Example to use momentum vector

A pion at rest decays into a muon plus a neutrino (assume massless)
What is the speed of muon in the rest frame of pion?

method 1. In the rest frame of pion

$$E_\pi = m_\pi$$

$$E_\mu = (m_\mu^2 + |\vec{P}_\mu|^2)^{\frac{1}{2}}$$

$$E_\nu = |\vec{P}_\nu| = |\vec{P}_\mu|$$

Use Conservation of energy (i.e., the 0-component of four-momentum conservation $P_\pi^\alpha = P_\mu^\alpha + P_\nu^\alpha$)

$$P_\pi^0 = P_\mu^0 + P_\nu^0, \text{ that is, } E_\pi = E_\mu + E_\nu$$

$$\Rightarrow m_\pi = (m_\mu^2 + |\vec{P}_\mu|^2)^{\frac{1}{2}} + |\vec{P}_\mu|$$

$$\Rightarrow |\vec{P}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

and $E_\mu = (m_\mu^2 + |\vec{P}_\mu|^2)^{\frac{1}{2}} = \dfrac{m_\pi^2 + m_\mu^2}{2m_\pi}$

$$\Rightarrow |\vec{v}_\mu| = \frac{|\vec{P}_\mu|}{E_\mu} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} \doteq \frac{139.57^2 - 105.66^2}{139.57^2 + 105.66^2} \doteq 0.27$$

(note: $p^\alpha = (p^0, p^i) = m\gamma(1, v^i)$

$$\Rightarrow v^i = \frac{p^i}{p^0} = \frac{p^i}{E}, \text{ i.e., } \vec{v} = \frac{\vec{P}}{E}$$)

Find the masses, etc. of a particle in pdg.lbl.gov.

Method 2.

$$P_\pi^\alpha = P_\mu^\alpha + P_\nu^\alpha \quad \text{, or} \quad P_\nu^\alpha = P_\pi^\alpha - P_\mu^\alpha$$

$$\Downarrow$$

$$P_\nu^2 = (P_\pi - P_\mu)^2 \quad \text{(dot product)}$$

$$\Downarrow$$

$$0 = m_\pi^2 + m_\mu^2 - 2P_\pi \cdot P_\mu .$$

$\uparrow$
(note that $P_\nu^2 = m_\nu^2 = 0$)

note that $P_\pi \cdot P_\mu$ is an invariant and can be evaluated in any reference frame and the result is the same. While in pion rest frame, $P_\pi \cdot P_\mu = m_\pi \cdot E_\mu$.

(a bit explanation: $P_\pi \cdot P_\mu = P_\pi^0 P_\mu^0 - P_\pi^1 P_\mu^1 - P_\pi^2 P_\mu^2 - P_\pi^3 P_\mu^3$
$= m_\pi E_\mu$

since
$P_\pi^\alpha = (m_\pi, 0, 0, 0) = m_\pi \gamma_\pi (1, V_\pi^1, V_\pi^2, V_\pi^3)$

$= m_\pi (1, 0, 0, 0)$
$\uparrow$
Since $\gamma_\pi = 1$, $V_\pi^{1,2,3} = 0$.)

$$0 = m_\pi^2 + m_\mu^2 - 2m_\pi E_\mu \Rightarrow E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}$$

Since $P_\mu^\alpha = P_\pi^\alpha - P_\nu^\alpha$, then $P_\mu^2 = (P_\pi - P_\nu)^2 \Rightarrow m_\mu^2 = m_\pi^2 + 0 - 2m_\pi E_\nu$

$$= m_\pi^2 - 2m_\pi |\vec{P}_\nu| = m_\pi^2 - 2m_\pi |\vec{P}_\mu|$$

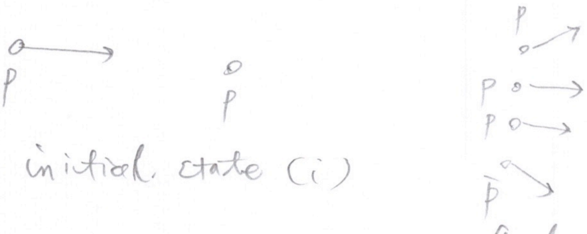
$$\Rightarrow |\vec{P}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

$$\Rightarrow |\vec{v}_\mu| = \frac{|\vec{P}_\mu|}{E_\mu} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}$$

Example 2.

In the lab frame, a high energy proton hit a proton at rest, and the process is $p + p \longrightarrow p + p + p + \bar{p}$. What is the threshold energy for this reaction (i.e, the minimum energy of the incident proton in the lab frame)?

in lab frame:



initial state (i)

final state (f)

in center of momentum frame
(CM)

initial (i)

final (f)

In the lab frame, the total four-momentum is

$$P^{\mu}_{i,lab} = (E + m,\ |\vec{P}|,\ 0,\ 0)$$

$$\Rightarrow\ P_i^2 = P^{\mu}_{i,lab} P_{\mu\, i,lab} = (E+m)^2 - |\vec{P}|^2 = E^2 - |\vec{P}|^2 + 2mE + m^2 = 2m^2 + 2mE$$

P_i^2 is an invariant, can be evaluated at any frame and the result is the same.

By energy-momentum conservation, $P^{\mu}_{i,lab} = P^{\mu}_{f,lab}$, $P^{\mu}_{i,CM} = P^{\mu}_{f,CM}$,
(note: $P^{\mu}_{i,lab} \neq P^{\mu}_{f,CM}$)

Therefore $P_i^2 = P_f^2$, note that P_i^2 and P_f^2 can be evaluated at different frame, since they are both Lorentz invariants.

The most convinient frame to evaluate P_f^2 is in the CM frame, since all the space components are zero, that is, $P^{\mu}_{f,CM} = (P^0_{f,CM}, 0, 0, 0)$

$$\Rightarrow\ P_f^2 = {P^0_{f,CM}}^2 = P_i^2 = 2m^2 + 2mE$$

Therefore the minimum of $P^0_{f,CM}$ gives the minimum of E

The minimum of $P^0_{f.\,CM}$ is $4m$ (i.e., all four final particles are at rest, no kinetic energy).

$$\Rightarrow \quad E_{min} = \frac{(4m)^2 - 2m^2}{2m} = 7m.$$

Question

Is that always possible to find a Center of momentum frame?

Suppose in an abitrary frame,

$$E_{tot} = \sum_i \gamma_i m_i \ , \ \vec{P}_{tot} = \sum_i \gamma_i m_i \vec{v}_i$$

note that for massless particle i, $\gamma_i m_i$ should be understood as $h\nu_i = \hbar \omega_i = \omega_i$, and $|\vec{v}_i| = 1$

Now make a Lorentz transformation along the direction of $\vec{P}_{tot}$

$$|\vec{P}'_{tot}| = \gamma(|\vec{P}_{tot}| - v E_{tot})$$

(note that since p^μ is a vector, it has the same Lorentz transformation as x^μ)

require $|\vec{P}'_{tot}| = 0$

$$\Rightarrow \quad v = \frac{|\vec{P}_{tot}|}{E_{tot}} = \frac{|\sum_i \gamma_i m_i \vec{v}_i|}{\sum_i \gamma_i m_i}$$

Using
$$|\sum_i \gamma_i m_i \vec{v}_i| = \Big[\big(\sum_i \gamma_i m_i |\vec{v}_i| \cos\theta_i\big)^2 + \big(\sum_i \gamma_i m_i |\vec{v}_i| \sin\theta_i \cos\varphi_i\big)^2 + \big(\sum_i \gamma_i m_i |\vec{v}_i| \sin\theta_i \sin\varphi_i\big)^2\Big]^{1/2}$$

$$= \Big(\sum_i (\gamma_i m_i |\vec{v}_i|)^2 + 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \cos\theta_i \cos\theta_j$$
$$+ 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \sin\theta_i \sin\theta_j \cos\varphi_i \cos\varphi_j$$
$$+ 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \sin\theta_i \sin\theta_j \sin\varphi_i \sin\varphi_j \Big]^{\frac{1}{2}}$$

$$= \Big[\sum_i (\gamma_i m_i |\vec{v}_i|)^2 + 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \cos\theta_i \cos\theta_j$$
$$+ 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |v_i| \gamma_j m_j |v_j| \sin\theta_i \sin\theta_j \cos(\varphi_i - \varphi_j)\Big]^{\frac{1}{2}}$$

$$\leq \Big[\sum_i (\gamma_i m_i |\vec{v}_i|)^2 + 2\sum_{\substack{i,j\\ i\neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| (\cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j)\Big]^{\frac{1}{2}}$$

$$\leq \left[\sum_i (\gamma_i m_i |\vec{v}_i|)^2 + 2 \sum_{\substack{i,j \\ i \neq j}} \gamma_i m_i |\vec{v}_i| \gamma_j m_j |\vec{v}_j| \right]^{\frac{1}{2}}$$

$$= \sum_i \gamma_i m_i |\vec{v}_i|$$

$$\leq \sum_i \gamma_i m_i$$

$$\Rightarrow \quad V \leq 1.$$

the "=" is achieved only when all particles are massless and they move in the same direction, otherwise $V < 1$ and it means a CM frame can be found.

Formally (a bit more tensor algebra)

For coordinate transformation $dx^\mu \longrightarrow dx^{\mu'} = \Lambda^\mu{}_\nu dx^\nu$.

its inverse transformation is $dx^{\mu'} \longrightarrow dx^\mu \equiv \bar{\Lambda}^\mu{}_\nu dx^{\nu'}$

(note that the prime '$'$' is for x', not for μ', and the two coordinate systems are $x^\mu = (t, x, y, z)$ and $x^{\mu'} = (t', x', y', z')$)

From $dx^{\mu'} = \Lambda^\mu{}_\nu dx^\nu = \Lambda^\mu{}_\nu \bar{\Lambda}^\nu{}_\alpha dx^{\alpha'}$

$$\Rightarrow \quad \Lambda^\mu{}_\nu \bar{\Lambda}^\nu{}_\alpha = \delta^\mu{}_\alpha$$

From $dx^\mu = \bar{\Lambda}^\mu{}_\nu dx^{\nu'} = \bar{\Lambda}^\mu{}_\nu \Lambda^\nu{}_\beta dx^\beta$

$$\Rightarrow \quad \bar{\Lambda}^\mu{}_\nu \Lambda^\nu{}_\beta = \delta^\mu{}_\beta$$

That is, $\bar{\Lambda}^\mu{}_\lambda \Lambda^\lambda{}_\nu = \Lambda^\mu{}_\lambda \bar{\Lambda}^\lambda{}_\nu = \delta^\mu{}_\nu$.

where $\delta^\mu{}_\nu = \begin{cases} 1, & \mu = \nu \\ 0, & \mu \neq \nu \end{cases}$

Therefore, without introducing $g_{\mu\nu}$, we can still define arbitrary covariant vector:

If the coordinate transformation is $dx^{\mu'} = \Lambda^\mu{}_\nu dx^\nu$.

and the inverse transformation is $dx^\mu = \bar{\Lambda}^\mu{}_\nu dx^{\nu'}$

where $\bar{\Lambda}^\mu{}_\lambda \Lambda^\lambda{}_\nu = \Lambda^\mu{}_\lambda \bar{\Lambda}^\lambda{}_\nu = \delta^\mu{}_\nu$

then we call A^μ a contravariant vector if it transform as

$$A^\mu \longrightarrow A^{\mu'} = \Lambda^\mu{}_\nu A^\nu \quad \text{(i.e., the same as } x^\mu \text{)}$$

and we call A_μ a covariant vector if it transform as

$$A_\mu \longrightarrow A_\mu' = \bar{\Lambda}^\nu{}_\mu A_\nu$$

note that $A^{\mu'} A_{\mu'} = A_{\mu'} A^{\mu'} = \Lambda^\mu{}_\lambda A^\lambda \bar{\Lambda}^\beta{}_\mu A_\beta = \delta^\beta{}_\lambda A^\lambda A_\beta = A^\beta A_\beta$

That is, $A^\mu A_\mu$ is an invariant under coordinate transformation.

Higher rank tensor can be defined as, e.g.,

$$A^{\mu\nu}{}_{\lambda} \longrightarrow A^{\mu\nu}{}_{\lambda}{}' = \Lambda^{\mu}{}_{\sigma} \Lambda^{\nu}{}_{\rho} \bar{\Lambda}^{\tau}{}_{\lambda} A^{\sigma\rho}{}_{\tau}.$$

Show that Kronecker symbol is a rank two mixed tensor.

Since $\Lambda^{\mu}{}_{\lambda} \bar{\Lambda}^{\beta}{}_{\tau} \delta^{\lambda}{}_{\beta} = \Lambda^{\mu}{}_{\lambda} \bar{\Lambda}^{\lambda}{}_{\tau} = \delta^{\mu}{}_{\tau}$

while in all coordinate systems the Kronecker is the same,

that is, $\delta^{\mu}{}_{\tau}{}' = \delta^{\mu}{}_{\tau}$

$$\Rightarrow \quad \delta^{\mu}{}_{\tau}{}' = \Lambda^{\mu}{}_{\lambda} \bar{\Lambda}^{\beta}{}_{\tau} \delta^{\lambda}{}_{\beta}$$

which means Kronecker symbol is a rank two mixed tensor.

Since $ds^2 = d\tau^2$ is an invariant under coordinate transformation, i.e.,

$ds^2 = d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2 = dt'^2 - dx'^2 - dy'^2 - dz'^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$

then $g_{\mu\nu}$ is a rank two covariant tensor $\qquad = g'_{\mu\nu} dx^{\mu\prime} dx^{\nu\prime}$

proof:

$$g'_{\mu\nu} dx^{\mu\prime} dx^{\nu\prime} = g'_{\mu\nu} \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\beta} dx^{\lambda} dx^{\beta}$$

$$= g_{\lambda\beta} dx^{\lambda} dx^{\beta}$$

$$\Rightarrow \quad g'_{\mu\nu} \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\beta} = g_{\lambda\beta}$$

$$\Rightarrow \quad g'_{\mu\nu} \Lambda^{\mu}{}_{\lambda} \Lambda^{\nu}{}_{\beta} \bar{\Lambda}^{\lambda}{}_{\kappa} \bar{\Lambda}^{\beta}{}_{\gamma} = g_{\lambda\beta} \bar{\Lambda}^{\lambda}{}_{\kappa} \bar{\Lambda}^{\beta}{}_{\gamma}$$

$$\Rightarrow \quad g'_{\kappa\gamma} = g_{\lambda\beta} \bar{\Lambda}^{\lambda}{}_{\kappa} \bar{\Lambda}^{\beta}{}_{\gamma} \quad \checkmark$$

The inverse of $g_{\mu\nu}$, which satisfies $g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}{}_{\nu}$ is a rank two contravariant tensor

proof:

$$g'^{\mu\lambda} g'_{\lambda\nu} = \delta^{\mu}{}_{\nu}$$

$$= g'^{\mu\lambda} g_{\alpha\beta} \bar{\Lambda}^{\alpha}{}_{\lambda} \bar{\Lambda}^{\beta}{}_{\nu}$$

times $\Lambda^{\nu}{}_{\kappa}$ on both sides $\Rightarrow$ $g'^{\mu\lambda} g_{\alpha\beta} \bar{\Lambda}^{\alpha}{}_{\lambda} \bar{\Lambda}^{\beta}{}_{\nu} \Lambda^{\nu}{}_{\kappa} = \delta^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\kappa} = \Lambda^{\mu}{}_{\kappa}$

$$= g'^{\mu\lambda} g_{\alpha\beta} \bar{\Lambda}^{\alpha}{}_{\lambda} \delta^{\beta}{}_{\kappa} = g'^{\mu\lambda} g_{\alpha\kappa} \bar{\Lambda}^{\alpha}{}_{\lambda}$$

times $g^{\kappa\beta}$ on both sides

$\Longrightarrow$

$$g'^{\mu\lambda} g_{\alpha\kappa} \bar{\Lambda}^{\alpha}{}_{\lambda}\, g^{\kappa\beta} = \Lambda^{\mu}{}_{\kappa} g^{\kappa\beta}$$

$$= g'^{\mu\lambda} \bar{\Lambda}^{\beta}{}_{\lambda}$$

times $\Lambda^{\alpha}{}_{\beta}$ on both sides

$\Longrightarrow$

$$g'^{\mu\lambda} \bar{\Lambda}^{\beta}{}_{\lambda} \Lambda^{\alpha}{}_{\beta} = \Lambda^{\mu}{}_{\kappa} g^{\kappa\beta} \Lambda^{\alpha}{}_{\beta}$$

$$= g'^{\mu\alpha}$$

✓

Since Kronecker symbol is a unit matrix, it is OK to write it as $\delta^{\mu}{}_{\nu}$, or $\delta^{\mu\nu}$, or $\delta_{\mu\nu}$, or $g^{\mu}{}_{\nu}$.

Note 1: $x^{\mu} = (t, x, y, z) = (x^0, x^1, x^2, x^3)$

$x_{\mu} = (t, -x, -y, -z) = (x^0, -x^1, -x^2, -x^3)$

$= (x_0, x_1, x_2, x_3)$

i.e., $x^0 = x_0$, $x^i = -x_i$, similarly, $p^0 = p_0$, $p^i = -p_i$

Note 2: $\partial_{\mu} \equiv \dfrac{\partial}{\partial x^{\mu}} = (\partial_0, \vec{\nabla}) = \left(\dfrac{\partial}{\partial x^0}, \dfrac{\partial}{\partial x^i}\right)$

$\partial^{\mu} = \dfrac{\partial}{\partial x_{\mu}} = (\partial_0, -\vec{\nabla})$

d'Alembert $\square \equiv \partial^2 \equiv \partial_{\mu}\partial^{\mu} = \left(\dfrac{\partial}{\partial x^0}\right)^2 - \vec{\nabla}^2$

(a bit more Lorentz transformation)

Lorentz Symmetry

$\boxed{\text{warm up.}}$ — write coordinate transformation using matrix.

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \overset{c=1}{=} (t, x, y, z) = (t, \vec{x}) = (x^0, \vec{x})$$

Usually, Greek indices take on the value 0, 1, 2, 3;

Latin indices $i, j, \ldots$, ——— 1, 2, 3, for space components.

Transformation of coordinate system from $\{x^\mu\}$ to $\{x'^\mu\}$

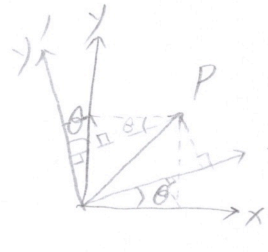
ν is dummy index

$$x'^\mu = a^\mu{}_\nu x^\nu = a^\mu{}_0 x^0 + a^\mu{}_1 x^1 + a^\mu{}_2 x^2 + a^\mu{}_3 x^3$$

(Einstein summation convention: when an index variable appear twice in a single term and is not otherwise defined, it implies summation of that term over all the values of the index.)

Example 1. Space Rotation (a rotation of coordinate system about the z axis by a positive angle θ in the counterclockwise direction) where $\theta \in [0, 2\pi)$

$$a^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$\Rightarrow \quad x'^0 = x^0$$
$$x'^1 = x^1\cos\theta + x^2\sin\theta$$
$$x'^2 = -x^1\sin\theta + x^2\cos\theta$$
$$x'^3 = x^3$$

Example 2. pure Lorentz transformation (relates two coordinate systems differ only by a uniform relative motion of velocity v)

$$a^\mu{}_\nu = \begin{pmatrix} \cosh w & -\sinh w & 0 & 0 \\ -\sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad x'^0 = \cosh w(x^0 - x^1\tanh w) = \gamma(x^0 - v x^1)$$
$$x'^1 = \cosh w(-x^0\tanh w + x^1) = \gamma(-v x^0 + x^1)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

where $-\infty < w < +\infty$, $\cosh w = \gamma = \frac{1}{\sqrt{1-v^2}}$, $\tanh w = v$

Note that the two coordinate systems satisfy

$(x', y', z') = (x, y, z) = (0, 0, 0)$ at $t = t' = 0$.

Meaning: an event happens at (t, x, y, z) happens at (t', x', y', z')

→ an event happens at $(t=0, x=0, y=0, z=0)$ happens at $(t'=0, x'=0, y'=0, z'=0)$

Note: the benefit to use hyperbolic functions compared to directly use v and $\gamma = \frac{1}{\sqrt{1-v^2}}$:

If there is three reference frame A & B & C, suppose B moves relative to A by a constant velocity v_1, and C moves relative to B by a constant velocity v_2, and suppose v_1 & v_2 are both along x-direction and the origins of A & B & C coincide at $t = t' = t'' = 0$.

Then for an event at (t, x) in A, will have (t', x') in B and (t'', x'') in C satisfying (no need to worry y and z directions)

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh w_1 & -\sinh w_1 \\ -\sinh w_1 & \cosh w_1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

and $$\begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh w_2 & -\sinh w_2 \\ -\sinh w_2 & \cosh w_2 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

Therefore $$\begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} \cosh w_2 & -\sinh w_2 \\ -\sinh w_2 & \cosh w_2 \end{pmatrix} \begin{pmatrix} \cosh w_1 & -\sinh w_1 \\ -\sinh w_1 & \cosh w_1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(w_1+w_2) & -\sinh(w_1+w_2) \\ -\sinh(w_1+w_2) & \cosh(w_1+w_2) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

That is, frame C moving with velocity $\tanh(w_1+w_2)$ relative to frame A.

where $\tanh(w_1+w_2) = \frac{\tanh w_1 + \tanh w_2}{1+\tanh w_1 \tanh w_2} = \frac{v_1+v_2}{1+v_1 v_2}$.

(note: $\cosh(w_1+w_2) = \cosh w_1 \cosh w_2 + \sinh w_1 \sinh w_2$

$\sinh(w_1+w_2) = \sinh w_1 \cosh w_2 + \cosh w_1 \sinh w_2$)

Therefore, use hyperbolic functions make it easier for velocity add up.

The above velocity add up can be derived alternatively:

Consider an object move with velocity V_2 in B frame, and B frame move relative to A frame with velocity V_1, then what is the velocity the object relative to A?

use
$$t' = \gamma(t - v_1 x)$$
$$x' = \gamma(-v_1 t + x)$$

$$\Rightarrow V_2 = \frac{dx'}{dt'} = \frac{\gamma(-v_1 dt + dx)}{\gamma(dt - v_1 dx)} = \frac{-v_1 + \frac{dx}{dt}}{1 - v_1 \frac{dx}{dt}}$$

$$\Rightarrow V_2 - V_2 V_1 \frac{dx}{dt} = -V_1 + \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{V_2 + V_1}{1 + V_1 V_2}$$ ✓

$\boxed{\text{end worm up}}$

Lorentz transformation

Lorentz transformations leave invariant the proper time interval $d\tau$ (or, write it as ds)

means it is the same in any inertial refrence frame

时空的四维间隔

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \overset{c=1}{=} dt^2 - dx^2 - dy^2 - dz^2.$$

$$ds^2 = \underbrace{g_{\mu\nu}}_{\text{度规 (metric)}} dx^\mu dx^\nu$$

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu.$$

in matrix form: $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$