Linear Algebra (I) Midterm Exam 1

Linear Algebra I Midterm Exam 1

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Time: 90 minutes

注意事項:

- 本題目紙背面也有試題。
- 請在答案紙的封面寫上自己的姓名。
- 請勿使用任何書籍、筆記或電子儀器。
- 請將答案寫在答案紙上並清楚標明題號。
- 答題順序不拘, 雙面皆可書寫, 但請勿將一題的答案分散在不連續的頁面上。
- 字跡請勿潦草, 以免批改者無法辨識。
- 每一題皆須邏輯正確無誤且論證完整清晰才能得到滿分。批改者可斟酌給予部分分數。
- 課堂上或習題中證明過的定理可以直接引用, 無須重新證明一次。
- 除了將英文翻譯成中文以外, 監考人員不回答任何跟試題有關的問題。
- 如有未盡事宜, 以監考人員的指示爲準。

1. In each of (a)–(c) there are two statements (i) and (ii), one true and one false. Identify the **false statements** and **provide explicit counterexamples** for them to show that they are false. Just claiming that a statement is false without giving a counterexample will not earn any points. You don't need to give proofs for the true statements.

(a) **(4 points)** Let V be an n-dimensional vector space, and let S be a finite subset of V. Which of the following two statements is false?

(i) If S generates V then $|S| \geq n$.

(ii) If S does not generate V then $|S| < n$.

(b) **(4 points)** Let V be a vector space over a field F, and let $S = \{u_1, \dots, u_n\}$ be a subset of V. Which of the following two statements is false?

(i) If S is linearly independent, then $a_1u_1 + \cdots + a_nu_n \neq 0$ for any $a_1, \dots, a_n \in F$ that are not all zero.

(ii) If S is linearly dependent, then $a_1u_1 + \cdots + a_nu_n = 0$ for any $a_1, \dots, a_n \in F$ that are not all zero.

(c) **(4 points)** Let V be a vector space over a field F, and let W be a subset of V. Which of the following two statements is false?

(i) If there exist $u, v \in W$ and $c \in F$ such that $u + v \in W$ and $cu \in W$, then W is a subspace of V.

(ii) If there exist $u, v \in W$ and $c \in F$ such that $u + v \notin W$ and $cu \notin W$, then W is not a subspace of V.

There are problems on the back page.

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2. **(5 points)** Show that $\{A \in M_{2\times 2}(\mathbb{R}) \mid A^2 = 0\}$ is **not** a subspace of $M_{2\times 2}(\mathbb{R})$.

3. **(5 points)** Let V be a vector space over a field F. Show that for any $u, v, w \in V$ and any $a, b, c \in F$, we have

$$\operatorname{span}\{u, v, w\} = \operatorname{span}\{u, au + v, bu + cv + w\}.$$

4. Let $S = \{x^3 - 2x^2 - 4x + 2, 3x^3 + x^2 + 2x - 1, x^3 + x^2 + 2x - 1\} \subseteq P_3(\mathbb{R})$.

   (a) **(5 points)** Show that S is linearly dependent.

   (b) **(5 points)** Find the dimension of $\operatorname{span}(S)$, and justify your answer.

5. **(6 points)** Let V be a vector space, and let S and T be finite subsets of V. Show that if $S \cup T$ is linearly independent, then $\operatorname{span}(S) \cap \operatorname{span}(T) = \{0\}$.

   $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$

6. **(6 points)** Let V be an n-dimensional vector space, and let $W \subsetneq V$ be a subspace of V. Show that if W contains a linearly independent subset S such that $|S| = n - 1$, then $\dim(W) = n - 1$ and S is a basis for W.

   $W = a_1v_1 + \cdots + a_{n-1}v_{n-1} = b_1v_1 + \cdots + b_{n-1}v_{n-1}$

7. **(6 points)** Let V be an n-dimensional vector space, and let $W \subseteq V$ be an m-dimensional subspace of V. Show that there exists an $(n-m)$-dimensional subspace U of V such that $W + U = V$.

Solutions:

1.

(a) (ii) is false

Let $V = \mathbb{R}^2, S = \{(1,0), (2,0), (3,0)\}$

$\therefore span(S) \neq \mathbb{R}^2$

Yet, $|S| = 3 > \dim \mathbb{R}^2 = 2$

(b) (ii) is false

Let $V = \mathbb{R}^2, S = \{(1,0), (2,0)\}$

Take $a_1 = 1, a_2 = 2$

$1 \cdot (1,0) + 2 \cdot (2,0) = (5,0) \neq (0,0)$

(c) (i) is false

Let $V = \mathbb{R}, W = \{1,2,3\}$

Take $u = 1, v = 2$

$\therefore u + v = 3 \in W$

Take $u = 1, c = 3$

$\therefore cv = 3 \in W$

Yet, W is not a subspace. ($\because 0 \notin W$)

2.

Let $S = \{A \in M_{2\times 2}(\mathbb{R}) | A^2 = 0\}$

Assume S is a subspace of $M_{2\times 2}(\mathbb{R})$

$\because \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}^2 = O$

$\therefore \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \in S$

$\because \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = O$

$\therefore \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in S$

$\because \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in S$

$\therefore \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \in S$

But, $\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \neq O$ (* contradicts)

$\therefore S$ is not a subspace of $M_{2\times 2}(\mathbb{R})$

3.

Claim 1: $\{u, v, w\} \subseteq span\{u, au + v, bu + cv + w\}$

$u = 1 \cdot u + 0 \cdot (au + v) + 0 \cdot (bu + cv + w)$

$v = (-a) \cdot u + 1 \cdot (au + v) + 0 \cdot (bu + cv + w)$

$w = (-b + ac) \cdot u + (-c) \cdot (au + v) + 1 \cdot (bu + cv + w)$

$\therefore \{u, v, w\} \subseteq span\{u, au + v, bu + cv + w\}$

Claim 2: $\{u, au + v, bu + cv + w\} \subseteq span\{u, v, w\}$

$u = a \cdot u + 0 \cdot v + 0 \cdot w$

$au + v = a \cdot u + 1 \cdot v + 0 \cdot w$

$bu + cv + w = b \cdot u + c \cdot v + 1 \cdot w$

$\therefore \{u, au + v, bu + cv + w\} \subseteq span\{u, v, w\}$

By 1 and 2, $span\{u, v, w\} = span\{u, au + v, bu + cv + w\}$

5.

Let $S = \{u_1, u_2, \dots, u_n\}, T = \{v_1, v_2, \dots, v_m\}$ for some $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m \in V$

$\forall w \in span(S) \cap span(T)$

$w = a_1 u_1 + \cdots + a_n u_n = b_1 v_1 + \cdots + b_m v_m$ for some $a_1, \dots, a_n, b_1, \dots, b_m \in F$

$\therefore a_1 u_1 + \cdots + a_n u_n + (-b_1) v_1 + \cdots + (-b_m) v_m = 0$

$\because S \cup T = \{u_1, \dots, u_n, v_1, \dots, v_m\}$ is linearly independent

$\therefore a_1 = \cdots = a_n = b_1 = \cdots = b_m = 0$

$\therefore w = 0$

$\therefore span(S) \cap span(T) = \{0\}$

6.

$\because W \subset V$ and $\dim V = n$

$\therefore \dim W < n$

$\because W$ contains a linear independent subset S with $|S| = n - 1$

$\because \dim W \geq n - 1$

$\therefore \dim W = n - 1$

$\because S$ is linear independent and $|S| = n - 1$

$\therefore S$ is a basis for W

7.

$\because \dim V = n, \dim W = m$ and $W \subset V$

$\therefore$ let $\{v_1, \dots, v_m\}$ be a basis for W and $\{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$ be a basis for V

$\therefore \{v_{m+1}, \dots, v_n\} \subset \{v_1, \dots, v_m, v_{m+1}, \dots, v_n\}$

Consider $U = span\{v_{m+1}, \dots, v_n\}$

Claim 1: $V \subseteq W + U$

$\forall v \in V$

$v = a_1 v_1 + \cdots + a_m v_m + a_{m+1} v_{m+1} + \cdots + a_n v_n$ for $a_1, \dots, a_m \in F$

$\because a_1 v_1 + \cdots + a_m v_m + a_{m+1} \in W$ and $a_{m+1} v_{m+1} + \cdots + a_n v_n \in U$

$\therefore v \in W + U$

Claim 2: $V \supseteq W + U$

$\forall t \in W + U, \exists w \in W, u \in U$ s.t. $t = w + u$

$= (a_1 v_1 + \cdots + a_m v_m) + (a_{m+1} v_{m+1} + \cdots + a_n v_n) \in span\{v_1, \dots, v_n\} = V$ for $a_1, \dots, a_m \in F$

$\therefore t \in V$

By 1 and 2, $V = W + U$