Math 110, Spring 2019

Contents

1 Inner Product Spaces

Motivation

Recall that

Definition 1.1
In $\mathbb{R}^n$, the dot product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \cdot \vec{y} := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

for $\vec{x} = (x_1, x_2, \ldots, x_n), \vec{y} = (y_1, y_2, \ldots, y_n)$.

1.1 Inner Product and Norms

Settings

V is a vector space over $\mathbb{F}$, we can define the following mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$.

Definition 1.2
$\langle \cdot, \cdot \rangle$ is called an inner product if it satisfying the following rules:

1. $\langle \vec{v} + \vec{u}, \vec{w} \rangle = \langle \vec{v} + \vec{w} \rangle + \langle \vec{u}, \vec{w} \rangle, \ \forall \vec{v}, \vec{u}, \vec{w} \in V$
2. $\langle \lambda \vec{v}, \vec{w} \rangle = \lambda \langle \vec{v} + \vec{w} \rangle, \ \forall \vec{v} \vec{u}, \vec{w} \in V, \lambda \in \mathbb{F}$
3. $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{v} \rangle}, \ \forall \vec{v}, \vec{w} \in V$
4. $\langle \vec{v}, \vec{v} \rangle \geq 0, \ \forall \vec{v} \in V$
5. $\langle \vec{v}, \vec{v} \rangle = 0$ iff $\vec{v} = \vec{0}$.

Question 1.3
What about linearity in the second slot?

Answer 1.4. We can compute

$$\langle \vec{v}, \vec{u} + \vec{w} \rangle = \overline{\langle \vec{u} + \vec{w}, \vec{v} \rangle} = \overline{\langle \vec{u}, \vec{v} \rangle + \rangle \vec{w}, \vec{v}} = \overline{\langle \vec{u}, \vec{v} \rangle} + \overline{\langle \vec{w}, \vec{v} \rangle} = \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{w} \rangle$$

$$\langle \vec{v}, \lambda \vec{u} \rangle = \overline{\langle \lambda u, \vec{v} \rangle} = \overline{\lambda \langle \vec{u}, \vec{v} \rangle} = \overline{\lambda} \ \overline{\langle \vec{u}, \vec{v} \rangle} = \overline{\lambda} \langle \vec{v}, \vec{u} \rangle$$

Not quite. ☹

Remark 1.5
If $\vec{v} \in V$ is fixed then the function $\langle \cdot, \vec{v} \rangle : \vec{u} \mapsto \langle \vec{u}, \vec{v} \rangle$ is a function functional.

Example 1.6

On $\mathbb{R}^n$, we could use any function of the type

$$c_1x_1y_1 + c_2x_2y_2 + \cdots + c_nx_ny_n$$

where all $c_j \in \mathbb{R}^+$.

Remark 1.7 (Generalization to $\mathbb{C}^n$)

The inner product of this form of the standard product to $\mathbb{C}^n$ can be defined as

$$\langle \vec{x}, \vec{y} \rangle = x_1\overline{y}_1 + x_2\overline{y}_2 + \cdots + x_n\overline{y}_n$$

Remark 1.8 (Generalization to any function space)

$$\langle f, g \rangle := \int_D f(t)\overline{g(t)}dt$$

or generally

$$\langle f, g \rangle := \int_D f(t)\overline{g(t)}w(t)dt$$

where $w(t)$ is the positive weight function. e.g. if $V = \mathcal{P}(\mathbb{R})$, or $V = \mathcal{P}(\mathbb{C})$, then

$$\langle f, g \rangle := \int_0^\infty f(t)\overline{g(t)}e^{-t}dt$$

Definition 1.9

For $v \in V$, the (Euclidean) Norm is defined as

$$||v|| := \sqrt{\langle v, v \rangle}$$

Theorem 1.10

$$||\lambda v|| = |\lambda|\ ||v||\ \forall v \in V, \forall \lambda \in \mathbb{F}$$

Last updated: April 1, 2019