Calculus Stewart Ch6 Sec6.3*

109. (a) Use mathematical induction to prove that for $x \geq 0$ and any positive integer n,

$$e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.

(c) Use part (a) to show that

$$\lim_{\mathrm{n}\to\infty} \frac{\mathrm{e}^{\mathrm{x}}}{x^k} = \infty$$

for any positive integer k

Proof:

(a)

Induction on n.

When $n = 1$, $(e^x)' = e^x > 0$ is increasing and $e^0 = 1 \Rightarrow e^x \geq 1$ for $x \geq 0$ is true

Assume $n = k$, $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$ for $x \geq 0$ is true

When $n = k + 1$, since $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} \geq 0$ for $x \geq 0$

$$\Rightarrow \int_0^x e^x dx \geq \int_0^x \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}\right) dx$$

$$\Rightarrow e^x - 1 \geq x + \frac{x^2}{2} + \cdots . + \frac{x^{k+1}}{(k+1)!} - 0$$

$e^x \geq 1 + x + \frac{x^2}{2} + \cdots . + \frac{x^{k+1}}{(k+1)!}$ is true

By M.I., $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ for $x \geq 0$ is true.

(b)

$x = 1$ and choose $n \geq 4$, then

$$e^1 = e > 2.7$$

(c)

$$\because e^x \geq 1 + x + \frac{x^2}{2} + \cdots . + \frac{x^{k+1}}{(k+1)!}$$

$$\lim_{\mathrm{n}\to\infty} \frac{\mathrm{e}^{\mathrm{x}}}{x^k} \geq \lim_{\mathrm{n}\to\infty} \left(\frac{1}{x^k} + \cdots + \frac{1}{k!} + \frac{1}{(k+1)!} x\right) = \infty$$

$$\Rightarrow \lim_{\mathrm{n}\to\infty} \frac{\mathrm{e}^{\mathrm{x}}}{x^k} = \infty$$