Calculus (I) Second Midterm Examination November 27, 2015

CALCULUS (I)

Second Midterm Examination; November 27, 2015

Show your work, otherwise no credit will be granted.

1. (20 points) Calculate

$$(i)\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}, \qquad (ii)\int_{-2}^{2}(e^{|x|}+x^6\sin x)dx,$$

$$(iii)\int_4^6 \frac{dx}{(x-3)\sqrt{x^2-6x+8}}, \qquad (iv)\int \frac{dx}{\sqrt{4x-x^2}}.$$

$4-(x-2)^2$

$(x-3)^2-1$

2. (10 points) Suppose that $G'(x)$ is continuous and $G(x)=\frac{1}{x}\int_3^x(2t-5G'(t))dt$. Find $G'(3)$.

$G(3)=0$

3. (10 points) Find the volume enclosed by the surface obtained by revolving the ellipse $x^2+2y^2=1$ about the y-axis.
4. (10 points) Let $P(x)=x^8-3x^7-5x^3+x^2-6$. Find $\lim_{x\to\infty}((P(x))^{1/8}-x)$.
5. (10 points) Differentiate the function $h(x)=x^{x^x}$, $x>1$.
6. (10 points) Prove the formula $\cosh^{-1}x=\ln(x+\sqrt{x^2-1})$, $x\geq 1$.
7. (10 points) Evaluate the integral $\int_0^{\pi/2}\frac{\sin^{0.8}x}{\sin^{0.8}x+\cos^{0.8}x}dx$.
8. (10 points) Find the limit

$$\lim_{x\to 0^+} x^{-2/15}\int_{x^{1/2}}^{x^{1/5}} t^{-1/3}\cos t\; dt.$$

9. (10 points) Let f, g be two continuous real-valued function on $[0,1]$. Show that

$$\left|\int_0^1 f(x)g(x)dx\right| \leq \left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}}\cdot\left(\int_0^1 g^2(x)dx\right)^{\frac{1}{2}}.$$

(Hint: You may consider $f(x)+tg(x)$, $t\in\mathbb{R}$ or apply Cauchy's inequality.)

Solutions:

3.

$$V = 2\int_0^{\frac{\sqrt{2}}{2}} \pi\sqrt{1-2y^2}^{\,2}\,dy = \frac{2}{3}\sqrt{2}\pi$$

4.

$$\lim_{x\to\infty}(x^8-3x^7-5x^3+x^2-6)^{\frac{1}{8}}-x=\lim_{x\to\infty}x\left((1-3x^{-1}-5x^{-5}+x^{-6}-6x^{-8})^{\frac{1}{8}}-1\right)$$
$$=\lim_{t\to 0^+}\frac{(1-3t-5t^5+t^6-6t^8)^{\frac{1}{8}}-1\to 0}{t\qquad\qquad\to 0}\ as\ t\to 0^+$$
$$=\lim_{t\to 0^+}\frac{1}{8}(1-3t-5t^5+t^6-6t^8)^{-\frac{7}{8}}(-3-25t^4+6t^5-48t^7)=-\frac{3}{8}$$

5.

$$\frac{d}{dx}x^{x^x}=\frac{d}{dx}e^{x^x\ln x}=e^{x^x\ln x}\left(\left(\frac{d}{dx}x^x\right)\ln x+x^x\cdot\frac{1}{x}\right)=x^{x^x}\left(e^{x\ln x}\left(\ln x+x\cdot\frac{1}{x}\right)\ln x+x^{x-1}\right)$$
$$=x^{x^x}(x^x(\ln x+1)\ln x+x^{x-1})$$

6.

$$x=\cosh y=\frac{e^y+e^{-y}}{2}=\frac{e^{2y}+1}{2e^y}$$

$$\Rightarrow 2xe^y=e^{2y}+1$$

$$\Rightarrow e^{2y}-2xe^y+1=0$$

$$\Rightarrow e^y=\frac{2x\pm\sqrt{4x^2-4}}{2}=x\pm\sqrt{x^2-1}$$

Also, $\lim_{y\to\infty}\cosh y=\infty$ and $\lim_{x\to\infty}\ln\left(x-\sqrt{x^2-1}\right)=\lim_{x\to\infty}\ln\left(\frac{1}{x+\sqrt{x^2-1}}\right)=-\ln\left(x+\sqrt{x^2-1}\right)=-\infty$

$$\Rightarrow y=\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right)$$

7.

Let $t=\frac{\pi}{2}-x$

$$\int_0^{\frac{\pi}{2}}\frac{\sin^{0.8}x}{\sin^{0.8}x+\cos^{0.8}x}dx=\int_{\frac{\pi}{2}}^{0}\frac{\cos^{0.8}t}{\cos^{0.8}t+\sin^{0.8}t}(-dt)=\int_0^{\frac{\pi}{2}}\frac{\cos^{0.8}t}{\sin^{0.8}t+\cos^{0.8}t}dt$$

Also

$$\int_0^{\frac{\pi}{2}}\left(\frac{\sin^{0.8}x}{\sin^{0.8}x+\cos^{0.8}x}+\frac{\cos^{0.8}x}{\sin^{0.8}x+\cos^{0.8}x}\right)dx=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}$$

$$\Rightarrow\int_0^{\frac{\pi}{2}}\frac{\sin^{0.8}x}{\sin^{0.8}x+\cos^{0.8}x}dx=\frac{\frac{\pi}{2}}{2}=\frac{\pi}{4}$$

8.

Sol1:

Let $f(x)=\int_0^x t^{-\frac{1}{3}}\cos t\,dt$

And $\lim\limits_{x\to0^+} x^{\frac{2}{15}}=0,\ \lim\limits_{x\to0^+} f\left(x^{\frac{1}{5}}\right)=0,\ \lim\limits_{x\to0^+} f\left(x^{\frac{1}{2}}\right)=0$

$$\lim_{x\to0^+}\frac{f\left(x^{\frac{1}{5}}\right)-f\left(x^{\frac{1}{2}}\right)}{x^{\frac{2}{15}}}=\lim_{x\to0^+}\frac{x^{-\frac{1}{15}}\cos x^{\frac{1}{5}}\frac{1}{5}x^{\frac{4}{5}}-x^{-\frac{1}{6}}\cos x^{\frac{1}{2}}\frac{1}{2}x^{-\frac{1}{2}}}{\frac{2}{15}x^{-\frac{3}{15}}}$$
$$=\lim_{x\to0^+}\frac{15}{2}\left(\frac{1}{5}\cos x^{\frac{1}{5}}-\frac{1}{2}x^{\frac{1}{5}}\cos x^{\frac{1}{2}}\right)=\frac{15}{2}\cdot\frac{1}{5}=\frac{3}{2}$$

Sol2:

Given $\varepsilon>0,\exists 0<\delta<1$ s.t. $1-\varepsilon<\cos t\le 1$ on $(0,\delta]$

$\because 1-\varepsilon<\cos t\le 1$

$\Rightarrow t^{-\frac{1}{3}}(1-\varepsilon)<t^{-\frac{1}{3}}\cos t\le t^{-\frac{1}{3}}$

$$\Rightarrow \int_{x^{\frac{1}{2}}}^{x^{\frac{1}{5}}} t^{-\frac{1}{3}}(1-\varepsilon)dt<\int_{x^{\frac{1}{2}}}^{x^{\frac{1}{5}}} t^{-\frac{1}{3}}\cos t\,dt<\int_{x^{\frac{1}{2}}}^{x^{\frac{1}{5}}} t^{-\frac{1}{3}}dt$$

$$\Rightarrow \frac{3}{2}(1-\varepsilon)\left(x^{\frac{2}{15}}-x^{\frac{1}{3}}\right)<\int_{x^{\frac{1}{2}}}^{x^{\frac{1}{5}}} t^{-\frac{1}{3}}\cos t\,dt\le\frac{3}{2}\left(x^{\frac{2}{15}}-x^{\frac{1}{3}}\right)$$

$$\Rightarrow \frac{3}{2}(1-\varepsilon)\left(1-x^{\frac{1}{5}}\right)<x^{-\frac{2}{15}}\int_{x^{\frac{1}{2}}}^{x^{\frac{1}{5}}} t^{-\frac{1}{3}}\cos t\,dt\le\frac{3}{2}\left(1-x^{\frac{1}{5}}\right)$$

$\because x\to0,\varepsilon\to0,x^{\frac{1}{5}}\to0$

By pinching theorem, $x^{-\frac{2}{15}}\int_{x^{\frac{1}{2}}}^{x^{\frac{1}{5}}} t^{-\frac{1}{3}}\cos t\,dt=\frac{3}{2}$

9.

Sol1:

$$\left|\int_0^1 f(x)g(x)dx\right|=\lim_{n\to\infty}\left|\sum_{k=1}^n f\left(\frac{k}{n}\right)g\left(\frac{k}{n}\right)\frac{1}{n}\right|\le\lim_{n\to\infty}\left(\sum_{k=1}^n f^2\left(\frac{k}{n}\right)\frac{1}{n}\right)^{\frac{1}{2}}\left(\sum_{k=1}^n g^2\left(\frac{k}{n}\right)\frac{1}{n}\right)^{\frac{1}{2}}$$
$$=\left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^1 g^2(x)dx\right)^{\frac{1}{2}}$$

Sol2:

$\big(f(x)+tg(x)\big)^2\ge 0$ for $t\in\mathbb{R}$

$$\Rightarrow \int_0^1 \left(f(x) + tg(x)\right)^2 dx \geq 0$$

$$\Rightarrow \int_0^1 f^2(x)dx + 2t \int_0^1 f(x)g(x)dx + t^2 \int_0^1 g^2(x)dx \geq 0$$

(1)

$$\int_0^1 g^2(x)dx = 0 \Rightarrow g \equiv 0$$

$\left|\int_0^1 f(x) \cdot 0dx\right| = 0 \leq \left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}} \left(\int_0^1 0^2 dx\right)^{\frac{1}{2}} = 0$ is true

(2)

$$\int_0^1 g^2(x)dx > 0$$

$\because at^2 + bt + c \geq 0, a > 0 \Rightarrow D = b^2 - 4ac \leq 0$

$$D = \left(2 \int_0^1 f(x)g(x)dx\right)^2 - 4\left(\int_0^1 f^2(x)dx\right)\left(\int_0^1 g^2(x)dx\right) \leq 0$$

$$\Rightarrow \left(\int_0^1 f(x)g(x)dx\right)^2 \leq \left(\int_0^1 f^2(x)dx\right)\left(\int_0^1 g^2(x)dx\right)$$

$$\Rightarrow \left|\int_0^1 f(x)g(x)dx\right| \leq \left(\int_0^1 f^2(x)dx\right)^{\frac{1}{2}} \left(\int_0^1 g^2(x)dx\right)^{\frac{1}{2}}$$