Solution.

1) $$\pi(\vec{x},t) = \frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)} = \dot{\phi}(\vec{x},t)$$

using $$\phi(x) = \phi(\vec{x},t) = \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3 2E(\vec{p})} \left(a(\vec{p}) e^{-ip\cdot x} + a^\dagger(\vec{p}) e^{ip\cdot x}\right)$$

$$\Rightarrow \pi(\vec{x},t) = \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3 2E(\vec{p})} (-iE(\vec{p}))\left(a(\vec{p}) e^{-ip\cdot x} - a^\dagger(\vec{p}) e^{ip\cdot x}\right)$$

$$= \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3} \left(-\frac{i}{2}\right)\left(a(\vec{p}) e^{-ip\cdot x} - a^\dagger(\vec{p}) e^{ip\cdot x}\right)$$

2) See derivition on the next page. If use the same way as in the lecture notes, then

$$a(\vec{p}) = \frac{1}{2}\left[\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\, (2\pi)^3 2E(\vec{p}) \frac{1}{iE(\vec{p})}\left[\phi(x)\left(\frac{\partial}{\partial t} e^{ip\cdot x}\right) - \left(\frac{\partial}{\partial t}\phi(x)\right) e^{ip\cdot x}\right]\right.$$

$$= -i\int_{-\infty}^{+\infty} d^3\vec{x}\left[\phi(x)\left(\frac{\partial}{\partial t} e^{ip\cdot x}\right) - \left(\frac{\partial}{\partial t}\phi(x)\right) e^{ip\cdot x}\right]$$

$$= -i\int_{-\infty}^{+\infty} d^3\vec{x}\left[iE(\vec{p})\phi(x) - \pi(x)\right] e^{ip\cdot x}$$

$$a^\dagger(\vec{p}) = i\int_{-\infty}^{+\infty} d^3\vec{x}\left[-iE(\vec{p})\phi(x) - \pi(x)\right] e^{-ip\cdot x}$$

3) $$[a(\vec{p}), a^\dagger(\vec{p}')] = \left[\int_{-\infty}^{+\infty} d^3\vec{x}\left(iE(\vec{p})\phi(t,\vec{x}) - \pi(t,\vec{x})\right) e^{iE(\vec{p})t - i\vec{p}\cdot\vec{x}}\right.,$$
$$\left.\int_{-\infty}^{+\infty} d^3\vec{y}\left(-iE(\vec{p}')\phi(t,\vec{y}) - \pi(t,\vec{y})\right) e^{-iE(\vec{p}')t + i\vec{p}'\cdot\vec{y}}\right]$$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{y}\Big(iE(\vec{p})(-i)\delta^3(\vec{x}-\vec{y})\, e^{iE(\vec{p})t - i\vec{p}\cdot\vec{x} - iE(\vec{p}')t + i\vec{p}'\cdot\vec{y}}$$
$$+ iE(\vec{p}')(-i)\delta^3(\vec{x}-\vec{y})\, e^{iE(\vec{p})t - i\vec{p}\cdot\vec{x} - iE(\vec{p}')t + i\vec{p}'\cdot\vec{y}}\Big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\left(E(\vec{p}) + E(\vec{p}')\right) e^{iE(\vec{p})t - iE(\vec{p}')t - i(\vec{p}-\vec{p}')\cdot\vec{x}}$$

$$= (2\pi)^3\, 2E(\vec{p})\cdot\delta^3(\vec{p}-\vec{p}')$$

$$[a(\vec{p}), a(\vec{p}')] = -\Big[\int_{-\infty}^{+\infty} d^3\vec{x}\,(iE(\vec{p})\phi(t,\vec{x}) - \pi(t,\vec{x}))e^{iE(\vec{p})t - i\vec{p}\cdot\vec{x}},$$
$$\int_{-\infty}^{+\infty} d^3\vec{y}\,(iE(\vec{p}')\phi(t,\vec{y}) - \pi(t,\vec{y}))e^{iE(\vec{p}')t - i\vec{p}'\cdot\vec{y}}\Big]$$
$$= -\int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{y}\,\big(iE(\vec{p})(-i)\delta^3(\vec{x}-\vec{y}) + iE(\vec{p}')\,i\delta^3(\vec{x}-\vec{y})\big)\cdot e^{iE(\vec{p})t - i\vec{p}\cdot\vec{x} + iE(\vec{p}')t - i\vec{p}'\cdot\vec{y}}$$
$$= -\int_{-\infty}^{+\infty} d^3\vec{x}\,\big(E(\vec{p}) - E(\vec{p}')\big)e^{iE(\vec{p})t - iE(\vec{p}')t - i(\vec{p}+\vec{p}')\cdot\vec{x}}$$
$$= -\int_{-\infty}^{+\infty} d^3\vec{x}\,(2\pi)^3\delta^3(\vec{p}+\vec{p}')\,(E(\vec{p}) - E(\vec{p}'))e^{iE(\vec{p})t - iE(\vec{p}')t}$$
$$= 0, \text{ since } E(\vec{p}) = E(-\vec{p})$$
$$[a^+(\vec{p}), a^+(\vec{p}')] = [a(\vec{p}'), a(\vec{p})]^+ = 0$$

4) $[\phi] = [E]^1$

$[\pi] = [E]^2$

$[a(\vec{p})] = [\phi] - [E]^2 = [E]^{-1}$

$[\delta^3(\vec{x}-\vec{x}')] = [E]^3$

Derive $a(\vec{p})$ from ϕ and π

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(t,\vec{x})e^{-ik\cdot x}d^3\vec{x} = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{p}\,\frac{1}{(2\pi)^3 2E(\vec{p})}(a(\vec{p})e^{-ip\cdot x} + a^+(\vec{p})e^{ip\cdot x})\cdot e^{-ik\cdot x}$$
$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{p}\,\frac{1}{(2\pi)^3 2E(\vec{p})}\big(a(\vec{p})e^{-iE(\vec{p})t - iE(\vec{k})t + i(\vec{p}+\vec{k})\cdot\vec{x}} + a^+(p)e^{iE(p)t - iE(k)t + i(\vec{k}-\vec{p})\cdot\vec{x}}\big)$$
$$= \int_{-\infty}^{+\infty} d^3\vec{p}\,\frac{1}{(2\pi)^3 2E(\vec{p})}\big(a(\vec{p})e^{-iE(\vec{p})t - iE(\vec{k})t}\delta^3(\vec{p}+\vec{k}) + a^+(\vec{p})e^{iE(\vec{p})t - iE(\vec{k})t}\delta^3(\vec{k}-\vec{p})\big)$$
$$= \frac{1}{(2\pi)^3 2E(\vec{k})}\big(a(-\vec{k})e^{-2iE(\vec{k})t} + a^+(k)\big)$$
$\uparrow$
$E(\vec{k}) = E(-\vec{k})$

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\pi(t,\vec{x})\,e^{-ik\cdot x}\,d^3\vec{x} = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{p}\,\frac{1}{(2\pi)^3}\left(-\frac{i}{2}\right)\left(a(\vec{p})e^{-ip\cdot x} - a^\dagger(\vec{p})e^{ip\cdot x}\right)\cdot e^{-ik\cdot x}$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} d^3\vec{x}\,d^3\vec{p}\,\frac{1}{(2\pi)^3}\left(-\frac{i}{2}\right)\Big(a(\vec{p})\,e^{-iE(\vec{p})t - iE(\vec{k})t + i(\vec{p}+\vec{k})\cdot\vec{x}} - a^\dagger(\vec{p})\,e^{iE(\vec{p})t - iE(\vec{k})t + i(\vec{k}-\vec{p})\cdot\vec{x}}\Big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\,\frac{1}{(2\pi)^3}\left(-\frac{i}{2}\right)\Big(a(\vec{p})\,e^{-iE(\vec{p})t - iE(\vec{k})t}\,\delta^3(\vec{p}+\vec{k}) - a^\dagger(p)\,e^{iE(\vec{p})t - iE(\vec{k})t}\,\delta^3(\vec{p}-\vec{k})\Big)$$

$$= \frac{1}{(2\pi)^3}\left(-\frac{i}{2}\right)\left(a(-\vec{k})\,e^{-2iE(\vec{k})t} - a^\dagger(\vec{k})\right)$$

$$\Rightarrow \quad a^\dagger(k) = \Big[\Big(\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(t,\vec{x})\,e^{-ik\cdot x}\,d^3\vec{x}\Big)(2\pi)^3\,2E(\vec{k}) - \Big(\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\pi(t,\vec{x})\,e^{-ik\cdot x}\,d^3\vec{x}\Big)(2\pi)^3\left(-\frac{2}{i}\right)\Big]\Big/2$$

$$\Rightarrow \quad a^\dagger(p) = i\int_{-\infty}^{+\infty} d^3\vec{x}\,\left[-iE(\vec{p})\,\phi(x) - \pi(x)\right]e^{-ip\cdot x}$$

$$\Rightarrow \quad a(p) = -i\int_{-\infty}^{+\infty} d^3\vec{x}\,\left[iE(\vec{p})\,\phi(x) - \pi(x)\right]e^{ip\cdot x}$$