Math 143, Spring 2021

Contents

1 Week 1

1.1 Logistics

- Asynchronous lectures are available Monday Wednesday night on Bcourse under "Media Gallery"
- The course website can be found at https://math.berkeley.edu/~braggdan/143
- Zoom meetings / Office Hour from 9-10 MTWth to answer any questions.

1.1.1 Grading

- Weekly Homework (15%)
- 2 Midterms (25 % each)
- Final (35 %)

1.1.2 Textbook Used

- "Elementary Algebraic Geometry" by Hulek. (Official Book)
- "Algebraic Curves: An Introduction to Algebraic Geometry" by Fulton. (Supplement material)

1.2 What is Algebraic Geometry?

Let k denote a field.

Recall that Linear Algebra is the study of solutions to systems of equations, for example

$$\begin{aligned} f_1 &= a_1x_1 + a_2x_2 + \cdots + a_xx_n \\ f_2 &= b_1x_1 + b_2x_2 + \cdots + b_nx_n \\ &\vdots \end{aligned}$$

Then algebraic geometry can be described as the study of solutions to systems of polynomial equations, for example

$$\begin{aligned} f_1(x_1, x_2, \dots x_n) &= 0 \\ f_2(x_1, x_2, \dots x_n) &= 0 \\ &\vdots \\ f_m(x_1, x_2, \dots x_n) &= 0 \end{aligned}$$

where $f_i \in k[x_1, x_2, \dots x_n]$ (polynomials in the variables $x_1, x_2, \dots x_n$ with coefficients in k).

This gets much much harder than the linear case because

1. Higher dimension (more variable) gets more complex really quickly
2. k might not be a nice field (e.g. let $k = \mathbb{Q}$ and find solutions of $x^n + y^n = z^n$)

1.3 Notation

Definition 1.1

Let $\mathbb{A}_k^n = \mathbb{A}^n = \{(a_1, a_2, \ldots a_n) \mid a_i \in k\} = k^n$

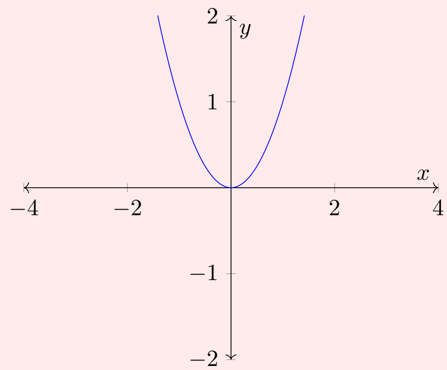
Definition 1.2

Given $f\ ink[x_1, x_2, \ldots x_n]$, we denote the **vanishing set of** f $V(f) \subset \mathbb{A}^n$ as

$$V(f) = \{a_1, a_2, \ldots a_n \in \mathbb{A}^n \mid f(a_1, a_2, \ldots a_n) = 0\}$$

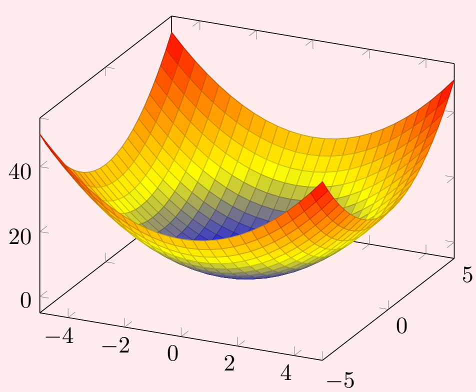
Example 1.3

Take $k = \mathbb{R}$, and $f(x, y) = y - x^2$, then can vanish set $V(f)$ can be describe by the following parabola graph



Example 1.4

Now consider $f = z - x^2 - y^2$, then $V(f)$ in $\mathbb{A}^3$ looks like this



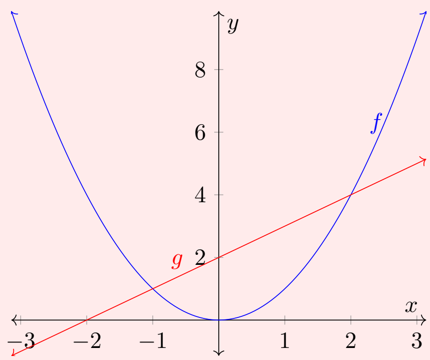
Definition 1.5

Given $f_1, f_2, \dots f_m \in k[x_1, x_2, \dots x_n]$, we let

$$\begin{aligned} V(f_1, f_2, \dots f_n) &= V(f_1) \cap V(f_2) \cdots V(f_n) \\ &= \{(a_1, a_2, \dots a_n) \in \mathbb{A}^n \mid f_1(a_1, a_2, \dots a_n) = 0 \text{ and } \cdots \text{ and } f_m(a_1, a_2, \dots a_n) = 0\} \end{aligned}$$

Example 1.6

Now consider $f = y - x^2$ and $g = y - x - 2$, then $V(f)$ and $V(g)$ are the following:



and the intersection of $V(f)$ and $V(g)$ is preciously $V(f, g) = \{(-1, 1), (2, 4)\}$

Lemma 1.7

Let $S \subset k[x_1, x_2, \dots x_n]$,

1. If $f \in S$ and $h \in k[x_1, x_2, \dots x_n]$, then if $(a_1, a_2, \dots a_n) \in V(f)$, then $(a_1, a_2, \dots a_n) \in V(hf)$.
2. If $f, g \in S$, then if $(a_1, a_2, \dots a_n) \in V(f) \cap V(g)$, then $(a_1, a_2, \dots a_n) \in V(f + g)$

Proof. Since $f(a_1, a_2, \dots a_n) = 0$, then

$$(hf)(a_1, a_2, \dots a_n) = h(a_1, a_2, \dots a_n) f(a_1, a_2, \dots a_n) = h(a_1, a_2, \dots a_n) \cdot 0 = 0.$$

Since $f(a_1, a_2, \dots a_n) = 0$ and $g(a_1, a_2, \dots a_n) = 0$, then

$$(f + g)(a_1, a_2, \dots a_n) = f(a_1, a_2, \dots a_n) + g(a_1, a_2, \dots a_n) = 0 + 0 = 0$$

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Definition 1.8

If $S = k[x_1, x_2, \dots x_n]$, we write I_S as the **ideal generated by** S.

$$I_S = \{h_1 f_1 + h2 f_2 + \cdots + h_m f_m \mid f_1, f_2, \dots f_m \in S, h_1, h_2, \dots h_m \in k[x_1, x_2, \dots x_n]\}$$

Corollary 1.9

If $S \subset k[x_1, x_2, \dots x_n]$, then $V(S) = V(I_S)$, then vanishing set of S is equal to the vanishing set of I_S, the ideal generated by S.

Definition 1.10

Let (f) denote the ideal generated by f, and let $(f_1, f_2, \dots f_n)$ denote the ideal generated by $f_1, f_2, \dots f_n$, i.e.

$$(f) = I_{\{f\}}, (f_1, f_2, \dots f_n) = I_{\{f_1, f_2, \dots f_n\}}.$$

Corollary 1.11

$$V(f) = V((f))$$

Definition 1.12

An **algebraic subset** of $\mathbb{A}^n$ is a subset of the form $V(I)$ for some ideal $I \subset k[x_1, x_2, \dots x_n]$.

Example 1.13

The *zero ideal* $0 = (0) = \{0\} \subset k[x_1, x_2, \dots x_n]$. Thne

$$\begin{aligned} V(0) &= \{(a_1, a_2, \dots a_n) \mid (0)(a_1, a_2, \dots a_n) = 0\} = \mathbb{A}^n \\ V(1) &= \{(a_1, a_2, \dots a_n) \mid (1)(a_1, a_2, \dots a_n) = 0\} = \varnothing \end{aligned}$$

where (0) and (1) are polynomials evaluated at $(a_1, a_2, \dots a_n)$.

Example 1.14

For any point $(a_1, a_2, \dots a_n) \in \mathbb{A}^n$, the point itself is an algebraic subset. This can be done by construction. Consider the vanishing set of the following polynomial:

$$\begin{aligned} V(x_1 - a_1, x_2 - a_2, \dots, x_n - a_n) &= \{(b_1, b_2, \dots b_n) \in \mathbb{A}^n \mid b_1 - a_1 = 0, \dots, b_n - a_n = 0\} \\ &= \{(a_1, a_2, \dots a_n)\} \end{aligned}$$

Proposition 1.15

Recall

If I and J are ideals of $k[x_1, x_2, \dots x_n]$, then

$$I + J := \{f + g \mid f \in I, g \in J\} \text{ and } I \cap J := \{f \mid f \in I \text{ and } f \in J\}$$

and both $I + J$ and $I \cap J$ are ideals.

Proof. It's easy to see that $I + J$ and $I \cap J$ are subrings of $k[x_1, x_2, \dots x_n]$. Now take $f \in I + J$, and let $f_I \in I$ and $f_J \in J$ such that $f_I + f_J = f$ and $h \in k[x_1, x_2, \dots x_n]$ we have

$$f \cdot h = (f_I + f_J) \cdot h = f_I \cdot h + f_J \cdot h \in I + J$$

Now take $g \in I \cap J$, then

$$g \cdot h \in I, g \cdot h \in J \implies g \cdot h \in I \cap J$$

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Proposition 1.16 (Properties of Ideals)

Let $I, J \in k[x_1, x_2, \dots x_n]$ be ideals.

1. $I \subset J \implies V(J) \subset V(I)$
2. $V(I \cap J) = V(I) \cup V(J)$
3. $V(I + J) = V(I) \cap V(J)$

Proof.

1. Say $p = (a_1, a_2, \dots a_n) \in V(J)$. Then for all $f \in J$ we have $f(p) = 0$. Take any $g \in I \subset J$ we have $g(P) = 0$, therefore $V(J) \subset V(I)$.
2. ($\subset$) Say $P \notin V(I) \cup V(J)$, then there exists $f \in I, g \in J$ such that $f(p) \neq 0, g(p) \neq 0$. We then have $fg(p) = f(p)g(p) \neq 0$. Since $fg \in I \cap J$, therefore $p \notin V(I \cap J)$, taking the contrapositive we have $p \in V(I \cap J) \implies p \in V(I) \cup V(J) \implies V(I \cap J) \subset V(I) \cup V(J)$
($\supset$) Take $p \in V(I) \cup V(J)$, then without the loss of generality say $p \in V(I)$, then for all $f \in I, f(p) = 0$. Then for all $q \in I \cap J, f(q) = 0 \implies f \in V(I \cap J)$.
3. ($\subset$) ($\supset$)

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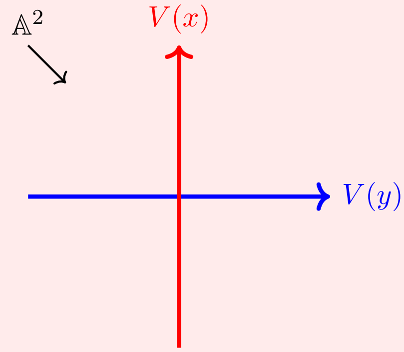
Example 1.17

Let $I = (x)$ and $J = (y)$, and $I, J \in k[x, y]$. Then we can visualize then on $\mathbb{A}^2$ in the following figure



where the red line denotes $V(x)$ and the blue line denotes $V(y)$.

We have $I \cap J = (x) \cap (y) = (xy)$ (all polynomial divisible by xy). Then

$$V(xy) = \{(x_0, y_0) \mid x_0 y_0 = 0\}$$

by inspection we can see that indeed $V(xy) = V(I) \cup V(J)$.

We have $I + J = (x) + (y)$, all polynomials of zero constant terms. Then it's clear to see that

$$V((x) + (y)) = \{(0, 0)\}$$

Since there are no constant term in any of the polynomials in $(x) + (y)$. We can also see this is preciously $V(I) \cap V(J)$ too.