Conserved current and Noether theorem

Will show that there is a close relationship between the invariance of the action in an arbitrary continuous global transformation and the existence of a conserved current. —— Noether's theorem.

(global: means that the transformation parameters are constants)

Consider infinitesimal coordinate transformations

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu$$

then for a generic field $\varphi(x) \to \varphi'(x') = \varphi(x) + \delta\varphi(x)$

(for translation, $\delta x^\mu = -\delta a^\mu$;
for Lorentz transformation, $\delta x^\mu = \varepsilon^\mu{}_\nu x^\nu$)

The total variation of the field is

$$\begin{aligned}
\delta\varphi(x) &= \varphi'(x') - \varphi(x) \\
&= \left(\varphi'(x') - \varphi'(x)\right) + \left(\varphi'(x) - \varphi(x)\right) \\
&\simeq \delta x^\mu \partial_\mu \varphi'(x) + \delta_0 \varphi(x) \\
&= \delta x^\mu \partial_\mu (\delta_0 \varphi(x) + \varphi(x)) + \delta_0\varphi(x) \\
&\simeq \delta x^\mu \partial_\mu \varphi(x) + \delta_0 \varphi(x)
\end{aligned}$$

$$\begin{aligned}
\delta\mathcal{L}(x) &= \mathcal{L}'(x') - \mathcal{L}(x) = \left(\mathcal{L}'(x') - \mathcal{L}'(x)\right) + \left(\mathcal{L}'(x) - \mathcal{L}(x)\right) \\
&\simeq \delta x^\mu \cdot \partial_\mu \mathcal{L}'(x) + \delta_0 \mathcal{L}(x) \\
&= \delta x^\mu \partial_\mu (\mathcal{L}(x) + \delta_0 \mathcal{L}(x)) + \delta_0 \mathcal{L}(x) \\
&\simeq \delta x^\mu \partial_\mu \mathcal{L}(x) + \delta_0 \mathcal{L}(x) \\
&= \delta x^\mu \partial_\mu \mathcal{L}(x) + \left(\frac{\partial \mathcal{L}}{\partial \varphi}\delta_0\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta_0(\partial_\mu\varphi)\right) \\
&= \delta x^\mu \partial_\mu \mathcal{L}(x) + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\delta_0\varphi\right) + \left[\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\right]\delta_0\varphi
\end{aligned}$$

while $\delta S = \int_V \delta(d^4x)\mathcal{L} + \int_V d^4x\, \delta\mathcal{L}(x)$

V ↳ four-dimensional integration volume, leave it arbitrary where $\delta(d^4x) = d^4x' - d^4x = \left(\left| \det\left(\frac{\partial x'^\mu}{\partial x^\nu} \right) \right| - 1 \right) d^4x$

$$= \left(\left| \det[\delta^\mu{}_\nu + \partial_\nu(\delta x^\mu)] \right| - 1 \right) d^4x$$

$$\simeq \left(1 + \partial_\mu \delta x^\mu - 1 \right) d^4x$$

$$= (\partial_\mu \delta x^\mu)\, d^4x$$

(note: $\frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} 1+\partial_0\delta x^0 & \partial_1\delta x^0 & \partial_2\delta x^0 & \partial_3\delta x^0 \\ \partial_0\delta x^1 & 1+\partial_1\delta x^1 & \partial_2\delta x^1 & \partial_3\delta x^1 \\ \partial_0\delta x^2 & \partial_1\delta x^2 & 1+\partial_2\delta x^2 & \partial_3\delta x^2 \\ \partial_0\delta x^3 & \partial_1\delta x^3 & \partial_2\delta x^3 & 1+\partial_3 x^3 \end{pmatrix} \Rightarrow \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)$
$= (1 + \partial_0\delta x^0 + \partial_1\delta x^1 + \partial_2\delta x^2 + \partial_3\delta x^3) + o((\delta x)^2)$)

$$\Rightarrow \delta S = \int_V d^4x (\partial_\mu \delta x^\mu) \mathcal{L} + \int_V d^4x \left\{ \delta x^\mu \partial_\mu \mathcal{L}(x) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 \varphi \right) + \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta_0 \varphi \right\}$$

$$= \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta_0 \varphi + \int_V d^4x\, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 \varphi + \mathcal{L} \delta x^\mu \right]$$

Note that we don't impose the constraints that $\delta_0 \varphi = 0$ on the integration limits.

If φ satisfies the equation of motion, then the first line of δS vanishes, then $\delta S = \int_V d^4x\, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 \varphi + \mathcal{L} \delta x^\mu \right]$

For global infinitesimal transformations defined by the small constant parameters $\delta \omega$, the invariance of S to the symmetry transformation defined by $\delta \omega$ means that

$$\frac{\delta S}{\delta \omega} = \int_V d^4x\, \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\delta_0 \varphi}{\delta \omega} + \mathcal{L} \frac{\delta x^\mu}{\delta \omega} \right] = 0$$

Since the integration volumn V is arbitrary, the integrand should be zero. That is, if define $j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\delta_0 \varphi}{\delta \omega} + \mathcal{L} \frac{\delta x^\mu}{\delta \omega}$, then $\partial_\mu j^\mu(x) = 0$.

↳ conserved current

Note that if $\mathcal{L} = \mathcal{L}(\varphi_1, \varphi_2, \ldots, \varphi_N, \partial_\mu \varphi_1, \partial_\mu \varphi_2, \ldots, \partial_\mu \varphi_N)$,

then $j^\mu = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \frac{\delta_0 \varphi_i}{\delta \omega} + \mathcal{L} \frac{\delta x^\mu}{\delta \omega}$

If we define $Q \equiv \int_V d^3\vec{x}\, j^0(t, \vec{x})$, then

$$\frac{dQ}{dt} = \int_V d^3\vec{x}\, \partial_0 j^0(t,\vec{x}) = \int_V d^3\vec{x}\, (\underbrace{\partial_\mu j^\mu}_{0} - \vec{\nabla}\cdot\vec{j}) \underset{\text{Gauss' theorem}}{=} -\int_S d\vec{S}\cdot\vec{j}$$

If we further assume that $\vec{j} = 0$ on the integration surface S, then

$\frac{dQ}{dt} = 0$, so we get a conserved charge Q (since it is a constant in time).

Examples

(a) Translation of a generic field.

Consider infinitesimal translation $x^\mu \to x'^\mu = x^\mu - \delta a^\mu$,

$\delta x^\mu = -\delta a^\mu$

For a scalar field, we have shown that $\delta_0 \phi(x) = \delta a^\mu \partial_\mu \phi(x)$.

For a vector field, $A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu(x) = \frac{\partial(x^\mu - \delta a^\mu)}{\partial x^\nu} A^\nu(x)$

$$= (\delta^\mu_\nu - \underbrace{\partial_\nu \delta a^\mu}_{0 \text{ since } \delta a^\mu \text{ is a constant.}}) A^\nu(x) = A^\mu(x)$$

$\Rightarrow \quad \delta_0 A^\mu(x) = \delta a^\nu \partial_\nu A^\mu(x)$

Actually, it is the same for a generic field φ.

$\delta_0 \varphi = \delta a^\mu \partial_\mu \varphi$

$\delta \omega \equiv \delta a^\nu$

$\Rightarrow \quad \frac{\delta x^\mu}{\delta \omega} = \frac{-\delta a^\mu}{\delta a^\nu} = -\delta^\mu_\nu$,

$\frac{\delta_0 \varphi}{\delta \omega} = \delta^\mu_\nu \partial_\mu \varphi = \partial_\nu \varphi$

$$\Rightarrow\ j^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu{}_\nu \mathcal{L}$$

(note that j^μ introduced before does not prevent it from having more indices, which are introduced by the parameter $\delta\omega$. That is, although

$$X \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta_0 \varphi + \mathcal{L}\,\delta x^\mu$$

only introduce the index "μ", more indices are introduced by the $\delta\omega$ when define $j^\mu = \frac{X}{\delta\omega}$.)

and
$$Q_\nu = \int d^3x\, j^0{}_\nu = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} \partial_\nu \varphi - \delta^0{}_\nu \mathcal{L} \right)$$
$$= \int d^3x \left(\pi \partial_\nu \varphi - \delta^0{}_\nu \mathcal{L} \right)$$
$$Q_0 = \int d^3x \left(\pi \partial_0 \varphi - \mathcal{L} \right) = \int d^3x\, \mathcal{H}.$$

Therefore, in fact, Q_0 is the Hamiltonian.

$$Q_i = \int d^3x\, \pi\, \partial_i \varphi$$

We will pick up the expressions of Q_0 and Q_i from here later when we quantize the fields. For now, we just claim that Q_i is the momentum operator (since Q_0 is the energy operator — Hamiltonian) and we get energy-momentum conservation ($\frac{dQ_\mu}{dt} = 0$) from the invariance of the action subject to space-time translations.

Let's just call $Q_0 \equiv P_0 = H$, $Q_i \equiv P_i$.

(By the way, $j^\mu{}_\nu$ is just the energy-momentum tensor, usually written as $T^\mu{}_\nu$, and its GR version appears in Einstein field equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$
)

Example (b) Lorentz transformation of a real scalar field.

As shown before, $\delta X^\mu = \varepsilon^\mu{}_\nu X^\nu = \varepsilon^{\mu\nu} X_\nu \quad \delta_0 \phi(x) = -\frac{i}{2}\varepsilon^{\mu\nu} L_{\mu\nu}\phi(x)$

let $\delta\omega = \varepsilon^{\rho\sigma}$, then

$$\frac{\delta x^\mu}{\delta\omega} = (\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) X_\nu = \delta^\mu_\rho X_\sigma - \delta^\mu_\sigma X_\rho$$

$$\frac{\delta_0\phi(x)}{\delta\omega} = (\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho)(-\frac{i}{2} L_{\mu\nu}\phi(x)) = -\frac{i}{2}(L_{\rho\sigma} - L_{\sigma\rho})\phi(x) = -iL_{\rho\sigma}\phi(x)$$

$$= (-i)\cdot i\,(X_\rho \partial_\sigma - X_\sigma \partial_\rho)\phi(x) = (X_\rho\partial_\sigma - X_\sigma\partial_\rho)\phi(x)$$

$$\Rightarrow j^\mu{}_{\rho\sigma} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}(X_\rho\partial_\sigma - X_\sigma\partial_\rho)\phi(x) + \mathcal{L}(\delta^\mu_\rho X_\sigma - \delta^\mu_\sigma X_\rho)$$

$$= X_\rho T^\mu{}_\sigma - X_\sigma T^\mu{}_\rho$$

and $Q_{\rho\sigma} = \int d^3x\, j^0_{\rho\sigma} = \int d^3x\,(X_\rho T^0{}_\sigma - X_\sigma T^0_\rho) = \int d^3x\,(X_\rho P_\sigma - X_\sigma P_\rho)$

and its spatial components are

$$Q_{ij} = \int d^3x\,(X_i P_j - X_j P_i)$$

so $\frac{dQ}{dt} = 0$ means angular momentum conservation.

Here $j^\mu{}_{\rho\sigma}$ is usually written as $M^\mu{}_{\rho\sigma}$, which is called the angular momentum density tensor.

So the invariance of the action to Lorentz transformation gives angular momentum conservation.

Example (c) Internal transformation of a complex Scalar field.

(so the space-time coordinates are not affected, $\delta x^\mu = 0$)

$$\delta\phi = \delta_0\phi \neq 0$$

$$\phi_{(x)} \to \phi'_{(x)} = e^{-i\alpha}\phi_{(x)} \simeq \phi - i\alpha\phi$$

$$\phi^*_{(x)} \to \phi'^*_{(x)} = e^{i\alpha}\phi^*_{(x)} \simeq \phi^* + i\alpha\phi^*$$

where α is a constant.

In such internal transformation, the action is invariant, which can be easily check as

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \lambda(\phi^*\phi)^2$$

$$\begin{aligned}\mathcal{L}' &= \partial_\mu\phi'^*\partial^\mu\phi' - m^2\phi'^*\phi' - \lambda(\phi'^*\phi')^2 \\ &= \partial_\mu(e^{i\alpha}\phi^*)\partial^\mu(e^{-i\alpha}\phi) - m^2(e^{i\alpha}\phi^*)(e^{-i\alpha}\phi) - \lambda[(e^{i\alpha}\phi^*)e^{-i\alpha}\phi]^2 \\ &= \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \lambda(\phi^*\phi)^2 \\ &= \mathcal{L}\end{aligned}$$

$$\Rightarrow \quad \delta\mathcal{L} = 0$$

also, since $\delta x^\mu = 0$, then $\delta(d^4x) = 0$

$$\Rightarrow \delta S = \int \delta(d^4x)\mathcal{L} + \int d^4x\,\delta\mathcal{L} = 0$$

Now let $\delta\omega = \delta\alpha$, (and infinitesimal transformation of the fields are

$$\phi \to \phi' = e^{-i\delta\alpha}\phi \simeq \phi - i(\delta\alpha)\phi$$
$$\phi^* \to \phi'^* = e^{i\delta\alpha}\phi^* \simeq \phi^* + i(\delta\alpha)\phi^*\ )$$

$$\Rightarrow \frac{\delta_0\phi}{\delta\omega} = -i\phi, \quad \frac{\delta_0\phi^*}{\delta\omega} = i\phi^*, \quad \text{and} \quad \frac{\delta x^\mu}{\delta\omega} = 0$$

$$\Rightarrow j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(-i\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}i\phi^*$$

and $Q = \int d^3x\, j^0 = i\int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0\phi^*)} \phi^* - \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} \phi \right)$

$= i\int d^3x \left(\dot{\phi}\phi^* - \dot{\phi}^*\phi \right)$

we will look at it when we quantize the field.