Linear Algebra (I) Final Exam

Linear Algebra I Final Exam

provided by chengscott 鄭余玄 at http://chengscott.github.io/

Time: 110 minutes

注意事項:

- 本題目紙背面也有試題。
- 請在答案紙的封面寫上自己的姓名。
- 請勿使用任何書籍、筆記或電子儀器。
- 請將答案寫在答案紙上並清楚標明題號。
- 答題順序不拘, 雙面皆可書寫, 但請勿將一題的答案分散在不連續的頁面上。
- 字跡請勿潦草, 以免批改者無法辨識。
- 每一題皆須邏輯正確無誤且論證完整清晰才能得到滿分。批改者可斟酌給予部分分數。
- 課堂上或習題中證明過的定理可以直接引用, 無須重新證明一次。
- 除了將英文翻譯成中文以外, 監考人員不回答任何跟試題有關的問題。
- 如有未盡事宜, 以監考人員的指示為準。

1. (5 points) Find the inverse of the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & c & 1 & 0 \\ b & d & e & 1 \end{pmatrix}$.

2. (5 points) Find all $t \in \mathbb{R}$ such that the matrix $\begin{pmatrix} 1 & t & 0 & 0 \\ t & 1 & t & 0 \\ 0 & t & 1 & t \\ 0 & 0 & t & 1 \end{pmatrix}$ is not invertible.

3. Let

$$A = \begin{pmatrix} -1 & 1 & 3 & 2 & 1 \\ 1 & 2 & 0 & 7 & 2 \\ -2 & 3 & 7 & 7 & 3 \\ 3 & 1 & -5 & 5 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 4 \\ -1 \\ 1 \end{pmatrix}.$$

(a) (5 points) Solve the system of linear equations $Ax = b$ over $\mathbb{R}$.

(b) (5 points) Find the rank of A.

(c) (5 points) Find a basis for the null space of L_A.

(d) (5 points) Let $W = \{w \in \mathbb{R}^4 \mid Ax = w \text{ has a solution}\}$. Find a basis for W.

There are problems on the back page.

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4. **(5 points)** Let $Ax = b$ be a system of n linear equations in n unknowns such that all entries of A and b are integers. Show that if $\det(A) = 1$, then all entries of the solution x are integers.

5. **(5 points)** Find the determinant of the $n \times n$ matrix whose diagonal entries are c and all the other entries are 1. Hint: Let $\{e_1, \dots, e_n\}$ be the standard basis for F^n, and let $v = e_1 + \cdots + e_n$. Then the determinant in question can be written as $\det\big((c-1)e_1 + v, (c-1)e_2 + v, \dots, (c-1)e_n + v\big)$.

6. **(5 points)** Let A and B be $n \times n$ invertible matrices. Show that $\det(A) = \det(B)$ if and only if there exists an $n \times n$ matrix C with $\det(C) = 1$ such that $AC = B$.

$C = A^{-1}B$

7. **(5 points)** Let $A \in M_{n\times n}(\mathbb{R})$ and let $b \in \mathbb{R}^n$. Show that if $\det(A) = 0$, then the system of linear equations $Ax = b$ either has no solution or has infinitely many solutions.

8. Let $A \in M_{m\times n}(F)$ be a matrix such that the linear system $Ax = b$ has a solution for any $b \in F^m$.

$F^n \to F^m$

(a) **(5 points)** Show that $n \geq m$.

(b) **(5 points)** Show that the solution space of the homogeneous system $Ax = 0$ is $(n-m)$-dimensional.

9. **(5 points)** Let $A \in M_{m\times n}(F)$. Show that the row vectors of A are linearly independent (and distinct) if and only if there exists a subset of the column vectors of A which is a basis for F^m.

10. (a) **(5 points)** Let $T\colon V \to V$ be a linear operator on a finite-dimensional vector space V. Show that if $T^2 = 0$, then $\operatorname{rank}(T) \leq \dim(V)/2$. Hint: the condition $T^2 = 0$ is equivalent to $R(T) \subseteq N(T)$.

(b) **(5 points)** Use the results of (a) to show that if an $n \times n$ matrix A satisfies $A^2 = 0$, then $\operatorname{rank}(A) \leq n/2$.

Solutions:

2.

$$\det\begin{pmatrix} 1 & t & 0 & 0 \\ t & 1 & t & 0 \\ 0 & t & 1 & t \\ 0 & 0 & t & 1 \end{pmatrix} = 0 \Rightarrow t = \frac{1 \pm \sqrt{5}}{2}, \frac{-1 \pm \sqrt{5}}{2}$$

4.

$\because \det A = 1 \neq 0$

$\therefore x = A^{-1}b = \dfrac{adj(A)}{\det A} b = adj(A)b$

$\because$ all entries of A are integers

$\therefore$ the cofactors of A must be integers

$\therefore$ all entries of $adj(A)$ are integers

$\because$ all entries of b are integers

$\therefore$ all entries of $adj(A)b$ are integers

$\therefore$ all entries of the solution x are integers

5.

Let $\{e_1, \dots, e_n\}$ be the standard basis for F^n

Let $v = e_1 + e_2 + \cdots + e_n$

$$\det\begin{pmatrix} c & 1 & \cdots & 1 \\ 1 & c & \ddots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & c \end{pmatrix} = \det(ce_1 + e_2 + \cdots + e_n, e_1 + ce_2 + e_3 + \cdots + e_n, \dots, e_1 + \cdots + e_{n-1} + ce_n)$$
$$\begin{aligned}
&= \det\big((c-1)e_1 + v, (c-1)e_2 + v, \dots, (c-1)e_n + v\big) \\
&= \det\big((c-1)e_1 + v, (c-1)(e_2 - e_1), \dots, (c-1)(e_n - e_1)\big) \\
&= (c-1)^n \det\big(e_1, (e_2 - e_1), \dots, (e_n - e_1)\big) \\
&+ (c-1)^{n-1} \det\big(e_1 + e_2 + \cdots + e_n, (e_2 - e_1), \dots, (e_n - e_1)\big) \\
&= (c-1)^n \det(e_1, \dots, e_n) + (c-1)^{n-1} \det\big(ne_1, (e_2 - e_1), \dots, (e_n - e_1)\big) \\
&= (c-1)^n + n(c-1)^{n-1} \det(e_1, \dots, e_n) = (c-1)^n + n(c-1)^{n-1}
\end{aligned}$$

6.

$(\Leftarrow)$

$\det AC = \det A \det C = \det A \cdot 1 = \det B$

$(\Rightarrow)$

Let $C = A^{-1}B \in M_{n\times n}(F)$

$$\det C = \det A^{-1}B = \det A^{-1} \det B = \frac{1}{\det A} \det B = \frac{1}{\det B} \det B = 1$$

7.

$\because \det A = 0$

$\therefore rank(A) < n$

$\therefore nullity(A) = n - rank(A) > 0$

$\therefore N(A) \neq \{0\}$

If $Ax = b$ has no solution, then proved!

Otherwise, say $Ax_0 = b$ for some x_0

The solution set $\{x|Ax = b\} = x_0 + \{x|Ax = 0\} = x_0 + N(A)$

$\Rightarrow Ax = b$ has infinitely many solutions

8.

(a)

$L_A: F^n \to F^m$

$\because \forall b \in F^m, L_A(x) = b$ has a solution

$\therefore L_A$ is surjective

$\therefore n \geq m$

(b)

$\because L_A$ is onto

$\therefore rank(L_A) = m$

$\therefore$ the solution space of $Ax = 0$ has dimension $n - rank(A) = n - rank(L_A) = n - m$

Type equation here.

9.

$(\Rightarrow)$

$\because$ row vectors of A are linear independent

$\therefore$ row vectors of A forms a basis fro F^m

$\dim row(A) = m = \dim col(A)$

$\therefore$ we can find a subset of m's column vectors of A which forms a basis for $col(A)$

$\Rightarrow$ which is a basis for F^m

$(\Leftarrow)$

$\because$ a basis of column vector is a basis for F^m

$\therefore \dim col(A) = \dim F^m = m$

$\therefore$ the row vectors of A are a basis for F^m

$\therefore$ the row vectors of A are linearly independent

10.

(a)

$\forall v \in V$

$\because T\big(T(v)\big) = 0$

$\therefore T(v) \in N(T)$

$\therefore R(T) \subseteq N(T)$

$\therefore rank(T) \leq nullity(T)$

$\dim V = rank(T) + nullity(T) \geq rank(T) + rank(T) = 2rank(T)$

$\therefore rank(T) \leq \dim V / 2$

(b)

Let $T = L_A, V = F^n$

$L_A: F^n \rightarrow F^n$

$\because A^2 = 0$

$\therefore L_A^2 = 0$

By (a),

$rank(A) = rank(L_A) \leq \dim F^n / 2 = n/2$