1) $n + (n-1) + \cdots + 1 = \dfrac{n(n+1)}{2}$

it is $\boxed{10}$ for $n=4$

2) $\dfrac{n(n+1)}{2} - n = \dfrac{n(n-1)}{2}$

it is $\boxed{6}$ for $n=4$

3)
$$s'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta s^{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta s^{\beta\alpha} = s'^{\nu\mu}$$
$$a'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta a^{\alpha\beta} = -\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta a^{\beta\alpha} = -a'^{\nu\mu}$$

4)
$$s_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} s^{\alpha\beta} = g_{\mu\alpha} g_{\nu\beta} s^{\beta\alpha} = s_{\nu\mu}$$
$$a_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} a^{\alpha\beta} = -g_{\mu\alpha} g_{\nu\beta} a^{\beta\alpha} = -a_{\nu\mu}$$

5)
$$s^{\mu\nu} a_{\mu\nu} = s^{\nu\mu} a_{\mu\nu} = -s^{\nu\mu} a_{\nu\mu} = -s^{\mu\nu} a_{\mu\nu}$$
$$\Rightarrow s^{\mu\nu} a_{\mu\nu} = 0$$

6)
$$t^{\mu\nu} = s^{\mu\nu} + a^{\mu\nu} \Rightarrow t^{\nu\mu} = s^{\nu\mu} + a^{\nu\mu} = s^{\mu\nu} - a^{\mu\nu}$$
$$\Rightarrow \boxed{s^{\mu\nu} = \frac{1}{2}(t^{\mu\nu} + t^{\nu\mu}), \quad a^{\mu\nu} = \frac{1}{2}(t^{\mu\nu} - t^{\nu\mu})}$$

2018/09/28, p5 (checked)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - j_\mu A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

1) $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu}$

$$= \partial^\lambda(\partial^\mu A^\nu - \partial^\nu A^\mu) + \partial^\mu(\partial^\nu A^\lambda - \partial^\lambda A^\nu) + \partial^\nu(\partial^\lambda A^\mu - \partial^\mu A^\lambda)$$

$$= \partial^\lambda\partial^\mu A^\nu - \partial^\lambda\partial^\nu A^\mu + \partial^\mu\partial^\nu A^\lambda - \partial^\mu\partial^\lambda A^\nu + \partial^\nu\partial^\lambda A^\mu - \partial^\nu\partial^\mu A^\lambda$$

since $\partial^\lambda\partial^\mu = \partial^\mu\partial^\lambda$

$= 0$

2) $$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}\frac{\partial[(\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial^\rho A^\sigma - \partial^\sigma A^\rho)]}{\partial(\partial_\mu A_\nu)}$$

$$= -\frac{1}{4}2(\partial^\rho A^\sigma - \partial^\sigma A^\rho)(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)$$

$$= -\frac{1}{2}(\partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu + \partial^\mu A^\nu)$$

$$= -F^{\mu\nu}$$

$$\Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial_\mu F^{\mu\nu}$$

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = m^2 A^\nu - j^\nu$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$$

3) From $\partial_\mu F^{\mu\nu} = j^\nu$

$\Rightarrow$ for $\nu = 0$, LHS $= \partial_\mu F^{\mu 0} = \partial_i F^{i0} = \vec{\nabla}\cdot\vec{E}$, RHS $= j^0 = \rho \Rightarrow \vec{\nabla}\cdot\vec{E} = \rho$

for $\nu = i$, LHS $= \partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji} = -\frac{\partial E^i}{\partial t} + \partial_j(-\varepsilon^{jik}B^k)$

$$= -\frac{\partial E^i}{\partial t} + \varepsilon^{ijk}\partial_j B^k = \left(-\frac{\partial\vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)^i$$

RHS $= j^i$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}$$

From $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0$

$\Rightarrow$ For $(\lambda, \mu, \nu) = (i, j, k)$

$$\partial^i F^{jk} = -\partial_i(-\varepsilon^{jkm} B^m) = \partial_i \varepsilon^{jkm} B^m$$

$$\partial^j F^{ki} = \partial_j \varepsilon^{kim} B^m$$

$$\partial^k F^{ij} = \partial_k \varepsilon^{ijm} B^m$$

For $i=1,\ j=2,\ k=3$

$$\partial^1 F^{23} = \partial_1 \varepsilon^{23m} B^m = \partial_1 \varepsilon^{231} B^1 = \partial_1 B^1$$

$$\partial^2 F^{31} = \partial_2 \varepsilon^{31m} B^m = \partial_2 B^2$$

$$\partial^3 F^{12} = \partial_3 \varepsilon^{12m} B^m = \partial_3 B^3$$

$\Rightarrow$ LHS $= \vec{\nabla}\cdot\vec{B}$, RHS $=0$ $\Rightarrow$ $\vec{\nabla}\cdot\vec{B} = 0$ ✓

For $(\lambda, \mu, \nu) = (0, i, j)$

$$\partial^0 F^{ij} = \partial_0(-\varepsilon^{ijm} B^m) = -\varepsilon^{ijm}\partial_0 B^m$$

$$\partial^i F^{j0} = -\partial_i E^j$$

$$\partial^j F^{0i} = +\partial_j E^i$$

$$\Rightarrow \text{LHS} = \partial^0 F^{ij} + \partial^i F^{j0} + \partial^j F^{0i} = -\left(\varepsilon^{mij}\frac{\partial B^m}{\partial t} + (\partial_i E^j - \partial_j E^i)\right)$$

times ε^{kij} on LHS,

$$\Rightarrow -\left(\varepsilon^{kij}\varepsilon^{mij}\frac{\partial B^m}{\partial t} + \varepsilon^{kij}(\partial_i E^j - \partial_j E^i)\right)$$

$$= -\left[2\delta^k_m \frac{\partial B^m}{\partial t} + (\vec{\nabla}\times\vec{E})^k + (\vec{\nabla}\times\vec{E})^k\right]$$

$$= -2\left(\frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times\vec{E}\right)^k$$

times ε^{kij} on 0 still gives 0

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times\vec{E} = 0$$ ✓

4)

From $\partial_\mu F^{\mu\nu} + m^2 A^\nu = j^\nu$

$\Rightarrow \quad \partial_\nu \partial_\mu F^{\mu\nu} + m^2 \partial_\nu A^\nu = \partial_\nu j^\nu$

Since $F^{\mu\nu} = -F^{\nu\mu}$, then $\partial_\nu \partial_\mu F^{\mu\nu} = 0$

Since $j^\nu = (q\delta^3(\vec{x}), 0, 0, 0)$, then $\partial_\nu j^\nu = \partial_0 j^0 + 0 = 0$.

$\Rightarrow \quad \partial_\nu A^\nu = 0 \quad$ for $m > 0$

put it back to the original equation and expand $F^{\mu\nu}$

$\Rightarrow \quad \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = j^\nu$

$\Rightarrow \quad (\partial_\mu \partial^\mu + m^2) A^\nu = j^\nu$

For $A^\nu = (A^0, 0, 0, 0)$ and $j^\nu = (q\delta^3(\vec{x}), 0, 0, 0)$

the $\nu = i$ $(i = 1, 2, 3)$ equations are obviously satisfied since $0 = 0$.

For $\nu = 0$, if put $A^0 = \frac{q}{(2\pi)^3} \iiint_{-\infty}^{+\infty} d^3\vec{k} \, \frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2 + m^2}$ on the LHS,

we get $(\partial_0 \partial^0 + \partial_i \partial^i + m^2) \, q \iiint_{-\infty}^{+\infty} d^3\vec{k} \, \frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2 + m^2}$

$$= \frac{q}{(2\pi)^3} \iiint_{-\infty}^{+\infty} d^3\vec{k} \, \frac{\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial (x^i)^2} + m^2\right) e^{i(k^1x^1 + k^2x^2 + k^3x^3)}}{|\vec{k}|^2 + m^2}$$

$$= \frac{q}{(2\pi)^3} \iiint_{-\infty}^{+\infty} d^3\vec{k} \, \frac{(0 - i^2(k^1)^2 - i^2(k^2)^2 - i^2(k^3)^2 + m^2)}{|\vec{k}|^2 + m^2} e^{i\vec{k}\cdot\vec{x}}$$

$$= \frac{q}{(2\pi)^3} \iiint_{-\infty}^{+\infty} d^3\vec{k} \, \frac{|\vec{k}|^2 + m^2}{|\vec{k}|^2 + m^2} e^{i\vec{k}\cdot\vec{x}}$$

$$= \frac{q}{(2\pi)^3} \iiint_{-\infty}^{+\infty} d^3\vec{k} \, e^{i\vec{k}\cdot\vec{x}}$$

$$= q\delta^3(\vec{x}) \quad ✓$$

5)

$$A^0 = \frac{q}{(2\pi)^3}\iiint_{-\infty}^{+\infty} d^3\vec{k}\,\frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2+m^2}$$

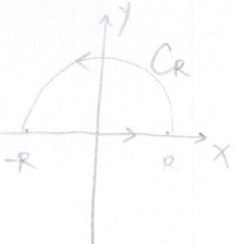
$$= \frac{q}{(2\pi)^3}\cdot 2\pi\int_0^{+\infty}|\vec{k}|^2 d|\vec{k}|\int_{-1}^{1} d(\cos\theta)\,\frac{e^{i|\vec{k}|r\cos\theta}}{|\vec{k}|^2+m^2}$$

$$= \frac{q}{(2\pi)^2}\int_0^{+\infty}|\vec{k}|^2 d|\vec{k}|\,\frac{e^{i|\vec{k}|r}-e^{-i|\vec{k}|r}}{(|\vec{k}|^2+m^2)\,i|\vec{k}|r}$$

$$= \frac{q}{(2\pi)^2 ir}\int_0^{+\infty}\frac{|\vec{k}|\,d|\vec{k}|}{|\vec{k}|^2+m^2}\left(e^{i|\vec{k}|r}-e^{-i|\vec{k}|r}\right)$$

$$= \frac{q}{(2\pi)^2 ir}\int_{-\infty}^{+\infty}\frac{|\vec{k}|\,e^{i|\vec{k}|r}}{|\vec{k}|^2+m^2}\,d|\vec{k}|.$$

Since $\frac{1}{|\vec{k}|^2+m^2} = \frac{1}{(|\vec{k}|+im)(|\vec{k}|-im)}$



We close the contour on the upper half of the complex plane, and the contour integral can be evaluated using the residue theorem

$$\oint\frac{z e^{izr}\,dz}{z^2+m^2} = (2\pi i)\,\frac{im\,e^{i\cdot imr}}{im+im} = \frac{2\pi i\,e^{-mr}}{2} = \pi i\,e^{-mr}$$

Since $\frac{z}{z^2+m^2} = \frac{Re^{i\varphi}}{(Re^{i\varphi})^2+m^2}\xrightarrow{R\to\infty}0$, then by Jordan's lemma

$$\lim_{R\to\infty}\int_{C_R}\frac{z e^{izr}}{z^2+m^2}\,dz = 0$$

$$\Rightarrow \oint\frac{z e^{izr}\,dz}{z^2+m^2} = \int_{-\infty}^{+\infty}\frac{|\vec{k}|\,e^{i|\vec{k}|r}}{|\vec{k}|^2+m^2}\,d|\vec{k}|$$

$$\Rightarrow A^0 = \frac{q}{(2\pi)^2 ir}\,\pi i\,e^{-mr} = \boxed{\frac{q}{4\pi r}e^{-mr}}$$

(using Mathematica to evaluate $\int_{-\infty}^{+\infty}\frac{|\vec{k}|\,e^{i|\vec{k}|r}}{|\vec{k}|^2+m^2}\,d|\vec{k}|$ confirms the result.

In[1]:=
```
Integrate[k / (k^2 + m^2) * Exp[I * k * r],
  {k, -∞, +∞}, Assumptions → {(r > 0) && (m > 0)}]
```

Out[1]= $i\, e^{-m r}\, \pi$