Quantization of a scalar field

2018年11月11日 21:20

A scalar field, a component of a vector field and a component of a spinor field all satisfy the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0$$

$$(\Box + m^2) A_\mu(x) = 0 \qquad \text{for } \mu = 0, 1, 2, 3$$

$$(\Box + m^2) \psi_a(x) = 0 \qquad \text{for } a = 1, 2, 3, 4$$

where $\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$

We have seen that the Klein-Gordon equation can be constructed by using the relativistic four-momentum relation

$$E^2 - \vec{p}^{\,2} = m^2 \qquad (c = 1), \text{ note that } \vec{p}^{\,2} = \vec{p}\cdot\vec{p} = |\vec{p}|^2$$

and the identifications

$$E \to i\hbar \frac{\partial}{\partial t}, \quad \vec{p} \to -i\hbar \vec{\nabla}$$

$$\Rightarrow \left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right)\varphi(x) = m^2 \varphi(x), \quad (\hbar = 1)$$

$$\Rightarrow (\Box + m^2)\varphi(x) = 0$$

$\hookleftarrow$ a generic field.

On the other hand, the Schrödinger equation, which describes the motion of a non-relativistic particle, can be constructed as

$$\frac{\vec{p}^{\,2}}{2m} \longrightarrow -\frac{\hbar^2}{2m}\vec{\nabla}^2$$

$$E \longrightarrow i\hbar \frac{\partial}{\partial t}$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t}\varphi(\vec{x}, t) = -\frac{\hbar^2}{2m}\vec{\nabla}^2 \varphi(\vec{x}, t)$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t}\varphi(\vec{x},t) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\varphi(\vec{x},t)$$

(Note that the above Schrödinger equation does not have the potential term, otherwise it is $i\hbar\frac{\partial}{\partial t}\varphi(\vec{x},t) = (-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V)\varphi(\vec{x},t)$)

The solution of a free particle (i.e, $V=0$) Schrödinger equation is

$$\varphi(\vec{x},t) \propto \exp(-iEt + i\vec{p}\cdot\vec{x}) \quad , \quad (\hbar = 1)$$

In general,

$$\varphi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\varphi(\vec{p})\exp(-iEt + i\vec{p}\cdot\vec{x})\, d^3\vec{p}$$

where $E = \frac{\vec{p}^{\,2}}{2m}$

$$\left(\begin{aligned}
\text{check: } \; & i\frac{\partial}{\partial t}\varphi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\varphi(\vec{p})\,E\exp(-iEt + i\vec{p}\cdot\vec{x})\, d^3\vec{p} \\
& -\frac{1}{2m}\vec{\nabla}^2\varphi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\varphi(\vec{p})\frac{\vec{p}^{\,2}}{2m}\exp(-iEt + i\vec{p}\cdot\vec{x})\, d^3\vec{p} \\
\Rightarrow \; & i\frac{\partial}{\partial t}\varphi(\vec{x},t) = -\frac{1}{2m}\vec{\nabla}^2\varphi(\vec{x},t) \quad \text{due to } E = \frac{\vec{p}^{\,2}}{2m}
\end{aligned}\right)$$

Similarly, for the Klein-Gordon equation,

$$(\Box + m^2)\varphi(x) = 0$$

if interpret $\varphi(x)$ as a single particle wave function, then the solutions are

$$\varphi(x) \propto \begin{cases} \exp(-iEt + i\vec{p}\cdot\vec{x}) = e^{-ip\cdot x} \\ \exp(+iEt + i\vec{p}\cdot\vec{x}) \\ \exp(-iEt - i\vec{p}\cdot\vec{x}) \\ \exp(+iEt - i\vec{p}\cdot\vec{x}) = e^{ip\cdot x} \end{cases}$$

In general,

$$\varphi(x) = \int_{-\infty}^{+\infty} \underbrace{C(E)}_{\text{normalization factor}}\left[g(\vec{p})e^{-ip\cdot x} + h(\vec{p})e^{ip\cdot x}\right] d^3\vec{p}$$

where $E^2 = \vec{p}^{\,2} + m^2$.

normalization factor

where $E^2 = \vec{p}^2 + m^2$.

note that we don't have to worry about the solution $\exp(-iEt - i\vec{p}\cdot\vec{x})$ once we have used $\exp(-iEt + i\vec{p}\cdot\vec{x}) = \exp(-ip\cdot x)$ to build the general solution, since

$$\int_a^b f(x)\,dx = \int_{-a}^{-b} f(-y)\,d(-y) = \int_{-b}^{-a} f(-y)\,dy = \int_{-b}^{-a} f(-x)\,dx$$

and therefore

$$\int_{-\infty}^{+\infty} C(E)\left[g(\vec{p})e^{-iEt+i\vec{p}\cdot\vec{x}} + g_1(\vec{p})e^{-iEt-i\vec{p}\cdot\vec{x}}\right]d^3\vec{p}$$

$$= \int_{-\infty}^{+\infty} C(E)\left[\underbrace{g(\vec{p}) + g_1(-\vec{p})}_{\text{redefine it as } g(\vec{p})}\right] e^{-iEt+i\vec{p}\cdot\vec{x}}\, d^3\vec{p}$$

Similarly, we don't need to worry about the solution $\exp(iEt + i\vec{p}\cdot\vec{x})$

since $$\int_{-\infty}^{+\infty} C(E)\left[h(\vec{p})e^{iEt-i\vec{p}\cdot\vec{x}} + h_1(\vec{p})e^{iEt+i\vec{p}\cdot\vec{x}}\right]d^3\vec{p}$$

$$= \int_{-\infty}^{+\infty} C(E)\left[\underbrace{h(\vec{p}) + h_1(-\vec{p})}_{\text{redefine it as } h(\vec{p})}\right] e^{iEt-i\vec{p}\cdot\vec{x}}\, d^3\vec{p}$$

Note that $g(\vec{p})$ and $h(\vec{p})$ should be understood as having Lorentz index when $\varphi(x)$ is a vector, and taking the form of 4×1 columns when $\varphi(x)$ is a Dirac spinor.

Now let's focus on the scalar field.

Let's look at the probability density and current density in quantum mechanics:

times ψ^* on $i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = [-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V]\psi(\vec{x},t)$

and minus ψ times on $-i\hbar\frac{\partial}{\partial t}\psi^* = [-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V^*]\psi^*$

and assume $V = V^*$

$$\Rightarrow \quad i\hbar\frac{\partial}{\partial t}(\psi\psi^*) = -\frac{\hbar^2}{2m}(\psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*)$$

$$= -\frac{\hbar^2}{2m}\vec{\nabla}\cdot(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$$

let $\rho \equiv \psi^*\psi$, $\vec{j} \equiv -\frac{i\hbar}{2m}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$

$$\Rightarrow \quad \frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0.$$

For a complex scalar field, the internal transformation gives $\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi^* - m^2\phi\phi^*$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(-i\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}i\phi^*$$

$$= \partial^\mu\phi^*(-i\phi) + \partial^\mu\phi\,(i\phi^*)$$

that is, $j^0 = \dot{\phi}^*(-i\phi) + \dot{\phi}(i\phi^*) = i(\dot{\phi}\phi^* - \dot{\phi}^*\phi)$

$\vec{j} = -\vec{\nabla}\phi^*(-i\phi) - \vec{\nabla}\phi\,(i\phi^*) = -i(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*)$

Also, $\partial_\mu j^\mu = 0$ from Noether theorem.

So the $\vec{j}$ here takes the same form as in the probability current in quantum mechanics up to a constant factor $\frac{1}{2m}$.

Also, the current j^μ for complex scalar field can also be derived in a similar way as in quantum mechanics:

times ϕ^* on $(\Box + m^2)\phi = 0$

and minus ϕ times on $(\Box + m^2)\phi^* = 0$

$$\Rightarrow \quad \phi^* \partial_\mu \partial^\mu \phi - \phi \partial_\mu \partial^\mu \phi^* = 0$$

$$\Rightarrow \quad \partial_\mu \left[i \left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^* \right) \right] = 0$$

so that $\partial_\mu j^\mu = 0$ if define $j^\mu = i\left(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*\right)$

Does it mean that we can interpret j^0 as the probability density as in quantum mechanics? No, we cannot.

In quantum mechanics,

$$\rho = |\psi|^2 = \psi^* \psi \geq 0$$

$\rightsquigarrow$ wave function is a complex function

However, if interpret the klein-Gordon equation as a wave equation, then for the wave function solution

$$\phi(x) = C e^{-ip \cdot x}$$

we have $\phi^*(x) = C^* e^{+ipx}$, $\dot{\phi}(x) = C(-iE) e^{-ipx} = (-iE)\phi(x)$

$$\dot{\phi}^*(x) = C^* (+iE) e^{+ip \cdot x} = (+iE)\phi^*$$

$$\Rightarrow j^0 = i(\dot{\phi}\phi^* - \dot{\phi}^*\phi) = i\left[(-iE)|\phi|^2 - (iE)|\phi|^2\right] = 2E|\phi|^2 \geq 0$$

But, for $\phi(x) = C e^{ip \cdot x}$

we have $\phi^*(x) = C^* e^{-ip \cdot x}$, $\dot{\phi}(x) = C(iE) e^{ipx} = (iE)\phi(x)$

$$\dot{\phi}^*(x) = C^*(-iE) e^{-ip \cdot x} = (-iE)\phi^*$$

$$\Rightarrow j^0 = i(\dot{\phi}\phi^* - \dot{\phi}^*\phi) = i\left[(iE)|\phi|^2 + iE|\phi|^2\right] = -2E|\phi|^2 \leq 0.$$

In the non-relativistic limit, $E \simeq m$

$\Rightarrow$ for $\phi(x) = C e^{-ip \cdot x}$, $j^0 \simeq 2m|\phi|^2$, so that up to the same constant factor $\frac{1}{2m}$ as for $\vec{j}$, j^0 is analogue to the ρ in quantum mechanics.

However, for $\phi(x) = Ce^{ip\cdot x}$, $j^0 \simeq -2m|\phi|^2$, the sign is opposite.

So, we cannot interpret j^0 as the probability density, which should be ≥ 0. It should be interpreted as a charge density, and it can be checked that $\vec{j}$ also take opposite signs for the two solutions:

$$\text{for } \phi(x) = Ce^{-ip\cdot x} = Ce^{-iEt+i\vec{p}\cdot\vec{x}} \Rightarrow \vec{\nabla}\phi(x) = i\vec{p}\,\phi(x)$$

$$\phi^*(x) = C^* e^{ipx} \quad , \vec{\nabla}\phi^*(x) = (-i\vec{p}\,)\,\phi^*(x)$$

$$\Rightarrow \vec{j} = -i(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*) = -i[(i\vec{p}\,)|\phi|^2 + (i\vec{p}\,)|\phi|^2] = 2\vec{p}\,|\phi|^2$$

$$\text{while for } \phi(x) = Ce^{ip\cdot x} = Ce^{iEt-i\vec{p}\cdot\vec{x}} \Rightarrow \vec{\nabla}\phi(x) = (-i\vec{p}\,)\,\phi(x)$$

$$\phi^*(x) = C^* e^{-ip\cdot x}, \quad \vec{\nabla}\phi^*(x) = (i\vec{p}\,)\,\phi^*(x)$$

$$\Rightarrow \vec{j} = -i(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*) = -i[(-i\vec{p}\,)|\phi|^2 - (i\vec{p}\,)|\phi|^2] = -2\vec{p}\,|\phi|^2$$

So, for both cases, j^0 and $\vec{j}$ form a four-vector.

$$j^\mu = 2p^\mu|\phi|^2, \text{ for } \phi = Ce^{-ipx}$$

$$j^\mu = -2p^\mu|\phi|^2, \text{ for } \phi = Ce^{ipx}$$

This also explains why we choose e^{ipx} and e^{-ipx} as the building blocks for the general solution $\phi(x) = \int_{-\infty}^{+\infty} C(E)[g(\vec{p}\,)\,e^{-ip\cdot x} + h(\vec{p}\,)\,e^{ip\cdot x}]\,d^3\vec{p}$.
If we choose the building blocks as $\phi = Ce^{-iEt-i\vec{p}\cdot\vec{x}}$, then $j^0 = 2E|\phi|^2$ and $\vec{j} = -2\vec{p}\,|\phi|^2$, then $j^\mu \neq 2p^\mu|\phi|^2$;
for $\phi = Ce^{+iEt+i\vec{p}\cdot\vec{x}}$, $\Rightarrow j^0 = -2E|\phi|^2$, $\vec{j} = 2\vec{p}\,|\phi|^2 \Rightarrow j^\mu \neq -2p^\mu|\phi|^2$

Quantization of a scalar field.

The Hamiltonian for a generic field was obtained before

$$H = \int d^3x \, (\pi \partial_0 \varphi - \mathcal{L}) \equiv \int d^3x \, \mathcal{H}$$

where $\pi = \dfrac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial t)}$

note that $\pi \partial_0 \varphi$ should be understood as $\sum_i (\pi_i \dfrac{\partial \varphi_i}{\partial t})$ if there are more than one field

① For a free real scalar field,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

$$\Rightarrow \pi = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)} = \partial_0 \phi = \dot{\phi}$$

$$\Rightarrow \mathcal{H} = \dot{\phi}^2 - \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2\right)$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2$$

② For a free complex scalar field,

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*$$

$$\Rightarrow \pi = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)} = \partial_0 \phi^* = \dot{\phi}^*$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial\phi^*/\partial t)} = \partial_0 \phi = \dot{\phi}$$

$$\Rightarrow \mathcal{H} = \dot{\phi}^* \dot{\phi} + \dot{\phi}\dot{\phi}^* - (\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi\phi^*)$$

$$= \dot{\phi}\dot{\phi}^* + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi^*) + m^2 \phi\phi^*$$

The $\underset{\delta x^\mu = 0}{\underline{\text{internal}}}$ $\underset{\phi'(x) = e^{-i\alpha}\phi(x),\ \phi'^*(x) = e^{i\alpha}\phi^*(x)}{\underline{\text{transformation}}}$ of free complex scalar field gives

$$Q = \int d^3x \, j^0 = i \int d^3x \, (\dot{\phi}\phi^* - \dot{\phi}^*\phi)$$

where $j^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(-i\phi) + \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} i\phi^*$.

Considering that we need to describe creating & destroying particles, and considering that the building block e^{ipx} does not give a non-negative j^0 which can be analogue to ρ in quantum mechanics, we re-interpret $\phi(x)$ as a quantum operator rather than a wave function (so that the Klein-Gordon equation becomes an equation for the field operator)

$$\phi(x) = \int_{-\infty}^{+\infty} C(E)\left[a_{\vec{p}}\, e^{-ip\cdot x} + b_{\vec{p}}^{+}\, e^{ip\cdot x}\right] d^3\vec{p}$$

and $$\phi^{+}(x) = \int_{-\infty}^{+\infty} C^{*}(E)\left[a_{\vec{p}}^{+}\, e^{ip\cdot x} + b_{\vec{p}}\, e^{-ip\cdot x}\right] d^3\vec{p}$$

For real scalar field, $\phi(x) = \phi^{+}(x)$

$$\Rightarrow \phi(x) = \int_{-\infty}^{+\infty} \underbrace{C(E)}_{\text{real}} \left[a_{\vec{p}}\, e^{-ip\cdot x} + a_{\vec{p}}^{+}\, e^{ip\cdot x}\right] d^3\vec{p}$$

The $a_{\vec{p}}$ & $b_{\vec{p}}$ are annihilation operators, and the $a_{\vec{p}}^{+}$ & $b_{\vec{p}}^{+}$ are creation operators.

Let's first consider the real scalar field.

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)} = \dot{\phi}$$

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

when interpret the field as operator, we expect the following commutation relations: → so, it's a postulate, need to be verified by experiments.

$$[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}')$$

$$[\phi(\vec{x},t), \phi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

(Note, recall that in quantum mechanics, $[X_i, P_j] = i\delta_{ij}$ $(i,j=1,2,3)$, $[X_i, X_j] = [P_i, P_j] = 0$

The Euler-Lagrangian equation for classical mechanics is $\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$, and the

one for field theory is $\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0$. The canonical momentum in classical mechanics is $p = \frac{\partial L}{\partial \dot{q}}$, while the one for field theory is $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$.
Therefore, when quantize the field, we can expect the above commutation relations.

With these commutation relations, let's derive the commutation relations for $a_{\vec{p}}$ and $a_{\vec{p}}^{+}$.

From $\phi(x) = \int_{-\infty}^{+\infty} C(E) \left[a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^{+} e^{ip \cdot x} \right] d^3\vec{p}$

$$\Rightarrow \dot{\phi}(x) = \int_{-\infty}^{+\infty} C(E)(-iE) \left[a_{\vec{p}} e^{-ip \cdot x} - a_{\vec{p}}^{+} e^{ip \cdot x} \right] d^3\vec{p}$$

using $\frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} e^{i(\vec{p}-\vec{p}') \cdot \vec{x}} d^3\vec{x} = \delta^3(\vec{p}-\vec{p}')$

$$\Rightarrow \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \phi(x) e^{-i\vec{p}' \cdot \vec{x}} d^3\vec{x} = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} C(E) \left[a_{\vec{p}} e^{-iEt + i\vec{p} \cdot \vec{x}} + a_{\vec{p}}^{+} e^{iEt - i\vec{p} \cdot \vec{x}} \right] e^{-i\vec{p}' \cdot \vec{x}'} d^3\vec{p} \, d^3\vec{x}$$

$$\Rightarrow \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \phi(x) e^{-i\vec{p}' \cdot \vec{x}} d^3\vec{x} = \int_{-\infty}^{+\infty} C(E) \left[a_{\vec{p}} e^{-iEt} \delta^3(\vec{p}-\vec{p}') + a_{\vec{p}}^{+} e^{iEt} \delta^3(\vec{p}+\vec{p}') \right] d^3\vec{p}$$

$$= C(E') \left[a_{\vec{p}'} e^{-iE't} + a_{-\vec{p}'}^{+} e^{iE't} \right]$$

$$\Rightarrow \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \phi(x) e^{iEt - i\vec{p} \cdot \vec{x}} = C(E) \left(a_{\vec{p}} + a_{-\vec{p}}^{+} e^{2iEt} \right)$$

while $\frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} e^{-i\vec{p}' \cdot \vec{x}} \dot{\phi}(x) d^3\vec{x} = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} C(E)(-iE) \left[a_{\vec{p}} e^{-ip \cdot x} - a_{\vec{p}}^{+} e^{ip \cdot x} \right] e^{-i\vec{p}' \cdot \vec{x}} d^3\vec{p} \, d^3\vec{x}$

$$\Rightarrow \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} e^{-i\vec{p}' \cdot \vec{x}} \dot{\phi}(x) d^3\vec{x} = \int_{-\infty}^{+\infty} C(E)(-iE) \left[a_{\vec{p}} \delta^3(\vec{p}-\vec{p}') e^{-iEt} - a_{\vec{p}}^{+} \delta^3(\vec{p}+\vec{p}') e^{iEt} \right] d^3\vec{p}$$

$$= C(E')(-iE') \left(a_{\vec{p}'} e^{-iE't} - a_{-\vec{p}'}^{+} e^{iE't} \right)$$

$$\Rightarrow \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} e^{iEt - i\vec{p} \cdot \vec{x}} \dot{\phi}(x) d^3x = C(E)(-iE) \left(a_{\vec{p}} - a_{-\vec{p}}^{+} e^{2iEt} \right)$$

$\Rightarrow$

$$a_{\vec{p}} = \frac{1}{2}\left[\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\, e^{ip\cdot x}\frac{d^3\vec{x}}{C(E)} + \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\dot{\phi}(x)\, e^{ip\cdot x}\frac{d^3\vec{x}}{C(E)(-iE)}\right]$$

since $e^{ip\cdot x} = \left[\frac{\partial}{\partial t}\left(e^{ip\cdot x}\right)\right]\frac{1}{iE}$

$$\Rightarrow a_{\vec{p}} = \frac{1}{2}\left[\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{d^3\vec{x}}{C(E)(iE)}\left[\phi(x)\left(\frac{\partial}{\partial t}e^{ipx}\right) - \left(\frac{\partial}{\partial t}\phi(x)\right)e^{ip\cdot x}\right]\right]$$

Similarly, start from

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\, e^{i\vec{p}'\cdot\vec{x}}\, d^3\vec{x} = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C(E)\left[a_{\vec{p}}\, e^{-ip\cdot x} + a_{\vec{p}}^{+} e^{ip\cdot x}\right] e^{i\vec{p}'\cdot x}\, d^3\vec{x}\, d^3\vec{p}$$

$$= \int_{-\infty}^{+\infty} C(E)\left[a_{\vec{p}}\, e^{-iEt}\,\delta^3(\vec{p}'+\vec{p}) + a_{\vec{p}}^{+} e^{iEt}\,\delta^3(\vec{p}'-\vec{p})\right] d^3\vec{p}$$

$$= C(E')\left[a_{-\vec{p}'}\, e^{-iE't} + a_{\vec{p}'}^{+} e^{iE't}\right]$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\, e^{-iE't + i\vec{p}'\cdot\vec{x}}\, d^3\vec{x} = C(E)\left[a_{-\vec{p}'}\, e^{-2iE't} + a_{\vec{p}'}^{+}\right]$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\, e^{-ip\cdot x}\, d^3\vec{x} = C(E)\left[a_{-\vec{p}}\, e^{-2iEt} + a_{\vec{p}}^{+}\right]$$

and

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\dot{\phi}(x)\, e^{i\vec{p}'\cdot\vec{x}}\, d^3\vec{x} = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C(E)(-iE)\left[a_{\vec{p}}\, e^{-ip\cdot x} - a_{\vec{p}}^{+} e^{ip\cdot x}\right] e^{i\vec{p}'\cdot\vec{x}}\, d^3\vec{p}\, d^3\vec{x}$$

$$= \int_{-\infty}^{+\infty} C(E)(-iE)\left[a_{\vec{p}}\,\delta^3(\vec{p}'+\vec{p})\, e^{-iEt} - a_{\vec{p}}^{+}\,\delta^3(\vec{p}'-\vec{p}')\, e^{iEt}\right]$$

$$= C(E')(-iE')\left[a_{\vec{p}'}\, e^{-iE't} - a_{\vec{p}'}^{+} e^{iE't}\right]$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\dot{\phi}(x)\, e^{-ip\cdot x}\, d^3\vec{x} = C(E)(-iE)\left[a_{-\vec{p}}\, e^{-2iEt} - a_{\vec{p}}^{+}\right]$$

$$\Rightarrow a_{\vec{p}}^{+} = \frac{1}{2}\left\{\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\, e^{-ip\cdot x}\frac{d^3\vec{x}}{C(E)} + \frac{1}{(iE)}\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\dot{\phi}(x)\, e^{-ip\cdot x}\frac{d^3\vec{x}}{C(E)}\right\}$$

using $e^{-ip\cdot x} = \left[\frac{\partial}{\partial t}\left(e^{-ip\cdot x}\right)\right]\frac{1}{(-iE)}$

$$\Rightarrow a_{\vec{p}}^{+} = \frac{1}{2}\left[\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{d^3\vec{x}}{C(E)(iE)}\left[\left(\frac{\partial}{\partial t}\phi\right)e^{-ipx} - \phi\frac{\partial}{\partial t}\left(e^{-ip\cdot x}\right)\right]\right.$$

$$\Rightarrow [a_{\vec{p}}, a_{\vec{p}'}^{+}] = \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)\,C(E')\,(iE)(iE')}\left[\int_{-\infty}^{+\infty}d^3\vec{x}\left(\phi(x)\frac{\partial}{\partial t}\left(e^{ip\cdot x}\right) - \left(\frac{\partial}{\partial t}\phi(x)\right)e^{ipx}\right)\right.,$$
$$\left.\int_{-\infty}^{+\infty}d^3\vec{x}\left(\left(\frac{\partial}{\partial t}\phi(x)\right)e^{-ip'\cdot x} - \phi(x)\frac{\partial}{\partial t}\left(e^{-ip'x}\right)\right)\right]$$

$$= \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)\,C(E')(iE)(iE')}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^3\vec{x}\,d^3\vec{x}'\left\{\left(\phi(\vec{x},t)\left(iE\,e^{iEt-i\vec{p}\cdot\vec{x}}\right) - \pi(\vec{x},t)\,e^{iEt-i\vec{p}\cdot\vec{x}}\right)\right.$$
$$\cdot\left(\pi(\vec{x}',t)\,e^{-iE't+i\vec{p}'\cdot\vec{x}'} + iE'\phi(\vec{x}',t)\,e^{-iE't+i\vec{p}'\cdot\vec{x}'}\right)$$
$$-\left(\pi(\vec{x}',t)\,e^{-iE't+i\vec{p}'\cdot\vec{x}'} + iE'\phi(\vec{x}',t)\,e^{-iE't+i\vec{p}'\cdot\vec{x}'}\right)$$
$$\left.\left(\phi(\vec{x},t)\left(iE\,e^{iEt-i\vec{p}\cdot\vec{x}}\right) - \pi(\vec{x},t)\,e^{iEt-i\vec{p}\cdot\vec{x}}\right)\right\}$$

$$= \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)\,C(E')\,(iE)\,(iE')}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^3\vec{x}\,d^3\vec{x}'\,e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{x}'}\,e^{iEt-iE't}$$
$$\times\left\{\begin{array}{l} iE\left(\phi(\vec{x},t)\,\pi(\vec{x}',t) - \pi(\vec{x}',t)\phi(\vec{x},t)\right) \\ -iE'\left(\pi(\vec{x},t)\,\phi(\vec{x}',t) - \phi(\vec{x}',t)\,\pi(\vec{x},t)\right)\end{array}\right\}$$

$$= \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)C(E')(iE)(iE')}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^3\vec{x}\,d^3\vec{x}'\,e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{x}'}\,e^{iEt-iE't}$$
$$\times\left\{iE\left[\phi(\vec{x},t),\pi(\vec{x}',t)\right] + iE'\left[\phi(\vec{x}',t),\pi(\vec{x},t)\right]\right\}$$

$$= \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)\,C(E')(iE)(iE')}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^3\vec{x}\,d^3\vec{x}'\,e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{x}'}\,e^{iEt-iE't}$$
$$\times\left\{iE\cdot i\,\delta^3(\vec{x}-\vec{x}') + iE'\underbrace{\delta^3(\vec{x}'-\vec{x})}_{\delta^3(\vec{x}-\vec{x}')}\right\}$$

$$= \left(\frac{1}{2(2\pi)^3}\right)^2 \frac{1}{C(E)\,C(E')\,(iE)(iE')} \int_{-\infty}^{+\infty} d^3\vec{x}\, e^{-i(\vec{p}-\vec{p}\,')\cdot\vec{x}}\, e^{iEt-iE't}\,(i^2E+i^2E')$$

$$= \left(\frac{1}{2(2\pi)^3}\right)^2 \frac{1}{C(E)\,C(E')\,iE(iE')} (2\pi)^3 \delta^3(\vec{p}-\vec{p}\,')\, e^{iEt-iE't}\,(i^2E+i^2E')$$

$$\underset{\uparrow}{=} \left(\frac{1}{2}\right)^2 \frac{1}{(2\pi)^3} \left(\frac{1}{C(E)}\right)^2 \frac{2E}{E^2}\, \delta^3(\vec{p}-\vec{p}\,') = \frac{1}{(2\pi)^3\, 2E} \left(\frac{1}{C(E)}\right)^2 \delta^3(\vec{p}-\vec{p}\,')$$

since $E=(\vec{p}^{\,2}+m^2)^{\frac{1}{2}}$, $E'=(\vec{p}\,'^2+m^2)^{\frac{1}{2}}$

The choice of the normalization factor $C(E)$ are different in the literature:

① if use $C(E)=\left[\frac{1}{(2\pi)^3\,2E}\right]^{\frac{1}{2}}$, then → e.g. 王正行《简明量子场论》

$$\phi(x)=\int_{-\infty}^{+\infty}\left(\frac{1}{(2\pi)^3\,2E}\right)^{\frac{1}{2}}\left(a_{\vec{p}}\,e^{-ip\cdot x}+a^{+}_{\vec{p}}\,e^{ip\cdot x}\right)$$

and $[a_{\vec{p}},\, a^{+}_{\vec{p}\,'}]=\delta^3(\vec{p}-\vec{p}\,')$

② if use $C(E)=\frac{1}{(2\pi)^3}\left(\frac{1}{2E}\right)^{\frac{1}{2}}$, then → e.g. Peskin & Schroeder "An Introduction to quantum field theory".

$$\phi(x)=\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^3}\frac{1}{\sqrt{2E}}\left(a_{\vec{p}}\,e^{-ip\cdot x}+a^{+}_{\vec{p}}\,e^{ip\cdot x}\right)$$

and $[a_{\vec{p}},\, a^{+}_{\vec{p}\,'}]=(2\pi)^3\delta^3(\vec{p}-\vec{p}\,')$

③ if use $C(E)=\frac{1}{(2\pi)^3\,2E}$, then → e.g. Ryder "Quantum field theory"

$$\phi(x)=\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^3\,2E}\left(a_{\vec{p}}\,e^{-ip\cdot x}+a^{+}_{\vec{p}}\,e^{ip\cdot x}\right)$$

and $[a_{\vec{p}},\, a^{+}_{\vec{p}\,'}]=(2\pi)^3\, \underset{\uparrow}{2E_{\vec{p}}}\, \delta^3(\vec{p}-\vec{p}\,')$

put the label $\vec{p}$ in E to make it clear that E depends on $\vec{p}$ through $E=(\vec{p}^{\,2}+m^2)^{\frac{1}{2}}$

It doesn't matter which one you prefer to choose. However, once you choose a normalization for the field operator $\phi(x)$, the state vector on which the field operator acts on should be defined accordingly, since the observable, which is the sandwich of the field operator(s) between the state vectors, should not depend on the normalization factor you choose.

For $[a_{\vec{p}}, a_{\vec{p}'}]$ and $[a^{+}_{\vec{p}}, a^{+}_{\vec{p}'}]$

$$[a_{\vec{p}}, a_{\vec{p}'}] = \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2 \frac{1}{C(E)\,C(E')\,(iE)(iE')}$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{x}' \Big[\phi(\vec{x},t)\, iE\, e^{iEt - i\vec{p}\cdot\vec{x}} - \pi(\vec{x},t)\, e^{iEt - i\vec{p}\cdot\vec{x}}\,,\ \phi(\vec{x}',t)\, iE'\, e^{iE't - i\vec{p}'\cdot\vec{x}'} - \pi(\vec{x}',t)\, e^{iE't - i\vec{p}'\cdot\vec{x}'}\Big]$$

where

$$\iint = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{iEt - i\vec{p}\cdot\vec{x} + iE't - i\vec{p}'\cdot\vec{x}'}$$

$$\times \Big\{(iE)(iE')\underbrace{[\phi(\vec{x},t), \phi(\vec{x}',t)]}_{0} + \underbrace{[\pi(\vec{x},t), \pi(\vec{x}',t)]}_{0} - iE'[\pi(\vec{x},t), \phi(\vec{x}',t)] - iE[\phi(\vec{x},t), \pi(\vec{x}',t)]\Big\}\, d^3\vec{x}\, d^3\vec{x}'$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{iEt - i\vec{p}\cdot\vec{x} + iE't - i\vec{p}'\cdot\vec{x}'} \Big\{-iE'(-i)\,\delta^3(\vec{x}-\vec{x}') - iE\, i\,\delta^3(\vec{x}-\vec{x}')\Big\}\, d^3\vec{x}\, d^3\vec{x}'$$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\; e^{iEt - i\vec{p}\cdot\vec{x} + iE't - i\vec{p}'\cdot\vec{x}} \Big\{(-i)^2 E' - i^2 E\Big\}$$

$$= (2\pi)^3\, \delta^3(\vec{p} + \vec{p}')\, e^{iEt + iE't}\, (-E' + E)$$

$$= \begin{cases} 0 & \text{when } \vec{p}' \neq -\vec{p} \\ 0 & \text{since when } \vec{p}' = -\vec{p} \Rightarrow E' = E \end{cases}$$

So $[a_{\vec{p}}, a_{\vec{p}'}] = 0$

$$[a^{+}_{\vec{p}}, a^{+}_{\vec{p}'}] = \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2 \frac{1}{C(E)\,C(E')\,(iE)(iE')}$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{x}' \Big[\pi(\vec{x},t)\, e^{-iEt + i\vec{p}\cdot\vec{x}} + iE\,\phi(\vec{x},t)\, e^{-iEt + i\vec{p}\cdot\vec{x}}\,,\ \pi(\vec{x}',t)\, e^{-iE't + i\vec{p}'\cdot\vec{x}'} + iE'\,\phi(\vec{x}',t)\, e^{-iE't + i\vec{p}'\cdot\vec{x}'}\Big]$$

where $\iint = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{x}'\, e^{-iEt+i\vec{p}\cdot\vec{x}-iE't+i\vec{p}'\cdot\vec{x}'}$

$$\times \Big\{ [\pi(\vec{x},t), \pi(\vec{x}',t)] + (iE)(iE')[\phi(\vec{x},t), \phi(\vec{x}',t)] + iE'[\pi(\vec{x},t), \phi(\vec{x}',t)] + iE[\phi(\vec{x},t), \pi(\vec{x}',t)] \Big\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{x}'\, e^{-iEt+i\vec{p}\cdot\vec{x}-iE't+i\vec{p}'\cdot\vec{x}'} \left(iE'(-i)\delta^3(\vec{x}-\vec{x}') + iE\, i\delta^3(\vec{x}-\vec{x}') \right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\, e^{-iEt+i\vec{p}\cdot\vec{x}-iE't+i\vec{p}'\cdot\vec{x}} (E'-E)$$

$$= (2\pi)^3 \delta^3(\vec{p}+\vec{p}')\, e^{-iEt-iE't} (E'-E)$$

$$= 0$$

So $[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}] = 0$

or, directly from $[a_{\vec{p}}, a_{\vec{p}'}] = 0$, do hermitian conjugate,

$$\Rightarrow (a_{\vec{p}} a_{\vec{p}'} - a_{\vec{p}'} a_{\vec{p}})^\dagger = 0$$

$$\Rightarrow a^\dagger_{\vec{p}'} a^\dagger_{\vec{p}} - a^\dagger_{\vec{p}} a^\dagger_{\vec{p}'} = 0$$

$$\Rightarrow [a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}] = 0$$