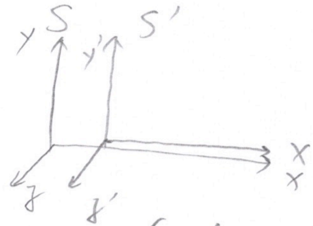
Lorentz transformation.

Two inertial frames S and S', with S' moving at uniform velocity $\vec{V}$
(↪ the frame in which Newton's first law is obeyed.)
with respect to S. Suppose also the velocity is in the common x & x' axis, and the two frames coincide (i.e., at $t=t'=0$, $x=x'=0$, $y=y'=0$, $z=z'=0$)

Then an event occurs at (x, y, z) and time t in S occurs in S' having

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \overset{c=1}{=} \gamma(t - vx).$$

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-v^2}}$



(note, x and x' axis actually overlap).

The inverse transformation (i.e., from S' to S) is (just solve x & t in the above equations)

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + vx')$$

Four consequences:

1. 同时的相对性 (the relativity of simultaneity)

If two events occurs at the same time in S, but at different locations, then they do not occur at the same time in S'.

That is, for two event A & B having $t_A = t_B$ but $x_A \neq x_B$ in frame S, then in S' frame

$$\left.\begin{aligned} t_A' &= \gamma(t_A - vx_A) \\ t_B' &= \gamma(t_B - vx_B) \end{aligned}\right\} \Rightarrow t_A' - t_B' = \gamma v(x_B - x_A) \neq 0.$$

2. Lorentz contraction (尺缩): 量运动的尺得到的长度变短

Suppose a stick lies on the x' axis, <u>at rest in S'</u>, say, one end is at the origin $x'=0$ and the other end is at L' (so its length in S' frame is L'). What is its length as <u>measured in S</u>?

(implies we need to measure the two ends at the same time in S)

Since
$$\Delta x = \gamma(\Delta x' + v\Delta t')$$
$$\Delta t = \gamma(\Delta t' + v\Delta x') \equiv 0 \Rightarrow \Delta t' = -v\Delta x'$$

(put it in)

$$\Rightarrow \Delta x = \gamma[\Delta x' + v(-v\Delta x')]$$
$$= \gamma \Delta x' (1 - v^2)$$
$$= \frac{\Delta x'}{\gamma}$$
$$= \frac{L'}{\gamma} \leq L'$$

(note that $\gamma = \frac{1}{\sqrt{1-v^2}} \geq 1$)

Note that y and z directions do not suffer Lorentz contraction.

3. Time dilation (钟慢)

$\longrightarrow$ 运动的钟走得慢.

Suppose a clock is at rect in S' frame and it measures a period T', then how long is this period as measured in S?

$$\Delta t = \gamma(\Delta t' + v\underbrace{\Delta x'}_{0\text{ since at rest in } S'})$$

$$= \gamma T'$$

$$\geqslant T'$$

4. velocity addition:

Suppose a particle P is moving in the x & x' direction at speed u', with respect to S', then what is its speed, u, with respect to S?

$$dx = \gamma(dx' + v\,dt')$$

$$dt = \gamma(dt' + v\,dx')$$

$$\Rightarrow \quad u = \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + v\,dx'} = \frac{u' + v}{1 + vu'}$$

Notice that if $u' = c = 1$, then $u = \dfrac{1+v}{1+v} = 1 = c$.

that is, speed of light is the same in all inertial frames.

Write the above formula in another way:

$$v_{PS} = \frac{v_{PS'} + v_{S'S}}{1 + v_{PS'} v_{S'S}/c^2}, \text{ or just write it as}$$

Galiean velocity addition rule

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + v_{AB} v_{BC}/c^2}, \quad (v_{AB} \text{ 表示 A 相对于 B 的速度, etc.})$$

Einstein's correction

Four-vectors

Now introduce position-time four vector x^μ

$$x^0 = ct = t \,, \quad x^1 = x \,, \quad x^2 = y \,, \quad x^3 = z.$$

So Lorentz transformation can be written as

$$x'^0 = \gamma(x^0 - \beta x^1) = \gamma(x^0 - v x^1) \,, \quad \text{note } \beta \equiv \frac{v}{c} = v$$

$$x'^1 = \gamma(x^1 - v x^0)$$

$$x'^2 = x^2$$

$$x'^3 = x^3.$$

Or just write $x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu$, $(\mu = 0, 1, 2, 3)$

where the coefficients $\Lambda^\mu{}_\nu$ are the elements of a matrix Λ:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix}$$

Introduce Einstein's summation convention to avoid writing Σ.

$$x'^\mu = \Lambda^\mu{}_{\underline{\nu}} x^{\underline{\nu}} = \Lambda^\mu{}_\alpha x^\alpha$$

dummy indices, whatever can be used

Invariant.

When go from S to S' frame, there is a particular combination remains the same:

$$I \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2.$$

(Similarly, the quantity $r^2 = x^2 + y^2 + z^2$ is invariant under rotation) check for the pure Lorentz transformation above

$$(x'^0)^2 = \gamma^2(x^0 - vx^1)^2 = \gamma^2[(x^0)^2 + v^2(x^1)^2 - 2vx^0x^1]$$

$$(x'^1)^2 = \gamma^2(x^1 - vx^0)^2 = \gamma^2[(x^1)^2 + v^2(x^0)^2 - 2vx^1x^0]$$

$$\Rightarrow (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2$$

$$= \gamma^2\left[(x^0)^2 + v^2(x^1)^2 - (x^1)^2 - v^2(x^0)^2\right] - (x^2)^2 - (x^3)^2$$

$$= (x^0)^2\gamma^2(1-v^2) - (x^1)^2\gamma^2(1-v^2) - (x^2)^2 - (x^3)^2$$

$$= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \quad \checkmark$$

Then how to write the $I = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ shorter?

Introduce metric $g_{\mu\nu}$ to help

$$g = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

(i.e., $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, $g_{\mu\nu} = 0$ for $\mu \neq \nu$)

(Caution: some books use opposite signs).

We can now write

$$I = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}x^\mu x^\nu = g_{\mu\nu}x^\mu x^\nu.$$

Note that $g_{\mu\nu} = g_{\nu\mu}$. (symmetric tensor)

Define x_μ now

$$x_\mu \equiv g_{\mu\nu} x^\nu .$$

We call the upper index guy, x^μ, contravariant four-vector, and the lower index guy, x_μ, covariant four-vector.

So $$I = x_\mu x^\mu = x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu .$$

(note that in GR, $g_{\mu\nu}$ is more general, and our $g_{\mu\nu}$ here is written as $\eta_{\mu\nu}$ in GR. But let's don't worry about it, since we will only work in flat spacetime in QFT I)

Define abitrary contravariant four-vector and covariant four-vector

A four-component object that transforms in the same way as x^μ does when go from one inertial frame to another,

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu$$

and for each a^μ we have a covariant four-vector.

$$a_\mu = g_{\mu\nu} a^\nu .$$

Introduce the inverse of $g_{\mu\nu}$, i.e., $g^{\mu\nu}$.

$$g^{\mu\nu} g_{\nu\alpha} = \underbrace{\delta^\mu_\alpha}_{\text{called kronecker symbol.}}$$

$$g^{-1} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Now we can get invariants from a contravariant and a covariant vectors.
(i.e., the same value in any inertial frame)

$$\underline{a^\mu b_\mu} = a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu = g^{\mu\nu} a_\mu b_\nu = \delta^\mu{}_\nu a^\nu b_\mu$$

$$= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

$a^\mu a_\mu$ is invariant, too

$b^\mu b_\mu$ is invariant, too.

We call it the "scalar product" of a and b. (it is simply the analog to the dot product of two three-vectors),

$$a \cdot b \equiv a_\mu b^\mu .$$

Usually we write

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

(note: the dot product of two three-vector is just
$\vec{a} \cdot \vec{b} = a^1 b^1 + a^2 b^2 + a^3 b^3$)

In particular,

$$a^2 \equiv a \cdot a = (a^0)^2 - \vec{a}^2 = a^0 a^0 - a^1 a^1 - a^2 a^2 - a^3 a^3$$

Note: if $a^2 > 0$, a^μ is called timelike (e.g., $a^\mu = (t, 0, 0, 0)$)
if $a^2 < 0$, a^μ is called spacelike (e.g., $a^\mu = (0, x, 0, 0)$)
if $a^2 = 0$, a^μ is called lightlike (e.g., $a^\mu = (t, x=ct, 0, 0)$)

Now define tensors

e.g., ① a second-rank tensor, $S^{\mu\nu}$, carries two indices, so has $4^2=16$ components, and transforms with two factors of Λ:

$$S'^{\mu\nu} = \Lambda^\mu_{\ \kappa}\Lambda^\nu_{\ \sigma} S^{\kappa\sigma}$$

② a third-rank tensor $t^{\mu\nu\lambda}$, has $4^3=64$ components, and transforms as

$$t'^{\mu\nu\lambda} = \Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta}\Lambda^\lambda_{\ \sigma} t^{\alpha\beta\sigma}$$

③ covariant tensors and mixed tensors can be obtained by lowering indices by $g_{\mu\nu}$, e.g.

$$S^\mu{}_\nu = g_{\nu\lambda} S^{\mu\lambda}, \quad S_{\mu\nu} = g_{\mu\kappa} g_{\nu\lambda} S^{\kappa\lambda}$$

④ a vector is a tensor of rank one;
a scalar (i.e., invariant) is a tensor of rank zero.

⑤ 张量与数相乘是同类张量, e.g., $p^\mu = m\eta^\mu$.
两个张量相乘得新张量, e.g., $a^\mu b^\nu$ is a second rank tensor.

⑥ Contraction of a $(n+2)$ rank tensor to get a n rank tensor, by summing like upper and lower indices.

e.g. $S^\mu{}_\mu$ is a scalar;
$t_\nu{}^{\mu\nu}$ is a vector;
$a_\mu t^{\mu\nu\lambda}$ is a second-rank tensor.

Energy & Momentum

Introduce <u>proper time $d\tau$</u>. (也叫四維間隔)

$$d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = dt^2 - dx^2 - dy^2 - dz^2$$

(note $dt^2 \equiv (dt)(dt)$, $dx^2 = (dx)(dx)$, just infinitesimal change of $I \equiv \tau^2 \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$

The proper time is the time measured by the watch moving together with the object. Therefore in the object's rest frame S',

$$d\tau = dt'$$

The velocity of the object respect to frame S is

$$\vec{v} = \frac{d\vec{x}}{dt} \quad \text{(or simply, } v = \frac{dx}{dt} \text{ if moving along } x\text{-direction)}$$

But, a transformation of $\vec{v}$ involves transformation in both $\vec{x}$ and t, so introduce $\boxed{\text{Proper velocity, } \eta^\mu \equiv \frac{dx^\mu}{d\tau}}$, so only need to worry about transformation for x^μ (τ is invariant)

or just call it "four-velocity".

From
$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + vx')$$

and use $\underline{dx' = 0,\ dy' = 0,\ dz' = 0},\ dt' = d\tau$

(since the object is at rest in S' frame)

$$\Rightarrow \quad dx = \gamma v\, d\tau$$
$$dt = \gamma\, d\tau$$

$$\Rightarrow \eta^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dx^0}{d\tau}, \frac{d\vec{x}}{d\tau}\right) = (\gamma, \gamma\vec{v}) = \gamma(1, \vec{v})$$
$$= \gamma(1, v_x, v_y, v_z) = \gamma(1, v^1, v^2, v^3)$$

$$\Rightarrow \eta^\mu \eta_\mu = \eta^\mu g_{\mu\nu} \eta^\nu = \gamma (1, v_x, v_y, v_z) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \gamma \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

$$= \gamma^2 (1 - v_x^2 - v_y^2 - v_z^2) = 1 = c^2$$

(yes, $\eta^\mu \eta_\mu$ is invariant, as expected).

Now define momentum in relativity.

$$p^\mu \equiv m \eta^\mu.$$

$$\Rightarrow p^0 = m\eta^0 = m\gamma = \frac{m}{\sqrt{1-v^2}} = \frac{mc}{\sqrt{1-v^2}}$$

$$\vec{p} = m\gamma\vec{v} = \frac{m\vec{v}}{\sqrt{1-v^2}}$$

Define energy as $E \equiv \gamma mc^2 = m\gamma$, then

$$\underset{\substack{\downarrow \\ \text{called four-momentum}}}{p^\mu} = \left(\frac{E}{c}, p_x, p_y, p_z\right) = (E, \vec{p})$$

So $p_\mu p^\mu = E^2 - |\vec{p}|^2 = (m\gamma)^2 - (m\gamma)^2 |\vec{v}|^2 = m^2$.

(yes. $p_\mu p^\mu$ is invariant, as expected)

Now we can try Taylor expand E in the non-relativistic regime ($v \ll c$)

$$E = m\gamma = m(1-v^2)^{-\frac{1}{2}} = m\left(1 + \tfrac{1}{2}v^2 + O(v^4)\right)$$

$$= \underbrace{m}_{\text{rest energy}} + \underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic energy}} + O(v^4)$$

For massless particle, $m=0$, $v=c$, $E = \underset{=|\vec{p}|}{|\vec{p}|c}$, and therefore $p^\mu p_\mu = 0$.

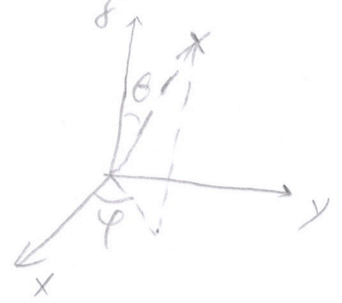
The energy and momentum of the massless particle is determined by its frequency $E = \hbar\nu$, not mass (0 anyway) or velocity (c anyway)

So for a massless particle

$$p^\mu = \hbar\omega\,(1,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

(since $(\sin\theta\cos\varphi)^2 + (\sin\theta\sin\varphi)^2 + (\cos\theta)^2 = 1$)

that is, the massless particle moving in a direction described by polar angle θ and azimuthal angle φ, with the speed $= c = 1$.



Example to use momentum vector

A pion at rest decays into a muon plus a neutrino (assume massless)
What is the speed of muon in the rest frame of pion?

method 1. In the rest frame of pion.

$$E_\pi = m_\pi$$

$$E_\mu = (m_\mu^2 + |\vec{P}_\mu|^2)^{\frac{1}{2}}$$

$$E_\nu = |\vec{P}_\nu| = |\vec{P}_\mu|$$

Use Conservation of energy (i.e., the 0-component of four-momentum conservation $P_\pi^\alpha = P_\mu^\alpha + P_\nu^\alpha$)

$P_\pi^0 = P_\mu^0 + P_\nu^0$, that is, $E_\pi = E_\mu + E_\nu$

$$\Rightarrow \quad m_\pi = (m_\mu^2 + |\vec{P}_\mu|^2)^{\frac{1}{2}} + |\vec{P}_\mu|$$

$$\Rightarrow \quad |\vec{P}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

and $E_\mu = (m_\mu^2 + |\vec{P}_\mu|^2)^{\frac{1}{2}} = \dfrac{m_\pi^2 + m_\mu^2}{2m_\pi}$

$$\Rightarrow \quad |\vec{v}_\mu| = \frac{|\vec{P}_\mu|}{E_\mu} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \doteqdot \frac{139.57^2 - 105.66^2}{139.57^2 + 105.66^2} \doteqdot 0.27$$

(note: $p^\alpha = (p^0, p^i) = m\gamma(1, v^i)$

$\Rightarrow \quad v^i = \dfrac{p^i}{p^0} = \dfrac{p^i}{E}$, i.e., $\vec{v} = \dfrac{\vec{P}}{E}$)

Find the masses. etc. of a particle in pdg.lbl.gov.

Method 2.

$$P_\pi^\alpha = P_\mu^\alpha + P_\nu^\alpha \quad , \text{ or } \quad P_\nu^\alpha = P_\pi^\alpha - P_\mu^\alpha$$

$$\Downarrow$$

$$P_\nu^2 = (P_\pi - P_\mu)^2 \quad (\text{dot product})$$

$$\Downarrow$$

$$0 = m_\pi^2 + m_\mu^2 - 2P_\pi \cdot P_\mu .$$

$\uparrow$

(note that $P_\nu^2 = m_\nu^2 = 0$)

note that $P_\pi \cdot P_\mu$ is an invariant and can be ~~While in pion rest frame, $P_\pi \cdot P_\mu = m_\pi \cdot E_\mu$.~~ evaluated in any reference frame and the result is the same.

(a bit explanation: $P_\pi \cdot P_\mu = P_\pi^0 P_\mu^0 - P_\pi^1 P_\mu^1 - P_\pi^2 P_\mu^2 - P_\pi^3 P_\mu^3$

$= m_\pi E_\mu$

since

$P_\pi^\alpha = (m_\pi, 0, 0, 0) = m_\pi \gamma_\pi (1, V_\pi^1, V_\pi^2, V_\pi^3)$

$= m_\pi (1, 0, 0, 0)$

$\uparrow$

Since $\gamma_\pi = 1$, $V_\pi^{1,2,3} = 0$.)

$$0 = m_\pi^2 + m_\mu^2 - 2m_\pi E_\mu \Rightarrow E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}$$

Since $P_\mu^\alpha = P_\pi^\alpha - P_\nu^\alpha$, then $P_\mu^2 = (P_\pi - P_\nu)^2 \Rightarrow m_\mu^2 = m_\pi^2 + 0 - 2m_\pi E_\nu$

$= m_\pi^2 - 2m_\pi |\vec{P}_\nu| = m_\pi^2 - 2m_\pi |\vec{P}_\mu|$

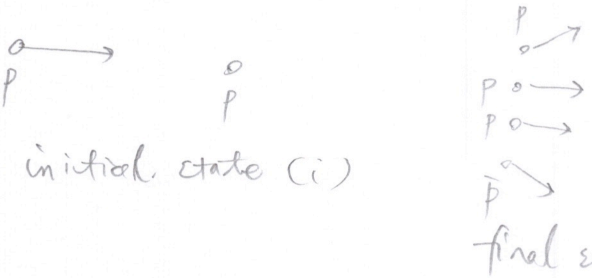
$$\Rightarrow |\vec{P}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

$$\Rightarrow |\vec{v}_\mu| = \frac{|\vec{P}_\mu|}{E_\mu} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}$$

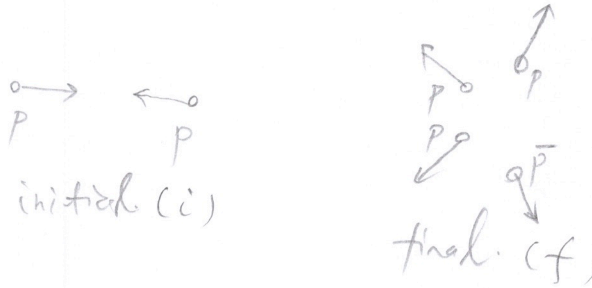
Example 2.

In the lab frame, a high energy proton hit a proton at rest, and the process is $p + p \to p + p + p + \bar{p}$. What is the threshold energy for this reaction (i.e, the minimum energy of the incident proton in the lab frame) ?

in lab frame:



in center of momentum frame
(CM)



In the lab frame, the total four-momentum is

$$P^{\mu}_{i,lab} = (E + m,\ |\vec{P}|,\ 0,\ 0)$$

$$\Rightarrow \quad P_i^2 = P^{\mu}_{i,lab} P_{\mu i,lab} = (E+m)^2 - |\vec{P}|^2 = E^2 - |\vec{P}|^2 + 2mE + m^2 = 2m^2 + 2mE$$

P_i^2 is an invariant, can be evaluated at any frame and the result is the same.

By energy-momentum conservation, $P^{\mu}_{i,lab} = P^{\mu}_{f,lab}$, $P^{\mu}_{i,CM} = P^{\mu}_{f,CM}$,
(note: $P^{\mu}_{i,lab} \neq P^{\mu}_{f,CM}$)

Therefore $P_i^2 = P_f^2$, note that P_i^2 and P_f^2 can be evaluated at different frame, since they are both Lorentz invariants.

The most convinient frame to evaluate P_f^2 is in the CM frame, since all the space components are zero, that is, $P^{\mu}_{f,CM} = (P^0_{f,CM}, 0, 0, 0)$

$$\Rightarrow P_f^2 = {P^0_{f,CM}}^2 = P_i^2 = 2m^2 + 2mE$$

Therefore the minimum of $P^0_{f,CM}$ gives the minimum of E

The minimum of $P^0_{f.CM}$ is $4m$ (i.e., all four final particles are at rest, no kinetic energy).

$$\Rightarrow \quad E_{min} = \frac{(4m)^2 - 2m^2}{2m} = 7m.$$

Question

Is that always possible to find a Center of momentum frame?

Suppose in an abitrary frame,

$$E_{tot} = \sum_i \gamma_i m_i \ , \ \vec{P}_{tot} = \sum_i \gamma_i m_i \vec{v}_i$$

note that for massless particle i, $\gamma_i m_i$ should be understood as $h\nu_i = \hbar\omega_i = \omega_i$, and $|\vec{v}_i| = 1$

Now make a Lorentz transformation along the direction of $\vec{P}_{tot}$

$$|\vec{P}'_{tot}| = \gamma(|\vec{P}_{tot}| - v E_{tot})$$

(note that since p^μ is a vector, it has the same Lorentz transformation as x^μ)

require $|\vec{P}'_{tot}| = 0$

$$\Rightarrow \quad v = \frac{|\vec{P}_{tot}|}{E_{tot}} = \frac{|\sum_i \gamma_i m_i \vec{v}_i|}{\sum_i \gamma_i m_i}$$

Using $|\sum_i \gamma_i m_i \vec{v}_i| = \Big[(\sum_i \gamma_i m_i |\vec{v}_i| \cos\theta_i)^2 + (\sum_i \gamma_i m_i |\vec{v}_i| \sin\theta_i \cos\varphi_i)^2 + (\sum_i \gamma_i m_i |\vec{v}_i| \sin\theta_i \sin\varphi_i)^2\Big]^{1/2}$

$$= \Big(\sum_i (\gamma_i m_i |\vec{v}_i|)^2 + 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \cos\theta_i \cos\theta_j$$
$$+ 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \sin\theta_i \sin\theta_j \cos\varphi_i \cos\varphi_j$$
$$+ 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \sin\theta_i \sin\theta_j \sin\varphi_i \sin\varphi_j \Big]^{\frac{1}{2}}$$

$$= \Big[\sum_i (\gamma_i m_i |\vec{v}_i|)^2 + 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \cos\theta_i \cos\theta_j$$
$$+ 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |v_i| \gamma_j m_j |v_j| \sin\theta_i \sin\theta_j \cos(\varphi_i - \varphi_j)\Big]^{\frac{1}{2}}$$
$$\leq \Big[\sum_i (\gamma_i m_i |\vec{v}_i|)^2 + 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| (\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j)\Big]^{\frac{1}{2}}$$

$$\leq \left[\sum_i (\gamma_i m_i |\vec{v}_i|)^2 + 2 \sum_{\substack{i,j \\ i \neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \right]^{\frac{1}{2}}$$

$$= \sum_i \gamma_i m_i |\vec{v}_i|$$

$$\leq \sum_i \gamma_i m_i$$

$$\Rightarrow \quad v \leq 1$$

the "=" is achieved only when all particles are massless and they move in the same direction, otherwise $v<1$ and it means a CM frame can be found.

Formally (a bit more tensor algebra)

For coordinate transformation $dx^\mu \longrightarrow dx^{\mu'} = \Lambda^\mu{}_\nu dx^\nu$.
its inverse transformation is $dx^{\mu'} \longrightarrow dx^\mu \equiv \bar{\Lambda}^\mu{}_\nu dx^{\nu'}$
(note that the prime ' is for x', not for μ', and the two coordinate systems are $x^\mu = (t, x, y, z)$ and $x^{\mu'} = (t', x', y', z')$)

From $dx^{\mu'} = \Lambda^\mu{}_\nu dx^\nu = \Lambda^\mu{}_\nu \bar{\Lambda}^\nu{}_\alpha dx^{\alpha'}$

$$\Rightarrow \quad \Lambda^\mu{}_\nu \bar{\Lambda}^\nu{}_\alpha = \delta^\mu{}_\alpha$$

From $dx^\mu = \bar{\Lambda}^\mu{}_\nu dx^{\nu'} = \bar{\Lambda}^\mu{}_\nu \Lambda^\nu{}_\beta dx^\beta$

$$\Rightarrow \quad \bar{\Lambda}^\mu{}_\nu \Lambda^\nu{}_\beta = \delta^\mu{}_\beta$$

That is, $\bar{\Lambda}^\mu{}_\lambda \Lambda^\lambda{}_\nu = \Lambda^\mu{}_\lambda \bar{\Lambda}^\lambda{}_\nu = \delta^\mu{}_\nu$.

where $\delta^\mu{}_\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}$

Therefore, without introducing $g_{\mu\nu}$, we can still define arbitrary covariant vector:

If the coordinate transformation is $dx^{\mu'} = \Lambda^\mu{}_\nu dx^\nu$.
and the inverse transformation is $dx^\mu = \bar{\Lambda}^\mu{}_\nu dx^{\nu'}$
where $\bar{\Lambda}^\mu{}_\lambda \Lambda^\lambda{}_\nu = \Lambda^\mu{}_\lambda \bar{\Lambda}^\lambda{}_\nu = \delta^\mu{}_\nu$

then we call A^μ a contravariant vector if it transform as

$$A^\mu \longrightarrow A^{\mu'} = \Lambda^\mu{}_\nu A^\nu \quad \text{(i.e., the same as } x^\mu\text{)}$$

and we call A_μ a covariant vector if it transform as

$$A_\mu \longrightarrow A_{\mu'} = \bar{\Lambda}^\nu{}_\mu A_\nu$$

note that $A^{\mu'} A_{\mu'} = A_{\mu'} A^{\mu'} = \Lambda^\mu{}_\lambda A^\lambda \bar{\Lambda}^\beta{}_\mu A_\beta = \delta^\beta{}_\lambda A^\lambda A_\beta = A^\beta A_\beta$

That is, $A^\mu A_\mu$ is an invariant under coordinate transformation.

Higher rank tensor can be defined as, e.g.,

$$A^{\mu\nu}{}_{\lambda} \longrightarrow A^{\mu\nu}{}_{\lambda}{}' = \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\rho} \bar{\Lambda}^{\tau}{}_{\lambda} A^{\sigma\rho}{}_{\tau}.$$

Show that Kronecker symbol is a rank two mixed tensor.

Since $\Lambda^{\mu}{}_{\lambda} \bar{\Lambda}^{\beta}{}_{\tau} \delta^{\lambda}_{\beta} = \Lambda^{\mu}{}_{\lambda} \bar{\Lambda}^{\lambda}{}_{\tau} = \delta^{\mu}{}_{\tau}$

while in all coordinate systems the Kronecker is the same,

that is, $\delta^{\mu}{}_{\tau}{}' = \delta^{\mu}{}_{\tau}$

$$\Rightarrow \quad \delta^{\mu}{}_{\tau}{}' = \Lambda^{\mu}{}_{\lambda} \bar{\Lambda}^{\beta}{}_{\tau} \delta^{\lambda}{}_{\beta}$$

which means Kronecker symbol is a rank two mixed tensor.

Since $ds^2 = d\tau^2$ is an invariant under coordinate transformation, i.e.,

$ds^2 = d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 = dt'^2 - dx'^2 - dy'^2 - dz'^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$

then $g_{\mu\nu}$ is a rank two covariant tensor $\qquad = g'_{\mu\nu} dx^{\mu\prime} dx^{\nu\prime}$

proof: $\quad g'_{\mu\nu} dx^{\mu\prime} dx^{\nu\prime} = g'_{\mu\nu} \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\beta} dx^{\lambda} dx^{\beta}$

$$= g_{\lambda\beta} dx^{\lambda} dx^{\beta}$$

$$\Rightarrow \quad g'_{\mu\nu} \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\beta} = g_{\lambda\beta}$$

$$\Rightarrow \quad g'_{\mu\nu} \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\beta} \bar{\Lambda}^{\lambda}{}_{\kappa} \bar{\Lambda}^{\beta}{}_{\gamma} = g_{\lambda\beta} \bar{\Lambda}^{\lambda}{}_{\kappa} \bar{\Lambda}^{\beta}{}_{\gamma}$$

$$\Rightarrow \quad g'_{\kappa\gamma} = g_{\lambda\beta} \bar{\Lambda}^{\lambda}{}_{\kappa} \bar{\Lambda}^{\beta}{}_{\gamma} \qquad \checkmark$$

The inverse of $g_{\mu\nu}$, which satisfies $g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu}$ is a rank two contravariant tensor

proof: $\quad g'^{\mu\lambda} g'_{\lambda\nu} = \delta^{\mu}{}_{\nu}$

$$= g'^{\mu\lambda} g_{\alpha\beta} \bar{\Lambda}^{\alpha}{}_{\lambda} \bar{\Lambda}^{\beta}{}_{\nu}$$

times $\Lambda^{\nu}{}_{\kappa}$ on both sides $\Rightarrow \quad g'^{\mu\lambda} g_{\alpha\beta} \bar{\Lambda}^{\alpha}{}_{\lambda} \bar{\Lambda}^{\beta}{}_{\nu} \Lambda^{\nu}{}_{\kappa} = \delta^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\kappa} = \Lambda^{\mu}{}_{\kappa}$

$$= g'^{\mu\lambda} g_{\alpha\beta} \bar{\Lambda}^{\alpha}{}_{\lambda} \delta^{\beta}{}_{\kappa} = g'^{\mu\lambda} g_{\alpha\kappa} \bar{\Lambda}^{\alpha}{}_{\lambda}$$

times $g^{\kappa\beta}$ on both sides

$$\Longrightarrow \quad g'^{\mu\lambda} g_{\alpha\kappa} \bar{\Lambda}^{\alpha}{}_{\lambda}\, g^{\kappa\beta} = \Lambda^{\mu}{}_{\kappa} g^{\kappa\beta}$$

$$= g'^{\mu\lambda} \bar{\Lambda}^{\beta}{}_{\lambda}$$

times $\Lambda^{\alpha}{}_{\beta}$ on both sides

$$\Longrightarrow \quad g'^{\mu\lambda} \bar{\Lambda}^{\beta}{}_{\lambda} \Lambda^{\alpha}{}_{\beta} = \Lambda^{\mu}{}_{\kappa} g^{\kappa\beta} \Lambda^{\alpha}{}_{\beta}$$

$$= g'^{\mu\alpha} \qquad \checkmark$$

Since Kronecker symbol is a unit matrix, it is OK to write it as $\delta^{\mu}{}_{\nu}$, or $\delta^{\mu\nu}$, or $\delta_{\mu\nu}$, or $g^{\mu}{}_{\nu}$.

Note 1: $x^{\mu} = (t, x, y, z) = (x^0, x^1, x^2, x^3)$

$$x_{\mu} = (t, -x, -y, -z) = (x^0, -x^1, -x^2, -x^3)$$

$$= (x_0, x_1, x_2, x_3)$$

i.e., $x^0 = x_0$, $x^i = -x_i$, similarly, $p^0 = p_0$, $p^i = -p_i$

Note 2: $\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} = (\partial_0, \vec{\nabla}) = (\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^i})$

$$\partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = (\partial_0, -\vec{\nabla})$$

d'Alembert $\Box \equiv \partial^2 \equiv \partial_{\mu}\partial^{\mu} = \left(\frac{\partial}{\partial x^0}\right)^2 - \vec{\nabla}^2$

(a bit more Lorentz transformation)

Lorentz Symmetry

$\boxed{\text{Warm up.}}$ —— write coordinate transformation using matrix.

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \overset{c=1}{=} (t, x, y, z) = (t, \vec{x}) = (x^0, \vec{x})$$

Usually, Greek indices take on the value 0, 1, 2, 3;

Latin indices $i, j, \ldots$, ———— 1, 2, 3, for space components.

ν is dummy index

Transformation of coordinate system from $\{x^\mu\}$ to $\{x'^\mu\}$

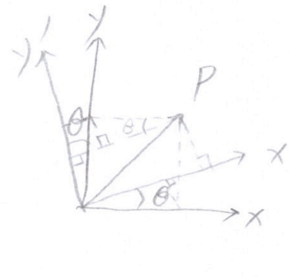
$$x'^\mu = a^\mu{}_\nu x^\nu = a^\mu{}_0 x^0 + a^\mu{}_1 x^1 + a^\mu{}_2 x^2 + a^\mu{}_3 x^3$$

(Einstein summation convention: when an index variable appear twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index.)

Example 1. Space Rotation (a rotation of coordinate system about the z axis by a positive angle θ in the counterclockwise direction)

$$a^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\theta \in [0, 2\pi)$



$$\Rightarrow \quad x'^0 = x^0$$
$$x'^1 = x^1 \cos\theta + x^2 \sin\theta$$
$$x'^2 = -x^1 \sin\theta + x^2 \cos\theta$$
$$x'^3 = x^3$$

Example 2. Pure Lorentz transformation (relates two coordinate systems differ only by a uniform relative motion of velocity v)

$$a^\mu{}_\nu = \begin{pmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad x'^0 = \cosh w \,(x^0 - x^1 \tanh w) = \gamma (x^0 - v x^1)$$
$$x'^1 = \cosh w \,(-x^0 \tanh w + x^1) = \gamma (-v x^0 + x^1)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

where $-\infty < \omega < +\infty$, $\cosh\omega = \gamma = \frac{1}{\sqrt{1-v^2}}$, $\tanh\omega = v$

Note that the two coordinate systems satisfy

$(x', y', z') = (x, y, z) = (0, 0, 0)$ at $t = t' = 0$.

Meaning: an event happens at (t, x, y, z) happens at (t', x', y', z')

→ an event happens at $(t=0, x=0, y=0, z=0)$ happens at $(t'=0, x'=0, y'=0, z'=0)$

Note: the benefit to use hyperbolic functions compared to directly use v and $\gamma = \frac{1}{\sqrt{1-v^2}}$:

If there is three reference frame A & B & C, suppose B moves relative to A by a constant velocity V_1, and C moves relative to B by a constant velocity V_2, and suppose V_1 & V_2 are both along x-direction and the origins of A & B & C coincide at $t = t' = t'' = 0$.

Then for an event at (t, x) in A, will have (t', x') in B and (t'', x'') in C satisfying (no need to worry y and z directions)

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\omega_1 & -\sinh\omega_1 \\ -\sinh\omega_1 & \cosh\omega_1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

and $$\begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh\omega_2 & -\sinh\omega_2 \\ -\sinh\omega_2 & \cosh\omega_2 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

Therefore $$\begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh\omega_2 & -\sinh\omega_2 \\ -\sinh\omega_2 & \cosh\omega_2 \end{pmatrix} \begin{pmatrix} \cosh\omega_1 & -\sinh\omega_1 \\ -\sinh\omega_1 & \cosh\omega_1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\omega_1+\omega_2) & -\sinh(\omega_1+\omega_2) \\ -\sinh(\omega_1+\omega_2) & \cosh(\omega_1+\omega_2) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

That is, frame C moving with velocity $\tanh(\omega_1+\omega_2)$ relative to frame A.

where $\tanh(\omega_1+\omega_2) = \frac{\tanh\omega_1 + \tanh\omega_2}{1+\tanh\omega_1 \tanh\omega_2} = \frac{V_1+V_2}{1+V_1 V_2}$.

(note: $\cosh(\omega_1+\omega_2) = \cosh\omega_1\cosh\omega_2 + \sinh\omega_1\sinh\omega_2$

$\sinh(\omega_1+\omega_2) = \sinh\omega_1\cosh\omega_2 + \cosh\omega_1\sinh\omega_2$)

Therefore, use hyperbolic functions make it easier for velocity add up.

The above velocity add up can be derived alternatively:

Consider an object move with velocity V_2 in B frame, and B frame move relative to A frame with velocity V_1, then what is the velocity the object relative to A?

use
$$t' = \gamma(t - v_1 x)$$
$$x' = \gamma(-v_1 t + x)$$

$$\Rightarrow V_2 = \frac{dx'}{dt'} = \frac{\gamma(-V_1\, dt + dx)}{\gamma(dt - V_1 dx)} = \frac{-V_1 + \frac{dx}{dt}}{1 - V_1 \frac{dx}{dt}}$$

$$\Rightarrow V_2 - V_2 V_1 \frac{dx}{dt} = -V_1 + \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{V_2 + V_1}{1 + V_1 V_2} \quad ✓$$

end worm up

Lorentz transformation

Lorentz transformations leave invariant the proper time interval $d\tau$ (or, write it as ds)

means it is the same in any inertial refrence frame

时空的四维间隔

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \overset{c=1}{=} dt^2 - dx^2 - dy^2 - dz^2.$$

$$ds^2 = \underbrace{g_{\mu\nu}}_{\text{度规 (metric)}} dx^\mu dx^\nu$$

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu.$$

in matrix form: $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Lorentz transformation (from GR section 2.1 by Weinberg).

The Principle of Special Relativity states that the laws of nature are invariant under a particular group of space-time coordinate transformations called Lorentz transformations.

Definition of Lorentz transformation.

$$x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta} + a^{\alpha}$$

$$\boxed{\text{Note that } \Lambda^{\alpha}{}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}}}$$

where a^{α} and $\Lambda^{\alpha}{}_{\beta}$ are constants, and $\Lambda^{\alpha}{}_{\beta}$ satisfy the conditions

$$\Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} \eta_{\alpha\beta} = \eta_{\gamma\delta}$$

where $\eta_{\alpha\beta} = \begin{cases} 1, & \alpha=\beta=0 \\ -1, & \alpha=\beta=1,2,3 \\ 0, & \alpha\neq\beta \end{cases}$

Lorentz transformations leave proper time invariant

proof: $d\tau^2 \equiv \eta_{\alpha\beta} dx^{\alpha} dx^{\beta} = dt^2 - dx^2 - dy^2 - dz^2$

since $x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta} + a^{\alpha}$,

then $dx'^{\alpha} = \Lambda^{\alpha}{}_{\beta} dx^{\beta}$

$$\Rightarrow d\tau'^2 = \eta'_{\alpha\beta} dx'^{\alpha} dx'^{\beta} = \eta'_{\alpha\beta} \Lambda^{\alpha}{}_{\gamma} dx^{\gamma} \Lambda^{\beta}{}_{\delta} dx^{\delta}$$

$$\underset{\eta_{\alpha\beta}=\eta'_{\alpha\beta}}{\overset{\text{since}}{=}} \eta_{\alpha\beta} \Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} dx^{\gamma} dx^{\delta}$$

$$\underset{=\eta_{\gamma\delta}}{\overset{\text{use } \Lambda^{\alpha}{}_{\gamma}\Lambda^{\beta}{}_{\delta}\eta_{\alpha\beta}}{=}} \eta_{\gamma\delta} dx^{\gamma} dx^{\delta}$$

$$= d\tau^2.$$

Done.

In particular, this means that the speed of light is the same in all inertial systems. (if the speed of light is $\left|\frac{d\vec{x}}{dt}\right| = 1$ in x frame, then $d\tau = 0$, then $d\tau' = 0$, then $\left|\frac{d\vec{x}'}{dt'}\right| = 1$).

The set of all Lorentz transformations $x'^{\alpha} = \Lambda^{\alpha}{}_{\beta} x^{\beta} + a^{\alpha}$ form a group called the inhomogeneous Lorentz group, or the Poincaré group. The subset with $a^{\alpha} = 0$ is called the homogeneous Lorentz group.

Both the inhomogeneous Lorentz group and homogeneous Lorentz group have subgroups called the proper inhomogeneous and homogeneous Lorentz group, defined by imposing

$$\Lambda^{0}{}_{0} \geq 1 \quad , \quad \operatorname{Det} \Lambda = +1$$

Note: ① $\Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta} \eta_{\alpha\beta} = \eta_{\gamma\delta} \Rightarrow (\Lambda^{0}{}_{0})^2 - \sum_{i=1}^{3} (\Lambda^{i}{}_{0})^2 = 1$

$\Rightarrow (\Lambda^{0}{}_{0})^2 = 1 + \sum_{i=1}^{3} (\Lambda^{i}{}_{0})^2 \geq 1$

② written in matrix form $\Lambda^{T} \eta \Lambda = \eta \Rightarrow \operatorname{Det}(\Lambda^{T} \eta \Lambda) = \operatorname{Det}(\eta)$

$\Downarrow$

$\operatorname{Det}(\Lambda^{T}) \operatorname{Det}(\eta) \underbrace{\operatorname{Det}(\Lambda)}_{= \operatorname{Det}(\eta)}$

$\Downarrow$

$(\operatorname{Det}(\Lambda))^2 \operatorname{Det}(\eta) = \operatorname{Det}(\eta)$

$\Downarrow$

$[\operatorname{Det}(\Lambda)]^2 = 1$

③ The identity matrix $\Lambda^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta}$ satisfy $\Lambda^{0}{}_{0} = 1$ and $\operatorname{Det} \Lambda = +1$

The proper homogeneous Lorentz group has a subgroup which is the rotation group: $\Lambda^{i}{}_{j} = R_{ij}$, $\Lambda^{i}{}_{0} = \Lambda^{0}{}_{i} = 0$, $\Lambda^{0}{}_{0} = 1$ where R_{ij} is a unimodular orthogonal matrix satisfying $\operatorname{Det} R = 1$ and $R^{T} R = 1$.

For example, rotation about z-axis counterclockwise by φ.

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}.$$

Derive a pure Lorentz boost in arbitrary direction.

Solution:

In class we gave the expression for a Lorentz boost in x-direction, which is

$$t' = \gamma(t - vx)$$
$$x' = \gamma(-vt + x)$$
$$y' = y$$
$$z' = z.$$

Since x-direction is of no special, the above expression can be understood as the change is in the boost direction only.

We can decompose the three-vector $\vec{x}$ as

$$\vec{x} = \vec{x}_{\parallel} + \vec{x}_{\perp}$$

where $\vec{x}_{\parallel}$ is in the direction of boost $\vec{v}$, and $\vec{x}_{\perp}$ is in the direction orthogonal to the boost direction $\vec{v}$

$$\Rightarrow \quad t' = \gamma(t - \vec{v}\cdot\vec{x}_{\parallel}) = \gamma(t - \vec{v}\cdot\vec{x}) = \gamma(t - v^1x^1 - v^2x^2 - v^3x^3)$$

where $\gamma = (1 - |\vec{v}|^2)^{-\frac{1}{2}}$

$$= [1 - (v^1)^2 - (v^2)^2 - (v^3)^2]^{-\frac{1}{2}}$$

$$\vec{x}'_{\parallel} = \gamma(-\vec{v}t + \vec{x}_{\parallel})$$
$$\vec{x}'_{\perp} = \vec{x}_{\perp}$$

$$\Rightarrow \quad \vec{x}' \equiv \vec{x}'_{\parallel} + \vec{x}'_{\perp} = \gamma(-\vec{v}t + \vec{x}_{\parallel}) + \vec{x}_{\perp}$$
$$= \gamma(-\vec{v}t + \vec{x}_{\parallel}) + \vec{x} - \vec{x}_{\parallel}$$
$$= \vec{x} + (\gamma - 1)\vec{x}_{\parallel} - \gamma\vec{v}t$$

Since $\vec{x}_{\parallel} = \vec{v}\,\dfrac{\vec{x}\cdot\vec{v}}{|\vec{v}|^2}$

then $$\vec{x}' = \vec{x} + (\gamma - 1)\frac{\vec{x}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} - \gamma\vec{v}t$$
$$= -\gamma\vec{v}t + \vec{x} + (\gamma - 1)\vec{v}\,\frac{v^1x^1 + v^2x^2 + v^3x^3}{|\vec{v}|^2}$$

$$\Rightarrow \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -v^1\gamma & -v^2\gamma & -v^3\gamma \\ -\gamma v^1 & 1+\frac{(\gamma-1)(v^1)^2}{|\vec{v}|^2} & (\gamma-1)\frac{v^1v^2}{|\vec{v}|^2} & (\gamma-1)\frac{v^1v^3}{|\vec{v}|^2} \\ -\gamma v^2 & (\gamma-1)\frac{v^1v^2}{|\vec{v}|^2} & 1+(\gamma-1)\frac{(v^2)^2}{|\vec{v}|^2} & (\gamma-1)\frac{v^2v^3}{|\vec{v}|^2} \\ -\gamma v^3 & (\gamma-1)\frac{v^1v^3}{|\vec{v}|^2} & (\gamma-1)\frac{v^2v^3}{|\vec{v}|^2} & 1+(\gamma-1)\frac{(v^3)^2}{|\vec{v}|^2} \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

See Mathematica code "general pure Lorentz boost.nb" for a different method.

How to derive a Lorentz boost in x-direction.

Solution:

Suppose $\bar{O}$ frame is moving with velocity v in the x-direction with respect to O, and the two frames overlap at $t=t'=0$.

[1] Consider a particle at rest in $\bar{O}$, $d\bar{x} = d\bar{y} = d\bar{z} = 0$.

Using $dx'^\alpha = \Lambda^\alpha{}_\beta \, dx^\beta$

$$\Rightarrow \begin{cases} \alpha=0, & dt' = \Lambda^0{}_0 \, dt \\ \alpha=1, & dx' = \Lambda^1{}_0 \, dt \\ \alpha=2, & dy' = \Lambda^2{}_0 \, dt \\ \alpha=3, & dz' = \Lambda^3{}_0 \, dt \end{cases}$$

Since $-v = \frac{dx'}{dt'}$, $\frac{dy'}{dt'} = 0$, $\frac{dz'}{dt'} = 0$

then $$\frac{\Lambda^1{}_0}{\Lambda^0{}_0} = -v, \quad \underbrace{\frac{\Lambda^2{}_0}{\Lambda^0{}_0} = 0, \quad \frac{\Lambda^3{}_0}{\Lambda^0{}_0} = 0}_{\Downarrow} \qquad ①$$

$$\Lambda^2{}_0 = 0, \quad \Lambda^3{}_0 = 0$$

Using $\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta}$

$$\Rightarrow \text{when } \gamma=\delta=0, \quad (\Lambda^0{}_0)^2 - (\Lambda^1{}_0)^2 = 1 \qquad ②$$

①② $\Rightarrow$ $(\Lambda^0{}_0)^2 - v^2(\Lambda^0{}_0)^2 = 1 \Rightarrow \Lambda^0{}_0 = \pm\frac{1}{\sqrt{1-v^2}} = \pm\gamma$

$\Lambda^1{}_0 = -v\gamma$

but take "+" sign for proper transformation.

[2] Consider a particle at rest in O', $dx' = dy' = dz' = 0$, $dy = 0$, $dz = 0$

Using $dx'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, dx^{\beta}$

$$\Rightarrow \begin{cases} \alpha = 0, & dt' = \Lambda^0{}_0\, dt + \Lambda^0{}_1\, dx \\ \alpha = 1, & dx' = \Lambda^1{}_0\, dt + \Lambda^1{}_1\, dx \\ \alpha = 2, & dy' = \Lambda^2{}_0\, dt + \Lambda^2{}_1\, dx \\ \alpha = 3, & dz' = \Lambda^3{}_0\, dt + \Lambda^3{}_1\, dx \end{cases}$$

since $\dfrac{dx}{dt} = v$

then $dt' = \gamma dt + \Lambda^0{}_1 v dt = (\gamma + \Lambda^0{}_1 v) dt$

$$0 = \Lambda^1{}_0 + \Lambda^1{}_1 v \Rightarrow \Lambda^1{}_1 = \gamma$$

$$0 = \Lambda^2{}_0 + \Lambda^2{}_1 v \Rightarrow \Lambda^2{}_1 = 0$$

$$0 = \Lambda^3{}_0 + \Lambda^3{}_1 v \Rightarrow \Lambda^3{}_1 = 0$$

using $\Lambda^{\alpha}{}_{\gamma} \Lambda^{\beta}{}_{\delta}\, \eta_{\alpha\beta} = \eta_{\gamma\delta}$

$\Rightarrow$ when $\gamma = \delta = 1$, $(\Lambda^0{}_1)^2 - (\Lambda^1{}_1)^2 = -1 \Rightarrow (\Lambda^0{}_1)^2 - \gamma^2 = -1$

$$\Rightarrow \Lambda^0{}_1 = \pm(\gamma^2 - 1)^{\frac{1}{2}}$$

$$= \pm\left(\frac{1 - 1 + v^2}{1 - v^2}\right)^{\frac{1}{2}}$$

$$= \pm v\gamma$$

On the other hand $d\tau = dt' = (dt^2 - dx^2)^{\frac{1}{2}} = dt(1 - v^2)^{\frac{1}{2}} = \gamma^{-1} dt$

$$\Rightarrow \gamma + \Lambda^0{}_1 v = \gamma^{-1}$$

$$\Rightarrow \Lambda^0{}_1 = \frac{\gamma^{-1} - \gamma}{v} = \frac{1 - \gamma^2}{\gamma v} = \frac{\frac{1 - v^2 - 1}{1 - v^2}}{\gamma v} = -\gamma v$$

By far, we have shown
$$\Lambda^0{}_0 = \gamma, \quad \Lambda^0{}_1 = -\gamma v,$$
$$\Lambda^1{}_0 = -v\gamma, \quad \Lambda^1{}_1 = \gamma$$
$$\Lambda^2{}_0 = \Lambda^3{}_0 = 0, \quad \Lambda^2{}_1 = \Lambda^3{}_1 = 0$$

Since the moving is along the x-axis, it has nothing to do with y- and z-axis, then
$\Lambda^0{}_2 = 0$, $\Lambda^0{}_3 = 0$, $\Lambda^1{}_2 = 0$, $\Lambda^1{}_3 = 0$, $\Lambda^2{}_2 = 1$, $\Lambda^2{}_3 = 0$, $\Lambda^3{}_2 = 0$, $\Lambda^3{}_3 = 1$

Therefore
$$\Lambda = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$