Calculus Stewart Ch16 Problem Plus

3. Let C be a simple closed piecewise-smooth space curve that lies in a plane with unit normal vector $\hat{n} = (a, b, c)$ and has positive orientation with respect to n. Show that the plane area enclosed by C is

$$\frac{1}{2}\int_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz$$

Solution:

Let $F = (bz - cy, cz - az, ay - bx)$

$curl\ F = (2a, 2b, 2c) = 2\hat{n}$

$$\iint_S (curl\ F) \cdot \hat{n} dS = \iint_S 2dS = 2\iint_S dS$$

$$\Rightarrow \iint_S dS = \frac{1}{2}\iint_S (curl\ F) \cdot \hat{n} dS = \frac{1}{2}\oint_C F \cdot dr = \frac{1}{2}\oint_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz$$