Homework 3
Due date: 2018.11.7

Problem 1. Start with the following defining properties of the γ-matrices,

$$\begin{aligned} \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu &= 2g_{\mu\nu}\,, \\ \gamma^\mu &= g^{\mu\nu}\gamma_\nu\,, \\ \gamma_\mu &= g_{\mu\nu}\gamma^\nu\,, \\ \gamma_\mu^\dagger &= \gamma_0\gamma_\mu\gamma_0\,, \\ \gamma^5 \equiv \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3\,, \end{aligned}$$

prove the following identities: [1 point for each, 2 points in total]
1) $(\gamma_0)^2 = (\gamma^0)^2 = 1$
2) $(\gamma_i)^2 = (\gamma^i)^2 = -1$, for i = 1,2,3
3) $(\gamma^5)^2 = 1$
4) $(\gamma^5)^\dagger = \gamma^5$
5) $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$
6) $\gamma_\lambda\gamma^\lambda = 4$
7) $\gamma_\lambda\gamma^\alpha\gamma^\lambda = -2\gamma^\alpha$
8) $\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda = 4g^{\alpha\beta}$
9) $\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\lambda = -2\gamma^\sigma\gamma^\beta\gamma^\alpha$
10) $\gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\sigma\gamma^\eta\gamma^\lambda = 2(\gamma^\eta\gamma^\alpha\gamma^\beta\gamma^\sigma + \gamma^\sigma\gamma^\beta\gamma^\alpha\gamma^\eta)$
11) A, B, ... denote four-vectors, and they commute with each other (i.e., for any two vectors, $A^\mu B^\nu = B^\nu A^\mu$ for all their indices μ and ν). Use the definition of the 'slashed' vector (i.e, $\not{A} \equiv \gamma^\alpha A_\alpha$), show that
$\not{A}\not{A} = A \cdot A$
$\not{A}\not{B} + \not{B}\not{A} = 2A \cdot B$
$\gamma_\lambda\not{A}\gamma^\lambda = -2\not{A}$
$\gamma_\lambda\not{A}\not{B}\gamma^\lambda = 4A \cdot B$
$\gamma_\lambda\not{A}\not{B}\not{C}\gamma^\lambda = -2\not{C}\not{B}\not{A}$
$\gamma_\lambda\not{A}\not{B}\not{C}\not{D}\gamma^\lambda = 2(\not{D}\not{A}\not{B}\not{C} + \not{C}\not{B}\not{A}\not{D})$
Note: If you want to use the identities derived in class, please re-derive them in your work.
Please use the convention $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$.

Problem 2. Prove the following trace identities for γ-matrices: [1 point for each, 2 points in total]
1) If $(\gamma^\alpha\gamma^\beta...\gamma^\mu\gamma^\nu)$ contains an odd number of γ-matrices, then $\mathrm{Tr}(\gamma^\alpha\gamma^\beta...\gamma^\mu\gamma^\nu) = 0$.
2) If $(\gamma^\alpha\gamma^\beta...\gamma^\mu\gamma^\nu)$ contains an even number of γ-matrices, then $\mathrm{Tr}(\gamma^\alpha\gamma^\beta...\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\nu\gamma^\mu...\gamma^\beta\gamma^\alpha)$.
3) $\mathrm{Tr}(\gamma^\alpha\gamma^\beta) = 4g^{\alpha\beta}$.
4) $\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})$.
5) $\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\gamma^5\gamma^\alpha) = \mathrm{Tr}(\gamma^5\gamma^\alpha\gamma^\beta) = \mathrm{Tr}(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu) = 0$.
Note: You can use the results in problem 1 if you need.

Problem 3. The γ-matrices in the standard representation are

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where $\mathbb{1}$ is the 2×2 identity matrix, 0 is the 2×2 zero matrix, and σ^i ($i = 1, 2, 3$) are the 2×2 Pauli matrices, namely,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1) Show that the γ-matrices in this representation satisfy the two defining properties, (a) $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ and (b) $\gamma^{\dagger\mu} = \gamma^0\gamma^\mu\gamma^0$. [3 points]
2) Calculate γ^5. [1 point]
Note: If you want to use the properties of the Pauli matrices to do this problem (for example, the commutation and/or the anticommutation relations of the Pauli matrices, the Hermitian property of them), please prove these properties first. If you would rather check (a) and (b) for all the μ and ν indices explicitly, you can either do it by hand or print your computer program.

Problem 4. Repeat the calculations of problem 3 (namely, check the two defining properties [1 point] and calculate γ^5 [1 point]), but this time for the γ-matrices in the Weyl representation, in which

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

Note: The spatial components of the γ-matrices in the standard representation and in the Weyl representation are the same. Therefore, you don't need to check the parts which you have already done in problem 3.