LOCAL OBSERVABILITY MATRIX AND ITS APPLICATION TO OBSERVABILITY ANALYSES

Zhe Chen and Ke Jiang
Automatic Control Department
Beijing University of Aeronautics and Astronautics
Beijing 100083, China

James C. Hung
Electrical and Computer Engineering Department
The University of Tennessee
Knoxville, TN 37996-2100, U.S.A.

Abstract

A local observability matrix is introduced for linear time-varying system. The matrix is very useful for the observability analysis. This matrix is much easier to use than the usual observability grammian for linear time-varying system. The matrix is applied to a terrain-aided inertial navigation system to reveal the observability of the system in terms of terrain nature. Simulation result is presented and discussed.

1. Introduced

Consider a discrete-time linear time-varying system represented by the following state space equations

$$\underline{x}(k+1) = F(k)\underline{x}(k) + G(k)\underline{u}(k) \tag{1}$$

$$\underline{m}(k) = H(k)\underline{x}(k) \tag{2}$$

It is well known that the observability of the system can be determined by using the observability grammian matrix

$$V = \sum_{k=1}^{N} F^T(k)H^T(k)H(k)F(k) \tag{3}$$

where N is a finite time index indicating the time period over which the system activity is taking place.[1,2] This grammian matrix is not convenient to use because of its summation nature. In addition, there are situations where a part of the system parameters may not be known, rendering the evaluation of the grammian summation impossible.

This paper presents a new observability measure called the "local observability matrix" for observability analysis. This observability measure will then be used to analyze the observability of a terrain-aided inertial navigation system. Simulation result will be presented to verify the effectiveness of the measure.

2. Local Observability Matrix

Definition: A system represented by (1) and (2) is said to be "locally observable" if the state $\underline{x}(k)$ can be determined from the knowledge of $\underline{m}(j)$ for j = k to k+n-1, where n is the order of the system.

Define the "local observability matrix" M as

$$M = \begin{bmatrix} H(k) \\ H(k+1)F(k) \\ H(k+2)F(k+1)F(k) \\ \vdots \\ H(k+n-1)F(k+n-2) \cdots F(k) \end{bmatrix} \tag{5}$$

This matrix resembles the "observability matrix" for linear time-invariant systems in appearance, but is different in that F and H matices are not constant.[2,3]

Assertion: For a system represented by (3) and (4) to be locally observable over the time period from k to k+n-1, it is necessary and sufficient that the local observabiity matrix M for the period is of rank n.

For a single output system the M matrix is square, therefore the local observability requires that the M matrix be nonsingular.

The local observability matrix is very useful in evaluating the workability of some on-line estimation and identification schemes. The matrix has been used, though not formaly defined, in evaluating the observability of a land navigator.[4] The matrix is also very useful in analyzing a terrain inertial navigation system, to be described next.

3. A Terrain-Aided Inertial Navigation System

An inertial navigarion system (INS) is a device for measuring and determining the orientation and position of a moving vehicle. It is widely used for navigation and guidance of aircrafts, missiles, ocean-vessels, satellites, and land vehicles. The key components of a modern INS consists of gyros, accelerometers and a computer. It is well known that an INS has error which is growing with time. The error is mainly due to the drift of gyros, the bias of accelerometers, and the changes of their scale factors. This error can be curbed by incorporating to the system one or more aiding sensors. Among the aiding sensors are odometers, altimeters, GPS (global positioning system) data, terrain profile data, and others. The choice depends on the application. A terrain-aided inertial navigation system will be considered here.

Figure 1 depicts the block diagram of a terrain-aided navigation system for an aircraft.[5-9] The system is modeled as a discrete time system having k as the time variable. The system has INS as its main sensor and a radar altimeter as an aiding sensor. In addition, prerecorded topographical data are stored on board, giving

087942-600-4/90/1100-0100$01.00 © 1990 IEEE

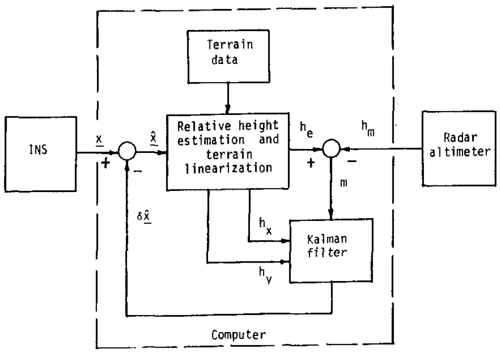

Figure 1. Block diagram of a terrain-aided inertial navigation system

the terrain profile at each location. A computer is used to execute a terrain-aided navigation scheme. In the figure, $\underline{x}(k)$ is the navigation state vector measured by INS. It is corrected by the navigation scheme to generate an updated navigation state $\hat{\underline{x}}(k)$. The scheme computes a navigation state correction $\delta\hat{\underline{x}}(k)$ using the inputs from the INS, radar altimeter, and topographical data. The required computations include the estimation of the height $h_e(k)$ of the aircraft with respect to ground based on a nonlinear relationship between the two, calculation of the discrepancy m(k) between $h_e(k)$ and the radar measured height $h_m(k)$, linearization of the nonlinear relationship between the height h and the navigation state $\underline{x}$, and performing Kalman filtering to generate the navigation state correction $\delta\hat{\underline{x}}(k)$. The output of the entire navigation system is the updated navigation state $\hat{\underline{x}}(k)$.

4. System Modeling

Kalman filtering is the main computation of the aided navigation scheme, therefore, modeling of the system is centered around the Kalman filter. The navigation state vector output by the INS is

$$\underline{x} = [x \;\; y \;\; z \;\; v_x \;\; v_y \;]^T$$

where x, y, and z denote the aircraft position in a three dimensional orthogonal coordinate system with the three axes pointed, respectively, eastward, westward, and vertically upward, and v_x and v_y are velocities along the x and y directions, respectively. The state vector for the Kalman filter is chosen to be the error in $\underline{x}$, denoted by $\delta\underline{x}$. The associated state equation is

$$\delta\dot{\underline{x}} = A\,\delta\underline{x} + \underline{w} \tag{6}$$

Where A is a constant system matrix given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7}$$

and $\underline{w}$ is the process noise which is white. Discretizing (6) for a sufficiently small sampling period T gives

$$\delta\underline{x}(k+1) = F(T)\,\delta\underline{x}(k) + \underline{w}(k) \tag{8}$$

where the transition matrix F(T) is given by

$$F(T) = \begin{bmatrix} 1 & 0 & 0 & T & 0 \\ 0 & 1 & 0 & 0 & T \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

For a fixed sampling period T, F(T) is a constant matrix.

Let the measurement be

$$m(k) = h_e(k) - h_m(k) \tag{10}$$

Note that

$$h_e(k) = h_a(k) - h_t(k) \tag{11}$$

where h_a and h_t are the absolute heights of the aircraft and the terrain, respectively, measured with respect to sea level. As mentioned earlier, h_t is a nonlinear function of position $\underline{x}$. Expand $h_t(k)$ into a Taylor series about the updated position $\hat{\underline{x}}$, and neglect the second and higher order terms,

$$[h_t(k)]_{\underline{x}} = [h_t(k)]_{\hat{\underline{x}}} + \left[\frac{\partial h_t(k)}{\partial \underline{x}(k)}\right]_{\hat{\underline{x}}} \delta\underline{x}(k) \tag{12}$$

where $h_t(k)$ is an abreviated notation for $h_t[\underline{x}(t)]$ since h_t is a function of location $\underline{x}$. Substituting Eq. (11) into Eq. (10) gives

$$m(k) = h_a(k) - [h_t(k)]_{\hat{\underline{x}}} - \left[\frac{\partial h_t(k)}{\partial \underline{x}(k)}\right]_{\hat{\underline{x}}} \delta\underline{x}(k) - h_m(k) \tag{13}$$

In reality,

$$h_a(k) - [h_t(k)]_{\hat{\underline{x}}} - h_m(k) \approx 0$$

Therefore Eq. (13) becomes

$$m(k) \approx H(k)\delta\underline{x}(k) \tag{14}$$

where H(k) is the measurement matrix given by

$$H(k) = -\left[\frac{\partial h_t(k)}{\partial \underline{x}(k)}\right]_{\hat{\underline{x}}} = -\left[\frac{\partial h_t}{\partial x} \;\; \frac{\partial h_t}{\partial y} \;\; \frac{\partial h_t}{\partial z} \;\; \frac{\partial h_t}{\partial V_x} \;\; \frac{\partial h_t}{\partial V_y}\right] \tag{15}$$

Examining the above matrix, one sees that its third element is equal to 1 and its fourth and fifth elements are equal to zero, giving

$$H(k) = \begin{bmatrix} -h_x(k) & -h_y(k) & 1 & 0 & 0 \end{bmatrix} \tag{16}$$

where $h_x(k)$ and $h_y(k)$ are the slopes of terrain along the x and y directions, respectively, evaluated at position $\hat{\underline{x}}$. The measurement contains errors, including the radar altimeter error, topographic map inaccuracy, and errors caused by terrain linearization. These errors are independent and practically white in nature, and can be lumped together as the white measurement noise. Therefore the measurement equation for the Kalman filter is

$$m(k) = H(k)\,\delta\underline{x}(k) + v(k) \tag{17}$$

Equations (8) and (17) form the needed model for Kalman filtering, while Eq. (10) is used to generate the measurement data. Notice that, while the system transition matrix F(T) is time-invariant, the measurement matrix H(k) is time-varying during the course of the aircraft flight.

5. Observability Analysis

A successful use of Kalman filtering requires that the system be observable. The local observability matrix is used to analyze the observability.

Substituting Equations (9) and (11) into (5), the local observability matrix is obtained as

$$M = \begin{bmatrix} -h_x(k) & -h_y(k) & 1 & 0 & 0 \\ -h_x(k+1) & -h_y(k+1) & 1 & -h_x(k+1)T & -h_y(k+1)T \\ -h_x(k+2) & -h_y(k+2) & 1 & -2h_x(k+2)T & -2h_y(k+2)T \\ -h_x(k+3) & -h_y(k+3) & 1 & -3h_x(k+3)T & -3h_y(k+3)T \\ -h_x(k+4) & -h_y(k+4) & 1 & -4h_x(k+4)T & -4h_y(k+4)T \end{bmatrix} \quad (18)$$

The system is locally observable if and only if the matrix M is nonsingular. Thus the observability is highly dependent of terrain slopes h_x and h_y. Examining Eq. (18), several remarks can be made.

1. Terrain slopes h_x and h_y cannot be too small. That is, local terrain cannot be too flat, such as plains and sea.
2. $h_x(k)$ and $h_y(k)$ must vary with k. That is, the terrain must fluctuate; the larger the fluctuation the stronger the observability.
3. $h_x(k)$ and $h_y(k)$ are not equal to each other. That is, the slopes along x and y directions are not the same.
4. From Eq. (16) one sees that, when the state $\delta\underline{x}$ is locally observable, state variable δz has the strongest observability, followed by δx and δy. State variables δv_x and δv_y have the weakest observability.

The analysis of local observability can be effected in two ways. One way is to evaluate the systems local observability matrix. The other way is to use computer simulation to do a covariance analysis of the state x under the given terrain profile. The computer simulation approach is used for the present case to verify the above remarks which have been made based on the study of the local observability matrix.

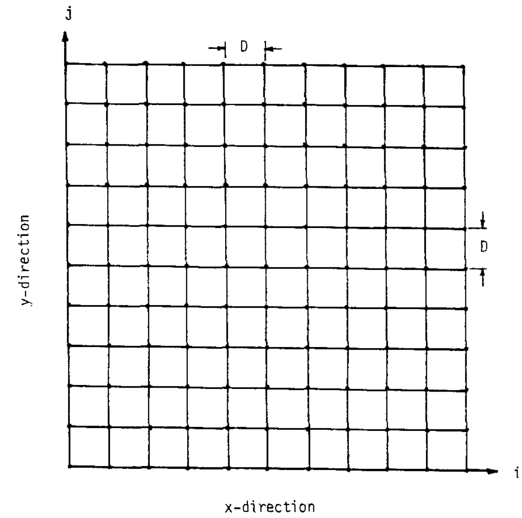

Figure 2. Grid geometry and grid points

Table 1

Terrain Slope Data For Two Different Paths

k	Path 1		Path 2	
	$-h_x$	$-h_y$	$-h_x$	$-h_y$
1	0.035	0.150	-0.700	-0.770
2	-0.095	-0.015	0.720	1.285
3	0.095	0.100	0.715	1.630
4	0.585	-0.585	0.530	0.005
5	0.305	-0.245	-0.730	-0.110
6	0.055	-0.285	-0.250	1.520
7	0.065	0.005	-0.095	1.765
8	0.010	-0.065	-0.315	-0.415
9	0.200	-0.017	-0.245	0.990
10	0.405	-0.320	0.755	1.150
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

6. Computer Simulation

An airplane flight over a mining area in Sichuan Province of China is chosen for the present simulation. The area is represented by a grid as shown in Figure 2. Topographical data at grid points are available for simulation. A simple linearization technique is used to obtain the elements of H(k) matrix. This is done by computing the slope $h_x(k)$ and $h_y(k)$ directly from $h_t(i,j)$, the terrain height at grid point (i,j).

$$h_x(k) = [h_t(i+1,j) - h_t(i-1,j)]/2D \quad (19)$$

$$h_y(k) = [h_t(i,j+1) - h_t(i,j-1)]/2D \quad (20)$$

where D is the grid spacing on the digital map. Values of $h_t(i,j)$ are available from the topographical data.

To reveal the affect of terrain slopes, two different paths on the grid are taken. The corresponding two set of terrain slope data are listed in Table 1. As shown in the table, both the slope values and the incremental slope values for path 2 are much larger than those of path 1. By covariance analysis simulation, standard deviation of the five state variables are obtained including σ_x, σ_y, σ_z, σ_{v_x}, and σ_{v_y} Their profiles as functions of time is shown in Figure 3.

Examining Figure 3, several remarks can be made:

1. The root-mean-square (rms) errors of all five state variables are bounded. However, their fluctuations and final trends are different.
2. The updating accuracy of the system depends on the local observability of the system, which, in turn, depends on the roughness of the terrain. The rougher is the terrain the better is the updating accuracy.
3. Among the standard deviations of the five state variables, σ_z is least dependent of terrain roughness, while the other four are heavily dependent.

4. When the terrain is sufficiently rough, the system's observability is good so the terrain-aided scheme can improve the position accuracy a great deal, while the improvement in velocity accuracy is less.

7. Conclusion

The concept of local observability and local observability matrix have been introduced. The matrix is a tool for analyzing the observability of a time-varying linear system. Application of the local observability matrix to a terrain-aided inertial navigation system has been given. The matrix helped to reveal the conditions under which the terrain-aiding is effective. Result of a simulated covariace analysis was presented and discussed. The usefulness of the local observability matrix has been illustrated.

References

1. H. Kwakernaak and R. Savin, Linear Optimal Control Systems, Wiley-Interscience, 1972, 67 and 463.

2. W. L. Brogan, Modern Control Theory, 2nd edition, Prentice-Hall, 1985, 308 and 311.

3. Z. Chen, Modern Control Theory, Metallurgical Industry Press, China, 1987, 189-190. (In Chinese)

4. P. D. Feng and J. C. Hung, "Cooperate Identification via Kalman Filtering", Proceedings of 1983 IEEE Southeastcon, IEEE Catalog No. CH1899-4, 11-14 April 1983, 339-343.

5. D. D. Boozer, M. K. Lau, and J. R. Fellerhoff, "The AFTI/F16 Terrain-Aided Navigation System", NAECON Record, IEEE, 1985, 351-357.

6. J. R. Fellerhoff, "SITAN Implementation in the SAINT System", PLANS Records, IEEE, 1986, 89-95.

7. E. J. Nova, E. E. Creel, J. R. Fellerhoff, and S. D. Martinez, "A Land Navigation Demonstration Vehicle with a Color Map Display for Tactical Use", PLANS Records, IEEE, 1986, 431-438.

8. R. D. Andreas, L. D. Hostetler, and R. C. Beckmann, "Continuous Kalman Updating of An Inertial Navigation System Using Terrain Measurement", NAECON Records, IEEE, 1978.

9. L. D. Hostetler and R. Andreas, "Nonlinear Kalman Filtering Techniques for Terrain-Aided Navigation", IEEE Transactions on Automatic Control, Vol. AC-28, No. 3, March 1983, 315-323.

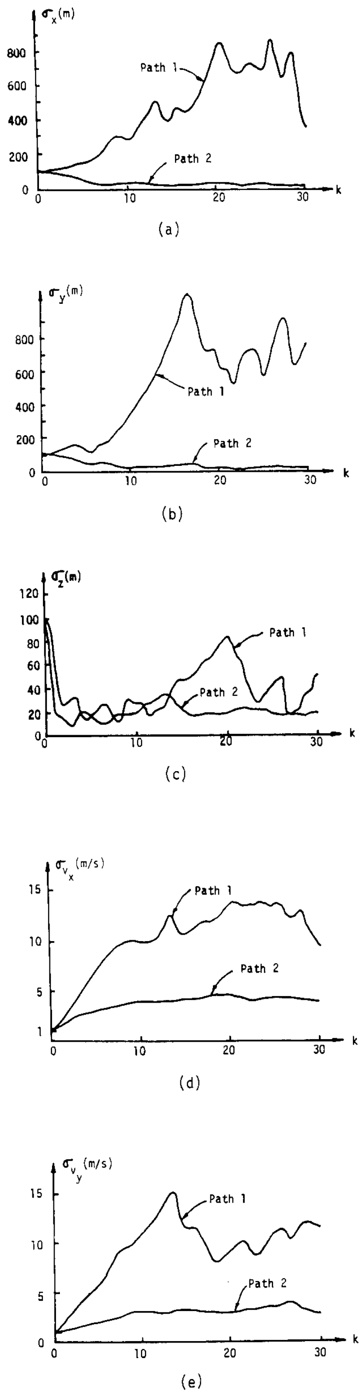

Figure 3. Standard deviation profiles from covariance analysis