Linear Algebra (I) Midterm Exam 2

Linear Algebra I Midterm Exam 2

Time: 90 minutes

注意事項:

- 本題目紙背面也有試題。
- 請在答案紙的封面寫上自己的姓名。
- 請勿使用任何書籍、筆記或電子儀器。
- 請將答案寫在答案紙上並清楚標明題號。
- 答題順序不拘, 雙面皆可書寫, 但請勿將一題的答案分散在不連續的頁面上。
- 字跡請勿潦草, 以免批改者無法辨識。
- 每一題皆須邏輯正確無誤且論證完整清晰才能得到滿分。批改者可斟酌給予部分分數。
- 課堂上或習題中證明過的定理可以直接引用, 無須重新證明一次。
- 除了將英文翻譯成中文以外, 監考人員不回答任何跟試題有關的問題。
- 如有未盡事宜, 以監考人員的指示為準。

1. Determine if each of the following mappings is a linear operator on the space of polynomials $P(\mathbb{R})$. Prove your answers.

(a) (3 points) $T\colon P(\mathbb{R}) \to P(\mathbb{R})$ given by $T\big(f(x)\big) = \big(f(x)\big)^2$ for all $f(x) \in P(\mathbb{R})$.

(b) (3 points) $S\colon P(\mathbb{R}) \to P(\mathbb{R})$ given by $S\big(f(x)\big) = f(x^2)$ for all $f(x) \in P(\mathbb{R})$.

2. Let $W \subseteq \mathbb{R}^3$ be a plane containing the origin, and let $S\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection about W. Let $T = S - I_{\mathbb{R}^3}$, where $I_{\mathbb{R}^3}$ is the identity mapping on $\mathbb{R}^3$. Determine if each of the following statements about T is true or false, and prove your answers.

(a) (4 points) The null space of T is equal to W. nullity T = dim W − rank T

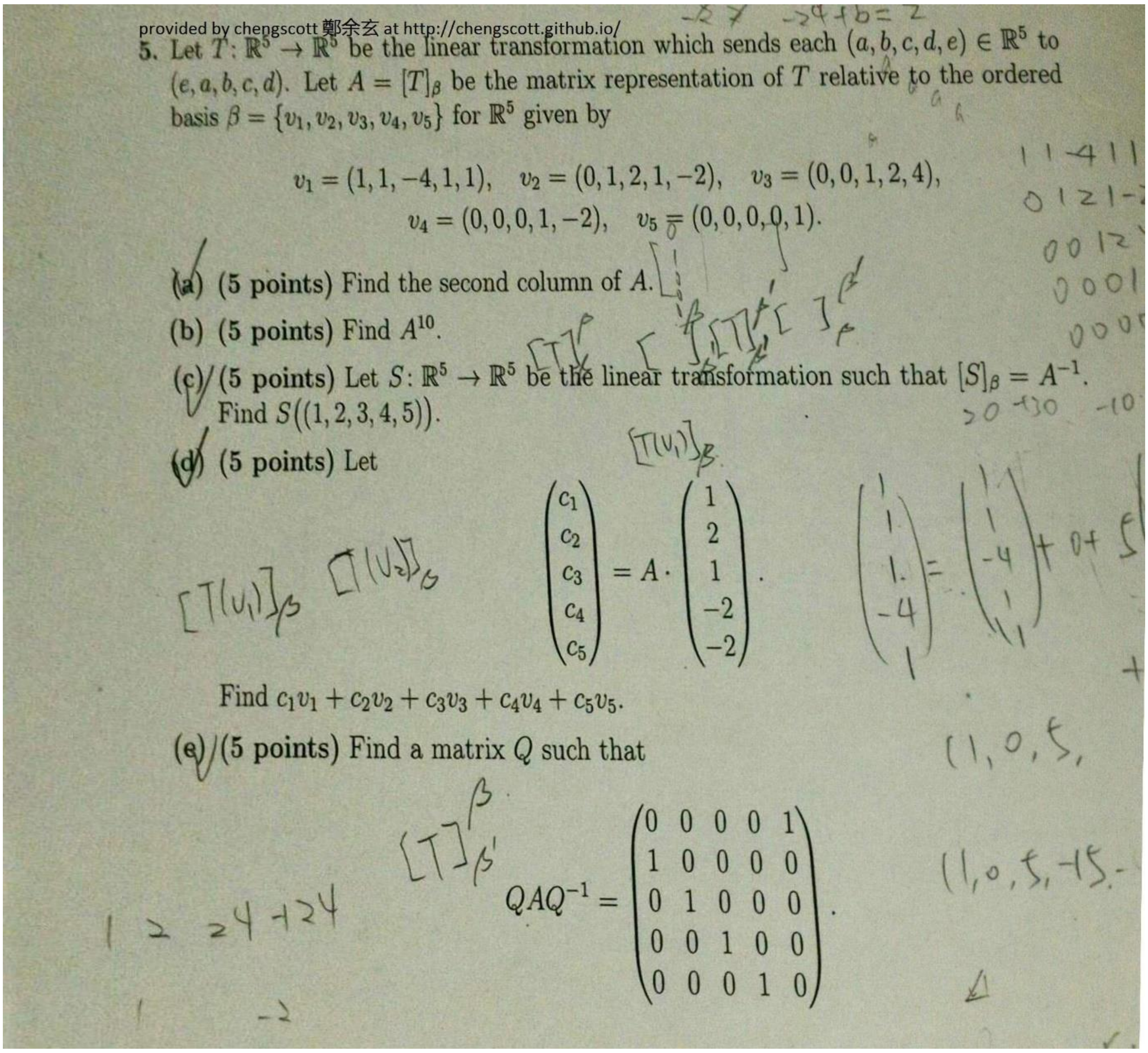
(b) (4 points) The rank of T is two.

3. (5 points) Let C be an $n\times n$ matrix. Define $T\colon M_{n\times n}(F) \to M_{n\times n}(F)$ by $T(A) = CA$. Show that T is an isomorphism if and only if C is invertible.

4. (6 points) Let V and W be finite-dimensional vector spaces over F. Let V_1 be a subspace of V, and let W_1 be a subspace of W. Show that there exists a linear transformation $T\colon V \to W$ such that $N(T) = V_1$ and $R(T) = W_1$ if and only if $\dim(V_1) + \dim(W_1) = \dim(V)$.

There are problems on the back page.

1



5. Let $T: \mathbb{R}^5 \to \mathbb{R}^5$ be the linear transformation which sends each $(a, b, c, d, e) \in \mathbb{R}^5$ to (e, a, b, c, d). Let $A = [T]_\beta$ be the matrix representation of T relative to the ordered basis $\beta = \{v_1, v_2, v_3, v_4, v_5\}$ for $\mathbb{R}^5$ given by

$$v_1 = (1, 1, -4, 1, 1), \quad v_2 = (0, 1, 2, 1, -2), \quad v_3 = (0, 0, 1, 2, 4),$$
$$v_4 = (0, 0, 0, 1, -2), \quad v_5 = (0, 0, 0, 0, 1).$$

(a) **(5 points)** Find the second column of A.

(b) **(5 points)** Find A^{10}.

(c) **(5 points)** Let $S: \mathbb{R}^5 \to \mathbb{R}^5$ be the linear transformation such that $[S]_\beta = A^{-1}$. Find $S((1, 2, 3, 4, 5))$.

(d) **(5 points)** Let

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = A \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \\ -2 \end{pmatrix}.$$

Find $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5$.

(e) **(5 points)** Find a matrix Q such that

$$QAQ^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Solutions:

2.

(a)

Claim 1: $N(T) \subseteq W$

$\forall x \in N(T), T(x) = 0$

$\therefore (S - I_{\mathbb{R}^3})(x) = 0$

$\therefore S(x) - x = 0$

$\therefore S(x) = x$

$\therefore x \in W$

Claim 2: $N(T) \supseteq W$

$\forall w \in W$

$T(w) = (S - I_{\mathbb{R}^3})(w) = S(w) - w = 0$

$\therefore w \in N(T)$

By 1 and 2, $N(T) = W$

(b)

$\because N(T) = W$

$\therefore nullity(T) = \dim W = 2$

By dimension theorem,

$rank(T) = 3 - nullity(T) = 1$

3.

$(\Rightarrow)$

Suppose C is not invertible

$\exists v \in F^n, v \neq O$ s.t. $Cv = 0$

Consider $B = \{v, \dots, v\}$

$\therefore T(B) = CB = O$

$\because T$ is an isomorphism

$\therefore T$ is injective (* contradict to $B \neq O$)

$\therefore C$ is invertible

$(\Leftarrow)$

$\forall A, B \in M_{n\times n}(F), \lambda \in F$

$T(A + \lambda B) = C(A + \lambda B) = CA + \lambda(CB) = T(A) + \lambda T(B)$

$\therefore T$ is linear

Let $T(A) = CA = O$

$\Rightarrow A = C^{-1}O = O$

$\therefore N(T) = \{O\}$

$\therefore T$ is injective

$\because \dim M_{n\times n}(F) = \dim M_{n\times n}(F)$

$\therefore T$ is bijective

$\therefore T$ is invertible

$\therefore T$ is an isomorphism

4.

(⇒)

$\dim V_1 + \dim W_1 = \dim N(T) + \dim R(T) = nullity(T) + rank(T) = \dim V$

(⇐)

Let $\dim V = n, \dim V_1 = m$

Let $\{v_1, \dots, v_m\}$ be a basis for V_1 and $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ be a basis for V

$\therefore \{v_{m+1}, \dots, v_n\}$ is a basis for W

$\forall v \in V, v = a_1 v_1 + \cdots + a_n v_n$ for $a_1, \dots, a_n \in F$

Define $T(v) = a_{m+1} v_{m+1} + \cdots + a_n v_n$

Claim: T is linear

$\forall u, v \in V, \lambda \in F, u = a_1 v_1 + \cdots + a_n v_n, v = b_1 v_1 + \cdots + b_n v_n$ for $a_1, \dots, a_n, b_1, \dots, b_n \in F$

$$\begin{aligned} T(u + \lambda v) &= (a_{m+1} + \lambda b_{m+1}) v_{m+1} + \cdots + (a_n + \lambda b_n) v_n \\ &= a_{m+1} v_{m+1} + \cdots + a_n v_n + \lambda (b_{m+1} v_{m+1} + \cdots + b_n v_n) = T(u) + \lambda T(v) \end{aligned}$$

$\therefore T$ is linear

$R(T) = span\{T(v_1), \dots, T(v_n)\} = span\{0, \dots, 0, v_{m+1}, \dots, v_n\} = W_1$

$\because T(v_i) = 0$ for $i = 1, \dots, m$

$\therefore V_1 \subseteq N(T)$

By Rank-Nullity theorem,

$\dim N(T) = n - \dim R(T) = n - \dim W_1 = n - m$

$\therefore N(T) = V_1$

5.

(a)

$[T(v_2)]_\beta = [(-2,0,1,2,1)]_\beta = (-2,2,-11,24,99)$

$$-2\begin{pmatrix}1\\1\\-4\\1\\1\end{pmatrix} + 2\begin{pmatrix}0\\1\\2\\1\\-2\end{pmatrix} - 11\begin{pmatrix}0\\0\\1\\2\\4\end{pmatrix} + 24\begin{pmatrix}0\\0\\0\\1\\-2\end{pmatrix} + 99\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}-2\\0\\1\\2\\1\end{pmatrix}$$

(b)

$A^{10} = [T]_\beta^{10} = [T^{10}]_\beta = [I_5]_\beta = I_5$

(c)

$S(1,2,3,4,5) = T^{-1}(1,2,3,4,5) = (2,3,4,5,1)$

(d)

$$[y]_\beta = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = A \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \\ -2 \end{pmatrix} = [T]_\beta [x]_\beta$$

$\Rightarrow x = v_1 + 2v_2 + v_3 - 2v_4 - 2v_5 = (1,3,1,3,3)$

$\because [T]_\beta [x]_\beta = [T(x)]_\beta = [y]_\beta$

$$\therefore y = T(x) = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$

(e)

Let β' be the standard basis for $\mathbb{R}^5$

$$[T]_\beta = [I_V]_{\beta'}^{\beta} [T]_{\beta'} [I_V]_{\beta}^{\beta'} \text{ where } [T]_{\beta'} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow Q = [I_V]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ -4 & 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 \\ 1 & -2 & 4 & -2 & 1 \end{pmatrix}$$