Homework 1
Due date: 2018.10.15

Problem 1. The critical density of the Universe is

$$\rho_c = \frac{3H_0^2}{8\pi G_N}\,,$$

where H_0 is the present day Hubble expansion rate, given as $H_0 = 100\,h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, where h is a dimensionless number. In the following questions, use $G_N = 6.674\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, $1\,\mathrm{Mpc} = 3.086\times10^{22}\,\mathrm{m}$, the Solar mass $M_\odot = 1.988\times10^{30}\,\mathrm{kg}$, and take $h = 0.7$.

Please round your results to two significant figures for 1), 2), 3) and 5), and don't worry about uncertainties.

1) What is the value of $1/H_0$ in Gyr? (1 Gyr = 10^9 year) [1 point]

2) What is the value of ρ_c in $\mathrm{g/cm^3}$? [1 point]

3) What is the value of ρ_c in $M_\odot/\mathrm{Mpc^3}$? [1 point]

4) In SI units, the Planck density (i.e., mass per unit volume), ρ_{Pl}, can be constructed as $\hbar^{n_1}\,c^{n_2}\,G_N^{n_3}$. Find n_1, n_2 and n_3. [1 point]

5) In $\hbar = c = 1$ units, what are the values of ρ_{Pl} and ρ_c in $\mathrm{GeV^4}$? [1 point] (Please note that we have given in class the relations between m & s & kg in SI units and the power of MeV or GeV in $\hbar = c = 1$ units. However, at that time we only kept one or two significant figures. Therefore, you might want to re-calculate them keeping more significant figures in order to achieve the correct two significant figure results of ρ_{Pl} and ρ_c.)

Problem 2. Given two four-vectors, $a^\mu = (2, 0, 1, 8)$ and $b^\mu = (0, 9, 2, 7)$, find: a_μ, b_μ, $\vec{a}\cdot\vec{a}$, $\vec{a}\cdot\vec{b}$, a^2, $a\cdot b$, $(a+b)^2$. [2 points]

Note: please use the convention $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$ & $g_{\mu\nu} = 0$ (when $\mu \neq \nu$) for the calculations. A three-vector, e.g., $\vec{a}$, is the one appearing in $a^\mu = (a^0, \vec{a})$. A four-vector dot product, e.g., $a\cdot b$, is defined as $a\cdot b \equiv a^\mu b_\mu$. In particular, $a^2 \equiv a\cdot a$.

Problem 3. In a two-body scattering event, A + B $\to$ C + D, it is convenient to introduce the *Mandelstam variables* $s \equiv (p_A + p_B)^2$, $t \equiv (p_A - p_C)^2$ and $u \equiv (p_A - p_D)^2$ (note that the square of a four-momentum should be always understood as, for example, $a^2 \equiv a\cdot a \equiv a^\mu a_\mu$). **The virtue of the Mandelstam variables is that they are Lorentz invariants, with the same value in any inertial system.**

1) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$. [1 point]

2) Find the energy of A in the center-of-momentum (CM) frame, in terms of s, m_A and m_B. [2 points]

3) Find the energy of A in B's rest frame, in terms of s, m_A and m_B. [2 points]

Problem 4. In the perfect vacuum, an electron and a positron can annihilate into two or more photons ($e^+ + e^- \to \mathrm{n}\,\gamma$, with n ≥ 2). From the view of conservation of energy and momentum, please explain why the single photon creation process cannot happen. [1 point]