Complex scalar field quantization.

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi\phi^*$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)} = \partial_0 \phi^* = \dot{\phi}^*$$

$$\pi^* = \frac{\partial \mathcal{L}}{\partial(\partial\phi^*/\partial t)} = \partial_0 \phi = \dot{\phi}$$

($\mathcal{H} = \sum_i (\pi_i \frac{\partial \varphi}{\partial t}) - \mathcal{L}$ for generic fields)

$$\mathcal{H} = \dot{\phi}^* \dot{\phi} + \dot{\phi}\dot{\phi}^* - \mathcal{L} = \dot{\phi}\dot{\phi}^* + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi^*) + m^2\phi\phi^*$$

Internal transformation of the complex scalar field:

$\delta x^\mu = 0$, $\phi'(x) = e^{-i\alpha}\phi(x)$, $\phi^{*\prime}(x) = e^{i\alpha}\phi^*(x)$

Noether theorem gives

conserved charge $Q = \int_{-\infty}^{+\infty} d^3\vec{x}\, j^0 = i\int_{-\infty}^{+\infty} d^3\vec{x}\,(\dot{\phi}\phi^* - \dot{\phi}^*\phi)$

The above expressions are for classical field theory.

Now let's quantize it.

$$\phi(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}})\left[a_{\vec{p}}\, e^{-ip\cdot x} + b_{\vec{p}}^{+}\, e^{ip\cdot x}\right],$$

$C(E_{\vec{p}})$ is real

we can put the phase factor in $a_{\vec{p}}$ and $b_{\vec{p}}^+$ to make $C(E_{\vec{p}})$ real.

Recall that the solution to the free Klein-Gordon equation (same for real and complex scalar field)

$$(\Box + m^2)\phi = 0.$$

are

$$\phi \sim e^{\pm ip\cdot x}, \text{ where } p^2 = E^2 - \vec{p}^{\,2} = m^2.$$

and we just use $a_{\vec{p}}$ and $b_{\vec{p}}^+$ as the expansion coefficients for the solutions. We'll see the physical reason to use $a_{\vec{p}}$ & $b_{\vec{p}}^+$ later.

$$\phi^+(x) = \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}})\left[a_{\vec{p}}^+\, e^{ip\cdot x} + b_{\vec{p}}\, e^{-ip\cdot x}\right]$$

Still starts from the postulations of Equal-Time Commutation Relations.
(ETCR)

$$[\phi(t,\vec{x}),\ \pi(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')$$

$$[\phi(t,\vec{x}),\ \phi(t,\vec{x}')] = [\pi(t,\vec{x}),\ \pi(t,\vec{x}')] = 0$$

and their Hermitian conjugate

(note that $[\phi,\pi]^+ = (\phi\pi)^+ - (\pi\phi)^+ = \pi^+\phi^+ - \phi^+\pi^+ = [\pi^+,\phi^+]$)

$$[\phi^+(t,\vec{x}),\ \pi^+(t,\vec{x}')] = i\delta^3(\vec{x}-\vec{x}')$$

$$[\phi^+(t,\vec{x}),\ \phi^+(t,\vec{x}')] = [\pi^+(t,\vec{x}),\ \pi^+(t,\vec{x}')] = 0$$

also $[\phi(t,\vec{x}),\ \phi^+(t,\vec{x}')] = [\pi(t,\vec{x}),\ \pi^+(t,\vec{x}')] = [\phi(t,\vec{x}),\pi^+(t,\vec{x}')]$
$= [\pi(t,\vec{x}),\ \phi^+(t,\vec{x}')] = 0$

$$\Rightarrow\quad [a_{\vec{p}},\ a^+_{\vec{p}'}] = \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[\frac{1}{C(E_{\vec{p}})}\right]^2\delta^3(\vec{p}-\vec{p}')$$

$$[b_{\vec{p}},\ b^+_{\vec{p}'}] = \frac{1}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[\frac{1}{C(E_{\vec{p}})}\right]^2\delta^3(\vec{p}-\vec{p}')$$

$$[a_{\vec{p}},\ a_{\vec{p}'}] = [b_{\vec{p}},\ b_{\vec{p}'}] = [a_{\vec{p}},\ b^+_{\vec{p}'}] = [b_{\vec{p}},\ a^+_{\vec{p}'}]$$
$$= [a_{\vec{p}},\ b_{\vec{p}'}] = [a^+_{\vec{p}},\ b^+_{\vec{p}'}] = [a^+_{\vec{p}},\ a^+_{\vec{p}'}] = [b^+_{\vec{p}},\ b^+_{\vec{p}'}] = 0$$

(homework)

$$H = \int_{-\infty}^{+\infty} d^3\vec{x}\ \mathcal{H}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\left(\dot{\phi}\dot{\phi}^+ + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi^+) + m^2\phi\phi^+\right)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\left[(2\pi)^3(C(E_{\vec{p}}))^2\, 2E_{\vec{p}}\right](a_{\vec{p}}a^+_{\vec{p}} + b^+_{\vec{p}}b_{\vec{p}})E_{\vec{p}}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\left[(2\pi)^3(C(E_{\vec{p}}))^2\, 2E_{\vec{p}}\right](a^+_{\vec{p}}a_{\vec{p}} + b^+_{\vec{p}}b_{\vec{p}})E_{\vec{p}}$$
$$+ \left(\int_{-\infty}^{+\infty}\frac{d^3\vec{p}}{(2\pi)^3}E_{\vec{p}}\right)\delta^3(0)\,(2\pi)^3$$

$$:H: = \int_{-\infty}^{+\infty} d^3\vec{p}\left[(2\pi)^3(C(E_{\vec{p}}))^2\, 2E_{\vec{p}}\right](a^+_{\vec{p}}a_{\vec{p}} + b^+_{\vec{p}}b_{\vec{p}})E_{\vec{p}}$$

note that $a_{\vec{p}} a^{\dagger}_{\vec{p}'} + b^{\dagger}_{\vec{p}'} b_{\vec{p}}$

$$= a^{\dagger}_{\vec{p}'} a_{\vec{p}} + \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[\frac{1}{C(E_{\vec{p}})}\right]^2 \delta^3(\vec{p}-\vec{p}')$$
$$+ b_{\vec{p}} b^{\dagger}_{\vec{p}'} - \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[\frac{1}{C(E_{\vec{p}})}\right]^2 \delta^3(\vec{p}-\vec{p}')$$
$$= a^{\dagger}_{\vec{p}'} a_{\vec{p}} + b_{\vec{p}} b^{\dagger}_{\vec{p}'}$$

$$\Rightarrow \quad a_{\vec{p}} a^{\dagger}_{\vec{p}} + b^{\dagger}_{\vec{p}} b_{\vec{p}} = a^{\dagger}_{\vec{p}} a_{\vec{p}} + b_{\vec{p}} b^{\dagger}_{\vec{p}}$$

From $P_i = \int d^3\vec{x}\, \pi_\ell \partial_i \varphi_\ell$ (φ_ℓ is a generic field)

$$\Rightarrow \quad \hat{\vec{P}} = -\int_{-\infty}^{+\infty} d^3\vec{x} \left(\pi \vec{\nabla}\phi + \pi^{\dagger} \vec{\nabla}\phi^{\dagger}\right)$$
$$= \int_{-\infty}^{+\infty} d^3\vec{p} \left[(2\pi)^3 (C(E_{\vec{p}}))^2\, 2E_{\vec{p}}\right] \left(a^{\dagger}_{\vec{p}} a_{\vec{p}} + b^{\dagger}_{\vec{p}} b_{\vec{p}}\right) \vec{p}$$

Recall for a real scalar field,

$$H = \int_{-\infty}^{+\infty} d^3\vec{p} \left[(2\pi)^3 (C(E_{\vec{p}}))^2\, 2E_{\vec{p}}\right] a^{\dagger}_{\vec{p}} a_{\vec{p}} E_{\vec{p}}$$
$$+ \left(\int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2} E_{\vec{p}}\right) \delta^3(0) (2\pi)^3$$

$$:H: = \int_{-\infty}^{+\infty} d^3\vec{p} \left[(2\pi)^3 (C(E_{\vec{p}}))^2\, 2E_{\vec{p}}\right] a^{\dagger}_{\vec{p}} a_{\vec{p}} E_{\vec{p}}$$

$$\hat{\vec{P}} = \int_{-\infty}^{+\infty} d^3\vec{p} \left[(2\pi)^3 (C(E_{\vec{p}}))^2\, 2E_{\vec{p}}\right] a^{\dagger}_{\vec{p}} a_{\vec{p}} \vec{p}$$

$$\hat{Q} = i\int_{-\infty}^{+\infty} d^3\vec{x}\ (\dot{\phi}\phi^{+} - \dot{\phi}^{+}\phi)$$

$$= i\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\ C(E_{\vec{p}})\, C(E_{\vec{k}})$$

$$\times \left\{ \begin{array}{l} (-iE_{\vec{p}})\ (a_{\vec{p}}\, e^{-ip\cdot x} - b^{+}_{\vec{p}}\, e^{ip\cdot x})\ (a^{+}_{\vec{k}}\, e^{ik\cdot x} + b_{\vec{k}}\, e^{-ik\cdot x}) \\ -(iE_{\vec{p}})\ (a^{+}_{\vec{p}}\, e^{ip\cdot x} - b_{\vec{p}}\, e^{-ip\cdot x})\ (a_{\vec{k}}\, e^{-ik\cdot x} + b^{+}_{\vec{k}}\, e^{ik\cdot x}) \end{array} \right\}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\ C(E_{\vec{p}})\, C(E_{\vec{k}})\, E_{\vec{p}}\ (2\pi)^3$$

$$\times \Big[a_{\vec{p}} a^{+}_{\vec{k}}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k}) + a_{\vec{p}} b_{\vec{k}}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k})$$

$$- b^{+}_{\vec{p}} a^{+}_{\vec{k}}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k}) - b^{+}_{\vec{p}} b_{\vec{k}}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k})$$

$$+ a^{+}_{\vec{p}} a_{\vec{k}}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k}) + a^{+}_{\vec{p}} b^{+}_{\vec{k}}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k})$$

$$- b_{\vec{p}} a_{\vec{k}}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\, \delta^3(\vec{p}+\vec{k}) - b_{\vec{p}} b^{+}_{\vec{k}}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\, \delta^3(\vec{p}-\vec{k}) \Big]$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2\, E_{\vec{p}}\ (2\pi)^3$$

$$\times \Big[a_{\vec{p}} a^{+}_{\vec{p}} + a_{\vec{p}} b_{-\vec{p}}\, e^{-2iE_{\vec{p}}t} - b^{+}_{\vec{p}} a^{+}_{-\vec{p}}\, e^{2iE_{\vec{p}}t} - b^{+}_{\vec{p}} b_{\vec{p}}$$

$$+ a^{+}_{\vec{p}} a_{\vec{p}} + a^{+}_{\vec{p}} b^{+}_{-\vec{p}}\, e^{2iE_{\vec{p}}t} - b_{\vec{p}} a_{-\vec{p}}\, e^{-2iE_{\vec{p}}t} - b_{\vec{p}} b^{+}_{\vec{p}} \Big]$$

Since $\int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2\, E_{\vec{p}}\, (2\pi)^3\, (a_{\vec{p}} b_{-\vec{p}}\, e^{-2iE_{\vec{p}}t} - b^{+}_{\vec{p}} a^{+}_{-\vec{p}}\, e^{2iE_{\vec{p}}t})$

$$\overset{\vec{p}\to -\vec{k}}{=} \int_{-\infty}^{+\infty} d^3\vec{k}\, (C(E_{\vec{k}}))^2\, E_{\vec{k}}\, (2\pi)^3\, (a_{-\vec{k}} b_{\vec{k}}\, e^{-2iE_{\vec{k}}t} - b^{+}_{-\vec{k}} a^{+}_{\vec{k}}\, e^{2iE_{\vec{k}}t})$$

$$\overset{\vec{k}\to \vec{p}}{=} \int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2\, E_{\vec{p}}\, (2\pi)^3\, (a_{-\vec{p}} b_{\vec{p}}\, e^{-2iE_{\vec{p}}t} - b^{+}_{-\vec{p}} a^{+}_{\vec{p}}\, e^{2iE_{\vec{p}}t})$$

$$\overset{[a_{\vec{p}}, b_{\vec{p}'}] = [a^{+}_{\vec{p}}, b^{+}_{\vec{p}'}] = 0}{=} \int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2\, E_{\vec{p}}\, (2\pi)^3\, (b_{\vec{p}} a_{-\vec{p}}\, e^{-2iE_{\vec{p}}t} - a^{+}_{\vec{p}} b^{+}_{-\vec{p}}\, e^{2iE_{\vec{p}}t})$$

then $\hat{Q} = \int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2\, E_{\vec{p}}\, (2\pi)^3\, [a_{\vec{p}} a^{+}_{\vec{p}} - b^{+}_{\vec{p}} b_{\vec{p}} + a^{+}_{\vec{p}} a_{\vec{p}} - b_{\vec{p}} b^{+}_{\vec{p}}]$

Since $$a_{\vec{p}} a_{\vec{p}}^{+} = a_{\vec{p}}^{+} a_{\vec{p}} + \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(0)$$

$$b_{\vec{p}} b_{\vec{p}}^{+} = b_{\vec{p}}^{+} b_{\vec{p}} + \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(0)$$

$$\Rightarrow a_{\vec{p}} a_{\vec{p}}^{+} - b_{\vec{p}} b_{\vec{p}}^{+} = a_{\vec{p}}^{+} a_{\vec{p}} - b_{\vec{p}}^{+} b_{\vec{p}}$$

$$\Rightarrow \hat{Q} = \int_{-\infty}^{+\infty} d^3\vec{p} \left[(2\pi)^3 (C(E_{\vec{p}}))^2 2E_{\vec{p}}\right] \left(a_{\vec{p}}^{+} a_{\vec{p}} - b_{\vec{p}}^{+} b_{\vec{p}}\right)$$

Define
$$\hat{N} = \int_{-\infty}^{+\infty} d^3\vec{p}\,[(2\pi)^3 (C(E_{\vec{p}}))^2\, 2E_{\vec{p}}]\; a^{\dagger}_{\vec{p}} a_{\vec{p}}$$
$$\hat{\bar{N}} \equiv \int_{-\infty}^{+\infty} d^3\vec{p}\,[(2\pi)^3 (C(E_{\vec{p}}))^2\, 2E_{\vec{p}}]\; b^{\dagger}_{\vec{p}} b_{\vec{p}}$$

$$\Rightarrow [\hat{N}, a_{\vec{k}}] = -a_{\vec{k}} \qquad \text{(same as the derivation for real scalar field)}$$
$$[\hat{N}, a^{\dagger}_{\vec{k}}] = a^{\dagger}_{\vec{k}}$$
$$[\hat{N}, b_{\vec{k}}] = [\hat{N}, b^{\dagger}_{\vec{k}}] = 0$$
$$[\hat{\bar{N}}, b_{\vec{k}}] = -b_{\vec{k}}$$
$$[\hat{\bar{N}}, b^{\dagger}_{\vec{k}}] = b^{\dagger}_{\vec{k}}$$
$$[\hat{\bar{N}}, a_{\vec{k}}] = [\hat{\bar{N}}, a^{\dagger}_{\vec{k}}] = 0$$
$$[\hat{\bar{N}}, \hat{N}] = 0$$

Let's define a common eigenstate of $\hat{N}$ and $\hat{\bar{N}}$ as $|s\rangle$, such that
$$\hat{N}|s\rangle = c|s\rangle$$
$$\hat{\bar{N}}|s\rangle = \bar{c}|s\rangle$$
where c & $\bar{c}$ are real numbers (since $\hat{N}$ and $\hat{\bar{N}}$ are Hermitian operators)

$$\Rightarrow \hat{N} a^{\dagger}_{\vec{k}}|s\rangle = (a^{\dagger}_{\vec{k}}\hat{N} + a^{\dagger}_{\vec{k}})|s\rangle = (c+1)\, a^{\dagger}_{\vec{k}}|s\rangle$$
$$\hat{N} a_{\vec{k}}|s\rangle = (a_{\vec{k}}\hat{N} - a_{\vec{k}})|s\rangle = (c-1)\, a_{\vec{k}}|s\rangle$$
$$\hat{\bar{N}} b^{\dagger}_{\vec{k}}|s\rangle = (\bar{c}+1)\, b^{\dagger}_{\vec{k}}|s\rangle$$
$$\hat{\bar{N}} b_{\vec{k}}|s\rangle = (\bar{c}-1)\, b_{\vec{k}}|s\rangle$$
$$\hat{N} b_{\vec{k}}|s\rangle = b_{\vec{k}}\hat{N}|s\rangle = c\, b_{\vec{k}}|s\rangle$$
$$\hat{N} b^{\dagger}_{\vec{k}}|s\rangle = c\, b^{\dagger}_{\vec{k}}|s\rangle$$
$$\hat{\bar{N}} a_{\vec{k}}|s\rangle = \bar{c}\, a_{\vec{k}}|s\rangle$$
$$\hat{\bar{N}} a^{\dagger}_{\vec{k}}|s\rangle = \bar{c}\, a^{\dagger}_{\vec{k}}|s\rangle$$
$$\hat{N}(a^{\dagger}_{\vec{k}_1})^{n_1}(a^{\dagger}_{\vec{k}_2})^{n_2}\cdots(a^{\dagger}_{\vec{k}_\ell})^{n_\ell}|s\rangle = (c+n_1+n_2+\cdots+n_\ell)(a^{\dagger}_{\vec{k}_1})^{n_1}(a^{\dagger}_{\vec{k}_2})^{n_2}\cdots(a^{\dagger}_{\vec{k}_\ell})^{n_\ell}|s\rangle$$
$$\hat{N}(a_{\vec{k}_1})^{n_1}(a_{\vec{k}_2})^{n_2}\cdots(a_{\vec{k}_\ell})^{n_\ell}|s\rangle = (c-n_1-n_2-\cdots-n_\ell)(a_{\vec{k}_1})^{n_1}(a_{\vec{k}_2})^{n_2}\cdots(a_{\vec{k}_\ell})^{n_\ell}|s\rangle$$

$$\hat{\bar{N}}\,(a^{+}_{\vec{k}_1})^{n_1}(a^{+}_{\vec{k}_2})^{n_2}\cdots(a^{+}_{\vec{k}_q})^{n_q}|s\rangle = \bar{c}\,(a^{+}_{\vec{k}_1})^{n_1}(a^{+}_{\vec{k}_2})^{n_2}\cdots(a^{+}_{\vec{k}_q})^{n_q}|s\rangle$$

$$\hat{\bar{N}}\,(a_{\vec{k}_1})^{n_1}(a_{\vec{k}_2})^{n_2}\cdots(a_{\vec{k}_q})^{n_q}|s\rangle = \bar{c}\,(a_{\vec{k}_1})^{n_1}(a_{\vec{k}_2})^{n_2}\cdots(a_{\vec{k}_q})^{n_q}|s\rangle$$

$$\hat{\bar{N}}(b^{+}_{\vec{k}_1})^{\bar{n}_1}(b^{+}_{\vec{k}_2})^{\bar{n}_2}\cdots(b^{+}_{\vec{k}_r})^{\bar{n}_r}|s\rangle = (\bar{c}+\bar{n}_1+\bar{n}_2+\cdots+\bar{n}_r)(b^{+}_{\vec{k}_1})^{\bar{n}_1}(b^{+}_{\vec{k}_2})^{\bar{n}_2}\cdots(b^{+}_{\vec{k}_r})^{\bar{n}_r}|s\rangle$$

$$\hat{\bar{N}}\,(b_{\vec{k}_1})^{\bar{n}_1}(b_{\vec{k}_2})^{\bar{n}_2}\cdots(b_{\vec{k}_r})^{\bar{n}_r}|s\rangle = (\bar{c}-\bar{n}_1-\bar{n}_2-\cdots-\bar{n}_r)(b_{\vec{k}_1})^{\bar{n}_1}(b_{\vec{k}_2})^{\bar{n}_2}\cdots(b_{\vec{k}_r})^{\bar{n}_r}|s\rangle$$

$$\hat{N}\,(b^{+}_{\vec{k}_1})^{\bar{n}_1}(b^{+}_{\vec{k}_2})^{\bar{n}_2}\cdots(b^{+}_{\vec{k}_r})^{\bar{n}_r}|s\rangle = c\,(b^{+}_{\vec{k}_1})^{\bar{n}_1}(b^{+}_{\vec{k}_2})^{\bar{n}_2}\cdots(b^{+}_{\vec{k}_r})^{\bar{n}_r}|s\rangle$$

$$\hat{N}(b_{\vec{k}_1})^{\bar{n}_1}(b_{\vec{k}_2})^{\bar{n}_2}\cdots(b_{\vec{k}_r})^{\bar{n}_r}|s\rangle = c\,(b_{\vec{k}_1})^{\bar{n}_1}(b_{\vec{k}_2})^{\bar{n}_2}\cdots(b_{\vec{k}_r})^{\bar{n}_r}|s\rangle$$

Define vaccum state $|0\rangle$, satisfying $a_{\vec{k}}|0\rangle = 0$, $b_{\vec{k}}|0\rangle = 0$. for any $\vec{k}$.

Then
$$\hat{N}|0\rangle = 0 = 0|0\rangle$$
$$\hat{\bar{N}}|0\rangle = 0 = 0|0\rangle$$

$$\hat{N}\,a^{+}_{\vec{k}}|0\rangle = (a^{+}_{\vec{k}}\hat{N} + a^{+}_{\vec{k}})|0\rangle = 1\cdot a^{+}_{\vec{k}}|0\rangle$$

$$\hat{\bar{N}}\,a^{+}_{\vec{k}}|0\rangle = a^{+}_{\vec{k}}\hat{\bar{N}}|0\rangle = 0 = 0|0\rangle$$

$$\hat{\bar{N}}\,b^{+}_{\vec{k}}|0\rangle = (b^{+}_{\vec{k}}\hat{\bar{N}} + b^{+}_{\vec{k}})|0\rangle = 1\cdot b^{+}_{\vec{k}}|0\rangle$$

$$\hat{N}\,b^{+}_{\vec{k}}|0\rangle = 0 = 0|0\rangle$$

$$\hat{N}(a^{+}_{\vec{k}_1})^{n_1}(a^{+}_{\vec{k}_2})^{n_2}\cdots(a^{+}_{\vec{k}_q})^{n_q}|0\rangle = (n_1+n_2+\cdots n_q)(a^{+}_{\vec{k}_1})^{n_1}(a^{+}_{\vec{k}_2})^{n_2}\cdots(a^{+}_{\vec{k}_q})^{n_q}|0\rangle$$

$$\hat{\bar{N}}\,(b^{+}_{\vec{k}_1})^{\bar{n}_1}(b^{+}_{\vec{k}_2})^{\bar{n}_2}\cdots(b^{+}_{\vec{k}_r})^{\bar{n}_r}|0\rangle = (\bar{n}_1+\bar{n}_2+\cdots+\bar{n}_r)(b^{+}_{\vec{k}_1})^{\bar{n}_1}(b^{+}_{\vec{k}_2})^{n_2}\cdots(b^{+}_{\vec{k}_r})^{\bar{n}_r}|0\rangle$$

$$\hat{\bar{N}}\,(a^{+}_{\vec{k}_1})^{n_1}(a^{+}_{\vec{k}_2})^{n_2}\cdots(a^{+}_{\vec{k}_q})^{n_q}|0\rangle = 0$$

$$\hat{N}(b^{+}_{\vec{k}_1})^{\bar{n}_1}(b^{+}_{\vec{k}_2})^{\bar{n}_2}\cdots(b^{+}_{\vec{k}_r})^{\bar{n}_r}|0\rangle = 0$$

$$:H:\,a^{+}_{\vec{k}_1}|0\rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\,[(2\pi)^3(C(E_{\vec{p}}))^2\,2E_{\vec{p}}]\,(a^{+}_{\vec{p}}a_{\vec{p}} + b^{+}_{\vec{p}}b_{\vec{p}})E_{\vec{p}}\,a^{+}_{\vec{k}_1}|0\rangle$$

$$= E_{\vec{k}_1}a^{+}_{\vec{k}_1}|0\rangle$$

$$:H:\ b^{+}_{\vec{k}_1}|0\rangle = E_{\vec{k}_1}b^{+}_{\vec{k}_1}|0\rangle$$

$$:H:(a^\dagger_{\vec{k}_1})^{n_1}(a^\dagger_{\vec{k}_2})^{n_2}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}|0\rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\left[(2\pi)^3(C(E_{\vec{p}}))^2 2E_{\vec{p}}\right](a^\dagger_{\vec{p}}a_{\vec{p}}+b^\dagger_{\vec{p}}b_{\vec{p}})E_{\vec{p}} \cdot (a^\dagger_{\vec{k}_1})^{n_1}(a^\dagger_{\vec{k}_2})^{n_2}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

the $b^\dagger_{\vec{p}}b_{\vec{p}}$ term does not contribute, so the derivition is the same as real scalar field

$$\overset{?}{=}(n_1E_{\vec{k}_1}+n_2E_{\vec{k}_2}+\cdots n_\ell E_{\vec{k}_\ell})(a^\dagger_{\vec{k}_1})^{n_1}(a^\dagger_{\vec{k}_2})^{n_2}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

Similarly, $:H:(b^\dagger_{\vec{k}_1})^{\bar{n}_1}(b^\dagger_{\vec{k}_2})^{\bar{n}_2}\cdots(b^\dagger_{\vec{k}_r})^{\bar{n}_r}|0\rangle$

$$=(\bar{n}_1E_{\vec{k}_1}+\bar{n}_2E_{\vec{k}_2}+\cdots+\bar{n}_rE_{\vec{k}_r})(b^\dagger_{\vec{k}_1})^{\bar{n}_1}(b^\dagger_{\vec{k}_2})^{\bar{n}_2}\cdots(b^\dagger_{\vec{k}_r})^{\bar{n}_r}|0\rangle$$

and $\hat{\vec{P}}\,a^\dagger_{\vec{k}_1}|0\rangle = \vec{k}\,a^\dagger_{\vec{k}_1}|0\rangle$

$$\hat{\vec{P}}\,b^\dagger_{\vec{k}_1}|0\rangle = \vec{k}_1\,b^\dagger_{\vec{k}_1}|0\rangle$$

$$\hat{\vec{P}}(a^\dagger_{\vec{k}_1})^{n_1}(a^\dagger_{\vec{k}_2})^{n_2}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}|0\rangle=(n_1\vec{k}_1+n_2\vec{k}_2+\cdots+n_\ell\vec{k}_\ell)(a^\dagger_{\vec{k}_1})^{n_1}(a^\dagger_{\vec{k}_2})^{n_2}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$\hat{\vec{P}}(b^\dagger_{\vec{k}_1})^{\bar{n}_1}(b^\dagger_{\vec{k}_2})^{\bar{n}_2}\cdots(b^\dagger_{\vec{k}_r})^{\bar{n}_r}|0\rangle=(\bar{n}_1\vec{k}_1+\bar{n}_2\vec{k}_2+\cdots+\bar{n}_r\vec{k}_r)(b^\dagger_{\vec{k}_1})^{\bar{n}_1}(b^\dagger_{\vec{k}_2})^{\bar{n}_2}\cdots(b^\dagger_{\vec{k}_r})^{\bar{n}_r}|0\rangle$$

while

$$\hat{Q}\,a^\dagger_{\vec{k}_1}|0\rangle=\int_{-\infty}^{+\infty}d^3\vec{p}\left[(2\pi)^3(C(E_{\vec{p}}))^2 2E_{\vec{p}}\right](a^\dagger_{\vec{p}}a_{\vec{p}}-b^\dagger_{\vec{p}}b_{\vec{p}})\,a^\dagger_{\vec{k}_1}|0\rangle$$

$$= a^\dagger_{\vec{k}_1}|0\rangle$$

$$\hat{Q}\,b^\dagger_{\vec{k}_1}|0\rangle = -b^\dagger_{\vec{k}_1}|0\rangle$$

$$\hat{Q}(a^\dagger_{\vec{k}_1})^{n_1}(a^\dagger_{\vec{k}_2})^{n_2}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}|0\rangle=\hat{N}(a^\dagger_{\vec{k}_1})^{n_1}(a^\dagger_{\vec{k}_2})^{n_2}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}|0\rangle=(n_1+n_2+\cdots n_\ell)(a^\dagger_{\vec{k}_1})^{n_1}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$\hat{Q}(b^\dagger_{\vec{k}_1})^{\bar{n}_1}(b^\dagger_{\vec{k}_2})^{\bar{n}_2}\cdots(b^\dagger_{\vec{k}_r})^{\bar{n}_r}|0\rangle=-\hat{\bar{N}}(b^\dagger_{\vec{k}_1})^{\bar{n}_1}(b^\dagger_{\vec{k}_2})^{\bar{n}_2}\cdots(b^\dagger_{\vec{k}_r})^{\bar{n}_r}|0\rangle=-(\bar{n}_1+\cdots+\bar{n}_r)(b^\dagger_1)^{\bar{n}_1}\cdots(b^\dagger_{\vec{k}_r})^{\bar{n}_r}|0\rangle$$

Moreover,

$$\hat{N}(a^\dagger_{\vec{k}_1})^{n_1}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}(b^\dagger_{\vec{p}_1})^{\bar{n}_1}\cdots(b^\dagger_{p_r})^{\bar{n}_r}|0\rangle=(n_1+n_2+\cdots+n_\ell)(a^\dagger_{\vec{k}_1})^{n_1}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}(b^\dagger_{\vec{p}_1})^{\bar{n}_1}\cdots(b^\dagger_{\vec{p}_r})^{\bar{n}_r}|0\rangle$$

$$\hat{\bar{N}}(a^\dagger_{\vec{k}})^{n_1}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}(b^\dagger_{\vec{p}_1})^{\bar{n}_1}\cdots(b^\dagger_{\vec{p}_r})^{\bar{n}_r}|0\rangle=(\bar{n}_1+\bar{n}_2+\cdots\bar{n}_r)(a^\dagger_{\vec{k}_1})^{n_1}\cdots(a^\dagger_{\vec{k}_\ell})^{n_\ell}(b^\dagger_{\vec{p}_1})^{\bar{n}_1}\cdots(b^\dagger_{\vec{p}_r})^{\bar{n}_r}|0\rangle$$

$$\hat{H}\,(a^+_{\vec{k}_1})^{n_1}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}(b^+_{\vec{p}_1})^{\bar{n}_1}\cdots(b^+_{\vec{p}_r})^{\bar{n}_r}|0\rangle$$

$$=\left[(n_1E_{\vec{k}_1}+n_2E_{\vec{k}_2}+\cdots+n_\ell E_{\vec{k}_\ell})+(\bar{n}_1E_{\vec{p}_1}+\bar{n}_2E_{\vec{p}_2}+\cdots+\bar{n}_rE_{\vec{p}_r})\right](a^+_{\vec{k}_1})^{n_1}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}(b^+_{\vec{p}_1})^{\bar{n}_1}\cdots(b^+_{\vec{p}_r})^{\bar{n}_r}|0\rangle$$

$$\hat{\vec{P}}\,(a^+_{\vec{k}_1})^{n_1}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}(b^+_{\vec{p}_1})^{\bar{n}_1}\cdots(b^+_{\vec{p}_r})^{\bar{n}_r}|0\rangle$$

$$=\left[(n_1\vec{k}_1+n_2\vec{k}_2+\cdots n_\ell\vec{k}_\ell)+(\bar{n}_1\vec{p}_1+\bar{n}_2\vec{p}_2+\cdots+\bar{n}_r\vec{p}_r)\right](a^+_{\vec{k}_1})^{n_1}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}(b^+_{\vec{p}_1})^{\bar{n}_1}\cdots(b^+_{\vec{p}_r})^{\bar{n}_r}|0\rangle$$

$$\hat{Q}\,(a^+_{\vec{k}_1})^{n_1}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}(b^+_{\vec{p}_1})^{\bar{n}_1}\cdots(b^+_{\vec{p}_r})^{\bar{n}_r}|0\rangle$$

$$=\left[(n_1+n_2+\cdots+n_\ell)-(\bar{n}_1+\bar{n}_2+\cdots+\bar{n}_r)\right](a^+_{\vec{k}_1})^{n_1}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}(b^+_{\vec{p}_1})^{\bar{n}_1}\cdots(b^+_{\vec{p}_r})^{\bar{n}_r}|0\rangle$$

Therefore, ① the state $a^+_{\vec{k}}|0\rangle$ can be interpreted as a one particle state having momentum $\vec{k}$ and energy $E_{\vec{k}}$ and charge one unit; the state $b^+_{\vec{p}}|0\rangle$ can be interpreted as a one anti-particle state having momentum $\vec{p}$ and energy $E_{\vec{p}}$ and charge negative one unit

NOTE that we have not specified how large is one unit charge, and we have not specified the type of charge.

The important point is that for any charge associated with internal transformation of a complex scalar field, the value is the same for particle and anti-particle, but with opposite signs.

② the state $(a^+_{\vec{k}_1})^{n_1}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}(b^+_{\vec{p}_1})^{\bar{n}_1}\cdots(b^+_{\vec{p}_r})^{\bar{n}_r}|0\rangle$ is a multi-particle state with n_1 particle having momentum $\vec{k}_1$ and energy $E_{\vec{k}_1}$, …, $\bar{n}_1$ anti-particle having momentum $\vec{p}_1$ and energy $E_{\vec{p}_1}$, … .

Similar to the real scalar field case, let's normalize the one-particle state $|p\rangle = f(\vec{p})\, a^{+}_{\vec{p}} |0\rangle$ by requiring

$\langle q|p\rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(\vec{p}-\vec{q})$ and $\langle 0|0\rangle = 1$ and $f(\vec{p})$ is real.

$$\Rightarrow \quad f(\vec{p}) = C(E_{\vec{p}})(2\pi)^3 2E_{\vec{p}}$$

Similarly, one-antiparticle state is normalized as

$|\bar{p}\rangle = \bar{f}(\vec{p})\, b^{+}_{\vec{p}} |0\rangle$ and $\bar{f}(\vec{p})$ is real

satisfying $\langle \bar{q}|\bar{p}\rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(\vec{p}-\vec{q})$

$$\Rightarrow \quad \bar{f}(\vec{p}) = C(E_{\vec{p}})(2\pi)^3 2E_{\vec{p}}$$

$$\Rightarrow \langle 0|\phi(x)|k\rangle = \langle 0| \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) (a_{\vec{p}} e^{-ip\cdot x} + b^{+}_{\vec{p}} e^{ip\cdot x})\, C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, a^{+}_{\vec{k}} |0\rangle$$

$$= e^{-ik\cdot x}$$

$$\langle k|\phi(x)|0\rangle = \langle 0|\, C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, a_{\vec{k}} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) (a_{\vec{p}} e^{-ip\cdot x} + b^{+}_{\vec{p}} e^{ip\cdot x})|0\rangle$$

$$= 0$$

$$\langle k|\phi^{+}(x)|0\rangle = \langle 0|\, C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, a_{\vec{k}} \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) (a^{+}_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{-ip\cdot x})|0\rangle$$

$$= e^{ik\cdot x}$$

$$\langle 0|\phi^{+}(x)|k\rangle = \langle 0| \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) (a^{+}_{\vec{p}} e^{ip\cdot x} + b_{\vec{p}} e^{-ip\cdot x})\, C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, a^{+}_{\vec{k}}|0\rangle$$

$$= 0$$

$$\langle 0|\phi(x)|\bar{k}\rangle = 0$$

$$\langle \bar{k}|\phi(x)|0\rangle = e^{ik\cdot x}$$

$$\langle \bar{k}|\phi^{+}(x)|0\rangle = 0$$

$$\langle 0|\phi^{+}(x)|\bar{k}\rangle = e^{-ik\cdot x}$$