Homework 5
Due date: 2018.12.12

Problem 1. Consider a complex scalar field, described by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* \, .$$

Do the following calculations starting from the decompositions of the field operators,

$$\begin{aligned} \phi(\vec{x}, t) &= \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left(a(\vec{p}) e^{-ip\cdot x} + b^\dagger(\vec{p}) e^{ip\cdot x} \right) , \\ \phi^\dagger(\vec{x}, t) &= \int_{-\infty}^{+\infty} d^3\vec{p}\, C(E_{\vec{p}}) \left(a^\dagger(\vec{p}) e^{ip\cdot x} + b(\vec{p}) e^{-ip\cdot x} \right) , \end{aligned}$$

where $E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$, and $C(E_{\vec{p}})$ is a real function of $E_{\vec{p}}$.
1) Find the decompositions of the corresponding canonical conjugate momentum operators, $\pi(\vec{x}, t)$ and $\pi^\dagger(\vec{x}, t)$. [1 point]
2) Find $a(\vec{p})$, $a^\dagger(\vec{p})$, $b(\vec{p})$ and $b^\dagger(\vec{p})$ in terms of ϕ, π, $\phi^\dagger$ and $\pi^\dagger$. [1 point]
3) From the equal-time commutation relations (ETCR)

$$\begin{gathered} [\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^3(\vec{x} - \vec{x}') \, , \; \left[\phi^\dagger(\vec{x}, t), \pi^\dagger(\vec{x}', t)\right] = i\delta^3(\vec{x} - \vec{x}') \, , \\ [\phi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \pi(\vec{x}', t)] = \left[\phi^\dagger(\vec{x}, t), \phi^\dagger(\vec{x}', t)\right] = \left[\pi^\dagger(\vec{x}, t), \pi^\dagger(\vec{x}', t)\right] = 0 \, , \\ \left[\phi(\vec{x}, t), \pi^\dagger(\vec{x}', t)\right] = \left[\pi(\vec{x}, t), \phi^\dagger(\vec{x}', t)\right] = 0 \, , \\ \left[\phi(\vec{x}, t), \phi^\dagger(\vec{x}', t)\right] = \left[\pi(\vec{x}, t), \pi^\dagger(\vec{x}', t)\right] = 0 \, , \end{gathered}$$

show that [2 points]

$$\begin{gathered} [a(\vec{p}), a^\dagger(\vec{p}')] = \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})} \right)^2 \delta^3(\vec{p} - \vec{p}') \, , \\ [b(\vec{p}), b^\dagger(\vec{p}')] = \frac{1}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\frac{1}{C(E_{\vec{p}})} \right)^2 \delta^3(\vec{p} - \vec{p}') \, , \\ [a(\vec{p}), a(\vec{p}')] = [b(\vec{p}), b(\vec{p}')] = [a(\vec{p}), b^\dagger(\vec{p}')] = [b(\vec{p}), a^\dagger(\vec{p}')] = 0 \, , \\ [a^\dagger(\vec{p}), a^\dagger(\vec{p}')] = [b^\dagger(\vec{p}), b^\dagger(\vec{p}')] = [a^\dagger(\vec{p}), b^\dagger(\vec{p}')] = [a(\vec{p}), b(\vec{p}')] = 0 \, . \end{gathered}$$

4) Show that the decomposition of the momentum operator, $\hat{\vec{P}} = -\int_{-\infty}^{+\infty} d^3\vec{x}\, (\pi \vec{\nabla} \phi + \pi^\dagger \vec{\nabla} \phi^\dagger)$, is [3 points]

$$\hat{\vec{P}} = \int_{-\infty}^{+\infty} d^3\vec{p} \left[(2\pi)^3 2E_{\vec{p}} (C(E_{\vec{p}}))^2 \right] \left[a^\dagger(\vec{p}) a(\vec{p}) + b^\dagger(\vec{p}) b(\vec{p}) \right] \vec{p} \, .$$

5) Show that $\int_{-\infty}^{+\infty} d^3\vec{x}\, \pi \vec{\nabla} \phi = \int_{-\infty}^{+\infty} d^3\vec{x}\, (\vec{\nabla} \phi) \pi$. [3 points]