Homework 2
Due date: 2018.10.22

Problem 1. A second-rank tensor is called symmetric if it is unchanged when you switch the indices ($s^{\mu\nu} = s^{\nu\mu}$); it is called antisymmetric if it changes sign ($a^{\mu\nu} = -a^{\nu\mu}$). Assume that the tensors are defined in the four dimensional Minkowski spacetime used in special relativity, answer the following questions.

1) How many independent elements are there in a symmetric tensor? (Since $s^{12} = s^{21}$, these would count as only one independent element.) [1 point]

2) How many independent elements are there in an antisymmetric tensor? [1 point]

3) Show that symmetry is preserved by Lorentz transformations. That is, if $s^{\mu\nu}$ is symmetric, so too is $s'^{\mu\nu}$; if $a^{\mu\nu}$ is antisymmetric, so too is $a'^{\mu\nu}$. (Hint: use the definition of Lorentz transformations between x and x' frames.) [1 point]

4) If $s^{\mu\nu}$ is symmetric, show that $s_{\mu\nu}$ is also symmetric. If $a^{\mu\nu}$ is antisymmetric, show that $a_{\mu\nu}$ is also antisymmetric. (Hint: use the metric tensor to lower the indices.) [1 point]

5) If $s^{\mu\nu}$ is symmetric and $a^{\mu\nu}$ is antisymmetric, show that $s^{\mu\nu}a_{\mu\nu} = 0$. [1 point]

6) Any second-rank tensor ($t^{\mu\nu}$), can be written as the sum of a symmetric part ($s^{\mu\nu}$) and an antisymmetric part ($a^{\mu\nu}$): $t^{\mu\nu} = s^{\mu\nu} + a^{\mu\nu}$. Construct $s^{\mu\nu}$ and $a^{\mu\nu}$ in terms of $t^{\mu\nu}$ and $t^{\nu\mu}$. [1 point]

Problem 2. Consider the Lagrangian for a massive vector field A_μ

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - j_\mu A^\mu \,, \tag{1}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, while j_μ does not depend on the vector field or its derivative.

1) Show that [1 point]

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0\,. \tag{2}$$

2) Use Euler-Lagrange equation to show that the equation of motion is [2 points]

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu\,. \tag{3}$$

Hint: $\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\rho A_\sigma)} = \delta^\rho_\mu \delta^\sigma_\nu$.

3) In the case $m = 0$, we construct $F^{\mu\nu}$ as $F^{i0} = -F^{0i} \equiv E^i$ and $F^{ij} \equiv -\epsilon^{ijk}B^k$, where ϵ^{ijk} are the Levi-Civita symbol, with the sign convention $\epsilon^{123} = 1$ and $\epsilon_{123} = 1$, and $j^\mu = (\rho, \vec{j})$. That is,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}\,, \tag{4}$$

where $\vec{E} = (E^1, E^2, E^3)$ and $\vec{B} = (B^1, B^2, B^3)$. Show that when $m = 0$, eqs. (2) and (3) give the four Maxwell equations: [3 points]

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \rho \,, \\
\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{j} \,, \\
\vec{\nabla} \cdot \vec{B} &= 0 \,, \\
\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \,.
\end{aligned}$$

Note that the indices of the three-dimensional Levi-Civita symbols used here are not lowered or raised by the metric tensor, $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. However, $F^{\mu\nu}$ and $F_{\mu\nu}$ are connected by the metric tensor.

If you need, you can use the properties of the three-dimensional Levi-Civita symbols, $\epsilon_{ijk}\epsilon^{imn} = \delta_j^m \delta_k^n - \delta_j^n \delta_k^m$, $\epsilon_{jmn}\epsilon^{imn} = 2\delta_j^i$ and $\epsilon_{ijk}\epsilon^{ijk} = 6$. Please prove them by yourself (you don't have to show the proof on your work).

4) In a reference frame with a static charge at the coordinate origin, i.e., $j^\mu = (q\delta^3(\vec{x}), 0, 0, 0)$, for $m > 0$ check that the corresponding static solution to eq. (3) is

$$A^\mu = (A^0, A^1, A^2, A^3) = (\frac{q}{(2\pi)^3} \int\int\int_{-\infty}^{+\infty} d^3\vec{k} \frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2 + m^2}, 0, 0, 0) \,.$$

Note that you do not have to derive this solution. You just need to check that it satisfies eq. (3) (or an equation which can be obtained by simplifying eq. (3) without loss of information). [2 points]

5) Evaluate the above integral to get an explicit form for $A^0(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. [2 points]

Please show all your steps. If you know how to do contour integration, that's great! However, if you don't know how to do it by hand, please print your computer program (for example, Mathematica or whatever software) which shows your work.