Quantization of a scalar field

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A scalar field, a component of a vector field and a component of a spinor field all satisfy the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0$$

$$(\Box + m^2)A_\mu(x) = 0 \quad \text{for } \mu = 0,1,2,3$$

$$(\Box + m^2)\psi_a(x) = 0 \quad \text{for } a = 1,2,3,4$$

where $\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$

We have seen that the Klein-Gordon equation can be constructed by using the relativistic four-momentum relation

$E^2 - \vec{p}^{\,2} = m^2$ $\quad (c=1)$, note that $\vec{p}^{\,2} = \vec{p}\cdot\vec{p} = |\vec{p}|^2$

and the identifications

$$E \to i\hbar\frac{\partial}{\partial t}, \quad \vec{p} \to -i\hbar\vec{\nabla}$$

$$\Rightarrow \left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right)\varphi(x) = m^2\varphi(x), \quad (\hbar = 1)$$

$$\Rightarrow (\Box + m^2)\varphi(x) = 0$$

$\varphi(x)$ ← a generic field.

On the other hand, the Schrödinger equation, which describes the motion of a non-relativistic particle, can be constructed as

$$\frac{\vec{p}^{\,2}}{2m} \longrightarrow -\frac{\hbar^2}{2m}\vec{\nabla}^2$$

$$E \longrightarrow i\hbar\frac{\partial}{\partial t}$$

$$\Rightarrow i\hbar\frac{\partial}{\partial t}\varphi(\vec{x},t) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\varphi(\vec{x},t)$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t} \varphi(\vec{x}, t) = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \varphi(\vec{x}, t)$$

(Note that the above Schrödinger equation does not have the potential term, otherwise it is $i\hbar \frac{\partial}{\partial t}\varphi(\vec{x},t) = (-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V)\varphi(\vec{x},t)$)

The solution of a free particle (i.e. $V=0$) Schrödinger equation is

$$\varphi(\vec{x}, t) \propto \exp(-iEt + i\vec{p}\cdot\vec{x}) \quad , \quad (\hbar = 1)$$

In general,

$$\varphi(\vec{x},t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \varphi(\vec{p}) \exp(-iEt + i\vec{p}\cdot\vec{x})\, d^3\vec{p}$$

where $E = \frac{\vec{p}^{\,2}}{2m}$

$$\left(\begin{aligned} \text{check: } i\frac{\partial}{\partial t}\varphi(\vec{x},t) &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \varphi(\vec{p})\, E \exp(-iEt + i\vec{p}\cdot\vec{x})\, d^3\vec{p} \\ -\frac{1}{2m}\vec{\nabla}^2\varphi(\vec{x},t) &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} \varphi(\vec{p})\, \frac{\vec{p}^{\,2}}{2m} \exp(-iEt + i\vec{p}\cdot\vec{x})\, d^3\vec{p} \\ \Rightarrow \quad i\frac{\partial}{\partial t}\varphi(\vec{x},t) &= -\frac{1}{2m}\vec{\nabla}^2\varphi(\vec{x},t) \quad \text{due to } E = \frac{\vec{p}^{\,2}}{2m} \end{aligned} \right)$$

Similarly, for the Klein-Gordon equation,

$$(\Box + m^2)\varphi(x) = 0$$

if interpret $\varphi(x)$ as a single particle wave function, then the solutions are

$$\varphi(x) \propto \begin{cases} \exp(-iEt + i\vec{p}\cdot\vec{x}) = e^{-ip\cdot x} \\ \exp(+iEt + i\vec{p}\cdot\vec{x}) \\ \exp(-iEt - i\vec{p}\cdot\vec{x}) \\ \exp(+iEt - i\vec{p}\cdot\vec{x}) = e^{ip\cdot x} \end{cases}$$

In general,

$$\varphi(x) = \int_{-\infty}^{+\infty} \underbrace{C(E)}_{\text{normalization factor}} [g(\vec{p})e^{-ip\cdot x} + h(\vec{p})e^{ip\cdot x}]\, d^3\vec{p}$$

where $E^2 = \vec{p}^{\,2} + m^2$.

normalization factor

where $E^2 = \vec{p}^2 + m^2$.

note that we don't have to worry about the solution $\exp(-iEt - i\vec{p}\cdot\vec{x})$ once we have used $\exp(-iEt + i\vec{p}\cdot\vec{x}) = \exp(-ip\cdot x)$ to build the general solution, since

$$\int_a^b f(x)\,dx = \int_{-a}^{-b} f(-y)\,d(-y) = \int_{-b}^{-a} f(-y)\,dy = \int_{-b}^{-a} f(-x)\,dx$$

and therefore

$$\int_{-\infty}^{+\infty} C(E)\left[g(\vec{p})e^{-iEt + i\vec{p}\cdot\vec{x}} + g_1(\vec{p})e^{-iEt - i\vec{p}\cdot\vec{x}}\right] d^3\vec{p}$$

$$= \int_{-\infty}^{+\infty} C(E)\left[\underbrace{g(\vec{p}) + g_1(-\vec{p})}_{\text{redefine it as } g(\vec{p})}\right] e^{-iEt + i\vec{p}\cdot\vec{x}}\, d^3\vec{p}$$

Similarly, we don't need to worry about the solution $\exp(iEt + i\vec{p}\cdot\vec{x})$

since $$\int_{-\infty}^{+\infty} C(E)\left[h(\vec{p})e^{iEt - i\vec{p}\cdot\vec{x}} + h_1(\vec{p})e^{iEt + i\vec{p}\cdot\vec{x}}\right] d^3\vec{p}$$

$$= \int_{-\infty}^{+\infty} C(E)\left[\underbrace{h(\vec{p}) + h_1(-\vec{p})}_{\text{redefine it as } h(\vec{p})}\right] e^{iEt - i\vec{p}\cdot\vec{x}}\, d^3\vec{p}$$

Note that $g(\vec{p})$ and $h(\vec{p})$ should be understood as having Lorentz index when $\varphi(x)$ is a vector, and taking the form of 4×1 columns when $\varphi(x)$ is a Dirac spinor.

Now let's focus on the scalar field.

Let's look at the probability density and current density in quantum mechanics:

times ψ^* on $i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = [-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V]\psi(\vec{x},t)$

and minus ψ times on $-i\hbar\frac{\partial}{\partial t}\psi^* = [-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V^*]\psi^*$

and assume $V = V^*$

$$\Rightarrow \quad i\hbar\frac{\partial}{\partial t}(\psi\psi^*) = -\frac{\hbar^2}{2m}(\psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*)$$

$$= -\frac{\hbar^2}{2m}\vec{\nabla}\cdot(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$$

let $\rho \equiv \psi^*\psi$, $\vec{j} \equiv -\frac{i\hbar}{2m}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)$

$$\Rightarrow \quad \frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} = 0.$$

For a complex scalar field, the internal transformation gives $\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - m^2\phi\phi^*$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(-i\phi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}i\phi^*$$

$$= \partial^\mu\phi^*(-i\phi) + \partial^\mu\phi\,(i\phi^*)$$

that is, $j^0 = \dot{\phi}^*(-i\phi) + \dot{\phi}(i\phi^*) = i(\dot{\phi}\phi^* - \dot{\phi}^*\phi)$

$$\vec{j} = -\vec{\nabla}\phi^*(-i\phi) - \vec{\nabla}\phi\,(i\phi^*) = -i(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*)$$

Also, $\partial_\mu j^\mu = 0$ from Noether theorem.

So the $\vec{j}$ here takes the same form as in the probability current in quantum mechanics up to a constant factor $\frac{1}{2m}$.

Also, the current j^μ for complex scalar field can also be derived in a similar way as in quantum mechanics:

times ϕ^* on $(\Box + m^2)\phi = 0$

and minus ϕ times on $(\Box + m^2)\phi^* = 0$

$$\Rightarrow \quad \phi^* \partial_\mu \partial^\mu \phi - \phi \partial_\mu \partial^\mu \phi^* = 0$$

$$\Rightarrow \quad \partial_\mu \left[i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \right] = 0$$

so that $\partial_\mu j^\mu = 0$ if define $j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$

Does it mean that we can interpret j^0 as the probability density as in quantum mechanics? No, we cannot.

In quantum mechanics,

$$\rho = |\psi|^2 = \psi^* \psi \geq 0$$

↳ wave function is a complex function

However, if interpret the klein-Gordon equation as a wave equation, then for the wave function solution

$$\phi(x) = C e^{-ip\cdot x}$$

we have $\phi^*(x) = C^* e^{+ipx}$, $\dot{\phi}(x) = C(-iE) e^{-ipx} = (-iE)\phi(x)$

$$\dot{\phi}^*(x) = C^*(+iE) e^{+ip\cdot x} = (+iE)\phi^*$$

$$\Rightarrow j^0 = i(\dot{\phi}\phi^* - \dot{\phi}^*\phi) = i\left[(-iE)|\phi|^2 - (iE)|\phi|^2\right] = 2E|\phi|^2 \geq 0$$

But, for $\phi(x) = C e^{ip\cdot x}$

we have $\phi^*(x) = C^* e^{-ip\cdot x}$, $\dot{\phi}(x) = C(iE) e^{ipx} = (iE)\phi(x)$

$$\dot{\phi}^*(x) = C^*(-iE) e^{-ip\cdot x} = (-iE)\phi^*$$

$$\Rightarrow j^0 = i(\dot{\phi}\phi^* - \dot{\phi}^*\phi) = i\left[(iE)|\phi|^2 + iE|\phi|^2\right] = -2E|\phi|^2 \leq 0.$$

In the non-relativistic limit, $E \simeq m$

$\Rightarrow$ for $\phi(x) = C e^{-ip\cdot x}$, $j^0 \simeq 2m|\phi|^2$, so that up to the same constant factor $\frac{1}{2m}$ as for $\vec{j}$, j^0 is analogue to the ρ in quantum mechanics.

However, for $\phi(x) = Ce^{ip\cdot x}$, $j^0 \simeq -2m|\phi|^2$, the sign is opposite,

So, we cannot interpret j^0 as the probability density, which should be ≥ 0. It should be interpreted as a charge density, and it can be checked that $\vec{j}$ also take opposite signs for the two solutions:

$$\text{for } \phi(x) = Ce^{-ip\cdot x} = Ce^{-iEt+i\vec{p}\cdot\vec{x}} \Rightarrow \vec{\nabla}\phi(x) = i\vec{p}\,\phi(x)$$

$$\phi^*(x) = C^* e^{ipx} \quad , \vec{\nabla}\phi^*(x) = (-i\vec{p}\,)\,\phi^*(x)$$

$$\Rightarrow \vec{j} = -i(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*) = -i[(i\vec{p}\,)|\phi|^2 + (i\vec{p}\,)|\phi|^2] = 2\vec{p}\,|\phi|^2$$

while for $\phi(x) = Ce^{ip\cdot x} = Ce^{iEt-i\vec{p}\cdot\vec{x}} \Rightarrow \vec{\nabla}\phi(x) = (-i\vec{p}\,)\,\phi(x)$

$$\phi^*(x) = C^* e^{-ip\cdot x}, \quad \vec{\nabla}\phi^*(x) = (i\vec{p}\,)\,\phi^*(x)$$

$$\Rightarrow \vec{j} = -i(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*) = -i[(-i\vec{p}\,)|\phi|^2 - (i\vec{p}\,)|\phi|^2] = -2\vec{p}\,|\phi|^2$$

So, for both cases, j^0 and $\vec{j}$ form a four-vector.

$$j^\mu = 2p^\mu|\phi|^2, \text{ for } \phi = Ce^{-ipx}$$

$$j^\mu = -2p^\mu|\phi|^2, \text{ for } \phi = Ce^{ipx}$$

This also explains why we choose e^{ipx} and e^{-ipx} as the building blocks for the general solution $\varphi(x) = \int_{-\infty}^{+\infty} C(E)[g(\vec{p})\,e^{-ip\cdot x} + h(\vec{p})\,e^{ip\cdot x}]d\vec{p}^{\,3}$.

If we choose the building blocks as $\phi = Ce^{-iEt-i\vec{p}\cdot\vec{x}}$, then $j^0 = 2E|\phi|^2$ and $\vec{j} = -2\vec{p}\,|\phi|^2$, then $j^\mu \neq 2p^\mu|\phi|^2$;

for $\phi = Ce^{+iEt+i\vec{p}\cdot\vec{x}}$, $\Rightarrow j^0 = -2E|\phi|^2$, $\vec{j} = 2\vec{p}\,|\phi|^2 \Rightarrow j^\mu \neq -2p^\mu|\phi|^2$

Quantization of a scalar field.

The Hamiltonian for a generic field was obtained before

$$H = \int d^3x\,(\pi\,\partial_0\varphi - \mathcal{L}) \equiv \int d^3x\,\mathcal{H}$$

where $\pi = \dfrac{\partial\mathcal{L}}{\partial(\partial\varphi/\partial t)}$

note that $\pi\,\partial_0\varphi$ should be understood as $\sum_i \left(\pi_i \dfrac{\partial\varphi_i}{\partial t}\right)$ if there are more than one field

① For a free real scalar field,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\Rightarrow\ \pi = \frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial t)} = \partial_0\phi = \dot{\phi}$$

$$\Rightarrow\ \mathcal{H} = \dot{\phi}^2 - \left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right)$$

$$= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

② For a free complex scalar field,

$$\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi^* - m^2\phi\phi^*$$

$$\Rightarrow\ \pi = \frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial t)} = \partial_0\phi^* = \dot{\phi}^*$$

$$\pi^* = \frac{\partial\mathcal{L}}{\partial(\partial\phi^*/\partial t)} = \partial_0\phi = \dot{\phi}$$

$$\Rightarrow\ \mathcal{H} = \dot{\phi}^*\dot{\phi} + \dot{\phi}\dot{\phi}^* - (\partial_\mu\phi\,\partial^\mu\phi^* - m^2\phi\phi^*)$$

$$= \dot{\phi}\dot{\phi}^* + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi^*) + m^2\phi\phi^*$$

The $\underset{\delta x^\mu = 0}{\underline{\text{internal}}}$ $\underset{\phi'(x) = e^{-i\alpha}\phi(x),\ \phi'^*(x) = e^{i\alpha}\phi^*(x)}{\underline{\text{transformation}}}$ of free complex scalar field gives

$$Q = \int d^3x\, j^0 = i\int d^3x\,(\phi\dot{\phi}^* - \dot{\phi}^*\phi)$$

where $j^\mu = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(-i\phi) + \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\, i\phi^*$.

Considering that we need to describe creating & destroying particles, and considering that the building block e^{ipx} does not give a non-negative j^0 which can be anologue to ρ in quantum mechanics, we re-interpret $\phi(x)$ as a quantum operator rather than a wave function (so that the Klein-Gordon equation becomes an equation for the field operator)

$$\phi(x) = \int_{-\infty}^{+\infty} C(E)\left[a_{\vec{p}}\, e^{-ip\cdot x} + b_{\vec{p}}^{+}\, e^{ip\cdot x}\right] d^3\vec{p}$$

and $$\phi^{+}(x) = \int_{-\infty}^{+\infty} C^{*}(E)\left[a_{\vec{p}}^{+}\, e^{ip\cdot x} + b_{\vec{p}}\, e^{-ip\cdot x}\right] d^3\vec{p}$$

For real scalar field, $\phi(x) = \phi^{+}(x)$

$$\Rightarrow \phi(x) = \int_{-\infty}^{+\infty} \underbrace{C(E)}_{\text{real}} \left[a_{\vec{p}}\, e^{-ip\cdot x} + a_{\vec{p}}^{+}\, e^{ip\cdot x}\right] d^3\vec{p}$$

The $a_{\vec{p}}$ & $b_{\vec{p}}$ are annihilation operators,
and the $a_{\vec{p}}^{+}$ & $b_{\vec{p}}^{+}$ are creation operators.

Let's first consider the real scalar field.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial t)} = \dot{\phi}$$

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

when interpret the field as operator, we expect the following commutation relations:

$$[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\delta^3(\vec{x}-\vec{x}')$$

$$[\phi(\vec{x},t), \phi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0$$

so, it's a postulate, need to be verified by experiments.

Note, recall that in quantum mechanics, $[X_i, P_j] = i\delta_{ij}$ $(i,j=1,2,3)$, $[X_i, X_j] = [P_i, P_j] = 0$
The Euler-Lagrangian equation for classical mechanics is $\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$, and the

one for field theory is $\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0$. The canonical momentum in classical mechanics is $p = \frac{\partial L}{\partial \dot{q}}$, while the one for field theory is $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$.
Therefore, when quantize the field, we can expect the above commutation relations.

With these commutation relations, let's derive the commutation relations for $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$.

From $\phi(x) = \int_{-\infty}^{+\infty} C(E) [a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^\dagger e^{ip\cdot x}] d^3\vec{p}$

$$\Rightarrow \dot{\phi}(x) = \int_{-\infty}^{+\infty} C(E)(-iE) [a_{\vec{p}} e^{-ip\cdot x} - a_{\vec{p}}^\dagger e^{ip\cdot x}] d^3\vec{p}$$

using $\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} e^{i(\vec{p}-\vec{p}')\cdot\vec{x}} d^3\vec{x} = \delta^3(\vec{p}-\vec{p}')$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} \phi(x) e^{-i\vec{p}'\cdot\vec{x}} d^3\vec{x} = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C(E)[a_{\vec{p}} e^{-iEt+i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{iEt-i\vec{p}\cdot\vec{x}}] e^{-i\vec{p}'\cdot\vec{x}'} d^3\vec{p}\, d^3\vec{x}$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} \phi(x) e^{-i\vec{p}'\cdot\vec{x}} d^3\vec{x} = \int_{-\infty}^{+\infty} C(E)\left[a_{\vec{p}} e^{-iEt} \delta^3(\vec{p}-\vec{p}') + a_{\vec{p}}^\dagger e^{iEt} \delta^3(\vec{p}+\vec{p}')\right] d^3\vec{p}$$

$$= C(E')[a_{\vec{p}'} e^{-iE't} + a_{-\vec{p}'}^\dagger e^{iE't}]$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} \phi(x) e^{iEt-i\vec{p}\cdot\vec{x}} = C(E)(a_{\vec{p}} + a_{-\vec{p}}^\dagger e^{2iEt})$$

while $\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} e^{-i\vec{p}'\cdot\vec{x}} \dot{\phi}(x) d^3\vec{x} = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C(E)(-iE)[a_{\vec{p}} e^{-ip\cdot x} - a_{\vec{p}}^\dagger e^{ip\cdot x}] e^{-i\vec{p}'\cdot\vec{x}} d^3\vec{p}\, d^3\vec{x}$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} e^{-i\vec{p}'\cdot\vec{x}} \dot{\phi}(x) d^3\vec{x} = \int_{-\infty}^{+\infty} C(E)(-iE)\left[a_{\vec{p}} \delta^3(\vec{p}-\vec{p}') e^{-iEt} - a_{\vec{p}}^\dagger \delta^3(\vec{p}+\vec{p}') e^{iEt}\right] d^3\vec{p}$$

$$= C(E')(-iE')(a_{\vec{p}'} e^{-iE't} - a_{\vec{p}'}^\dagger e^{iE't})$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} e^{iEt-i\vec{p}\cdot\vec{x}} \dot{\phi}(x) d^3x = C(E)(-iE)(a_{\vec{p}} - a_{-\vec{p}}^\dagger e^{2iEt})$$

$\Rightarrow$

$$a_{\vec p} = \frac{1}{2}\left[\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\,e^{ip\cdot x}\frac{d^3\vec x}{C(E)} + \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\dot\phi(x)\,e^{ip\cdot x}\frac{d^3\vec x}{C(E)(-iE)}\right]$$

Since $e^{ip\cdot x} = \left[\frac{\partial}{\partial t}(e^{ip\cdot x})\right]\frac{1}{iE}$

$$\Rightarrow a_{\vec p} = \frac{1}{2}\left[\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{d^3\vec x}{C(E)(iE)}\left(\phi(x)\left(\frac{\partial}{\partial t}e^{ipx}\right) - \left(\frac{\partial}{\partial t}\phi(x)\right)e^{ip\cdot x}\right]$$

Similarly, start from

$$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\,e^{i\vec p'\cdot\vec x}d^3\vec x = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}C(E)[a_{\vec p}e^{-ip\cdot x}+a^\dagger_{\vec p}e^{ip\cdot x}]\,e^{i\vec p'\cdot x}\,d^3\vec x\,d^3\vec p$$

$$= \int_{-\infty}^{+\infty}C(E)[a_{\vec p}e^{-iEt}\delta^3(\vec p'+\vec p) + a^\dagger_{\vec p}e^{iEt}\delta^3(\vec p'-\vec p)]\,d^3\vec p$$

$$= C(E')[a_{-\vec p'}e^{-iE't} + a^\dagger_{\vec p'}e^{iE't}]$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\,e^{-iE't+i\vec p'\cdot\vec x}d^3\vec x = C(E)[a_{-\vec p'}e^{-2iE't} + a^\dagger_{\vec p'}]$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\,e^{-ip\cdot x}d^3\vec x = C(E)[a_{-\vec p}e^{-2iEt} + a^\dagger_{\vec p}]$$

and $$\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\dot\phi(x)\,e^{i\vec p'\cdot\vec x}d^3\vec x = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}C(E)(-iE)[a_{\vec p}e^{-ip\cdot x} - a^\dagger_{\vec p}e^{ip\cdot x}]\,e^{i\vec p'\cdot\vec x}d^3\vec p\,d^3\vec x$$

$$= \int_{-\infty}^{+\infty}C(E)(-iE)[a_{\vec p}\,\delta^3(\vec p'+\vec p)\,e^{-iEt} - a^\dagger_{\vec p}\,\delta^3(\vec p'-\vec p')\,e^{iEt}]$$

$$= C(E')(-iE')[a_{\vec p'}e^{-iE't} - a^\dagger_{\vec p'}e^{iE't}]$$

$$\Rightarrow \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\dot\phi(x)\,e^{-ip\cdot x}d^3\vec x = C(E)(-iE)[a_{-\vec p}e^{-2iEt} - a^\dagger_{\vec p}]$$

$$\Rightarrow a^\dagger_{\vec p} = \frac{1}{2}\left\{\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\phi(x)\,e^{-ip\cdot x}\frac{d^3\vec x}{C(E)} + \frac{1}{(iE)}\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\dot\phi(x)\,e^{-ip\cdot x}\frac{d^3\vec x}{C(E)}\right\}$$

using $$e^{-ip\cdot x} = \left[\frac{\partial}{\partial t}\left(e^{-ip\cdot x}\right)\right]\frac{1}{(-iE)}$$

$$\Rightarrow a^{+}_{\vec{p}} = \frac{1}{2}\left[\frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{d^3\vec{x}}{C(E)(iE)}\left\{\left(\frac{\partial}{\partial t}\phi\right)e^{-ip\cdot x} - \phi\frac{\partial}{\partial t}\left(e^{-ip\cdot x}\right)\right]\right.$$

$$\Rightarrow [a_{\vec{p}}, a^{+}_{\vec{p}'}] = \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)\,C(E')\,(iE)\,(iE')}\Big[\int_{-\infty}^{+\infty}d^3\vec{x}\left(\phi(x)\frac{\partial}{\partial t}(e^{ip\cdot x}) - \left(\frac{\partial}{\partial t}\phi(x)\right)e^{ip\cdot x}\right),$$
$$\int_{-\infty}^{+\infty}d^3\vec{x}\left(\left(\frac{\partial}{\partial t}\phi(x)\right)e^{-ip'\cdot x} - \phi(x)\frac{\partial}{\partial t}(e^{-ip'\cdot x})\right)\Big]$$

$$= \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)\,C(E')(iE)(iE')}$$
$$\times\int\!\!\int_{-\infty\,-\infty}^{+\infty\,+\infty} d^3\vec{x}\,d^3\vec{x}'\Big\{\left(\phi(\vec{x},t)\left(iE\,e^{iEt-i\vec{p}\cdot\vec{x}}\right) - \pi(\vec{x},t)\,e^{iEt-i\vec{p}\cdot\vec{x}}\right)$$
$$\cdot\left(\pi(\vec{x}',t)\,e^{-iE't+i\vec{p}'\cdot\vec{x}'} + iE'\phi(\vec{x}',t)\,e^{-iE't+i\vec{p}'\cdot\vec{x}'}\right)$$
$$-\left(\pi(\vec{x}',t)\,e^{-iE't+i\vec{p}'\cdot\vec{x}'} + iE'\phi(\vec{x}',t)\,e^{-iE't+i\vec{p}'\cdot\vec{x}'}\right)$$
$$\left(\phi(\vec{x},t)\left(iE\,e^{iEt-i\vec{p}\cdot\vec{x}}\right) - \pi(\vec{x},t)\,e^{iEt-i\vec{p}\cdot\vec{x}}\right)\Big\}$$

$$= \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)\,C(E')\,(iE)\,(iE')}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^3\vec{x}\,d^3\vec{x}'\,e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{x}'}\,e^{iEt-iE't}$$
$$\times\left\{\begin{array}{l} iE\left(\phi(\vec{x},t)\,\pi(\vec{x}',t) - \pi(\vec{x}',t)\phi(\vec{x},t)\right) \\ -iE'\left(\pi(\vec{x},t)\,\phi(\vec{x}',t) - \phi(\vec{x}',t)\,\pi(\vec{x},t)\right)\end{array}\right\}$$

$$= \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)C(E')(iE)(iE')}$$
$$\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^3\vec{x}\,d^3\vec{x}'\,e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{x}'}\,e^{iEt-iE't}$$
$$\times\left\{iE[\phi(\vec{x},t),\pi(\vec{x}',t)] + iE'[\phi(\vec{x}',t),\pi(\vec{x},t)]\right\}$$

$$= \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2\frac{1}{C(E)C(E')(iE)(iE')}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d^3\vec{x}\,d^3\vec{x}'\,e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{x}'}\,e^{iEt-iE't}$$
$$\times\left\{iE\cdot i\,\delta^3(\vec{x}-\vec{x}') + iE'\underbrace{\delta^3(\vec{x}'-\vec{x})}_{\delta^3(\vec{x}-\vec{x}')}\right\}$$

$$= \left(\frac{1}{2(2\pi)^3}\right)^2 \frac{1}{C(E)\,C(E')\,(iE)(iE')} \int_{-\infty}^{+\infty} d^3\vec{x}\; e^{-i(\vec{p}-\vec{p}\,')\cdot\vec{x}}\; e^{iEt-iE't}\,(i^2E+i^2E')$$

$$= \left(\frac{1}{2(2\pi)^3}\right)^2 \frac{1}{C(E)\,C(E')\,iE(iE')} (2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,')\; e^{iEt-iE't}\,(i^2E+i^2E')$$

$$= \left(\frac{1}{2}\right)^2 \frac{1}{(2\pi)^3} \left(\frac{1}{C(E)}\right)^2 \frac{2E}{E^2}\,\delta^3(\vec{p}-\vec{p}\,') = \frac{1}{(2\pi)^3\,2E}\left(\frac{1}{C(E)}\right)^2 \delta^3(\vec{p}-\vec{p}\,')$$

$\uparrow$ since $E=(\vec{p}^{\,2}+m^2)^{\frac{1}{2}}$, $E'=(\vec{p}\,'^{2}+m^2)^{\frac{1}{2}}$

The choice of the normalization factor $C(E)$ are different in the literature:

① if use $C(E)=\left[\frac{1}{(2\pi)^3\,2E}\right]^{\frac{1}{2}}$, then $\longrightarrow$ e.g. 王正行《简明量子场论》, Hokim & Yem "Elementary particles and Their Interactions".

$$\phi(x)=\int_{-\infty}^{+\infty}\left(\frac{1}{(2\pi)^3\,2E}\right)^{\frac{1}{2}}\left(a_{\vec{p}}\,e^{-ip\cdot x}+a^{+}_{\vec{p}}\,e^{ip\cdot x}\right)$$

and $[a_{\vec{p}},\, a^{+}_{\vec{p}\,'}]=\delta^3(\vec{p}-\vec{p}\,')$

② if use $C(E)=\frac{1}{(2\pi)^3}\left(\frac{1}{2E}\right)^{\frac{1}{2}}$, then $\longrightarrow$ e.g. Peskin & Schroeder "An Introduction to quantum field theory".

$$\phi(x)=\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^3}\frac{1}{\sqrt{2E}}\left(a_{\vec{p}}\,e^{-ip\cdot x}+a^{+}_{\vec{p}}\,e^{ip\cdot x}\right)$$

and $[a_{\vec{p}},\, a^{+}_{\vec{p}\,'}]=(2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,')$

③ if use $C(E)=\frac{1}{(2\pi)^3\,2E}$, then $\longrightarrow$ e.g. Ryder "Quantum field theory"

$$\phi(x)=\int_{-\infty}^{+\infty}\frac{1}{(2\pi)^3\,2E}\left(a_{\vec{p}}\,e^{-ip\cdot x}+a^{+}_{\vec{p}}\,e^{ip\cdot x}\right)$$

and $[a_{\vec{p}},\, a^{+}_{\vec{p}\,'}]=(2\pi)^3\,2E_{\vec{p}}\,\delta^3(\vec{p}-\vec{p}\,')$

$\uparrow$ put the label $\vec{p}$ in E to make it clear that E depends on $\vec{p}$ through $E=(\vec{p}^{\,2}+m^2)^{\frac{1}{2}}$

It doesn't matter which one you prefer to choose. However, once you choose a normalization for the field operator $\phi(x)$, the state vectors on which the field operator acts on are usually defined accordingly, so that the Feynman rules can be written without including the normalization factor. But in any case, the observables do not depend on the normalization factor.

For $[a_{\vec{p}}, a_{\vec{p}'}]$ and $[a^+_{\vec{p}}, a^+_{\vec{p}'}]$

$$[a_{\vec{p}}, a_{\vec{p}'}] = \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2 \frac{1}{C(E)\,C(E')(iE)(iE')}$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{x}' \Big[\phi(\vec{x},t)\, iE\, e^{iEt - i\vec{p}\cdot\vec{x}} - \pi(\vec{x},t)\, e^{iEt-i\vec{p}\cdot\vec{x}}$$

$$, \phi(\vec{x}',t)\, iE' e^{iE't - i\vec{p}'\cdot\vec{x}'} - \pi(\vec{x}',t)\, e^{iE't - i\vec{p}'\cdot\vec{x}'}\Big]$$

where

$$\iint = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{iEt - i\vec{p}\cdot\vec{x} + iE't - i\vec{p}'\cdot\vec{x}'}$$

$$\times \Big\{ (iE)(iE')\,\underbrace{[\phi(\vec{x},t), \phi(\vec{x}',t)]}_{0}$$

$$+ \underbrace{[\pi(\vec{x},t), \pi(\vec{x}',t)]}_{0}$$

$$- iE'[\pi(\vec{x},t), \phi(\vec{x}',t)]$$

$$- iE[\phi(\vec{x},t), \pi(\vec{x}',t)] \Big\}\, d^3\vec{x}\, d^3\vec{x}'$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{iEt - i\vec{p}\cdot\vec{x} + iE't - i\vec{p}'\cdot\vec{x}'} \Big\{ -iE'(-i)\,\delta^3(\vec{x}-\vec{x}') - iE\, i\,\delta^3(\vec{x}-\vec{x}') \Big\}\, d^3\vec{x}\, d^3\vec{x}'$$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\, e^{iEt - i\vec{p}\cdot\vec{x} + iE't - i\vec{p}'\cdot\vec{x}} \Big\{ (-i)^2 E' - i^2 E \Big\}$$

$$= (2\pi)^3\,\delta^3(\vec{p} + \vec{p}')\, e^{iEt + iE't}(-E' + E)$$

$$= \begin{cases} 0 & \text{when } \vec{p}' \neq -\vec{p} \\ 0 & \text{since when } \vec{p}' = -\vec{p} \Rightarrow E' = E \end{cases}$$

So $[a_{\vec{p}}, a_{\vec{p}'}] = 0$

$$[a^+_{\vec{p}}, a^+_{\vec{p}'}] = \left(\frac{1}{2}\frac{1}{(2\pi)^3}\right)^2 \frac{1}{C(E)\,C(E')(iE)(iE')}$$

$$\times \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{x}' \Big[\pi(\vec{x},t)\, e^{-iEt + i\vec{p}\cdot\vec{x}} + iE\,\phi(\vec{x},t)\, e^{-iEt + i\vec{p}\cdot\vec{x}}$$

$$, \pi(\vec{x}',t)\, e^{-iE't + i\vec{p}'\cdot\vec{x}'} + iE'\phi(\vec{x}',t)\, e^{-iE't + i\vec{p}'\cdot\vec{x}'}\Big]$$

where $$\iint = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{x}'\, e^{-iEt + i\vec{p}\cdot\vec{x} - iE't + i\vec{p}'\cdot\vec{x}'}$$
$$\times \Big\{ [\pi(\vec{x},t), \pi(\vec{x}',t)]$$
$$+ (iE)(iE')[\phi(\vec{x},t), \phi(\vec{x}',t)]$$
$$+ iE'[\pi(\vec{x},t), \phi(\vec{x}',t)]$$
$$+ iE[\phi(\vec{x},t), \pi(\vec{x}',t)] \Big\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{x}'\, e^{-iEt + i\vec{p}\cdot\vec{x} - iE't + i\vec{p}'\cdot\vec{x}'} \big(iE'(-i)\delta^3(\vec{x}-\vec{x}') + iE\, i\delta^3(\vec{x}-\vec{x}') \big)$$

$$= \int_{-\infty}^{+\infty} d^3\vec{x}\, e^{-iEt + i\vec{p}\cdot\vec{x} - iE't + i\vec{p}'\cdot\vec{x}} (E' - E)$$

$$= (2\pi)^3 \delta^3(\vec{p}+\vec{p}')\, e^{-iEt - iE't} (E' - E)$$

$$= 0$$

So $[a^{+}_{\vec{p}}, a^{+}_{\vec{p}'}] = 0$

or, directly from $[a_{\vec{p}}, a_{\vec{p}'}] = 0$, do hermitian conjugate,

$$\Rightarrow (a_{\vec{p}}a_{\vec{p}'} - a_{\vec{p}'}a_{\vec{p}})^{+} = 0$$

$$\Rightarrow a^{+}_{\vec{p}'}a^{+}_{\vec{p}} - a^{+}_{\vec{p}}a^{+}_{\vec{p}'} = 0$$

$$\Rightarrow [a^{+}_{\vec{p}}, a^{+}_{\vec{p}'}] = 0$$

The Hamiltonian is

$$H = \int_{-\infty}^{+\infty} d^3\vec{x}\ \mathcal{H}$$

where the Hamiltonian density for a real scalar field is

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

Now let's write it in terms of creation & annihilation operators

using $\phi(x) = \int_{-\infty}^{+\infty} C(E_{\vec{p}})[a_{\vec{p}}e^{-ip\cdot x} + a_{\vec{p}}^{+}e^{ip\cdot x}]d^3\vec{p}$

$$\dot{\phi}(x) = \int_{-\infty}^{+\infty} C(E_{\vec{p}})(-iE_{\vec{p}})[a_{\vec{p}}e^{-ip\cdot x} - a_{\vec{p}}^{+}e^{ip\cdot x}]d^3\vec{p}$$

$$\vec{\nabla}\phi = \int_{-\infty}^{+\infty} C(E_{\vec{p}})(i\vec{p})[a_{\vec{p}}e^{-ip\cdot x} - a_{\vec{p}}^{+}e^{ip\cdot x}]d^3\vec{p}$$

$$\Rightarrow H = \frac{1}{2}\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\ C(E_{\vec{p}})C(E_{\vec{k}})$$

$$\times \begin{Bmatrix} (-iE_{\vec{p}})(-iE_{\vec{k}})(a_{\vec{p}}e^{-ip\cdot x} - a_{\vec{p}}^{+}e^{ip\cdot x})(a_{\vec{k}}e^{-ik\cdot x} - a_{\vec{k}}^{+}e^{ik\cdot x}) \\ + (i\vec{p})\cdot(i\vec{k})(a_{\vec{p}}e^{-ip\cdot x} - a_{\vec{p}}^{+}e^{ip\cdot x})(a_{\vec{k}}e^{-ik\cdot x} - a_{\vec{k}}^{+}e^{ik\cdot x}) \\ + m^2(a_{\vec{p}}e^{-ip\cdot x} + a_{\vec{p}}^{+}e^{ip\cdot x})(a_{\vec{k}}e^{-ik\cdot x} + a_{\vec{k}}^{+}e^{ik\cdot x}) \end{Bmatrix}$$

$$= \frac{1}{2}\int_{\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\ C(E_{\vec{p}})C(E_{\vec{k}})$$

$$\times \begin{Bmatrix} [(-iE_{\vec{p}})(-iE_{\vec{k}}) + (i\vec{p})\cdot(i\vec{k}) + m^2](a_{\vec{p}}a_{\vec{k}}e^{-ip\cdot x - ik\cdot x} + a_{\vec{p}}^{+}a_{\vec{k}}^{+}e^{ip\cdot x + ik\cdot x}) \\ + [-(-iE_{\vec{p}})(-iE_{\vec{k}}) - (i\vec{p})\cdot(i\vec{k}) + m^2](a_{\vec{p}}a_{\vec{k}}^{+}e^{-ip\cdot x + ik\cdot x} + a_{\vec{p}}^{+}a_{\vec{k}}e^{ip\cdot x - ik\cdot x}) \end{Bmatrix}$$

$$= \frac{(2\pi)^3}{2}\int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\ C(E_{\vec{p}})C(E_{\vec{k}}) \times \begin{Bmatrix} [(-iE_{\vec{p}})(-iE_{\vec{k}}) + (i\vec{p})\cdot(i\vec{k}) + m^2](a_{\vec{p}}a_{\vec{k}}e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\delta^3(\vec{p}+\vec{k}) \\ + a_{\vec{p}}^{+}a_{\vec{k}}^{+}e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\delta^3(\vec{p}+\vec{k})) \\ + [-(-iE_{\vec{p}})(-iE_{\vec{k}}) - (i\vec{p})\cdot(i\vec{k}) + m^2](a_{\vec{p}}a_{\vec{k}}^{+}e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\delta^3(\vec{p}-\vec{k}) \\ + a_{\vec{p}}^{+}a_{\vec{k}}e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\delta^3(\vec{p}-\vec{k})) \end{Bmatrix}$$

$$= \frac{(2\pi)^3}{2}\int_{-\infty}^{+\infty} d^3\vec{p}\,(C(E_{\vec{p}}))^2 \begin{Bmatrix} [(-iE_{\vec{p}})^2 + (i\vec{p})(-i\vec{p}) + m^2](a_{\vec{p}}a_{-\vec{p}}e^{-2iE_{\vec{p}}t} + a_{\vec{p}}^{+}a_{-\vec{p}}^{+}e^{2iE_{\vec{p}}t}) \\ + [-(-iE_{\vec{p}})^2 - (i\vec{p})^2 + m^2](a_{\vec{p}}a_{\vec{p}}^{+} + a_{\vec{p}}^{+}a_{\vec{p}}) \end{Bmatrix}$$

$\uparrow$

note $E_{\vec{p}} = E_{-\vec{p}}$

$$= \frac{(2\pi)^3}{2} \int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2\, 2E_{\vec{p}}^2\, \underbrace{(a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}})}_{= a_{\vec{p}}^\dagger a_{\vec{p}} + \frac{1}{(2\pi)^3 2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(0)}$$

use $E_{\vec{p}}^2 = \vec{p}^2 + m^2$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3\, 2E_{\vec{p}} \right) a_{\vec{p}}^\dagger a_{\vec{p}}\, E_{\vec{p}}$$

$$+ \underbrace{\left(\int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2} E_{\vec{p}} \right)}_{\hookrightarrow \text{ zero-point energy density, which is infinite}} \underbrace{\delta^3(0)(2\pi)^3}_{\hookrightarrow \text{ from } \int_{-\infty}^{+\infty} d^3\vec{x} \;\left(\text{note } \int_{-\infty}^{+\infty} e^{-i\vec{p}\cdot\vec{x}} d^3\vec{x} = (2\pi)^3 \delta^3(\vec{p})\right)}$$

Recall in Quantum Mechanics, the energy eigenvalues for a harmonic oscillator is

$$E_n = (n + \tfrac{1}{2})\hbar\omega$$

$E_0 = \frac{1}{2}\hbar\omega$ is the zero point energy.

Moreover, in terms of a & $a^\dagger$, the Hamiltonian of a harmonic oscillator is

$$\hat{H} = (a^\dagger a + \tfrac{1}{2})\hbar\omega.$$

where $a^\dagger a$ is defined as $\hat{N}$ — the number operator

We can get rid of the infinite zero-point energy by introducing normal ordering — in every term, all the $a_{\vec{p}}$'s are to the right of all the $a_{\vec{p}}^\dagger$'s, i.e., $:aa^\dagger: = a^\dagger a$

Then

$$:H: = \frac{(2\pi)^3}{2} \int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2\, 2E_{\vec{p}}^2 : (a_{\vec{p}} a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}}):$$

$$= \frac{(2\pi)^3}{2} \int_{-\infty}^{+\infty} d^3\vec{p}\, (C(E_{\vec{p}}))^2\, 2E_{\vec{p}}^2\, 2a_{\vec{p}}^\dagger a_{\vec{p}}$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3\, 2E_{\vec{p}} \right) a_{\vec{p}}^\dagger a_{\vec{p}}\, E_{\vec{p}}$$

We can also make the argument that the experiments measure only energy differences from the ground state (i.e., vaccum state) of H, so we can ignore the infinite constant term. (However, ignoring infinity is uncomfortable mathe... still)

For momentum, again from Noether theorem, $P_i = \int d^3\vec{x}\, \Pi\, \partial_i \phi$

$$\Rightarrow \hat{\vec{P}} = -\int_{-\infty}^{+\infty} d^3\vec{x}\, \pi\, \vec{\nabla}\phi$$

(note that $(\vec{P})^i = P^i = -P_i$, $\vec{\nabla} = (\partial_1, \partial_2, \partial_3)$)

$$\Rightarrow \hat{\vec{P}} = -\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}$$
$$\times\; C(E_{\vec{p}})(-iE_{\vec{p}})C(E_{\vec{k}})(i\vec{k})$$
$$\times \left(a_{\vec{p}}\, e^{-ip\cdot x} - a^{+}_{\vec{p}}\, e^{ip\cdot x}\right)\left(a_{\vec{k}} e^{-ik\cdot x} - a^{+}_{\vec{k}}\, e^{ik\cdot x}\right)$$

$$= -\int_{-\infty}^{+\infty} d^3\vec{x}\, d^3\vec{p}\, d^3\vec{k}\; C(E_{\vec{p}})C(E_{\vec{k}})\,E_{\vec{p}}\,\vec{k}$$
$$\times \Big(a_{\vec{p}} a_{\vec{k}}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\, e^{i(\vec{p}+\vec{k})\cdot\vec{x}} + a^{+}_{\vec{p}} a^{+}_{\vec{k}}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\, e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}$$
$$- a^{+}_{\vec{p}} a_{\vec{k}}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\, e^{-i(\vec{p}-\vec{k})\cdot\vec{x}} - a_{\vec{p}} a^{+}_{\vec{k}}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\, e^{i(\vec{p}-\vec{k})\cdot\vec{x}}\Big)$$

$$= -(2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\; C(E_{\vec{p}})C(E_{\vec{k}})E_{\vec{p}}\,\vec{k}$$
$$\times \Big(a_{\vec{p}} a_{\vec{k}}\, e^{-iE_{\vec{p}}t - iE_{\vec{k}}t}\,\delta^3(\vec{p}+\vec{k}) + a^{+}_{\vec{p}} a^{+}_{\vec{k}}\, e^{iE_{\vec{p}}t + iE_{\vec{k}}t}\,\delta^3(\vec{p}+\vec{k})$$
$$- a^{+}_{\vec{p}} a_{\vec{k}}\, e^{iE_{\vec{p}}t - iE_{\vec{k}}t}\,\delta^3(\vec{p}-\vec{k}) - a_{\vec{p}} a^{+}_{\vec{k}}\, e^{-iE_{\vec{p}}t + iE_{\vec{k}}t}\,\delta^3(\vec{p}-\vec{k})\Big)$$

$$= -(2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p}\, [C(E_{\vec{p}})]^2 E_{\vec{p}} \Big(a_{\vec{p}} a_{-\vec{p}}(-\vec{p})\, e^{-2iE_{\vec{p}}t} + a^{+}_{\vec{p}} a^{+}_{-\vec{p}}(-\vec{p})\, e^{2iE_{\vec{p}}t}$$
$$- a^{+}_{\vec{p}} a_{\vec{p}}\,\vec{p} - a_{\vec{p}} a^{+}_{\vec{p}}\,\vec{p}\Big)$$

note $E_{\vec{p}} = E_{-\vec{p}}$

$$\equiv **$$

Since $E_{\vec{p}}$ is even for $\vec{p} \to -\vec{p}$, $a_{\vec{p}} a_{-\vec{p}} = a_{-\vec{p}} a_{\vec{p}}$, $a^{+}_{\vec{p}} a^{+}_{-\vec{p}} = a^{+}_{-\vec{p}} a^{+}_{\vec{p}}$

then the $a_{\vec{p}} a_{-\vec{p}}$ term and $a^{+}_{\vec{p}} a^{+}_{-\vec{p}}$ term vanish after the integration.

$$\Rightarrow ** = (2\pi)^3 \int_{-\infty}^{+\infty} d^3\vec{p}\, [C(E_{\vec{p}})]^2 E_{\vec{p}}\,\vec{p}\,\big(a^{+}_{\vec{p}} a_{\vec{p}} + \underbrace{a_{\vec{p}} a^{+}_{\vec{p}}}_{= a^{+}_{\vec{p}} a_{\vec{p}} + \frac{1}{(2\pi)^3 2E_{\vec{p}}}\left(\frac{1}{C(E_{\vec{p}})}\right)^2 \delta^3(0)}\big)$$

(By this step, it's already easy to see that compared to H, the only change is $E_{\vec{p}}$ inside the integration in H to $\vec{p}$ inside the integration in $\vec{P}$)

$$= \int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3\, 2E_{\vec{p}}\right) a^{+}_{\vec{p}} a_{\vec{p}}\,\vec{p} + \underbrace{\left(\int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{1}{2}\vec{p}\right)}_{=0} \delta^3(0)(2\pi)^3$$

So, $$\hat{\vec{P}} = \int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3\, 2E_{\vec{p}}\right) a^{+}_{\vec{p}} a_{\vec{p}}\,\vec{p}.$$

Unequal-time commutators

$$
\begin{aligned}
[\phi(\vec{x},t),\ \pi(\vec{x}',t')] &= \Big[\int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec p})\left(a_{\vec p}\, e^{-ip\cdot x} + a^+_{\vec p}\, e^{ip\cdot x}\right),\ \int_{-\infty}^{+\infty} d^3\vec{k}\, C(E_{\vec k})(-iE_{\vec k}) \\
&\qquad \left(a_{\vec k}\, e^{-ik\cdot x'} - a^+_{\vec k}\, e^{ik\cdot x'}\right)\Big] \\
&= \iint_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\ C(E_{\vec p})\, C(E_{\vec k})(-iE_{\vec k})\left[a_{\vec p}\, e^{-ip\cdot x} + a^+_{\vec p}\, e^{ip\cdot x},\ a_{\vec k}\, e^{-ik\cdot x'} - a^+_{\vec k}\, e^{ik\cdot x'}\right] \\
&= \iint_{-\infty}^{+\infty} d^3\vec{p}\, d^3\vec{k}\ C(E_{\vec p})\, C(E_{\vec k})(-iE_{\vec k})\Big\{ -\frac{1}{(2\pi)^3 2E_{\vec p}}\left(\frac{1}{C(E_{\vec p})}\right)^2 \delta^3(\vec p-\vec k)\, e^{-ip\cdot x+ik\cdot x'} \\
&\qquad - \frac{1}{(2\pi)^3 2E_{\vec p}}\left(\frac{1}{C(E_{\vec p})}\right)^2 \delta^3(\vec p-\vec k)\, e^{ip\cdot x-ik\cdot x'}\Big\} \\
&= \int_{-\infty}^{+\infty} d^3\vec{p}\,(-i)\left(-\frac{1}{2}\frac{1}{(2\pi)^3} e^{-ip\cdot(x-x')} - \frac{1}{2}\frac{1}{(2\pi)^3} e^{ip\cdot(x-x')}\right)
\end{aligned}
$$

$\vec p=\vec k \Rightarrow E_{\vec p}=E_{\vec k}$

$$
= \frac{i}{2}\int_{-\infty}^{+\infty} \frac{d^3\vec p}{(2\pi)^3}\left(e^{-ip\cdot(x-x')} + e^{ip\cdot(x-x')}\right)
$$

If $t=t'$, then $e^{-ip\cdot(x-x')} = +i\vec p\cdot(\vec x-\vec x')$ and $e^{ip\cdot(x-x')} = e^{-i\vec p\cdot(\vec x-\vec x')}$

$$
\begin{aligned}
\Rightarrow [\phi(\vec x,t),\ \pi(\vec x',t)] &= \frac{i}{2}\int_{-\infty}^{+\infty}\frac{d^3\vec p}{(2\pi)^3}\left(e^{i\vec p\cdot(\vec x-\vec x')} + e^{-i\vec p\cdot(\vec x-\vec x')}\right) \\
&= i\delta^3(x-x')
\end{aligned}
$$

as expected

$$
\begin{aligned}
&[\phi(\vec x,t),\ \phi(\vec x',t')] \\
&= \Big[\int_{-\infty}^{+\infty} d^3\vec p\ C(E_{\vec p})\left(a_{\vec p}\, e^{-ip\cdot x} + a^+_{\vec p}\, e^{ip\cdot x}\right),\ \int_{-\infty}^{+\infty} d^3\vec k\ C(E_{\vec k})\left(a_{\vec k}\, e^{-ik\cdot x'} + a^+_{\vec k}\, e^{ik\cdot x'}\right)\Big] \\
&= \iint_{-\infty}^{+\infty} d^3\vec p\, d^3\vec k\ C(E_{\vec p})C(E_{\vec k})\left[a_{\vec p}\, e^{-ip\cdot x} + a^+_{\vec p}\, e^{ip\cdot x},\ a_{\vec k}\, e^{-ik\cdot x'} + a^+_{\vec k}\, e^{ik\cdot x'}\right] \\
&= \iint_{-\infty}^{+\infty} d^3\vec p\, d^3\vec k\ C(E_{\vec p})C(E_{\vec k})\Big([a_{\vec p}, a^+_{\vec k}]\, e^{-ip\cdot x+ik\cdot x'} + [a^+_{\vec p}, a_{\vec k}]\, e^{ip\cdot x-ik\cdot x'}\Big) \\
&= \iint_{-\infty}^{+\infty} d^3\vec p\, d^3\vec k\ C(E_{\vec p})C(E_{\vec k})\Big\{\frac{1}{(2\pi)^3 2E_{\vec p}}\left(\frac{1}{C(E_{\vec p})}\right)^2\delta^3(\vec p-\vec k)\, e^{-ip\cdot x+ik\cdot x'} \\
&\qquad - \frac{1}{(2\pi)^3 2E_{\vec p}}\left(\frac{1}{C(E_{\vec p})}\right)^2\delta^3(\vec p-\vec k)\, e^{ip\cdot x-ik\cdot x'}\Big\}
\end{aligned}
$$

$$= \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left(e^{-ip\cdot(x-x')} - e^{ip\cdot(x-x')} \right)$$

We can check that for $t = t'$,

Since
$$\int_{-\infty}^{+\infty} \frac{d^3\vec{p}\, e^{+i\vec{p}\cdot(\vec{x}-\vec{x}')}}{(2\pi)^3 2E_{\vec{p}}} \underset{\vec{p}\to-\vec{k}}{=} \int_{+\infty}^{-\infty} \frac{d^3(-\vec{k})\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}}{(2\pi)^3 2E_{-\vec{k}}} \underset{E_{-\vec{k}}=E_{\vec{k}}}{=} \int_{-\infty}^{+\infty} \frac{d^3\vec{k}\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}}{(2\pi)^3 2E_{\vec{k}}}$$

$$\underset{\vec{k}\to\vec{p}}{=} \int_{-\infty}^{+\infty} \frac{d^3\vec{p}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')}}{(2\pi)^3 2E_{\vec{p}}}$$

then $[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0$, as expected.

Note that
$$\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} = \frac{d^3\vec{p}}{(2\pi)^3} \frac{dp^0}{2p^0} \delta(p^0 - E_{\vec{p}})$$

→ it should be understood as an integration for p^0 in this and the following steps

$$= \frac{d^4p}{(2\pi)^4} \frac{2\pi}{2p^0} [\delta(p^0 - E_{\vec{p}}) + \delta(p^0 + E_{\vec{p}})] \theta(p^0)$$

→ step function

$$= \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^{0\,2} - \underbrace{E_{\vec{p}}^2}_{|\vec{p}|^2+m^2}) \theta(p^0)$$

$$\delta(x^2 - \alpha^2) = \frac{1}{2|\alpha|} [\delta(x+\alpha) + \delta(x-\alpha)]$$

$$= \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^2 - m^2)\, \theta(p^0)$$

A Lorentz transformation from p^μ to p'^μ makes $d^4p \to d^4p' = |\det(\Lambda)| d^4p = d^4p$.

$$p'^\mu = \Lambda^\mu{}_\nu p^\nu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right) p^\nu$$

since $|\det(\Lambda)| = 1$

and $p'^2 = p^2$ since it is a scalar.

$e^{-ip\cdot(x-x')}$ and $e^{ip\cdot(x-x')}$ are Lorentz scalar.

Also, we can write $\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}$ alternatively as

$$\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} = \frac{d^3\vec{p}}{(2\pi)^3} \frac{dp^0}{(-2p^0)} \delta(p^0 + E_{\vec{p}})$$

$$= \frac{d^4p}{(2\pi)^4} \frac{2\pi}{(-2p^0)} [\delta(p^0 + E_{\vec{p}}) + \delta(p^0 - E_{\vec{p}})] \theta(-p^0)$$

$$= \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^{0\,2} - E_{\vec{p}}^2)\, \theta(-p^0)$$

$$= \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^2 - m^2)\, \theta(-p^0)$$

Therefore, it is clear that the role of $\theta(p^0)$ or $\theta(-p^0)$ together with $\delta(p^2 - m^2)$ when do the integration is just select one of the two possible value $p^0 = \pm\sqrt{\vec{p}^2 + m^2} = \pm E_{\vec{p}}$, and the result of the integration doesn't depends on which one to choose.

So, for a Lorentz transformation, it does not matter whether the sign of $p^{0'}$ and p^0 are the same or not, where $p^{0'} = \Lambda^0{}_\nu p^\nu = \Lambda^0{}_0 p^0 + \Lambda^0{}_i p^i$, and note that both p^0 and $p^{0'}$ need to satisfy the δ function condition (i.e., on-shell condition).

So a Lorentz transformation make

$$\frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^2 - m^2)\, \theta(p^0) \longrightarrow \frac{d^4p'}{(2\pi)^4} 2\pi\, \delta(p'^2 - m^2)\, \theta(p^{0'})$$

$$= \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^2 - m^2)\, \theta(p^0)$$

$$\text{or } \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^2 - m^2)\, \theta(-p^0)$$

and they are the same.

So $[\phi(\vec{x}, t), \phi(\vec{x}', t')] = \int_{-\infty}^{+\infty} \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} (e^{-ip\cdot(x-x')} - e^{ip\cdot(x-x')})$ is Lorentz invariant.

We've already seen that for a special space-like case, i.e., the equal time case,

$[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0$. when $\vec{x} \neq \vec{x}'$ (we don't care when $\vec{x} = \vec{x}'$)

For any arbitrary $(\vec{x},t)$ and $(\vec{x}',t')$ seperated space-likely,

that is, $(t-t')^2 - (\vec{x}-\vec{x}')^2 < 0$

we can always find a reference frame in which $t \to t''$, $t' \to t'''$ satisfying $t'' = t'''$: let's first setup the frame so that $y = y'$ & $z = z'$. then we boost along the x-axis

using $t'' = \gamma(t - v x)$

$t''' = \gamma(t' - v x')$

$\Rightarrow$ $t'' - t''' = \gamma(t-t') - \gamma v (x - x')$

if $t'' = t'''$, then $v = \dfrac{t-t'}{x-x'}$

Since $(t-t')^2 - (x-x')^2 < 0$ $(y = y'$ & $z = z')$

then $-1 < \dfrac{t-t'}{x-x'} < 1$

So such a Lorentz boost can be found.

Therefore, we can use the equal-time commutator result such that

$[\phi(\vec{x}'',t''), \phi(\vec{x}''',t'')] = 0$

The Lorentz invariance of $[\phi(\vec{x},t), \phi(\vec{x}',t')]$ means any spacelike seperated point $(\vec{x},t)$ & $(\vec{x}',t')$ satisfy $[\phi(\vec{x},t), \phi(\vec{x}',t')] = 0$.

This is called microcausality.

Let's look at the meaning of a & a^+

In

$$:H: = \int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3 \, 2E_{\vec{p}} \right) a_{\vec{p}}^+ a_{\vec{p}} \, E_{\vec{p}}$$

and

$$:\hat{\vec{P}}: = \hat{\vec{P}} = \int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3 \, 2E_{\vec{p}} \right) a_{\vec{p}}^+ a_{\vec{p}} \, \vec{p}$$

If we define the common part $\int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3 \, 2E_{\vec{p}} \right) a_{\vec{p}}^+ a_{\vec{p}} = \hat{N}$,

then

$$[\hat{N}, a_{\vec{k}}] = \left[\int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3 \, 2E_{\vec{p}} \right) a_{\vec{p}}^+ a_{\vec{p}} \, , \, a_{\vec{k}}\right]$$

$$= -\int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3 \, 2E_{\vec{p}} \right) \frac{1}{(2\pi)^3 \, 2E_{\vec{p}}} \left[\frac{1}{C(E_{\vec{p}})}\right]^2 \delta^3(\vec{p}-\vec{k}) \, a_{\vec{p}}$$

$$= -a_{\vec{k}}$$

and $[\hat{N}, a_{\vec{k}}^+] = a_{\vec{k}}^+$

If we define an eigenstate of $\hat{N}$ as $|s\rangle$, such that

$$\hat{N}|s\rangle = c|s\rangle$$

where c is the eigenvalue and c is a real number (since $\hat{N}$ is a Hermitian operator)

then $\hat{N} a_{\vec{k}}^+ |s\rangle = (a_{\vec{k}}^+ \hat{N} + a_{\vec{k}}^+)|s\rangle = (c+1) \, a_{\vec{k}}^+ |s\rangle$

therefore $a_{\vec{k}}^+|s\rangle$ is also an eigenstate of $\hat{N}$, with eigenvalue $(c+1)$.

Also, $\hat{N} a_{\vec{k}} |s\rangle = (a_{\vec{k}} \hat{N} - a_{\vec{k}})|s\rangle = (c-1) \, a_{\vec{k}}|s\rangle$

therefore $a_{\vec{k}}|s\rangle$ is also an eigenstate of $\hat{N}$, with eigenvalue $(c-1)$.

We can repeat the processes

$$\hat{N} (a_{\vec{k}_1}^+)^{n_1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_q}^+)^{n_q} |s\rangle = (a_{\vec{k}_1}^+ \hat{N} + a_{\vec{k}}^+)(a_{\vec{k}_1}^+)^{n_1-1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_q}^+)^{n_q}|s\rangle$$

$$a_{\vec{k}_1}^+ \hat{N} (a_{\vec{k}_1}^+)^{n_1-1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_q}^+)^{n_q} |s\rangle + (a_{\vec{k}_1}^+)^{n_1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_q}^+)^{n_q} |s\rangle$$

$$= a_{\vec{k}_1}^+ (a_{\vec{k}_1}^+ \hat{N} + a_{\vec{k}_1}^+)(a_{\vec{k}_1}^+)^{n_1-2} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_q}^+)^{n_q} |s\rangle + (a_{\vec{k}_1}^+)^{n_1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_q}^+)^{n_q} |s\rangle$$

$$= (a_{\vec{k}_1}^+)^2 \hat{N} (a_{\vec{k}_1}^+)^{n_1-2} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_\ell}^+)^{n_\ell} |s\rangle + 2 (a_{\vec{k}_1}^+)^{n_1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_\ell}^+)^{n_\ell} |s\rangle$$

$$= \cdots = (a_{\vec{k}_1}^+)^{n_1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_\ell}^+)^{n_\ell - 1} \hat{N} a_{\vec{k}_\ell}^+ |s\rangle + (n_1 + n_2 + \cdots n_\ell - 1) (a_{\vec{k}_1}^+)^{n_1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_\ell}^+)^{n_\ell} |s\rangle$$

$$= (C + n_1 + n_2 + \cdots n_\ell) (a_{\vec{k}_1}^+)^{n_1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_\ell}^+)^{n_\ell} |s\rangle$$

So, $(a_{\vec{k}_1}^+)^{n_1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_\ell}^+)^{n_\ell} |s\rangle$ is also an eigenstate of $\hat{N}$, with eigenvalue $(C + n_1 + n_2 + \cdots n_\ell)$.

Similarly,

$$\hat{N} a_{\vec{k}_1}^{n_1} a_{\vec{k}_2}^{n_2} \cdots a_{\vec{k}_\ell}^{n_\ell} |s\rangle = (a_{\vec{k}_1} \hat{N} - a_{\vec{k}_1}) a_{\vec{k}_1}^{n_1 - 1} a_{\vec{k}_2}^{n_2} \cdots a_{\vec{k}_\ell}^{n_\ell} |s\rangle$$

$$= a_{\vec{k}_1} \hat{N} a_{\vec{k}_1}^{n_1 - 1} a_{\vec{k}_2}^{n_2 - 1} \cdots a_{\vec{k}_\ell}^{n_\ell} |s\rangle - a_{\vec{k}_1}^{n_1} a_{\vec{k}_2}^{n_2} \cdots a_{\vec{k}_\ell}^{n_\ell} |s\rangle$$

$$= \cdots = a_{\vec{k}_1}^{n_1} a_{\vec{k}_2}^{n_2} \cdots a_{\vec{k}_\ell}^{n_\ell - 1} \hat{N} a_{\vec{k}_\ell} |s\rangle - (n_1 + n_2 + \cdots n_\ell - 1) a_{\vec{k}_1}^{n_1} a_{\vec{k}_2}^{n_2} \cdots a_{\vec{k}_\ell}^{n_\ell} |s\rangle$$

$$= (C - n_1 - n_2 - \cdots - n_\ell) a_{\vec{k}_1}^{n_1} a_{\vec{k}_2}^{n_2} \cdots a_{\vec{k}_\ell}^{n_\ell} |s\rangle$$

So, $a_{\vec{k}_1}^{n_1} a_{\vec{k}_2}^{n_2} \cdots a_{\vec{k}_\ell}^{n_\ell} |s\rangle$ is also an state of $\hat{N}$, with eigenvalue $(C - n_1 - n_2 - \cdots - n_\ell)$.

Therefore, it is reasonable to take $\hat{N}$ as the particle number operator. $\hat{a}$ is the operator destroy a particle (so number decreases by 1), and $\hat{a}^+$ is the operator create a particle (so number increases by 1).

However, this interpretation requires that the eigenvalues are non-negative. (since we don't know the meaning of negative number of particles).

So, we define the vaccum state $|0\rangle$, satisfying $a_{\vec{k}} |0\rangle = 0$ for any $\vec{k}$.

Then $$\hat{N} |0\rangle = \int_{-\infty}^{+\infty} d^3\vec{p} \left([C(E_{\vec{p}})]^2 (2\pi)^3 2E_{\vec{p}} \right) a_{\vec{p}}^+ a_{\vec{p}} |0\rangle = 0 = 0|0\rangle$$

Starting from vaccume state

$$\hat{N} a_{\vec{k}}^+ |0\rangle = (a_{\vec{k}}^+ \hat{N} + a_{\vec{k}}^+) |0\rangle = (0 + 1) a_{\vec{k}}^+ |0\rangle = a_{\vec{k}}^+ |0\rangle$$

$$\hat{N} (a_{\vec{k}_1}^+)^{n_1} (a_{\vec{k}_2}^+)^{n_2} \cdots (a_{\vec{k}_\ell}^+)^{n_\ell} |0\rangle = (n_1 + n_2 + \cdots + n_\ell) |0\rangle$$

So, $a^+_{\vec{k}}|0\rangle$ state has 1 particle,

$(a^+_{\vec{k}_1})^{n_1}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$ state has $(n_1+n_2+\cdots n_\ell)$ particles.

Now let's look at $:H:$ and $\hat{\vec{P}}$.

$$:H:a^+_{\vec{k}}|0\rangle = \int_{-\infty}^{+\infty} d^3\vec{p}\left([C(E_{\vec{p}})]^2(2\pi)^3 2E_{\vec{p}}\right) a^+_{\vec{p}}a_{\vec{p}}E_{\vec{p}}\, a^+_{\vec{k}}|0\rangle$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\left([C(E_{\vec{p}})]^2(2\pi)^3 2E_{\vec{p}}\right) a^+_{\vec{p}}E_{\vec{p}}\frac{1}{(2\pi)^3 2E_{\vec{k}}}\left[\frac{1}{C(E_{\vec{k}})}\right]^2\delta^3(\vec{k}-\vec{p})|0\rangle$$

$$= E_{\vec{k}}a^+_{\vec{k}}|0\rangle$$

$$:H:(a^+_{\vec{k}_1})^{n_1}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\left([C(E_{\vec{p}})]^2(2\pi)^3 2E_{\vec{p}}\right) a^+_{\vec{p}}a_{\vec{p}}E_{\vec{p}}(a^+_{\vec{k}_1})^{n_1}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}(\qquad)E_{\vec{p}}a^+_{\vec{p}}\frac{1}{(2\pi)^3 2E_{\vec{p}}}\left[\frac{1}{C(E_{\vec{p}})}\right]^2\delta^3(p-\vec{k}_1)(a^+_{\vec{k}_1})^{n_1-1}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$+\int_{-\infty}^{+\infty} d^3\vec{p}(\qquad)E_{\vec{p}}a^+_{\vec{p}}a^+_{\vec{k}_1}a_{\vec{p}}(a^+_{\vec{k}_1})^{n_1-1}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$= E_{k_1}(a^+_{\vec{k}_1})^{n_1}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$+\int_{-\infty}^{+\infty} d^3\vec{p}(\qquad)E_{\vec{p}}a^+_{\vec{p}}a^+_{\vec{k}_1}\frac{1}{(2\pi)^3 2E_{\vec{p}}}\left[\frac{1}{C(E_{\vec{p}})}\right]^2\delta^3(\vec{p}-\vec{k}_1)(a^+_{\vec{k}_1})^{n_1-2}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$+\int_{-\infty}^{+\infty} d^3\vec{p}(\qquad)E_{\vec{p}}a^+_{\vec{p}}(a^+_{\vec{k}_1})^2a_{\vec{p}}(a^+_{\vec{k}_1})^{n_1-2}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$= \cdots$$

$$= n_1E_{k_1}(a^+_{\vec{k}_1})^{n_1}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$+\int_{-\infty}^{+\infty} d^3\vec{p}(\qquad)E_{\vec{p}}(a^+_{\vec{k}_1})^{n_1}a^+_{\vec{p}}a_{\vec{p}}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$= \cdots$$

$$= [n_1E_{\vec{k}_1}+n_2E_{\vec{k}_2}+(n_\ell-1)E_{\vec{k}_\ell}](a^+_{\vec{k}_1})^{n_1}(a^+_{\vec{k}_2})^{n_2}\cdots(a^+_{\vec{k}_\ell})^{n_\ell}|0\rangle$$

$$+\int_{-\infty}^{+\infty} d^3\vec{p}(\qquad)E_{\vec{p}}(a^+_{\vec{k}_1})^{n_1}(a^+_{\vec{k}_2})^{n_2}\cdots a^+_{\vec{p}}(a^+_{\vec{k}_\ell})^{n_\ell-1}a_{\vec{p}}a^+_{\vec{p}}|0\rangle$$

$$= (n_1 E_{\vec{k}_1} + n_2 E_{\vec{k}_2} + \cdots + n_\ell E_{\vec{k}_\ell}) (a^{+}_{\vec{k}_1})^{n_1} (a^{+}_{\vec{k}_2})^{n_2} \cdots (a^{+}_{\vec{k}_\ell})^{n_\ell} |0\rangle$$

Similarly,

$$\hat{\vec{P}} a^{+}_{\vec{k}} |0\rangle = \vec{k}\, a^{+}_{\vec{k}} |0\rangle$$

$$\hat{\vec{P}} (a^{+}_{\vec{k}_1})^{n_1} (a^{+}_{\vec{k}_2})^{n_2} \cdots (a^{+}_{\vec{k}_\ell})^{n_\ell} |0\rangle = (n_1 \vec{k}_1 + n_2 \vec{k}_2 + \cdots + n_\ell \vec{k}_\ell) (a^{+}_{\vec{k}_1})^{n_1} (a^{+}_{\vec{k}_2})^{n_2} \cdots (a^{+}_{\vec{k}_\ell})^{n_\ell} |0\rangle$$

Therefore, ① the state $a^{+}_{\vec{k}}|0\rangle$ can be interpreted as a one particle state having momentum $\vec{k}$ and energy $E_{\vec{k}}$;

② the state $(a^{+}_{\vec{k}_1})^{n_1} (a^{+}_{\vec{k}_2})^{n_2} \cdots (a^{+}_{\vec{k}_\ell})^{n_\ell} |0\rangle$ is a multi-particle state with n_1 particles having momentum $\vec{k}_1$ and energy $E_{\vec{k}_1}$, n_2 particles having momentum $\vec{k}_2$ and energy $E_{\vec{k}_2}$, ..., and n_ℓ particles having momentum $\vec{k}_\ell$ and energy $E_{\vec{k}_\ell}$;

③ any arbitrary number particle state $|S\rangle$ can be built from the vaccum state by acting on the latter powers of $\hat{a}^{+}_{\vec{k}_1}$; $\hat{a}^{+}_{\vec{k}_2}$ .... Also, since $[a^{+}_{\vec{k}}, a^{+}_{\vec{p}}] = 0$, the order of $a^{+}_{\vec{k}_1}, a^{+}_{\vec{k}_2} \cdots$ acting on the vaccum state doesn't matter. Moreover, the number of particles having the same momentum can be any value. So, the particles are bosons.

④ If we act $a_{\vec{p}}$ on $a^{+}_{\vec{k}}|0\rangle$, we get $a_{\vec{p}} a^{+}_{\vec{k}} |0\rangle = a^{+}_{\vec{k}} a_{\vec{p}} |0\rangle = 0$ if $\vec{p} \neq \vec{k}$, and $a_{\vec{p}} a^{+}_{\vec{k}} |0\rangle = \frac{1}{(2\pi)^3 2E_{\vec{p}}} \left[\frac{1}{C(E_{\vec{p}})}\right]^2 \delta^3(0) |0\rangle + a^{+}_{\vec{k}} \underbrace{\cancel{a_{\vec{p}}}|0\rangle}_{0} \propto |0\rangle$.

This means that $a_{\vec{p}}$ change the one particle state having momentum $\vec{p}$ and energy $E_{\vec{p}}$ to vaccum state, or if directly acts on the vaccum state and get zero if the one particle state does not have momentum $\vec{p}$ and energy $E_{\vec{p}}$

(5) If we act on $a_{\vec p}$ on $(a^\dagger_{\vec k_1})^{n_1}(a^\dagger_{\vec k_2})^{n_2}\cdots(a^\dagger_{\vec k_\ell})^{n_\ell}|0\rangle$, we get 0 if none of the $\vec k$'s equals $\vec p$, and we get $n_r \frac{1}{(2\pi)^3 2E_{\vec p}}\left[\frac{1}{c(E_{\vec p})}\right]^2 \delta^3(0) \cdot (a^\dagger_{\vec k_1})^{n_1}(a^\dagger_{\vec k_2})^{n_2}\cdots(a^\dagger_{\vec p})^{n_r-1}\cdots(a^\dagger_{\vec k_\ell})^{n_\ell}|0\rangle$

$\propto (a^\dagger_{\vec k_1})^{n_1}(a^\dagger_{\vec k_2})^{n_2}\cdots(a^\dagger_{\vec p})^{n_r-1}(a^\dagger_{\vec k_\ell})^{n_\ell}|0\rangle$.

This means that $a_{\vec p}$ removes one particle with momentum $\vec p$ and energy $E_{\vec p}$ from the original state, or it directly acts on the vaccum state and get zero if there is no particle having momentum $\vec p$ and energy $E_{\vec p}$ in the original state.

For later convinience when we derive the Feynman rules, let's normalize the one particle state $|p\rangle \equiv f(\vec{p})\, a^{+}_{\vec{p}}|0\rangle$ by requiring

$$\langle q|p\rangle = (2\pi)^3 2E_{\vec{p}}\, \delta^3(\vec{p}-\vec{q}) \quad \text{and} \quad \langle 0|0\rangle = 1 \text{ and } f(\vec{p}) \text{ is real.}$$

$$\Rightarrow \quad \langle 0|\, a_{\vec{q}}\, f(\vec{q})\, f(\vec{p})\, a^{+}_{\vec{p}}\, |0\rangle$$

$$= \quad (f(\vec{p}))^2 \frac{1}{(2\pi)^3 2E_{\vec{p}}} \left[\frac{1}{C(E_{\vec{p}})}\right]^2 \delta^3(\vec{p}-\vec{q})$$

$$\Rightarrow \quad f(\vec{p}) = C(E_{\vec{p}})(2\pi)^3 2E_{\vec{p}}$$

and therefore

$$\langle 0|\phi(x)|k\rangle = \langle 0| \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) (a_{\vec{p}} e^{-ip\cdot x} + a^{+}_{\vec{p}} e^{ip\cdot x})\, C(E_{\vec{k}})(2\pi)^3 2E_{\vec{k}}\, a^{+}_{\vec{k}}|0\rangle$$

$$= \int_{-\infty}^{+\infty} d^3\vec{p}\; C(E_{\vec{p}})C(E_{\vec{k}})(2\pi)^3\, 2E_{\vec{k}} \cdot \frac{1}{(2\pi)^3 2E_{\vec{p}}} \left[\frac{1}{C(E_{\vec{p}})}\right]^2 \delta^3(\vec{p}-\vec{k})\, e^{-ip\cdot x}$$

$$= e^{-ip\cdot x}$$

In Peskin & Schroeder, $C(E_{\vec{p}}) = \frac{1}{(2\pi)^3}\left(\frac{1}{2E_{\vec{p}}}\right)^{\frac{1}{2}}$

$$|p\rangle = \sqrt{2E_{\vec{p}}}\, a^{+}_{\vec{p}}|0\rangle$$

In Ryder "Quantum Field Theory", $C(E_{\vec{p}}) = \frac{1}{(2\pi)^3 2E_{\vec{p}}}$

$$|p\rangle = a^{+}_{\vec{p}}|0\rangle$$

In Ho-Kim & Yem "Elementary Particles and their Interactions",

$$C(E_p) = \left[\frac{1}{(2\pi)^3 2E_{\vec{p}}}\right]^{\frac{1}{2}},$$

$$|p\rangle = \left[(2\pi)^3 2E_{\vec{p}}\right]^{\frac{1}{2}} a^{+}_{\vec{p}}|0\rangle.$$