

Solution

1.

$$\text{From } \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \Rightarrow \begin{cases} 2\gamma_i^2 = -2 \Rightarrow \gamma_i^2 = -1 \\ 2\gamma_0^2 = 2 \Rightarrow \gamma_0^2 = 1 \end{cases}$$

$$\gamma^\mu = g^{\mu\nu} \gamma_\nu \Rightarrow \gamma^0 = \gamma_0, \gamma^i = -\gamma_i$$

$$\Rightarrow (\gamma^0)^2 = (\gamma_0)^2 = 1,$$

$$(\gamma^i)^2 = (-\gamma_i)^2 = (\gamma_i)^2 = -1$$

Done 1) & 2) ✓

$$(\gamma^5)^2 = (i \gamma^0 \gamma^1 \gamma^2 \gamma^3) (i \gamma^0 \gamma^1 \gamma^2 \gamma^3)$$

$$\text{From } \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} \Rightarrow \begin{cases} \gamma_0 \gamma_i = -\gamma_i \gamma_0 \Rightarrow \gamma^0 \gamma^i = -\gamma^i \gamma^0 \\ \gamma_i \gamma_j = -\gamma_j \gamma_i \text{ (for } i \neq j), \Rightarrow \gamma^i \gamma^j = -\gamma^j \gamma^i \end{cases}$$

$$\Rightarrow ((\gamma^0 \gamma^1 \gamma^2 \gamma^3) ((\gamma^0 \gamma^1 \gamma^2 \gamma^3)) \quad \text{(for } i \neq j)$$

$$= (-1) \underbrace{\gamma^0 \gamma^1 \gamma^2 \gamma^3}_{\gamma^0 \gamma^1 \gamma^2 \gamma^3} \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$= (-1) \cdot (-1)^3 (\gamma^0)^2 \underbrace{\gamma^1 \gamma^2 \gamma^3 \gamma^1 \gamma^2 \gamma^3}_{\gamma^1 \gamma^2 \gamma^3 \gamma^1 \gamma^2 \gamma^3}$$

$$= (-1)^2 (\gamma^1)^2 \underbrace{\gamma^2 \gamma^3 \gamma^2 \gamma^3}_{\gamma^2 \gamma^3 \gamma^2 \gamma^3}$$

$$= (-1) \cdot (-1)^1 (\gamma^2)^2 \cdot \gamma^3 \gamma^3$$

$$= -1$$

Done 3) ✓

$$(\gamma^5)^+ = (i \gamma^0 \gamma^1 \gamma^2 \gamma^3)^+ = (-i) \cdot (\gamma^3)^+ (\gamma^2)^+ (\gamma^1)^+ (\gamma^0)^+$$

$$\text{From } \gamma_\mu^+ = \gamma_0 \gamma_\mu \gamma_0 \Rightarrow \begin{cases} \gamma_0^+ = (\gamma_0)^3 = \gamma_0 \Rightarrow (\gamma^0)^+ = \gamma^0 \\ \gamma_i^+ = \gamma_0 \gamma_i \gamma_0 = -\gamma_0 \gamma_0 \gamma_i = -\gamma_i \Rightarrow (\gamma^i)^+ = \gamma^i \end{cases}$$

$$\Rightarrow (\gamma^5)^+ = (-i) (-1)^3 \cdot \gamma^3 \gamma^2 \gamma^1 \gamma^0 = i \underbrace{\gamma^3 \gamma^2 \gamma^1 \gamma^0}_{\gamma^3 \gamma^2 \gamma^1 \gamma^0}$$

$$= i \underbrace{\gamma^2 \gamma^1 \gamma^0 \gamma^3}_{\gamma^2 \gamma^1 \gamma^0 \gamma^3} (-1)^3 = -i (-1)^5 \underbrace{\gamma^1 \gamma^0 \gamma^2 \gamma^3}_{\gamma^1 \gamma^0 \gamma^2 \gamma^3} = -i (-1)^5 \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$= i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5$$

Done 4)

$$\gamma_5 \gamma_\mu = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_\mu =$$

$$\Rightarrow \gamma_5 \gamma_\mu = -\gamma_\mu \gamma_5$$

Done 5)

(checked)

$$\begin{cases} (\text{for } \mu=0) : i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_0 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -\gamma_0 \gamma_5 \\ (\text{for } \mu=1) : i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_1 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -i \gamma_1 \gamma_0 \gamma_2 \gamma_3 = -\gamma_1 \gamma_5 \\ (\text{for } \mu=2) : i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_2 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -i \gamma_2 \gamma_0 \gamma_1 \gamma_3 = -\gamma_2 \gamma_5 \\ (\text{for } \mu=3) : i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_3 = -i \gamma_3 \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -\gamma_3 \gamma_5 \end{cases}$$

$$\gamma_\lambda \gamma^\lambda = \gamma_0 \gamma^0 + \sum_{i=1}^3 \gamma_i \gamma^i = (\gamma_0)^2 - (\gamma_1)^2 - (\gamma_2)^2 - (\gamma_3)^2 = 1 + 1 + 1 + 1 = 4$$

(or, just from  $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} = \gamma_\mu \gamma^\mu + \gamma_\nu \gamma^\nu = 2 \times g_{\mu}^\mu = 8 \Rightarrow \gamma_\lambda \gamma^\lambda = 4$ )  
 Done 6)

From  $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$  and  $\gamma^\mu = g^{\mu\nu} \gamma_\nu \Rightarrow \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$

$$\Rightarrow \gamma_\lambda \gamma^\alpha \gamma^\lambda = \gamma_\lambda (2g^{\alpha\lambda} - \gamma^\lambda \gamma^\alpha) = 2\gamma^\alpha - 4\gamma^\alpha = -2\gamma^\alpha$$

Done 7)

$$\begin{aligned} \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\lambda &= \gamma_\lambda \gamma^\alpha (2g^{\beta\lambda} - \gamma^\lambda \gamma^\beta) = 2\gamma^\beta \gamma^\alpha - \gamma_\lambda \gamma^\alpha \gamma^\lambda \gamma^\beta \\ &= 2\gamma^\beta \gamma^\alpha + 2\gamma^\alpha \gamma^\beta \\ &= 4g^{\alpha\beta} \end{aligned}$$

Done 8)

$$\begin{aligned} \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\sigma \gamma^\lambda &= \gamma_\lambda \gamma^\alpha \gamma^\beta (2g^{\sigma\lambda} - \gamma^\lambda \gamma^\sigma) = 2\gamma^\sigma \gamma^\alpha \gamma^\beta - \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\lambda \gamma^\sigma \\ &= 2\gamma^\sigma \gamma^\alpha \gamma^\beta - 4g^{\alpha\beta} \gamma^\sigma \\ &= 2\gamma^\sigma (2g^{\alpha\beta} - \gamma^\beta \gamma^\alpha) - 4g^{\alpha\beta} \gamma^\sigma \\ &= -2\gamma^\sigma \gamma^\beta \gamma^\alpha \end{aligned}$$

Done 9)

$$\begin{aligned} \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\sigma \gamma^\tau \gamma^\lambda &= \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\sigma (2g^{\tau\lambda} - \gamma^\lambda \gamma^\tau) = 2\gamma^\tau \gamma^\alpha \gamma^\beta \gamma^\sigma + 2\gamma^\sigma \gamma^\alpha \gamma^\beta \gamma^\tau \\ &= 2(\gamma^\tau \gamma^\alpha \gamma^\beta \gamma^\sigma + \gamma^\sigma \gamma^\alpha \gamma^\beta \gamma^\tau) \end{aligned}$$

Done 10)

$$\begin{aligned} \text{II}) \quad AA &= A^\alpha A^\beta \gamma_\alpha \gamma_\beta = \frac{1}{2} (A^\alpha A^\beta \gamma_\alpha \gamma_\beta + A^\beta A^\alpha \gamma_\alpha \gamma_\beta) \\ &= \frac{1}{2} A^\alpha A^\beta (\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha) = \frac{1}{2} A^\alpha A^\beta 2g_{\alpha\beta} = A^\alpha A_\alpha = A \cdot A \quad \checkmark \end{aligned}$$

$$AB + BA = A^\alpha B^\beta (\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha) = A^\alpha B^\beta 2g_{\alpha\beta} = 2A \cdot B. \quad \checkmark$$

$$\gamma_\lambda A \gamma^\lambda = A_\alpha \gamma_\lambda \gamma^\alpha \gamma^\lambda = -2A_\alpha \gamma^\alpha = -2A \quad \checkmark$$

$$\gamma_\lambda AB \gamma^\lambda = A_\alpha B_\beta \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\lambda = 4A_\alpha B_\beta g^{\alpha\beta} = 4A \cdot B \quad \checkmark$$

$$\gamma_\lambda AB \not C \gamma^\lambda = A_\alpha B_\beta C_\gamma \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\sigma \gamma^\tau \gamma^\lambda = -2A_\alpha B_\beta C_\gamma \gamma^\sigma \gamma^\tau \gamma^\lambda = -2C \not B A \quad \checkmark$$

$$\begin{aligned} \gamma_\lambda A \not C \not D \gamma^\lambda &= A_\alpha B_\beta C_\gamma D_\delta \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\tau \gamma^\lambda = A_\alpha B_\beta C_\gamma D_\delta 2(\gamma^\tau \gamma^\alpha \gamma^\beta \gamma^\gamma + \gamma^\tau \gamma^\beta \gamma^\alpha \gamma^\gamma) \\ &= 2(D \not C \not A + \not C \not B D) \quad \checkmark \end{aligned}$$

(checked)

2.

$$1) \text{ use } (\gamma_5)^2 = 1 \text{ and } \gamma_5 \gamma^\mu = -\gamma^\mu \gamma_5$$

$$\text{and } \text{Tr}(AB) = \text{Tr}(BA).$$

$$(\text{proof: } \text{Tr}(AB) = \sum_{i=1}^N \sum_{k=1}^M A_{ik} B_{ki} \text{ (for } N \times M \text{ matrix } A \text{ and } M \times N \text{ matrix } B)$$

$$\text{if all } A_{ik} B_{ki} = B_{ki} A_{ik}, \text{ then}$$

$$\text{Tr}(AB) = \sum_{i=1}^N \sum_{k=1}^M B_{ki} A_{ik} = \text{Tr}(BA) )$$

$$\Rightarrow \text{Tr}(\gamma^a \gamma^b \dots \gamma^m \gamma^n) = \text{Tr}(\gamma^a \gamma^b \dots \gamma^m \gamma^n \gamma_5^2) = \text{Tr}(\underbrace{\gamma_5 \gamma^a \gamma^b \dots \gamma^m \gamma^n}_{\text{Do permutation}} \gamma_5)$$

Do permutation to move the  $\gamma_5$  in front  $\gamma^2$  to the position after  $\gamma^n$ . Since  $\gamma_5 \gamma^\mu = -\gamma^\mu \gamma_5$  and there're odd number of permutations due to there're odd number of  $\gamma$ 's, then

$$\begin{aligned} \text{Tr}(\gamma^a \gamma^b \dots \gamma^m \gamma^n) &= \text{Tr}(\gamma_5 \gamma^a \gamma^b \dots \gamma^m \gamma^n \gamma_5) = (-1)^n \text{Tr}(\gamma^a \gamma^b \dots \gamma^m \gamma^n \gamma_5^2) \\ &= (-1)^n \text{Tr}(\gamma^a \gamma^b \dots \gamma^m \gamma^n), \text{ where } n \text{ is an odd number,} \end{aligned}$$

$$\Rightarrow \text{Tr}(\gamma^a \gamma^b \dots \gamma^m \gamma^n) = -\text{Tr}(\gamma^a \gamma^b \dots \gamma^m \gamma^n)$$

$$\Rightarrow \text{Tr}(\gamma^a \gamma^b \dots \gamma^m \gamma^n) = 0$$

$$2) \text{ Since } \text{Tr}(A) = \text{Tr}(A^\top)$$

$$(\text{Proof: } \text{Tr}(A) = \sum_{i=1}^N A_{ii}, \text{Tr}(A^\top) = \sum_{i=1}^N (A^\top)_{ii} = \sum_{i=1}^N A_{ii}, \text{ for } N \times N \text{ matrix } A \\ \text{then } \text{Tr}(A) = \text{Tr}(A^\top).)$$

$$\Rightarrow \text{Tr}(\gamma^a \gamma^b \dots \gamma^m \gamma^n) = \text{Tr}[(\gamma^a \gamma^b \dots \gamma^m \gamma^n)^\top]$$

$$\text{Since } A^\dagger = (A^\top)^*, \text{ then } A^\top = (A^\dagger)^\dagger.$$

$$\text{Since } \text{Tr}(A^*) = \sum_{i=1}^N (A^*)_{ii} = \left( \sum_{i=1}^N A_{ii} \right)^* = (\text{Tr}(A))^*$$

$$\text{then } \text{Tr}[(\gamma^a \gamma^b \dots \gamma^m \gamma^n)^\top] = \left\{ \text{Tr}[(\gamma^a \gamma^b \dots \gamma^m \gamma^n)^+] \right\}^* \\ = \left\{ \text{Tr}(\gamma^{+a} \gamma^{+b} \dots \gamma^{+m} \gamma^{+n}) \right\}^*$$

$$\text{use } \gamma^{+n} = \gamma^n \gamma^n \gamma^0$$

$$\Rightarrow \left\{ \text{Tr}(\gamma^{+a} \gamma^{+b} \dots \gamma^{+m} \gamma^{+n}) \right\}^* = \left\{ \text{Tr}(\gamma^0 \gamma^{+a} \gamma^{+b} \gamma^{+c} \dots \gamma^{+n} \gamma^{+m} \gamma^{+l} \gamma^{+d} \gamma^{+e}) \right\}^*$$

(checked)

Since  $(\gamma^a)^2 = 1$ , then

$$\left\{ \text{Tr} (\gamma^a \gamma^v \gamma^u \gamma^s \gamma^t \gamma^r \dots \gamma^b \gamma^c \gamma^d \gamma^e \gamma^f) \right\}^* = \left\{ \text{Tr} (\gamma^v \gamma^u \gamma^s \gamma^t \gamma^r \gamma^a) \right\}^*$$

use  $\text{Tr}(AB) = \text{Tr}(BA)$  to put the  $\gamma^a$  in the front to the end position, then

$$\begin{aligned} \left\{ \text{Tr} (\gamma^a \gamma^v \gamma^u \gamma^s \gamma^t \gamma^r \gamma^f) \right\}^* &= \left\{ \text{Tr} (\gamma^v \gamma^u \gamma^s \gamma^t \gamma^r \gamma^f \gamma^a) \right\}^* \\ &= \left\{ \text{Tr} (\gamma^v \gamma^u \gamma^s \gamma^t \gamma^r \gamma^f) \right\}^* (\gamma^a \gamma^a = 1 \text{ is used}) \end{aligned}$$

For the trace of even number of  $\gamma$ -matrices,

$$\text{Tr} (\gamma^{M_1} \gamma^{M_2} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}})$$

$$\text{use } \gamma^{M_1} \gamma^{M_2} + \gamma^{M_2} \gamma^{M_1} = 2g^{M_1 M_2}$$

$$\begin{aligned} \Rightarrow \text{Tr} (\gamma^{M_1} \gamma^{M_2} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}}) &= 2g^{M_1 M_2} \text{Tr} (\gamma^{M_3} \gamma^{M_4} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}}) \\ &\quad - \text{Tr} (\gamma^{M_2} \gamma^{M_1} \gamma^{M_3} \gamma^{M_4} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}}) \end{aligned}$$

Do the same thing to move  $\gamma^{M_1}$  to the end of the  $\gamma$ -matrices,

$$\text{Tr} (\gamma^{M_1} \gamma^{M_2} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}})$$

$$\begin{aligned} &= 2g^{M_1 M_2} \text{Tr} (\gamma^{M_3} \gamma^{M_4} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}}) - 2g^{M_1 M_3} \text{Tr} (\gamma^{M_2} \gamma^{M_4} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}}) \\ &\quad + \text{Tr} (\gamma^{M_2} \gamma^{M_3} \gamma^{M_1} \gamma^{M_4} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}}) \end{aligned}$$

$$\begin{aligned} &= 2g^{M_1 M_2} \text{Tr} (\gamma^{M_3} \gamma^{M_4} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}}) - 2g^{M_1 M_3} \text{Tr} (\gamma^{M_2} \gamma^{M_4} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}}) \\ &\quad + 2g^{M_1 M_4} \text{Tr} (\gamma^{M_2} \gamma^{M_3} \gamma^{M_5} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}}) \end{aligned}$$

$$- 2g^{M_1 M_5} \text{Tr} (\gamma^{M_2} \gamma^{M_3} \gamma^{M_4} \gamma^{M_6} \dots \gamma^{M_{2N-1}} \gamma^{M_{2N}})$$

+ ...

$$- 2g^{M_1 M_{2N-1}} \text{Tr} (\gamma^{M_2} \gamma^{M_3} \dots \gamma^{M_{2N-2}} \gamma^{M_{2N}})$$

$$+ \text{Tr} (\gamma^{M_2} \gamma^{M_3} \dots \gamma^{M_{2N-2}} \gamma^{M_{2N-1}} \gamma^{M_1} \gamma^{M_{2N}})$$

Note that the last term is

$$\begin{aligned} \text{Tr} (\gamma^{M_2} \gamma^{M_3} \dots \gamma^{M_{2N-2}} \gamma^{M_{2N-1}} \gamma^{M_1} \gamma^{M_{2N}}) &= 2g^{M_1 M_{2N}} \text{Tr} (\gamma^{M_2} \gamma^{M_3} \dots \gamma^{M_{2N-2}} \gamma^{M_{2N-1}}) \\ &- \text{Tr} (\gamma^{M_2} \gamma^{M_3} \dots \gamma^{M_{2N-2}} \gamma^{M_{2N-1}} \gamma^{M_{2N}} \gamma^{M_1}) \end{aligned}$$

(checked)

use  $\text{Tr}(AB) = \text{Tr}(BA)$  to put  $\gamma^{\mu_1}$  at the end position in the second term to the front position, then

$$\begin{aligned}
& \text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2} \dots \gamma^{\mu_{2n-2}}\gamma^{\mu_{2n-1}}\gamma^{\mu_1}\gamma^{\mu_{2n}}) \\
&= 2g^{\mu_1\mu_{2n}} \text{Tr}(\gamma^{\mu_2}\gamma^{\mu_3} \dots \gamma^{\mu_{2n-2}}\gamma^{\mu_{2n-1}}) - \text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2} \dots \gamma^{\mu_{2n-1}}\gamma^{\mu_{2n}}) \\
\Rightarrow & \text{Tr}(\gamma^{\mu_1}\gamma^{\mu_2} \dots \gamma^{\mu_{2n-1}}\gamma^{\mu_{2n}}) \\
= & g^{\mu_1\mu_2} \text{Tr}(\gamma^{\mu_3}\gamma^{\mu_4} \dots \gamma^{\mu_{2n-1}}\gamma^{\mu_{2n}}) \\
&- g^{\mu_1\mu_3} \text{Tr}(\gamma^{\mu_2}\gamma^{\mu_4} \dots \gamma^{\mu_{2n-1}}\gamma^{\mu_{2n}}) \\
&+ g^{\mu_1\mu_4} \text{Tr}(\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_5} \dots \gamma^{\mu_{2n-1}}\gamma^{\mu_{2n}}) \\
&- g^{\mu_1\mu_5} \text{Tr}(\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_6} \dots \gamma^{\mu_{2n-1}}\gamma^{\mu_{2n}}) \\
&+ \dots \\
&- 2g^{\mu_1\mu_{2n-1}} \text{Tr}(\gamma^{\mu_2}\gamma^{\mu_3} \dots \gamma^{\mu_{2n-2}}\gamma^{\mu_{2n}}) \\
&+ g^{\mu_1\mu_{2n}} \text{Tr}(\gamma^{\mu_2}\gamma^{\mu_3} \dots \gamma^{\mu_{2n-2}}\gamma^{\mu_{2n-1}})
\end{aligned}$$

that is ; the trace of  $(2N)$   $\gamma$ -matrices can be written as the sum of  $(2N-1)$  traces of  $(2N-2)$   $\gamma$ -matrices.

We can repeat the process so that the trace of  $(2N)$   $\gamma$ -matrices can be written as the sum of  $\text{Tr}(\gamma^{\mu_i}\gamma^{\mu_j})$ .

$$\begin{aligned}
\text{Since } \text{Tr}(\gamma^{\mu_i}\gamma^{\mu_j}) &= \text{Tr}(2g^{\mu_i\mu_j} - \gamma^{\mu_i}\gamma^{\mu_j}) \\
&= 2g^{\mu_i\mu_j} \text{Tr}(1) - \text{Tr}(\gamma^{\mu_i}\gamma^{\mu_j}) \\
&= 8g^{\mu_i\mu_j} - \text{Tr}(\gamma^{\mu_i}\gamma^{\mu_j})
\end{aligned}$$

then  $\text{Tr}(\gamma^{\mu_i}\gamma^{\mu_j}) = 4g^{\mu_i\mu_j}$ , which is a real number. Therefore, the trace of a product of even number of  $\gamma$ -matrices is real, that is

$$\begin{aligned}
[\text{Tr}(\gamma^\nu\gamma^\mu \dots \gamma^\delta\gamma^\alpha)]^* &= \text{Tr}(\gamma^\nu\gamma^\mu \dots \gamma^\delta\gamma^\alpha) \\
\Rightarrow \text{Tr}(\gamma^\alpha\gamma^\beta \dots \gamma^\nu\gamma^\mu) &= \text{Tr}(\gamma^\nu\gamma^\mu \dots \gamma^\beta\gamma^\alpha) \quad \checkmark
\end{aligned}$$

(checked)

3) As already shown in 2)

$$\begin{aligned}\text{Tr}(\gamma^\alpha \gamma^\beta) &= \text{Tr}(2g^{\alpha\beta} - \gamma^\beta \gamma^\alpha) \\ &= 2g^{\alpha\beta} \text{Tr}(1) - \text{Tr}(\gamma^\beta \gamma^\alpha) = 2g^{\alpha\beta} \times 4 - \text{Tr}(\gamma^\alpha \gamma^\beta) \\ \Rightarrow \text{Tr}(\gamma^\alpha \gamma^\beta) &= 4g^{\alpha\beta}\end{aligned}$$

4) As already shown in 2)

$$\begin{aligned}\text{Tr}(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta) &= g^{\alpha\beta} \text{Tr}(\gamma^\gamma \gamma^\delta) - g^{\alpha\gamma} \text{Tr}(\gamma^\beta \gamma^\delta) + g^{\alpha\delta} \text{Tr}(\gamma^\beta \gamma^\gamma) \\ &= 4g^{\alpha\beta} g^{\gamma\delta} - 4g^{\alpha\gamma} g^{\beta\delta} + 4g^{\alpha\delta} g^{\beta\gamma} \\ &= 4(g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma})\end{aligned}$$

5)  $\text{Tr}(\gamma^5) = \text{Tr}(\gamma^5 \gamma^0 \gamma^0) = \text{Tr}(\overbrace{-\gamma^0 \gamma^5 \gamma^0}^{\text{use } (\gamma^0)^2 = 1}) = -\text{Tr}(\gamma^0 \gamma^0 \gamma^5) \stackrel{\text{use } \gamma^5 \gamma^0 = \gamma^0 \gamma^5}{=} -\text{Tr}(\gamma^5) \stackrel{\text{use } \text{Tr}(AB) = \text{Tr}(BA)}{=} \text{Tr}(\gamma^5)$

$$\begin{aligned}\text{Tr}(\gamma^5 \gamma^\alpha) &= \text{Tr}(\gamma^\alpha \gamma^5) \stackrel{\text{use } \gamma^5 \gamma^\alpha = -\gamma^\alpha \gamma^5}{=} \text{Tr}(-\gamma^5 \gamma^\alpha) = -\text{Tr}(\gamma^5 \gamma^\alpha) \\ \Rightarrow \text{Tr}(\gamma^5 \gamma^\alpha) &= 0 \quad \checkmark\end{aligned}$$

For  $\text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta)$ ,

- ① if  $\alpha = \beta = 0$ , then  $\text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta) = \text{Tr}(\gamma^5 (\gamma^0)^2) = \text{Tr}(\gamma^5) = 0$
- ② if  $\alpha = \beta = i$ , then  $\text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta) = \text{Tr}(\gamma^5 (\gamma^i)^2) = -\text{Tr}(\gamma^5) = 0$
- ③ if  $\alpha \neq \beta$  and  $\alpha \neq 0$  and  $\beta \neq 0$ , then

$$\begin{aligned}\text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta) &= \text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^0 \gamma^0) \stackrel{\text{use } \text{Tr}(AB) = \text{Tr}(BA)}{=} -\text{Tr}(\gamma^0 \gamma^5 \gamma^\alpha \gamma^\beta \gamma^0) \\ &= -\text{Tr}(\gamma^0 \gamma^0 \gamma^5 \gamma^\alpha \gamma^\beta) \stackrel{\text{since } \gamma^0 \gamma^i = -\gamma^i \gamma^0, \gamma^0 \gamma^5 = -\gamma^5 \gamma^0}{=} -\text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta) \\ \Rightarrow \text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta) &= 0\end{aligned}$$

(checked)

④ if  $\alpha \neq \beta$  and either  $\alpha=0$  or  $\beta=0$ , then choose  $i \neq \alpha \& i \neq \beta$ ,

$$\Rightarrow \text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta) = -\text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^i \gamma^i) = \text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^i)$$

use  $(\gamma^i)^2 = -1$

$\uparrow$   
use  $\gamma^\alpha \gamma^i = -\gamma^i \gamma^\alpha$ ,  $\gamma^\beta \gamma^i = -\gamma^i \gamma^\beta$   
when  $i \neq \alpha \& i \neq \beta$   
and  $\gamma^i \gamma^5 = -\gamma^5 \gamma^i$

$$= \text{Tr}(\gamma^i \gamma^i \gamma^5 \gamma^\alpha \gamma^\beta) = -\text{Tr}(\gamma^i \gamma^\alpha \gamma^\beta)$$

$\uparrow$   
use  $\text{Tr}(AB) = \text{Tr}(BA)$

$$\Rightarrow \text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta) = 0$$

Therefore, for all cases,  $\text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta) = 0$

$$\text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\mu) = i \underbrace{\text{Tr}(\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\alpha \gamma^\beta \gamma^\mu)}_{\text{there're 7 \gamma-matrices (odd number of \gamma-matrices), so}}$$

$\text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\mu) = 0$  ✓

So,  $\text{Tr}(\gamma^5) = \text{Tr}(\gamma^5 \gamma^\alpha) = \text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta) = \text{Tr}(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\mu) = 0$ .

(checked)

$$\begin{aligned}
 3. 1) \sigma^1 \sigma^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i \sigma^3 \\
 \sigma^2 \sigma^1 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i \sigma^3 \\
 \sigma^2 \sigma^3 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i \sigma^1 \\
 \sigma^3 \sigma^2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i \sigma^1 \\
 \sigma^3 \sigma^1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \sigma^2 \\
 \sigma^1 \sigma^3 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i \sigma^2 \\
 \sigma^1 \sigma^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 \sigma^2 \sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 \sigma^3 \sigma^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
 \end{aligned}$$

that is  $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$  where  $\epsilon^{123} = 1$   
 also  $\sigma^i \sigma^j + \sigma^j \sigma^i = 2\delta^{ij}$

$$(\sigma^1)^+ = \sigma^1$$

$$(\sigma^2)^+ = \sigma^2$$

$$(\sigma^3)^+ = \sigma^3$$

For  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu$ .

$$\textcircled{1} \quad \mu=\nu=0, \quad 2(\gamma^0)^2 = 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2g^{00} \mathbb{I}_{4 \times 4}$$

$$\textcircled{2} \quad (\mu=0 \& \nu=i) \text{ or } (\mu=i \& \nu=0), \quad \gamma^0 \gamma^i + \gamma^i \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} + \begin{pmatrix} a & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = a = 2g^{0i}$$

$$\textcircled{3} \quad \mu=i \& \nu=j, \quad \gamma^i \gamma^j + \gamma^j \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} + (i \leftrightarrow j) \\ = \begin{pmatrix} -\sigma^i \sigma^j - \sigma^j \sigma^i & 0 \\ 0 & -\sigma^i \sigma^j - \sigma^j \sigma^i \end{pmatrix} = -2 \begin{pmatrix} \delta^{ij} & 0 \\ 0 & \delta^{ij} \end{pmatrix} = -2g^{ij}$$

(checked)

$$\text{So, } \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

For  $\gamma^{+\mu}$

$$\textcircled{1} \quad \mu=0, \quad \gamma^0+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^+ = \begin{pmatrix} 1 & 0 \\ 0 & -1^+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0$$

$$\gamma^0 \gamma^0 \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0$$

$$\textcircled{2} \quad \mu=i, \quad \gamma^i+ = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & -\sigma^{i+} \\ \sigma^{i+} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} = -\gamma^i$$

$$\gamma^0 \gamma^i \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\text{So, } \gamma^i+ = \gamma^0 \gamma^i \gamma^0 = -\gamma^i$$

$$\begin{aligned} \textcircled{3} \quad \gamma^5 &= i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \\ &= i \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \begin{pmatrix} -\sigma^2 \sigma^3 & 0 \\ 0 & -\sigma^2 \sigma^3 \end{pmatrix} = i \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \begin{pmatrix} -i \sigma^1 & 0 \\ 0 & -i \sigma^1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} = \begin{pmatrix} 0 & (\sigma^1)^2 \\ (\sigma^1)^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

(checked)

$$4. \gamma^0\gamma^i + \gamma^i\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \tau^i \\ -\tau^i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -\tau^i & 0 \\ 0 & \tau^i \end{pmatrix} + \begin{pmatrix} \tau^i & 0 \\ 0 & -\tau^i \end{pmatrix}$$

$$= 0 = 2g^{0i}$$

$$\gamma^0\gamma^0 + \gamma^0\gamma^0 = 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2g^{00}$$

Since the spatial part  $\gamma^i\gamma^j + \gamma^j\gamma^i = 2g^{ij}$  has been checked in problem 3,  
then done the check of  $\gamma^v\gamma^v + \gamma^v\gamma^v = 2g^{vv}$ .

$$\gamma^{+0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^0\gamma^0\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \gamma^{+0} = \gamma^0\gamma^0\gamma^0$$

$$\text{Since } \gamma^v\gamma^v + \gamma^v\gamma^v = 2g^{vv}.$$

$$\text{Then } \gamma^0\gamma^i + \gamma^i\gamma^0 = 0 \Rightarrow \gamma^0\gamma^i = -\gamma^i\gamma^0$$

$$\text{also } (\gamma^0)^2 = 1$$

$$\Rightarrow \gamma^0\gamma^i\gamma^0 = -\gamma^i(\gamma^0)^2 = -\gamma^i$$

and  $\gamma^{+i} = -\gamma^i$  has been checked in problem 3.

$$\text{So, done the check of } \gamma^{+u} = \gamma^0\gamma^u\gamma^0$$

$$\begin{aligned} \gamma^b &= i\gamma^0\gamma^1\gamma^2\gamma^3 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \tau^1 \\ -\tau^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \tau^2 \\ -\tau^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \tau^3 \\ -\tau^3 & 0 \end{pmatrix} \\ &= i \begin{pmatrix} -\tau^1 & 0 \\ 0 & \tau^1 \end{pmatrix} \begin{pmatrix} -i\tau^1 & 0 \\ 0 & -i\tau^1 \end{pmatrix} = \begin{pmatrix} -(\tau^1)^2 & 0 \\ 0 & (\tau^1)^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

For problem 3&4, the Mathematica code is also attached.

(checked)

This code checks the defining properties of gamma matrices in the standard and Weyl representations

This code is checked on Nov. 1, 2018

check  $\gamma$ -matrices properties in the standard representation

define  $\gamma$ -matrices and  $g^{\mu\nu}$

```
In[1]:= gamma[0] = ArrayFlatten[{{IdentityMatrix[2], ConstantArray[0, {2, 2}]}, {ConstantArray[0, {2, 2}], -IdentityMatrix[2]}}];
MatrixForm[%]
Out[1]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$


In[3]:= gamma[1] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[1]}, {-PauliMatrix[1], ConstantArray[0, {2, 2}]}}];
MatrixForm[%]
Out[3]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$


In[5]:= gamma[2] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[2]}, {-PauliMatrix[2], ConstantArray[0, {2, 2}]}}];
MatrixForm[%]
Out[5]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$


In[7]:= gamma[3] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[3]}, {-PauliMatrix[3], ConstantArray[0, {2, 2}]}}];
MatrixForm[%]
Out[7]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$


In[9]:= gmunu = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}};
MatrixForm[%]
Out[9]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```

check  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$  (note that in Mathematica the column and row start from 1, rather than

0)

```
In[1]:= Do[If[(gamma[mdo].gamma[ndo] + gamma[ndo].gamma[mdo] -  
2 * gmunu[[mdo + 1, ndo + 1]] * IdentityMatrix[4]) !=  
ConstantArray[0, {4, 4}], Print["error"], Print["ok"]],  
{mdo, 0, 3, 1}, {ndo, 0, 3, 1}]
```

ok

find gamma[5]

```
In[13]:= MatrixForm[I * gamma[0].gamma[1].gamma[2].gamma[3]]
```

Out[13]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

In[14]:=

check  $\gamma$ -matrices properties in the Weyl representation (note that the only difference compared to the standard representation is  $\gamma^0$ )

define  $\gamma$ -matrices and  $g^{\mu\nu}$

```
In[15]:= gamma[0] = ArrayFlatten[{{ConstantArray[0, {2, 2}], IdentityMatrix[2]}, {IdentityMatrix[2], ConstantArray[0, {2, 2}]}}];
MatrixForm[%]

Out[16]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$


In[17]:= gamma[1] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[1]}, {-PauliMatrix[1], ConstantArray[0, {2, 2}]}];
MatrixForm[%]

Out[18]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$


In[19]:= gamma[2] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[2]}, {-PauliMatrix[2], ConstantArray[0, {2, 2}]}];
MatrixForm[%]

Out[20]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & -\frac{i}{2} \\ 0 & 0 & \frac{i}{2} & 0 \\ 0 & \frac{i}{2} & 0 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 \end{pmatrix}$$


In[21]:= gamma[3] = ArrayFlatten[{{ConstantArray[0, {2, 2}], PauliMatrix[3]}, {-PauliMatrix[3], ConstantArray[0, {2, 2}]}];
MatrixForm[%]

Out[22]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$


In[23]:= gmunu = {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -1, 0}, {0, 0, 0, -1}};
MatrixForm[%]

Out[24]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```

check  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$  (note that in Mathematica the column and row start from 1, rather than 0)

```
In[25]:= Do[If[(gamma[mdo].gamma[ndo] + gamma[ndo].gamma[mdo] -  
2 * gmunu[[mdo + 1, ndo + 1]] * IdentityMatrix[4]) !=  
ConstantArray[0, {4, 4}], Print["error"], Print["ok"]],  
{mdo, 0, 3, 1}, {ndo, 0, 3, 1}]  
ok  
check ( $\gamma^\mu$ )+ =  $\gamma^0 \gamma^\mu \gamma^0$   
In[26]:= Do[If[(ConjugateTranspose@gamma[mdo]) - gamma[0].gamma[mdo].gamma[0] !=  
ConstantArray[0, {4, 4}], Print["error"], Print["ok"]], {mdo, 0, 3, 1}]  
ok  
ok  
ok  
ok  
ok  
find gamma[5]  
In[27]:= MatrixForm[I * gamma[0].gamma[1].gamma[2].gamma[3]]  
Out[27]//MatrixForm= 
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
  
In[28]:= Date[]  
Out[28]= {2018, 11, 1, 16, 15, 44.082165}
```