Math 250, Fall 2019

Groups, Rings and Fields

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Contents

1 Groups

1.1 Groups

There are two definitions to define a group.

Definition 1.1 (Concrete definition)
A group is a symmetries of something 1:1 map preserving "structure".

Example 1.2
Consider the rotation of a rectangle, we have a group of order 4.

Example 1.3
Consider the rotation of a icosahedron, we are able to obtain a group of order 60.

Example 1.4
Let V be a n-dimensional over $\mathbb{R}$. The general linear group $GL_n(\mathbb{R})$, all matrices with $\det \neq 0$ from a group.

Definition 1.5 (Abstract Definition, Cayley)
A group is a set G with a binary operation $a+b$ or $a \times b$ or $a \circ b$ or ab (notation sucks) such that

1. Identity element 0, 1, or e, i,e $a1 = 1a = a$.
2. Each element has inverse a^{-1}, i,e $aa^{-1} = a^{-1}a = 1$.
3. Associative $(ab)c = a(bc)$ for all $a, b, c \in G$

Definition 1.6
A group G acts on S means given operation

$$G \times S \to S$$

such that $1s = s$ and $a(bs) = ab(s)$.

Example 1.7
Let G be the icosahedron group and let S be the icosahedron.

Question 1.8

How does G acts on G?

Definition 1.9

There are 8 different types of actions

1. $g(s) = s$, left action (trivial)
2. $g(s) = gs$
3. $g(s) = sg^{-1}$
4. $g(s) = gsg^{-1}$, adjoint action

Note that all of these are left actions of the group, then similarly there are also 4 right group actions. $S \times G \to S$

1. $sg = s$
2. $sg = sg$
3. $sg = g^{-1}s$
4. $sg = g^{-1}sg$

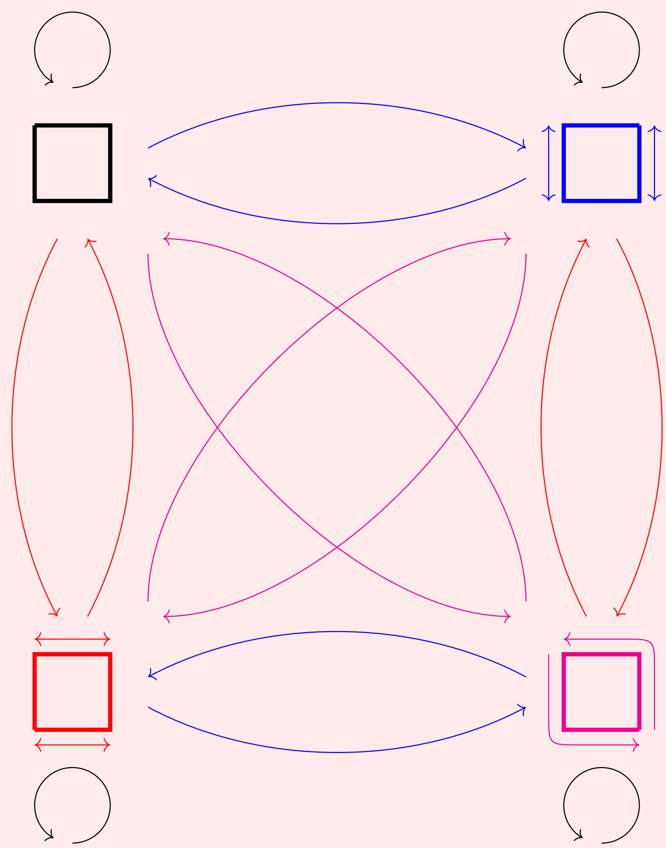
Remark 1.10

$g(s) = gs, g(s) = sg^{-1}, sg = sg, sg = g^{-1}s$ does not preserve group operation of S.

We let G act on $S(= G)$ by $g(s) = gs$. This means G into subset of all permutations of $S(= G)$. Now we want to add extra "structure" to S so G is exactly symmetries of S with this structure.
Extra structure is **right** action of G on S. We now have 3 copies of G

1. Set $S(= G)$.
2. G acting on **right** on $S \leftarrow$ part of structure
3. G acting on **left** on $S \leftarrow$ symmetry group

Example 1.11 (Cayley Graph of 4 elements)



We get colored (directed) graph arrow gives **right** action of G, which is not the same as the **left** action.

Remark 1.12

Goals of group theory

1. Classify all groups
2. Given a group G, find all ways G acts on something.

Example 1.13

Linear representation = actions of G over vector space.
Permutation = actions of G over on a set.

Definition 1.14

A homomorphism $f : G \to H$ map preserving group structure. i.e. $f(gh) = f(g)f(h)$.
A isomorphism is a homomorphism that is a bijection.
The kernel of f is the set of elements such that it maps to the trivial element of H

Example 1.15

Consider the function

$$\exp : \langle \mathbb{R}, + \rangle \to \langle \mathbb{R}^*, \cdot \rangle$$

exp is a isomorphism from $\mathbb{R}$ to $R_{>0}$

Example 1.16

Consider the function

$$\exp : \langle \mathbb{C}, + \rangle \to \langle \mathbb{C}^*, \cdot \rangle$$

kernel = elements $2\pi in, n \in \mathbb{Z}$.

Example 1.17 (Number Theory)

Consider $\mathbb{Z}/4\mathbb{Z}$ integers mod 4. and $(\mathbb{Z}/5\mathbb{Z})^*$ nonzero integers mod 5 under multiplication.

Example 1.18

Consider the function :

$$\det : GL_n(R) = \mathbb{R}^*$$

is a homomorphism.
kernel $= SL_n(\mathbb{R})$ = special linear group.

Theorem 1.19 (Lagrange's Theorem)

If H is a subgroup of G, order of H divides order of G. (G is finite)

Lemma 1.20

2 cosets either are the same or disjoint.

Proof. If $aH \cap bH = \varnothing$, then the proof is done.
Now we suppose $aH \cap bH \neq \varnothing$, then we know that $ah_1 = bh_2$ for some element in aH and bH. We compute $ah_1 = bh_2 \implies h_1 = a^{-1}bh_2 \implies a^{-1}b = h_1h_2^{-1} \implies a^{-1}b \in H$. By proposition 1 we know that $aH = bH$.
We can use a similar argument to show that this works for the right cosets as well. This is left as an exercises to the reader. ∎

Lemma 1.21

Any cosets have the same size.

Proof. We can simply prove that $\phi : h \mapsto bh$ is bijective, therefore $|H| = |bH|$.
This proof is trivial and is left as an exercise to the reader. (Hint: prove that ϕ is injective and say it's surjective by construction) ∎

Proof. Suppose G acts on S. Pick $s \in S$, put H =set of elements fixing s such that $hs = s$, then H is a subgroup of G.
Given a subgroup H of G, we can find set S acted on by G, $s \in S$. H = things fixed in S.
Given $g, h(H \subseteq G)$. S = left cosets of H.
we get action of G on set of cosets by putting $g(aH) = (ga)H$. (well-defined left as an exercise)
Therefore $|G| = |H| \times$ number of cosets. Therefore the order of H divides the order of H. ∎

Theorem 1.22

If $g \in G$, then the order of g divides divides order of G.

Corollary 1.23

If G is prime order, it is cyclic

Proof. Pick any element $g \neq 1$. Order divided p, so p is primes. so G = powers of g ∎

Example 1.24

List of all groups

1. Order 1 : Trivial group
2. Order 2 : 1 group $\mathbb{Z}/2\mathbb{Z}$, $0, 1$.
3. Order prime p : Integer $\mod p$.
4. Order 4 : Cyclic group $\mathbb{Z}/4\mathbb{Z}$ and symmetry of a rectangle. These are not isomorphic as the symmetry group of rectangle does not have a element of order 4.
5. Classify all groups with $g^2 = 1$ for all g. Group is abelian. $gh = hg$. This follows because $ghgh = (gh)^2 = 1 = h^2g^2 = hhgg \implies hg = gh$. Since G is abelian, we can write group operation. Notice that G is a vector space over field of order 2, namely $\mathbb{Z}/2\mathbb{Z}$. so h has a basis, and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$($n$-dimensional vector space) to some n. So only 1 other group of order 4.

Definition 1.25

Suppose G, H are groups, then the product(sum) of the group is defined as follows

$$G \times H = \text{set of pairs}(g, h)$$

and the operation is defined as

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

Example 1.26

symmetry of rectangle is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Example 1.27

$\mathbb{C}^* = S \times \mathbb{R}_{>0}$, where S is the circle group. Notice that this is the polar decomposition of complex numbers.

Definition 1.28

The product(sum) of groups are elements $(g_1, g_2, \ldots)$ such that all but finite number of g_i are trivial.

Example 1.29

$\mathbb{Q}^* =$ infinite sum of groups $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \ldots$. This follows by fundamental theorem of arithmetic.