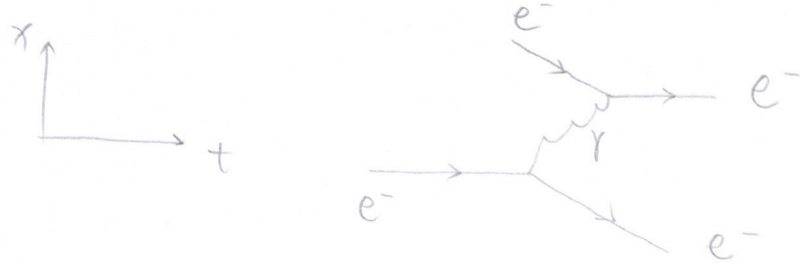
# 复习第一节课




相互作用

玻色子 $0, 1, 2, \cdots$

费米子 $\frac{1}{2}, \frac{3}{2}, \cdots$

our main focus：标量场(0)，矢量场(1)，旋量场($\frac{1}{2}$)．
这三种场的相互作用

参考书，只要包含上述三种场，然后讲它们的相互作用，然后给计算最低阶（tree level）的Feynman图，得到散射截面和衰变速率（寿命）就够了。

不计算圈图。

$\hbar = c (= k_B) = 1$ unit SI unit. m, kg, sec.

$1\,eV = 1.6\times10^{-19}\,J$ 电子伏特 (eV)

$W = UQ$

$[J] = [V][C]$

$1\,MeV = 10^6\,eV$

$1\,keV = 10^3\,eV$

$1\,GeV = 10^9\,eV$

$1\,TeV = 10^{12}\,eV$

$C = 3\times10^8\ m\cdot sec^{-1}$

$\hbar = \frac{h}{2\pi} = 1.05\times10^{-34}\ J\cdot Sec = 1.05\times10^{-34}\times 1eV\,\frac{1J}{1eV}\cdot sec$

$= 1.05\times10^{-34}\,eV\,\frac{J}{1.6\times10^{-19}J}\cdot sec = 6.6\times10^{-16}\ eV\cdot sec$

$\Rightarrow \hbar c = 20\times10^{-8}\ eV\cdot m = 200\ MeV\cdot fm$

$k_B = 1.38\times10^{-23}\ J\cdot K^{-1}$

If set $c=1$, then $1\,sec = 3\times10^8\,m$, so $[L]=[T]$, i.e., length and time are equivalent dimensions. (so velocity becomes dimensionless)

If set $\hbar=1$, then $1\,sec = \frac{1}{1.05\times10^{-34}J}$, so $[T]=[E]^{-1}$ (so action becomes dimensionless) $= 1.5\times10^{21}\,MeV^{-1}$

In Newtonian physics $P = mv$, $KE = \frac{1}{2}mv^2$, then $[P]=[M]=[E]$.

In special relativity $E^2 = P^2c^2 + m^2c^4$, so $c=1$ makes $[E]=[P]=[m]$

$1\,m = \frac{1\,sec}{3\times10^8} = \frac{1.5\times10^{21}}{3\times10^8}\,MeV^{-1} = 5\times10^{12}\,MeV^{-1}$

$1\,kg = \frac{1J}{1\,m^{-2}\,sec^{2}} = \frac{\frac{1}{1.6\times10^{-19}}\,eV\cdot(1.5\times10^{21}MeV^{-1})^2}{(5\times10^{12}MeV^{-1})^2} = 5.6\times10^{29}\,MeV$

mass of electron: $9.1\times10^{-31}kg = 9.1\times10^{-31}\times5.6\times10^{29}\,MeV = 0.51\,MeV$

$G$: (from $F = \frac{G m_1 M_2}{r^2} \Rightarrow [F] = [J]\cdot[m]^{-1} \Rightarrow [G] = [J][m]^{-1}\cdot[m]^2[kg]^{-2}$)

$G = 6.67\times10^{-11}\ J\cdot m\cdot kg^{-2} = 6.67\times10^{-11}\ kg\cdot m^2\cdot sec^{-2}\cdot m\cdot kg^{-2} = 6.67\times10^{-11}\,m^3\cdot kg^{-1}\,sec^{-2}$

$$\Rightarrow G = 6.67\times10^{-11}\,(5\times10^{12}\,MeV^{-1})^3\cdot(5.6\times10^{29}\,MeV)^{-1}\,(1.5\times10^{21}\,MeV^{-1})^{-2} = 6.7\times10^{-39}\,GeV^{-2}$$

So we usually use $\omega$ as energy (since $\hbar\omega = E$), $\vec{k}$ (波数) as momentum (since $\vec{p} = \hbar\vec{k}$)

If set $k_B = 1$, then $1K = 1.38\times10^{-23}J = 1.38\times10^{-23}\frac{1}{1.6\times10^{-19}}eV$

$$= 8.6\times10^{-11}\,MeV$$

Fine structure constant

$$\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \quad \text{where } \varepsilon_0 = 8.85\times10^{-12}\,F\cdot m^{-1} = 8.85\times10^{-12}\cdot C^2 J^{-1}\cdot m^{-1}$$

$$\left(1F = \frac{1\,Coulomb}{1V} = \frac{1\,Coulomb^2}{1J}\right)$$

$$\equiv \frac{k_e e^2}{\hbar c}$$

(note: $\varepsilon_0$ is vacuum permittivity, also called electric constant.)

and $k_e \simeq 9\times10^9\,J\cdot m\cdot C^{-2}$

$$= \frac{9\times10^9\,J\cdot m\cdot Coulomb^{-2}\times(1.6\times10^{-19}\,Coulomb)^2}{1}$$

$$= 9\times10^9\times(1.6\times10^{-19})^2\,J\cdot m = 9\times10^9\times(1.6\times10^{-19})^2\times\frac{1eV}{1.6\times10^{-19}}\,5\times10^{12}\,MeV^{-1}$$

$$= \frac{1}{137}$$

If we take $\varepsilon_0 = 1 = \hbar = c$, then $\alpha = \frac{e^2}{4\pi}$

then $1\,Coulomb = (8.85\times10^{-12})^{-\frac{1}{2}}(J\cdot m)^{\frac{1}{2}}$

$$= (8.85\times10^{-12})^{-\frac{1}{2}}\left(\frac{1eV}{1.6\times10^{-19}}\,5\times10^{12}\,MeV^{-1}\right)^{\frac{1}{2}}$$

$$= 1.9\times10^{18}$$

$$\Rightarrow \quad e = 1.6\times10^{-19}\,coulomb = 0.3$$

this dimensionless number "$e$" is what appeared in interaction Lagrangian, e.g. $\mathcal{L}_{int} = e\bar{\psi}\gamma^\mu\psi A_\mu$, or just write it as

$$\mathcal{L}_{int} = \sqrt{4\pi\alpha}\,\bar{\psi}\gamma^\mu\psi A_\mu$$

Note: $\alpha$ is dimensionless by definition, while $e$ is dimensionless by taking $\underbrace{\varepsilon_0\hbar c}_{\text{the product}}$ dimensionless.

计算静电力和引力的比值

以两电子为例.

$$F_{电} = \frac{k_e e^2}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} \left( \text{i.e., } F = k_e q^2/r^2 \right)$$

$$F_{引} = \frac{G m_e^2}{r^2}$$

$$\Rightarrow \frac{F_{电}}{F_{引}} = \frac{k_e e^2}{G m_e^2} = \frac{\overset{J\cdot m\cdot C^{-2}}{9\times10^9} \times (1.6\times10^{-19}C)^2}{\underset{J\cdot m\cdot kg^{-2}}{6.67\times10^{-11}} \times (9.1\times10^{-31}kg)^2} \simeq 4\times10^{42}$$

or directly use natrul unit ($\varepsilon_0 = \hbar = c = 1$)

$$F_{电} = \frac{e^2}{4\pi r^2} = \frac{\alpha \hbar c}{r^2} = \frac{\alpha}{r^2}$$

$$\Rightarrow \frac{F_{电}}{F_{引}} = \frac{\frac{1}{137}}{6.7\times10^{-39}GeV^{-2} \cdot (0.51\,MeV)^2} \simeq 4\times10^{42}$$

or

$$\frac{F_{电}}{F_{引}} = \frac{\alpha \hbar c}{G m_e^2} = \frac{\frac{1}{137}\times 1.05\times10^{-34}\,J\cdot sec \cdot 3\times10^8\,m\cdot sec^{-1}}{6.67\times10^{-11}\,J\cdot m\cdot kg^{-2}\times(9.1\times10^{-31}kg)^2}$$

$$= 9\times10^9\,J\cdot m\cdot C^{-2}\,(1.6\times10^{-19}C)^2$$

$$\simeq 4\times10^{42}$$

or

$$\frac{F_{电}}{F_{引}} = \frac{9\times10^9\cdot J\cdot m\,C^{-2}\times(1.6\times10^{-19}C)^2}{6.7\times10^{-39}GeV^{-2}\,(0.51\,MeV)^2}$$

use $1J = \frac{1}{1.6\times10^{-19}}\,eV$,

$1\,m = 5\times10^{12}\,MeV^{-1}$

$$\Rightarrow \frac{F_{电}}{F_{引}} = \frac{9\times10^9\,\frac{1}{1.6\times10^{-19}}\,eV\cdot 5\times10^{12}MeV^{-1}\cdot(1.6\times10^{-19})^2}{6.7\times10^{-39}\,GeV^{-2}\,(0.51\,MeV)^2}$$

$$= \frac{\frac{1}{137}}{6.7\times10^{-39}GeV^{-2}\,(0.51\,MeV)^2} \simeq 4\times10^{42}$$

氢原子 电离能为 13.6eV,

then calculate wave length:

$$\lambda = \frac{c}{\nu} = \frac{c}{E/h} = \frac{2\pi\hbar c}{E} = \frac{2\pi}{13.6\,eV}$$

$$= \frac{2\pi}{13.6\times10^{-6}\,MeV} = \frac{2\pi}{13.6\times10^{-6}}\,\frac{1}{MeV}$$

$$= \frac{2\pi}{13.6\times10^{-6}}\,\frac{1\,m}{5\times10^{12}}$$

$$\simeq 9\times10^{-8}\,m$$

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Construct Planck mass from $\hbar$, $c$, $G$

start by writing the dimension of $\hbar$, $c$ and $G$ in terms of m, kg, sec,

$$[\hbar] = kg\cdot m^2\cdot sec^{-1}$$

$$[c] = m\cdot sec^{-1}$$

$$[G] = m^3\cdot kg^{-1}\cdot sec^{-2}$$

then $$[\hbar]^{n_1}[c]^{n_2}[G]^{n_3} = kg^{n_1}\cdot m^{2n_1}\, sec^{-n_1}\, m^{n_2}\, sec^{-n_2}\cdot m^{3n_3}\, kg^{-n_3}\, sec^{-2n_3}$$

need

$$n_1 - n_3 = 1 \quad ①$$

$$2n_1 + n_2 + 3n_3 = 0 \quad ②$$

$$-n_1 - n_2 - 2n_3 = 0 \quad ③$$

$$\Rightarrow -n_2 - 3n_3 = 1 \quad (① + ③)$$

$$\Rightarrow 2(1+n_3) + (-1-3n_3) + 3n_3 = 0 \quad (③)$$

$$n_3 = -\frac{1}{2}$$

$$n_1 = \frac{1}{2}$$

$$n_2 = \frac{1}{2}$$

$$\Rightarrow m_{pl} = \left(\frac{\hbar c}{G}\right)^{\frac{1}{2}} \simeq 1.2\times10^{19}\,GeV$$

$$\simeq 2\times10^{-8}\,kg$$

Planck length:

$$n_1 - n_3 = 0 \quad ①$$
$$2n_1 + n_2 + 3n_3 = 1 \quad ②$$
$$-n_1 - n_2 - 2n_3 = 0 \quad ③$$

$$\Rightarrow \quad n_1 = n_3 \quad (①)$$
$$n_1 + n_3 = 1 \quad (② + ③)$$
$$\Rightarrow \quad n_1 = n_3 = \frac{1}{2}$$
$$n_2 = -\frac{3}{2}$$
$$\Rightarrow \quad l_{pl} = \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}} = 8 \times 10^{-20} \, GeV^{-1}$$
$$= 1.6 \times 10^{-35} \, m$$

Planck energy.

$$E_{pl} = m_{pl} c^2 = 1.2 \times 10^{18} \, GeV = 2 \times 10^9 \, J$$
$$= \left(\frac{\hbar c^5}{G}\right)^{\frac{1}{2}}$$

Planck time. ( or directly $t_{pl} = \frac{l_{pl}}{c} = \left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}}$

$$n_1 - n_3 = 0 \quad ①$$
$$2n_1 + n_2 + 3n_3 = 0 \quad ②$$
$$-n_1 - n_2 - 2n_3 = 1 \quad ③$$

$$\Rightarrow \quad -n_2 - 3n_3 = 1 \quad (① + ③)$$
$$\Rightarrow \quad 2n_3 + (-3n_3 - 1) + 3n_3 = 0$$
$$\Rightarrow \quad n_3 = \frac{1}{2}$$
$$n_1 = \frac{1}{2}$$
$$n_2 = -\frac{5}{2}$$
$$\Rightarrow \quad t_{pl} = \left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}} = 8 \times 10^{-20} \, GeV^{-1} = 5 \times 10^{-44} \, sec$$

Note that sometimes reduced planck mass is used

$$\sqrt{\frac{\hbar c}{8\pi G}} \simeq 2.4\times 10^{18}\,\text{GeV} = 4\times 10^{-9}\,\text{kg}$$

this $8\pi$ simplifies equations in general relativity.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

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Reconstruct the correct power of $c$, $\hbar$.

Bohr radius of Hydrogen.

$$a = \frac{1}{\alpha\mu} \qquad \text{where } \mu = \frac{m_e m_p}{m_e + m_p} \simeq m_e$$

$$= \frac{1}{\frac{1}{137}\,0.511\,\text{MeV}} = \frac{1}{\frac{1}{137}\times 0.511}\,\frac{1\,\text{m}}{5\times 10^{12}} = 0.53\times 10^{-10}\,\text{m}.$$

or start from

SI unit. $$a = \frac{\hbar^2 4\pi\varepsilon_0}{\mu e^2} = \frac{(1.05\times 10^{-34}\,\text{J}\cdot\text{sec})^2\,4\pi\,8.85\times 10^{-12}\,\text{C}^2\,\text{J}^{-1}\,\text{m}^{-1}}{9.1\times 10^{-31}\,\text{kg}\cdot(1.6\times 10^{-19}\,\text{C})^2}$$

$$= 0.53\times 10^{-10}\,\underbrace{\underbrace{\underbrace{\frac{\text{J}}{\text{kg}}\,\text{sec}^2\,\text{m}^{-1}}\;\frac{\text{kg}\,\text{m}^2\cdot\text{sec}^{-2}\cdot\text{sec}^2\cdot\text{m}^{-1}}{\text{kg}}}}_{\text{m}}$$

Question. if we are given $a = \frac{1}{\alpha\mu}$, how to go back $a = \frac{\hbar^2 4\pi\varepsilon_0}{\mu e^2}$ ?

From $a = \frac{\hbar^2 \cdot 4\pi\varepsilon_0}{\mu e^2}$ to $a = \frac{1}{\alpha\mu}$ is easy.

( Just use $\frac{e^2}{4\pi\varepsilon_0 \hbar c} = \alpha$, and then set $\hbar = c = 1$ )

we get $a = \frac{\hbar^2}{\mu\alpha\hbar c} = \frac{1}{\mu\alpha}$. )

From $a = \frac{1}{\alpha\mu}$, first put $\varepsilon_0$ in by using

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$$

So $a = \frac{4\pi\varepsilon_0 \hbar c}{e^2 \mu}$

then write $a = \frac{4\pi\varepsilon_0 \hbar c}{e^2\mu} \hbar^{n_1} c^{n_2}$

and let the dimension by $[m]$ to get $n_1$ & $n_2$

using $[\varepsilon_0] = [\text{Coulomb}]^2 [J]^{-1} [m]^{-1}$

$[e] = [\text{Coulomb}]$

$[\mu] = [kg]$

$[\hbar] = kg \cdot m^2\, sec^{-1}$

$[c] = m \cdot sec^{-1}$

$\Rightarrow$

$$[kg]^{n_1+1} [m]^{2(n_1+1)} [sec]^{-(n_1+1)} [m]^{n_2+1} [sec]^{-(n_2+1)} [J]^{-1} [m]^{-1} [kg]^{-1} \equiv [m]$$

and using $[J] = [kg]\,[m]^2 [sec]^{-2}$

$\Rightarrow$

$$n_1 + 1 - 1 - 1 = 0$$
$$2(n_1+1) + (n_2+1) - 1 - 2 = 1$$
$$-(n_1+1) - (n_2+1) + 2 = 0$$

$\Rightarrow$

$$n_1 = 1$$
$$n_2 = -1$$

$\Rightarrow$

$$a = \frac{4\pi\varepsilon_0 \hbar^2}{e^2\mu}$$

In $8\pi G = \hbar = c = k_B = 1$ unit, the black hole temperature is $T = \frac{1}{M}$. , get back the version in SI unit.

$$T = \frac{1}{M}(8\pi G)^{n_1}(\hbar)^{n_2}(c)^{n_3}(k_B)^{n_4}.$$

$$[G] = m^3 \cdot kg^{-1} \cdot sec^{-2}$$

$$[\hbar] = kg \cdot m^2 \cdot sec^{-1}$$

$$[c] = m \cdot sec^{-1}$$

$$[k_B] = J \cdot K^{-1} = kg \cdot m^2 \cdot sec^{-2} \cdot K^{-1}.$$

$$[T] = K.$$

$$[M] = kg$$

$$m^{3n_1} kg^{-n_1} sec^{-2n_1} \cdot kg^{n_2} m^{2n_2} sec^{-n_2} \cdot m^{n_3} sec^{-n_3} \cdot kg^{n_4} m^{2n_4} sec^{-2n_4} K^{-n_4} kg^{-1} = K$$

$\Rightarrow$

$$n_4 = -1$$

$$3n_1 + 2n_2 + n_3 + 2n_4 = 0$$

$$-n_1 + n_2 + n_4 - 1 = 0$$

$$-2n_1 - n_2 - n_3 - 2n_4 = 0$$

$\Rightarrow$

$$3n_1 + 2n_2 + n_3 = 2 \quad ①$$

$$-n_1 + n_2 = 2 \quad ②$$

$$-2n_1 - n_2 - n_3 = -2 \quad ③$$

$\Rightarrow$

$$5n_1 + n_3 = -2 \quad (① - 2②)$$

$$-3n_1 - n_3 = 0 \quad (② + ③)$$

$\Rightarrow$

$$5n_1 + (-3n_1) = -2 \Rightarrow n_1 = -1$$

$$n_3 = 3$$

$$n_2 = 1$$

$$\Rightarrow T = \frac{\hbar c^3}{M 8\pi G k_B}$$

WIMP miracle

$$\langle \sigma v \rangle = 3 \times 10^{-26}\ cm^3 \cdot sec^{-1}.$$

Now convert it to $GeV^{-2}$.

$$cm^3 = (10^{-2} m)^3 = (10^{-2} \times 5 \times 10^{12}\ MeV^{-1})^3.$$

$$sec^{-1} = (1.5 \times 10^{21}\ MeV^{-1})^{-1}$$

$$\Rightarrow \quad cm^3 \cdot sec^{-1} = (10^{-2} \times 5 \times 10^{12})^3 \times (1.5 \times 10^{21})^{-1} \cdot \underbrace{MeV^{-2}}_{\underbrace{(10^{-3} GeV)^{-2}}_{10^{6}\ GeV^{-2}}}$$

$$\Rightarrow \langle \sigma v \rangle$$

$$= 2.5 \times 10^{-9}\ GeV^{-2}$$

$$= \frac{1}{(100\ GeV)^2} \quad 100^2 \times 2.5 \times 10^{-9}$$

$$= \frac{2.5 \times 10^{-5}}{(100\ GeV)^2}$$

$$= \frac{\alpha^2 (2.5 \times 10^{-5}/\alpha^2)}{(100\ GeV)^2}$$

$$= 0.5 \frac{\alpha^2}{(100\ GeV)^2}.$$

# Estimate the number of CMB photons in $1cm^3$.

From blackbody radiation of CMB, we know

energy density $\rho \propto T^4$.

the average photon energy is $\sim T$

so the number density is $T^3$.

Since it is 2.73K temperature, we have

$$n \sim (2.73K)^3 = (2.73 \times 8.6 \times 10^{-11} MeV)^3$$

$$= (2.73 \times 8.6 \times 10^{-11})^3 \cdot (5 \times 10^{12})^3 \cdot m^{-3}$$

$$\frac{1}{1m^3} = \frac{1}{(10^2 cm)^3} = 10^{-6} \frac{1}{cm^3}$$

$$= (2.73 \times 8.6 \times 10^{-11} \times 5 \times 10^{12} \times 10^{-2})^3 \ cm^{-3}$$

$$= 1.6 \times 10^3 \ cm^{-3}.$$

The order one coefficient is $\frac{2 \times \zeta(3)}{\pi^2} = 0.25$

and the result is $n \sim 400 \ cm^{-3}$.

the SI unit formula is

$$n = \frac{2 \times \zeta(3)}{\pi^2} \frac{k^3}{\hbar^3 c^3} T^3$$

check unit:

$[k] = J \cdot K^{-1}$, $[\hbar] = J \cdot sec$ $[c] = m \cdot sec^{-1}$

$$\Rightarrow \left[\frac{k^3}{\hbar^3 c^3} T^3\right] = \left(\frac{J \cdot K^{-1} \cdot K}{J \cdot sec \cdot m \cdot sec^{-1}}\right)^3 = m^{-3} \ \checkmark$$

but the point is that in natrual unit, you don't need to worry about $\hbar$, $c$ & $k_B$.

In $\hbar = c = 1$ unit.

$v = 0.3$ means $v$ is 0.3 speed of light

In $\hbar = c = G = 1$ unit.

$[L]^3 [M]^{-1} [T]^{-2}$ is the SI unit of $G$. means

$[M] = [L]$ since $c = 1$ gives $[L] = [T]$

(However $\hbar = 1$ means $[E] = [T^{-1}]$, then together with
$[E] = [M][L]^2[T]^{-2} \Rightarrow [E] = [M]$)

or from $\hbar = 1.05 \times 10^{-34}$ J·sec $= 1.05 \times 10^{-34}$ kg·m²·sec⁻¹

$\Rightarrow$ $[M][L]^2[T]^{-1}$ is dimensionless when let $\hbar = 1$

$\Rightarrow$ $[M] = [L]^{-1}$

Therefore in $\hbar = c = G = 1$ unit,

$[M] = [L]$

$[M] = [L]^{-1}$

$\Rightarrow$ $[M]$ is dimensionless.

$[L]$ is dimensionless

$[T]$ is dimensionless

By solving
$$\begin{cases} \hbar = 1.05 \times 10^{-34}\ \mathrm{kg \cdot m^2 \cdot sec^{-1}} = 1 & ① \\ c = 3 \times 10^{8}\ \mathrm{m \cdot sec^{-1}} = 1 & ② \\ G = 6.67 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1}\, sec^{-2}} = 1 & ③ \end{cases}$$

$\Rightarrow$ $1\ \mathrm{sec} = 3 \times 10^{8}\ \mathrm{m}$ (from ②)

$$1.05 \times 10^{-34} \cdot \mathrm{kg} \cdot \mathrm{m^2} \frac{1}{3 \times 10^{8}\ \mathrm{m}} = 1 \quad (\text{from ①})$$

$$\Rightarrow 1\ \mathrm{kg} = \frac{3 \times 10^{8}}{1.05 \times 10^{-34}}\ \mathrm{m^{-1}}$$

$$\Rightarrow 6.67 \times 10^{-11}\ \mathrm{m^3} \cdot \left(\frac{1.05 \times 10^{-34}}{3 \times 10^{8}}\ \mathrm{m}\right)(3 \times 10^{8}\ \mathrm{m})^{-2} = 1 \quad (\text{from ③})$$

$$\Rightarrow 1\ \mathrm{m} = 6.2 \times 10^{34}$$

Recall planck length $l_{pl} = \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}} = 1.6 \times 10^{-35}\ \mathrm{m}$ ($= 1$ in $c = \hbar = G$ unit)

So $1\ \mathrm{m} = 6.2 \times 10^{34}$ means $1\ \mathrm{m} = 6.2 \times 10^{34}\ l_{pl}$

$c = 1 \Rightarrow [L] = [T]$

$\hbar = 1 \Rightarrow [E] = [T]^{-1}$

Since $[E] = [M]\cdot[L]^2[T]^{-2} \Rightarrow [E] = [M]$

That is $\Rightarrow [L] = [E]^{-1}$, $[T] = [E]^{-1}$, $[M] = [E]^{-1}$.

so all length, time and energy can be given unit of $GeV^{-1}$ and GeV.

$C = 1 \Rightarrow 1\,sec = 3\times10^8\,m$

$\hbar = 1 \Rightarrow 1.05\times10^{-34}\,kg\cdot m^2\cdot sec^{-1} = 1 \Rightarrow 1.05\times10^{-34}\,kg\cdot m\cdot\frac{1}{3\times10^8} = 1$

$\Rightarrow 1\,kg = 3\times10^8\cdot\frac{1}{1.05\times10^{-34}}\,m^{-1}$

$8\pi G = 1 \Rightarrow 6.67\times10^{-11}\,kg^{-1}\cdot m^3\cdot sec^{-2} = \frac{1}{8\pi}$

$\Rightarrow 6.67\times10^{-11}\,\frac{1.05\times10^{-34}}{3\times10^8}\,m\cdot m^3\cdot\left(\frac{1}{3\times10^8\,m}\right)^2 = \frac{1}{8\pi}$

$\Rightarrow$

$1\,m = 1.2\times10^{34}$

( Recall $l_{pl} = 1.6\times10^{-35}\,m = \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}}$

If define reduced planck length as $\left(\frac{\hbar\cdot 8\pi G}{c^3}\right)^{\frac{1}{2}}$

then $l_{pl,reduced} = 8.0\times10^{-35}\,m$ )

Therefore, $1\,m = 1.2\times10^{34}$ means $1\,m = 1.2\times10^{34}\,l_{pl,reduced}$.

Similarly, $c = \hbar = 8\pi G = 1$

$\Rightarrow 1\,kg = 3\times10^8\,\frac{1}{1.05\times10^{-34}}\,\frac{1}{1.2\times10^{34}} = 2.4\times10^8$

( Recall the reduced planck mass is defined as $\left(\frac{\hbar c}{8\pi G}\right)^{\frac{1}{2}} = 4\times10^{-9}\,kg$ )

Therefore, $1\,kg = 2.4\times10^8$ means $1\,kg = 2.4\times10^8\,m_{pl,reduced}$.

$1\,sec = 3\times10^8\,m = 3\times10^8\times1.2\times10^{34} = 3.6\times10^{42}$

( recall that the Planck time is $\left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}} = 5\times10^{-44}\,sec$, then define reduced Planck time as $\left(\frac{\hbar\cdot 8\pi G}{c^5}\right)^{\frac{1}{2}} = 2.5\times10^{-43}\,sec$ )

Therefore $1\,sec = 3.6\times10^{42}$ means $1\,sec = 3.6\times10^{42}\,t_{pl,reduced}$

Similarly

$1\ \text{sec} = 3\times10^{8}\ \text{m} = 3\times10^{8}\times 6.2\times10^{34} = 2\times10^{43}$

recall $t_{pl} = \left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}} = 5\times10^{-44}\ \text{sec}$ ($=1$ in $c=\hbar=G=1$ unit.)

So $1\ \text{sec} = 2\times10^{43}$ means $1$ sec is $2\times10^{43}\ t_{pl}$.

$1\ \text{kg} = \frac{3\times10^{8}}{1.05\times10^{-34}}\ \frac{1}{6.2\times10^{34}} = 5\times10^{7}$

recall $m_{pl} = \left(\frac{\hbar c}{G}\right)^{\frac{1}{2}} = 2\times10^{-8}$

so $1\ \text{kg} = 5\times10^{7}\ m_{pl}$

Therefore, in $G=\hbar=c=1$ unit, the length is measured in terms of Planck length, mass is measured in terms of Planck mass, and time is measured in terms of Planck time.

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However, choose natural unit need to avoid inconsistency.

For example, we can not simultaneously choose both the electron and proton mass to be 1, because the proton/electron mass ratio is a fixed dimensionless number which should not depend on the choice of unit. (i.e., $\frac{m_p}{m_e} = 1836$, $\frac{m_p}{m_e}$ never equals 1, so cannot choose $m_p = 1$ & $m_e = 1$ simultaneously)

# Atomic units

$$\boxed{\begin{aligned} m_e &\equiv 1 \\ e &\equiv 1 \\ \hbar &\equiv 1 \\ k_e = \frac{1}{4\pi\varepsilon_0} &\equiv 1 \end{aligned}}$$

Therefore, since $\alpha = \dfrac{e^2}{4\pi\varepsilon_0 \hbar c} \simeq \dfrac{1}{137}$, then $c \simeq 137$

NOTE that dimensionless physical constants retain their values in any system of units.

Since in SI unit,
$$k_e = 9\times10^{9}\ \text{Coulomb}^{-2}\cdot kg\cdot m^3\cdot sec^{-2}$$
$$\hbar = 1.05\times10^{-34}\ kg\cdot m^2\cdot sec^{-1}$$
$$e = 1.6\times10^{-19}\ \text{Coulomb}$$
$$m_e = 9.1\times10^{-31}\ kg$$

$\Rightarrow$ In Atomic Units,

$$\boxed{\begin{aligned} 1\,kg &= \frac{1}{9.1\times10^{-31}} = 1.1\times10^{30} \\ 1\,\text{Coulomb} &= \frac{1}{1.6\times10^{-19}} = 6.3\times10^{18} \end{aligned}}$$

$$\hbar = 1.05\times10^{-34}\cdot\frac{1}{9.1\times10^{-31}}\ m^2\cdot sec^{-1} = 1 \quad ①$$

$$k_e = 9\times10^{9}\ \frac{(1.6\times10^{-19})^2}{9.1\times10^{-31}}\ m^3\cdot sec^{-2} \equiv 1 \quad ②$$

$$\Rightarrow \boxed{\begin{aligned} 1\,m &= 1.9\times10^{10} \\ 1\,sec &= 4.2\times10^{16} \end{aligned}} \qquad \begin{aligned} &(①^2/②) \\ &(①^3/②^2) \end{aligned}$$

$\Rightarrow$ check: $3\times10^{8}\ m\cdot sec^{-1} = 3\times10^{8}\cdot\dfrac{1.9\times10^{10}}{4.2\times10^{16}} \simeq 137$ ✓

Note that Bohr radius is $a_0 \equiv \dfrac{4\pi\varepsilon_0\hbar^2}{m_e e^2} \simeq 5.3\times10^{-11}\ m$ in SI unit

$\qquad = 1$ in Atomic unit

So $1\,m = 1.9\times10^{10}\ a_0$.

Check dimension:

$$\left[\frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2}\right] = \frac{\text{Coulomb}^2\ \text{kg}^{-1}\ \text{m}^{-3}\ \text{sec}^2 \cdot \text{kg}^2\ \text{m}^4\ \text{sec}^{-2}}{\text{kg}\cdot\text{Coulomb}^2} = \text{m} \checkmark$$

Now try to find a combination of $1/(4\pi\varepsilon_0)$, $\hbar$, $m_e$ and $e$ with dimension second, so that we can understand what is 1sec in terms of.

$$\left[\frac{1}{4\pi\varepsilon_0}\right]^{n_1} [\hbar]^{n_2} [m_e]^{n_3} [e]^{n_4} \equiv \text{sec}$$

$$\text{Coulomb}^{-2n_1} \cdot \text{kg}^{+n_1}\ \text{m}^{+3n_1}\ \text{sec}^{-2n_1} \cdot \text{kg}^{n_2}\ \text{m}^{2n_2}\ \text{sec}^{-n_2} \cdot \text{kg}^{n_3}\ \text{Coulomb}^{n_4} \equiv \text{sec}$$

$\Rightarrow$

$$-2n_1 + n_4 = 0 \quad ①$$
$$+n_1 + n_2 + n_3 = 0 \quad ②$$
$$+3n_1 + 2n_2 = 0 \quad ③$$
$$-2n_1 - n_2 = 1 \quad ④$$

$\Rightarrow$ put ④ in ③ $\Rightarrow$ $+3n_1 + 2(-2n_1 - 1) = 0 \Rightarrow n_1 = -2$

$\Rightarrow$ $n_2 = 3$, $n_3 = -1$, $n_4 = -4$

$\Rightarrow$ 1 unit of time is $\left(\frac{1}{4\pi\varepsilon_0}\right)^{-2} \hbar^3 m_e^{-1} e^{-4}$

$$= \frac{(4\pi\varepsilon_0)^2 \cdot \hbar^2 \cdot \hbar}{e^4 \cdot m_e}$$

what is this?

In Bohr model, the potential energy of the electron in Bohr radius is

$$\frac{e^2}{4\pi\varepsilon_0 \cdot a_0} = \frac{e^2\ m_e e^2}{4\pi\varepsilon_0\ 4\pi\varepsilon_0 \hbar^2} \equiv E_h$$

(called Hartree energy $= 2 \times 13.6$ eV)

So 1 unit of time is $\frac{\hbar}{E_h}$

What about the meaning of $c \simeq 137$?

It is in unit of electron speed in Bohr orbit.

check: $\frac{e^2}{4\pi\varepsilon_0 a_0^2} = \frac{m_e v^2}{a_0} \Rightarrow v = \left(\frac{e^2}{4\pi\varepsilon_0 a_0 m_e}\right)^{\frac{1}{2}} = 2.2 \times 10^6$ m/s in SI unit.

$\parallel$ 1 in Atomic unit