Homework 1 Solution

problem 1.

1) $$\frac{1}{H_0} = \frac{1}{100\times 0.7\ km\cdot s^{-1}\ Mpc^{-1}} = \frac{1\,s}{100\times 0.7\times\frac{1}{3.086\times 10^{19}}}$$

use $$1s = \frac{1s}{1Gyr}\times 1\,Gyr = \frac{1\,Gyr}{10^9\times 365\times 24\times 60\times 60}$$

$$\Rightarrow \frac{1}{H_0} = 1.4\times 10^1\,Gyr = \boxed{14\,Gyr.}$$

2) $$\rho_c = \frac{3H_0^2}{8\pi G_N} = \frac{3\times(100\times 0.7\cdot km\cdot s^{-1}\cdot Mpc^{-1})^2}{8\pi\times 6.674\times 10^{-11}\ m^3\,kg^{-1}\cdot s^{-2}}$$

collect the units: $$\frac{(km\cdot s^{-1}\cdot Mpc^{-1})^2}{m^3\cdot kg^{-1}\cdot s^{-2}} = \frac{s^{-2}\cdot\left(\frac{1}{3.086\times 10^{19}}\right)^2}{m^3\cdot kg^{-1}\cdot s^{-2}}$$

$$= \left(\frac{1}{3.086\times 10^{19}}\right)^2 kg/m^3$$

$$= \left(\frac{1}{3.086\times 10^{19}}\right)^2 \times \frac{1\,kg}{1\,g}\,1g\cdot\frac{1\,cm^3}{1\,m^3}\,\frac{1}{1\,cm^3}$$

$$= \left(\frac{1}{3.086\times 10^{19}}\right)^2 \times \frac{10^3}{10^6}\,g/cm^3$$

$$\Rightarrow \rho_c = \boxed{9.2\times 10^{-30}\,g/cm^3}$$

3) $$\frac{g}{cm^3} = \frac{g}{M_\odot}M_\odot\,\frac{Mpc^3}{cm^3}\,\frac{1}{Mpc^3} = \frac{1}{1.988\times 10^{33}}\times(3.086\times 10^{24})^3\,\frac{M_\odot}{Mpc^3}$$

$$\Rightarrow \rho_c = \boxed{1.4\times 10^{11}\,M_\odot/Mpc^3}$$

4) $$[\hbar] = kg\cdot m^2\cdot s^{-1},\ [c] = m\cdot s^{-1},\ [G_N] = m^3\cdot kg^{-1}\cdot s^{-2}$$

$$\Rightarrow [\hbar^{n_1}c^{n_2}G_N^{n_3}] = kg^{n_1}\,m^{2n_1}\,s^{-n_1}\cdot m^{n_2}\cdot s^{-n_2}\,m^{3n_3}\,kg^{-n_3}\cdot s^{-2n_3} \overset{\text{require}}{=} kg\cdot m^{-3}$$

$$\Rightarrow \begin{cases} n_1 - n_3 = 1 & ① \\ 2n_1 + n_2 + 3n_3 = -3 & ② \\ -n_1 - n_2 - 2n_3 = 0 & ③ \end{cases} \Longrightarrow \quad ②+③ \Rightarrow \begin{cases} n_1 + n_3 = -3 \\ n_1 - n_3 = 1 \end{cases} ① \quad \Rightarrow \quad \begin{aligned} n_1 &= -1 \\ n_3 &= -2 \\ n_2 &= 5 \end{aligned}$$

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$$\Rightarrow \quad \boxed{n_1 = -1, \quad n_2 = 5, \quad n_3 = -2}$$

The Planck density is $\rho_{pl} \equiv \dfrac{c^5}{\hbar G_N^2}$

5) In $\hbar = c = 1$ units, $\rho_{pl} = \dfrac{1}{G_N^2}$

use $1\,\text{eV} = 1.6022 \times 10^{-19}\,\text{J}$

From $\hbar = 1.0546 \times 10^{-34}\,\text{J}\cdot\text{s} \equiv 1$

$$\Rightarrow \quad 1\,\text{s} = \frac{1}{1.0546 \times 10^{-34}\,\text{J}} = \frac{1}{1.0546 \times 10^{-34}} \frac{\text{eV}}{\text{J}} \frac{\text{GeV}}{\text{eV}} \frac{1}{\text{GeV}}$$

$$= \frac{1.6022 \times 10^{-19}}{1.0546 \times 10^{-34}} \times 10^{9}\,\text{GeV}^{-1}$$

From $c = 2.9979 \times 10^{8}\,\text{m}\cdot\text{s}^{-1} \equiv 1$

$$\Rightarrow \quad 1\,\text{m} = \frac{1\,\text{s}}{2.9979 \times 10^{8}} = \frac{1}{2.9979 \times 10^{8}} \frac{1.6022 \times 10^{-19}}{1.0546 \times 10^{-34}} \times 10^{9}\,\text{GeV}^{-1}$$

From $1\,\text{J} = 1\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-2}$

$$\Rightarrow \quad 1\,\text{kg} = \frac{1\,\text{J}}{\text{m}^2\cdot\text{s}^{-2}} = \frac{1\,\text{eV}}{1.6022 \times 10^{-19}} (2.9979 \times 10^{8})^2$$

$$= \frac{(2.9979 \times 10^{8})^2}{1.6022 \times 10^{-19}} \times 10^{-9}\,\text{GeV}$$

$$\Rightarrow \quad \rho_{pl} = \frac{1}{G_N^2} = \frac{1}{(6.674 \times 10^{-11})^2}\,\text{m}^{-6}\cdot\text{kg}^{2}\,\text{s}^{4}$$

$$= \frac{1}{(6.674 \times 10^{-11})^2} \left(\frac{1}{2.9979 \times 10^{8}} \frac{1.6022 \times 10^{-19}}{1.0546 \times 10^{-34}} \times 10^{9} \right)^{-6}$$

$$\times \left(\frac{(2.9979 \times 10^{8})^2}{1.6022 \times 10^{-19}} \times 10^{-9} \right)^2 \left(\frac{1.6022 \times 10^{-19}}{1.0546 \times 10^{-34}} \times 10^{9} \right)^4 \cdot \text{GeV}^4$$

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$$= \frac{1}{(6.674\times10^{-11})^2} (2.9979\times10^{8})^{10} \times (1.6022\times10^{-19})^{-4} \times (1.0546\times10^{-34})^2$$

$$\times (10^{9})^{-4} \ \mathrm{GeV}^4$$

$$= \boxed{2.2\times10^{76}\ \mathrm{GeV}^4}$$

$$\rho_c = \frac{3H_0^2}{8\pi G_N} = \frac{3\times\left(100\times0.7\times\frac{1}{3.086\times10^{19}}\ \mathrm{s}^{-1}\right)^2}{8\pi\times6.674\times10^{-11}}\ \mathrm{m}^{-3}\,\mathrm{kg}\,\mathrm{s}^2$$

where $\mathrm{kg}\cdot\mathrm{m}^{-3} = \frac{(2.9979\times10^{8})^2}{1.6022\times10^{-19}}\times10^{-9}\times\left(\frac{1}{2.9979\times10^{8}}\frac{1.6022\times10^{-19}}{1.0546\times10^{-34}}\times10^{9}\right)^{-3}\ \mathrm{GeV}^4$

$$= (2.9979\times10^{8})^{5}\times(1.6022\times10^{-19})^{-4}$$
$$\times(1.0546\times10^{-34})^{3}\times(10^{9})^{-4}\ \mathrm{GeV}^4$$

$$\Rightarrow \quad \rho_c = \boxed{4.0\times10^{-47}\ \mathrm{GeV}^4}$$

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problem 2.

$$a^\mu = (2, 0, 1, 8), \quad b^\mu = (0, 9, 2, 7)$$

$$\Rightarrow \boxed{a_\mu = g_{\mu\nu} a^\nu = (2, 0, -1, -8)}$$

$$\boxed{b_\mu = (0, -9, -2, -7)}$$

$$\vec{a} \cdot \vec{a} = 0^2 + 1^2 + 8^2 = \boxed{65}$$

$$\vec{a} \cdot \vec{b} = 0 \times 9 + 1 \times 2 + 8 \times 7 = \boxed{58}$$

$$a^2 = 2^2 - 0^2 - 1^2 - 8^2 = \boxed{-61}$$

$$a \cdot b = 2 \times 0 - (0 \times 9 + 1 \times 2 + 8 \times 7) = \boxed{-58}$$

$$(a+b)^2 = (2+0)^2 - (0+9)^2 - (1+2)^2 - (8+7)^2 = \boxed{-311}$$

problem 3

1) $$s+t+u = P_A^2 + P_B^2 + 2P_A\cdot P_B + P_A^2 + P_C^2 - 2P_A\cdot P_C + P_A^2 + P_D^2 - 2P_A\cdot P_D$$

$$= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 + 2P_A\cdot(P_B - P_C - P_D)$$

$$= 3m_A^2 + m_B^2 + m_C^2 + m_D^2 - 2P_A\cdot P_A$$

$$= m_A^2 + m_B^2 + m_C^2 + m_D^2$$

2) In the CM frame,

$$S = (P_A + P_B)^2 = (E_A + E_B)^2 \Rightarrow \sqrt{s} = E_A + E_B$$

$$m_B^2 = P_B^2 = [(P_A+P_B) - P_A]^2 = s + m_A^2 - 2(P_A+P_B)\cdot P_A = s + m_A^2 - 2\sqrt{s}\,E_A$$

$$\Rightarrow \boxed{E_{A,CM} = \frac{s + m_A^2 - m_B^2}{2\sqrt{s}}}$$

3) In B's rest frame,

$$S = (P_A + P_B)^2 = m_A^2 + m_B^2 + 2P_A\cdot P_B = m_A^2 + m_B^2 + 2m_B E_A$$

$$\Rightarrow \boxed{E_{A,B\,rest} = \frac{s - m_A^2 - m_B^2}{2m_B}}$$

problem 4

For the initial state, a reference frame can be found in which the three-momenta sum of the initial e^- and e^+ is zero, i.e., the CM frame. However, in this CM frame, a single photon has non-zero three-momentum. Therefore, it is clear that the conservation of momentum is violated.

$\Rightarrow$ The process $e^+ + e^- \to \gamma$ cannot happen.

Another way to see this is that the final state photon has $P_\gamma^2 = 0$ while the initial state four-momentum satisfies $(P_{e^-} + P_{e^+})^2 \geq (2m_e)^2 > 0$

$\Rightarrow$ $P_\gamma^2 \neq (P_{e^-} + P_{e^+})^2$, so the process cannot happen.

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