Math 110, Spring 2019

Contents

1. Vector Space

1.a. Vector Space over a field and subspace

Recall that $(\mathbb{F}, +, \cdot)$ or $\langle \mathbb{F}, +, \cdot \rangle$, where $\mathbb{F}$ is a set, and $+, \cdot$ are binary operations.
We know that $(\mathbb{F}, +)$ and $(\mathbb{F} \setminus \{0\}, \cdot)$ and $+, \cdot$ satisfy distributivity.

Definition 1.1

V is a vector space over a field $\mathbb{F}$ is V is equipped with vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{F} \times V \to V$.

Lists (and vector spaces of lists)

Example 1.2

$\mathbb{R}^n, \mathbb{C}^n$, or generally $\mathbb{F}^n$.

$$\mathbb{R}^n = \{(x_1, x_2, \cdots x_n) : x_i \in \mathbb{R} \ \forall \ j = 1, 2 \cdots n\}$$

$$\mathbb{F}^n = \{(x_1, x_2, \cdots x_n) : x_i \in \mathbb{F} \ \forall \ j = 1, 2 \cdots n\}$$

We claim that $\mathbb{F}^n$ is a vector space over $\mathbb{F}$ provided $\mathbb{F}$ is a field. We can define addition and scalar multiplication as

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) := (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$

$$\alpha \cdot (x_1, x_2, \ldots, x_n) = (\alpha \cdot x_1, \alpha \cdot x_2, \cdots, \alpha \cdot x_n) \quad \alpha \cdot x_i \in \mathbb{F}$$

What rules / axioms should we impose?

- Commutativity
  $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \ \forall \ \mathbf{v}, \mathbf{w} \in V$.
- Associativity
  $(\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u}) \ \forall \ \mathbf{v}, \mathbf{w}, \mathbf{u} \in V$.
- Additive Identity
  $\exists \ \mathbf{0} \in V : \mathbf{v} + \mathbf{0} = \mathbf{v} + \mathbf{0} = \mathbf{v}$
- Additive Inverse $\forall \mathbf{v} \in V \ \exists \ \mathbf{w} \in V : \mathbf{v} + \mathbf{w} = \mathbf{0}$.
- (Mixed) Scalar Multiplication Rules
  $1 \cdot \mathbf{v} \in \mathbf{v} \qquad \forall v \in V$
- Distributivity:
  $(\alpha + \beta) \cdot \mathbf{v} = \alpha \cdot \mathbf{v} + \beta \cdot \mathbf{v} \qquad \forall \ a, b, \in \mathbb{F} \qquad \forall \ \mathbf{v} \in V$
  $\alpha \cdot (\mathbf{v} + \mathbf{w}) = \alpha \cdot \mathbf{v} + \alpha \cdot \mathbf{w} \qquad \forall \ a \in \mathbb{F} \qquad \forall \ \mathbf{v}, \mathbf{w} \in V$

Now we can check that these rules hold in $\mathbb{F}^2$:

$$(0,0,\cdots,0)+(x_1,x_2,\ldots,x_n)=(x_1,x_2,\ldots,x_n)$$
$$(x_1,x_2,\ldots,x_n)+(-x_1,-x_2,\ldots,-x_n)=(0,0,\cdots,0)$$

Basic Observation $\mathbf{0}$ is unique

Proof. Suppose $\mathbf{0}_1$ and $\mathbf{0}_2$ are both identity element with respect to $+$:

$$\mathbf{0}_1=\mathbf{0}_1+\mathbf{0}_2+\mathbf{0}_2$$

A contradiction. ■

Additive inverse ate unique, i.e., if $\mathbf{v}+\mathbf{w}=\mathbf{0}$ and $\mathbf{v}+\mathbf{w}=0$, then $\mathbf{u}=\mathbf{w}$.

Proof. Suppose $\mathbf{v}+\mathbf{w}=\mathbf{0}$ and $\mathbf{v}+\mathbf{w}=\mathbf{0}$, then

$$\mathbf{w}=\mathbf{w}+\mathbf{0}=\mathbf{w}+(\mathbf{v}+\mathbf{u})=(\mathbf{w}+\mathbf{v})+\mathbf{u}=\mathbf{0}+\mathbf{u}=\mathbf{u}$$

A contradiction. ■

Additive Inverse

$$\begin{aligned}(-1)\cdot\mathbf{v}+\mathbf{v}&=(-1)\cdot\mathbf{v}+1\cdot\mathbf{v}\\&=((-1)+1)\cdot\mathbf{v}\\&=0\cdot\mathbf{v}\\0\cdot\mathbf{v}+0\cdot\mathbf{v}&=(0+0)\cdot\mathbf{v}\\&=0\mathbf{v}\implies\boxed{0\cdot\mathbf{v}=\mathbf{0}}\end{aligned}$$

Additive inverse $\implies$ $0\cdot\mathbf{v}=\mathbf{0}$ on both sides.

1.b. Subspaces

Definition 1.3

V is a vector space over a field $\mathbb{F}$, Let $W\subseteq V$.
W is called a subspace of V if W equipped with the same operations $+,\cdot$ inherited from V is still a vector space.

Is it enough for W to be just a subset of V?
Suppose $V=\mathbb{R}^3$ is a vector space over $\mathbb{R}$. Let $W:=\{(1,1,1)\}$, the additive inverse doesn't exist. Note that W is not closed in addition and scalar multiplication.

$$W:=\{(x,0,0)\ :\ x_1\in\mathbb{R}\}$$

Why is 0 in every subspace?
We know that a vector space is a *non empty* set, and W is closed under multiplication, so since $0\in\mathbb{F}$, therefore $0\cdot\mathbf{v}=\mathbf{0}\in W$.

Remark 1.4

If $\mathbf{v} + \mathbf{w} = \mathbf{v}$, for some $\mathbf{v} \in V$, then $\mathbf{w} = \mathbf{0}$.

Proof. Suppose $\mathbf{v} + \mathbf{w} = \mathbf{v}$, then $-\mathbf{v} + \mathbf{v} + \mathbf{w} = -\mathbf{v} + \mathbf{v} \implies \mathbf{0} + \mathbf{w} = \mathbf{0} \implies \mathbf{w} = \mathbf{0}$ ∎

We recall that $\mathbb{F}^n$ is defined as

$$\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) \ : \ x_i \in \mathbb{F}\}$$

We can define $\mathbb{F}^S$ for S being a set as $\mathbb{F}^S = \left\{ f : \underbrace{S}_{\text{no structure needed}} \to \underbrace{\mathbb{F}}_{\text{field}} \right\}$

We can define addition and multiplication as

$$(f + g)(s) \coloneqq f(s) + g(s) \ \forall s \in S$$

$$(\lambda \cdot f)(s) \coloneqq \lambda \cdot f(s) \in \mathbb{F}$$

Suppose $S = \{1, 2, 3\}$, what is $\mathbb{F}^S$ or $\mathbb{R}^S$? We can thought of $\mathbb{R}^S$ as $\mathbb{R}^3$..... why?

Remark 1.5

We can conclude $\mathbb{F}^S \cong \mathbb{F}^{|S|}$, where $|S|$ is the cardinality of S. If S is finite.

What is $\mathbb{R}^{\mathbb{N}}$? ← the set of all of all real sequences.

Remark 1.6

In the book we uses $\mathbb{R}^\infty$, we can conclude that

$$\mathbb{R}^\infty \cong \mathbb{R}^{\mathbb{N}}$$

We say that W is a subspace of $\mathbb{R}^\infty$ with $+, \cdot$

$$W \coloneqq \left\{ s \ : \ \lim_{n\to\infty} s(n) = 0 \right\}$$

Proof. We can see that if $\lim_{n\to\infty} s(n) = 0$ and $\lim_{n\to\infty} t(n) = 0$, then $\lim_{n\to\infty} (s + t)(n) = 0$.
If $\lambda \in \mathbb{R}$, then $\lim_{n\to\infty} s(n) = 0 \implies \lim_{n\to\infty} (\lambda \cdot s)(n) = 0$.
The zero sequence is in W so $\mathbf{0} \in V$. Therefore W is a subspace of V. ∎

Theorem 1.7

W is a subspace of V iff W is closed under addition, multiplication by scalar multiplication by scalars, and $\mathbf{0} \in V$.

Since the operation in inherent from vector space V, we do not need to verify the other property since they all for all V and W is a subspace of V.
How do we form new subspaces from existing ones?

Theorem 1.8

Suppose W_1, W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.

Proof. We know that W_1, W_2 are subspaces of V, therefore $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$, then $0 \in W_1 \cap W_2$.
Suppose $\mathbf{v}, \mathbf{u} \in W_1 \cap W_2$, we know that $\mathbf{v}, \mathbf{u} \in W_1$ and $\mathbf{v}, \mathbf{u} \in W_2$. Since W_1, W_2 is a subspace, therefore $\mathbf{u} + \mathbf{v} \in W_1 \wedge \mathbf{u} + \mathbf{v} \in W_2 \implies \mathbf{u} + \mathbf{v} \in W_1 \cap W_2$, therefore $W_1 \cap W_2$ is closed under vector addition.
Suppose $\alpha \in \mathbb{F}$ and $\mathbf{v} \in W_1 \cap W_2$. We know that $\alpha \cdot \mathbf{v} \in W_1$ and $\alpha \cdot \mathbf{v} \in W_2$ since they are both subspaces of V. Therefore we conclude $\alpha \cdot \mathbf{v} \in W_1 \cap W_2$, therefore $W_1 \cap W_2$ is closed under multiplication.
Therefore $W_1 \cap W_2$ is a subspace of V. ■

Proposition 1.9

The union of two subspace of V are generally not a subspace of V

Proof. We can see that span $\{\mathbf{e}_1\}$ and span $\{\mathbf{e}_2\}$ is not a subspace if $\mathbb{R}^2$ as $(1,1) \notin W_1 \cup W_2$ ■

Theorem 1.10

Union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. The proof is left as an exercise. ■

Theorem 1.11

$W_1 + W_2$ is a subspace of V.

Proof. (identity) $\mathbf{0} \in W_1 \wedge \mathbf{0} \in W_2 \implies \mathbf{0} + \mathbf{0} = \mathbf{0} \in W_1 + W_2$.
(closure under addition) Suppose $\mathbf{w}_1 + \mathbf{w}_2 \in W_1 + W_2$ and $\tilde{\mathbf{w}}_1 + \tilde{\mathbf{w}}_2 \in W_1 + W_2$. We compute $(\mathbf{w}_1 + \mathbf{w}_2) + (\tilde{\mathbf{w}}_1 + \tilde{\mathbf{w}}_2) = \underbrace{(\mathbf{w}_1 + \tilde{\mathbf{w}}_1)}_{\in W_1} + \underbrace{(\mathbf{w}_2 + \tilde{\mathbf{w}}_2)}_{\in W_2} \implies (\mathbf{w}_1 + \mathbf{w}_2) + (\tilde{\mathbf{w}}_1 + \tilde{\mathbf{w}}_2) \in W_1 + W_2$.

(closure under scalar multiplication) Suppose $\mathbf{w}_1 + \mathbf{w}_2 \in W_1 + W_2$, and $\lambda \in \mathbb{F}$, we compute $\lambda \cdot (\mathbf{w}_1 + \mathbf{w}_2) = \underbrace{(\lambda \cdot \mathbf{w}_1)}_{\in W_1} + \underbrace{(\lambda \cdot \mathbf{w}_2)}_{\in W_2} \implies \lambda \cdot (\mathbf{w}_1 + \mathbf{w}_2) W_1 + W_2$ ■

Remark 1.12

$W_1 + W_2 + \cdots + W_n$ if the smallest subspace containing $W_1, W_2, \ldots, W_n$.
If $\tilde{\mathbf{v}}$ is a subspace of $V \supseteq W_j \ \forall j$, since $\tilde{\mathbf{v}}$ is closed under $+$, $\mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_n \in W_n$

Example 1.13
Suppose $V = \mathbb{R}^3$. Let

$$W_1 = \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}, W_2 = \text{span}\{(0,1,1)\}, W_3 = \text{span}\{(x,y,z) : x+y+z=0\}$$

What is $W_1 + W_2 + W_3$?

Note that $(0,0,1) = \underbrace{\left(0, \frac{1}{2}, \frac{1}{2}\right)}_{\in W_2} + \underbrace{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}_{\in W_3}$. We also know that $(1,0,0) \in W_1$ and $(0,1,0) \in W_2$, therefore $W_1 + W_2 + W_3 = \mathbb{R}^3$

Discussion

Definition 1.14
A vector space, is often denoted as $(\underbrace{\mathbb{F}}_{\text{scalars}}, \underbrace{V}_{\text{vectors}}, \underbrace{\cdot : \mathbb{F} \to B}_{scaling})$

Example 1.15
$(\mathbb{R}, \mathbb{R}^n, \cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n)$ is a vector space.

Example 1.16
$(\mathbb{R}, \mathbb{R}, \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ is also a vector space.

Notion of a field

Suppose $F = \{0,1,2,3\}$. Can F be a field?

Definition 1.17
A subset W of the vector space V is a subspace of V if it satisfy the following:

1) $\mathbf{0} \in W$
2) $+ : W \times W \to W \subseteq V$ (closure under addition)
3) $\cdot : \mathbb{F} \times W \to W \subseteq V$ (closure under scalar multiplication)

Example 1.18
Can we find a subset W of V such that W satisfy property 1), 2) but not 3)?
Suppose $W = \{(x,0) : x \in \mathbb{Z}\}$ the proof is trivial and is left as an exercise.

Example 1.19

The set of functions $\{f : (0,\infty) \to \mathbb{R}\} = \mathbb{R}^{(0,\infty)}$ is a vector space. We claim that W is a subspace of V.

$$W = \{f : (0,\infty) \to \mathbb{R} : f'(1) = 0\}$$

Proof. We begin by verifying the three properties

1) The zero function is in W
2) Suppose $f, g \in W$, then $(f+g)'(1) = f'(1) + g'(1) = 0 + 0 = 0 \implies f(x) + g(x) \in V$
3) Suppose $f \in W$ and $\lambda \in \mathbb{F}$, then $\lambda \cdot f'(1) = \lambda \cdot 0 = 0 \implies \lambda \cdot f(x) \in V$

Therefore W is a subspace of V. ∎

1.c. Direct Sum

Definition 1.20

Let $(\mathbb{F}, V, \cdot : \mathbb{F} \times V \to V)$ be a vector space. Given that $U_1, U_2, \ldots, U_n \subseteq V$ are subspaces of V, we can define the sum of the subspaces as

$$U_1 + U_2 + \cdots + U_n = \{\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n : \mathbf{u}_i \in U_i\}$$

Proof. 1) We can see that $\mathbf{0} \in U_i \ \forall i$, and $\mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}$

2) Suppose $\mathbf{x} = \sum_{i=1}^{n} \mathbf{x}_i \in U_i$ and $\mathbf{y} = \sum_{j=1}^{n} \mathbf{y}_j \in U_j$, we can see that $\mathbf{x} + \mathbf{y} = \sum_{k=0}^{n} \mathbf{x}_k + \mathbf{y}_k \in U_k$, therefore it's closed under addition.
3) Suppose $\lambda \in \mathbb{F}$ and $\mathbf{x} = \sum_{i=1}^{n} \mathbf{x}_i \in U_i$, we compute $\lambda \cdot \mathbf{x} = \lambda \cdot \sum_{i=1}^{n} \mathbf{x}_i \in U_i$, therefore it's closed under scalar multiplication.

∎

Definition 1.21

We say that $U_1 + U_2 + \cdots + U_n$ is a direct sum, denoted as $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ if for every $\mathbf{v} \in U_1 + U_2 + \cdots + U_n$, $\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_n$ has a unique representation.

Remark 1.22

How best to check $U_1 + U_2 + \cdots + U_n$ is a direct sum?
Check that $U_i \cap U_j = \{\mathbf{0}\}$. We will go over in depth later.

What about $W_1 + W_2 + \cdots W_n$ being a direct sum?

Theorem 1.23

The sum of subspaces $W_1, W_2, \ldots, W_n$:

$$W_1 + W_2 + \cdots + W_n$$

is a direct sum iff $\mathbf{0}$ can be written in only **one way** as a sum

$$\mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_n = 0$$

namely $\mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}$.

Remark 1.24

If $W_1 \cap W_2 \neq \{\mathbf{0}\}, W_1 \cap W_3 \neq \{\mathbf{0}\}, W_2 \cap W_3 \neq \{\mathbf{0}\}$, it is not possible for W_1, W_2, W_3 to be a direct sum, However, the opposite of the proposition is not sufficient for being a direct sum as demonstrated in Remark 1.25.

Remark 1.25

$W_1 \cap W_2 = \{\mathbf{0}\}$, $W_1 \cap W_2 = \{\mathbf{0}\}$, $W_2 \cap W_3 = \{\mathbf{0}\}$ and $W_1 + W_2 + W_3$ being not a direct sum is possible. For example, consider $\mathbb{R}_2$, for line $x = y, y = 0$ and $x = 0$, we can see that they only have the trivial intersection but they are not a direct sum. (credit: Catherine)

2. Finite Dimensional Vector Spaces

2.a. Linear Dependence and Independence

Definition 2.1

We will works with lists of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, then the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ can be defined as

$$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) := \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k\} \, \forall \alpha_i \in \mathbb{F}$$

If the the list happens to cover the entire vector space V, we call the list a spanning list of V.

Definition 2.2

V is finite dimensional if V is a span of finitely many vectors.

Remark 2.3

V is not finite dimensional is logically equivalent to V is infinite dimensional.

Example 2.4

Consider the vector spaces: $\mathcal{P}(x) := \{\alpha_0 + \alpha_1 x + \cdots + a_k x^k : a_j \in \mathbb{F} \text{ for some } k\}$. We can see that $\mathcal{P}(x) \subseteq \mathbb{F}^{\mathbb{F}}$, and $\mathcal{P}(x)$ is infinite dimensional.

Definition 2.5

We can define the degree of a polynomial, denoted as $\deg(f(x))$, is the highest power of x where hose coefficient (α_k) is nonzero. The zero function $f(x) = 0$ has $-\infty$ degree.

Example 2.6

$\mathcal{P}(x)$ is infinite dimensional.

Proof. Suppose $\mathcal{P}(x) = \operatorname{span}(f_1, f_2, \ldots, f_k)$, where f_j is polynomials, for all j. Let

$$D := \max\{\deg(f_1), \deg(f_2), \cdots, \deg(f_k)\}$$

Suppose $f(x) = x^{D+1} \in \mathcal{P}(x)$ however, $x \notin \operatorname{span}(f_1, f_2, \ldots, f_k)$. Since $f(x)$ is not a linear combination of $f_1, f_2, \ldots, f_k$. A contradiction, therefore $\mathcal{P}(x)$ is a infinite dimensional vector space. ■

Definition 2.7
V has dimension k over $\mathbb{F}$ if you can find vectors $v_1, v_2, \ldots, v_k$ such that

$$\forall \mathbf{v} \in V : \mathbf{v} = \sum f_i \mathbf{v}_i \text{ uniquely}$$

Definition 2.8
$\mathcal{P}_d(x) :=$ all polynomials in $g(x)$ of degree $\leq d$.
Note that $\{1, x, x^2, \cdots, x^d\}$ is a spanning list for $\mathcal{P}_d(x)$

Definition 2.9
A list $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ is called **linearly independent** if the equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = 0 \implies \alpha_1, \alpha_2, \ldots, \alpha_k = 0$$

Definition 2.10
A list $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ is called **linearly dependent** if it is not independent.

Digression on Logic

Logic: $A \implies B$ is equivalent to $\neg A \vee B$. Then we know that

$$\neg(A \implies B) \iff (\neg(\neg A \vee B)) \iff A \wedge \neg B$$

Definition 2.11 (The better definition)
A list $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ is called **linearly dependent** if for equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = 0$$

has a nontrivial solution such that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \neq 0$

Example 2.12

Is $\{\}$ linearly independent?
By definition, it is linearly independent, because it is not linearly dependent. A set S is linearly dependent if there exists a finite set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and corresponding scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that there exists at least one $\alpha_i \neq 0$ so that

$$\sum_{i=0}^{n} \alpha_i \mathbf{v}_i = 0$$

since α_i doesn't exist, we know that $\{\}$ is linearly independent.

Example 2.13

Is $\{(1,0,0),(0,1,0),(0,0,1)\}$ linearly independent on $\mathbb{R}^3$?

$$\begin{aligned} \alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) &= (0,0,0) \\ \implies (\alpha_1, \alpha_2, \alpha_3) &= (0,0,0) \\ \implies \alpha_1 = \alpha_2 = \alpha_3 = 0 \end{aligned}$$

Remark 2.14

We can remove vectors from a linearly independent list can still remain independent, however, we cannot guarantee the result if we are still adding vectors; In mathematical terms, any sublist of the list is linearly independent, since $\{\}$ is a sublist of any list, therefore its linearly independent.

Lemma 2.15 (Linear Dependence Lemma)

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_h$ is linearly independent. Then there exists j between 1 and k such that

- $v_j \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{j-1}\}$
- $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_h\} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \cdots, \mathbf{v}_k\}$

Proof. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a linearly dependent list, there are coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ not all 0, such that

$$\alpha_1 \mathbf{v}_1 + \alpha \mathbf{v}_2 + \cdots \alpha_k \mathbf{v}_k = 0$$

Take j such that α_j is the largest index with $\alpha_j \neq 0$. Then $\alpha_{j+1} = \alpha_{j+2} = \cdots = \alpha_k = 0$ and

$$\mathbf{v}_j = \frac{-1}{\alpha_j}(\alpha_1 \mathbf{v}_1 + \alpha \mathbf{v}_2 + \cdots + \alpha_{j-1} \mathbf{v}_{j-1})$$

hence $\mathbf{v}_j \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}\}$. ∎

Lemma 2.16 (Very Important, a.k.a. Magic Lemma)

The length of the independent list $\leq$ length of any spanning list.

Proof. Say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is linearly independent say $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ is spanning. Then we want to establish that $m \leq n$.

Step 1. Take the list $\mathbf{u}_1, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ It is linearly independent since $\mathbf{u}_1 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$. By the linear dependence lemma, there is a j such that $\mathbf{v}_j$ can be removed (noted that $\mathbf{u}_1$ cannot be subject to removal since $\mathbf{u}_1$ comes from a linearly independent list). Consider the new list $\{\mathbf{u}_1, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1}, \cdots, \mathbf{v}_m\}$

Step 2. We can continue this process by bringing $\mathbf{u}_2, \mathbf{u}_3, \cdots \mathbf{u}_n$, we know that $\mathbf{u}_i$ since they are linearly independent.

Note that this process preserves linear span of the whole list.
We know that this list contains all the $\mathbf{u}_i$ (plus possibly some remaining $\mathbf{v}_j$) and the length of the list is always n. So $\boxed{m \leq n}$. ■

2.b. Bases and Dimension

Definition 2.17

A basis a linearly independent spanning list.

Theorem 2.18

Any two basis in a finite dimensional space have the same number of vectors.

Remark 2.19

The span of $\{\}$ is the zero vector.

Theorem 2.20

Suppose V is a finite dimensional vector space. Let W be a subspace of V, then W is finite dimensional.

Proof. V is finite dimensional means that V is spanned by some k vectors. Consider W. If $W = \{\mathbf{0}\}$, then w is spanned by the empty list $\mathbf{0}$. If $W \neq \mathbf{0}$, there exists $\mathbf{w}_1 \in W$ such that $W = \operatorname{span}\{\mathbf{u}_1\}$, done. Otherwise take $\mathbf{w}_2 \in W \setminus \operatorname{span}\{\mathbf{w}_1\}$. Repeat this algorithm until it terminates. Now we want to show that this algorithm will terminate at $\mathbf{w}_k$, we know that $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j$ is linearly independent by construction and the linearly dependence lemma. By remark 2.16, we know that the length of any such list will not exceed length k, therefore we know the algorithm will terminate in finite steps. This implies that W is finitely spanned, or W is finite dimension. ■

Dimension

Definition 2.21

Dimension of a vector space V is the cardinality of any basis in a finite dimensional space.

Proposition 2.22 (Criterion for a Basis)

$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a basis for V if and only if any $v \in V$ can be uniquely written as a lineat combination

$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda \mathbf{v}_n$$

Proof. We know that "can be written as linear combination" is logically equivalent a $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is a spanning list for V. "uniqueness" is logically equivalent as linear independence. Suppose

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_k \mathbf{v}_k$$

Not all $\alpha_j = \beta_j$. Then $(\alpha_1 - \beta_1)\mathbf{v}_1 + (\alpha_2 - \beta_2)\mathbf{v}_2 + \cdots (\alpha_k - \beta_k)\mathbf{v}_k = \mathbf{0}$ is a nontrivial linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and vice versa. ■

Theorem 2.23

Any spanning set for a finite dimensional space can be shrink down to a basis.

Proof. Trivial by the linear dependence lemma. ■

Example 2.24

Consider $\mathcal{P}_2(x)$ is spanned by $\{x^2, (x-1)^2, (x-3)^2, (x-3)^2\}$, we can see that this can be thinned down to $\{x^2, (x-1)^2, (x-2)^2\}$.

Corollary 2.25

Any linearly independent list in a finite dimensional space can be enlarged to a basis.

Proof. Add a spanning list at the back of our given list, then do removal for the linearly independent lemma. ■

Theorem 2.26

Suppose V is finite dimensional and W is a subspace, then there is a subspace U such that $V = W \oplus U$.

Proof. We already know by proceeding stuff W is finite dimensional an its dimensional does not exceed that of V. Take any basis of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ of W. It's linearly independent so can be enlarged to a basis for V. Suppose the resulting basis is $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_l$. Take $U = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_l)$. Then $W + U = V$ and $W \cap U = \{0\}$. ■

Remark 2.27

$\operatorname{span}(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_n)$ is a subspace of V by construction.

Example 2.28

Consider $\mathcal{P}(x)T$. We define W as

$$W := \{f \in \mathcal{P}_3(x) : f'(5) = 0\}$$

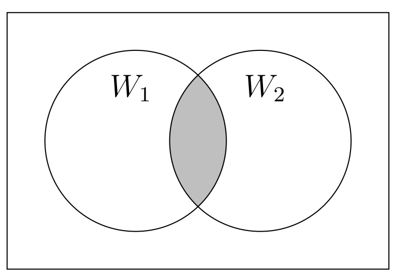
A basis for W can be taken as $\{1, (x-5)^2, (x-5)^3\}$.
Now consider

$$\tilde{W} := \{f \in \mathcal{P}_3(x) : f''(5)\}$$

. A basis for W can be taken as $\{1, (x-5)^2, (x-5)^3\}$.

Dimension of a Sum

Principal of Inclusion for subspaces



$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Suppose $W_1 \cap W_2$ forms a basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$. We can extend the basis to

$$\mathbf{w}_1^{(1)}, \mathbf{w}_2^{(1)}, \ldots, \mathbf{w}_l^{(1)}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$$

is a basis for W_1. Similarly, we can extend the basis to

$$\mathbf{w}_1^{(2)}, \mathbf{w}_2^{(2)}, \ldots, \mathbf{w}_m^{(2)}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$$

is a basis for W_2.
We want to establish that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) = l + m + k$$

We want to prove that

$$\mathbf{w}_1^{(1)}, \mathbf{w}_2^{(1)}, \dots, \mathbf{w}_l^{(1)}, \mathbf{w}_1^{(2)}, \mathbf{w}_2^{(2)}, \dots, \mathbf{w}_m^{(2)}, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k$$

is a basis for $W_1 + W_2$. We can see that

$$\operatorname{span}(\mathbf{w}^1{}_1, \mathbf{w}^1{}_2, \dots, \mathbf{w}^1{}_l, \mathbf{w}_1^{(2)}, \mathbf{w}_2^{(2)}, \dots, \mathbf{w}_m^{(2)}, \mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k) \supseteq U_1, U_2$$

Hence $\operatorname{span}(\dots) \supseteq U_1 + U_2$. Suppose the equation

$$\alpha_1 \mathbf{w}_1^{(1)} + \alpha_2 \mathbf{w}_2^{(1)} + \dots + \alpha_l \mathbf{w}_l^{(1)} + \beta_1 \mathbf{w}_1^{(2)} + \beta_1 \mathbf{w}_2^{(2)} + \dots + \beta_m \mathbf{w}_m^{(2)} + \gamma_1 \mathbf{w}_1 + \gamma_2 \mathbf{w}_2 + \dots + \gamma_k \mathbf{w}_k = \mathbf{0}$$

Manipulate the equation and we can see

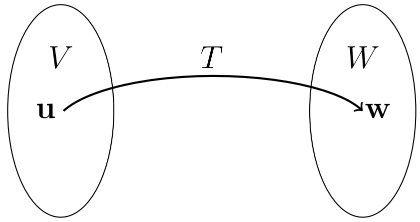
$$\underbrace{\left(\alpha_1 \mathbf{w}_1^{(1)} + \alpha_2 \mathbf{w}_2^{(1)} + \dots + \alpha_l \mathbf{w}_l^{(1)} + \beta_1 \mathbf{w}_1^{(2)} + \beta_1 \mathbf{w}_2^{(2)} + \dots + \beta_m \mathbf{w}_m^{(2)}\right)}_{\in W_1} = -\underbrace{\left(\gamma_1 \mathbf{w}_1 + \gamma_2 \mathbf{w}_2 + \dots + \gamma_k \mathbf{w}_k\right)}_{\in W_2 \setminus W_1}$$

Since they belongs to different sets, clearly they cannot span each other. Therefore

$$\alpha_1 = \alpha_2 = \dots = \alpha_l = \beta_1 = \beta_2 = \dots = \beta_m = \gamma_1 = \gamma_2 = \dots = \gamma_n = 0$$

Hence the list of vectors is also linearly independent.

3. Linear Maps



3.a. Linear Maps as Vector Space

Suppose V and W are two linear spaces over $\mathbb{F}$. T is a function with domain V and codomain W. T is called linear iff

1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$
2. $T(\lambda \mathbf{u}) = \lambda \cdot T(\mathbf{u})$

$\forall \mathbf{v}_1, \mathbf{v}_2 \in V$ and $\forall \mathbf{v}_1 \mathbf{v}_2 \in V, \forall \lambda \in \mathbb{F}$.

Example 3.1

Let $V = \mathbb{R}^3, W = \mathbb{R}^4$. Define T as $(x_1, x_2, x_3) \mapsto (x_1, 0, 0, 0)$

Example 3.2

$T : \mathcal{P}(x) \to \mathcal{P}(x)$, where $f(x) \mapsto \displaystyle\int_{10}^{x} f(x)\ dx$ is a linear map.

Definition 3.3

$\mathcal{L}\{V, W\}$ denotes the set of all linear maps from V to W. Note that $\mathcal{L}\{V, W\}$ with $+$ and $\cdot$ becomes a vector space over $\mathbb{F}$. This requires the additions of functions and multiplications of linear maps by scalars (from $\mathbb{F}$). Given $T_1, T_2 \in \mathcal{L}(V, W)$ we define addition as $(T_1 + T_2)(\mathbf{u}) := T_1(\mathbf{u}) + T_2(\mathbf{u})$, multiplication as $(\lambda T)(\mathbf{u}) := \lambda \cdot T(\mathbf{u})$.

Theorem 3.4

In finite vector space V, W, let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be a basis for V, let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be any vectors in W. Then there exist a unique linear map $T \in \mathcal{L}\{V, W\}$ such that $T(\mathbf{u}_j) = \mathbf{w}_j \forall j$.

Proof. Any vector in V has a unique representation $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{u}$.
Define $T(u) := \underbrace{\alpha_1 T(\mathbf{u})_1 + \alpha_2 T(\mathbf{u})_2 + \cdots + \alpha_n T(\mathbf{u})_n}_{\in W}$ This makes T a linear map from V to W. Indeed if $\lambda \in \mathbb{F}$, then $T(\lambda u) = T(\sum_{j=1}^{n} \lambda \alpha_j \mathbf{u}_j) = \lambda \sum_{j=1}^{n} \alpha_j \mathbf{w} j$. Suppose $\tilde{T}(\mathbf{u}_j) = \mathbf{w}_j$ for all j, then $T = \tilde{T}$ as a map function by linearity and basis. ■

3.b. Null Space and Range

Theorem 3.5

Let Null $(T) := \{\mathbf{u} \in V : T(\mathbf{u}) = 0\}$. Null (T) is a subspace of V.

Theorem 3.6

Let Range $(T) := \{\mathbf{w} \in W : T(\mathbf{u}) = \mathbf{w}\}$. Range (T) is a subspace of W.

Proof. The proof is trivial and is left as an exercise for the reader. ∎

Example 3.7

Let $T : f \to f', V := \mathcal{P}(x), W := \mathcal{P}(x)$. Null $(T) = \mathcal{P}_0(x)$, Range $(T) = \mathcal{P}_2(x)$.
Let $T : f \to f'', V := \mathcal{P}(x), W := \mathcal{P}(x)$. Null $(T) = \mathcal{P}_1(x)$, Range $(T) = \mathcal{P}_1(x)$.

Example 3.8

Find a basis of $\mathcal{L}(V, W)$ given bases $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$ of V and W. The basis consists of $m \times n$ vectors as follows:

$$T_{11} = T(\mathbf{u}_1) = \mathbf{w}_1, T(\mathbf{u}_2) = \mathbf{0}, T(\mathbf{u}_3) = \mathbf{0}, \dots, T(\mathbf{u}_m) = \mathbf{0}$$

$$T_{12} = T(\mathbf{u}_1) = \mathbf{w}_2, T(\mathbf{u}_2) = \mathbf{0}, T(\mathbf{u}_3) = \mathbf{0}, \dots, T(\mathbf{u}_m) = \mathbf{0}$$

$$\dots$$

$$T_{mn} = T(\mathbf{u}_1) = \mathbf{0}, T(\mathbf{u}_2) = \mathbf{0}, T(\mathbf{u}_3) = \mathbf{0}, \dots, T(\mathbf{u}_m) = \mathbf{w}_n$$

Example 3.9

Let $\mathcal{U} = \{f : \mathbb{R} \to \mathbb{R} : f(x) = f(1 - x)\ \forall x\}$.

1. Show that $\mathcal{U}$ is a subspace of $f : \mathbb{R} \to \mathbb{R}$.
2. Find a complement.

$$\mathcal{W} = \{g : \mathbb{R} \to \mathbb{R} : g(x) = -g(1 - x)\ \forall x\}$$

Proof. We can see that the zero function $f(x) = 0$ satisfies the requirement since $0 = 0$ for all values of x.

Suppose $f(x), g(x) \in \mathcal{U}$, then we compute

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ &= f(1 - x) + g(1 - x) \\ &= (f + g)(1 - x)\end{aligned}$$

Therefore we can see that $\mathcal{U}$ is closed under addition.

Suppose $f(x) \in \mathcal{U}, \lambda \in \mathbb{R}$, then we compute

$$\begin{aligned}(\lambda \cdot f)(x) &= \lambda \cdot f(x) \\ &= \lambda \cdot f(1-x) \\ &= (\lambda \cdot f)(1-x)\end{aligned}$$

Therefore we can see that $\mathcal{U}$ is closed addition. Hence $\mathcal{U}$ is a vector space. ∎

Proof. The proof for subspace is similar to part (i) and is omitted here.
We now want to show that $\mathcal{U} + \mathcal{W} = \mathbb{R}^{\mathbb{R}}$. We can see that for $f(x) \in \mathbb{R}^{\mathbb{R}}$, we can rewrite $f(x)$ as

$$f(x) = \frac{f(x) + f(1-x)}{2} + \frac{f(x) - f(1-x)}{2}$$

Clearly $\dfrac{f(x) + f(1-x)}{2} \in \mathcal{U}$ and $\dfrac{f(x) - f(1-x)}{2} \in \mathcal{W}$. For uniqueness, suppose that a nonzero $h(x) \in \mathcal{U} \cap \mathcal{W}$, therefore $h(x) = h(1-x) = -h(1-x)$, and the only solution is $f(x) = 0$, a contradiction, therefore $\mathcal{U} \cap \mathcal{W} = \{0\}$. Hence $\boxed{\mathbb{R}^{\mathbb{R}} = \mathcal{U} \oplus \mathcal{W}}$ ∎

Theorem 3.10 (Rank-Nullity Theorem also known as the Fundamental Theorem of Linear Maps)
Let V, M be finite dimensional vector spaces, let $T \in \mathcal{L}(V, W)$. Then

$$\dim V = \dim \operatorname{Null} T + \dim \operatorname{Range} T$$

Proof. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ to be the the basis for the basis for Null T. By the linear independent list extension theory, this list can be extended to a basis of V. Say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_l$ is asujc an extension to a basis of V. We can see that dim $= k + l$. We want to show that Range $T = l$. Consider $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_l$. We want to show that $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_l$ is basis for Range T. Notice that $\mathbf{v} \in V$ can be written as a linear combination of $\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k + \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_l\mathbf{v}_l$. Then we compute

$$\begin{aligned}T\mathbf{v} &= \alpha_1 T\mathbf{u}_1 + \alpha_2 T\mathbf{u}_2 + \cdots + \alpha_k T\mathbf{u}_k + \beta_1 T\mathbf{v}_1 + \beta_2 T\mathbf{v}_2 + \cdots + \beta_l T\mathbf{v}_l \\ &= \beta_1 T\mathbf{v}_1 + \beta_2 T\mathbf{v}_2 + \cdots + \beta_l T\mathbf{v}_l\end{aligned}$$

hence $T\mathbf{u} \in \operatorname{span}(T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_l)$.
Suppose $\beta_1 T\mathbf{v}_1 + \beta_2 T\mathbf{v}_2 + \cdots + \beta_l T\mathbf{v}_l = 0$. Then $\beta_1 T\mathbf{v}_1 + \beta_2 T\mathbf{v}_2 + \cdots + \beta_l T\mathbf{v}_l \in \operatorname{Null} T$. So

$$\beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_l\mathbf{v}_l = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_k$ since $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ form a basis for Null T.
But $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_l, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ form a basis for V, all of the coefficient has to be 0. Therefore $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_k$ is indeed a basis for Range T. ∎

Example 3.11 (Direct consequences of the Theorem)
Suppose $\dim W < \dim V$ (both finite), and $T \in \mathcal{L}(V, W)$. Then T cannot be injective.

Proof. T is injective implies that $\text{Null } T = \{\mathbf{0}\}$. So $\dim V = 0 + \dim \text{Range } T \leq \dim W < \dim V$, a contradiction. ■

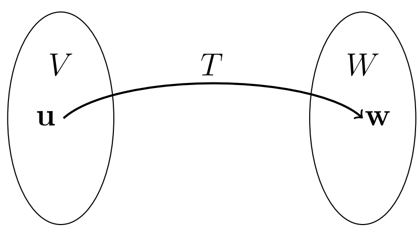
Example 3.12 (Direct consequences of the Theorem)
Suppose $\dim W > \dim V$ (both finite), and $T \in \mathcal{L}(V, W)$. Then T cannot be surjective.

Proof. T is surjective implies that $\text{Range } T = W$. So $\dim V = \dim \text{Null } T + \dim \text{Range } T \geq \dim W > \dim V$, a contradiction. ■

Example 3.13 (Fun Question)
Suppose that $p \in \mathcal{P}(\mathbb{R})$, prove that $\exists q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.
[*This exercise can be done without linear algebra, but it's more fun to do it using linear algebra.*]

Proof. Let $d = \deg p$. Define linear transformation $T : \mathcal{P}_{d+1}(\mathbb{R}) \to P_d(\mathbb{R})$ as $T : q \to 5q'' + 3q'$. We can see that $\dim \text{Null } T = 1$, by the rank nullity theorem, know that T must be surjective as $\dim \mathcal{P}_{d+1}(\mathbb{R}) = \dim \text{Null } T + \dim \text{Range } T = 1 + \dim \text{Range} \implies \dim \text{Range } T = \dim P_d(\mathbb{R})$. ■

3.c. Matrix Notation



Recall this diagram, we want to understand T "correctly". Pick a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for V and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ for W. We can see that $\dim V = n$ and $\dim W = m$. We can define T as

$$Tv_j = A_{1,j}\mathbf{w}_1 + A_{2,j}\mathbf{w}_2 + \cdots + A_{m,j}\mathbf{w}_m$$

Notice that A has the following form

$$\begin{array}{c} \mathbf{w}_1 \to \\ \mathbf{w}_2 \to \\ \vdots \\ \mathbf{w}_m \to \end{array} \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{bmatrix}$$

This is called the matrix representation of T.

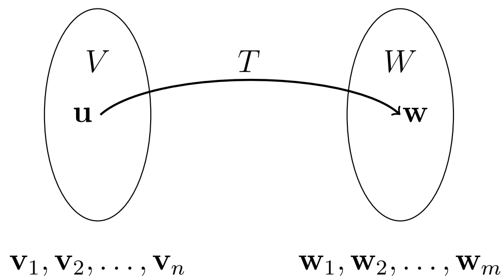
Example 3.14

Let $D : V \to W$ be defined as $D := p \to p'$. Let $V := \text{span}(1, \cos x, \sin x, \cos 2x, \sin 2x) = W$. We can see that

$$\begin{aligned} 1 &\mapsto 0 \\ \cos x &\mapsto -\sin x \\ \sin x &\mapsto \cos x \\ \cos 2x &\mapsto -2\sin 2x \\ \sin 2x &\mapsto \cos 2x \end{aligned} \qquad A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$$

3.d. Matrix Representation

Recall that if T is a linear transformation



$$T\mathbf{v}_u = \sum_{k=1}^{m} A_{i,k}\mathbf{w}_k$$

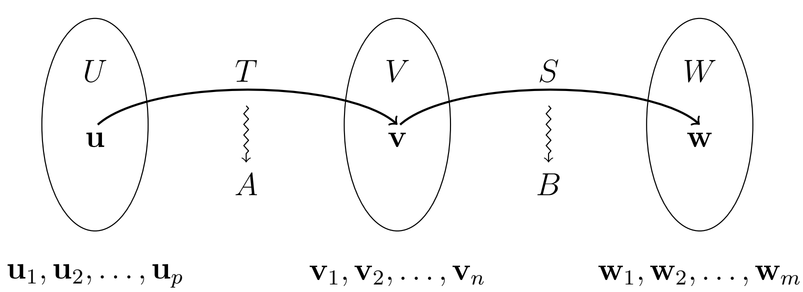
Note that Matrix $A = [A_{i,k}]$ has m rows n columns.

$$\begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{bmatrix}$$

Suppose $v = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, T\mathbf{v} = c_1T\mathbf{v}_1 + c_2T\mathbf{v}_2 + \cdots + c_nT\mathbf{v}_n$.

$$T\mathbf{v} = c_1\sum_{i=1}^{m} A_{i,1}\mathbf{w}_1 + c_2\sum_{i=1}^{m} A_{i,2}\mathbf{w}_2 + \cdots + c_n\sum_{i=1}^{m} A_{i,n}\mathbf{w}_n = \sum_{i=1}^{m}\left(\sum_{i=1}^{n} A_{i,j}c_j\right)\mathbf{w}_j$$

Notice that the operation is the equivalent as the matrix-vector multiplication.



$$ST\mathbf{u}_k = S(T\mathbf{u}_k) = S\left(\sum_{j=1}^{n} A_{j,k}\mathbf{v}_j\right) = \sum_{j=1}^{n} A_{j,k}\left(S\mathbf{v}_j\right) = \sum_{j=1}^{n} A_{j,k}\sum_{i=1}^{m} B_{i,j}\mathbf{w}_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} B_{i,j}A_{j,k}\right)\mathbf{w}_i$$

Use name $\mathcal{M}(S) := B$, $\mathcal{M}(T) := A$, $\mathcal{M}(ST) = BA = \mathcal{M}(S) \cdot \mathcal{M}(T)$. So matrix representation multiply as matrices to produce a composition map or product.

Remark 3.15 (Book Keeping)
$A_{*,j}$ denotes the jth column of A.
$A_{i,*}$ denotes the ith row of A.

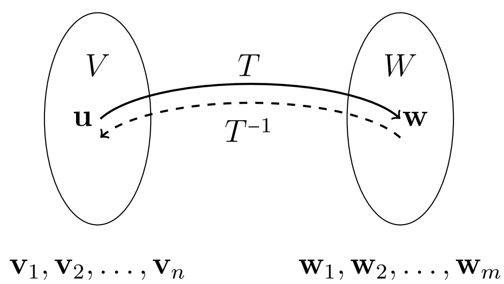
Notice that $\mathcal{M}$ is a linear map, $\mathcal{L}(V, W) \xrightarrow{\mathcal{M}} \mathbb{F}^{m,n}$.

Proposition 3.16
$\mathcal{M}$ is a linear map.

Proposition 3.17
$\mathbb{F}^{m,n}$ has a basis.

Proof. Consider $E_{i,j}$, the matrix consists of all zeros with the exception of 1 in position (i, j). This can be done for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. Also notice that $\dim \mathbb{F}^{m,n} = m \cdot n$ ■

3.e. Invertibility and Isomorphism



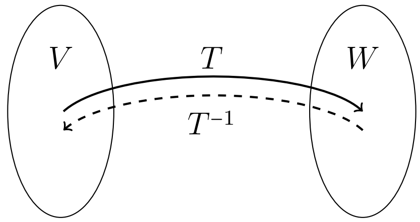
Definition 3.18
$T \in \mathcal{L}(V, W)$ is invertible provided that there exists a mapping T^{-1} from W to V (not necessarily linear) such that

$$T^{-1} \circ T = \mathbb{I}_V$$
$$T \circ T^{-1} = \mathbb{I}_W$$

Where $\mathbb{I}_V, \mathbb{I}_W$ is the identity map on V and W.

Theorem 3.19
T is invertible if and only if T is both injective and surjective.

Proof. Suppose T is invertible, then $T(T^{-1}\mathbf{w}) = \mathbf{w}\ \forall \mathbf{w} \in W$, so Range $T = W$. Also we know that $T^{-1}(T\mathbf{v}) = \mathbf{v}$. Suppose $T\mathbf{v}_1 = T\mathbf{v}_2$, apply the left inverse and we have $T^{-1}(T\mathbf{v}_1) = T^{-1}(T\mathbf{v}_2) \implies \mathbf{v}_1 = \mathbf{v}_2$. Hence T is injective. Therefore T is bijective.
Now suppose T is bijective. We want to construct T^{-1}



We need to take $\mathbf{w} \in W$, there is a $\mathbf{v} \in V$ such that $T\mathbf{v} = \mathbf{w}$ and such $\mathbf{v}$ is unique since T is injective. We declare $T^{-1}\mathbf{w}$ to be $\mathbf{v}$. So $T^{-1} \circ T = \mathbb{I}_V$. We compute

$$(T \circ T^{-1})\mathbf{w} = T(T^{-1}\mathbf{w}) = T\mathbf{v} = \mathbf{w}\ \forall \mathbf{w} \in W$$

So $T \circ T^{-1} = \mathbb{I}_W$ ■

Definition 3.20

If V, W are vector spaces, such that there exists a invertible linear map $T \in \mathcal{L}(V, W)$ then V, W are isomorphic.

Remark 3.21

Before we proceed, we want to check that T^{-1} is a linear map when $T \in \mathcal{L}(V, W)$ and T^{-1} exists.

Proof. Take $\mathbf{w}_1, \mathbf{w}_2 \in W, \lambda \in \mathbb{F}$. We compute $T^{-1}(\lambda\mathbf{w}_1 + \mathbf{w}_2)$. We know that $\mathbf{w}_1 = T\mathbf{v}_1$ and $\mathbf{w}_2 = T\mathbf{v}_2$. Then we know that $T(\lambda\mathbf{v}_1 + \lambda\mathbf{v}_2) = \lambda T\mathbf{v}_1 + T\mathbf{v}_2 = \lambda\mathbf{w}_1 + \mathbf{w}_2$. Subsitute this into T^{-1} and we get

$$T^{-1}(\lambda\mathbf{w}_1 + \mathbf{w}_2) = T^{-1} \circ T(\lambda\mathbf{v}_1 + \mathbf{v}_2) = \mathbb{I}(\lambda\mathbf{v}_1 + \mathbf{v}_2) = \lambda\mathbf{v}_1 + \mathbf{v}_2 = \lambda T^{-1}\mathbf{w}_1 + T^{-1}\mathbf{w}_2$$

Hence T^{-1} is linear. ■

Corollary 3.22

$\mathcal{M}$ is actually a bijection between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$, therefore $\mathcal{L}(V, W)$ is isomorphic to $\mathbb{F}^{m,n}$.

Theorem 3.23

Suppose $T \in \mathcal{L}(V, W)$ is linear and invertible, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be a basis for V. Then $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n$ is a basis for W.

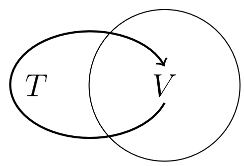
Proof. Suppose $\alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2 + \cdots + \alpha_n T\mathbf{v}_n = 0$. Then $T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) = 0$. Since T is injective, this implies $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = 0$. Therefore $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis. Take $\mathbf{w} \in W$, then there exists a unique $\mathbf{v} \in V$ such that $T\mathbf{v} = \mathbf{w}$, and $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n$, so $T\mathbf{v} = \mathbf{w} = \alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2 + \cdots + \alpha_n T\mathbf{v}_n$, hence span. ∎

Corollary 3.24

dim is invariant under isomorphism.

Linear Operators

We are dealing with a specific case where $\mathcal{L}(V, W)$ is replaced by $\mathcal{L}(V, V)$.



Recall that $\dim V = \dim \operatorname{Null} T + \dim \operatorname{Range} T$. This gives a better test for invertablity if $W = V$.

Theorem 3.25

Let $T \in \mathcal{L}(V, V)$. If V is finite dimensional vector space, then the following are equivalent:

(a) T is injective.

(b) T is surjective.

(c) T is invertible.

Proof.
(a) $\implies$ (c). Trivial by definition.
(b) $\implies$ (c). Suppose T is injective $\xRightarrow[T \text{ being linear}]{} \operatorname{Null} T = \{0\} \iff \dim \operatorname{Null} T = 0$. Therefore

$$\dim V = \dim \operatorname{Null} T = \dim \operatorname{Range} T = 0$$

So $\dim V = \dim \operatorname{Range} T$, hence $\operatorname{Range} T = V$, so T is surjective.
(c) $\implies$ (a).

$$\dim V = \dim \operatorname{Null} T + \dim \operatorname{Range} T = \dim \operatorname{Null} T + \dim V \implies \dim \operatorname{Null} T = 0 \implies \operatorname{Null} T = \{\mathbf{0}\}$$

So T is injective. T is already known to be surjective, so T is bijective, or T is invertable. ∎

Example 3.26

The theorem does not hold for infinite dimension vector spaces, for example:
The differentiation map $T : f(x) \mapsto f'(x)$ is surjective but not invertible over $\mathcal{P}(\mathbb{R})$. From calculus we know that for every $f(x) \in \mathcal{P}(\mathbb{R})$, there exists $g(x) \in \mathcal{P}(\mathbb{R})$ such that $g'(x) = f(x)$, however, we can see that Null $T \neq \{0\}$ as $1 \in$ Null T. Hence T is not injective.
The integration map $T : f(x) \mapsto \int_0^x f(t)dt$ is injective but not invertible over $\mathcal{P}(\mathbb{R})$. We can see that Null $T = \{0\}$, hence T is injective, however, $1 \notin$ Range T.

3.f. Duality

Definition 3.27

Given a vector space V, we define its dual space as $V' = \mathcal{L}(V, \mathbb{F})$.

Remark 3.28

Objects in $\mathcal{L}(V, \mathbb{F})$ are also called linear functional.

Example 3.29

Linear Functional on $\mathbb{R}^3$: $(x_1, x_2, x_3) \mapsto \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \ \forall \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Definition 3.30

Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is a basis of V. The list of linear functionals $\varphi_1, \varphi_2, \dots, \varphi_n \in V'$ such that

$$\varphi_i(\mathbf{v}_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

We claim that $\varphi_1, \varphi_2, \dots, \varphi_n$ is the dual basis of V'.

Lemma 3.31

A dual basis is a basis of V'.

Proof. Suppose there exists $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$\alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \cdots + \alpha_n \varphi_n = 0$$

We compute for v_1

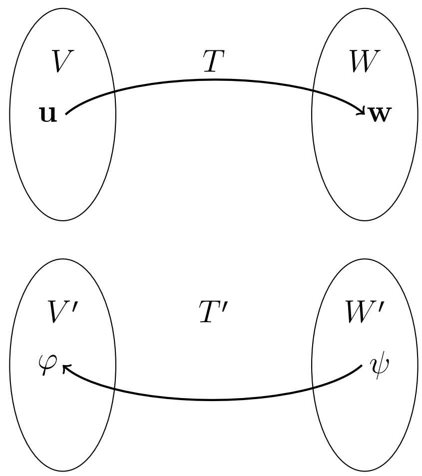
$$(\alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \cdots + \alpha_n \varphi_n)(n) = \alpha_1 \cdot 1 + 0 \implies \alpha_1 = 0$$

Similarly, if we plug in an arbitrary v_j

$$(\alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots + \alpha_n\varphi_n)\mathbf{v}_j) \implies v_j = 0$$

This shows that $\varphi_1, \varphi_2, \ldots, \varphi_n$ is linear independent, and the count is n. So $\varphi_1, \varphi_2, \ldots, \varphi_n$ form a basis for V'. ■

Dual Maps



"upside down"

Definition 3.32

Given $T \in \mathcal{L}(V, W)$, we define $T' : \varphi \mapsto \varphi \circ T$. $\phi \in W'$ i.e. $\varphi \in \mathcal{L}(W, \mathbb{F})$. Notice that $\varphi \circ T \in \mathcal{L}(V, \mathbb{F})$.

Example 3.33

Define V, W, φ, T as $\varphi : f \mapsto \int_0^1 f(t)dt, V = \mathcal{P}_3(\mathbb{R}), W = \mathcal{P}_2(\mathbb{R}), T : f \mapsto f'$. What is $\varphi \circ T$?

$$(\varphi \circ T)(f) = \varphi(T(f)) = \varphi(f') = \int_0^1 f'(t)dt = f(1) - f(0)$$

Remark 3.34 (Algebraic Property of dual maps)

1. $(S+T)' = S' + T'$

   Proof. For any $S, T \in \mathcal{L}(V, W), S', T' \in \mathcal{L}(W', V')$, we compute

   $$(S+T)'(\varphi) = \varphi \circ (S+T) = \varphi \circ S + \varphi \circ T = S'(\varphi) + T'(\varphi)$$

   ∎

2. $(\lambda S)' = \lambda S'$

   Proof.

   $$(\lambda S)'(\varphi) = \varphi \circ (\lambda S) = \lambda(\varphi \circ S) = \lambda \cdot S'(\varphi)$$

   ∎

3. $(ST)' = T'S'$

   Proof. Sanity check: $S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$

   $$(ST)' : \varphi \mapsto \varphi \circ (ST) = (\varphi \circ S) \circ T = T'(\varphi \circ S) = T'(S'(\varphi)) = T' \circ S'$$

   ∎

Definition 3.35 (Annihilators)
Let a set S to be a subset of a vector space V. We can define S^0 as

$$S^0 := \{\varphi \in V' : \varphi(v) = 0 \ \forall \mathbf{v} \in S\}$$

Example 3.36
Consider $\mathbb{R}^3$, let $S := \{(1,0,0),(1,1,0)\}$. We know that any $\varphi \in \mathbb{R}^3$ will have the form of $(x_1, x_2, x_3) \mapsto a_1x_1 + a_2x_2 + a_3x_3$ for some constant a_1, a_2, a_3. Plug in $(1,0,0)$ and $(1,1,0)$ and we get

$$a_1 \cdot 1 + a_2 \cdot 0 + a_3 \cdot 0 = 0 \implies a_1 = 0$$

$$a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 0 = 0 \implies a_2 = 0$$

We can see that $S^0 = \{\varphi(x_1, x_2, x_3) = a_3x_3 \text{ for some } a_3 \in \mathbb{R}\}$ and forms a subspace for $\mathbb{R}^{3}$'.

Lemma 3.37
Regardless of the nature of S, S^0 is always a subspace.

Proof. 1. The zero functional is clearly in S^0.

2. Suppose $\varphi \in S^0$, take $\lambda \in \mathbb{F}$, then $(\lambda\varphi)(\mathbf{v}) = \lambda \cdot \varphi(\mathbf{v}) = 0$ for all $\mathbf{v} \in S$. So $\lambda\varphi \in S$.

3. Suppose $\varphi, \psi \in S^0$, then $(\varphi + \psi)(\mathbf{v}) = \varphi(\mathbf{v}) + \psi(\mathbf{v}) = 0$ for all $\mathbf{v} \in S$. Therefore $\varphi + \psi \in S^0$.

■

Theorem 3.38

Suppose $S = U$, where U is a subspace of V, then

$$\dim U + \dim U^0 = \dim V$$

Proof. Consider the inclusion map:

$$i : U \to V : \mathbf{u} \to \mathbf{u} \ \forall \mathbf{u} \in U$$

Take a look at the dual of U: $i' \in \mathcal{L}(V', U')$. Apply Rank-Nullity to the dual map and we can see that $\dim V' = \dim \text{Null } i' + \dim \text{Range } i'$. We also know that

$$\text{Null } i' = \{\varphi \in V' : \varphi \circ i = 0\}$$

Notice that $\varphi \circ i = 0$ as a functional implies

$$(\varphi \circ i)(\mathbf{u}) = 0 \implies \varphi(i(\mathbf{u})) = 0 \implies \varphi(\mathbf{u}) = 0 \ \forall \mathbf{u} \in U$$

Therefore we can see that Range $i' = \{\varphi \circ i : \varphi \in V'\} = U'$ since any linear functional on U extends to V. I have a clever proof for this but it does not fit in the margin of the page and is left as an exercise for the reader. ■

Theorem 3.39

Let V, W be finite dimensional vector space and let $T \in \mathcal{L}(V, W)$. Then

(a) $\text{Null } T' = (\text{Range } T)^0$

(b) $\dim \text{Null } T' = \dim \text{Null } T + \dim W - \dim V$

Proof.

(a) $\varphi \in \text{Null } T' \iff \varphi \circ T = 0 \iff (\varphi \circ T)(\mathbf{v}) = 0 \ \forall \mathbf{v} \in V \iff \varphi(T\mathbf{v}) = 0 \ \forall \mathbf{v} \in V$ $\iff \varphi \in (\text{Range } T)^0$

(b) $\dim \text{Null } T' = \dim(\text{Range } T)^0 = \dim W - \dim \text{Range } T = \dim W - (\dim V - \dim \text{Null } T) =$ $\dim W - \dim V + \dim \text{Null } T$

■

Corollary 3.40

T' is injective if and only if T is surjective.

Theorem 3.41

Suppose V and W are finite dimensional and $T \in \mathcal{L}(V, W)$, then

(a) $\dim \operatorname{Range} T' = \dim \operatorname{Range} T$

(b) $\operatorname{Range} T' = (\operatorname{Null} T)^0$

Proof.

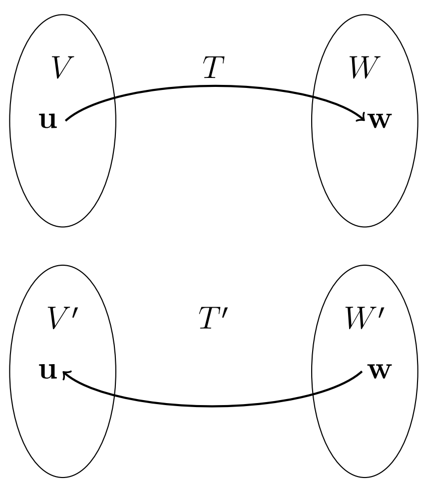
(a) $\dim \operatorname{Range} T' = \dim W' - \dim \operatorname{Null} T' = \dim W - (\dim W - \dim V + \dim \operatorname{Null} T) = \dim V - \dim \operatorname{Null} T = \dim \operatorname{Range} T$

(b) $\psi \in \operatorname{Range} T' \iff \exists \varphi : \varphi \circ T = \psi \iff \varphi \circ T(\mathbf{v}) = \psi(\mathbf{v})\ \forall \mathbf{v} \in V \iff \varphi(T\mathbf{v}) = \psi(\mathbf{v}) \forall \mathbf{v} \in V$.
So $T\mathbf{v} = 0 \implies \psi(\mathbf{v}) = 0$. This shows $\operatorname{Range} T' \subseteq (\operatorname{Null} T)^0$. But $\dim \operatorname{Range} T' = \dim \operatorname{Range} T = \dim V - \dim \operatorname{Null} T = \dim (\operatorname{Null} T)^0$. Hence $\operatorname{Range} T = \operatorname{Null} T$.

■

3.i. Matrix Representation of the dual map

Recall that



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We also recall that $T' : \varphi \mapsto \varphi \circ T$.

Question 3.42

How do we get $\mathcal{M}(T')$ given $\mathcal{M}(T)$?

Answer 3.43. For $\mathcal{M}(T)$, we need 2 bases $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for V and $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ for W.
Take $\varphi_1, \varphi_2, \ldots, \varphi_n$, a dual basis to $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, it is a basis for V'.
Tale $\psi_1, \psi_2, \ldots, \psi_m$, a dual basis to $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$, it is a basis for W'.
Given $\mathcal{M}(T)$, we want to construct / understand $\mathcal{M}(T')$with regard to the basis $\psi_1, \psi_2, \ldots, \psi_m$ of W' and $\varphi_1, \varphi_2, \ldots, \varphi_n$ of V'.
We know $\mathcal{M}(T)$ has m rows n columns, and $\mathcal{M}(T')$ has n rows and m columns.
Suppose $\mathcal{M}(T) = A, \mathcal{M}(T') = C$, we then know

$$T\mathbf{v}_j = \sum_{i=1}^{m} A_{i,j}\mathbf{w}_i \ \forall j = 1, 2, \ldots, n, T'\psi_l = \sum_{l=1}^{n} C_{l,k}\varphi_l \ \forall l = 1, 2, \ldots, m$$

$$T'\psi_k = \psi_k \circ T \implies (\psi_k \circ T)(\mathbf{v}_j) = \psi_k(T\mathbf{v}_j) = \psi_k\left(\sum_{i=1}^{m} A_{i,j}\mathbf{w}_j\right) = \sum_{i=1}^{m} A_{i,j}\psi_k(\mathbf{w}_i) = \sum_{i=1}^{m} A_{i,j}\delta_{ki} = A_{k,j}$$

$$(T'\psi_k)(\mathbf{v}_j) = \left(\sum_{l=1}^{n} C_{l,k}\varphi_l\right)(\mathbf{v}_j) = \sum_{l=1}^{n} C_{l,k}\varphi(\mathbf{v}_j) = \sum_{l=1}^{n} C_{l,k}\delta_{l,j} = C_{j,k}$$

Notice that $A_{k,j} = C_{j,k} \ \forall j, k$.

Conjecture 3.44. So we obtained that

$$M(T') = M(T)^T$$

provided that the basis of V' and W' are chosen to be the dual to the bases of V and W, respectively.

Example 3.45

Let $T : p \mapsto p'$ for $V = \mathcal{P}_3(\mathbb{R})$ with basis $1, x, x^2, X^3$, and $W = \mathcal{P}_2(\mathbb{R})$ with $1, x, x^2$.
We can see that the dual basis for $1, x, x^2, x^3$ is

$$\varphi_0 : p \mapsto p(0), \varphi_1 : p \mapsto p'(0), \varphi_2 : p \mapsto \frac{p''(0)}{2}, \varphi_3 : p \mapsto \frac{p'''(0)}{3!}$$

Dual basis for $1, x, x^2$ is

$$\psi_0 : p \mapsto p(0), \psi_1 : p \mapsto p'(0), \psi_2 : p \mapsto \frac{p''(0)}{2},$$

Notice that

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \text{ and } \mathcal{M}(T') = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4. Polynomials

Recall we call consider polynomials over $\mathbb{C}$ or $\mathbb{R}$.

Theorem 4.1

For any $z_1, z_2 \in \mathbb{C}$, we define $|z| = \sqrt{a^2 + b^2}$, we know that

1. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
2. $|z_1 \cdot z_2| \leq |z_1| + |z_2|$

Proof. Left as an execise. ■

4.a. Axler's Recap on Polynomial

Theorem 4.2

Suppose $p(x) \in \mathcal{P}(\mathbb{F})$, is identically zero. Then all of its coefficient must be 0.

Proof. If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $a_j = \frac{p^{(j0}(0)}{j!}$, If $p(x) \equiv 0$, then $p^{(j)}(x) = 0$, so $a_j = \frac{0}{j!} = 0 \forall j$ ■

Corollary 4.3

Suppose $p(x) \equiv q(x)$ for $p, q \in \mathcal{P}(\mathbb{F})$, then all coefficients of p are the same as all coefficients of q.

4.b. Zero of polynomials and their algebraic manifestations

Algorithm 4.4 (Euclidean Algorithm for polynomials)

Given $p(x), s(x)$, without the loss of generality, $\deg p(x) > \deg s(x)$, otherwise it's boring; we can always find $q(x), r(x)$ such that $p(x) = s(x)q(x) + r(x)$, where $\deg r(x) < \deg s(x)$.

Corollary 4.5

$p(a) \iff p(x) = (x - a)q(x)$ for some $a \in \mathbb{F}$.

Proof. If $p(a) = (x - a)q(x)$, then $p(a) = 0 \cdot q(a) = 0$.
Conversely, suppose $p(a) = 0$, by division algorithm we have $p(x) = (x - a)q(x) + r(x)$, where $\deg r \leq \deg(x - a)$, therefore $r(x) = c$ for some $c \in \mathbb{F}$. Plug in a and we get $(a - a)q(a) + c = 0 \implies 0 + c = 0 \implies c = 0$. Therefore $p(x) = (r - a)q(x)$. ■

Theorem 4.6

Let $p(x)$ be a nonzero polynomial with coefficients in $\mathbb{F}$ have degree n. Then p has at most n zeros in $\mathbb{F}$.

Proof.
Base case: $\deg p = 1$, i.e. $p(x) = a_1x + a_0$ for some $a_1 \in \mathbb{F}^\times, a_0 \in \mathbb{F}$. Then $p\left(\frac{-b}{a}\right) = 0$, so p has exactly one zero.
Inductive Hypothesis: Suppose the statement is true for all polynomials for all polynomials of degree less than m.
Inductive Step: Take $p(x)$ to be a degree m polynomial. If p has no zeros in $\mathbb{F}$, we are done. If p has a zero, by corollary we have $p(x) = (x-a)q(x)$, where $\deg q = m-1$. So the inductive hypothesis applies and q at most $n-1$ distinct zeros in $\mathbb{F}$. ∎

Theorem 4.7 (Fundamental Theorem of Algebra)

Every nonconstant polynomial with complex coefficients has a zero.

Proof with "Black Box" from Complex Analysis.
Assume $\deg p \geq 1$. Assume that $p(a) \neq 0$ $\forall a \in \mathbb{C}$. Consider the function $\frac{1}{p(x)}$ is well-defined $\forall x \in \mathbb{C}$ and is analytic in $\mathbb{C}$, more over $\lim_{|z|\to\infty} \frac{1}{p(z)} = 0$. We know that

$$\begin{aligned} p(x) &= a_0 + a_1x + \cdots + a_nx^n \\ &= x^n\left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + a_n\right) \\ \frac{1}{p(x)} &= \frac{1}{x^n\left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + a_n\right)} \end{aligned}$$

As $|x| \to \infty$, $\dfrac{1}{x^n} \to 0$. Since $\left|\dfrac{1}{x^n}\right| = \dfrac{1}{|x|^n} \to 0$. But $\dfrac{a_0}{x^n} + \dfrac{a_1}{x^{n-1}} + \cdots + a_n \to a_n \neq 0$. Hence $\dfrac{1}{p(x)} \to 0$ as $|x| \to -\infty$.
By Louisville's theorem, any analytic function with this property has to be constant. But $\frac{1}{p(x)}$ is non-constant, so p must have at least 1 zero in $\mathbb{C}$. ∎

Corollary 4.8

Any polynomial $p(x)$ with coefficients in $\mathbb{C}$ factors as follows

$$p(x) = c(x-a_1)(x-a_2)\cdots(x-a_m), c \neq 0$$

Proof. By Induction it's clear for degree 1 and if $\deg p = m$ then factor $p(x) = (x-a)q(x)$ and repeat the process for q. ∎

Question 4.9

What happens over $\mathbb{R}$?

Theorem 4.10
If $p(x)$ has coefficient in $\mathbb{R}$, and $c \in \mathbb{C}$ is a zero of p, then $\overline{c}$ is also a zero of p.

Proof. $p(c) = 0$ means

$$a_0 + a_1 c + a_2 c^2 + \cdots + a_n c^n = 0$$

We then can see

$$\begin{aligned}
\overline{a_0} + \overline{a_1 c} + \overline{a_2 c^2} + \cdots + \overline{a_n c^n} &= \overline{0} = 0 \\
\overline{a_0} + \overline{a_1}\,\overline{c} + \overline{a_2}\,\overline{c^2} + \cdots + \overline{a_n}\,\overline{c^n} &= 0 \\
a_0 + a_1 \overline{c} + a_2 \overline{c^2} + \cdots + a_n \overline{c^n} &= 0
\end{aligned}$$

Hence $p(\overline{c}) = 0$ as well. ∎

So over $\mathbb{C}$, a polynomial with real coefficient factors as follows

$$p(x) = (x - a_1)(x - a_2)\cdots(x - a_n)(x - \lambda_1)(x - \overline{\lambda_1})(x - \lambda_2)(x - \overline{\lambda_2})\cdots(x - \lambda_m)(x - \overline{\lambda_m})$$

For some $c \in \mathbb{R}, a_1, a_2, \ldots, a_n \in \mathbb{R}, \lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$.
To translate this into a factorization over $\mathbb{R}$, we can see that $x^2 - (\lambda + \overline{\lambda}) + |\lambda|^2$. These are quadratic with $\Delta < 0$. Indeed,

$$(\lambda + \overline{\lambda})^2 - 4|\lambda|^2 = \lambda^2 - 2|\lambda|^2 + \overline{\lambda}^2 = 2\mathrm{Re}\lambda^2 - 2|\lambda|^2$$

Notice that $\mathrm{Re}\lambda^2 \leq |\lambda|^2$ and $\mathrm{Re}\lambda^2 = |\lambda|^2$ iff $\lambda \in \mathbb{R}$, therefore $\Delta < 0$.

Question 4.11
Why do we study polynomials?

Answer 4.12.

1. We will form polynomials in linear operators
2. We will associate special polynomials with linear operators.

Remark 4.13
An operator has he same co-domain as its domain.

5. Eigenvalues, Eigenvectors, and Invariant Subspaces

5.a. Invariant Subspaces

Definition 5.1

Let $T \in \mathcal{L}(V, V)$ on a vector space $V \neq \{0\}$. A subspace $U \subseteq V$ is called an invariant subspace is invariant under T if $T\mathbf{u} \in U \ \forall \mathbf{u} \in U$.

Example 5.2

For any $T \in \mathcal{L}(V, V)$, the following subspaces are invariant:

1. $\{0\}$
2. V
3. Null $T = \{\mathbf{v} \in V : T\mathbf{v} = 0\}$
   If $T\mathbf{v} \in$ Null T, then $T\mathbf{v} = 0 \in$ Null T.
4. Range $T = \{\mathbf{w} \in W : \mathbf{w} = T\mathbf{v}$ for some $\mathbf{v} \in V\}$
   So $Tw \in$ Range T.

Question 5.3

What are 1-dimensional invariant subspaces?

Answer 5.4. Then $U = \text{span}(\mathbf{u})$ for some $\mathbf{u} \neq 0$. Invariant means $T\mathbf{u} = \lambda\mathbf{u}$ for some $\lambda \in \mathbb{F}$, where $\mathbf{u}$ is the eigenvector of T and λ is the eigenvalues.

Remark 5.5

$\mathbf{u} \neq 0$ if $\mathbf{u}$ is a eigenvector is T. $\lambda = 0$ is possible.

Proposition 5.6

Let T be a linear operator in V, then the following are equivalent

1. λ is a eigenvalue of T.
2. $T - \lambda\mathbb{I}$ is not invertible.
3. $T - \lambda\mathbb{I}$ is not injective.
4. $T - \lambda\mathbb{I}$ is not surjective.

We have already proven that statement $2, 3, 4$ are logically equivalent.

Theorem 5.7

Suppose $v_1, v_2, \ldots, v_m$ are eigenvectors of $T \in \mathcal{L}(V)$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ will be linearly independent.

Proof. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent. By linear dependence lemma, we find a the minimum index $k \leq m$ such that $\mathbf{v}_k \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1})$. i.e.

$$\mathbf{v}_k = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{k-1}\mathbf{v}_{k-1} \tag{1}$$

Apply linear transformation on both sides

$$T\mathbf{v}_k = T\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{k-1}\mathbf{v}_{k-1} \tag{2}$$

$$\lambda \mathbf{v}_k = \alpha_1 \lambda_1 \mathbf{v}_1 + \alpha_2 \lambda_2 \mathbf{v}_2 + \cdots + \alpha_n \lambda_n \mathbf{v}_n \tag{3}$$

We multiply by equation 1 by λ_m and subtract by from 3 and we get

$$0 = \alpha_1(\lambda_1 - \lambda_k)\mathbf{v}_1 + \alpha_2(\lambda_2 - \lambda_k)\mathbf{v}_2 + \cdots + \alpha_{k-1}(\lambda_{k-1} - \lambda_k)\mathbf{v}_{k-1}$$

A contradiction since k is not the minimum index with the property chosen above. Therefore the list $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ must be linearly independent. ■

Corollary 5.8

An operator $T \in \mathcal{L}(V)$ has at most $\boxed{\dim V}$ distinct eigenvalues.

5.i. Restriction Operators

Definition 5.9

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Then the restriction operator $T|_U \in \mathcal{L}(U)$ is deifned by $T|_U(\mathbf{u}) = T\mathbf{u}$ for all $\mathbf{u} \in U$.

5.b. Eigenvectors and Upper-Triangular Matrices

5.i. Polynomials in T

Definition 5.10

Suppose $T \in \mathcal{L}(V)$, then T^k is defined as

$$T^k := \underbrace{k \circ k \circ \cdots \circ k}_{k \text{ times}}$$

Notice that $T^0 = \mathbb{I}, T^1 = T$.

Definition 5.11

If $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, then we can define $p(T)$ as $a_o \mathbb{I} + a_1 T + a_2 T + \cdots + a_n T^n$.

Example 5.12

Let $V := \mathcal{P}(\mathbb{R}), S : p \mapsto 3p'' + 2p' + p, D : p \mapsto p'$. We can see that S can be expressed as $S = D^0 + 2D + 3D^2$. Therefore

$$\mathcal{M}(S) = 3\mathcal{M}^2(D) + 2\mathcal{M}(D) + M(\mathbb{I})$$

we need to have to take the same basis for inputs and output when forming $\mathcal{M}(\cdot)$. Let's use our favorite basis $1, x, x^2, x^3$. We then can see

$$\mathcal{M}(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{M}(S) = \begin{bmatrix} 1 & 2 & 6 & 0 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 5.13

What is the best matrix representation for an operator?

Question 5.14

What information about eigenvalues/eigenvectors can be read off from a matrix representation?

Theorem 5.15

Suppose $T \in \mathcal{L}(V)$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis of V. Then the following are logically equivalent:

1. $\mathcal{M}(T)$ is upper triangular.
2. $T\mathbf{v}_j \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j)\ \forall j = 1, 2, \ldots, n$.
3. $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j)$ is invariant under $T\ \forall j = 1, 2, \ldots, n$.

Proof. 1) $\implies$ 2)

$$\begin{bmatrix} * & * & * & * & \cdots & * \\ & * & * & * & \cdots & * \\ & & * & * & \cdots & * \\ & & & * & \cdots & * \\ & & & & \ddots & \vdots \\ & & & & & * \end{bmatrix}$$

We can see that 2) holds true by inspection.
2) $\implies$ 3) Consider $T\mathbf{v}_h$ for $h \leq j$, by 2) we have $T\mathbf{v}_k \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_h) \subseteq \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j)$. So $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j)$ is invariant under T.
3) $\implies$ 2) Consider $T\mathbf{v}_j$, by 3) it is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j$ because $T\mathbf{v}_j \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j)$ so $\mathcal{M}(T)(i,j) = 0$ if $i > j$. ∎

Question 5.16

What about conditions for lower-triangular matrices?

Lemma 5.17

Over $\mathbb{C}$, every linear operator has at least one eigenvalue.

Proof. Take $\mathbf{v} \in V \setminus \{0\}$, and consider the list $\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \ldots, T^n\mathbf{v}$ where $n = \dim V$. There is a nontrivial linear combination of these vectors which is 0. Suppose the equation

$$a_0\mathbf{v}_1 + a_1T\mathbf{v} + a_2T^2\mathbf{v} + \cdots + a_nT^n\mathbf{v} = 0$$

i.e. $p(T)v = 0$ for nonconstant $p(x) := a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. By the fundamental theorem of algebra p splits into linear factors over $\mathbb{C}$.

$$p(x) = c(x - \lambda_1)(x - \lambda_2)\cdots(x - \lambda_m)$$

for some $m \leq n$. Therefore

$$p(T)v = c(T - \lambda_1\mathbb{I})(T - \lambda_2\mathbb{I})\cdots(T - \lambda_m\mathbb{I})$$

Therefore at least one of these factors is not injective. This shows that T has at least 1 eigenvalue. ∎

Theorem 5.18

For any $T \in \mathcal{L}(V)$, V is finite dimensional vector space over $\mathbb{C}$, there exists its matrix representation $\mathcal{M}(T)$ which is upper-triangular.

Proof. We can induct on the dimension of V. *Base Step.* $n = 1$ is trivially true.
Inductive Hypothesis. Suppose Theorem holds for all vector spaces of dimension less than $\dim V$.
Inductive Step. Consider $\lambda \in \mathbb{C}$ an eigenvalue of T by lemma. We can define

$$U := \text{Range } (T - \lambda\mathbb{I})$$

U is a subspace of V. By the characterization of eigenvalues, $T - \lambda\mathbb{I}$ is not surjective, hence Range $T - \lambda\mathbb{I} \notin V$, hence $\dim \text{Range } (T - \lambda\mathbb{I}) < \dim V$. We want to show that U is invariant under T. Suppose $\mathbf{v} \in U$, then

$$T\mathbf{v} = \underbrace{(T - \lambda\mathbb{I})\mathbf{v}}_{\in U} + \underbrace{\lambda\mathbf{v}}_{\in U}$$

therefore we know that U is invariant under T. Consider

$$T|_U \in \mathcal{L}(U) : (T|_U)(\mathbf{v}) := T\mathbf{v} \forall \mathbf{v} \in U$$

If $U \neq \{0\}$, then there is a basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ of U $(m < n)$ such that the matrix representation of T/U with respect to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ is upper triangular by the inductive hypothesis. Extend $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ to a basis of V, $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. We compute

$$T\mathbf{v}_j = \underbrace{(T - \lambda\mathbb{I})\mathbf{v}_j}_{\in U = \text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m)} + \lambda\mathbf{v}_j$$

We also know that $T\mathbf{u}_l \in \text{span}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{l-1})$. We can see the matrix representation and hence we are done

$$m \left[\begin{array}{cccc|ccc} * & * & \cdots & * & * & * & * \\ 0 & * & \cdots & * & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * & * & * \\ \hline 0 & 0 & \cdots & 0 & \lambda & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \lambda & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \lambda \end{array}\right]$$

■

Question 5.19

What about eigenvalues of a upper-triangular matrix?

Theorem 5.20

An upper triangular matrix is invertible if and only if all its diagonal entries are nonzero.

Proof. Suppose all diagonal entries are nonzero. Prove surjectivity.

$$\begin{aligned}
&T\mathbf{v}_1 = A_{1,1}v_1, A_{1,1} \neq 0 \implies \mathbf{v}_1 \in \text{Range } T \\
&T\mathbf{v}_2 = A_{1,2}\mathbf{v}_1 + A_{2,2}\mathbf{v}_2, A_{2,2} \neq 0 \implies \mathbf{v}_2 \in \text{Range } T \\
&\quad \vdots \qquad\qquad\qquad\qquad \implies \\
&T\mathbf{v}_n = A_{1,n}\mathbf{v}_1 + A_{2,n}\mathbf{v}_2 + \cdots + A_{n,n}\mathbf{v}_n \neq 0 \implies \mathbf{v}_n \in \text{Range } T
\end{aligned}$$

Therefore Range $T = V$, so T is surjective, hence T is invertible. Suppose at least one one diagonal entry is 0 we want to show that T is not invertible. Say $A_{j,j} = 0$ for some j and upper triangular matrix A. If $j = 1$, then $v_1 \in T$, hence T is not invertible, and we are done. If $j > 1$, consider $U := \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j)$. T maps U to $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{j-1})$. This shows $T|_U$ us not surjective, then we know that $T|_U$ is not injective and there exists $\mathbf{u} \in U$ such that $\mathbf{u} \in T|_U \implies \mathbf{u} \in T$. Therefore T is not injective. Hence T is not invertible. ■

Corollary 5.21

An upper triangular matrix / operator in upper triangular form has the diagonal elements / entries as its eigenvalues.

Example 5.22

The matrix

$$A = \begin{bmatrix} 5 & * & * & * & * \\ 0 & 9 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 8 & * \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$$

has eigenvalue $1, 5, 9, 8, 10$.

Example 5.23

$T : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R}) : p \mapsto 3p'' - 5'p' + 7p$ has eigenvalues $3, -5, 7$.

5.c. Eigenspaces and Diagonal Matrices

Definition 5.24

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The eigenspace of T correspoding to λ, denoted as $E(\lambda, T)$ is defined as

$$E(\lambda, T) := \{\mathbf{v} \in V : T\mathbf{v} = \lambda\mathbf{v}\} = \text{Null } (T - \lambda I)$$

Definition 5.25

An operator $T \in \mathcal{L}(T)$ is called diagonalizable if the operator has a diagonal matrix with repsect to some basis of V.

Theorem 5.26

For $T \in \mathcal{L}(V)$, where V is a finite dimensional vector space, then the following are equivalent

1. $\mathcal{M}(T)$ is a diagonal matrix.
2. the corresponding basis for V consists of eigenvalue of T.
3. $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$ where $\dim U_j = 1$ and U_j is invariant under T for all j.
4. $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$, where $V/W_l = \lambda_l \mathbb{I}$ for all l and W_l is invariant under T.
5. $\dim V = \dim W_1 + \dim W_2 + \cdots + W_k$, where $W_e = (T - \lambda_e \mathbb{I})$.

Proof. Refer to Axler Page 157. ■

6. Inner Product Spaces

Motivation

Definition 6.1
In $\mathbb{R}^n$, the dot product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n)$.

6.a. Inner Product and Norms

Settings

V is a vector space over $\mathbb{F}$, we can define the following mapping $\langle *, * \rangle : V \times V \to \mathbb{F}$.

Definition 6.2
$\langle \cdot, \cdot \rangle$ is called an inner product if it satisfying the following rules:

1. (additivity in the first slot) $\langle \mathbf{v} + \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v} + \mathbf{w} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle,\ \forall \mathbf{v}, \mathbf{u}, \mathbf{w} \in V$
2. (homogeneity in the first slot) $\langle \lambda \mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{v} + \mathbf{w} \rangle,\ \forall \mathbf{v}\mathbf{u}, \mathbf{w} \in V, \lambda \in \mathbb{F}$
3. (conjugate symmetry) $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle},\ \forall \mathbf{v}, \mathbf{w} \in V$
4. (positivity) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0,\ \forall \mathbf{v} \in V$
5. (definiteness) $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\mathbf{v} = \mathbf{0}$.

Question 6.3
What about linearity in the second slot?

Answer 6.4. We can compute

$$\langle \mathbf{v}, \mathbf{u} + \mathbf{w} \rangle = \overline{\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle} = \overline{\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle} = \overline{\langle \mathbf{u}, \mathbf{v} \rangle} + \overline{\langle \mathbf{w}, \mathbf{v} \rangle} = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$$

$$\langle \mathbf{v}, \lambda \mathbf{u} \rangle = \overline{\langle \lambda \mathbf{u}, \mathbf{v} \rangle} = \overline{\lambda \langle \mathbf{u}, \mathbf{v} \rangle} = \overline{\lambda}\ \overline{\langle \mathbf{u}, \mathbf{v} \rangle} = \overline{\lambda} \langle \mathbf{v}, \mathbf{u} \rangle$$

Not quite. ☹

Remark 6.5
If $\mathbf{v} \in V$ is fixed then the function $\langle *, \mathbf{v} \rangle : \mathbf{u} \mapsto \langle \mathbf{u}, \mathbf{v} \rangle$ is a function functional.

Example 6.6

On $\mathbb{R}^n$, we could use any function of the type

$$c_1 x_1 y_1 + c_2 x_2 y_2 + \cdots + c_n x_n y_n$$

where all $c_j \in \mathbb{R}^+$.

Remark 6.7 (Generalization to $\mathbb{C}^n$)

The inner product of this form of the standard product to $\mathbb{C}^n$ can be defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 \overline{y}_1 + x_2 \overline{y}_2 + \cdots + x_n \overline{y}_n$$

Remark 6.8 (Generalization to any function space)

$$\langle f, g \rangle := \int_D f(t)\overline{g(t)}dt$$

or generally

$$\langle f, g \rangle := \int_D f(t)\overline{g(t)}w(t)dt$$

where $w(t)$ is the positive weight function. e.g. if $V = \mathcal{P}(\mathbb{R})$, or $V = \mathcal{P}(\mathbb{C})$, then

$$\langle f, g \rangle := \int_0^\infty f(t)\overline{g(t)}e^{-t}dt$$

Definition 6.9

For $\mathbf{v} \in V$, the (Euclidean) Norm is defined as

$$||\mathbf{v}|| := \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Theorem 6.10

(Properties of Norms)

1. $||\lambda \mathbf{v}|| = |\lambda|\ ||\mathbf{v}||\ \forall \mathbf{v} \in V, \forall \lambda \in \mathbb{F}$
2. $||\mathbf{v}|| > 0$ for all $\mathbf{v} \in V$
3. $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

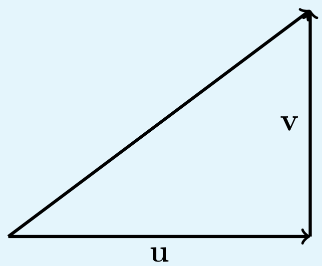
Definition 6.11

An inner product space is a vector space V along with and inner product on V.

Definition 6.12

For $\mathbf{u}, \mathbf{v} \in V$, we say $\mathbf{u}$ and $\mathbf{v}$ is orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$

Theorem 6.13 (Pythagorean Theorem)



$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \iff \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle$$

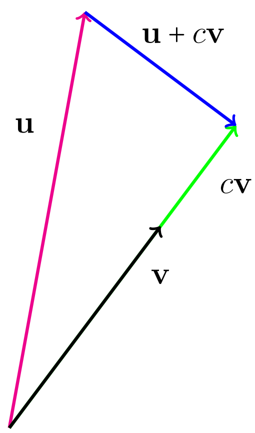
Proof. We compute

$$\langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + 0 + 0 + \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{u}\| + \|\mathbf{v}\|$$

■

Obeservation

Given $\mathbf{u}, \mathbf{v} \in V$ such that $\mathbf{v} \neq 0$, we want to modify $\mathbf{u}$ such that $\mathbf{u} + c\mathbf{v}$ is orthogonal to $\mathbf{v}$. We know that $\langle \mathbf{v} + c\mathbf{v}, \mathbf{v} \rangle = 0$, solve for c gives $c = \frac{-\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$.



An orthogonal decompostion

Theorem 6.14 (Cauchy-Schwarz Inequality)

For any $u, v \in V$ where V is a inner product space, the following holds

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$$

Proof. Given $\mathbf{u}, \mathbf{v} \in V$, we can assume without the loss of generality that $\mathbf{v} \neq 0$. So we can consider vectors $\mathbf{u} + c\mathbf{v}$ and $\mathbf{v}$ that are orthogonal for the choice that

$$c := \frac{-\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}$$

By Pathgrathrean theorem, $||\mathbf{u} + c\mathbf{v}||^2 + ||c\mathbf{v}||^2 = ||\mathbf{u}||^2$. But $||c\mathbf{v}||^2 = |c|^2||\mathbf{v}||^2$ and recall

$$c = \frac{-\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}, \text{ so } c^2 = \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{\langle \mathbf{v}, \mathbf{v} \rangle^2} = \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{||\mathbf{v}||^4}, \text{ therefore } ||c\mathbf{v}||^2 = \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{||\mathbf{v}||^4}||\mathbf{v}||^2 = \frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{||\mathbf{v}||^2}$$

So by dropping $||\mathbf{u} + c\mathbf{v}||^2 > 0$, we obtain $||c\mathbf{v}||^2 \leq ||\mathbf{u}||$, i.e,

$$\frac{|\langle \mathbf{u}, \mathbf{v} \rangle|^2}{||\mathbf{v}||^2} \leq ||\mathbf{u}||^2 \implies |\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq ||\mathbf{u}||^2 \cdot ||\mathbf{v}||^2 \implies |\langle \mathbf{u}, \mathbf{v} \rangle|^2 \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||$$

∎

Theorem 6.15 (Triangle Inequality)

$$||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

Proof. We have

$$\begin{aligned} ||\mathbf{u} + \mathbf{v}||^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u} + \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + \langle \mathbf{u} + \mathbf{v} \rangle + \overline{\langle \mathbf{u} + \mathbf{v} \rangle} = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle \\ &\leq ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2|\langle \mathbf{u}, \mathbf{v} \rangle| \leq ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2||\mathbf{u}||\ ||\mathbf{v}|| \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2 \end{aligned}$$

∎

Theorem 6.16 (Alternative Version of Triangle Inequality)

$$\big|||\mathbf{u}|| - ||\mathbf{v}||\big| \leq ||\mathbf{u} - \mathbf{v}||$$

Proof. Notice that

$$||\mathbf{u}|| - ||\mathbf{v}|| \leq ||\mathbf{u} - \mathbf{v}|| \iff ||\mathbf{u}|| \leq ||\mathbf{u} - \mathbf{v}|| + ||\mathbf{v}||$$

Which is the triangle inequaliy. Swapping out $\mathbf{u}$ and $\mathbf{v}$ gives us

$$||\mathbf{v}|| - ||\mathbf{u}|| \leq ||\mathbf{v} - \mathbf{u}|| \iff ||\mathbf{u}|| \leq ||\mathbf{u} - \mathbf{v}|| + ||\mathbf{v}||$$

Combining these equations gives us

$$\big|||\mathbf{u}|| - ||\mathbf{v}||\big| \leq ||\mathbf{u} - \mathbf{v}||$$

∎

Fact 6.17 (Fun inequalities)

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}+\mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right)$$

6.b. Orthogonality

Definition 6.18
A list $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is V is called orthonormal if

$$\langle \mathbf{v}_i \mathbf{v}_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Lemma 6.19
Any list of orthonormal vectors is necessarily linearly indepedent.

Proof. Suppose $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = \mathbf{0}$. We can compute on the standard inner product

$$\langle \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k, \mathbf{v}_1 \rangle = \alpha_1\langle \mathbf{v}_1, \mathbf{v}_1\rangle + \alpha_2\langle \mathbf{v}_2, \mathbf{v}_1\rangle + \cdots + \alpha_k\langle \mathbf{v}_k, \mathbf{v}_1\rangle \implies \alpha_1 = 0$$

$$\langle \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k, \mathbf{v}_2 \rangle = \alpha_1\langle \mathbf{v}_2, \mathbf{v}_2\rangle + \alpha_2\langle \mathbf{v}_2, \mathbf{v}_2\rangle + \cdots + \alpha_k\langle \mathbf{v}_k, \mathbf{v}_2\rangle \implies \alpha_1 = 0$$

$$\vdots$$

$$\langle \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k, \mathbf{v}_k \rangle = \alpha_1\langle \mathbf{v}_1, \mathbf{v}_k\rangle + \alpha_2\langle \mathbf{v}_2, \mathbf{v}_k\rangle + \cdots + \alpha_k\langle \mathbf{v}_k, \mathbf{v}_k\rangle \implies \alpha_k = 0$$

Hence $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ is linearly indepedent. ∎

Question 6.20
What is nice about orthonormal basis?

Answer 6.21. If $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ is an orthonormal basis, then an arbitary vector can be written as

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{v}, \mathbf{v}_2\rangle\mathbf{v}_2 + \cdots + \langle \mathbf{v}, \mathbf{v}_n\rangle\mathbf{v}_n$$

Furthermore, we can conclude the following theorem:

Theorem 6.22 (Generalized Pythagorean Theorem)

$$\|\mathbf{v}\|^2 = \sum_{j=1}^{n} \left|\langle \mathbf{v}, \mathbf{v}_j\rangle\right|^2$$

Algorithm 6.23 (Gram-Schmidt Algorithm)

`Input`: Any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ that is linearly independent.
`Output`: $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ such that $\mathbf{e}_j \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_j)$ for all $j \leq n$.

Process.

$$\begin{aligned}
\mathbf{e}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \\
\mathbf{e}_2 &= \frac{\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{e}_1 \rangle \mathbf{e}_1}{\|\mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{e}_1 \rangle \mathbf{e}_1\|} \\
\mathbf{e}_3 &= \frac{\mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{e}_1 \rangle \mathbf{e}_1 - \langle \mathbf{v}_3, \mathbf{e}_2 \rangle \mathbf{e}_2}{\|\mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{e}_1 \rangle \mathbf{e}_1 - \langle \mathbf{v}_3, \mathbf{e}_2 \rangle \mathbf{e}_2\|} \\
&\vdots \\
\mathbf{e}_n &= \frac{\mathbf{v}_n - \langle \mathbf{v}_n, \mathbf{e}_1 \rangle \mathbf{e}_1 - \langle \mathbf{v}_n, \mathbf{e}_2 \rangle \mathbf{e}_n \cdots - \langle \mathbf{v}_n - \mathbf{e}_{n-1} \rangle \mathbf{e}_{n-1}}{\|\mathbf{v}_n - \langle \mathbf{v}_n, \mathbf{e}_1 \rangle \mathbf{e}_1 - \langle \mathbf{v}_n, \mathbf{e}_2 \rangle \mathbf{e}_n \cdots - \langle \mathbf{v}_n - \mathbf{e}_{n-1} \rangle \mathbf{e}_{n-1}\|}
\end{aligned}$$

■

Proposition 6.24

Every finite inner product vector space has a orthonormal basis.

Proof. Suppose V is a finite dimensional vector space. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V. We then apply Gram-Schmidt Algorithm to the basis to obtain a orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. ■

Remark 6.25 (Projection orthonal with the repect to inner product)

Given a subsace U of V for finie dimensioanl vector space V, there is a projector P_V that project all vectors in V on V orthogonally.

Remark 6.26 (Relations between inner product and linear functionals)

Suppose V is finite dimensional vector space. Given any $\mathbf{u} \in V$, then function $\langle \cdot, \mathbf{u} \rangle$ is a linear functional (i.e. an element of $V' = \mathcal{L}(V, \mathbb{F})$)

Theorem 6.27 (Riesz Representation Theorem)

For any $\varphi \in V'$ there exists a *unique* $\mathbf{u} \in V$ such that $\langle \mathbf{v}, \mathbf{u} \rangle = \varphi(\mathbf{v}) = \varphi(\mathbf{v})$ for al $\mathbf{v} \in V$.

Proof. **Existence**
Take an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. of V. Any $\mathbf{v}$ can be written as a linear combination the basis (To preserve linearity we want to put $\mathbf{v}$ into the first slot)

$$\begin{aligned}
\mathbf{v} &= \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{v}, \mathbf{e}_2 \rangle + \cdots + \langle \mathbf{v}, \mathbf{e}_n \rangle \\
\varphi(v) &= \langle \mathbf{v}, \mathbf{e}_1 \rangle \varphi(\mathbf{e}_1) + \langle \mathbf{v}, \mathbf{e}_2 \rangle \varphi(\mathbf{e}_2) + \cdots + \langle \mathbf{v}, \mathbf{e}_n \rangle \varphi(\mathbf{e}_n) \\
&= \langle \mathbf{v} + \overline{\varphi(\mathbf{e}_1)} e_1 + \overline{\varphi(\mathbf{e}_2)} e_2 + \cdots + \overline{\varphi(\mathbf{e}_n)} \mathbf{e}_n \rangle \\
&= \mathbf{u}
\end{aligned}$$

Uniqueness
Suppose there are $\mathbf{u}_1, \mathbf{u}_2$ such that $\langle \mathbf{v}, \mathbf{u}_1 \rangle = \langle \mathbf{v}, \mathbf{u}_2 \rangle$ for all $\mathbf{v} \in V$. Take $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$. We can see that

$$\langle \mathbf{v}_1, \mathbf{u}_1 \rangle + \langle \mathbf{v}, \mathbf{v}_2 \rangle \iff \langle \mathbf{v}, \mathbf{u}_1 - \mathbf{u}_2 \rangle$$

Plug in v and we get

$$\langle \mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2 \rangle = ||\mathbf{u}_1 - \mathbf{u}_2||^2 = 0$$

This means that $\mathbf{u}_1 - \mathbf{u}_2 = 0$, or $\mathbf{u}_1 = \mathbf{u}_2$, hence such $\mathbf{u}$ is unique. ∎

Example 6.28

Let $V := \mathcal{P}(\mathbb{C})$ and let $\varphi(p) := \int_{-1}^{1} p(t) \sin t \cdot dt$. Find a reprsentation in V with with respect to

$$\langle f, g \rangle = \int_{-1}^{1} f(t) \overline{g(t)} dt$$

Suppose $p(t) = a_0 + a_1 t + a_2 t^2$.

6.c. Orthogonality and Orthogonal Projections

Definition 6.29

Given an inner produce space V and its subset U of V we can define

$$U^{\perp} := v \in V : \{\mathbf{u}, \mathbf{v}\rangle = 0 \ \forall \mathbf{u} \in U\}$$

Theorem 6.30

Basic facts about orthogonal complement

(a) If U is a susbet of V, then $U^\perp$ is a subspace of V.

(b) $\{\mathbf{0}\}^\perp = V$

(c) $V^\perp = \{\mathbf{0}\}$

(d) $U \cap U^\perp \subseteq \{\mathbf{0}\}$

(e) If $U \subseteq W$ then $U^\perp \supseteq W^\perp$

Proof.

(a) Clearly $\mathbf{0} \in U^\perp$ as $\langle \mathbf{v}, \mathbf{0} \rangle = 0 \ \forall \mathbf{v} \in V$. Take $\mathbf{v}_1, \mathbf{v}_2 \in U^\perp$ and $\lambda \in \mathbb{F}$, then $\langle \mathbf{v}_1 + \lambda \mathbf{v}_2, \mathbf{u} \rangle = \langle \mathbf{v}_1, \mathbf{u} \rangle + \lambda \langle \mathbf{v}_2, \mathbf{u} \rangle = 0 \ \forall \mathbf{u} \in U$.

(b) Trivial by part (c).

(c) Trivial by part (b).

(d) Suppose $\mathbf{v} \in U \cap U^\perp$, then $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \implies \mathbf{v} = \mathbf{0}$.

(e) Suppose $\mathbf{v} \in W^\perp$, then $\langle \mathbf{v}, \mathbf{w} \rangle = 0 \ \forall \mathbf{w} \in W$. Since $U \subseteq W$, $\langle \mathbf{v}, \mathbf{u} \rangle = 0 \ \forall \mathbf{u} \in U$, hence $U^\perp \supseteq W^\perp$.

■

Theorem 6.31

If V is a finite dimensional inner product space and U is a subspace of V, then

$$U \oplus U^\perp = V$$

Proof. We already know that the sum is direct by $U \cap U^\perp = \{0\}$. By Gram Schmidt we can construct an orthonormal basis of U and extend it to a normal normal basis of V we have $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We claim that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis of $U^\perp$ as $\langle \mathbf{u}_i, \mathbf{v}_j \rangle = 0$ for all i in $1, 2, \ldots, k$ and j in $1, 2, \ldots, n$. Hence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in U^\perp$. On the other hand,

$$U^\perp \in \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_k \mathbf{u}_k + \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_n \mathbf{v}_n$$

must satisify $\alpha_j = \langle \mathbf{v}_i, \mathbf{u}_j \rangle = \mathbf{0}$ for all i in $1, 2, \ldots, k$. Therefore we have $U^\perp = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$. Hence $U \oplus U^\perp = V$. ■

Theorem 6.32

Suppose U is a finite dimensional subspace of V. Then

$$U = (U^\perp)^\perp$$

Definition 6.33

Suppose U is a finite-dimensional subspace of V. The orthogonal projection of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For $\mathbf{v} \in V$, write $\mathbf{v} = \mathbf{u} + \mathbf{w}$, where $\mathbf{u} \in U$ and $\mathbf{w} \in U^\perp$ Then $P_U v = u$.

Theorem 6.34 (Properties of the orthogonal projection P_U)

Suppose U is finite dimensional subspace of V and $\mathbf{v} \in V$. Then

(a) $P_U \in \mathcal{L}(V)$

(b) Range $P_U = U$

(c) Null $P_U = U^\perp$

(d) $P_U^2 = P_U$

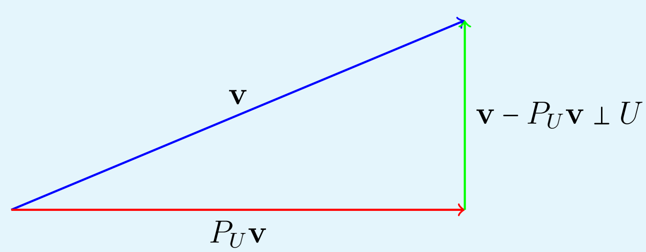
(e) Range $(\mathbb{I} - P_U) = U^\perp$

(f) $(\mathbb{I} - P_U)^2 = (\mathbb{I} - P_U)$

(g) $||P_U\mathbf{v}|| \leq ||\mathbf{v}||$

(h) For every orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of U, we have

$$P_U\mathbf{v} = \langle \mathbf{v}, \mathbf{e}_1\rangle\mathbf{e}_1 + \langle \mathbf{v}, \mathbf{e}_2\rangle\mathbf{e}_2 + \cdots + \langle \mathbf{v}, \mathbf{e}_n\rangle\mathbf{e}_n$$



The red vector is the projection of the blue vector $\mathbf{v}$ onto subspace U.

7. Operators on Inner Product Spaces

7.a. Self-Adjoint and Normal Operators

Definition 7.1

Suppose $T \in L(V, W)$, we define $T*$ by this formula

$$\langle T\mathbf{v}, \mathbf{w}\rangle_W = \langle \mathbf{v}, T^*\mathbf{w}\rangle_V$$

We can think og $\mathbf{w}$ as fixed, we note that $\langle T*, \mathbf{w}\rangle$ is a linear functiona;. hence it has a representer by Riesz, so we are entitled to call it $T^*\mathbf{w}$ such that

$$\langle T\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, T^*\mathbf{w}\rangle$$

Hence we have $T^* \in \mathcal{L}(W, V)$. We want to verify this property. Consider $T * (\mathbf{w}_1 + \lambda\mathbf{w}_2)$. We compute

$$\begin{aligned}
\langle \mathbf{v}, T * (\mathbf{w}_1 + \lambda\mathbf{w}_2)\rangle &= \langle T\mathbf{v}, \mathbf{w}_1 + \lambda\mathbf{w}_2\rangle \\
&= \langle T\mathbf{v}, \mathbf{w}_1\rangle + \overline{\lambda}\langle T\mathbf{v} + \mathbf{w}_2\rangle \\
&= \langle \mathbf{v}, T^*\mathbf{w}_1\rangle + \overline{\lambda}\langle \mathbf{v}, T^*\mathbf{w}_2\rangle \\
&= \langle \mathbf{v}, T^*\mathbf{w}_1\rangle + \langle \mathbf{v}, \lambda T^*\mathbf{w}_2\rangle \\
&= \langle \mathbf{v}, T^*\mathbf{w}_1 + \lambda T^* w_2 \ \rangle \forall \mathbf{v} \in V, \mathbf{w}_1, \mathbf{w}_2 \in W, \lambda \in \mathbb{F}
\end{aligned}$$

So $T^*(\mathbf{w}_1 + \lambda\mathbf{w}_2) = T^*\mathbf{w}_1 + \lambda T^*\mathbf{w}_2$.

Theorem 7.2 (Properties of the adjoint)

Let $T \in \mathcal{L}(V, W)$, we have

1. $(S + T)^* = S^* + T^*$
2. $(\lambda T)^* = \overline{\lambda} T^*$
3. $(S \cdot T)^* = T^* S^*$
4. $(\lambda T)^* = \overline{\lambda} T^*$
5. $\mathbb{I}^* = \mathbb{I}$

Proof. Refer to book page 206 ■

Theorem 7.3 (Null space and range of adjoint)
Let $T \in \mathcal{L}(V, W)$, then

1. Null $T^* =$ (Range $T)^\perp$
2. Range $T^* =$ (Null $T)^\perp$
3. Null $T =$ (Range $T*)^\perp$
4. Range $T =$ (Null $T^*)^\perp$

Proof. Let $\mathbf{w} \in W$. Then

$$\begin{aligned}\mathbf{w} \in \text{Null } T^* &\iff T^*\mathbf{w} = 0 \\ &\iff \langle \mathbf{v}, T^*\mathbf{w}\rangle = 0 \ \forall \mathbf{v} \in V \\ &\iff \langle T\mathbf{v}, \mathbf{w}\rangle \ \forall \mathbf{v} \in V \\ &\iff \mathbf{w} \in (\text{Range } T)^\perp\end{aligned}$$

(b),(c),(d) follows by a similar logic and is left as an exercise. ∎

Definition 7.4
Let $T \in \mathcal{L}(V)$. T is called self-adjoint if $T^* = T$.

Definition 7.5
T is called normal if $TT^* = T^*T$.

Example 7.6
Suppose $T \in \mathcal{L}(V)$, we can define $T : \mathbf{v} \mapsto \langle ,\mathbf{x}\rangle\mathbf{y}$ for some fixed $\mathbf{x}, \mathbf{y}$ in V. Compute T^*.
We compute

$$\begin{aligned}\langle \mathbf{v}, T^*\mathbf{w}\rangle &= \langle T\mathbf{v}, \mathbf{w}\rangle \\ &= \langle \langle \mathbf{v}, \mathbf{x}\rangle\mathbf{y}, \mathbf{w}\rangle \\ &= \langle \mathbf{v}, \mathbf{x}\rangle\langle \mathbf{y}, \mathbf{w}\rangle \\ &= \langle \mathbf{v}, \overline{\langle \mathbf{y}, \mathbf{w}\rangle}\mathbf{x}\rangle \\ &= \langle \mathbf{v}, \langle \mathbf{w}, \mathbf{y}\rangle\mathbf{x}\rangle\end{aligned}$$

hence we can concldue $T^*\mathbf{w} = \langle \mathbf{w}, \mathbf{y}\rangle\mathbf{x}$ for all $\mathbf{w} \in W$.

7.i. Matrix representation

Suppose $T \in \mathcal{L}(V,W)$, where V,W are finite-dimensional vector spaces. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be an orthonormal basis for V and $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m$ be a orthonormal basis for W. We can see that $\mathcal{M}(T)$ is obtained through

$$T\mathbf{e}_j = \langle T\mathbf{e}_j, \mathbf{f}_1\rangle \mathbf{f}_1 + \langle T\mathbf{e}_j, \mathbf{f}_2\rangle \mathbf{f}_2 + \cdots + \langle T\mathbf{e}_j, \mathbf{f}_m\rangle \mathbf{f}_m$$

$$T^*\mathbf{f}_k = \langle T^*\mathbf{f}_k, \mathbf{e}_1\rangle \mathbf{e}_1 + \langle T^*\mathbf{f}_k, \mathbf{e}_2\rangle \mathbf{e}_2 + \cdots + \langle T^*\mathbf{f}_k, \mathbf{e}_n\rangle \mathbf{e}_n$$

Then

$$\mathcal{M}(T^*)(l,k) = \langle T^*\mathbf{f}_k, \mathbf{e}_l\rangle \implies \mathcal{M}(T^*)(j,i) = \langle T^*\mathbf{f}_i, \mathbf{e}_j\rangle$$

Therefore we have $\mathcal{M}(T^*) = \overline{\mathcal{M}(T)}^T$

Remark 7.7

The above statement only holds if the basis for V and W are orthonormal.

Remark 7.8

If an operator T is self-adjoint, then T is normal, but not the converse.

Proposition 7.9

The eigenvalue of any self-adjoint operator is real.

Proof. Suppose self-adjoint $T \in \mathcal{L}(V)$. and λ is an eigenvalue of T and let $\mathbf{v}$ be the eignevector corespond to λ. We compute

$$\lambda\langle \mathbf{v}, \mathbf{v}\rangle = \langle \lambda\mathbf{v}, \mathbf{v}\rangle = \langle T\mathbf{v}, \mathbf{v}\rangle = \langle \mathbf{v}, T\mathbf{v}\rangle = \langle \mathbf{v}, \lambda\mathbf{v}\rangle = \overline{\lambda}\langle \mathbf{v}, \mathbf{v}\rangle$$

We can see that $\lambda = \overline{\lambda} \implies \lambda \in \mathbb{R}$. ■

Question 7.10

Suppose $\langle T\mathbf{v}, \mathbf{v}\rangle = 0 \forall \mathbf{v} \in V$. Does the statement implies T is the zero map?

Answer 7.11 (Surprisingly)**.** Yes over $\mathbb{C}$ and no over $\mathbb{R}$.

Proof. Supoose $\mathbb{F} = \mathbb{C}$, the following holds.

$$\langle T(\mathbf{v}+\mathbf{w}), \mathbf{v}+\mathbf{w}\rangle = \langle T\mathbf{v}, \mathbf{v}\rangle + \langle T\mathbf{v}, \mathbf{w}\rangle + \langle T\mathbf{w}, \mathbf{v}\rangle + \langle T\mathbf{w}, \mathbf{w}\rangle$$

$$\langle T(\mathbf{v}-\mathbf{w}), \mathbf{v}+\mathbf{w}\rangle = \langle T\mathbf{v}, \mathbf{v}\rangle - \langle T\mathbf{v}, \mathbf{w}\rangle - \langle T\mathbf{w}, \mathbf{v}\rangle + \langle T\mathbf{w}, \mathbf{w}\rangle$$

Subtratc the first equation by the second we have

$$\langle T(\mathbf{v}+\mathbf{w}), (\mathbf{v}+\mathbf{w})\rangle - \langle T(\mathbf{v}-\mathbf{w}), (\mathbf{v}+\mathbf{w})\rangle = \boxed{2\langle T\mathbf{v}, \mathbf{w}\rangle + 2\langle T\mathbf{w}, \mathbf{v}\rangle}$$

We also compute

$$\langle T(\mathbf{v}+i\mathbf{w}),(\mathbf{v}+i\mathbf{w})\rangle - \langle T(\mathbf{v}-i\mathbf{w}),(\mathbf{v}-i\mathbf{w})\rangle = \boxed{2i\left(\langle T\mathbf{w},\mathbf{v}\rangle - \langle T\mathbf{v},\mathbf{w}\rangle\right)}$$

Take the two boxed equation and divide the second one by i then subtract from first gives us

$$4\langle T\mathbf{v},\mathbf{w}\rangle = 0$$

Suppose $\mathbb{F}=\mathbb{R}$. Consider $\mathbb{R}^2$. Take $T\mathbf{v}$ and rotate $\pi/2$ gives us $T(x_1,x_2) := (-x_2,x_1)$. We can see that $\langle T\mathbf{v},\mathbf{v}\rangle = 0 \forall \mathbf{v}$ but $T \neq 0$. However, if T is self-adjoint then T is 0. ■

Remark 7.12

Suppose $\mathbb{F}=\mathbb{R}$ and $T=T^*$. We have

$$4\langle T\mathbf{v},\mathbf{w}\rangle = \langle T(\mathbf{v}+\mathbf{w}),(\mathbf{v}+\mathbf{w})\rangle - \langle T(\mathbf{v}-\mathbf{w}),(\mathbf{v}-\mathbf{w})\rangle$$

Hence $T=0$.

Corollary 7.13

$\langle T\mathbf{v},\mathbf{v}\rangle \in \mathbb{R}$ in a complex space is equivalent to T being self adjoint.

Proof.

$$\langle T\mathbf{v},\mathbf{v}\rangle \in \mathbb{R} \iff \langle T\mathbf{v},\mathbf{v}\rangle = \langle T^*\mathbf{v},\mathbf{v}\rangle \implies \langle (T-T^*)\mathbf{v},\mathbf{v}\rangle = 0 \implies T-T^* = 0$$

We can see that $T=T^*$. Hence T is self-adjoint. ■

Theorem 7.14

T is normal if anf only if $||T\mathbf{v}|| = ||T^*\mathbf{v}||\ \forall \mathbf{v} \in V$.

Proof.

$$||T\mathbf{v}|| = ||T^*\mathbf{v}|| \implies \langle T\mathbf{v},T\mathbf{v}\rangle = \langle T^*\mathbf{v},T^*\mathbf{v}\rangle \implies \langle T^*T\mathbf{v},\mathbf{v}\rangle = \langle TT^*\mathbf{v},\mathbf{v}\rangle$$

Hence T is normal sicne $TT^* = T^*T$. ■

Theorem 7.15

Say $\lambda,\mathbf{v}$ is an eigenpair of a normal oeprator T, then

$$||(T-\lambda\mathbb{I})\mathbf{v}|| = ||(T^*-\overline{\lambda}\mathbb{I})\mathbf{v}||$$

7.b. Spectral Theorem

Over Complex Vector Space

Theorem 7.16 (Spectral Theorem over Complex Vector Space)
Suppose $T \in \mathcal{L}(V)$ where V is finite dimensioanl vector space and $\mathbb{F} = \mathbb{C}$ and T is normal. Then V is a orthonormal basis of eigenvactors of T, and vice versa, if T has a diagonal representation with respect to some orthonormal basis, then T is normal.

Proof. Suppose T has a diagonal matrix representation with some repsect to some orthonormal basis. i.e.

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \qquad \mathcal{M}(T^*) = \begin{bmatrix} \overline{\lambda}_1 & & & 0 \\ & \overline{\lambda}_2 & & \\ & & \ddots & \\ 0 & & & \overline{\lambda}_n \end{bmatrix}$$

Since any two diagonal matrices commute, we can see that

$$\mathcal{M}(T)\mathcal{M}(T^*) = \mathcal{M}(T^*)\mathcal{M}(T) = \begin{bmatrix} |\lambda_1|^2 & & & 0 \\ & |\lambda_2|^2 & & \\ & & \ddots & \\ 0 & & & |\lambda_n|^2 \end{bmatrix}$$

We have $TT^* = T^*T$, hence T is normal.
Suppose T is normal. By Schur's Theorem, there exists an orthonormal basis such that

$$\mathcal{M}(T) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \Longrightarrow \mathcal{M}(T^*) = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} & \cdots & \overline{a}_{1n} \\ 0 & \overline{a}_{22} & \cdots & \overline{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{a}_{nn} \end{bmatrix}$$

Rall that $||T\mathbf{v}|| = ||T^*\mathbf{v}|| \quad \forall \mathbf{v} \in V$. Call this this orthornormal basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We have $T\mathbf{e}_1 = a_{11}\mathbf{e}_1$, so $||T\mathbf{e}_1|| = |a_{11}|$, we then compute

$$T^*\mathbf{e}_1 = \overline{a}_{11}\mathbf{e}_1 + \overline{a}_{12}\mathbf{e}_2 + \cdots + \overline{a}_{1n}\mathbf{e}_n$$
$$||T^*\mathbf{e}_1|| = \sqrt{|a_{11}|^2 + |a_{12}|^2 + \cdots + |a_{1n}|^2}$$

Since $||T\mathbf{e}_1|| = ||T^*\mathbf{e}_1||$, we get $|a_{12}| = |a_{13}| = \cdots = |a_{1n}| = 0$.
Using a similar logic, we have $||T\mathbf{e}_j|| = ||T^*\mathbf{e}_j||$ implies $|a_{jj+1}| = |a_{jj+2}| = \cdots = |a_{jn}| = 0$. Hence T is diagonal ■

Remark 7.17
So actaully the schur form of a normal operator is neccisarily diagonal.

Over Real Vector Space

Lemma 7.18

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $\beta, \gamma \in \mathbb{R}$ such that $b\beta^2 - 4\gamma$ then

$$T^2 + \beta T + \gamma I$$

is invertible.

Proof. Consider nonzero $\mathbf{v} \in V$. We can factor

$$\begin{aligned}
\langle (T^2 + \beta T - \gamma I) \rangle &= \langle T^2\mathbf{v}, \mathbf{v} \rangle + \langle \beta T\mathbf{v}, \mathbf{v} \rangle + \gamma \langle \mathbf{v}, \mathbf{v} \rangle \\
&= \langle T\mathbf{v}, T\mathbf{v} \rangle + \beta \langle T\mathbf{v}, \mathbf{v} \rangle + \gamma ||\mathbf{v}||^2 \\
&\geq ||T\mathbf{v}||^2 - |\beta| \cdot ||T\mathbf{v}|| \cdot ||\mathbf{v}|| + \gamma ||\mathbf{v}||^2 \\
&= \left(||T\mathbf{v}|| - \frac{|\beta| \cdot ||\mathbf{v}||}{2} \right)^2 + \left(\gamma - \frac{\beta^2}{4} \right) ||\mathbf{v}||^2 \\
&> 0
\end{aligned}$$

hence we can see that Null $(T^2 + \beta T - \gamma I) = \{0\}$. Hence it's injective. Since $T^2 + \beta T + \gamma I \in \mathcal{L}(V)$, we know $(T^2 + \beta T + \gamma I)$ is invertible. ∎

Theorem 7.19

T has a eigenvalue if T is self-adjoint in any vector space.

Proof. Assume $\dim V = n$. Consider any $\mathbf{v} \in V$. Then $\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \ldots, T^n\mathbf{n}$ are linearly dependent (I.e, there exist $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that

$$a_0\mathbf{v} + a_1 T\mathbf{v} + \cdots + a_n T^{\mathbf{v}} = 0$$

Consider $f(x) = a_0 x + a_1 x + \cdots + a_n x^n$. We know from chapter 4 we can factor $f(x)$ as

$$f(x) = c(x^2 + \beta_1 x + \gamma_1) \cdots (x^2 + \beta_m x + \gamma_m)(x - \lambda_1) \cdots (x - \lambda_n)$$

where all coefficients are real and $\beta_i^2 - 4\gamma < 0$. By lemma we know that the qaudratice term is invertible, then we can simply factor them out. Therefore we have

$$0 = (T - \lambda_1 I) \cdots (T - \lambda_m I)\mathbf{v}$$

Hence one of the $(T - \lambda_j I)$ is not injective. Hence T has an eigenvalue. ∎

Theorem 7.20

Suppose $T \in \mathcal{L}(V)$, where V is a fininite dimensional vector space and $\mathbb{F} = \mathbb{R}$ and T is self-adjoint. Then T has a digona matrix representation with some orthonorma basis for V. And conversely, if T has a diagonal matrix representation eith repsetc to some orthonormal basis, then $T = T^*$.

Proof. Suppose T has a diagonal matrix representation with some repsect to some orthonormal basis. i.e.

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \qquad \mathcal{M}(T^*) = \begin{bmatrix} \overline{\lambda}_1 & & & 0 \\ & \overline{\lambda}_2 & & \\ & & \ddots & \\ 0 & & & \overline{\lambda}_n \end{bmatrix}$$

We know $\mathcal{M}(T) = \mathcal{M}(T^*)$ since $\lambda = \overline{\lambda}$ in reals. Hence T is self-adjoint. Conversely, suppose $T = T^*$. We just found out T has at least one eigenvalue eigenvector pair. Say $T\mathbf{u} = \lambda\mathbf{u}$. Without the loss of generality $\|\mathbf{u}\| = 1$. If $\mathbf{w} \perp \mathbf{u}$, then $\langle T\mathbf{u}, \mathbf{w}\rangle = 0 = \langle \mathbf{u}, T\mathbf{w}\rangle$. So $T\mathbf{w} \perp \mathbf{u}$. Notice that $T|_{\text{span}(\mathbf{u})^\perp}$ is still self-adjoint.

$$\langle T_{\text{span}(\mathbf{u})^\perp}\mathbf{w}_1, \mathbf{w}_2\rangle = \langle T\mathbf{w}_1, \mathbf{w}_2\rangle = \langle \mathbf{w}_1, T\mathbf{w}_2\rangle = \langle \mathbf{w}_1, T|_{\text{span}(\mathbf{u})^\perp} w_2\rangle \qquad \forall \mathbf{w}_1, \mathbf{w}_2 \in \text{span}(\mathbf{u})^\perp$$

Hence $\mathcal{M}(T) = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ 0 & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$ Now the problem is reduce to that for $T|_{\text{span}(\mathbf{u})^\perp}$, which has a dimensional of $\dim V - 1$. By induction we can build a orthonomal basis of V which consists of eigenvectors. ■

7.c. Positive Operators and Isometries

Definition 7.21

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and satisfies

$$\langle T\mathbf{v}, \mathbf{v}\rangle \geq 0 \qquad \forall \mathbf{v} \in V$$

Then T is called **nonnegative**.
If instead $\langle T\mathbf{v}, \mathbf{v}\rangle > 0 \quad \forall \mathbf{v} \in V$ then T is called **positive**.

Theorem 7.22 (Characterzation Theorem)

The following are equivalent

1. T is nonnegative
2. $T = T^*$ and all its eigenvalyes are nonnegative.
3. T has a nonnegative square root, i.e. there exists $T = R^* \in \mathcal{L}(V)$ such that $R^2 = T$.
4. T has a self-adjoint square root. i.e. $\exists S = S^*$ such that $S^2 = T$.
5. There exists $Q \in \mathcal{L}(V)$ such that $Q^*Q = T$.

Proof. (e) $\implies$ (a). Suppose $T = Q^*Q$, so i

$$\langle T\mathbf{v}, \mathbf{v}\rangle = \langle Q^*Q\mathbf{v}, \mathbf{v}\rangle = \langle Q\mathbf{v}, Q\mathbf{v}\rangle = ||Q\mathbf{v}||^2 \geq 0$$

(a) $\implies$ (b) Suppose T is nonnegative. We know that nonnegative already satisfies self-adjoint. TODO ■

Definition 7.23

Suppose $S \in \mathcal{L}(V)$. S is called an **isometry** if

$$||S\mathbf{v}|| = ||\mathbf{v}|| \qquad \forall \mathbf{v} \in V$$

Remark 7.24

Observe that isometry necessarily preserves all inner products.

$$\langle S\mathbf{u}, S\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle \qquad \forall \mathbf{u}, \mathbf{v} \in V$$

This following from polar polarization from 7.11. for $\mathbb{F} = \mathbb{R}$ we have

$$4\langle T\mathbf{u}, \mathbf{v}\rangle = \langle T(\mathbf{u} + \mathbf{v}), (\mathbf{u} + \mathbf{v})\rangle - \langle T(\mathbf{u} - \mathbf{v}), (\mathbf{u} - \mathbf{v})\rangle$$

Corollary 7.25

An isometry maps an orthonormal to another orthonormal basis.

Proof. If $\langle \mathbf{e}_i, \mathbf{e}_j\rangle = \delta_{ij}$. Then $\langle S\mathbf{e}_i, S\mathbf{e}_j\rangle = \langle \mathbf{e}_i, \mathbf{e}_j\rangle = \delta_{ij}$ where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.
So if $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ is an northnormal basis then $S\mathbf{e}_1, S\mathbf{e}_2, \ldots, S\mathbf{e}_n$ is an orthonormal basis. ■

7.d. Polar Decomposition and Singular Value Decomposition

Theorem 7.26 (Polar Decomposition)
Take $T \in \mathcal{L}(V)$. There exist an isometry $S \in \mathcal{L}(V)$ such that

$$T = S\sqrt{T^*T}$$

where $\sqrt{T^*T}$ is the nonnegative square root of T^*T.

Proof. Observe that $\|T\mathbf{v}\| = \left\|\sqrt{T^*T}\mathbf{v}\right\| \qquad \forall \mathbf{v} \in V$. Indeed

$$\begin{aligned} \|T\mathbf{v}\| &= \langle T\mathbf{v}, T\mathbf{v}\rangle \\ &= \langle T^*T\mathbf{v}, \mathbf{v}\rangle \\ &= \langle \sqrt{T^*T}\cdot\sqrt{T^*T}\mathbf{v}, \mathbf{v}\rangle \\ &= \langle \sqrt{T^*T}\mathbf{v}, \sqrt{T^*T}\mathbf{v}\rangle \\ &= \left\|\sqrt{T^*T}\mathbf{v}\right\| \end{aligned}$$

It's clearly that there exists an isometry between T and $\sqrt{T^*T}$. ■

Remark 7.27 (Construction of S)
For any $\mathbf{v} \in V$, define

$$S_1\left(\sqrt{T^*T}\mathbf{v}\right) := T\mathbf{v}$$

We first need to check this is well-defined. That is if

$$\sqrt{T^*T}\mathbf{v}_1 = \sqrt{T^*T}\mathbf{v}_2 \implies \mathbf{v}_1 = \mathbf{v}_1$$

This is true because

$$\sqrt{T^*T}(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0} \implies 0 = \left\|\sqrt{T^*T}(\mathbf{v}_1 - \mathbf{v}_2)\right\| = \|T(\mathbf{v}_1 - \mathbf{v}_2)\|$$

hence $T\mathbf{v}_1 = T\mathbf{v}_2$.
So S_1 is now defined as an element of $\mathcal{L}\left(\text{Range }\sqrt{T^*T}, \text{Range } T\right)$ and S_1 is actually invertiable and an isometry.
So $\dim \text{Range }\sqrt{T^*T} = \dim \text{Range } T$. Now we need to extend S_1 to an operator on V.
Take $\left(\text{Range }\sqrt{T^*T}\right)^\perp$ and $(\text{Range } T)^\perp$. Send any orthonormal basis of $\left(\text{Range }\sqrt{T^*T}\right)^\perp$ to any orthonormal basis of $(\text{Range } T)^\perp$. This defines another isometry S_2. Finally define

$$S\mathbf{v} = S_1\mathbf{u} + S_2\mathbf{w}$$

where $\mathbf{v} = \mathbf{u} + \mathbf{w}, \mathbf{u} \in \text{Range }(\sqrt{(T^*T)}), \mathbf{w} \in \text{Range }\left(\sqrt{(T^*T)}\right)^\perp$. This creates S which is now an isometry on entire V. Hence $T = S\sqrt{T^*T}$.

Theorem 7.28

Let $T \in \mathcal{L}(V)$, for finite dimensional V. Then there exists orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ and $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ and values $s_1, s_2, \dots, s_n$, all nonnegative such that

$$T\mathbf{v} = s_1\langle \mathbf{v}, \mathbf{e}_1\rangle \mathbf{f}_1 + s_2\langle \mathbf{v}, \mathbf{e}_2\rangle \mathbf{f}_2 + \cdots + s_n\langle \mathbf{v}, \mathbf{e}_n\rangle \mathbf{f}_n$$

The s_j are called singular values of T.

Proof. (derivation for polar decomposition). Say $T = S\sqrt{T^*T}$. By the charaterization theorem w eknow that V has an orthonormal eigenbasis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ consisiting of eigenvectors of $\sqrt{T^*T}$ corresponding to (nonnegative) eigenvalues $s_1, s_2, \dots, s_n$

$$\begin{aligned}
\mathbf{v} &= \langle \mathbf{v}, \mathbf{e}_1\rangle + \langle \mathbf{v}, \mathbf{e}_2\rangle + \cdots + \langle \mathbf{v}, \mathbf{e}_n\rangle \\
\sqrt{T^*T}\mathbf{v} &= s_1\langle \mathbf{v}, \mathbf{e}_1\rangle + s_2\langle \mathbf{v}, \mathbf{e}_2\rangle + \cdots + s_n\langle \mathbf{v}, \mathbf{e}_n\rangle \\
S\sqrt{T^*T}\mathbf{v} &= s_1\langle \mathbf{v}, \mathbf{e}_1\rangle \mathbf{f}_1 + s_2\langle \mathbf{v}, \mathbf{e}_2\rangle \mathbf{f}_2 + \cdots + s_n\langle \mathbf{v}, \mathbf{e}_n\rangle \mathbf{f}_n \\
T\mathbf{v} &= s_1\langle \mathbf{v}, \mathbf{e}_1\rangle \mathbf{f}_1 + s_2\langle \mathbf{v}, \mathbf{e}_2\rangle \mathbf{f}_2 + \cdots + s_n\langle \mathbf{v}, \mathbf{e}_n\rangle \mathbf{f}_n
\end{aligned}$$

and $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$ is also orthonormal. ∎

Example 7.29

Take $T(x_1, x_2) = (2x_1 + x_2, -x_1 + 2x_2)$. Find its polar decomposition.

Answer 7.30.

$$T = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \qquad T^* = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \implies T^*T = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Therefore we have $s_1 = s_2 = \sqrt{5}$ and $f_1 = \left(2/\sqrt{5}, -1/\sqrt{5}\right), f_2 = \left(1/\sqrt{5}, 2/\sqrt{5}\right)$.

8. Operators on Complex Vector Spaces

8.c. Characteristic and Minimal Polynomial

Definition 8.1

The number of times an eignevalue λ appears in the matrix is called the algebragic multiplicity of λ.

Example 8.2

Suppose V is a complex vector space and let $T \in \mathcal{L}(V)$. Supppose T has the following matrix presentation

$$\begin{bmatrix} 2 & 1 & 0 & & & \\ 0 & 2 & 1 & & & \\ 0 & 0 & 2 & & & \\ & & & 3 & 1 & \\ & & & 0 & 3 & \\ & & & & & 2 \end{bmatrix}$$

We can see that $\lambda = 2$ has a multiplicity of 4 and $\lambda = 3$ has a algebraic multiplicaity of 2.

Definition 8.3

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Suppose T has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with algebraic multiplicity of $d_1, d_2, \ldots, d_n$. Then the polynomial

$$p_{\text{char}}(z) = \prod_j (z - \lambda_j)^{d_j}$$

is the characteristic polynomial of T.

Theorem 8.4 (The Cayley-Hamilton Theorem)

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Then $p(T) = 0$, where p is the characteristic polynomial.

Proof. Trivial by Jordan Normal Form in section 8.d. ■

Definition 8.5

A minimal polynomial for $T \in \mathcal{L}(V)$ os a monoic polynomial of the smallest degree that annihilates T. i.e. $q(T) = 0$ and q is of smallest degree with this property of leading coefficient 1.

Example 8.6

Consider T in example 8.2. Take the largest block of eahc eigenvalue and raise each term to the size of the block will yield the minimal polynomial

$$p_{textmin}(z) = (z-2)^3(z-3)^2$$

Corollary 8.7

Suppose $h(T) = 0$ for some polynomial $h \not\equiv 0$. Then $h(z) = p_{\min}(z)q(x)$ for some q.

Proof. By the remainder theorem we have

$$h(z) = p_{\min}(z)q(z) + r(z)$$

where $\deg r < \deg p_{\min}$. By the minimality of $p_{\min}$, $r \equiv 0$. ∎

8.d. Jordan Form

Goal

To find te the sparest matrix representation for an arbitary linear operator on a finite dimensional vector space over $\mathbb{C}$.

8.i. Observation

"Rough" decomposition 1

$$\left[\begin{array}{ccc|ccc} * & \cdots & * & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{array}\right] = \mathcal{M}(T)$$

Notice that T has two invariant subspaces that are direct sums of each other.

Definition 8.8

An operator is called nilpotent if some power of it equals to 0.

Proposition 8.9

For any $T \in \mathcal{L}(V)$, there exists two subsapces, V_s and V_r such that $V = V_s \oplus V_r$ and V_s, V_r are both T-invariant, and $T|_{V_s}$ is nilpotent and $T|_{V_r}$ is invertible.

Proof. Consider

$$\{v\} \subseteq \text{Null } T \subseteq \text{Null } T^2 \subseteq \cdots$$

Because $\dim V \le \infty$, we must be able to find $q \in \mathbb{N}$ such that T^q and T^{q+h} for any $h \in \mathbb{N}$ have the same null space. In other words

$$\exists q \in \mathbb{N} : \text{Null } T^q = \text{Null } T^{q+h} \qquad \forall k \in \mathbb{N}$$

Take $V_s := \text{Null } T^q$ and $V_r = \text{Range } T^q$. Obsrve that V_s and V_r ae T-invariant.
Next we want to check that $V_s \cap V_r = \{\mathbf{0}\}$.
Suppose $\mathbf{v} \in V_s \cap V_r$. Then $T^q\mathbf{v} = \mathbf{0}$, and $T^q\mathbf{w} = \mathbf{v}$ for some $\mathbf{w} \in V$. So $T^{2q}\mathbf{w} = \mathbf{0}$. Hnec eby the choice of q we have $\mathbf{w} \in \text{Null } T^q$, so

$$T^q\mathbf{w} = \mathbf{0} = \mathbf{v}$$

So $\mathbf{v} = 0$, and $V_s \cap V_r = \{\mathbf{0}\}$. By Rank-Nullity, $V = V_s \oplus V_r$.
$T|_{V_s}$ is nilpotent sicne $(T|_{V_s})^2$ is zero.
$T|_{V_r}$ is invertible since for any $\mathbf{w} \in V_r$ such that $T\mathbf{w} = \mathbf{0}$ will also satify $T^q\mathbf{w} = \mathbf{0}$, hence $\mathbf{w} = \mathbf{0}$, and $T|_{V_r}$ being injective implies invertiablity. ∎

Zoom in to the nilpotent part

Say, the whole space V satifies the condition $T^q = 0$ and without the loss of generality we can take q minimal with this property. This means there exists $\mathbf{v}_0 \in V$ such that $T^{q-1}\mathbf{v}_0 \neq \mathbf{0}$. Take

$$\mathbf{v}_0 = \text{span}\left\{\mathbf{v}_0, T\mathbf{v}_0, \ldots, T^{q-1}\mathbf{v}_0\right\}$$

Since there exists a vector such that $T^{q-1}\mathbf{v}_0 \neq \mathbf{0}$ we can also there exists $\mathbf{w}_0 \in V$ such that $\langle T^{q-1}\mathbf{v}_0, \mathbf{w}_0\rangle \neq 0$. Take the following matrix

$$\left(\langle T^{j-1}\mathbf{v}_0, {T^*}^{q-i}\mathbf{w}_0\rangle\right)_{i,j=1}^{q} = \left(\langle T^{q+j-i-1}\mathbf{v}_0, \mathbf{w}_0\rangle\right)_{i,j=1}^{q}$$

Notice that this is a lower triangular matrix with nonzero diagonal matrix.

Corollary 8.10
The list $\mathbf{v}_0, T\mathbf{v}_0, \ldots, T^{q-1}\mathbf{v}_0$ is linearly indepedent and so is the list $\mathbf{w}_0, T^*\mathbf{w}_0, \ldots, {T^*}^{q-1}\mathbf{w}_0$

Proof. Take $V_1 := \left(\text{span}\left(\mathbf{w}_0, T^*\mathbf{w}_0, \ldots, {T^*}^{q-1}\mathbf{w}_0\right)\right)^{\perp}$. Notice that if W is $T*$-invariant, $W^{\perp}$ is T-invariant. Indeed, for any $\mathbf{v} \in W^{\perp}$ and any $\mathbf{w} \in W$, we have

$$\langle T\mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{v}, T^*\mathbf{w}\rangle = 0$$

Hence we have $V = V_0 \oplus V_1$, where V_0, V_1 are both T-invariant. To see that the sum is direct Suppose

$$\alpha_0\mathbf{v}_0 + \alpha_1 T\mathbf{v}_0 + \cdots + \alpha^{q-1}T^{q-1}\mathbf{v}$$

is orthogonal to $\mathbf{w}_0, T^*\mathbf{w}_0, \ldots, {T^*}^{q-1}\mathbf{w}_0$. Thn the matrix

$$\left(\langle T^{j-1}\mathbf{v}_0, {T^*}^{q-1}\mathbf{w}_0\rangle\right)$$

being invertible gurantees that

$$\alpha_0 = \alpha_1 = \cdots = \alpha_{q-1} = 0$$

■

fine decomposition

We look at $\mathcal{M}\left(T|_{V_1}\right)$ with respect to the basis $\mathbf{v}_0, T\mathbf{v}_0, \ldots, T\mathbf{v}_0$. Hence we have

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Warp-up

Repeat the process many gives

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

This guarantees a bock-diagonal form where each block looks like

$$\begin{bmatrix} \lambda_j & 1 & & & & & \\ & \lambda_j & 1 & & & & \\ & & \lambda_j & 1 & & & \\ & & & \lambda_j & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \lambda_j & 1 \\ & & & & & & \lambda_j \end{bmatrix}$$

Example 8.11

Consider

$$\mathcal{M}(T) = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where the empty entries are zero. We can see that the T has eigenvalue $3, -2, 1, 0$.
We can see that

$$\begin{aligned} \dim \text{Null } (T - 3\mathbb{I})^j &= 3, 5, 6, 6, 6, \ldots \\ \dim \text{Null } (T + 2\mathbb{I})^j &= 1, 2, \ldots \\ \dim \text{Null } (T - 1\mathbb{I})^j &= 1, 2, 2, \ldots \\ \dim \text{Null } (T - 0\mathbb{I})^j &= 1, 2, 2, 2, \ldots \\ j = 1, 2, 3, \ldots & \quad \forall \ j \in \mathbb{N} \end{aligned}$$

Example 8.12

Suppose $V := \mathcal{P}_4(\mathbb{C})$. Let D be the differetiation operator. Construct the Jordan Normal Form of D and the Jordan Basis of V.
We can compute for $\mathcal{M}(T)$ with the standard basis

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with some algebraic manipulation we then can see that D has jodran normal form of

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with basis $1, x, \frac{1}{2}x^2, \frac{1}{3}x^3, \frac{1}{4}x^4$.

Example 8.13

Suppose $V = \text{span}\left(e^{ikt} : |k| \le 3\right)$. Let D be the differetiation operator. Construct the Jordan Normal Form of D and the Jordan Basis of V.
We can see that $V = \text{span}(e^{-i3t}, e^{-i2t}, e^{-it}, e^{0}, e^{it}, e^{i2t}, e^{i3t})$. We can compute for the matrxi representation with repsect to the standard basis

$$\mathcal{M}(T) = \begin{bmatrix} -3 & & & & & & \\ & -2 & & & & & \\ & & -1 & & & & \\ & & & 0 & & & \\ & & & & 1 & & \\ & & & & & 2 & \\ & & & & & & 3 \end{bmatrix}$$

Notice that if we use basis $V = \text{span}\left(\frac{1}{-3i}e^{-i3t}, \frac{1}{-2i}e^{-i2t}, \frac{1}{-i}e^{-it}, e^{0}, \frac{1}{i}e^{it}, \frac{1}{2i}e^{i2t}, \frac{1}{3i}e^{i3t}\right)$ we can obtain the Jordan Normal Form

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 0 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$$

Reverse Engineering of the Jordan Normal Form

	$\dim(T-\lambda\mathbb{I})$	$\dim(T-\lambda\mathbb{I})^2$	$\dim(T-\lambda\mathbb{I})^3$	$\dim(T-\lambda\mathbb{I})^4$	$\dim(T-\lambda\mathbb{I})^j, j\geq 5$
$\lambda = i$	3	6	7	7	7
$\lambda = -i$	2	4	6	8	8
$\lambda = 1$	1	2	3	$\cancel{5}$ 4	5

What is the Jordan Normal form og T based on this info? Assume all eigenvalues of T are given above.

Solution. Notice that the difference in the sequence denotes the number of 1's that get send to 0, hence we have the following matrix

$$\begin{bmatrix}
i & 1 \\
& i & 1 \\
& & i \\
& & & i & 1 \\
& & & & i \\
& & & & & i & 1 \\
& & & & & & i \\
& & & & & & & -i & 1 \\
& & & & & & & & -i & 1 \\
& & & & & & & & & -i & 1 \\
& & & & & & & & & & -i \\
& & & & & & & & & & & -i & 1 \\
& & & & & & & & & & & & -i & 1 \\
& & & & & & & & & & & & & -i & 1 \\
& & & & & & & & & & & & & & -i \\
& & & & & & & & & & & & & & & 1 & 1 \\
& & & & & & & & & & & & & & & & 1 & 1 \\
& & & & & & & & & & & & & & & & & 1 & 1 \\
& & & & & & & & & & & & & & & & & & 1 & 1 \\
& & & & & & & & & & & & & & & & & & & 1
\end{bmatrix}$$

■

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