The second assumption guarantees that our model is valid for the control design, as the system is actually hybrid due to unilateral tension constraints. As we will see, our results validate that the assumption holds for a large class of trajectories.

In order to distinguish the reduced system and the full system, we add the subscript "r" to the states of the reduced system. Our control design is split into the following steps:

1) Compute a feedback wrench applied to the load as:
$$W_d = \begin{bmatrix} \ddot{x}_{Ld} + ge_3 \\ -\hat{\Omega}_L R_{Lr}^T R_{Ld} \Omega_{Ld} + R_{Lr}^T R_{Ld} \dot{\Omega}_{Ld} \end{bmatrix} - \begin{bmatrix} k_{x_L} e_{x_L r} + k_{v_L} e_{v_L r} \\ k_{R_L} e_{R_L r} + k_{\Omega_L} e_{\Omega_L r} \end{bmatrix}$$
where the load's translational error is defined as
$$e_{x_{Lr}} = x_{Lr} - x_{Ld}, \quad e_{v_{Lr}} = \dot{x}_{Lr} - \dot{x}_{Ld},$$
and its orientation error is defined as
$$e_{R_{Lr}} = \frac{1}{2}(R_{Ld}^T R_{Lr} - R_{Lr}^T R_{Ld})^{\vee},\ e_{\Omega_{Lr}} = \Omega_{Lr} - R_{Lr}^T R_{Ld} \Omega_{Ld}.$$
2) Use the above expression to obtain $u_{iv}^{\parallel}$ which is the *parallel component of* v_i *along* q_{ir} as the following
$$\begin{bmatrix} u_{1v}^{\parallel} \\ u_{2v}^{\parallel} \\ \vdots \\ u_{nv}^{\parallel} \end{bmatrix} = u_v^{\parallel} = G^{\dagger}(-d + AW_d),$$
where the terms G, d, A are computed using the dynamics based on states of reduced system.
3) Based on this, we are able to cancel out the effects of load's accelerations on the cable attitude using the perpendicular part of v_i as:
$$u_{iv}^{\perp} = \hat{q}_{ir}((q_{ir} \cdot \omega_{id})\dot{q}_{ir} - \hat{q}_{ir}^2 \dot{\omega}_{id} - (k_{q_i} e_{q_{ir}} + k_{\omega_i} e_{\omega_{ir}})) - \frac{1}{L_i} \hat{q}_{ir}^2 (\ddot{x}_{Lbr} + ge_3 + R_{Lr}(\hat{\Omega}_{Lr}^2 + \dot{\hat{\Omega}}_{Lbr}) r_i)$$
where $i = 1, 2, \cdots, n$ and
$$e_{q_{ir}} = q_{id} \times q_{ir}, \quad e_{\omega_{ir}} = \omega_{ir} + \hat{q}_{ir}^2 \omega_{id},$$
are the position and velocity error functions in $\mathbb{S}^2$.

We then have the following proposition:

***Proposition** 2: (Almost Global Exponential Tracking of the Reduced System)* Consider the reduced system shown in Figure 2. Also consider the desired force to be applied by each quadrotor as,
$$v_i = m_i L_i (u_{iv}^{\parallel} + u_{iv}^{\perp}),$$
where $u_{iv}^{\parallel}, u_{iv}^{\perp}$ are as defined above. Then, there exist gain parameters $k_{x_L}, k_{v_L}, k_{R_L}, k_{\Omega_L}$ and $k_{q_i}, k_{\omega_i}, k_{R_i}, k_{\Omega_i}$ $i = 1, 2, \cdots, n$ such that the reduced system tracks the reference output $(x_{Ld}(t), R_{Ld}(t), q_{id}(t))$ exponentially.

Proof: See Appendix B. ■

Then by selecting the virtual control v_i as specified by Prop. 2 and the reference yaw angle θ_{id} and inputs (f_i, M_i) for each quadrotor as specified by Proposition 1, we have completed the full controller design. The stability properties are established by the following proposition.

***Proposition** 3: (Exponential Tracking for the Full System)* Consider the full model of the system which includes the quadrotor dynamics, and consider the virtual control v_i as specified by Prop. 2, along with the reference yaw angle θ_{id} and the quadrotor inputs (f_i, M_i) as specified by Prop. 1. Then there exists $\varepsilon^* > 0$ such that $\forall \varepsilon < \varepsilon^*$, the reference outputs $(x_{Ld}, R_{Ld}, q_{id}, \theta_{id})$ is exponentially tracked for the closed-loop full system.

Proof: See Appendix C. ■

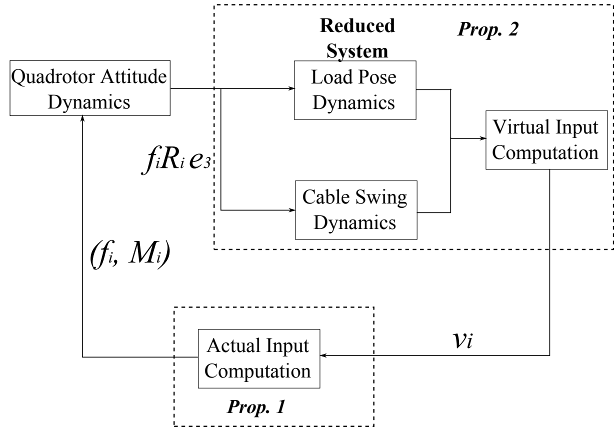


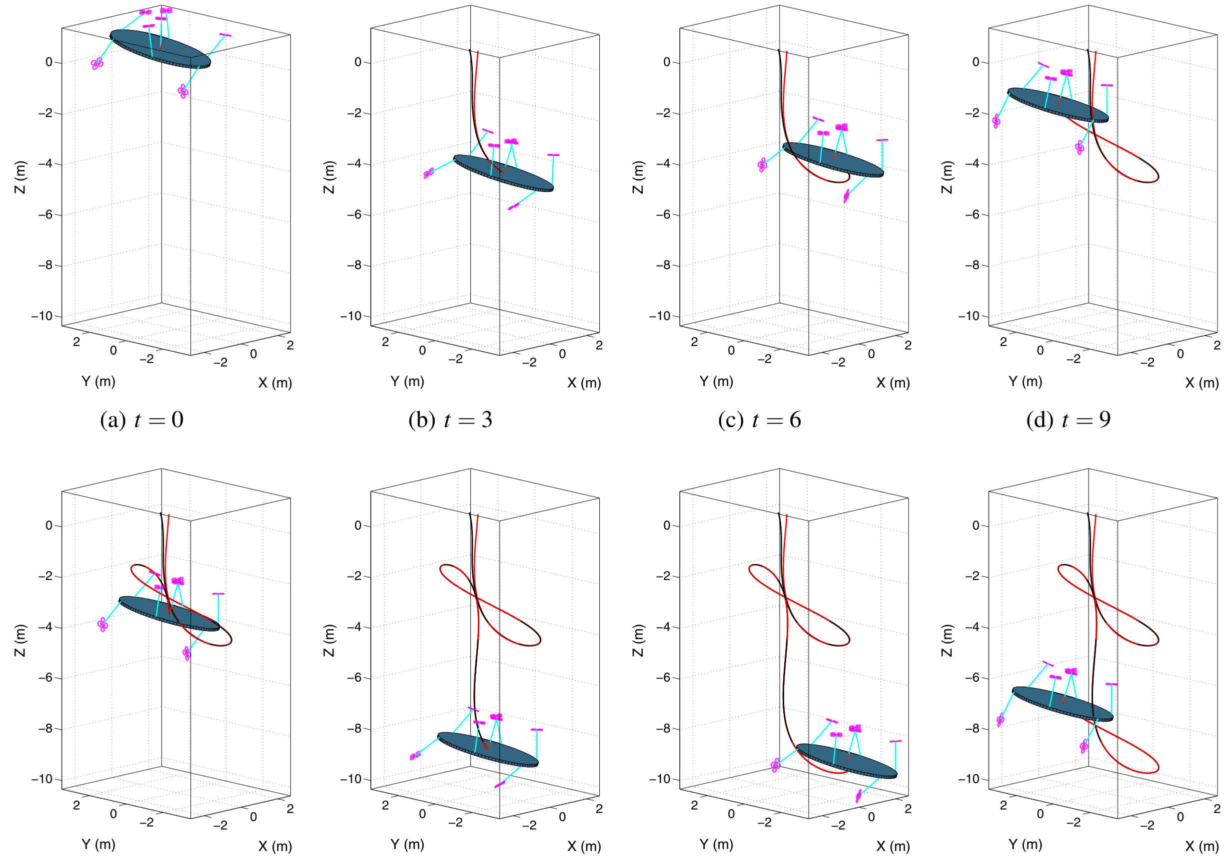
Fig. 3: Block diagram of the controller's structure.

TABLE II: System Parameters for Simulation (Units in SI)

Parameter	Value	Parameter	Value
n	7	L	1.2
m_L	3.0	m_i	0.55
J_L	$diag[3,3,6]$	J_Q	$diag[2.32, 2.32, 7.6] \times 10^{-3}$
r_{bi}	$[2\cos\frac{k\pi}{3}, 2\sin\frac{k\pi}{3}, 0.05]$ $(i = 1, 2, \cdots, 6)$		
r_{b7}	$[0, 0, 0.05]$	ε	0.005

IV. Simulation Results and Discussion

In order to validate the stability of our controller and shed some light on its limitations, we perform a numerical simulation in Matlab. We consider a cylinder-shaped load suspended with cables of equal length from 7 identical quadrotors. The values of the important system parameters are shown in Table II.



(a) $t=0$ (b) $t=3$ (c) $t=6$ (d) $t=9$

(e) $t=12$ (f) $t=15$ (g) $t=18$ (h) $t=21$

Fig. 5: Snapshots of a dynamic load manipulation trajectory. The load, cables, and quadrotors are shown at intervals of 3 seconds. The black line is the nominal trajectory to be followed and red line is actual trajectory for the load's CoM.

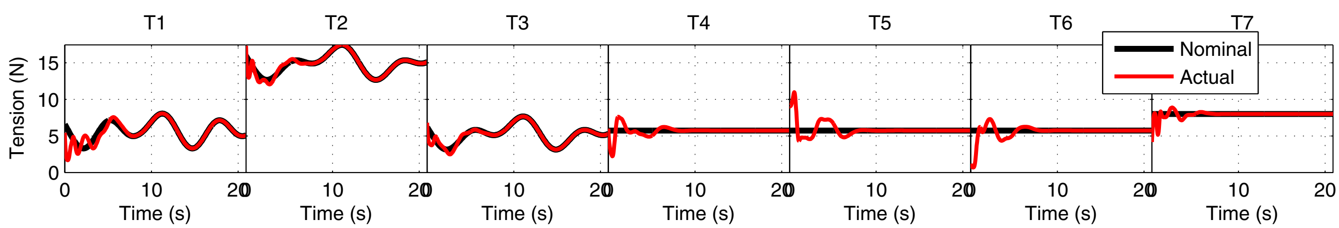


Fig. 6: Plots of the tension in each of the cables that are between the suspended load and the quadrotors. As is clearly seen, the tension is strictly positive indicating that the controller did not cause any of the cables to go slack.

Since the system is shown to be differentially flat in [15], we are able to generate a smooth reference based on the flat outputs that consist of the the load's pose, the tensions for the last four quadrotors, and the yaw angles for all quadrotors. These flat outputs are listed below,

$$x_{Ld} = [1.2\cos t, 2.0\sin(0.5t+\frac{\pi}{6}), 2.5\cos(0.5t+\frac{\pi}{4}) - 0.4t]^T$$

$$R_{Ld} = e^{\hat{\Omega}_0}, \quad \text{where} \quad \Omega_0 = [0.2, 0.2, 0.5]^T,$$

$$T_4 = [2,2,5]^T,\ T_5 = [-2,2,5]^T,\ T_6 = [2,-2,5]^T$$

$$T_7 = [0,0,8]^T$$

where the four tensions ($T4-T7$) are specified as constants with respect to the load's body frame shown in Figure 6, and the quadrotor yaw angles are specified as zero.

The simulation is performed using Matlab, with the full system state vector containing 144 variables (18 states for the load and each of the seven quadrotors). We also consider two quadrotors in an inverted configuration that pull the load down to illustrate the flexibility of the proposed controller.

Simulation results are presented in Figures 4-6. Figure 4 illustrates the load position and orientation errors going down to zero exponentially. Figure 5 illustrates snapshots of the system comprising of the load, suspension cables, and quadrotors, as the controller drives the system to follow the reference output (black and red traces represent the nominal and actual trajecotries for the load's CoM respectively.) A

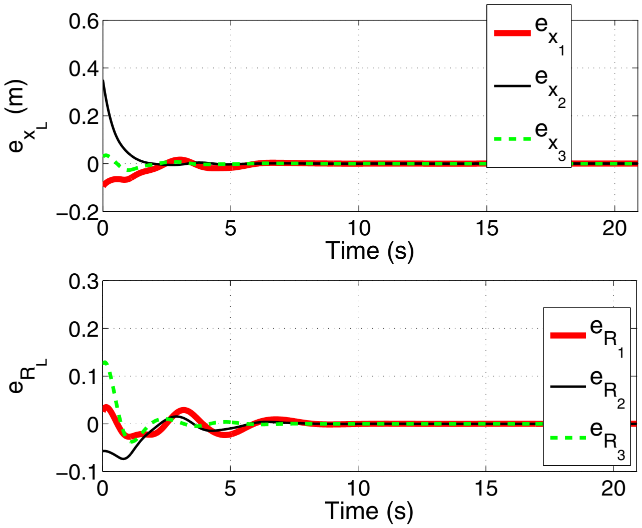


Fig. 4: Position error (e_{x_L}) and orientation error (e_{R_L}) of the load with respect to time.

large initial position error along with an orientation error for the load are specified and the controller drives them to zero. Finally, in order to test the validness of our assumption that the cables do not go slack, each cable tension is computed from the simulation and illustrated in Figure 6. As can be seen, the tensions in all cables are strictly positive.

Having presented numerical results demonstrating the performance of the proposed controller, we now discuss some of its shortcomings. Our geometric control design requires a perfect model, which could cause problems on real-world systems with uncertainties. Moreover, although the controller demonstrates good performance with cable tensions that are strictly positive, the controller does not explicitly guarantee this. There potentially exists initial conditions that could cause the cables to go slack. Furthermore, the controller does not place any state or input constraints, such as collision constraints between the various quadrotors and constraints for actuator limits. Finally, the experimental implementation of the proposed controller will be potentially hard due to noisy state estimates and limited communication bandwidth.

V. Conclusion

We have presented a coordinate-free dynamical model for a system comprising of a rigid-body payload suspended through cables from n-quadrotor UAVs. A geometric controller is designed to realize the tracking of the load's pose (position and orientation), as well as the yaw angle of each quadrotor and the orientation of each cable. The controller is developed initially for a reduced system that ignores the attitude dynamics of the quadrotor, and is then extended to the full system through singular perturbation. Numerical results establishing the performance, as well as rigorous proofs demonstrating stability properties of the controller are presented.

VI. Appendix

A. *Proof of Prop. 1*

The key idea of proof comes from [17]. We first write out the closed-loop error dynamics for each quadrotor's orientation as

$$\varepsilon \begin{bmatrix} \dot{\tilde{e}}_{R_i} \\ \dot{e}_{\Omega_i} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(tr(R_i^T R_{ic})I_3 - R_i^T R_{ic})e_{\Omega_i} \\ -k_{R_i}\tilde{e}_{R_i} - k_{\Omega_i}e_{\Omega_i} \end{bmatrix}$$

where the new error function $\tilde{e}_{R_i} = e_{R_i}/\varepsilon$.

By *Prop. 1* in [17], there exists suitable values for $k_{R_i}, k_{\Omega_i}, \bar{\varepsilon}$ such that the quadrotor's orientation R_i could track the reference R_{ic} exponentially. In addition, the exponential rate is directly related to $\frac{1}{\varepsilon}$. On the other hand side, the error function for the external force

$$f_i R_i e_3 - v_i = (v_i \cdot R_i e_3) R_i e_3 - v_i = (I - e_3^i (e_3^i)^T) v_i$$

Then rewrite this expression in terms of e_3^{ic} as:

$$f_i R_i e_3 - v_i = [e_3^{ic} - (e_3^i \cdot e_3^{ic}) e_3^i] \cdot ||v||$$

Due to exponential tracking of attitude, it holds that e_{3i} tends to e_{3ic} exponentially. Based on the previous expression and boundedness of $||v||$, we have that the external force $f_i R_i e_3$ tends to v_i exponentially.

B. *Proof of Prop. 2*

For the reduced system, what we could directly control is the force v_i applied to each point-mass. Based on the previous discussion in III, we have that only the parallel part of v_i denoted as $u_{iv}^{\parallel}$ could affect the load's pose dynamics while its perpendicular part $u_{iv}^{\perp}$ could change the cable's attitude.Thus, with the given feedback law:

$$\begin{bmatrix} u_{1v}^{\parallel} \\ u_{2v}^{\parallel} \\ \vdots \\ u_{nv}^{\parallel} \end{bmatrix} = u_v^{\parallel} = G^{\dagger}(-d + AW_d),$$

$$\begin{aligned} u_{iv}^{\perp} = \; & \hat{q}_i((q_i \cdot \omega_{id})\dot{q}_i - \hat{q}_i^2 \dot{\omega}_{id} - (k_{q_i} e_{q_i} + k_{\omega_i} e_{\omega_i})) \\ & - \frac{1}{L_i} \hat{q}_i^2 (\ddot{x}_{Lc} + g e_3 + R_L(\hat{\Omega}_L^2 + \dot{\hat{\Omega}}_{Lc}) r_i), \end{aligned}$$

we can compute the closed-loop accelerations of the load directly as:

$$\begin{aligned} A \begin{bmatrix} \ddot{x}_{Lr} + g e_3 \\ \dot{\Omega}_{Lr} \end{bmatrix} &= G(q) \begin{bmatrix} u_{1v}^{\parallel} \\ u_{2v}^{\parallel} \\ \vdots \\ u_{nv}^{\parallel} \end{bmatrix} + d, \\ &= G \cdot G^{\dagger}(AW_d - d) + d, \\ &= AW_d, \end{aligned}$$

where we have utilized the fact that $G \cdot G^{\dagger} = I$. Thus, the error dynamics of the load's pose for the reduced system is,

$$\begin{aligned} \ddot{e}_{x_{Lr}} &= -k_{x_L} e_{x_{Lr}} - k_{v_L} e_{v_{Lr}} \\ \dot{e}_{\Omega_{Lr}} &= -k_{R_L} e_{R_{Lr}} - k_{\Omega_L} e_{\Omega_{Lr}} \end{aligned}$$

Similarly, for the tracking of cable swing dynamics, we have:

$$\dot{e}_{\omega_{ir}} = -k_{q_i} e_{q_{ir}} - k_{\omega_i} e_{\omega_{ir}}$$

Thus, the errors of load's translational, rotational and cable attitude are totally decoupled from each other.Thus we could treat each of them as independent subsystems. As well-studied in the [2], [4], [18], there exist suitable gains $k_{x_L}, k_{v_L}, k_{R_L}, k_{\Omega_L}$ and $k_{q_i}, k_{\omega_i}, k_{R_i}, k_{\Omega_i}$ $i = 1, 2, \cdots, n$ for the reduced system such that almost global exponential stability is guaranteed.

C. Proof of Prop. 3

We start the proof by first rearranging the terms in the hsystem dynamics:

$$\begin{bmatrix} \ddot{x}_L + g e_3 \\ \dot{\Omega}_L \end{bmatrix} = -W_d + A^{-1} G \begin{bmatrix} f_1 R_1 e_3 - v_1 \\ f_2 R_2 e_3 - v_2 \\ \vdots \\ f_n R_n e_3 - v_n \end{bmatrix},$$

$$\begin{aligned} \dot{\omega}_i = & -\hat{q}_i^2 [(q_i \cdot \omega_{id}) \dot{q}_i - \hat{q}_i^2 \dot{\omega}_{id} - (k_{q_i} e_{q_i} + k_{\omega_i} e_{\omega_i})] \\ & - \hat{q}_i (f_i R_i e_3 - v_i) \\ & + q_i \times \frac{1}{L_i} [\ddot{x}_L - \ddot{x}_{Lr} + R_L (\dot{\hat{\Omega}}_L - \dot{\hat{\Omega}}_{Lr}) r_i]. \end{aligned}$$

Next, rewriting the equation in terms of the errors for the full and reduced system, we have,

$$\begin{bmatrix} \ddot{e}_{x_L} \\ \dot{e}_{\Omega_L} \end{bmatrix} = \begin{bmatrix} \ddot{e}_{x_{Lr}} \\ \dot{e}_{\Omega_{Lr}} \end{bmatrix} + A^{-1} G \begin{bmatrix} \delta F_1 \\ \delta F_2 \\ \vdots \\ \delta F_n \end{bmatrix},$$

$$\begin{aligned} \dot{e}_{\omega_i} = & \dot{e}_{\omega_{ir}} - \hat{q}_i (\delta F_i) \\ & + \frac{1}{L_i} q_i \times [\ddot{e}_{x_L} - \ddot{e}_{x_{Lr}} - R_L \hat{r}_i (\dot{e}_{\Omega_L} - \dot{e}_{\Omega_{Lr}})], \end{aligned}$$

where $\delta F_i = f_i R_i e_3 - v_i$. Moreover, the fast-changing dynamics of quadrotor's orientation are given by,

$$\varepsilon \begin{bmatrix} \dot{\tilde{e}}_{R_i} \\ \dot{e}_{\Omega_i} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(tr(R_i^T R_{ic}) I_3 - R_i^T R_{ic}) e_{\Omega_i} \\ -k_{R_i} \tilde{e}_{R_i} - k_{\Omega_i} e_{\Omega_i} \end{bmatrix}.$$

By Prop. 1, for proper selected gain parameters k_{R_i}, k_{Ω_i}, there exists $\bar{\varepsilon}$ such that whenever $\varepsilon < \bar{\varepsilon}$, the *quadrotor's yaw angles* can be tracked exponentially. Next, treating ε as the perturbation parameter of the error dynamic system, we have, when $\varepsilon \to 0$, we have $f_i R_i e_3 \to v_i$ (i.e., $\delta F_i \to 0$.) In this case, the error of the full system becomes the error of the reduced system, and thus it will decay exponentially by Prop. 2. However, we know that for a physical system, ε can never become zero, but the method of *singular perturbations* indicates that for sufficiently small ε, the exponential stability of the reduced system can still be preserved for the full system under certain conditions [3, Thm. 11.4]. Now we are going to check these conditions step by step.

- It's obvious that zero is an isolated equilibrium for the error system.
- From Prop. 2, we know that the origin of the error dynamics for the reduced system is exponentially stable under a properly selected parameter set, $k_{x_L}, k_{v_L}, k_{R_L}, k_{\Omega_L}$ and $k_{q_i}, k_{\omega_i}, k_{R_i}, k_{\Omega_i}$.
- Since all the expressions involved in the dynamics are smooth, their partial derivatives are continuous functions. So *according to boundedness of state, we can conclude that all the partial derivatives up to the second order are bounded*.
- The fast dynamics, i.e quadrotor orientation dynamics, is also exponentially stable when $\varepsilon < \bar{\varepsilon}$ by Prop. 1.

Thus, applying Theorem 11.4 in [3] yields that there exists a $\tilde{\varepsilon}$ such that whenever $\varepsilon \le \tilde{\varepsilon}$, the error for the full dynamical model would tend to zero exponentially. We now select the value $\varepsilon^* = \min\{\bar{\varepsilon}, \tilde{\varepsilon}\}$, and the conclusion of Prop. 3 holds accordingly.

REFERENCES

[1] F. Bullo and A. D. Lewis, *Geometric Control of Mechanical Systems*. New York-Heidelberg-Berlin: Springer-Verlag, 2004.

[2] ——, *Geometric Control of Mechanical Systems*. New York-Heidelberg-Berlin: Springer-Verlag, 2004.

[3] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[4] T. Lee, M. Leok, and N. McClamroch, "Stable manifolds of saddle equilibria for pendulum dynamics on s^2 and $so(3)$," in *IEEE Conference on Decision and Control and European Control Conference*, December 2011, pp. 3915–3921.

[5] ——, "Control of complex maneuvers for a quadrotor uav using geometric methods on $se(3)$," *Asian Journal of Control*, vol. 15, no. 3, pp. 1–18, 2013.

[6] T. Lee, M. Leok, and N. H. McClamroch, "Discrete Control Systems," *Springer Encyclopedia of Complexity and Systems Science*, pp. 2002—-2019, 2008.

[7] ——, "Geometric Tracking Control of a Quadrotor UAV on SE (3)," in *IEEE Conference on Decision and Control*, no. 3, Atlanta, GA, 2010, pp. 5420–5425.

[8] T. Lee, K. Sreenath, and V. Kumar, "Geometric control of cooperating multiple quadrotor UAVs with a suspended payload," in *IEEE Conference on Decision and Control (CDC)*, Florence, Italy, December 2013, pp. 5510–5515.

[9] Q. Lindsey, D. Mellinger, and V. Kumar, "Construction of cubic structures with quadrotor teams," in *Robotics: Science and Systems*, Los Angeles, CA, June 2011.

[10] D. Mellinger, N. Michael, and V. Kumar, "Trajectory generation and control for precise aggressive maneuvers with quadrotors," *The International Journal of Robotics Research*, vol. 31, no. 5, pp. 664–674, Jan. 2012.

[11] D. Mellinger, Q. Lindsey, M. Shomin, and V. Kumar, "Design, modeling, estimation and control for aerial grasping and manipulation," *2011 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pp. 2668–2673, Sep. 2011.

[12] D. Mellinger, M. Shomin, N. Michael, and V. Kumar, "Cooperative grasping and transport using multiple quadrotors," in *Distributed Autonomous Robotic Systems*, Lausanne, Switzerland, November 2010, pp. 545–558.

[13] J. Milnor, *Morse Theory*. Princeton: Princeton University Press, 1963.

[14] K. Sreenath and V. Kumar, "Dynamics, control and planning for cooperative manipulation of payloads suspended by cables from multiple quadrotor robots," in *Robotics: Science and Systems (RSS)*, 2013.

[15] ——, "Dynamics, control and planning for cooperative manipulation of payloads suspended by cables from multiple quadrotor robots," in *Robotics: Science and Systems (RSS)*, 2013.

[16] K. Sreenath, T. Lee, and V. Kumar, "Geometric control and differential flatness of a quadrotor UAV with a cable-suspended load," in *IEEE Conference on Decision and Control (CDC)*, Florence, Italy, December 2013, pp. 2269–2274.

[17] ——, "Geometric control and differential flatness of a quadrotor UAV with a cable-suspended load," in *IEEE Conference on Decision and Control (CDC)*, Florence, Italy, December 2013, pp. 2269–2274.

[18] T. Wu, "Spacecraft relative attitude formation tracking on so(3) based on line-of-sight measurements," Master's thesis, The George Washington University, January 2013.