Calculus Stewart Ch1 Sec8

63. For what value of x is f continuous?

$$f(x) = \begin{cases} 0, \text{if } x \text{ is rational} \\ 1, \text{if } x \text{ is irrational} \end{cases}$$

Proof:

(1)

$\forall L > 0, c \in \mathbb{R}$

$\exists \varepsilon = \frac{L}{2} > 0 \; \forall \delta > 0$

$\because$ real number is dense

$\therefore \exists x_\delta \in \mathbb{Q}: 0 < |x_\delta - c| < \delta$

$|f(x_\delta) - L| = |0 - L| \geq \varepsilon = \frac{L}{2}$

(2)

$\forall L \leq 0, c \in \mathbb{R}$

$\exists \varepsilon = \frac{1}{2} > 0 \; \forall \delta > 0$

$\because$ real number is dense

$\therefore \exists x_\delta \in \mathbb{R} \backslash \mathbb{Q}: 0 < |x_\delta - c| < \delta$

$|f(x_\delta) - L| = |1 - L| \geq 1 + |L| > \varepsilon = \frac{1}{2}$

$\Rightarrow f$ has no limits on $\mathbb{R}$

$\Rightarrow f$ is continuous nowhere

68. (a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.

(b) Prove that if f is a continuous function on an interval, then so is $|f|$.

(c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If sp, prove it. If not, find a counterpart example.

Proof:

(a)

Given $\varepsilon > 0 \; \exists \delta = \varepsilon > 0$ s.t.

$\big||x| - |c|\big| \leq |x - c| < \epsilon = \delta$ for $0 < |x - c| < \delta$

$\therefore \lim_{x \to c} |x| = |c|$

(b)

$|f|$ is a composite of two continuous function $|y|$ and $y = f(x)$

$\Rightarrow |f(x)|$ is a continuous function

(c) NO

Consider $f = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

$|f(x)| = 1$ is a continuous function

Yet, $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$

$\Rightarrow f(x)$ is a not continuous function