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2.2 MERGESORT

- ▶ *mergesort*
- ▶ *bottom-up mergesort*
- ▶ *sorting complexity*
- ▶ *comparators*
- ▶ *stability*

Two classic sorting algorithms

Critical components in the world's computational infrastructure.

- Full scientific understanding of their properties has enabled us to develop them into practical system sorts.
- Quicksort honored as one of top 10 algorithms of 20th century in science and engineering.

Mergesort. [this lecture]

- Java sort for objects.
- Perl, C++ stable sort, Python stable sort, Firefox JavaScript, ...

Quicksort. [next lecture]

- Java sort for primitive types.
- C qsort, Unix, Visual C++, Python, Matlab, Chrome JavaScript, ...

Algorithms

ROBERT SEDGEWICK | KEVIN WAYNE

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2.2 MERGESORT

- ▶ *mergesort*
- ▶ *bottom-up mergesort*
- ▶ *sorting complexity*
- ▶ *comparators*
- ▶ *stability*

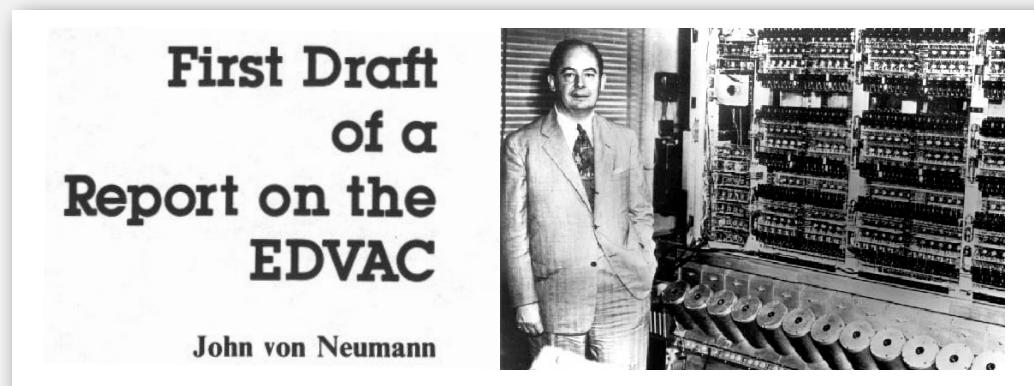
Mergesort

Basic plan.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves.

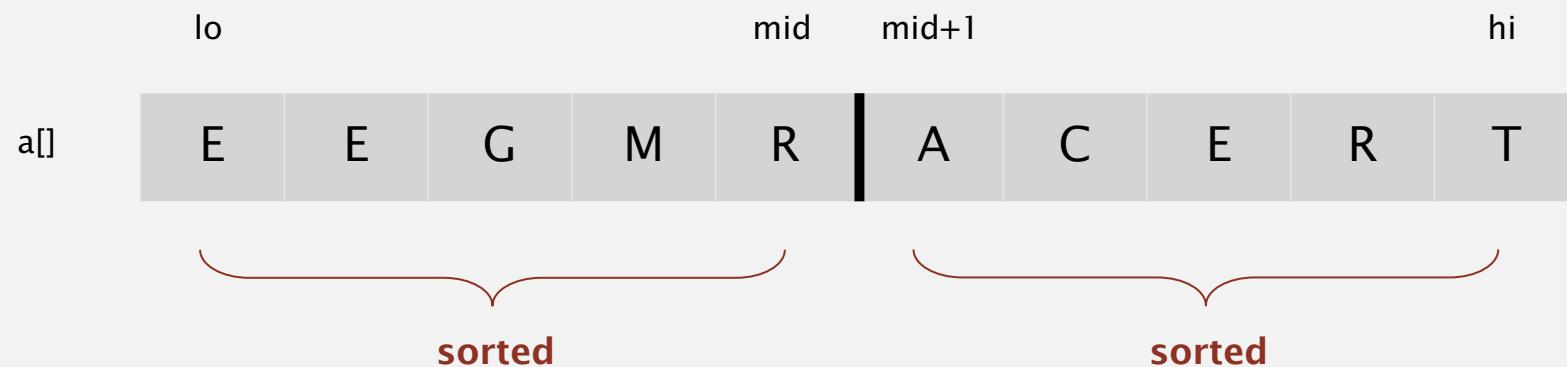
| | | | | | | | | | | | | | | | | | |
|-----------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| input | M | E | R | G | E | S | O | R | T | E | X | A | M | P | L | E | |
| sort left half | E | E | G | M | M | O | R | R | S | T | E | X | A | M | P | L | E |
| sort right half | E | E | G | M | M | O | R | R | S | A | E | E | L | M | P | T | X |
| merge results | A | E | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |

Mergesort overview



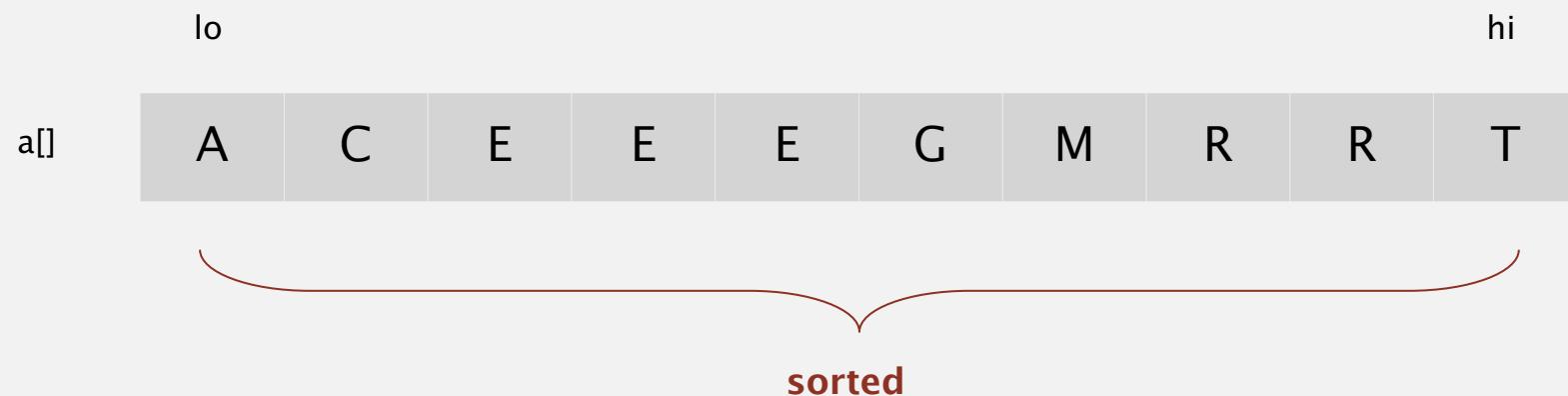
Abstract in-place merge demo

Goal. Given two sorted subarrays $a[lo]$ to $a[mid]$ and $a[mid+1]$ to $a[hi]$, replace with sorted subarray $a[lo]$ to $a[hi]$.



Abstract in-place merge demo

Goal. Given two sorted subarrays $a[lo]$ to $a[mid]$ and $a[mid+1]$ to $a[hi]$, replace with sorted subarray $a[lo]$ to $a[hi]$.



Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
    assert isSorted(a, lo, mid);      // precondition: a[lo..mid] sorted
    assert isSorted(a, mid+1, hi);   // precondition: a[mid+1..hi] sorted

    for (int k = lo; k <= hi; k++)
        aux[k] = a[k];                                copy

    int i = lo, j = mid+1;
    for (int k = lo; k <= hi; k++)                  merge
    {
        if (i > mid)                      a[k] = aux[j++];
        else if (j > hi)                  a[k] = aux[i++];
        else if (less(aux[j], aux[i]))    a[k] = aux[j++];
        else                            a[k] = aux[i++];
    }

    assert isSorted(a, lo, hi);      // postcondition: a[lo..hi] sorted
}
```



Assertions

Assertion. Statement to test assumptions about your program.

- Helps detect logic bugs.
- Documents code.

Java assert statement. Throws exception unless boolean condition is true.

```
assert isSorted(a, lo, hi);
```

Can enable or disable at runtime. ⇒ No cost in production code.

```
java -ea MyProgram    // enable assertions  
java -da MyProgram    // disable assertions (default)
```

Best practices. Use assertions to check internal invariants;
assume assertions will be disabled in production code. ←

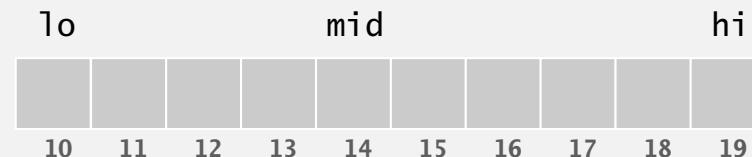
do not use for external
argument checking

Mergesort: Java implementation

```
public class Merge
{
    private static void merge(...)

    private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
    {
        if (hi <= lo) return;
        int mid = lo + (hi - lo) / 2;
        sort(a, aux, lo, mid);
        sort(a, aux, mid+1, hi);
        merge(a, aux, lo, mid, hi);
    }

    public static void sort(Comparable[] a)
    {
        aux = new Comparable[a.length];
        sort(a, aux, 0, a.length - 1);
    }
}
```



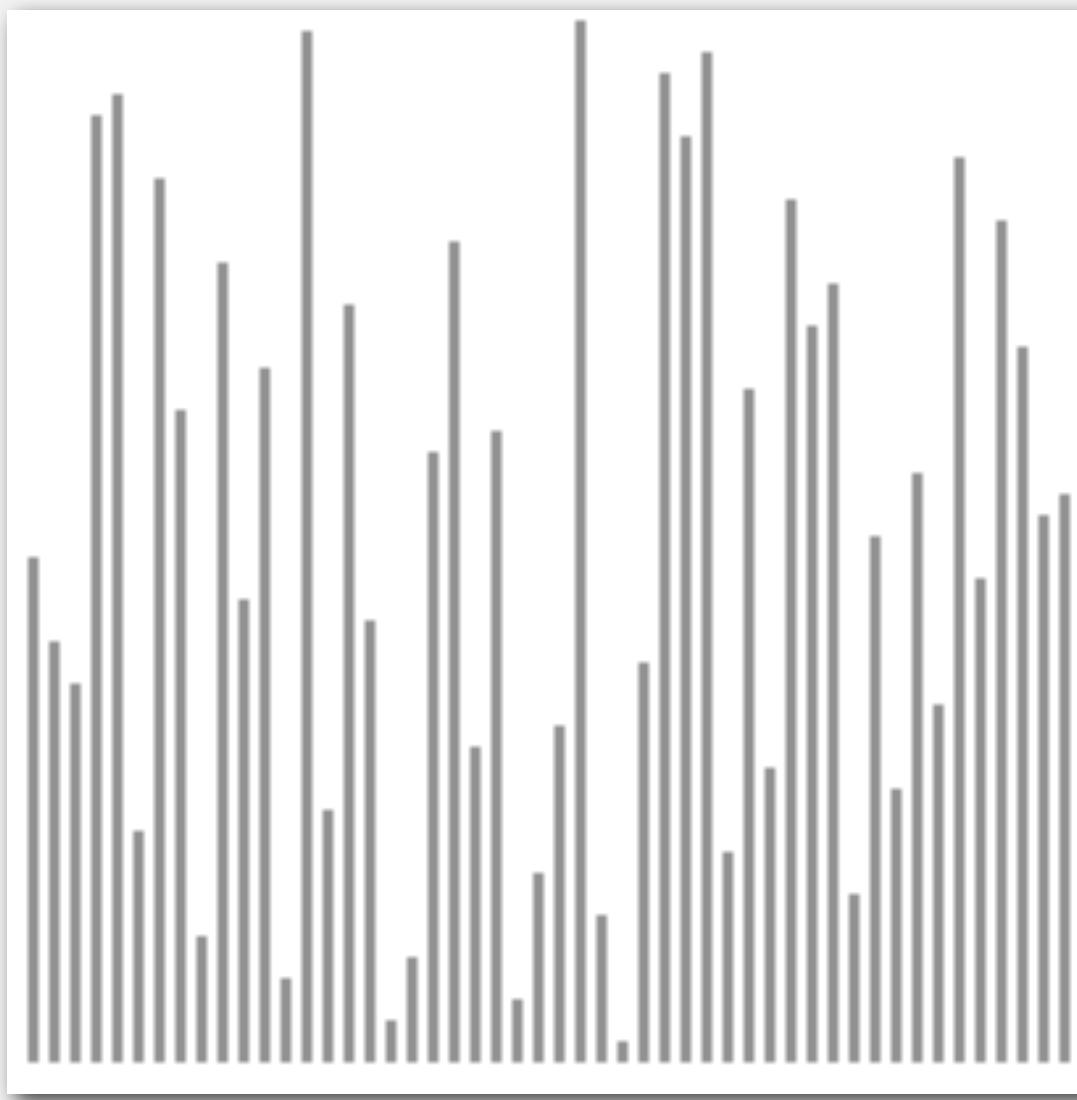
Mergesort: trace

| | a[] | | | | | | | | | | | | | | | | |
|---------------------------|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | |
| lo | M | E | R | G | E | S | O | R | T | E | X | A | M | P | L | E | |
| hi | E | M | R | G | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 0, 0, 1) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 2, 2, 3) | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 0, 1, 3) | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 4, 4, 5) | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 6, 6, 7) | E | G | M | R | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 4, 5, 7) | E | G | M | R | E | O | R | S | T | E | X | A | M | P | L | E | |
| merge(a, aux, 0, 3, 7) | E | E | G | M | O | R | R | S | T | E | X | A | M | P | L | E | |
| merge(a, aux, 8, 8, 9) | E | E | G | M | O | R | R | S | E | T | X | A | M | P | L | E | |
| merge(a, aux, 10, 10, 11) | E | E | G | M | O | R | R | S | E | T | A | X | M | P | L | E | |
| merge(a, aux, 8, 9, 11) | E | E | G | M | O | R | R | S | A | E | T | X | M | P | L | E | |
| merge(a, aux, 12, 12, 13) | E | E | G | M | O | R | R | S | A | E | T | X | M | P | L | E | |
| merge(a, aux, 14, 14, 15) | E | E | G | M | O | R | R | S | A | E | T | X | M | P | E | L | |
| merge(a, aux, 12, 13, 15) | E | E | G | M | O | R | R | S | A | E | T | X | E | L | M | P | |
| merge(a, aux, 8, 11, 15) | E | E | G | M | O | R | R | S | A | E | E | L | M | P | T | X | |
| merge(a, aux, 0, 7, 15) | A | E | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |

result after recursive call

Mergesort: animation

50 random items

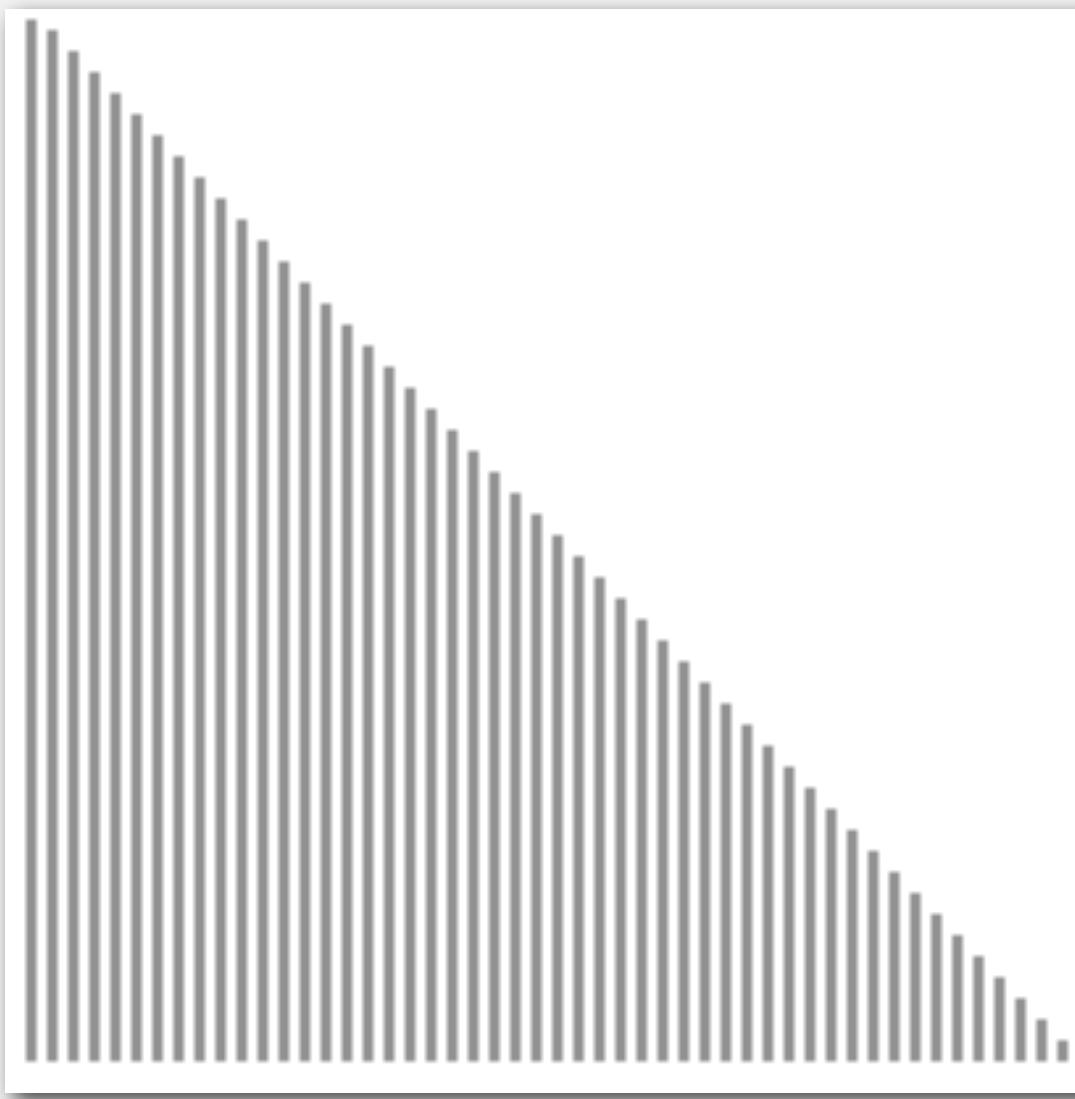


<http://www.sorting-algorithms.com/merge-sort>

- ▲ algorithm position
- █ in order
- ▒ current subarray
- ░ not in order

Mergesort: animation

50 reverse-sorted items



<http://www.sorting-algorithms.com/merge-sort>

- ▲ algorithm position
 - █ in order
 - █ current subarray
 - █ not in order

Mergesort: empirical analysis

Running time estimates:

- Laptop executes 10^8 compares/second.
- Supercomputer executes 10^{12} compares/second.

| computer | insertion sort (N^2) | | | mergesort ($N \log N$) | | |
|----------|--------------------------|-----------|-----------|--------------------------|----------|---------|
| | thousand | million | billion | thousand | million | billion |
| home | instant | 2.8 hours | 317 years | instant | 1 second | 18 min |
| super | instant | 1 second | 1 week | instant | instant | instant |

Bottom line. Good algorithms are better than supercomputers.

Mergesort: number of compares and array accesses

Proposition. Mergesort uses at most $N \lg N$ compares and $6N \lg N$ array accesses to sort any array of size N .

Pf sketch. The number of compares $C(N)$ and array accesses $A(N)$ to mergesort an array of size N satisfy the recurrences:

$$C(N) \leq C(\lceil N/2 \rceil) + C(\lfloor N/2 \rfloor) + N \text{ for } N > 1, \text{ with } C(1) = 0.$$



$$A(N) \leq A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N \text{ for } N > 1, \text{ with } A(1) = 0.$$

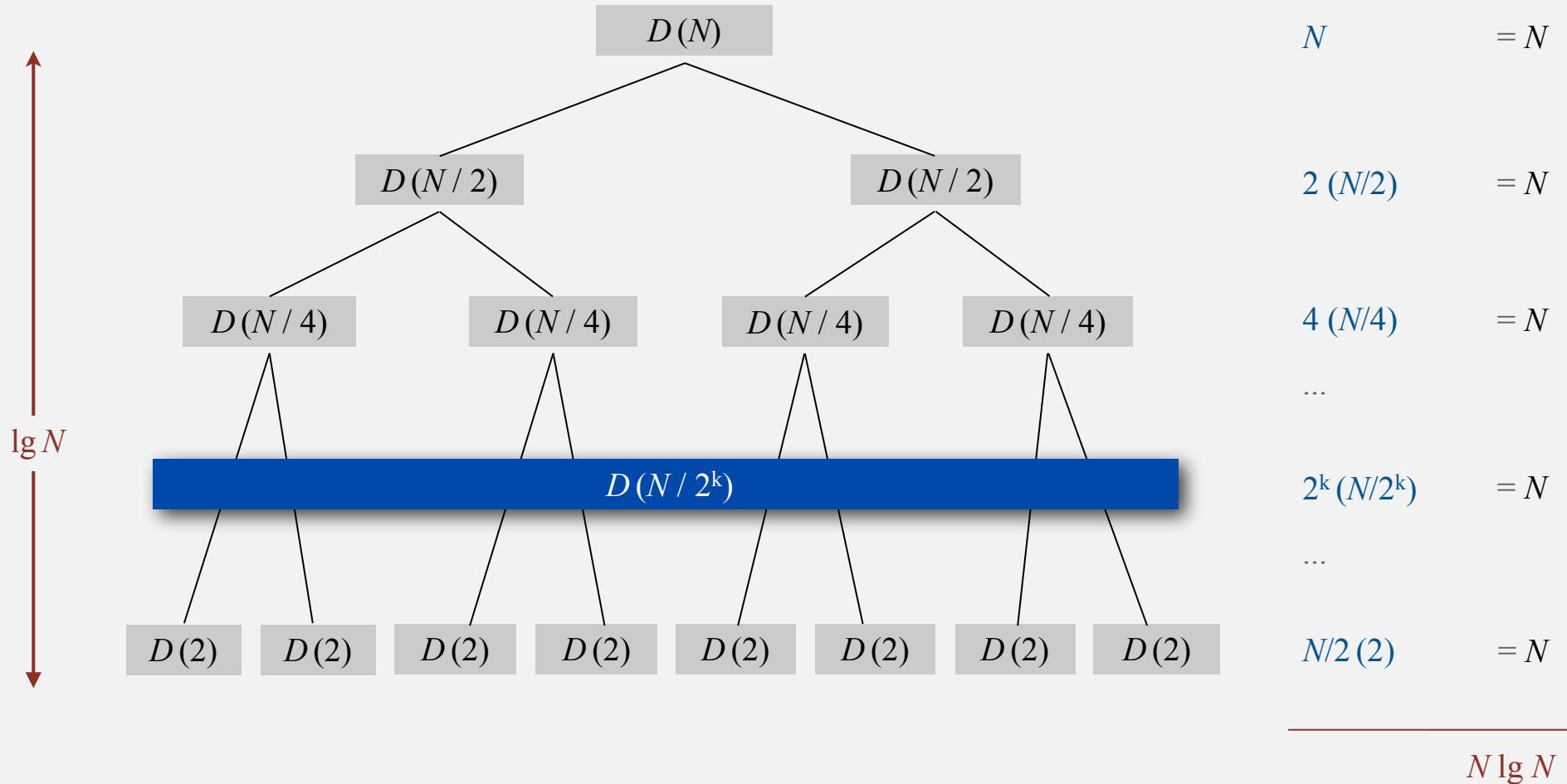
We solve the recurrence when N is a power of 2. \leftarrow result holds for all N

$$D(N) = 2D(N/2) + N, \text{ for } N > 1, \text{ with } D(1) = 0.$$

Divide-and-conquer recurrence: proof by picture

Proposition. If $D(N)$ satisfies $D(N) = 2 D(N / 2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 1. [assuming N is a power of 2]



$$N \lg N$$

Divide-and-conquer recurrence: proof by expansion

Proposition. If $D(N)$ satisfies $D(N) = 2D(N/2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 2. [assuming N is a power of 2]

$$D(N) = 2D(N/2) + N$$

given

$$D(N)/N = 2D(N/2)/N + 1$$

divide both sides by N

$$= D(N/2)/(N/2) + 1$$

algebra

$$= D(N/4)/(N/4) + 1 + 1$$

apply to first term

$$= D(N/8)/(N/8) + 1 + 1 + 1$$

apply to first term again

...

$$= D(N/N)/(N/N) + 1 + 1 + \dots + 1$$

stop applying, $D(1) = 0$

$$= \lg N$$

Divide-and-conquer recurrence: proof by induction

Proposition. If $D(N)$ satisfies $D(N) = 2 D(N / 2) + N$ for $N > 1$, with $D(1) = 0$, then $D(N) = N \lg N$.

Pf 3. [assuming N is a power of 2]

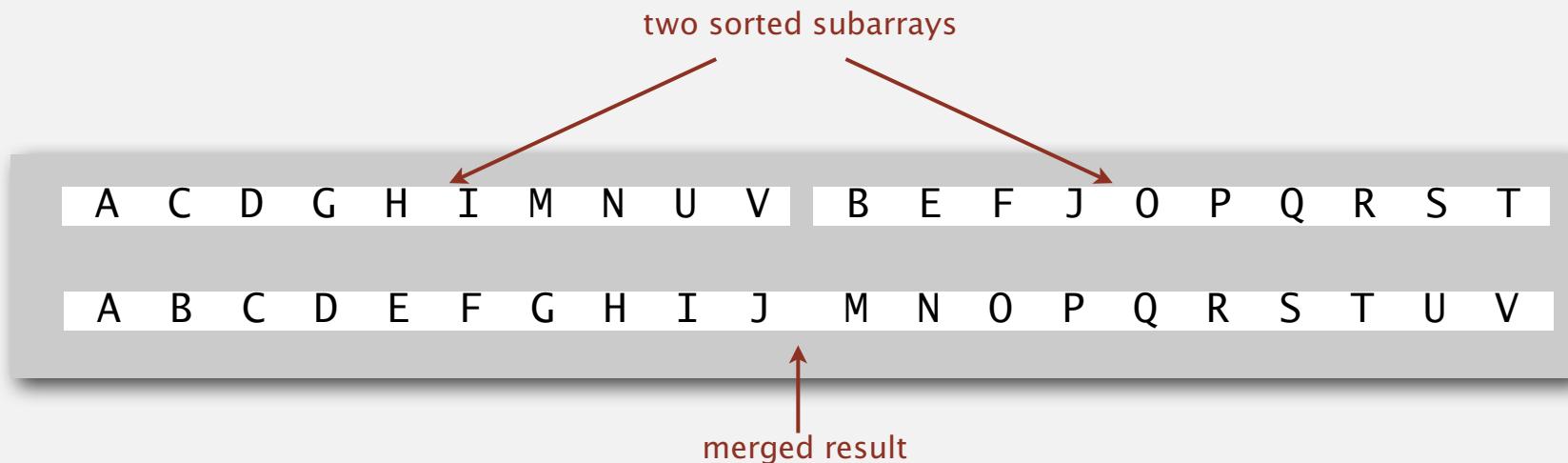
- Base case: $N = 1$.
- Inductive hypothesis: $D(N) = N \lg N$.
- Goal: show that $D(2N) = (2N) \lg (2N)$.

$$\begin{aligned} D(2N) &= 2 D(N) + 2N && \text{given} \\ &= 2 N \lg N + 2N && \text{inductive hypothesis} \\ &= 2 N (\lg (2N) - 1) + 2N && \text{algebra} \\ &= 2 N \lg (2N) && \text{QED} \end{aligned}$$

Mergesort analysis: memory

Proposition. Mergesort uses extra space proportional to N .

Pf. The array $\text{aux}[]$ needs to be of size N for the last merge.



Def. A sorting algorithm is **in-place** if it uses $\leq c \log N$ extra memory.

Ex. Insertion sort, selection sort, shellsort.

Challenge for the bored. In-place merge. [Kronrod, 1969]

Mergesort: practical improvements

Use insertion sort for small subarrays.

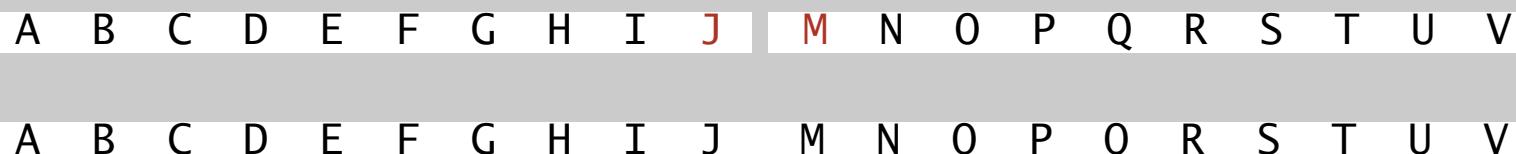
- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 7 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
    if (hi <= lo + CUTOFF - 1)
    {
        Insertion.sort(a, lo, hi);
        return;
    }
    int mid = lo + (hi - lo) / 2;
    sort (a, aux, lo, mid);
    sort (a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvements

Stop if already sorted.

- Is biggest item in first half \leq smallest item in second half?
- Helps for partially-ordered arrays.



```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
    if (hi <= lo) return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    if (!less(a[mid+1], a[mid])) return;
    merge(a, aux, lo, mid, hi);
}
```

Mergesort: practical improvements

Eliminate the copy to the auxiliary array. Save time (but not space) by switching the role of the input and auxiliary array in each recursive call.

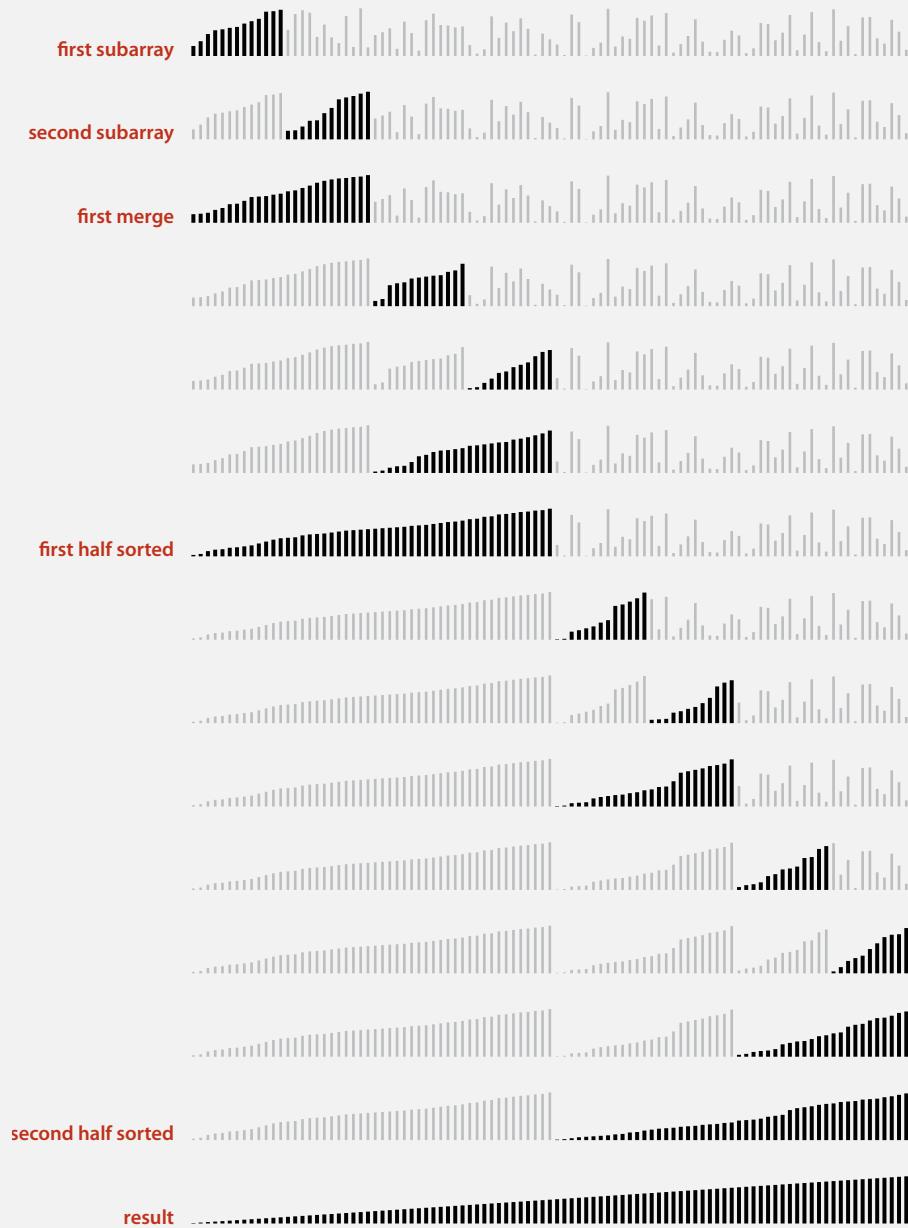
```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
    int i = lo, j = mid+1;
    for (int k = lo; k <= hi; k++)
    {
        if (i > mid)          aux[k] = a[j++];
        else if (j > hi)       aux[k] = a[i++];
        else if (less(a[j], a[i])) aux[k] = a[j++]; ← merge from a[] to aux[]
        else                   aux[k] = a[i++];
    }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
```

```
    if (hi <= lo) return;
    int mid = lo + (hi - lo) / 2;
    sort (aux, a, lo, mid);
    sort (aux, a, mid+1, hi);      Note: sort(a) initializes aux[] and sets
    merge(a, aux, lo, mid, hi);    aux[i] = a[i] for each i.
}
```

switch roles of aux[] and a[]

Mergesort: visualization



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- ▶ *bottom-up mergesort*
- ▶ *sorting complexity*
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- ▶ *stability*

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Bottom-up mergesort

Basic plan.

- Pass through array, merging subarrays of size 1.
- Repeat for subarrays of size 2, 4, 8, 16, ...

| a[i] | | | | | | | | | | | | | | | | | |
|---------------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|---|
| sz = 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | |
| merge(a, aux, 0, 0, 1) | E | M | R | G | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 2, 2, 3) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 4, 4, 5) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 6, 6, 7) | E | M | G | R | E | S | O | R | T | E | X | A | M | P | L | E | |
| merge(a, aux, 8, 8, 9) | E | M | G | R | E | S | O | R | E | T | X | A | M | P | L | E | |
| merge(a, aux, 10, 10, 11) | E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E | |
| merge(a, aux, 12, 12, 13) | E | M | G | R | E | S | O | R | E | T | A | X | M | P | L | E | |
| merge(a, aux, 14, 14, 15) | E | M | G | R | E | S | O | R | E | T | A | X | M | P | E | L | |
| sz = 2 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | |
| merge(a, aux, 0, 1, 3) | E | G | M | R | E | S | O | R | E | T | A | X | M | P | E | L | |
| merge(a, aux, 4, 5, 7) | E | G | M | R | E | O | R | S | E | T | A | X | M | P | E | L | |
| merge(a, aux, 8, 9, 11) | E | G | M | R | E | O | R | S | A | E | T | X | M | P | E | L | |
| merge(a, aux, 12, 13, 15) | E | G | M | R | E | O | R | S | A | E | T | X | E | L | M | P | |
| sz = 4 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | |
| merge(a, aux, 0, 3, 7) | E | E | G | M | O | R | R | S | A | E | T | X | E | L | M | P | |
| merge(a, aux, 8, 11, 15) | E | E | G | M | O | R | R | S | A | E | E | L | M | P | T | X | |
| sz = 8 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | |
| merge(a, aux, 0, 7, 15) | A | E | E | E | E | E | G | L | M | M | O | P | R | R | S | T | X |

Bottom-up mergesort: Java implementation

```
public class MergeBU
{
    private static void merge(...)
    { /* as before */ }

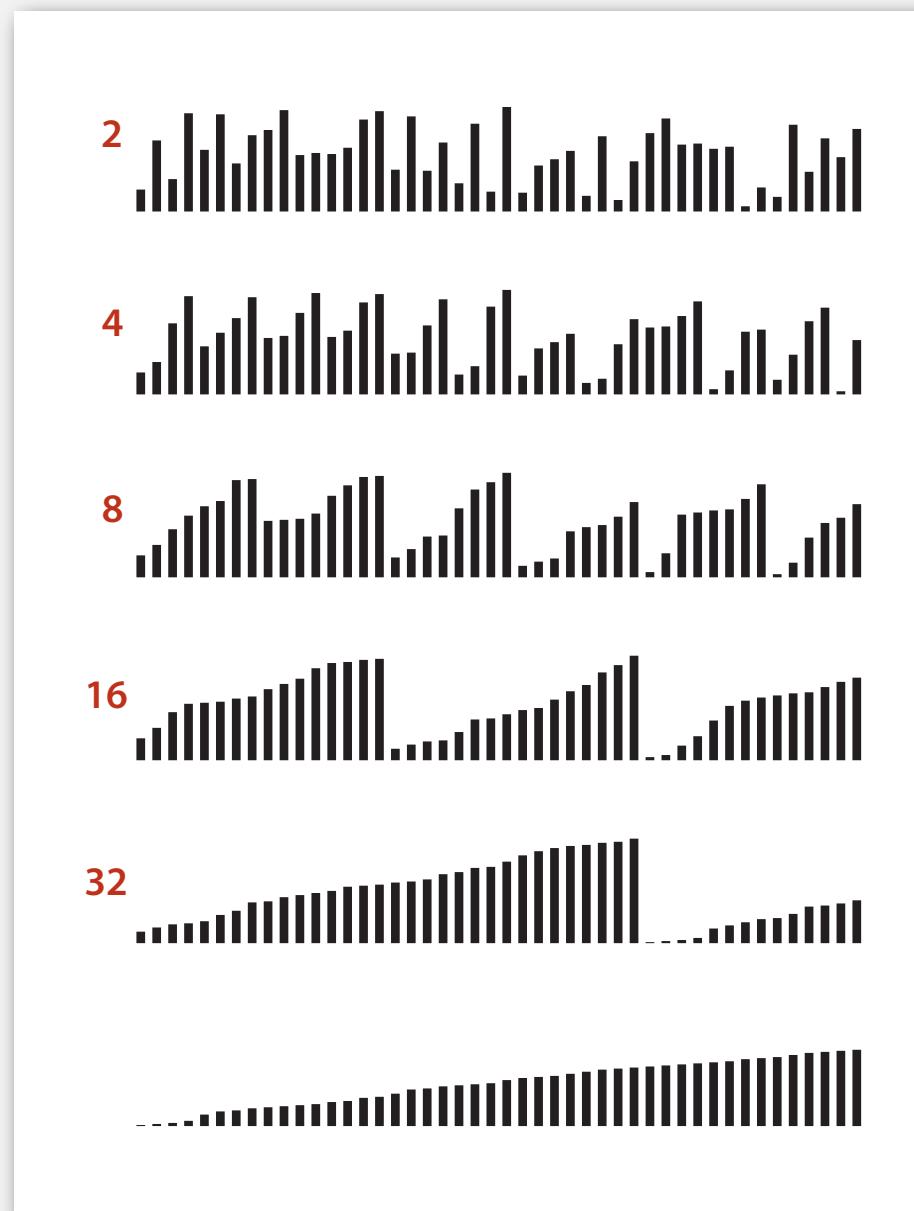
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        Comparable[] aux = new Comparable[N];
        for (int sz = 1; sz < N; sz = sz+sz)
            for (int lo = 0; lo < N-sz; lo += sz+sz)
                merge(a, aux, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
    }
}
```

but about 10% slower than recursive,
top-down mergesort on typical systems

Bottom line. Simple and non-recursive version of mergesort.



Bottom-up mergesort: visual trace



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Complexity of sorting

Computational complexity. Framework to study efficiency of algorithms for solving a particular problem X .

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by **some** algorithm for X .

Lower bound. Proven limit on cost guarantee of **all** algorithms for X .

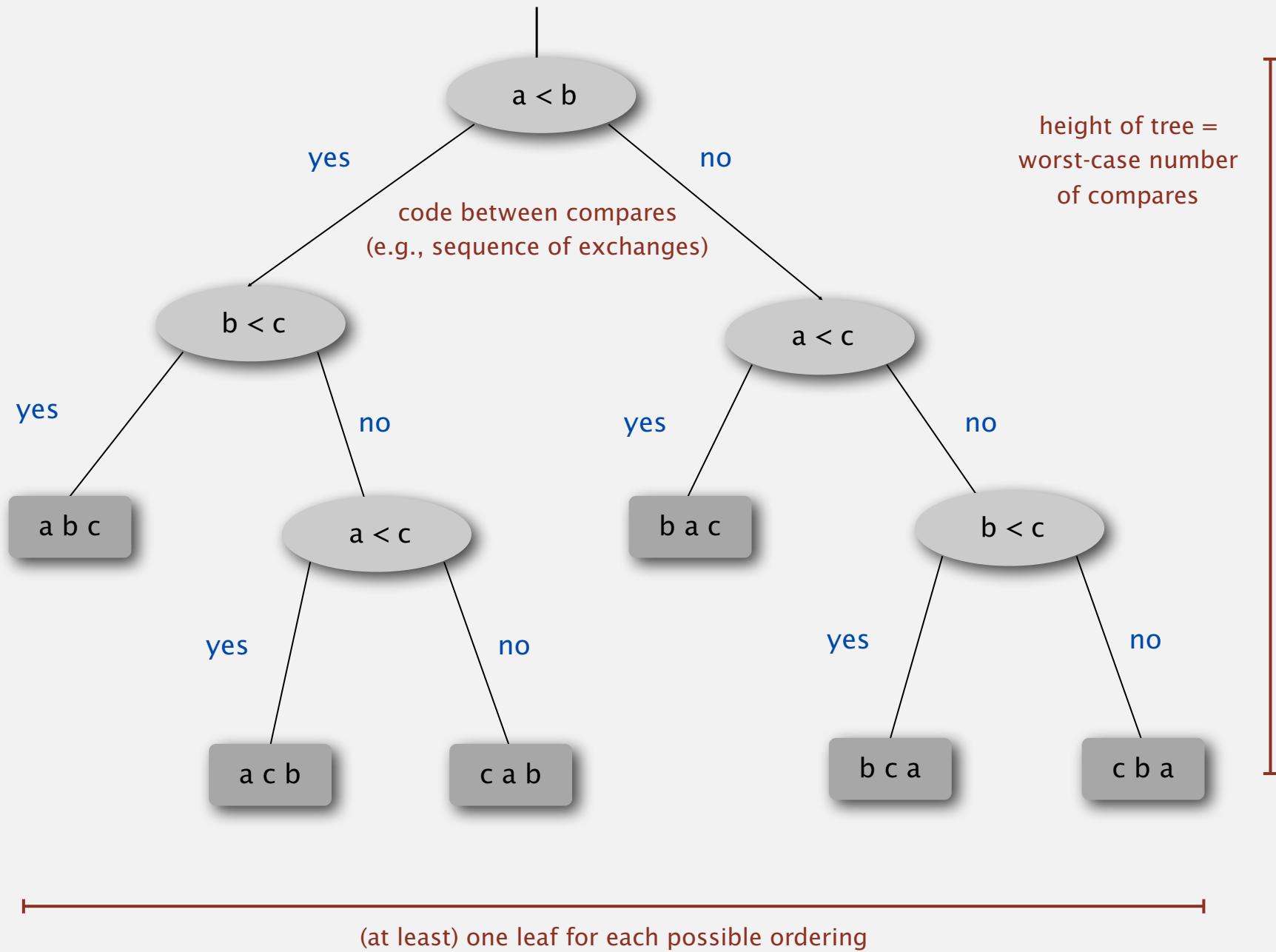
Optimal algorithm. Algorithm with best possible cost guarantee for X .

lower bound \sim upper bound

Example: sorting.

- Model of computation: decision tree. ← can access information only through compares
(e.g., Java Comparable framework)
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: ?
- Optimal algorithm: ?

Decision tree (for 3 distinct items a, b, and c)

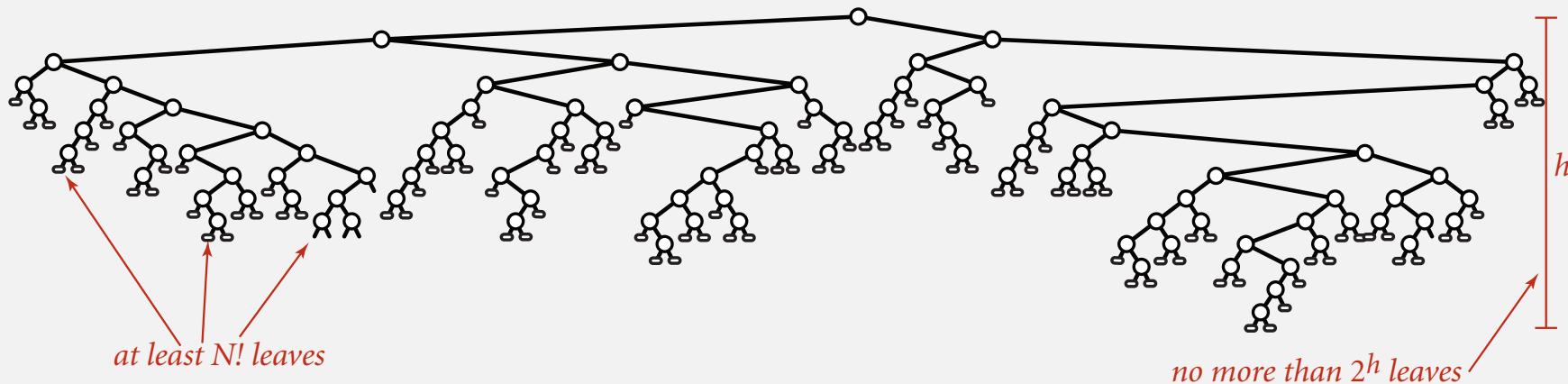


Compare-based lower bound for sorting

Proposition. Any compare-based sorting algorithm must use at least $\lg(N!) \sim N \lg N$ compares in the worst-case.

Pf.

- Assume array consists of N distinct values a_1 through a_N .
- Worst case dictated by **height** h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N!$ different orderings \Rightarrow at least $N!$ leaves.



Compare-based lower bound for sorting

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Pf.

- Assume array consists of N distinct values a_1 through a_N .
- Worst case dictated by **height** h of decision tree.
- Binary tree of height h has at most 2^h leaves.
- $N!$ different orderings \Rightarrow at least $N!$ leaves.

$$2^h \geq \# \text{leaves} \geq N!$$
$$\Rightarrow h \geq \lg(N!) \sim N \lg N$$

↑
Stirling's formula

Stirling's formula:

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$



$$\lg n! = \frac{\lg 2\pi n}{2} + n \lg \frac{n}{e}$$

Complexity of sorting

Model of computation. Allowable operations.

Cost model. Operation count(s).

Upper bound. Cost guarantee provided by some algorithm for X .

Lower bound. Proven limit on cost guarantee of all algorithms for X .

Optimal algorithm. Algorithm with best possible cost guarantee for X .

Example: sorting.

- Model of computation: decision tree.
- Cost model: # compares.
- Upper bound: $\sim N \lg N$ from mergesort.
- Lower bound: $\sim N \lg N$.
- Optimal algorithm = mergesort.

First goal of algorithm design: optimal algorithms.

Complexity results in context

Compares? Mergesort **is** optimal with respect to number compares.
Space? Mergesort **is not** optimal with respect to space usage.



Lessons. Use theory as a guide.

- Ex. Design sorting algorithm that guarantees $\frac{1}{2} N \lg N$ compares?
- Ex. Design sorting algorithm that is both time- and space-optimal?

Complexity results in context (continued)

Lower bound may not hold if the algorithm has information about:

- The initial order of the input.
- The distribution of key values.
- The representation of the keys.

Partially-ordered arrays. Depending on the initial order of the input,
we may not need $N \lg N$ compares.

insertion sort requires only $N-1$
compares if input array is sorted

Duplicate keys. Depending on the input distribution of duplicates,
we may not need $N \lg N$ compares.

stay tuned for 3-way quicksort

Digital properties of keys. We can use digit/character compares instead of
key compares for numbers and strings.

stay tuned for radix sorts

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Sort music library by artist name

| | Name | Artist | Time | Album |
|----|--|-------------------|------|---|
| 12 | <input checked="" type="checkbox"/> Let It Be | The Beatles | 4:03 | Let It Be |
| 13 | <input checked="" type="checkbox"/> Take My Breath Away | BERLIN | 4:13 | Top Gun – Soundtrack |
| 14 | <input checked="" type="checkbox"/> Circle Of Friends | Better Than Ezra | 3:27 | Empire Records |
| 15 | <input checked="" type="checkbox"/> Dancing With Myself | Billy Idol | 4:43 | Don't Stop |
| 16 | <input checked="" type="checkbox"/> Rebel Yell | Billy Idol | 4:49 | Rebel Yell |
| 17 | <input checked="" type="checkbox"/> Piano Man | Billy Joel | 5:36 | Greatest Hits Vol. 1 |
| 18 | <input checked="" type="checkbox"/> Pressure | Billy Joel | 3:16 | Greatest Hits, Vol. II (1978 – 1985) (Disc 2) |
| 19 | <input checked="" type="checkbox"/> The Longest Time | Billy Joel | 3:36 | Greatest Hits, Vol. II (1978 – 1985) (Disc 2) |
| 20 | <input checked="" type="checkbox"/> Atomic | Blondie | 3:50 | Atomic: The Very Best Of Blondie |
| 21 | <input checked="" type="checkbox"/> Sunday Girl | Blondie | 3:15 | Atomic: The Very Best Of Blondie |
| 22 | <input checked="" type="checkbox"/> Call Me | Blondie | 3:33 | Atomic: The Very Best Of Blondie |
| 23 | <input checked="" type="checkbox"/> Dreaming | Blondie | 3:06 | Atomic: The Very Best Of Blondie |
| 24 | <input checked="" type="checkbox"/> Hurricane | Bob Dylan | 8:32 | Desire |
| 25 | <input checked="" type="checkbox"/> The Times They Are A-Changin' | Bob Dylan | 3:17 | Greatest Hits |
| 26 | <input checked="" type="checkbox"/> Livin' On A Prayer | Bon Jovi | 4:11 | Cross Road |
| 27 | <input checked="" type="checkbox"/> Beds Of Roses | Bon Jovi | 6:35 | Cross Road |
| 28 | <input checked="" type="checkbox"/> Runaway | Bon Jovi | 3:53 | Cross Road |
| 29 | <input checked="" type="checkbox"/> Rasputin (Extended Mix) | Boney M | 5:50 | Greatest Hits |
| 30 | <input checked="" type="checkbox"/> Have You Ever Seen The Rain | Bonnie Tyler | 4:10 | Faster Than The Speed Of Night |
| 31 | <input checked="" type="checkbox"/> Total Eclipse Of The Heart | Bonnie Tyler | 7:02 | Faster Than The Speed Of Night |
| 32 | <input checked="" type="checkbox"/> Straight From The Heart | Bonnie Tyler | 3:41 | Faster Than The Speed Of Night |
| 33 | <input checked="" type="checkbox"/> Holding Out For A Hero | Bonny Tyler | 5:49 | Meat Loaf And Friends |
| 34 | <input checked="" type="checkbox"/> Dancing In The Dark | Bruce Springsteen | 4:05 | Born In The U.S.A. |
| 35 | <input checked="" type="checkbox"/> Thunder Road | Bruce Springsteen | 4:51 | Born To Run |
| 36 | <input checked="" type="checkbox"/> Born To Run | Bruce Springsteen | 4:30 | Born To Run |
| 37 | <input checked="" type="checkbox"/> Jungleland | Bruce Springsteen | 9:34 | Born To Run |
| 38 | <input checked="" type="checkbox"/> Tug! Tug! Tug! (To Everything) | The Rude | 3:57 | Forrest Gump The Soundtrack (Disc 2) |

Sort music library by song name

| | Name | Artist | Time | Album |
|----|-----------------------------------|-----------------------|-------------|---|
| 1 | Alive | Pearl Jam | 5:41 | Ten |
| 2 | All Over The World | Pixies | 5:27 | Bossanova |
| 3 | All Through The Night | Cyndi Lauper | 4:30 | She's So Unusual |
| 4 | Allison Road | Gin Blossoms | 3:19 | New Miserable Experience |
| 5 | Ama, Ama, Ama Y Ensancha El ... | Extremoduro | 2:34 | Deltoya (1992) |
| 6 | And We Danced | Hooters | 3:50 | Nervous Night |
| 7 | As I Lay Me Down | Sophie B. Hawkins | 4:09 | Whaler |
| 8 | Atomic | Blondie | 3:50 | Atomic: The Very Best Of Blondie |
| 9 | Automatic Lover | Jay-Jay Johanson | 4:19 | Antenna |
| 10 | Baba O'Riley | The Who | 5:01 | Who's Better, Who's Best |
| 11 | Beautiful Life | Ace Of Base | 3:40 | The Bridge |
| 12 | Beds Of Roses | Bon Jovi | 6:35 | Cross Road |
| 13 | Black | Pearl Jam | 5:44 | Ten |
| 14 | Bleed American | Jimmy Eat World | 3:04 | Bleed American |
| 15 | Borderline | Madonna | 4:00 | The Immaculate Collection |
| 16 | Born To Run | Bruce Springsteen | 4:30 | Born To Run |
| 17 | Both Sides Of The Story | Phil Collins | 6:43 | Both Sides |
| 18 | Bouncing Around The Room | Phish | 4:09 | A Live One (Disc 1) |
| 19 | Boys Don't Cry | The Cure | 2:35 | Staring At The Sea: The Singles 1979–1985 |
| 20 | Brat | Green Day | 1:43 | Insomniac |
| 21 | Breakdown | Deerheart | 3:40 | Deerheart |
| 22 | Bring Me To Life (Kevin Roen Mix) | Evanescence Vs. Pa... | 9:48 | |
| 23 | Californication | Red Hot Chili Pepp... | 1:40 | |
| 24 | Call Me | Blondie | 3:33 | Atomic: The Very Best Of Blondie |
| 25 | Can't Get You Out Of My Head | Kylie Minogue | 3:50 | Fever |
| 26 | Celebration | Kool & The Gang | 3:45 | Time Life Music Sounds Of The Seventies - C |
| 27 | Chaiwa Chaiwa | Sukhwinder Singh | 5:11 | Bombay Dreams |

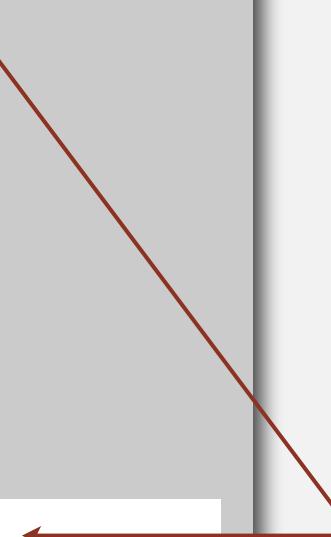
Comparable interface: review

Comparable interface: sort using a type's natural order.

```
public class Date implements Comparable<Date>
{
    private final int month, day, year;

    public Date(int m, int d, int y)
    {
        month = m;
        day   = d;
        year  = y;
    }

    ...
    public int compareTo(Date that)
    {
        if (this.year < that.year) return -1;
        if (this.year > that.year) return +1;
        if (this.month < that.month) return -1;
        if (this.month > that.month) return +1;
        if (this.day   < that.day)  return -1;
        if (this.day   > that.day)  return +1;
        return 0;
    }
}
```



natural order

Comparator interface

Comparator interface: sort using an alternate order.

```
public interface Comparator<Key>
    int compare(Key v, Key w)           compare keys v and w
```

Required property. Must be a total order.

Ex. Sort strings by:

- Natural order. Now is the time pre-1994 order for digraphs ch and ll and rr
- Case insensitive. is Now the time
- Spanish. café cafetero cuarto churro nube ñoño
- British phone book. McKinley Mackintosh
- . . .



Comparator interface: system sort

To use with Java system sort:

- Create Comparator object.
- Pass as second argument to Arrays.sort().

```
String[] a;           uses natural order
...
Arrays.sort(a);      uses alternate order defined by
...
Arrays.sort(a, String.CASE_INSENSITIVE_ORDER);    Comparator<String> object
...
Arrays.sort(a, Collator.getInstance(new Locale("es")));
...
Arrays.sort(a, new BritishPhoneBookOrder());
...
```

Bottom line. Decouples the definition of the data type from the definition of what it means to compare two objects of that type.

Comparator interface: using with our sorting libraries

To support comparators in our sort implementations:

- Use Object instead of Comparable.
- Pass Comparator to sort() and less() and use it in less().

insertion sort using a Comparator

```
public static void sort(Object[] a, Comparator comparator)
{
    int N = a.length;
    for (int i = 0; i < N; i++)
        for (int j = i; j > 0 && less(comparator, a[j], a[j-1]); j--)
            exch(a, j, j-1);
}

private static boolean less(Comparator c, Object v, Object w)
{ return c.compare(v, w) < 0; }

private static void exch(Object[] a, int i, int j)
{ Object swap = a[i]; a[i] = a[j]; a[j] = swap; }
```

Comparator interface: implementing

To implement a comparator:

- Define a (nested) class that implements the Comparator interface.
- Implement the compare() method.

```
public class Student
{
    public static final Comparator<Student> BY_NAME      = new ByName();
    public static final Comparator<Student> BY_SECTION = new BySection();
    private final String name;
    private final int section;
    ...
    private static class ByName implements Comparator<Student>
    {
        public int compare(Student v, Student w)
        { return v.name.compareTo(w.name); }
    }

    private static class BySection implements Comparator<Student>
    {
        public int compare(Student v, Student w)
        { return v.section - w.section; }
    }
}
```

one Comparator for the class

this technique works here since no danger of overflow

Comparator interface: implementing

To implement a comparator:

- Define a (nested) class that implements the Comparator interface.
- Implement the compare() method.

`Arrays.sort(a, Student.BY_NAME);`

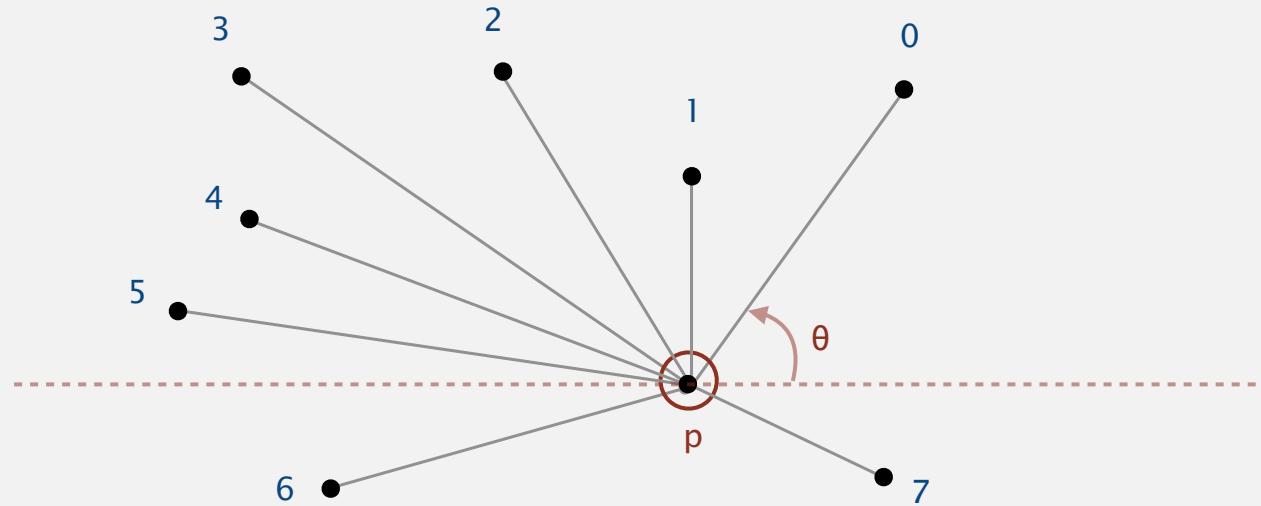
| | | | | |
|---------|---|---|--------------|--------------|
| Andrews | 3 | A | 664-480-0023 | 097 Little |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
| Fox | 3 | A | 884-232-5341 | 11 Dickinson |
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Gazsi | 4 | B | 766-093-9873 | 101 Brown |
| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |

`Arrays.sort(a, Student.BY_SECTION);`

| | | | | |
|---------|---|---|--------------|--------------|
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |
| Andrews | 3 | A | 664-480-0023 | 097 Little |
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| Battle | 4 | C | 874-088-1212 | 121 Whitman |
| Gazsi | 4 | B | 766-093-9873 | 101 Brown |

Polar order

Polar order. Given a point p , order points by polar angle they make with p .



```
Arrays.sort(points, p.POLAR_ORDER);
```

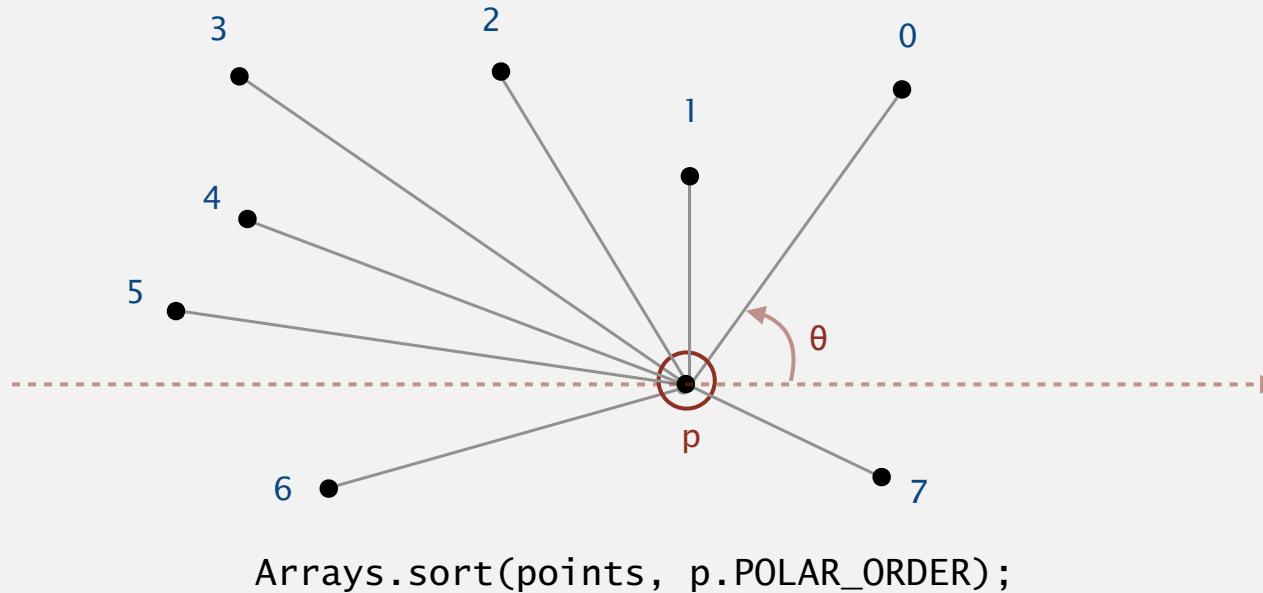
Application. Graham scan algorithm for convex hull. [see previous lecture]

High-school trig solution. Compute polar angle θ w.r.t. p using `atan2()`.

Drawback. Evaluating a trigonometric function is expensive.

Polar order

Polar order. Given a point p , order points by polar angle they make with p .



A ccw-based solution.

- If q_1 is above p and q_2 is below p , then q_1 makes smaller polar angle.
- If q_1 is below p and q_2 is above p , then q_1 makes larger polar angle.
- Otherwise, $ccw(p, q_1, q_2)$ identifies which of q_1 or q_2 makes larger angle.

Comparator interface: polar order

```
public class Point2D
{
    public final Comparator<Point2D> POLAR_ORDER = new PolarOrder();
    private final double x, y;
    ...
    private static int ccw(Point2D a, Point2D b, Point2D c)
    { /* as in previous lecture */ }

    private class PolarOrder implements Comparator<Point2D>
    {
        public int compare(Point2D q1, Point2D q2)
        {
            double dy1 = q1.y - y;
            double dy2 = q2.y - y;

            if (dy1 == 0 && dy2 == 0) { ... }
            else if (dy1 >= 0 && dy2 < 0) return -1;
            else if (dy2 >= 0 && dy1 < 0) return +1;
            else return -ccw(Point2D.this, q1, q2);
        }
    }
}
```

one Comparator for each point (not static)

p, q1, q2 horizontal

q1 above p; q2 below p

q1 below p; q2 above p

both above or below p

to access invoking point from within inner class

Algorithms

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2.2 MERGESORT

- ▶ *mergesort*
- ▶ *bottom-up mergesort*
- ▶ *sorting complexity*
- ▶ **comparators**
- ▶ *stability*

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- ▶ *mergesort*
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- ▶ *comparators*
- ▶ ***stability***

Stability

A typical application. First, sort by name; **then** sort by section.

`Selection.sort(a, Student.BY_NAME);`

| | | | | |
|---------|---|---|--------------|--------------|
| Andrews | 3 | A | 664-480-0023 | 097 Little |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
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| Gazsi | 4 | B | 766-093-9873 | 101 Brown |
| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |

`Selection.sort(a, Student.BY_SECTION);`

| | | | | |
|---------|---|---|--------------|--------------|
| Furia | 1 | A | 766-093-9873 | 101 Brown |
| Rohde | 2 | A | 232-343-5555 | 343 Forbes |
| Chen | 3 | A | 991-878-4944 | 308 Blair |
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| Kanaga | 3 | B | 898-122-9643 | 22 Brown |
| Gazsi | 4 | B | 766-093-9873 | 101 Brown |
| Battle | 4 | C | 874-088-1212 | 121 Whitman |

@#%&@! Students in section 3 no longer sorted by name.

A **stable** sort preserves the relative order of items with equal keys.

Stability

Q. Which sorts are stable?

A. Insertion sort and mergesort (but not selection sort or shellsort).

| sorted by time | sorted by location (not stable) | sorted by location (stable) |
|------------------|---------------------------------|-----------------------------|
| Chicago 09:00:00 | Chicago 09:25:52 | Chicago 09:00:00 |
| Phoenix 09:00:03 | Chicago 09:03:13 | Chicago 09:00:59 |
| Houston 09:00:13 | Chicago 09:21:05 | Chicago 09:03:13 |
| Chicago 09:00:59 | Chicago 09:19:46 | Chicago 09:19:32 |
| Houston 09:01:10 | Chicago 09:19:32 | Chicago 09:19:46 |
| Chicago 09:03:13 | Chicago 09:00:00 | Chicago 09:21:05 |
| Seattle 09:10:11 | Chicago 09:35:21 | Chicago 09:25:52 |
| Seattle 09:10:25 | Chicago 09:00:59 | Chicago 09:35:21 |
| Phoenix 09:14:25 | Houston 09:01:10 | Houston 09:00:13 |
| Chicago 09:19:32 | Houston 09:00:13 | Houston 09:01:10 |
| Chicago 09:19:46 | Phoenix 09:37:44 | Phoenix 09:00:03 |
| Chicago 09:21:05 | Phoenix 09:00:03 | Phoenix 09:14:25 |
| Seattle 09:22:43 | Phoenix 09:14:25 | Phoenix 09:37:44 |
| Seattle 09:22:54 | Seattle 09:10:25 | Seattle 09:10:11 |
| Chicago 09:25:52 | Seattle 09:36:14 | Seattle 09:10:25 |
| Chicago 09:35:21 | Seattle 09:22:43 | Seattle 09:22:43 |
| Seattle 09:36:14 | Seattle 09:10:11 | Seattle 09:22:54 |
| Phoenix 09:37:44 | Seattle 09:22:54 | Seattle 09:36:14 |

Note. Need to carefully check code ("less than" vs. "less than or equal to").

Stability: insertion sort

Proposition. Insertion sort is **stable**.

```
public class Insertion
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
            for (int j = i; j > 0 && less(a[j], a[j-1]); j--)
                exch(a, j, j-1);
    }
}
```

| i | j | 0 | 1 | 2 | 3 | 4 |
|---|---|----------------|----------------|----------------|----------------|----------------|
| 0 | 0 | B ₁ | A ₁ | A ₂ | A ₃ | B ₂ |
| 1 | 0 | A ₁ | B ₁ | A ₂ | A ₃ | B ₂ |
| 2 | 1 | A ₁ | A ₂ | B ₁ | A ₃ | B ₂ |
| 3 | 2 | A ₁ | A ₂ | A ₃ | B ₁ | B ₂ |
| 4 | 4 | A ₁ | A ₂ | A ₃ | B ₁ | B ₂ |
| | | A ₁ | A ₂ | A ₃ | B ₁ | B ₂ |

Pf. Equal items never move past each other.

Stability: selection sort

Proposition. Selection sort is **not** stable.

```
public class Selection
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
        {
            int min = i;
            for (int j = i+1; j < N; j++)
                if (less(a[j], a[min]))
                    min = j;
            exch(a, i, min);
        }
    }
}
```

| i | min | 0 | 1 | 2 |
|---|-----|----------------|----------------|----------------|
| 0 | 2 | B ₁ | B ₂ | A |
| 1 | 1 | A | B ₂ | B ₁ |
| 2 | 2 | A | B ₂ | B ₁ |

Pf by counterexample. Long-distance exchange might move an item past some equal item.

Stability: shellsort

Proposition. Shellsort sort is **not** stable.

```
public class Shell
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        int h = 1;
        while (h < N/3) h = 3*h + 1;
        while (h >= 1)
        {
            for (int i = h; i < N; i++)
            {
                for (int j = i; j > h && less(a[j], a[j-h]); j -= h)
                    exch(a, j, j-h);
            }
            h = h/3;
        }
    }
}
```

| h | 0 | 1 | 2 | 3 | 4 |
|---|----------------|----------------|----------------|----------------|----------------|
| | B ₁ | B ₂ | B ₃ | B ₄ | A ₁ |
| 4 | A ₁ | B ₂ | B ₃ | B ₄ | B ₁ |
| 1 | A ₁ | B ₂ | B ₃ | B ₄ | B ₁ |

Pf by counterexample. Long-distance exchanges.

Stability: mergesort

Proposition. Mergesort is **stable**.

```
public class Merge
{
    private static Comparable[] aux;
    private static void merge(Comparable[] a, int lo, int mid, int hi)
    { /* as before */ }

    private static void sort(Comparable[] a, int lo, int hi)
    {
        if (hi <= lo) return;
        int mid = lo + (hi - lo) / 2;
        sort(a, lo, mid);
        sort(a, mid+1, hi);
        merge(a, lo, mid, hi);
    }

    public static void sort(Comparable[] a)
    { /* as before */ }
}
```

Pf. Suffices to verify that merge operation is stable.

Stability: mergesort

Proposition. Merge operation is stable.

```
private static void merge(...)  
{  
    for (int k = lo; k <= hi; k++)  
        aux[k] = a[k];  
  
    int i = lo, j = mid+1;  
    for (int k = lo; k <= hi; k++)  
    {  
        if (i > mid) a[k] = aux[j++];  
        else if (j > hi) a[k] = aux[i++];  
        else if (less(aux[j], aux[i])) a[k] = aux[j++];  
        else a[k] = aux[i++];  
    }  
}
```

| | | | | | | | | | | |
|----------------|----------------|----------------|---|---|----------------|----------------|---|---|---|----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| A ₁ | A ₂ | A ₃ | B | D | A ₄ | A ₅ | C | E | F | G |

Pf. Takes from left subarray if equal keys.

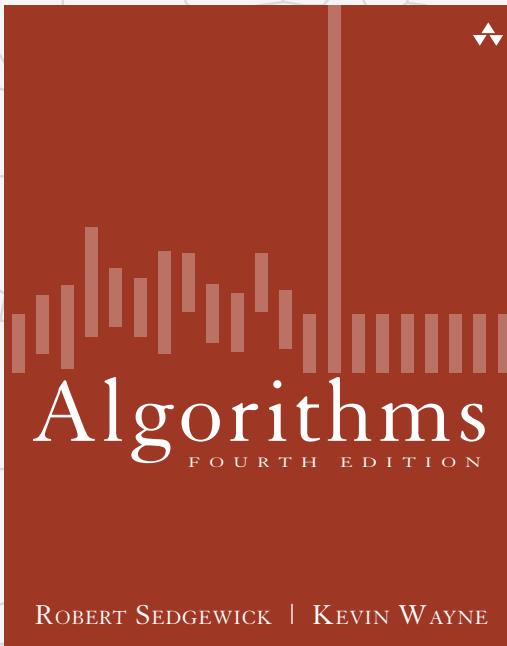
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