Calculus Stewart Ch11 Sec 2

50. A sequence of terms is defined by

$$a_1 = 1, a_n = (5-n)a_{n-1}$$

Calculate $\sum_{n=1}^{\infty} a_n$.

Solution:

$a_1 = 1, a_2 = 3, a_3 = 2 \cdot 3 = 6, a_4 = 1 \cdot 6 = 6, a_5 = 0 \cdot 6 = 0$

$\Rightarrow a_k = 0$ for $k \geq 5$

$$\sum_{n=1}^{\infty} a_n = 1 + 3 + 6 + 6 = 16$$

73. Find value of c if

$$\sum_{n=2}^{\infty} (1+c)^{-n} = 2$$

Solution:

$$\sum_{n=2}^{\infty} (1+c)^{-n} = \frac{\left(\frac{1}{1+c}\right)^2}{1 - \frac{1}{1+c}} = 2$$

$$\Rightarrow 2c^2 + 2c - 1 = 0 \Rightarrow c = \frac{1-\sqrt{3}}{2} < 1$$

83. If $\sum a_n$ is convergent and $\sum b_n$ is divergent. Show that the series $\sum (a_n + b_n)$ is divergent.

Proof:

Suppose $\sum (a_n + b_n)$ is convergent. Then

$\sum (a_n + b_n) - \sum a_n$ is convergent

$= \sum (a_n + b_n) - a_n = \sum b_n$ is convergent (* contradicts)

$\therefore \sum (a_n + b_n)$ is divergent

Calculus Stewart Ch11 Sec 3

42. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^2}$

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt[4]{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{4\sqrt[4]{n^3}}} = \lim_{n\to\infty} \frac{4}{\sqrt[4]{n}} = 0$$

$\therefore \ln n < \sqrt[4]{n}$ for large n

$$\frac{(\ln n)^2}{n^2} < \frac{\left(\sqrt[4]{n}\right)^2}{n^2} = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$$

$\Rightarrow \sum \frac{(\ln n)^2}{n^2} < \sum \frac{1}{n^{\frac{3}{2}}}$ converges

45. For all positive values of b for which the series $\sum_n^\infty b^{\ln n}$ converges.

Solution:

$b^{\ln n} = n^{\ln b}$

$\sum_n^\infty b^{\ln n}$ converges iff. $-\ln b > 1 \Leftrightarrow 0 < b < \frac{1}{e}$

Calculus Stewart Ch11 Sec 4

44. Show that if $a_n > 0$ and $\sum a_n$ is convergent, then $\sum \ln(1 + a_n)$ is convergent.

Hint:

$x > \ln(1 + x)$

45. If $\sum a_n$ is a convergent series with positive terms, is $\sum \sin(a_n)$ also converges?

Solution:

$\because \sum a_n$ converges $\therefore \lim_{n\to\infty} a_n = 0$

By Comparison Test, $\lim_{n\to\infty} \frac{|\sin(a_n)|}{|a_n|} = 1 > 0$

$\therefore \sum \sin(a_n)$ converges.

Calculus Stewart Ch11 Sec 5

31. The terms of a series are defined recursively by the equations

$$a_1 = 2, a_{n+1} = \frac{5n+1}{4n+3} a_n$$

Determine whether $\sum a_n$ converges or diverges.

Solution:

Since a_n is positive definite, by ratio test,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{5}{4} > 1$$

$\therefore \sum a_n$ diverges.

Calculus Stewart Ch11 Sec 7

23. $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

Solution:

By Comparison test, $\lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \frac{1}{\cos\left(\frac{1}{n}\right)} = 1 > 0.$

$\therefore \sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ diverges

27. $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$

Solution:

$$\lim_{k\to\infty} \frac{\ln k}{\sqrt{k}} = 0$$

$\therefore \ln k < \sqrt{k}$ for large k

$$\therefore \frac{k \ln k}{(k+1)^3} < \frac{k\sqrt{k}}{(k+1)^3} < \frac{k\sqrt{k}}{k^3} = \frac{1}{k^{\frac{3}{2}}}$$

$\therefore \sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ converges

36. $\sum_{k=1}^{\infty} \frac{1}{(\ln k)^{\ln k}}$

Solution:

$\ln k > e^2$ for large k

$$\frac{1}{(\ln k)^{\ln k}} < \frac{1}{(e^2)^{\ln k}} = \frac{1}{k^2}$$

$\therefore \sum_{k=1}^{\infty} \frac{1}{(\ln k)^{\ln k}}$ converges