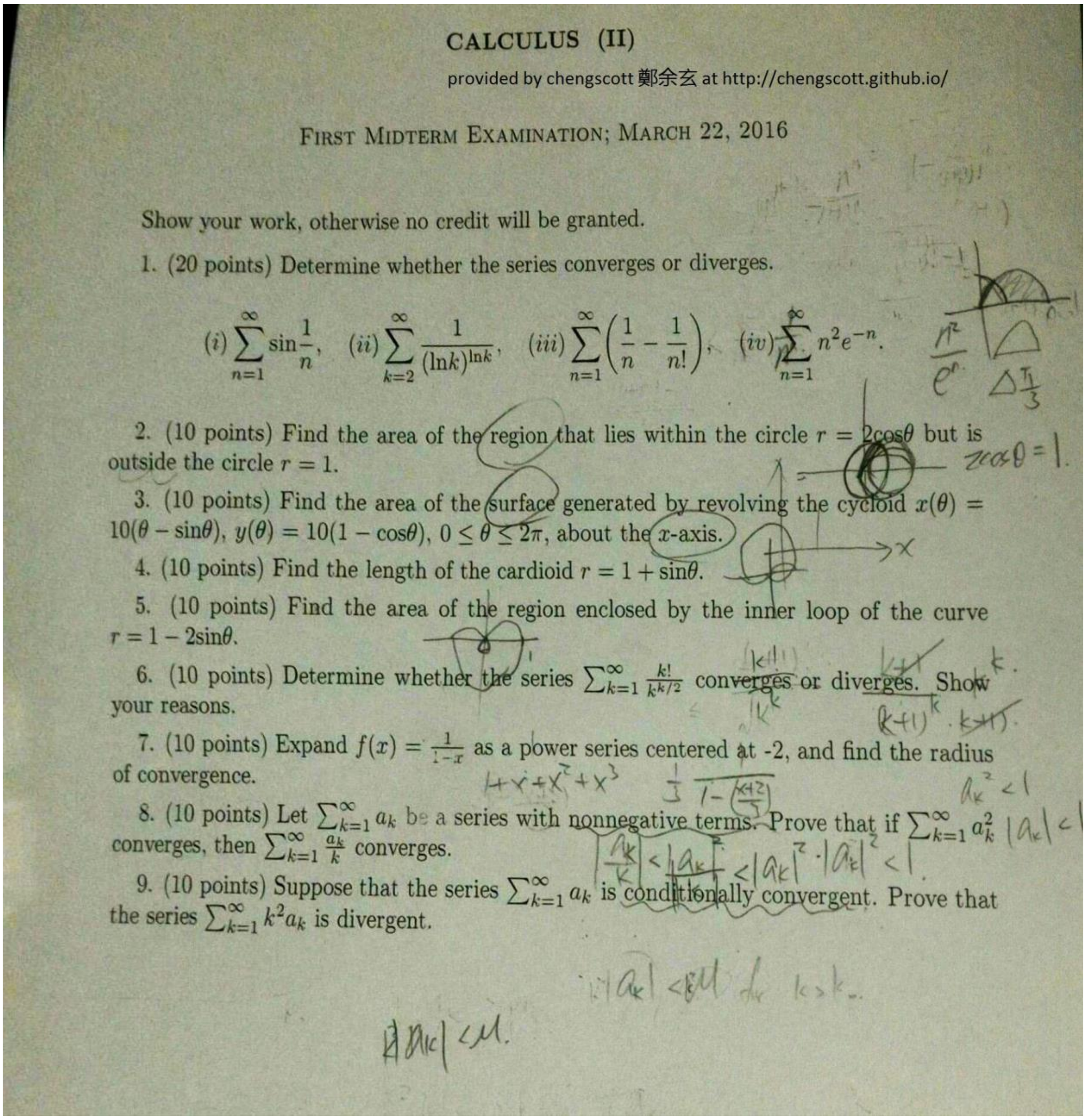
Calculus (II) First Midterm Examination March 22, 2016



CALCULUS (II)

FIRST MIDTERM EXAMINATION; MARCH 22, 2016

Show your work, otherwise no credit will be granted.

1. (20 points) Determine whether the series converges or diverges.

$$(i)\sum_{n=1}^{\infty}\sin\frac{1}{n},\quad (ii)\sum_{k=2}^{\infty}\frac{1}{(\ln k)^{\ln k}},\quad (iii)\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n!}\right),\quad (iv)\sum_{n=1}^{\infty}n^2e^{-n}.$$

2. (10 points) Find the area of the region that lies within the circle $r = 2\cos\theta$ but is outside the circle $r = 1$.

3. (10 points) Find the area of the surface generated by revolving the cycloid $x(\theta) = 10(\theta - \sin\theta)$, $y(\theta) = 10(1-\cos\theta)$, $0 \le \theta \le 2\pi$, about the x-axis.

4. (10 points) Find the length of the cardioid $r = 1 + \sin\theta$.

5. (10 points) Find the area of the region enclosed by the inner loop of the curve $r = 1 - 2\sin\theta$.

6. (10 points) Determine whether the series $\sum_{k=1}^{\infty}\frac{k!}{k^{k/2}}$ converges or diverges. Show your reasons.

7. (10 points) Expand $f(x) = \frac{1}{1-x}$ as a power series centered at -2, and find the radius of convergence.

8. (10 points) Let $\sum_{k=1}^{\infty}a_k$ be a series with nonnegative terms. Prove that if $\sum_{k=1}^{\infty}a_k^2$ converges, then $\sum_{k=1}^{\infty}\frac{a_k}{k}$ converges.

9. (10 points) Suppose that the series $\sum_{k=1}^{\infty}a_k$ is conditionally convergent. Prove that the series $\sum_{k=1}^{\infty}k^2a_k$ is divergent.

Solutions:

1.

(i)

$$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \neq 0$$

By Comparison test,

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty}\sin\frac{1}{n}$ diverges

(ii)

$\ln k > e^2$ for large $k > k_0 = \left\lceil e^{e^2} \right\rceil$

$\frac{1}{(\ln k)^{\ln k}} < \frac{1}{\left(e^{e^2}\right)^{\ln k}} = \frac{1}{k^2}$ for large k

By Comparison test,

$\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges $\Rightarrow \sum_{k=2}^{\infty} \frac{1}{(\ln k)^{\ln k}}$ converges

(iii)

$$\lim_{n\to\infty} \frac{\frac{1}{n} - \frac{1}{n!}}{\frac{1}{n}} = \lim_{n\to\infty} \left(1 - \frac{1}{(n-1)!}\right) = 1 \neq 0$$

By Comparison test,

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n!}\right)$ diverges

(iv)

Let $a_n = \frac{n^2}{e^2}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2}\frac{1}{e} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 \frac{1}{e} = \frac{1}{e} < 1$$

By Ratio test,

$\sum_{n=1}^{\infty} a_n$ converges

2.

$2\cos\theta = 1$ for $\theta \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow \theta = \frac{\pi}{3}$

$$\begin{aligned}\pi \cdot 1^2 - \pi \cdot 1^2 \cdot \frac{1}{3} - 2 \cdot \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (2\cos\theta)^2 d\theta &= \frac{2}{3}\pi - 4\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \frac{2}{3}\pi - 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta \\ &= \frac{2}{3}\pi - 2\left(\frac{\pi}{6} - \frac{1}{2}\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}\end{aligned}$$

3.

$$S = 2\pi \int_0^{2\pi} 10(1 - \cos\theta) \cdot 10\sqrt{(1 - \cos\theta)^2 + \sin^2\theta} \, d\theta = \frac{6400}{3}\pi$$

4.

$$L = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta = 8$$

5.

$1 - 2\sin\theta = 0$ for $\theta \in [0, 2\pi]$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 - 2\sin\theta)^2\, d\theta = \pi - \frac{3}{2}\sqrt{3}$$

6.

7.

$$f(x) = \frac{1}{1-x} = \frac{1}{3}\frac{1}{1 - \left(\frac{x+2}{3}\right)} = \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{x+2}{3}\right)^k$$

$\because$ f is geometric series

$\therefore$ f converges when $\left|\frac{x+2}{3}\right| < 1$ i.e. $|x+2| < 3$

$\therefore$ the radius of convergence $R = 3$

8.

$$\sum_{k=1}^{n} \frac{a_k}{k} \le \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \frac{1}{k^2}\right)^{\frac{1}{2}}$$

9.

Suppose $\sum_{k=1}^{\infty} k^2 a_k$ converges.

$\therefore \lim_{k\to\infty} k^2 a_k = 0$

$\therefore |k^2 a_k| < M$ for some $M > 0$ and large k

$$\Rightarrow |a_k| < \frac{M}{k^2}$$

By Comparison test,

$\sum_{k=1}^{\infty} \frac{M}{k^2} = M \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $\Rightarrow \sum_{k=1}^{\infty} |a_k|$ converges

$\therefore \sum_{k=1}^{\infty} a_k$ absolutely converges (* contradicts)

$\therefore \sum_{k=1}^{\infty} k^2 a_k$ diverges