Calculus (I) First Midterm Examination October 20, 2015

CALCULUS (I)

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FIRST MIDTERM EXAMINATION, OCTOBER 20, 2015

Show your work, otherwise no credit will be granted.

1. (10 points) Evaluate $\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$.

2. (10 points) Differentiate $f(x) = \sqrt{x^2 + 2\sqrt{2+\sin x}}$.

3. (10 points) Define the sequence $\{a_n\}_{n=1}^{\infty}$ as follows: $a_1 = 1$ and $a_n = \sqrt{6 + a_{n-1}}$ for $n = 2, 3, 4, \cdots$. Does the sequence $\{a_n\}$ converge? Prove or disprove it.

4. (10 points) Let $g(x) = x^4 - 2x^3 + 6x + \cos x$. Show that, for any two $x, y \in [0, 1]$, we have $|g(x) - g(y)| \le 7|x - y|$.

5. (10 points) Find equation of the tangent line to the curve $x^2 + 4xy + y^2 = 13$ at the point $(2, 1)$.

6. (10 points) Let $g(x)$ be a bounded function on $[-1, 1]$. Define $f(x) = xg(x)$ on $[-1, 1]$.
   (i) Is $f(x)$ continuous at 0? Prove or disprove it.
   (ii) Is $f(x)$ differentiable at 0? Prove or disprove it.

7. (10 points) Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions. Define $h(x) = \max\{f(x), g(x)\}$. Show that h is also a continuous function on $[a, b]$.

8. (10 points) Let $g : [0, 1] \to [0, 1]$ be a continuous function. Prove that g has at least one fixed point.

9. (10 points) Find the local and global extreme values of the function $f(x)$ on $[-4, 10]$, where $f(x) = |x^3 - 6x^2 + 15x + 2|$.

10. (10 points) If $a_0 + a_1 + \cdots + a_k = 0$, show that

$$\lim_{n\to\infty} (a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}) = 0.$$

Solutions:

1.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3} &= \lim_{x\to 0} \frac{\tan x - \sin x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)} \\
&= \lim_{x\to 0} \frac{\sin x}{x^3} \frac{1-\cos x}{\cos x} \frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}} \\
&= \lim_{x\to 0} \frac{\sin x}{x^3} \frac{1-\cos^2 x}{\cos x} \frac{1}{1+\cos x} \frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}} \\
&= \lim_{x\to 0} \frac{\sin^3 x}{x^3} \frac{1}{\cos x} \frac{1}{1+\cos x} \frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}} = \frac{1}{4}
\end{aligned}$$

3.

(1) Claim $a_n < 3 \ \forall n \in \mathbb{N}$

When $n = 1$, $a_1 = 1 < 3$ is true

Assume $n = k$, $a_k < 3$ is true

When $n = k + 1$, $a_{k+1} = \sqrt{6 + a_k} < \sqrt{6 + 3} = 3$ is true

By M.I, $a_n < 3$ is true ($\Rightarrow a_n$ is bounded)

(2) Claim $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$

When $n = 1$, $a_1 = 1 < a_2 = \sqrt{6 + 1} = \sqrt{7}$ is true

Assume $n = k$, $a_k < a_{k+1}$ is true

When $n = k + 1$, $a_{k+1} = \sqrt{6 + a_k} < \sqrt{6 + a_{k+1}} = a_{k+2}$ is true

By M.I, $a_n < a_{n+1}$ is true ($\Rightarrow a_n$ is increasing)

By (1), (2), $\lim_{n\to\infty} a_n$ exists $\Rightarrow \{a_n\}$ converges

4.

Since g is continuous on $[x, y]$ and differentable on (x, y),

By M.V.T., $\exists c \in (x, y) \subseteq [0,1]$ s.t.

$$g'(c) = \frac{g(x) - g(y)}{x - y}$$

$$g'(x) = 4x^3 - 6x^2 + 6 - \sin x$$

Let $f(x) = 4x^3 - 6x^2 + 6$

$$f'(x) = 12x^2 - 12x = 12x(x - 1)$$

$f(x)$ is decreasing on $[0,1] \Rightarrow$ maximum of f is $f(0) = 6$

$$g'(x) = f(x) - \sin x \le f(0) - \sin x \le 6 - (-1) = 7$$

$\Rightarrow |g'(c)| \le 7$ for $c \in [0,1]$, that is

$$|g'(c)| = \frac{|g(x) - g(y)|}{|x - y|} \le 7 \Leftrightarrow |g(x) - g(y)| \le 7|x - y|$$

6.

(i)

Suppose $|g(x)| \le M$ $\forall x \in [-1,1]$ for some $M \in \mathbb{R}$

$$0 \le |f(x)| = |x||g(x)| \le M|x|$$

$\lim_{x\to 0} M|x| = 0 \xRightarrow{pinching} \lim_{x\to 0} f(x) = 0$ and also $f(0) = 0g(0) = 0$

$\Rightarrow f$ is a continuous function

(ii)

Suppose $g(x) = \begin{cases} 1, x \in [0,1] \\ -1, x \in [-1,0) \end{cases}$ is bounded

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{xg(x)}{x} = \lim_{x\to 0} g(x)$$

However, $\lim_{x\to 0^+} g(x) = 1$ and $\lim_{x\to 0^-} g(x) = -1 \Rightarrow \lim_{x\to 0} g(x)$ does not exist

$\Rightarrow f'(0)$ does not exist

$\Rightarrow f$ is not differentiable at 0

7.

Sol1:

Consider $h = \frac{f+g}{2} + \frac{|f-g|}{2}$

Sol2:

$f, g: [a,b] \to \mathbb{R}$ are continuous

$h(p) = \max\{f(p), g(p)\}, p \in [a,b]$

(1) $f(p) > g(p)$

(2) $f(p) < g(p)$

(3) $f(p) = g(p)$

Given $\varepsilon > 0$,

$\exists \delta_1 > 0$ s.t. $|f(x) - f(p)| < \varepsilon$ for $|x - p| < \delta_1$

$\exists \delta_2 > 0$ s.t. $|g(x) - g(p)| < \varepsilon$ for $|x - p| < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$

If $|x - p| < \delta$, then $|h(x) - h(p)| < \varepsilon$

8.

If $g(0) = 0$ or $g(1) = 1$, then we get a fixed point. Otherwise,

Let $f(x) = g(x) - x$ for $x \in [0,1]$

Since f is continuous on $[0,1]$ and

$f(0)f(1) = (g(0) - 0)(g(1) - 1) < 0$

By I.V.T., $\exists c \in (0,1)$ s.t.

$f(c) = 0 \Rightarrow g(c) = c$

i.e. c is a fixed point

10.

$a_0 = -(a_1 + \cdots + a_k)$

$$\Rightarrow \lim_{n\to\infty}\left(a_1\left(\sqrt{n+1}-\sqrt{n}\right)+\cdots+a_k\left(\sqrt{n+k}-\sqrt{n}\right)\right) = \lim_{n\to\infty}\left(\frac{a_1}{\sqrt{n+1}+\sqrt{n}}+\cdots+\frac{a_k}{\sqrt{n+k}+\sqrt{n}}\right)$$
$$= 0$$