Homework 1
Due date: 2018.10.15

Note: in problem 1, 2 and 3, please round your results to two significant figures, and don't worry about uncertainties.

Problem 1. The critical density of the Universe is

$$\rho_c = \frac{3H_0^2}{8\pi G_N}\,,$$

where H_0 is the present day Hubble expansion rate, given as $H_0 = 100\,h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, where h is a dimensionless number. In the following questions, use $G_N = 6.674\times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, $1\,\mathrm{Mpc} = 3.086\times 10^{22}\,\mathrm{m}$, the Solar mass $M_\odot = 1.988\times 10^{30}\,\mathrm{kg}$, and take $h = 0.7$.

1) What is the value of $1/H_0$ in Gyr? (1 Gyr = 10^9 year) [1 point]

2) What is the value of ρ_c in g/cm^3? [1 point]

3) What is the value of ρ_c in $M_\odot/\mathrm{Mpc}^3$? [1 point]

4) In SI units, the Planck density (i.e., mass per unit volume), ρ_{Pl}, can be constructed as $\hbar^{n_1}\,c^{n_2}\,G_N^{n_3}$. Find n_1, n_2 and n_3. [1 point]

5) In $\hbar = c = 1$ units, what are the values of ρ_{Pl} and ρ_c in GeV^4? [1 point] (Please note that we have given in class the relations between m & s & kg in SI units and the power of MeV or GeV in $\hbar = c = 1$ units. However, at that time we only kept one or two significant figures. Therefore, you might want to re-calculate them keeping more significant figures in order to achieve the correct two significant figure results of ρ_{Pl} and ρ_c.)

Problem 2. In $\hbar = c = 1$ units, the lifetime of a positronium in the ground state is given as

$$\tau = \frac{2}{m_e\alpha^5}\,,$$

where m_e is the mass of the electron and α is the fine structure constant.

1) In $\hbar = c = 1$ units, what is the value of τ in GeV^{-1}? [1 point]

2) In SI units, what is the value of τ in second? [1 point]

Problem 3. (in $\hbar = c = 1$ units) In a version of large extra spatial dimension models, the common size of the extra spatial dimensions, R, is related to the reduced Planck mass, $M_P \equiv 1/\sqrt{8\pi G_N}$, and the electroweak scale, $m_{EW} \equiv 1\,\mathrm{TeV}$, as

$$M_P^2 = R^n m_{EW}^{2+n}\,,$$

where n is the number of extra spatial dimensions. What are the values of R in cm for n = 1, 2, 3 and 6? [2 point]

Problem 4. Given two four-vectors, $a^\mu = (2, 0, 1, 8)$ and $b^\mu = (0, 9, 2, 7)$, find: a_μ, b_μ, $\vec{a} \cdot \vec{a}$, $\vec{a} \cdot \vec{b}$, a^2, $a \cdot b$, $(a + b)^2$. [2 points]

Note: please use the convention $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ & $g_{\mu\nu} = 0$ (when $\mu \neq \nu$) for the calculations. A three-vector, e.g., $\vec{a}$, is the one appearing in $a^\mu = (a^0, \vec{a})$. A four-vector dot product, e.g., $a \cdot b$, is defined as $a \cdot b \equiv a^\mu b_\mu$. In particular, $a^2 \equiv a \cdot a$.

Problem 5. A second-rank tensor is called symmetric if it is unchanged when you switch the indices ($s^{\mu\nu} = s^{\nu\mu}$); it is called antisymmetric if it changes sign ($a^{\mu\nu} = -a^{\nu\mu}$). Assume that the tensors are defined in the four dimensional Minkowski spacetime used in special relativity, answer the following questions.

1) How many independent elements are there in a symmetric tensor? (Since $s^{12} = s^{21}$, these would count as only one independent element.) [1 point]

2) How many independent elements are there in an antisymmetric tensor? [1 point]

3) Show that symmetry is preserved by Lorentz transformations. That is, if $s^{\mu\nu}$ is symmetric, so too is $s'^{\mu\nu}$; if $a^{\mu\nu}$ is antisymmetric, so too is $a'^{\mu\nu}$. (Hint: use the definition of Lorentz transformations between x and x' frames.) [1 point]

4) If $s^{\mu\nu}$ is symmetric, show that $s_{\mu\nu}$ is also symmetric. If $a^{\mu\nu}$ is antisymmetric, show that $a_{\mu\nu}$ is also antisymmetric. (Hint: use the metric tensor to lower the indices.) [1 point]

5) If $s^{\mu\nu}$ is symmetric and $a^{\mu\nu}$ is antisymmetric, show that $s^{\mu\nu} a_{\mu\nu} = 0$. [1 point]

6) Any second-rank tensor ($t^{\mu\nu}$), can be written as the sum of a symmetric part ($s^{\mu\nu}$) and an antisymmetric part ($a^{\mu\nu}$): $t^{\mu\nu} = s^{\mu\nu} + a^{\mu\nu}$. Construct $s^{\mu\nu}$ and $a^{\mu\nu}$ in terms of $t^{\mu\nu}$ and $t^{\nu\mu}$. [1 point]

Problem 6. Particle A, at rest, decays into particles B and C (A $\to$ B + C). Do the following calculations in particle A's rest frame, and give your results in terms of the masses m_A, m_B and m_C.

1) Find the energy of the outgoing particles E_B and E_C. [1 point]

2) Find the magnitudes of the outgoing momenta $|\vec{p}_B|$ and $|\vec{p}_C|$. [1 point]

Problem 7. In a two-body scattering event, A + B $\to$ C + D, it is convenient to introduce the *Mandelstam variables* $s \equiv (p_A + p_B)^2$, $t \equiv (p_A - p_C)^2$ and $u \equiv (p_A - p_D)^2$ (note that the square of a four-momentum should be always understood as, for example, $a^2 \equiv a \cdot a \equiv a^\mu a_\mu$). **The virtue of the Mandelstam variables is that they are Lorentz invariants, with the same value in any inertial system.**

1) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$. [1 point]

2) Find the energy of A in the center-of-momentum (CM) frame, in terms of s, m_A and m_B. [2 points]

3) Find the energy of A in B's rest frame, in terms of s, m_A and m_B. [2 points]

Problem 8. In the perfect vacuum, an electron and a positron can annihilate into two or more photons ($e^+ + e^- \to \mathrm{n}\,\gamma$, with n ≥ 2). From the view of conservation of energy and momentum, please explain why the single photon creation process cannot happen. [1 point]