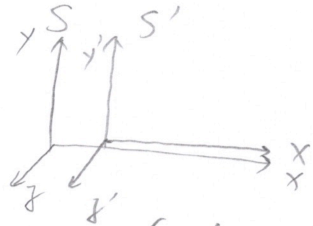
Lorentz transformation

Two inertial frames S and S', with S' moving at uniform velocity $\vec{V}$
(↪ the frame in which Newton's first law is obeyed.)
with respect to S. Suppose also the velocity is in the common x & x' axis, and the two frames coincide (i.e., at $t=t'=0$, $x=x'=0$, $y=y'=0$, $z=z'=0$)

Then an event occurs at (x, y, z) and time t in S occurs in S' having

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \overset{c=1}{=} \gamma(t - vx).$$

where $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}} = \dfrac{1}{\sqrt{1-v^2}}$



(note, x and x' axis actually overlap).

The inverse transformation (i.e., from S' to S) is (just solve x & t in the above equations)

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + vx')$$

Four consequences:

1 同时的相对性 (the relativity of simultaneity)

If two events occurs at the same time in S, but at different locations then they do not occur at the same time in S'.

That is, for two event A & B having $t_A = t_B$ but $x_A \neq x_B$ in frame S, then in S' frame

$$\left.\begin{aligned} t_A' &= \gamma(t_A - v x_A) \\ t_B' &= \gamma(t_B - v x_B) \end{aligned}\right\} \Rightarrow \quad t_A' - t_B' = \gamma v (x_B - x_A) \neq 0.$$

2. Lorentz contraction (尺缩): → 量运动的尺得到的长度变短

Suppose a stick lies on the x' axis, at rest in S', say, one end is at the origin $x'=0$ and the other end is at L' (so its length in S' frame is L'). What is its length as measured in S?

↓

(implies we need to measure the two ends at the same time in S)

Since $\Delta x = \gamma(\Delta x' + v\Delta t')$ (put it in)

$$\Delta t = \gamma(\Delta t' + v\Delta x') \equiv 0 \Rightarrow \Delta t' = -v\Delta x'$$

$$\Rightarrow \Delta x = \gamma[\Delta x' + v(-v\Delta x')]$$

$$= \gamma\Delta x'(1-v^2)$$

$$= \frac{\Delta x'}{\gamma}$$

$$= \frac{L'}{\gamma} \leq L'$$

(note that $\gamma = \frac{1}{\sqrt{1-v^2}} \geq 1$)

Note that y and z directions do not suffer Lorentz contraction.

3. Time dilation (钟慢)
运动的钟走得慢.

Suppose a clock is at rest in S' frame and it measures a period T', then how long is this period as measured in S ?

$$\Delta t = \gamma(\Delta t' + v\underbrace{\Delta x'}_{0 \text{ since at rest in } S'})$$

$$= \gamma T'$$

$$\geq T'$$

4. velocity addition:

Suppose a particle P is moving in the x & x' direction at speed u', with respect to S', then what is its speed, u, with respect to S ?

$$dx = \gamma(dx' + v\,dt')$$

$$dt = \gamma(dt' + v\,dx')$$

$$\Rightarrow \quad u = \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + v\,dx'} = \frac{u' + v}{1 + vu'}$$

Notice that if $u' = c = 1$, then $u = \frac{1+v}{1+v} = 1 = c$.

that is, speed of light is the same in all inertial frames.

Write the above formula in another way:

$$v_{PS} = \frac{v_{PS'} + v_{S'S}}{1 + v_{PS'}v_{S'S}/c^2}$$, or just write it as

Galilean velocity addition rule

$$V_{AC} = \frac{V_{AB} + V_{BC}}{1 + V_{AB}V_{BC}/c^2}$$, (V_{AB} 表示A相对于B的速度, etc.)

Einstein's correction

Four-vectors

Now introduce position-time four vector x^μ

$$x^0 = ct = t \,, \quad x^1 = x \,, \; x^2 = y \,, \; x^3 = z .$$

So Lorentz transformation can be written as

$$x'^0 = \gamma(x^0 - \beta x^1) = \gamma(x^0 - v x^1) \quad , \text{ note } \beta \equiv \frac{v}{c} = v$$

$$x'^1 = \gamma(x^1 - v x^0)$$

$$x'^2 = x^2$$

$$x'^3 = x^3 .$$

Or just write $x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x^\nu$, $(\mu = 0, 1, 2, 3)$

where the coefficients $\Lambda^\mu{}_\nu$ are the elements of a matrix Λ:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v & & \\ -\gamma v & \gamma & & 0 \\ & & 1 & 0 \\ & 0 & 0 & 1 \end{pmatrix}$$

Introduce Einstein's summation convention to avoid writing Σ.

$$x'^\mu = \Lambda^\mu{}_{\underline{\nu}} \, x^{\underline{\nu}} = \Lambda^\mu{}_\alpha \, x^\alpha$$

dummy indices, whatever can be used

Invariant.

when go from S to S' frame, there is a particular combination remains the same:

$$I \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 .$$

(Similarly, the quantity $r^2 = x^2 + y^2 + z^2$ is invariant under rotation) check for the pure Lorentz transformation above

$$(x'^0)^2 = \gamma^2(x^0 - v x^1)^2 = \gamma^2[(x^0)^2 + v^2(x^1)^2 - 2v x^0 x^1]$$

$$(x'^1)^2 = \gamma^2(x^1 - v x^0)^2 = \gamma^2[(x^1)^2 + v^2(x^0)^2 - 2v x^1 x^0]$$

$$\Rightarrow (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2$$

$$= \gamma^2[(x^0)^2 + v^2(x^1)^2 - (x^1)^2 - v^2(x^0)^2] - (x^2)^2 - (x^3)^2$$

$$= (x^0)^2\gamma^2(1-v^2) - (x^1)^2\gamma^2(1-v^2) - (x^2)^2 - (x^3)^2$$

$$= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \quad \checkmark$$

Then how to write the $I = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ shorter?

Introduce metric $g_{\mu\nu}$ to help

$$g = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ & 0 & -1 & \\ & & & -1 \end{pmatrix}$$

(i.e., $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$, $g_{\mu\nu} = 0$ for $\mu \neq \nu$)

(Caution: some books use opposite signs).

We can now write

$$I = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu = g_{\mu\nu} x^\mu x^\nu.$$

Note that $g_{\mu\nu} = g_{\nu\mu}$. (symmetric tensor)

Define x_μ now

$$x_\mu \equiv g_{\mu\nu} x^\nu .$$

We call the upper index guy, x^μ, contravariant four-vector, and the lower index guy, x_μ, covariant four-vector.

$$\text{So} \quad I = x_\mu x^\mu = x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu .$$

(note that in GR, $g_{\mu\nu}$ is more general, and our $g_{\mu\nu}$ here is written as $\eta_{\mu\nu}$ in GR. But let's don't worry about it, since we will only work in flat spacetime in QFT I)

Define abitrary contravariant four-vector and covariant four-vector

A four-component object that transforms in the same way as x^μ does when go from one inertial frame to another,

$$a'^\mu = \Lambda^\mu{}_\nu a^\nu$$

and for each a^μ we have a covariant four-vector.

$$a_\mu = g_{\mu\nu} a^\nu .$$

Introduce the inverse of $g_{\mu\nu}$, i.e., $g^{\mu\nu}$.

$$g^{\mu\nu} g_{\nu\alpha} = \underbrace{\delta^\mu_\alpha}_{\text{called Kronecker symbol.}}$$

$$g^{-1} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \delta = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Now we can get invariants (i.e., the same value in any inertial frame) from a contravariant and a covariant vectors.

$$a^\mu b_\mu = a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu = g^{\mu\nu} a_\mu b_\nu = \delta^\mu{}_\nu a^\nu b_\mu$$
$$= a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

$a^\mu a_\mu$ is invariant, too
$b^\mu b_\mu$ is invariant, too.

We call it the "scalar product" of a and b. (it is simply the analog to the dot product of two three-vectors),

$$a \cdot b \equiv a_\mu b^\mu .$$

Usually we write

$$a \cdot b = a^0 b^0 - \vec{a} \cdot \vec{b}$$

(note: the dot product of two three-vector is just
$\vec{a} \cdot \vec{b} = a^1 b^1 + a^2 b^2 + a^3 b^3$)

In particular,

$$a^2 \equiv a \cdot a = (a^0)^2 - \vec{a}^2 = a^0 a^0 - a^1 a^1 - a^2 a^2 - a^3 a^3$$

Note: if $a^2 > 0$, a^μ is called timelike (e.g., $a^\mu = (t, 0, 0, 0)$)
if $a^2 < 0$, a^μ is called spacelike (e.g., $a^\mu = (0, x, 0, 0)$)
if $a^2 = 0$, a^μ is called lightlike (e.g., $a^\mu = (t, x = ct, 0, 0)$)

Now define tensors

e.g., ① a second-rank tensor, $S^{\mu\nu}$, carries two indices, so has $4^2=16$ components, and transforms with two factors of Λ:

$$S'^{\mu\nu} = \Lambda^{\mu}_{\ \kappa}\Lambda^{\nu}_{\ \sigma} S^{\kappa\sigma}$$

② a third-rank tensor $t^{\mu\nu\lambda}$, has $4^3=64$ components, and transforms as

$$t'^{\mu\nu\lambda} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\Lambda^{\lambda}_{\ \sigma} t^{\alpha\beta\sigma}$$

③ covariant tensors and mixed tensors can be obtained by lowering indices by $g_{\mu\nu}$, e.g.

$$S^{\mu}_{\ \nu} = g_{\nu\lambda} S^{\mu\lambda}, \quad S_{\mu\nu} = g_{\mu\kappa} g_{\nu\lambda} S^{\kappa\lambda}$$

④ a vector is a tensor of rank one;
a scalar (i.e., invariant) is a tensor of rank zero.

⑤ 张量与数相乘是同类张量, e.g., $p^{\mu} = m\eta^{\mu}$.
两个张量相乘得新张量, e.g., $a^{\mu}b^{\nu}$ is a second rank tensor.

⑥ <u>Contraction</u> of a $(n+2)$ rank tensor to get a n rank tensor, by summing like upper and lower indices.

e.g. $S^{\mu}_{\ \mu}$ is a scalar;
$t_{\nu}^{\ \mu\nu}$ is a vector;
$a_{\mu} t^{\mu\nu\lambda}$ is a second-rank tensor.

Energy & Momentum

Introduce proper time $d\tau$. (也叫四維間隔)

$$d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = dt^2 - dx^2 - dy^2 - dz^2$$

(note $dt^2 \equiv (dt)(dt)$, $dx^2 = (dx)(dx)$, just infinitesimal change of $I \equiv \tau^2 \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$

The proper time is the time measured by the watch moving together with the object. Therefore in the object's rest frame S',

$$d\tau = dt'$$

The velocity of the object respect to frame S is

$$\vec{v} = \frac{d\vec{x}}{dt} \quad (\text{or simply, } v = \frac{dx}{dt} \text{ if moving along } x\text{-direction})$$

But, a transformation of $\vec{v}$ involves transformation in both $\vec{x}$ and t, so introduce $\boxed{\text{Proper velocity, } \eta^\mu \equiv \frac{dx^\mu}{d\tau}}$, so only need to worry about transformation for x^μ (τ is invariant)

→ or just call it "four-velocity".

From

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + vx')$$

and use $dx' = 0,\ dy' = 0,\ dz' = 0,\ dt' = d\tau$

↳ (since the object is at rest in S' frame)

$$\Rightarrow \quad dx = \gamma v \, d\tau$$
$$dt = \gamma \, d\tau$$

$$\Rightarrow \eta^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dx^0}{d\tau}, \frac{d\vec{x}}{d\tau} \right) = (\gamma, \gamma\vec{v}) = \gamma(1, \vec{v})$$
$$= \gamma(1, v_x, v_y, v_z) = \gamma(1, v^1, v^2, v^3)$$

$$\Rightarrow \eta^\mu \eta_\mu = \eta^\mu g_{\mu\nu} \eta^\nu = \gamma (1, v_x, v_y, v_z) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \gamma \begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

$$= \gamma^2 (1 - v_x^2 - v_y^2 - v_z^2) = 1 = c^2$$

(yes, $\eta^\mu \eta_\mu$ is invariant, as expected)

<u>Now define momentum in relativity</u>

$$p^\mu \equiv m \eta^\mu$$

$$\Rightarrow p^0 = m\eta^0 = m\gamma = \frac{m}{\sqrt{1-v^2}} = \frac{mc}{\sqrt{1-v^2}}$$

$$\vec{p} = m\gamma\vec{v} = \frac{m\vec{v}}{\sqrt{1-v^2}}$$

Define energy as $E \equiv \gamma m c^2 = m\gamma$, then

$$\underbrace{p^\mu}_{\text{called four-momentum}} = \left(\frac{E}{c}, p_x, p_y, p_z\right) = (E, \vec{p})$$

So $p_\mu p^\mu = E^2 - |\vec{p}|^2 = (m\gamma)^2 - (m\gamma)^2 |\vec{v}|^2 = m^2$

(yes. $p_\mu p^\mu$ is invariant, as expected)

Now we can try Taylor expand E in the non-relativistic regime ($v \ll c$)

$$E = m\gamma = m(1-v^2)^{-\frac{1}{2}} = m\left(1 + \tfrac{1}{2}v^2 + O(v^4)\right)$$

$$= \underbrace{m}_{\text{rest energy}} + \underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic energy}} + O(v^4)$$

For massless particle, $m=0$, $v=c$, $E = \underbrace{|\vec{p}|c}_{=|\vec{p}|}$, and therefore $p^\mu p_\mu = 0$.

The energy and momentum of the massless particle is determined by its frequency $E = \hbar\nu$, not mass (0 anyway) or velocity (c anyway)

So for a massless particle

$$p^\mu = \hbar\omega\,(1,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

$\left(\text{since } (\sin\theta\cos\varphi)^2 + (\sin\theta\sin\varphi)^2 + (\cos\theta)^2 = 1\right)$

that is, the massless particle moving in a direction described by polar angle θ and azimuthal angle φ, with the speed $= c = 1$.