Homework 4
Due date: 2018.11.28

Problem 1. Consider a real scalar field, described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \, .$$

Do the following calculations starting from the decompositions of the field operator,

$$\phi(x) \equiv \phi(\vec{x}, t) = \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3 2E(\vec{p})} \, \left(a(\vec{p}) e^{-ip\cdot x} + a^\dagger(\vec{p}) e^{ip\cdot x} \right) \, ,$$

where $E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$, and the definition of its corresponding canonical conjugate momentum operator, $\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)}$.
1) Find the decompositions of the operator $\pi(x) \equiv \pi(\vec{x}, t)$. [1 point]
2) Find $a(\vec{p})$ and $a^\dagger(\vec{p})$ in terms of ϕ and π. [3 points]
3) From the equal-time commutation relations (ETCR)

$$[\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^3(\vec{x} - \vec{x}') \, , \; [\phi(\vec{x}, t), \phi(\vec{x}', t)] = 0 \, , \; [\pi(\vec{x}, t), \pi(\vec{x}', t)] = 0 \, ,$$

show that [5 points]

$$[a(\vec{p}), a^\dagger(\vec{p}')] = (2\pi)^3 2E(\vec{p}) \delta^3(\vec{p} - \vec{p}') \, , \; [a(\vec{p}), a(\vec{p}')] = 0 \, , \; [a^\dagger(\vec{p}), a^\dagger(\vec{p}')] = 0 \, .$$

4) In the $c = \hbar = 1$ units, in terms of the powers of energy, the dimension of the Lagrangian density is $[E]^4$. In the above decompositions of the field operator, what is the dimension of $\phi(x)$, $\pi(x)$, $a(\vec{p})$ and $\delta^3(\vec{x} - \vec{x}')$, in terms of the powers of energy? [1 point]