# Calculus (I) Final Examination January 8, 2016

**CALCULUS (I)**

provided by chengscott 鄭余玄 at http://chengscott.github.io/

FINAL EXAMINATION; JANUARY 8, 2016

Show your work, otherwise no credit will be granted.

1. (20 points) Evaluate the following integrals.

$$(i)\int_0^{\pi/2}\frac{1}{3-2\cos\theta}d\theta,\quad (ii)\int\frac{1}{\sqrt{1+\sqrt[3]{x}}}dx,\quad (iii)\int\frac{dx}{x(1+x^4)},$$

2. (10 points) Suppose that both $g$ and $g'$ are continuous on $[a,b]$. Find the limit: $\lim_{n\to\infty}\int_a^b g(x)\cos(nx)dx$.

3. (10 points) Find the arc length of the curve: $\vec{\gamma}(t)=(1+\tan^{-1}t, 1-\ln\sqrt{1+t^2})$, $t\in[0,1]$.

4. (10 points) Find the area of the surface generated by revolving the curve $x=\sqrt{y}(y-3)$, $1\le y\le 9$, about the $y$-axis.

5. (10 points) Let $\Omega$ be the region bounded by $y=x+x^2$, $x$-axis, $x=0$ and $x=1$. Calculate the volume generated by revolving $\Omega$ about the line $L$: $x-y-2=0$.

6. (10 points) Solve the initial value problem: $x^2y''-3xy'+4y=0$, $y(1)=1$, $y'(1)=4$.

7. (10 points) Find general solution of the differential equation: $y''+2y'+3y=x+\sin x$.

8. (10 points) Find the limit:

$$\lim_{m\to\infty}\left(\int_0^2\left(\frac{1}{x^2-2x+3}\right)^m dx\right)^{\frac{1}{m}}.$$

9. (10 points) Evaluate the integral

$$\int\frac{3x^4-2x^3+6x^2-x+2}{x^5-x^4+2x^3-2x^2+x-1}dx.$$

Solutions:

2.

Teacher Sol:

$$\lim_{n\to\infty}\int_a^b g(x)\cos nx\,dx=\lim_{n\to\infty}\frac{1}{n}\int_a^b g(x)d\sin nx$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}\sin nx\,g(x)|_{x=a}^b-\int_a^b\frac{1}{n}\sin nx\,g'(x)dx\right)=0$$

$$0\le\left|\frac{1}{n}\sin nx\,g(x)|_{x=a}^b-\int_a^b\frac{1}{n}\sin nx\,g'(x)dx\right|\le\frac{1}{n}|g(b)|+\frac{1}{n}|g(a)|+\frac{1}{n}\int_a^b|g'(x)|dx$$

My sol:

Let $I = \lim_{n\to\infty} \int_a^b g(x) \cos nx \, dx$

$$I = \lim_{n\to\infty} \frac{1}{n} \int_a^b g(x) d \sin nx = \lim_{n\to\infty} \frac{1}{n} \left( g(x) \sin nx |_{x=a}^{b} - \int_a^b \sin nx \, g'(x) dx \right)$$

$$= g(b) \lim_{n\to\infty} \frac{\sin nb}{n} - g(a) \lim_{n\to\infty} \frac{\sin na}{n} - \lim_{n\to\infty} \frac{1}{n} \int_a^b \sin nx \, g'(x) dx$$

$$= - \lim_{n\to\infty} \frac{1}{n} \int_a^b \sin nx \, g'(x) dx$$

$\because$ $g'$ is continuous, $\exists M > 0$ s.t. $|g'(x)| < M$ is bounded

$$|I| = \left| \lim_{n\to\infty} \frac{1}{n} \int_a^b \sin nx \, g'(x) dx \right| \leq \lim_{n\to\infty} \frac{M}{n} \int_a^b |\sin nx| dx < \lim_{n\to\infty} \frac{M}{n} \int_a^b dx = 0$$

$$\Rightarrow I = \lim_{n\to\infty} \int_a^b g(x) \cos nx \, dx = 0$$

5.

$$M_y = \int_0^1 x(x + x^2) dx = \frac{7}{12}$$

$$M_x = \frac{1}{2} \int_0^1 (x + x^2)^2 dx = \frac{31}{60}$$

$$M = \int_0^1 (x + x^2) dx = \frac{5}{6}$$

$$C = (\bar{x}, \bar{y}) = \left( \frac{\frac{7}{12}}{\frac{5}{6}}, \frac{\frac{31}{60}}{\frac{5}{6}} \right) = (\frac{7}{10}, \frac{31}{50})$$

$$d(C, L) = \frac{1}{\sqrt{2}} \left| \frac{7}{10} - \frac{31}{50} - 2 \right| = \frac{1}{\sqrt{2}} \frac{48}{25}$$

Pappus: $V = 2\pi \cdot \frac{5}{6} \cdot \frac{1}{\sqrt{2}} \frac{48}{25} = \frac{8}{5} \pi \sqrt{2}$

6.

Characteristic equation: $k(k-1) - 3k + 4 = 0 \Rightarrow k = 2$

General solution: $y = c_1 x^2 + c_2 (\ln x) x^2$ for $c_1, c_2$ are constants

$y(1) = c_1 = 1$

$y'(1) = 4 \Rightarrow c_2 = 2$

Solution: $y = x^2 + 2(\ln x) x^2$

7.

Characteristic equation: $r^2 + 2r + 3 = 0 \Rightarrow -1 \pm \sqrt{2}i$

Homogeneous solution: $y_h = c_1 e^{-x} \sin \sqrt{2}x + c_2 e^{-x} \cos \sqrt{2}x$ for $c_1, c_2$ are constants

Solve for $y'' + 2y' + 3y = x$

Consider $y_{p1} = ax + b$

$$y_{p1} = \frac{1}{3}x - \frac{2}{9}$$

Consider $y'' + 2y' + 3y = \sin x$

$$y_{p2} = \frac{1}{4}\sin x - \frac{1}{4}\sin x$$

The general solution of $y'' + 2y' + 3y = x + \sin x$ is

$$y = y_h + y_{p1} + y_{p2} = c_1 e^{-x} \sin \sqrt{2}x + c_2 e^{-x} \cos \sqrt{2}x + \frac{1}{3}x - \frac{2}{9} + \frac{1}{4}\sin x - \frac{1}{4}\sin x$$

for $c_1, c_2$ are constants

8.

$$f(x) = \frac{1}{x^2 - 2x + 3} = \frac{1}{(x-1)^2 + 2}$$

$$\max_{x\in[0,2]} f(x) = \frac{1}{2}$$

Given $\varepsilon > 0, \exists \delta > 0$ s.t.

$$\frac{1}{2} - \varepsilon \le f(x), x \in (1-\delta, 1+\delta)$$

$$\int_{1-\delta}^{1+\delta} \left(\frac{1}{2} - \varepsilon\right)^m dx = (2\delta)\left(\frac{1}{2} - \varepsilon\right)^m \le \int_0^2 \left(\frac{1}{x^2 - 2x + 3}\right)^m dx \le \int_0^2 \left(\frac{1}{2}\right)^m dx = 2 \cdot \left(\frac{1}{2}\right)^m$$

$$\left(\frac{1}{2} - \varepsilon\right)(2\delta)^{\frac{1}{m}} \le \left(\int_0^2 \left(\frac{1}{x^2 - 2x + 3}\right)^m dx\right)^{\frac{1}{m}} \le \frac{1}{2} 2^{\frac{1}{m}}$$

$$\lim_{m\to\infty} \left(\int_0^2 \left(\frac{1}{x^2 - 2x + 3}\right)^m dx\right)^{\frac{1}{m}} = \frac{1}{2}$$

9.

$$\int \frac{3x^4 - 2x^3 + 6x^2 - x + 2}{x^5 - x^4 + 2x^3 - 2x^2 + x - 1} dx = \int \left(\frac{2}{x-1} + \frac{x-1}{x^2+1} + \frac{1}{(x^2+1)^2}\right) dx$$