spin $\frac{1}{2}$ fields

First, we know the Klein-Gordon equation makes sense. why?

non-relativistic : $\frac{P^2}{2m} = E$, $\hat{\vec{P}} \to -i\hbar\vec{\nabla}$, $E \to i\hbar\frac{\partial}{\partial t}$

$\Rightarrow -\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi$ (Schrödinger eq.)

relativistic : $P^2+m^2=E^2 \Rightarrow (-\nabla^2+m^2)\psi = -\frac{\partial^2}{\partial t^2}\psi \Rightarrow (\Box+m^2)\psi=0$ ($\hbar=1$)

Recall that for a free scalar field, the EoM is
↳ no interaction.

real : $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \partial_\mu\partial^\mu\phi = \Box\phi$

$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$

$\Rightarrow (\Box+m^2)\phi = 0$

complex : $\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \partial_\mu\partial^\mu\phi^* = \Box\phi^*$

$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi^*$

$\Rightarrow (\Box+m^2)\phi^* = 0$

$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right) = \partial_\mu\partial^\mu\phi = \Box\phi$

$\frac{\partial\mathcal{L}}{\partial\phi^*} = -m^2\phi$

$\Rightarrow (\Box+m^2)\phi = 0$

and Recall that for a massive real vector field, the EoM is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu \Rightarrow \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\frac{1}{4}\partial_\mu\left(\frac{(\partial_\rho A_\sigma - \partial_\sigma A_\rho)(\partial^\rho A^\sigma - \partial^\sigma A^\rho)}{\partial(\partial_\mu A_\nu)}\right)$$

$$= -\frac{2}{4}\partial_\mu(\partial^\rho A^\sigma - \partial^\sigma A^\rho)(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho)$$

$$= -\frac{1}{2}\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu - \partial^\nu A^\mu + \partial^\mu A^\nu)$$

$$= -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\partial_\mu F^{\mu\nu}$$

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = m^2 A^\nu$$

$\Rightarrow \partial_\mu F^{\mu\nu} + m^2A^\nu = 0$

$\Rightarrow \partial_\nu\partial_\mu F^{\mu\nu} + m^2\partial_\nu A^\nu = 0 \Rightarrow \partial_\nu A^\nu = 0$ (for $m \neq 0$)

$\Rightarrow (\Box + m^2)A^\nu = 0$

What about the spinor fields? Do they also satisfy Klein-Gordon equation?

Dirac equation

Dirac found the equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \quad \text{— Dirac equation}$$

apply $(i\gamma^\nu \partial_\nu + m)$ on both sides.

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$\Rightarrow (-\gamma^\nu\gamma^\mu \partial_\nu \partial_\mu - m^2)\psi(x) = 0$$

$$\Rightarrow \left[\frac{1}{2}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)\partial_\mu\partial_\nu + m^2\right]\psi(x) = 0$$

as long as $\frac{1}{2}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu) = g^{\mu\nu}$, then we get

$$(\Box + m^2)\psi(x) = 0 \quad \text{— Klein-Gordon equation}$$

Indeed it is the case.

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \qquad (\{\ \} \text{ is anticommutator})$$

note that $\gamma_\mu = g_{\mu\nu}\gamma^\nu$.

However, it turns out that $\psi(x)$ is a column $\psi(x) = \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix}$, not a simple number.

Therefore, γ's are 4X4 matrices.

So, the Dirac equation is actually

$$i\left[\begin{pmatrix}\gamma^0_{11} & \gamma^0_{12} & \gamma^0_{13} & \gamma^0_{14}\\ \gamma^0_{21} & \gamma^0_{22} & \gamma^0_{23} & \gamma^0_{24}\\ \gamma^0_{31} & \gamma^0_{32} & \gamma^0_{33} & \gamma^0_{34}\\ \gamma^0_{41} & \gamma^0_{42} & \gamma^0_{43} & \gamma^0_{44}\end{pmatrix}\partial_0 + \begin{pmatrix}\gamma^1_{11} & \gamma^1_{12} & \gamma^1_{13} & \gamma^1_{14}\\ \gamma^1_{21} & \gamma^1_{22} & \gamma^1_{23} & \gamma^1_{24}\\ \gamma^1_{31} & \gamma^1_{32} & \gamma^1_{33} & \gamma^1_{34}\\ \gamma^1_{41} & \gamma^1_{42} & \gamma^1_{43} & \gamma^1_{44}\end{pmatrix}\partial_1 + \begin{pmatrix}\gamma^2_{11} & \gamma^2_{12} & \gamma^2_{13} & \gamma^2_{14}\\ \gamma^2_{21} & \gamma^2_{22} & \gamma^2_{23} & \gamma^2_{24}\\ \gamma^2_{31} & \gamma^2_{32} & \gamma^2_{33} & \gamma^2_{34}\\ \gamma^2_{41} & \gamma^2_{42} & \gamma^2_{43} & \gamma^2_{44}\end{pmatrix}\partial_2\right.$$

$$\left.+ \begin{pmatrix}\gamma^3_{11} & \gamma^3_{12} & \gamma^3_{13} & \gamma^3_{14}\\ \gamma^3_{21} & \gamma^3_{22} & \gamma^3_{23} & \gamma^3_{24}\\ \gamma^3_{31} & \gamma^3_{32} & \gamma^3_{33} & \gamma^3_{34}\\ \gamma^3_{41} & \gamma^3_{42} & \gamma^3_{43} & \gamma^3_{44}\end{pmatrix}\partial_3 - m\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right]\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix} = 0$$

and the anticommutator relation is actually

$$\begin{pmatrix} \gamma^\mu_{11} & \gamma^\mu_{12} & \gamma^\mu_{13} & \gamma^\mu_{14} \\ \gamma^\mu_{21} & \gamma^\mu_{22} & \gamma^\mu_{23} & \gamma^\mu_{24} \\ \gamma^\mu_{31} & \gamma^\mu_{32} & \gamma^\mu_{33} & \gamma^\mu_{34} \\ \gamma^\mu_{41} & \gamma^\mu_{42} & \gamma^\mu_{43} & \gamma^\mu_{44} \end{pmatrix} \begin{pmatrix} \gamma^\nu_{11} & \gamma^\nu_{12} & \gamma^\nu_{13} & \gamma^\nu_{14} \\ \gamma^\nu_{21} & \gamma^\nu_{22} & \gamma^\nu_{23} & \gamma^\nu_{24} \\ \gamma^\nu_{31} & \gamma^\nu_{32} & \gamma^\nu_{33} & \gamma^\nu_{34} \\ \gamma^\nu_{41} & \gamma^\nu_{42} & \gamma^\nu_{43} & \gamma^\nu_{44} \end{pmatrix} - (\mu \leftrightarrow \nu) = 2g_{\mu\nu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that the dimensions of the Dirac spinor is 4. But, this has nothing to do with the dimensions of spacetime. That is, we cannot say that

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \text{ are the } \begin{pmatrix} \text{time component} \\ x\text{-component} \\ y\text{-component} \\ z\text{-component} \end{pmatrix}$$

It's just a coincidence that the space-time dimension and the spinor dimension are equal.

From the (matrices) relation

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}.$$

$$\Rightarrow \quad g^{\mu\alpha}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = 2g^{\mu\alpha} g_{\mu\nu} = 2\delta^\alpha{}_\nu$$

$$\Rightarrow \quad \gamma^\alpha \gamma_\nu + \gamma_\nu \gamma^\alpha = 2\delta^\alpha_\nu$$

$$\Rightarrow \quad \delta^\nu_\alpha(\gamma^\alpha \gamma_\nu + \gamma_\nu \gamma^\alpha) = 2\delta^\nu_\alpha \delta^\alpha_\nu = 2\delta^\nu_\nu = 2\times 4 = 8$$

$$\Rightarrow \quad 2\gamma^\alpha \gamma_\alpha = 8$$

$$\Rightarrow \quad \gamma^\alpha \gamma_\alpha = 4.$$

or, directly

$$g^{\mu\nu}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = 2g^{\mu\nu} g_{\mu\nu} = 2\delta^\mu_\mu = 8$$

$$\Rightarrow \quad \gamma^\alpha \gamma_\alpha = 4.$$

Also, from $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$

$$\Rightarrow \quad 2(\gamma_0)^2 = 2g_{00} = 2 \Rightarrow (\gamma_0)^2 = 1$$

$$2(\gamma_1)^2 = 2g_{11} = -2 \Rightarrow (\gamma_1)^2 = -1$$

similarly, $(\gamma_2)^2 = (\gamma_3)^2 = -1$

from $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

$$\Rightarrow \begin{cases} (\gamma^0)^2 = 1 \\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \end{cases}$$

or directly, $(\gamma^0)^2 = \gamma^0\gamma^0 = g^{0\mu}g^{0\nu}\gamma_\mu\gamma_\nu = g^{00}g^{00}\gamma_0\gamma_0$

$$= (\gamma_0)^2 = 1$$

$$(\gamma^1)^2 = \gamma^1\gamma^1 = g^{1\mu}g^{1\nu}\gamma_\mu\gamma_\nu = g^{11}g^{11}\gamma_1\gamma_1 = -1$$

and similar for $(\gamma^2)^2$ and $(\gamma^3)^2$.

Moreover, from $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$, we can derive

$$\gamma_0\gamma_i + \gamma_i\gamma_0 = 0 \Rightarrow \gamma_0\gamma_i = -\gamma_i\gamma_0$$

$$\gamma_i\gamma_j + \gamma_j\gamma_i = \begin{cases} -2, & \text{if } i=j \\ 0, & \text{if } i\neq j \end{cases}$$

i.e., $\gamma_i^2 = -1 \quad (i=1,2,3)$

$\gamma_i\gamma_j = -\gamma_j\gamma_i \quad (i\neq j)$.

$$\Rightarrow \quad \underbrace{\gamma_i\gamma_0\gamma_i}_{\text{not sum over } i} = -\gamma_i\gamma_i\gamma_0 = \gamma_0$$

$$\gamma_0\gamma_i\gamma_0 = -\gamma_0\gamma_0\gamma_i = -\gamma_i$$

The same for upper indices: $\gamma^0\gamma^i = -\gamma^i\gamma^0$

$\gamma^i\gamma^j = -\gamma^j\gamma^i \quad (\text{if } i\neq j)$

Use the property of trace: for a $N \times M$ matrix A and a $M \times N$ matrix B, if for all $i=1,\cdots,N$, $j=1,\cdots,M$; $A_{ij}B_{ji} = B_{ji}A_{ij}$,

then $$Tr(AB) = Tr(BA).$$

proof: $$Tr(AB) = \sum_{i=1}^{N}\sum_{j=1}^{M} A_{ij}B_{ji}$$

$$Tr(BA) = \sum_{j=1}^{M}\sum_{i=1}^{N} B_{ji}A_{ij}$$

since $A_{ij}B_{ji} = B_{ji}A_{ij}$

then $Tr(AB) = Tr(BA)$

Since the elements of the gamma matrices are pure numbers,

and $\gamma_0\gamma_i\gamma_0 = -\gamma_i$,

then
$$Tr(\gamma_0\gamma_i\gamma_0) = -Tr(\gamma_i)$$
$$\parallel$$
$$Tr(\gamma_0\gamma_0\gamma_i)$$
$$\parallel$$
$$Tr(\gamma_i)$$

$\Rightarrow$ $Tr(\gamma_i) = 0$

also, since $\gamma_i\gamma_0\gamma_i = \gamma_0$ (not sum over i)

then
$$Tr(\gamma_i\gamma_0\gamma_i) = Tr(\gamma_0)$$
$$\parallel$$
$$Tr(\gamma_i\gamma_i\gamma_0)$$
$$\parallel$$
$$Tr(-\gamma_0)$$
$$\parallel$$
$$-Tr(\gamma_0)$$

$\Rightarrow$ $Tr(\gamma_0) = 0$

Therefore, for $\mu = 0, 1, 2, 3$, $Tr(\gamma_\mu) = 0$

For later convinience, let's introduce γ^5 ($\gamma_5 \equiv \gamma^5$)

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

If we define the completely antisymmetric symbol $\varepsilon_{\lambda\mu\nu\pi}$ as

$\varepsilon_{0123} = +1$ (i.e., $\varepsilon_{\lambda\mu\nu\pi}$ is equal to $+1$ for (λ, μ, ν, π) an even permutation of $(0,1,2,3)$, is equal to -1 for an odd permutation, and vanishes if two or more indices are the same).

then $\gamma^5 = \frac{i}{4!} \varepsilon_{\lambda\mu\nu\pi} \gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\pi$

In addition to the anticommutation relation

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}.$$

there is one more property of γ_μ

$$\gamma_\mu^+ = \gamma_0\gamma_\mu\gamma_0$$

These two properties, together with the definition of $\gamma^\mu \equiv g^{\mu\nu}\gamma_\nu$, can be used to derive all the other properties of γ matrices independent of representations.

Convention:

$$\not{A} \equiv \gamma^\mu A_\mu$$

$$\not{\partial} \equiv \gamma^\mu \partial_\mu = \gamma^0 \frac{\partial}{\partial t} + \vec{\gamma}\cdot\vec{\nabla}.$$

Therefore, the Dirac equation can be written as

$$(i\not{\partial} - m)\psi(x) = 0$$

Let's find the Hermitian conjugation of this equation:

$$[(i\gamma^\mu\partial_\mu - m)\psi]^+ = 0^+ = 0$$

$$\Rightarrow \psi^+(-i\gamma^{\mu+}\overleftarrow{\partial}_\mu - m) = 0$$

times γ^0 from the right $\Rightarrow$ $\psi^+(-i\gamma^{\mu+}\overleftarrow{\partial}_\mu\gamma^0 - m\gamma^0) = 0$

use $\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 \Rightarrow \gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu\gamma^0\gamma^0 = \gamma^0\gamma^\mu$

$$\Rightarrow \quad \psi^\dagger \gamma^0 \gamma^\mu (-i) \overleftarrow{\partial}_\mu - \psi^\dagger \gamma^0 m = 0$$

Define $\bar\psi(x) \equiv \psi^\dagger(x)\gamma^0$ and $\bar\psi \overleftarrow{\not\partial} \equiv \partial_\mu \bar\psi \gamma^\mu$

$$\Rightarrow \quad \bar\psi (i\overleftarrow{\not\partial} + m) = 0$$

So $\bar\psi$ is a row vector.

Some popular representations of gamma matrices

① Dirac-Pauli representation, also called the standard representation

$$\gamma^0 = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \text{ i.e., } \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \gamma_1 = g_{1\mu}\gamma^\mu = g_{11}\gamma^1 = -\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = -\gamma^2 = \cdots$$

$$\gamma_3 = -\gamma^3 = \cdots$$

$$\gamma_0 = g_{0\mu}\gamma^\mu = g_{00}\gamma^0 = \gamma^0 = \cdots$$

$$\Rightarrow \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

② Weyl repr[e]sentation

$$\gamma^0 = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \Rightarrow \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

where σ^i are Pauli-matrices.

③ Majorana repr[e]sentation.

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

$$\Rightarrow \gamma^5 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$$

In the standard repr[e]sentation, if write $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$,

then $$\bar{\psi} = \psi^\dagger \gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$