Calculus Stewart Ch11 Problem Plus

1. If $f(x) = \sin(x^3)$, find $f^{(15)}(0)$.

Solution:

$$\sin(x^3) = 1 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots$$

$$\Rightarrow f^{(15)}(0) = \frac{15!}{5!}$$

3. (b) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n}$$

Solution:

Partial sum $s_n = \sum_{k=1}^{n} \frac{1}{2^k} \tan \frac{x}{2^k} = \sum_{k=1}^{n} \left(\frac{1}{2^k} \cot \frac{x}{2^{k+1}} - \frac{1}{2^k} \cot \frac{x}{2^k} \right) = \frac{1}{2^n} \cot \frac{x}{2^{n+1}} - \cot x$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} = \lim_{n\to\infty} s_n = \frac{1}{x} - \cot x$$

6.Find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$$

where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

Solution:

$$\sum_{k,j=0}^{\infty} \frac{1}{2^k} \frac{1}{3^j} = \left(\sum_{k=0}^{\infty} \frac{1}{2^k} \right) \left(\sum_{j=0}^{\infty} \frac{1}{3^j} \right) = \frac{1}{1 - \frac{1}{3}} \frac{1}{1 - \frac{1}{2}} = 3$$

7. (c) Deduce the following formula of John Machin:

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

Solution:

Let $\frac{1}{5} = \tan u$

$$\tan 2u = \frac{\frac{2}{5}}{1 - \left(\frac{1}{5}\right)^2} = \frac{5}{12}$$

$$\tan 4u = \frac{120}{119} \Rightarrow 4\arctan\frac{1}{5} = \arctan\frac{120}{119}$$

$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \arctan\frac{120}{119} - \arctan\frac{1}{239} = \arctan\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\frac{1}{239}} = \arctan 1 = \frac{\pi}{4}$$

8. (a) Prove a formula similar to the one in Problem 7(a) but involving arccot instead of arctan.

(b) Find the sum of the series $\sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1)$.

Solution:

(a)

$$\cot^{-1} x - \cot^{-1} y = \cot^{-1}\left(\frac{yx + 1}{y - x}\right)$$

(b)

$$\cot^{-1}(n^2 + n + 1) = \cot^{-1}\frac{n(n+1) - 1}{(n+1) - n} = \cot^{-1} n - \cot^{-1}(n+1)$$

$$\sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1) = \cot^{-1} 0 = \frac{\pi}{2}$$

9. Find the interval of convergence of $\sum_{n=0}^{\infty} n^3 x^n$ and find its sum.

Solution:

By ratio test, it converges when $|x| < 1$.

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}, \sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{2}{(1-x)^3}$$

$$\sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3} = \frac{6}{(1-x)^4}$$

$$k(k-1)(k-2) = k^3 - 3k^2 + 2k = k^3 - (3k^2 - 2k) = k^3 - 3k(k-1) - k$$

$$\frac{1}{x^3}\sum_{k=3}^{\infty} k^3 x^k = \frac{6}{(1-x)^4} + \frac{3}{x}\sum_{k=3}^{\infty} k(k-1)x^{k-2} + \frac{1}{x^2}\sum_{k=3}^{\infty} kx^{k-1}$$

$$\Rightarrow \sum_{k=1}^{\infty} k^3 x^k = x^3\left(\frac{6}{(1-x)^4} + \frac{3}{x}\left(\frac{2}{(1-x)^3} - 2\right) + \frac{1}{x^2}\left(\frac{1}{(1-x)^2} - 1 - 2x\right)\right) + 2^3x^2 + x$$

$$= \frac{6x^3}{(1-x)^4} + \frac{6x^2}{(1-x)^3} - 6x^2 + \frac{x}{(1-x)^2} - x - 2x^2 + 8x^3 + x$$

$$= \frac{6x^3 + 6x^2(1-x) + x(1-x)^2}{(1-x)^4} = \frac{x(x^2+4x+1)}{(1-x)^4}$$

11. Find the sum of the series

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$$

Solution:

$$\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) = \sum_{n=2}^{\infty} (\ln(n^2-1) - \ln n^2) = \sum_{n=2}^{\infty} \big((\ln(n+1) - \ln n) - (\ln n - \ln(n-1))\big)$$
$$= \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right) - \ln 2 - \ln 1 = -\ln 2$$

17. Taking the value of x^x at 0 be 1 and integrating a series term by term, show that

$$\int_0^1 x^x dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$$

Solution:

$$x^x = e^{x\ln x} = \sum_{n=0}^{\infty} \frac{(x\ln x)^n}{n!}$$

$$\int_0^1 x^x dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(x\ln x)^n}{n!}\right) dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(x\ln x)^n}{n!} dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (x\ln x)^n dx$$

$$\int_0^1 (x\ln x)^n dx = \int_0^1 (\ln x)^n \frac{dx^{n+1}}{n+1} = (\ln x)^n \frac{x^{n+1}}{n+1}\bigg|_{x=0}^{1} - \int_0^1 \frac{n}{n+1} (\ln x)^{n-1} x^{n+1} dx$$
$$= -\frac{n}{n+1} \int_0^1 (\ln x)^{n-1} x^{n+1} dx = \cdots = \frac{(-1)^n n!}{(n+1)^{n+1}}$$

$$\Rightarrow \int_0^1 x^x dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 (x\ln x)^n dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$$

19. Find the sum if the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

Solution:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = \sqrt{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)} \left(\frac{1}{\sqrt{3}}\right)^{2n+1} = -1 + \sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}\sqrt{3} - 1$$

25. Let

$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots$$

$$v = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots$$

$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} +$$

Show that $u^3 + v^3 + w^3 - 3uvw = 1$

Solution:

$u' = w, u(0) = 1$

$v' = u, v(0) = 0$

$w' = v, w(0) = 0$

$$\begin{aligned}(u^3 + v^3 + w^3 - 3uvw)' &= 3u^2u' + 3v^2v' + 3w^2w' - 3u'vw - 3uv'w - 3uvw' \\ &= 3u^2w + 3v^2u + 3w^2v - 3v^2w - 3u^2w - 3uv^2 = 0\end{aligned}$$

$\therefore u^3 + v^3 + w^3 - 3uvw = 1$