Evidentialist Logic

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1 Philosophy

1.1 Forward

The idea of applying modal logic to the study of knowledge more or less began with Hintikka (1969). In this it is suggested that one can use the possible world framework of modal logic to model ideal logical agents, and reason about concepts like knowledge and belief as modal $\Box$s. In Hintikka's original text, some philosophical emphasis is put on the ideas of *introspection*, which have two formulations:

- Positive: $\Box\phi \rightarrow \Box\,\Box\,\phi$ - "If the agent knows a fact, then she knows that she knows this."
- Negative: $\Diamond\phi \rightarrow \Box\Diamond\phi$ - "If the agent does not know a fact, she knows that she does not know this."

Intuitively, the second idea seems like something one ought to reject outright. Many will recall the somewhat famous piece of sophistry put forward by former US secretary of defense Donald Rumsfeld (Rumsfeld, 2002):

> Reports that say that something hasn't happened are always interesting to me, because as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns – the ones we don't know we don't know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones.

This quote was ultimately part of a larger, more diabolical justification for criminal military action. Still, it is undeniably at variance with negative introspection; and despite its malicious intent behind it, it is compelling. Furthermore, Hintikka also rejects negative introspection; and while it would seem from the above quote that Rumsfeld does not reject positive introspection, Hintikka does so explicitly (Hintikka, 1969).

Despite philosophical objections, the received view in modern epistemic logic embraces both negative and positive introspection. In addition, the following axiom is also considered:

- Reflection: $\Box\phi \rightarrow \phi$ - "If the agent knows a some statement, that statement is true"

These three axioms together, along with the axioms and rules of elementary modal logic, form C.I. Lewis' system $S5$ (Lewis and Langford, 1951). Under correspondence theory, these axioms express that the underlying accessibility modal relation is an equivalence relation. That is, they express that the ideal agent under investigation has partitioned their state space into *information states*. It is well known that game theory shares an equivalent notion of information states (see, for instance, Halpern (1999) and Rubinstein (1998, chapter 3)).

And while this view of knowledge finds industrial application, such as in Agray et al. (2002) and Hommersom et al. (2005, 2004), it presents an perspective on knowledge which is not very human.[1]

[1] Admittedly, most of the criticisms I shall levy against mainstream epistemic logic in the subsequent discussion are known to the epistemic logic community. I understand it is preferred that they be swept under the carpet and never

1.2 Thermometers

Imagine a 1 m^3 box with a thermometer sealed hermetically inside, as in Fig. 1. Further, pretend that the thermometer reads 290 Kelvin. How many moles of gas are in the chamber?

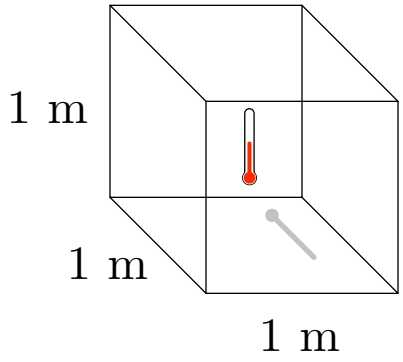


Figure 1: A thermometer in a box

The answer is indeterminate. Recall that the the *ideal gas law* was originally discovered by Émile Clapeyon (Clapeyron, 1834); in modern parlance it reads:

$$PV = nRT$$

Where:

- P is the pressure in pascals
- V is the volume in cubic meters
- n is the number of moles of gas
- T is the temperature in Kelvins
- R is the *ideal gas constant*, ≈ 8.3 $\mathrm{J{\cdot}K^{-1}{\cdot}mol^{-1}}$

With the ideal gas law, we can see that the thermometer is effectively in an epistemic space. To be explicit, consider the basic modal language with the following grammar:

$$\phi ::= x \text{ pascals} \mid y \text{ moles} \mid z \text{ Kelvin} \mid \phi \to \psi \mid \bot \mid \Box\phi$$

This can be seen to be an ordinary modal language with three kinds of letters. Under this language, we can naturally understand that the thermometer is an epistemic agent in an $S5$ model for this language. Explicitly, the model is the triple $\langle W, V, \sim \rangle$ where:

mentioned, as they are fairly painful to remember for epistemic logic enthusiasts. I am not particularly apologetic in recounting the basic failures of epistemic logic as an intellectual exercise. The discussion I present in this section is intended for people new to epistemic logic, who may be unaware of its basic inadequacies and irrelevance.

- W is pairs (P, n) where P is some positive pressure in pascals and n is some positive number of moles.
- V is defined as follows:
  - $(P, n) \in V(x \text{ pascals})$ if $P = x$
  - $(P, n) \in V(y \text{ moles})$ if $n = y$
  - $(P, n) \in V(z \text{ Kelvin})$ if $z = \frac{P}{n \cdot R}$
- Finally, $(P, n) \sim (P', n')$ holds if and only if $P \cdot n' = P' \cdot n$

We can also visualize the information states in Fig. 2; they form rays emanating from the origin.

Now, I give this example because it is representative of the perspective provided by epistemic logic - agents are essentially elaborate *sensor networks* in the received view. Imagine we were to go up to an agent and ask her why she believes some proposition ϕ; what could she possibly say? She'd say she feels ϕ with every fiber of her being, that it's true in every conceivable world she can think of. The reason that ϕ occurs to the agent is because it's what her sensory instruments tell her. Methodologically, this is exactly the same as the way the thermometer was modeled in the previous thought experiment. To this end I shall abbreviate the received view on epistemic logic as the *thermometer theory of knowledge.*

This isn't knowledge. The thermometer theory is inadequate: if I were to ask a person why she believes a proposition ϕ, I probably wouldn't accept appeals that she cannot conceive of the contrary as possible. I'd want some kind of explanation, especially if ϕ were a piece of mathematics, to give principle example. It would certainly make the enterprise of mathematics far simpler if proving theorems amounted to exhibiting that their negation is not imaginable. This gives rise to the following philosophical observation:

> **Thermometer Principle**: *Traditional epistemic agents, like thermometers, don't really have knowledge, since one must have reasons for the things they believe.*

As I will elaborate, the above assertion follows from more general concerns I have with epistemic logic.

1.3 Explicit Justification

I hold that the *Thermometer Principle* is an expression of the following more fundamental idea:

> **Justification Principle**: *In order to know something, one must have a some kind of sufficient justification.*

I contend that derivation in a logical calculus is suitable justification for the above principle. In light of this, all of my future developments shall center around the idea that in order to have knowledge, one must have a special derivation. Since knowledge entails belief (Gettier, 1963, under

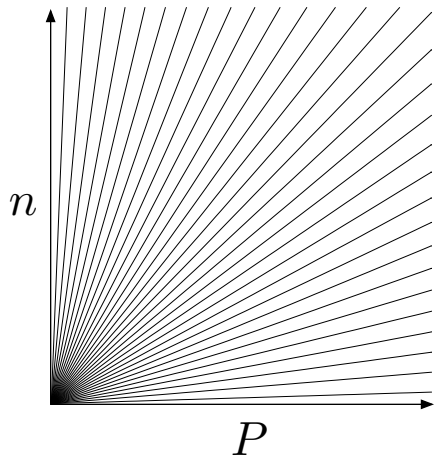


Figure 2: Thermometer information states

the traditional view in), I have decided to simply equate belief with having a derivation. Hence knowledge can then be thought of a species of this kind of derivational belief. And while it is perhaps as bad an oversimplification as the thermometer view I have previously criticized, my modeling technique assumes *doxastic omniscience*, which is to say that I assume an agent's beliefs are closed under logical consequence[2]. This will be illustrated shortly.

The *Justification Principle* is similar to demanding *explicit justification* for beliefs in epistemic logic. I am not the first person to suggest this. The modern hunt for logics of explicit justification appears to have been initiated in van Benthem (1991)[3]. One framework which has been proposed to achieve this is *Justification Logic* (Artemov and Nogina, 2005, Artemov, 2007, Fitting, 2004, 2005). Alternative frameworks for reasoning about implicit/explicit information have also been proposed in van Benthem and Velzquez-Quesada (2009) and Velzquez-Quesada (2009).

1.4 Sketch

These analyses mentioned in the previous section are all quite sophisticated; I have in mind something comparatively naïve. I shall present a sketch in this section – that being said I intend to be extremely informal. A formal development of the ideas shall be given in §2.1.1. With this proviso, consider the basic modal language $\mathcal{L}(\Phi)$:

$$\phi ::= p \in \Phi \mid \phi \rightarrow \psi \mid \bot \mid \Box \phi$$

Further, let $\mathfrak{M} \subseteq \wp\Phi \times \wp\mathcal{L}(\Phi)$, that is, let $\mathfrak{M}$ be pairs of sets of letters and formulae. Define the following truth predicate $\Vdash$ recursively as follows:

[2]Hintikka (1969) assumes a similar perspective; for modeling purposes, the agent is assumed to have adequate time to realize the consequences of their beliefs. Similarly, we might interpret belief as what is refered to in Levesque (1984) as "implicit belief."

[3]While this paper is considered seminal, it should be remarked that research into this subject began prior to it. Specifically, the *phrase* "explicit belief" appears to have its origins in (Levesque, 1984).

$\mathfrak{M}, (a, A) \Vdash p \iff p \in a$

$\mathfrak{M}, (a, A) \Vdash \phi \to \psi \iff \mathfrak{M}, (a, A) \Vdash \phi$ implies $\mathfrak{M}, (a, A) \Vdash \psi$

$\mathfrak{M}, (a, A) \Vdash \bot \iff False$

$\mathfrak{M}, (a, A) \Vdash \Box\phi \iff$ for all $(b, B) \in \mathfrak{M}$, $\mathfrak{M}, (b, B) \Vdash A$ implies $\mathfrak{M}, (b, B) \Vdash \phi$ for all $(b, B) \in \mathfrak{M}$

Since the semantics like the above shall be the principle objects of study, I will give how I read them philosophically. In these semantics, instead of thinking of every world individually, I think of every world as containing facts and a part of the agent's mind. This part of the agent's mind is represented by what I shall refer to as propositions which she ascents to. I shall refer to these interchangeably as *premises*, *assumptions*, *basic beliefs*, *experiences* or *evidence*. On the basis of her evidence, I imagine the agent composes arguments using the logic of semantics suggested and all of the worlds that are possible. In this way, I consider this approach to be roughly in line with the *evidentialist* view on epistemology, which Conee and Feldman (2004) describe as follows:

> [E]videntialism is a supervenience thesis according to which facts about whether or not a person is justified in believing a proposition supervene on facts describing the evidence that the person has.

...however, while my sympathies are with this perspective on epistemology, they differ foundationally - while evidentialism develops intuitions using analytical philosophy, our approach shall be founded in formal semantics like the one above.

That is, if a proper foundation can be provided at all. Admittedly, the above formulation of truth immediately runs into a *paradox* - for instance, if

- $a := \varnothing$
- $A := \{\Box\bot\}$
- and $\mathfrak{M} := \{(a, A)\}$

...then $\mathfrak{M}, (a, A) \Vdash \Box\bot$ has an indeterminate truth value. So let $\mathcal{L}_0(\Phi)$ be the propositional fragment of $\mathcal{L}$; we shall restrict the truth value to $\mathfrak{M} \subseteq \Phi \times \wp\mathcal{L}_0(\Phi)$. This suffices to make every truth value of this logic determinate.

We may observe that the logic of these semantics is familiar:

Proposition 1.4.1. *Assuming that the set of proposition letters Φ is infinite*

$$\vdash_K \phi \text{ if and only if } \mathfrak{M}, (a, A) \Vdash \phi \text{ for all finite } \mathfrak{M} \text{ for all } (a, A) \in \mathfrak{M}$$

...where K is basic modal logic.

Proof. Left to right is trivial, so we shall focus on right to left. Assume that $\nvdash_K \phi$, then we know from completeness and the finite model property that there's some finite model $\mathbb{M} = \langle W^{\mathbb{M}}, V^{\mathbb{M}}, R^{\mathbb{M}} \rangle$

and world $w \in W^{\mathbb{M}}$ such that $\mathbb{M}, w \nVdash \phi$ (see Blackburn et al. (2001, chapters 2 & 4) for details of these facts).

Now let Λ_ϕ be the proposition letters that occur as subformulae of ϕ, and let $\rho_\phi : W^{\mathbb{M}} \hookrightarrow \Phi \backslash \Lambda_\phi$ be an injection. Define $\theta_\phi : W^{\mathbb{M}} \to \wp\Phi \times \wp\mathcal{L}_0(\Phi)$ as follows[4]

$$\begin{aligned}\theta_\phi(w) := (\ & \{p \in \Phi \mid \mathbb{M}, w \Vdash p\} \cup \{\rho_\phi(w)\}, \\ & \{\bigvee\{\rho_\phi(v) \mid v \in W^{\mathbb{M}} \wedge wR^{\mathbb{M}}v\}\})\end{aligned}$$

Now let $\Theta := \theta[W^{\mathbb{M}}]$. An induction on the complexity of subformulae ψ of ϕ shows that $\mathbb{M}, w \Vdash \psi \iff \Theta, \theta(w) \Vdash \psi$ for all $w \in W^{\mathbb{M}}$. Since $\mathbb{M}, w \nVdash \phi$ then we know that $\Theta, \theta(w) \nVdash \phi$, which completes the proof. QED

Armed with this, we can see that these semantics are adequate for modeling agents according to my declared intentions, since we have the following:

Proposition 1.4.2. *Let A be finite and define*

$$Th(\mathfrak{M}) := \{\phi \in \mathcal{L}(\Phi) \mid \mathfrak{M}, (a, A) \Vdash \phi \text{ for all } (a, A) \in \mathfrak{M}\}$$

... then $\mathfrak{M}, (a, A) \Vdash \Box\phi$ if and only if $Th(\mathfrak{M}) \cup A \vdash_K \phi$.

Proof. To see left to right, just note that since A is finite then if $\mathfrak{M}, (a, A) \Vdash \Box\phi$ evidently for all $(b, B) \in \mathfrak{M}$ we have $\mathfrak{M}, (b, B) \Vdash \bigwedge A \to \phi$, which just means that $Th(\mathfrak{M}) \cup A \vdash_K \phi$ by the deduction rule.

On the other hand if $Th(\mathfrak{M}) \cup A \vdash_K \phi$, then we know that $Th(\mathfrak{M}) \vdash_K \bigwedge A \to \phi$, and since $Th(\mathfrak{M})$ is sound for $\mathfrak{M}$ we then have that for all $(b, B) \in \mathfrak{M}$ that $\mathfrak{M}, (b, B) \Vdash \bigwedge A \to \phi$. Since A is finite we know that $\mathfrak{M}, (b, B) \Vdash A$ if and only if $\mathfrak{M}, (b, B) \Vdash \bigwedge A$, and thus we can deduce that $\mathfrak{M}, (a, A) \Vdash \Box\phi$ by definition. QED

A natural way to read $Th(\mathfrak{M})$ is the background knowledge the agent has about the universe she lives in. This approach presents an analysis of modal logic whereby an idealized agent is modeled as closed under deduction; this is the *doxastic omniscience* I have mentioned earlier. Under this view, evidently the agent's beliefs correspond to those things for which she has proofs. This shall be the basis of my future investigations.

1.5 The Human Condition

To supplement to this basic framework, I shall try to illustrate how further inspiration and desiderata can be drawn from the philosophical literature. It should be remarked that I do this in stark contrast to the received view in epistemic logic (Lenzen, 1978, pg. 34):

[4] I am indebted to Johan van Benthem for the invention of this particular function.

> The search for the correct analysis of knowledge, while certainly of extreme importance and interest to epistemology, seems not significantly to affect the object of epistemic logic, the question of the validity of certain epistemic-logical principles.

. . . quite to the contrary, I feel epistemic logic should not turn it's back on philosophy. Philosophy critically provides guidance for the intuitions behind how knowledge should be correctly modeled. It also provides a solid grounding in a proper treatment of knowledge. However, engaging with philosophy is evidently not the thrust of mainstream epistemic logic.

Most mainstream epistemic logic, the object of study is really the nature of information, not human knowledge. It applies equally well to robots, *homo economicus*, or thermometers as suggested in 1.2. It's inspiration is not really in what it's like to be a living person; it's more naturally based in artificial intelligence, automata theory, algebra, topology, and other abstract disciplines.

In contrast, I propose the following principle:

> **The Human Condition**: *The analysis of knowledge should strive for a basis in human experience*

. . . this principle indeed underpins the Justification Principle provided in §1.3. This is because I feel that the belief in a proposition can be thought of human only if the agent has a reason associated with it. Otherwise, it seems that in the absence of reason, no account can be given for how the belief came about other than through instrumentation, which is the thermometer view.

Embracing this principle, I shall turn to the development of my thoughts from their philosophical origins.

1.6 Soundness

So to give a shallow example of a basic application of a philosophical idea, it is natural to insist that if knowledge is based on beliefs generated via deduction from some set of premises, then those premises have to be *sound*. I suggest this can be done by introducing a new operator $\circlearrowleft$ with the following semantics:

$$\mathfrak{M}, (a, A) \Vdash\!\circlearrowleft \iff \mathfrak{M}, (a, A) \Vdash A$$

Armed with these semantics, a first guess at what constitutes knowledge suggests it might be nothing more than possession of a belief based on a sound set of premises. So a first approximation of knowledge might be equated with the formula:

$$\circlearrowleft \wedge \Box\, \phi.$$

But is this anything like an adequate analysis of knowledge?

No. To illustrate why I shall resort to a thought experiment to motivate why I think to the contrary. Imagine that Charlotte suspects, correctly, that if John has tried to murder on Alex, then Alex has survived. She further learns, correctly, that John has indeed tried to murder Alex. But later, she "learns" some erroneous information asserting Vietnam is south of Malaysia. If we

codify all of this as a set C, and let the real world be denoted c and the universe $\mathfrak{M}$, evidently we have $\mathfrak{M}, (c, C) \nVdash \circlearrowleft$, so this previous definition of knowledge fails. But should it? I don't think so; Charlotte's knowledge about John's unspeakable betrayal of Alex is correct, as well as her inference that Alex is tough as nails. Just because she has been deluded regarding irrelevant facts about geography shouldn't have any bearing on her knowledge about Alex.

1.7 Descartes

In reflection on the previous section, it should be remarked that philosophers have historically been concerned with defeasible experiential data, going back at least as early as Plato's *The Republic VII* (Plato, 1998). In answer to the problem faced by the above analysis of knowledge, I think guidance can be found in Descartes' *Meditations* (Vietch et al., 2005). In *Meditations I*, Descartes suggests that he might be in an enlightenment era version of *The Matrix* created by an all powerful demon. In *Meditations II*, he famously suggests how one might escape this trap:

> The Meditation of yesterday has filled my mind with so many doubts, that it is no longer in my power to forget them. Nor do I see, meanwhile, any principle on which they can be resolved; and, just as if I had fallen all of a sudden into very deep water, I am so greatly disconcerted as to be unable either to plant my feet firmly on the bottom or sustain myself by swimming on the surface. I will, nevertheless, make an effort, and try anew the same path on which I had entered yesterday, that is, proceed by casting aside all that admits of the slightest doubt, not less than if I had discovered it to be absolutely false; and I will continue always in this track until I shall find something that is certain, or at least, if I can do nothing more, until I shall know with certainty that there is nothing certain.

This tactic proposes a natural solution to the problem the previous thought experiment: *Charlotte can know that Alex survives if she argues **only** from her experience involving Alex and John.* If like Descartes she can forget some of what she has come to believe that's a little suspicious, she might be able to compose an argument with a sound basis that Alex is alive. Taking Descartes as inspiration, I would suggest a new semantic operation:

$$\mathfrak{M}, (a, A) \Vdash \boxminus\phi \iff \text{for all } (b, B) \in \mathfrak{M} \text{ such that } a = b \text{ and } B \subseteq A \text{ then } \mathfrak{M}, (b, B) \Vdash \phi$$

This mechanism lets Charlotte access subsets of her beliefs, which would then form the basis for various arguments she might compose. Provided that $(c, C') \in \mathfrak{M}$, where C' is the same as C but doesn't mention erroneous beliefs about geographical data, it might serve as a basis for Charlotte's knowledge that Alex survives. This suggests that the following equation might reasonably express a more adequate notion of knowledge:

$$\diamondminus(\circlearrowleft \wedge \boxminus\, \phi)$$

1.8 Contradictions

There's hidden virtue in the previous analysis. To see what it is, I am inspired by the 19th century philosopher Ralph Waldo Emerson, who writes in his essay *Self-Reliance* (Emerson, 2008):

> Why drag about this corpse of your memory, lest you contradict somewhat you have stated in this or that public place? Suppose you should contradict yourself; what then? It seems to be a rule of wisdom never to rely on your memory alone, scarcely even in acts of pure memory, but to bring the past for judgment into the thousand-eyed present, and live ever in a new day. ...
>
> A foolish consistency is the hobgoblin of little minds, adored by little statesmen and philosophers and divines. With consistency a great soul has simply nothing to do. He may as well concern himself with his shadow on the wall. Speak what you think now in hard words and to-morrow speak what to-morrow thinks in hard words again, though it contradict every thing you said to-day. – 'Ah, so you shall be sure to be misunderstood.' – Is it so bad then to be misunderstood?

A healthy lack of consistency is just part of what makes up the day to day life of any living, sane person. Isn't error-prone reasoning a hallmark of human thought? And if a love sick epistemic agent $\exists$ is getting mixed signals from another epistemic agent $\forall$, why can't she draw inconsistent conclusions about $\forall$'s feelings on the one hand, but still have basic knowledge that $734 \times 12 = 8808$ and other such irrelevant facts? I don't see why not. Under these considerations, I'd embrace the following:

> **Emerson's Principle**: *One can be inconsistent and still have knowledge*

Permit me illustrate how the framework I have given accommodates this. My treatment is further inspired by a friend and contemporary of Emerson's, the poet *Walt Whitman*. In *Leaves of Grass* (Whitman, 2008), he writes:

> Do I contradict myself?
> Very well then I contradict myself,
> (I am large, I contain multitudes.)

So consider the model $\mathfrak{M}$ in Fig. 3; this is intended to be a toy model of how I interpret Walt Whitman in the above stanza. This figure should be read as follows:

- if one point (a, A) is above another point (b, B) and connected by a densely dotted line ⋮, this means that $a = b$ and $B \subset A$.
- if one point (a, A) is connected to another point (b, B) by a line with an arrow ↗, this means that $\mathfrak{M}, (b, B) \Vdash A$

Observe that $\mathfrak{M}, (\{p\}, \{p, \neg p\}) \Vdash \Box\bot$; it's obvious that in this state Walt is being inconsistent since he clearly believes contradictory things. Simultaneously, we have that $\mathfrak{M}, (\{p\}, \{p, \neg p\}) \Vdash \Diamond\!\!\!-(\circlearrowleft \wedge \Box\, p)$; so we figure that Walt has a sound argument that p. Walt might be inconsistent, but it would appear that *at least one* of his arguments makes sense. And this is naturally because Walt contains a multiplicity of inner selves, just like he says, which the the $\boxminus$ modality gives access to.

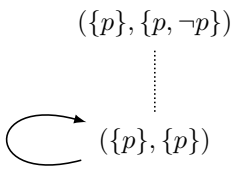


Figure 3: Inconsistent, yet still has knowledge

1.9 Irrationality

Embracing contradiction runs contrary to the received view on epistemic logic. For instance, Hendricks and Symons (2006) write:

> Epistemic logic does carry epistemological significance but in an inevitably idealized sort of way: One restricts attention to a class of rational agents where rationality is defined by certain postulates. Thus, agents have to satisfy at least some minimal conditions to simply qualify as rational. This is by and large what Lemmon originally suggests (Lemmon and Henderson, 1959).

Furthermore, it is conventional to think that rational agents do not hold contradictions.[5] For instance, in (Kraus and Lehmann, 1986), $\neg \Box \bot$ is taken as an axiom (it is A9 in their numbering).

This is similar to the thermometer concept of knowledge we provided in §1.3, since like the thermometer view, it's incompatible with a human perspective. Hence I extend the following:

> **Irrationality Principle**: *Since humans are not rational, views on epistemic logic that postulate this should be rejected*

I should mention that while this perspective is not typically embraced in epistemic logic[6], it finds sympathy in other logical traditions, namely in *relevance logic* and *paraconsistent logic*, as already noted (see Gabbay and Guenthner, 2002, chapters 1 & 4).

Apart from inconsistency, I do not really accommodate very much irrationality; I will freely admit that frameworks like (Rantala, 1982) and and Levesque (1984) employing *impossible world* semantics are far more accommodating to irrationality than the semantics I am proposing. Regardless, allowing for an agent's beliefs to naturally be inconsistent is already orthogonal to the assumption that agent's are rational.

[5] It should be remarked that Priest (2006) explicitly rejects this perspective on rationality. Priest points out that in times of scientific revolution, rational people naturally hold contradictory views. He suggests that a paraconsistent logic framework could account for a rational agent holding contradictory beliefs. I profess sympathy for Priest's perspective; however, I am confident that this does not represent the received view which I am arguing against.

[6] Noted exceptions to this are Rantala (1982) and Levesque (1984).

1.10 Quine

I shall now return to developing my framework. To recap, so far I have suggested adding a novel modality $\boxminus$ which corresponds to taking subsets of an agent's set of beliefs. In the context of conventional modal logic, this means a shift in perspective - instead of thinking of each world as a situation where the agent can imagine other situations, now each world corresponds to a network of beliefs ordered by inclusion. These networks of beliefs form a poset, or partially ordered set. Thus the choice to visually represent them as *Hasse diagrams*, as I have done in Fig. 3, follows the standard practice in lattice theory.

Furthermore, I should point out the following phenomenon - as higher nodes in a belief network are considered, the agent is employing more premises for the arguments they are composing, and using less pure logic to come to conclusions. I feel this suggests the following - namely, that as we consider levels higher and higher in the poset of an agent's beliefs, this sort of corresponds to embracing an agent's experience and interpretation of their sensory data. But arguments that rest on more premises are prima facie more fallible than arguments that rely on fewer assumptions.

A similar perspective has been presented before, however in a different setting, in *Two Dogmas of Empiricism* (Quine, 1951):

> Certain statements, though about physical objects and not sense experience, seem peculiarly germane to sense experience – and in a selective way: some statements to some experiences, others to others. Such statements, especially germane to particular experiences, I picture as near the periphery. But in this relation of "germaneness" I envisage nothing more than a loose association reflecting the relative likelihood, in practice, of our choosing one statement rather than another for revision in the event of recalcitrant experience. For example, we can imagine recalcitrant experiences to which we would surely be inclined to accommodate our system by re-evaluating just the statement that there are brick houses on Elm Street, together with related statements on the same topic. We can imagine other recalcitrant experiences to which we would be inclined to accommodate our system by re-evaluating just the statement that there are no centaurs, along with kindred statements. A recalcitrant experience can, I have already urged, be accommodated by any of various alternative re-evaluations in various alternative quarters of the total system; but, in the cases which we are now imagining, our natural tendency to disturb the total system as little as possible would lead us to focus our revisions upon these specific statements concerning brick houses or centaurs. These statements are felt, therefore, to have a sharper empirical reference than highly theoretical statements of physics or logic or ontology. **The latter statements may be thought of as relatively centrally located within the total network, meaning merely that little preferential connection with any particular sense data obtrudes itself.**

The emphasis on the last sentence is my addition. The above paragraph importantly anticipates ideas in belief revision theory (such as in Alchourron et al. (1985) and subsequent studies), as well as recent trends in probabilistic dynamic epistemic logic (such as in van Benthem, 2003, van Benthem et al., 2009, Baltag and Smets, 2008, Kooi, 2003, etc.). However, in the framework that I have so

far been developing, what Quine refers to as the "periphery" of his web of belief corresponds to a higher node in a belief poset, while what Quine referes to as the "center" reflects something like a lower node. This is visually depicted in Fig. 4. Beliefs that are members of lower nodes, and the ideas that follow from them, can be thought of as belonging to the agent's world-view.

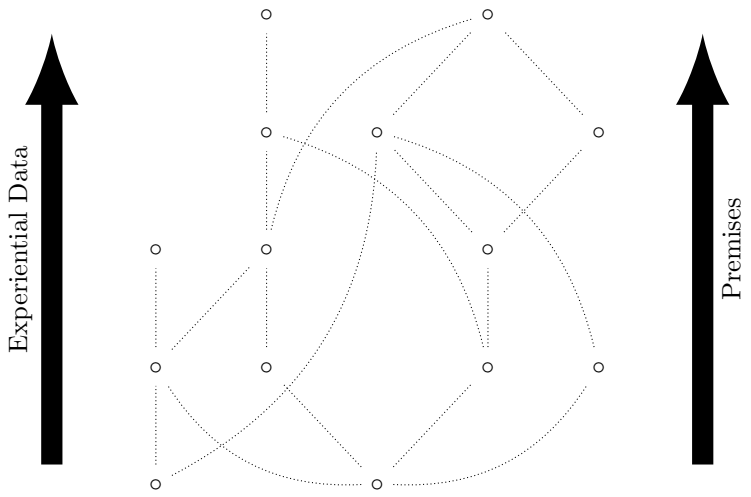


Figure 4: A network of beliefs

I feel the above observation informs a corresponding perspective on epistemology. If an agent's world view largely rested legends about the Norse gods, I'd be reluctant to say she knows various facts about nature, such as why lightning strikes. This is because all of her explanations would inevitably be based upon myths in one way or another, which would all occupy lower nodes in her belief network. This dictates that *sanity* plays a role in how much knowledge an agent can have - it is permissible to grant that an inconsistent agent has knowledge provided that the inconsistency follows only shallowly from her experiential data, and it is something she would readily give up. However, if a contradiction is intrinsic to the agent's psychology, and thus follows from a lower node in her belief poset, my analysis suggests she doesn't really have knowledge. So while I believe that irrational agents can possess knowledge, as I have argued in §1.9, I do not contend that they *always* posses knowledge. Moreover, I hold that the sort of irrationality that I am considering needn't be superficial - both mundane as well as deeply demented characters can be modeled.

I'll admit that the above essentially presents my own interpretation of Quine's web of belief, which might be contentious. On the other hand, I feel both the quote from Quine and the quote from Whitman in §1.8 suggest the following principle without too much controversy:

> **Quine/Whitman Principle**: *Epistemic agents are compound entities, which invite compositional analysis.*

The above presents the final philosophical principle that I intend to extend. Apart from this, I would say from the previous discussion, I would like to extract an additional thing: Figure 4

naturally suggests that we might think of *going up* in a belief net, in a manner similar to how $\boxminus$ allows one to *go down* as I suggested in §1.7. Along these lines, I would suggest the introduction of a new operator $\boxplus$. The semantics for $\boxplus$ are given as follows:

$$\mathfrak{M}, (a, A) \Vdash \boxplus\phi \iff \text{for all } (b, B) \in \mathfrak{M} \text{ if } b = a \text{ and } A \subseteq B \text{ then } \mathfrak{M}, (a, A) \Vdash \phi$$

Just as $\boxminus$ corresponds to the agent casting assumptions into doubt, or disregarding their premises, $\boxplus$ corresponds to the agent embracing their experience, suspending disbelief and accepting her intuitions and senses.

This essentially concludes the sketch of novelties I propose in the practice of modelling knowledge.

1.11 Closing Remarks

The various principles extended in the previous sections are not independent - some of them are more basic than others. Their relationship is summarized in Fig. 5 - here the lower a principle is depicted, the more basic I feel it is. Dotted lines indicate that I feel the philosophical justification for the higher principle supervenes on the justification of the lower principle. In addition, in further

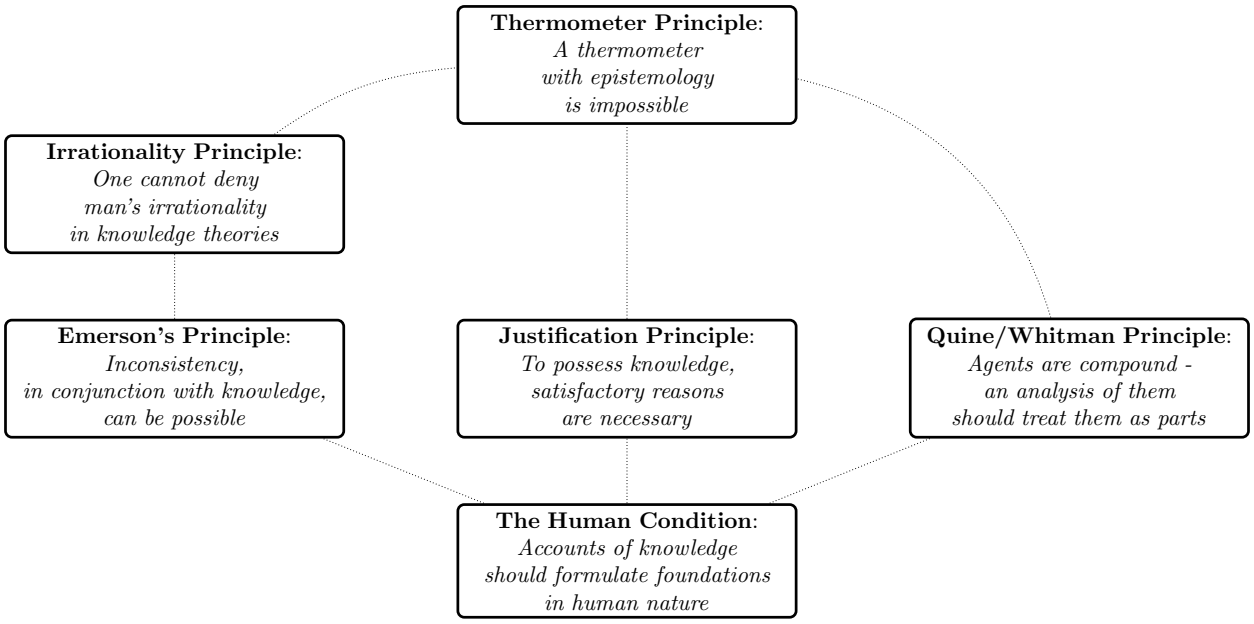


Figure 5: A visualization of the relationship of the principles I have suggested

development of the framework sketched in §1.3, I shall want the following criteria, based on my ideas given in relevant sections:

§1.3 • Agents shall be modelled with proofs for the things they believe.

- To avoid paradoxes, correct foundations must be provided. Idealy, I would like my semantics to correspond to a provably terminating computation, granting certain non-deterministic operations such as a *choice operator* ε, as described in (Hilbert, 1922).

For a set of beliefs A:

§1.6 It should be expressible whether everything in A is sound

§1.7 Certain subsets $B \subseteq A$ should be accessible

§1.10 Certain extensions $B \supseteq A$ could also be accessed

In line with evidentialist epistemology, as mentioned in §1.4, I have decided to call the logic I shall develop *Evidentialist Logic*, or EvIL for brevity.

2 Introduction to EviL

With the philosophical intuitions and scaffolding provided from §1, I shall now turn to giving a precise account of my previously developed ideas. This shall be done in three movements:

§2.1 In the first section I will provide the basic grammar and semantics for EvIL with a single agent; the presentation in this section will remain primarily philosophical and light.

§2.2 In the second section I develop several topics in the pure theory of EvIL which I consider a bit beyond the bare essentials.

§2.3 In this section, completeness and decidability is discussed in relation to EvIL and two sublogics.

2.1 Elementary EviL

2.1.1 Grammar & Semantics

In this section I turn to developing the formal semantics for EvIL with a single agent. I imagine the object of study in EvIL is an agent, which I call the EvIL agent. In §2.2.3, the semantic framework offered here is extended to incorporate multiple agents. In Appendix A, yet another framework is offered employing gamelike semantics, which avoids the grammar restriction suggested in §1.4.

The grammar restriction imposed on EvIL was introduced to avoid paradoxes. That being the case, I shall discard the previous definition of $(\Vdash)$ I suggested, in favor of demonstrably well-defined semantics. This shall be achieved in two steps.

Definition 2.1.1. *Let $\mathcal{L}_0(\Phi)$ be the language of classical propositional logic, defined by the following Backus-Naur form grammar:*

$$\phi ::= p \in \Phi \mid \phi \to \psi \mid \bot$$

Models for classical propositional logic can be thought of as sets $S \subseteq \Phi$; thus the truth predicate $(\models) : \wp\Phi \to \mathcal{L}_0(\Phi) \to \mathsf{bool}$[7] for classical propositional logic can be given recursively as follows:

Definition 2.1.2. *Define* $(\models)$ *such that*

$$\begin{aligned} S \models p &\iff p \in S \\ S \models \phi \to \psi &\iff S \models \phi \text{ implies } S \models \psi \\ S \models \bot &\iff False \end{aligned}$$

Further, observe that the language $\mathcal{L}_0$ is extended by EvIL

Definition 2.1.3. *Define* $\mathcal{L}(\Phi)$ *by the following Backus-Naur grammar:*

$$\phi ::= p \in \Phi \mid \phi \to \psi \mid \bot \mid \Box\,\phi \mid \boxminus\,\phi \mid \boxplus\,\phi \mid \circlearrowleft$$

EvIL models are sets $\mathfrak{M} \subseteq \wp\Phi \times \wp\mathcal{L}_0(\Phi)$. Like classical propositional logic, semantics for EvIL are given recursively by a predicate

$$(\Vdash) : \wp(\wp\Phi \times \wp\mathcal{L}_0(\Phi)) \to \wp\Phi \times \wp\mathcal{L}_0(\Phi) \to \mathcal{L}(\Phi) \to \mathsf{bool}.$$

That is, $(\Vdash)$ is a function that:

- Takes as input:
  - An EvIL model
  - A pair (a, A) where
    - $a \subseteq \Phi$ is a set of proposition letters
    - $A \subseteq \mathcal{L}_0(\Phi)$ is a set of propositional formulae.
  - A formula in the language $\mathcal{L}(\Phi)$
- Gives as output: a truth value in bool

I can now provide a formal definition of the semantics for EvIL:

Definition 2.1.4. *Define* $(\Vdash)$ *recursively such that:*

$$\begin{aligned} \mathfrak{M}, (a, A) \Vdash p &\iff p \in a \\ \mathfrak{M}, (a, A) \Vdash \phi \to \psi &\iff \mathfrak{M}, (a, A) \Vdash \phi \text{ implies } \mathfrak{M}, (a, A) \Vdash \psi \\ \mathfrak{M}, (a, A) \Vdash \bot &\iff False \\ \mathfrak{M}, (a, A) \Vdash \Box\,\phi &\iff \forall (b, B) \in \mathfrak{M}. (\forall \psi \in A. b \models \psi) \text{ implies } \mathfrak{M}, (b, B) \Vdash \phi \\ \mathfrak{M}, (a, A) \Vdash \boxminus \phi &\iff \forall (b, B) \in \mathfrak{M}. a = b \text{ and } B \subseteq A \text{ implies } \mathfrak{M}, (b, B) \Vdash \phi \\ \mathfrak{M}, (a, A) \Vdash \boxplus \phi &\iff \forall (b, B) \in \mathfrak{M}. a = b \text{ and } B \supseteq A \text{ implies } \mathfrak{M}, (b, B) \Vdash \phi \\ \mathfrak{M}, (a, A) \Vdash \circlearrowleft &\iff \forall \psi \in A. a \models \psi \end{aligned}$$

[7] ... where $\mathsf{bool} := \{True, False\}$. This is more commonly written as $(\models) : \wp\Phi \to \mathsf{bool}^{\mathcal{L}_0(\Phi)}$. My notation reflects the notation common to the typed functional programming languages *Haskell* and *OCaml*. I will use both notations interchangeably.

I will write $\mathfrak{M} \Vdash \phi$ *to mean* $\mathfrak{M}, (a, A) \Vdash \phi$ *for all* $(a, A) \in \mathfrak{M}$*. Further, I will write* $\Vdash \phi$ *to mean* $\mathfrak{M} \Vdash \phi$ *for all* $\mathfrak{M}$*.*

These semantics are well defined, since apart from relying on the semantics for propositional logic they may be observed to be compositional.[8] Moreover, the following relationship can be observed:

Lemma 2.1.5 (Truthiness)**.** *Let* $\phi \in \mathcal{L}_0(\Phi)$*. Then:*

$$a \models \phi \Longleftrightarrow \mathfrak{M}, (a, A) \Vdash \phi$$

. . .for any $\mathfrak{M}$ *and* A*.*

Proof. This may be seen immediately by induction on ϕ. QED

. . .with this, we have the following, mirroring Prop. 1.4.2:

Definition 2.1.6. *Define the following:*

$$Th(\mathfrak{M}) := \{\phi \in \mathcal{L}(\Phi) \mid \mathfrak{M} \Vdash \phi\}$$

Theorem 2.1.7 (Theorem Theorem)**.** *If* A *is finite, then* $\mathfrak{M}, (a, A) \Vdash \Box\phi$ *if and only if* $Th(\mathfrak{M}) \cup A \vdash_{\text{EvIL}} \phi$*.*

I shall present $\vdash_{\text{EvIL}}$, the logical consequence turnstile for EvIL, in §2.3.1.

I chose the name "Theorem Theorem" because it means that for every belief the EvIL agent has, it is a theorem she has derived from her premises. Theorem 2.1.7 establishes one of the central desiderata outlined in §1.11 is achieved by EvIL. With this result the foundation is set for the the central intuition driving EvIL - that beliefs are the consequences of logical deductions. It is a peculiarity of EvIL that these deductions are carried on in EvIL itself. This was achieved, primarily, by my previous flirtation with paradox. And as a consequence, I have tried to design EvIL to eat its own tail. This is my favorite kind of self-reference. As a modeler using logic, it establishes that the EvIL agent is herself also a modeler just like me, using the same logic I am using to think about her herself, to think about the state space she lives in. It reminds me of a quote regarding the wonderful circularity of mathematics, due to Browne (1736):

> All things began in order, so shall they end, and so shall they begin again; according to the ordainer of order and mystical Mathematicks of the City of Heaven.

[8] In fact, I have provided a formulation of these semantics in the same manner as above in the computer proof assistant Isabelle/HOL (Nipkow et al., 2002); I shall give my remarks on formal verification in §4. In the case of Isabelle/HOL, the function ($\Vdash$) was defined inductively, and automatically proven to be well-defined. Specifically, the conditions given above specify that ($\Vdash$) is determined by a *monotonic* predicate over suitable tupples, and similarly for ($\models$). Hence the result that ($\Vdash$) is well-defined ultimately relies on an application of the *Knaster-Tarski Fixpoint Theorem* (Roman, 2008, chapter 12). Further, since I have given an inductive definition, these recursive definitions rely on the *least* fixpoint of their associated monotonic operators. This is not really particular to EvIL; rather, this is central to the basic "logic engineering" philosophy of Isabelle/HOL (Berghofer, 2009).

2.1.2 Intuitions

In this section, I shall illustrate how I intuitively read the operators in EvIL, and provide a number of validities.

As per the traditional doxastic reading of $\Box\phi$, I read this as asserting "The EvIL agent believes ϕ". Because of Theorem 2.1.7, the Theorem Theorem, I shall freely conflate this with the assertion "The EvIL agent has an argument for ϕ," which I take to be a proof.

My intuition for how to read $\Diamond\!\!\!\!-\phi$ was first mentioned in §1.7 with respect to Descartes' Meditation II – it means "If the EvIL agent were to set aside some of her beliefs, or cast some of her beliefs into doubt, then ϕ would hold." Dually, I tend to read $\boxminus\phi$ as saying something like "For all the ways that the EvIL agent might use her imagination, ϕ holds." I recognize that these interpretations might seem inconsistent – however, I regard casting beliefs into doubt and embracing one's imagination as part of the same coin. For, naturally, when one doubts more things, then for a fleeting moment their dreams take flight as the inconceivable turns around into the conceivable, if only for a little while. To give an example, if I set aside for a moment my belief that

$$\text{the law of gravity is an exceptionless regularity of the universe} \qquad (g)$$

...then it seems natural to imagine that

$$\text{a propulsion device exploiting some exception to gravitation might be constructable.} \qquad (p)$$

In the symbology of EvIL formulae, I would code this intuition as

$$\boxminus(\Box\neg g \rightarrow \Diamond p).$$

To give another example, if I pretend that it isn't the case that:

$$\text{the canals of Amsterdam are filthy} \qquad (f)$$

I might be able to imagine a scenario where

$$\text{I am swimming comfortably in the Amstel river} \qquad (r)$$

But not really. I really can't really swim at ease in the Amstel, not just because it has tons of garbage, but also because

$$\text{I don't own a bathing suit,} \qquad (b)$$

Frankly, I am not so bold that I could go skinny dipping in Amstel without that being awkward. Hence I would say in the language of EvIL that:

$$\neg \boxminus (\Box\neg f \rightarrow \Diamond r)$$

This is because I can cast into doubt the assumption of the filthiness of the canals of Amsterdam, while still retaining my belief that I don't have a bathingsuit, so swimming in Amstel would still be awkward for me. In symbols, I would write express this sentiment as the following expression:

$$\Diamond\!\!\!\!-(\Diamond\neg f \wedge \Box b \wedge \neg\Diamond r)$$

Further, my intuition for how to read $\oplus\!\!\!\!-\phi$ is "If the EvIL agent were to remember something, then ϕ would hold." For instance, I can think of an instance where I woke up and searched myself for

my bike keys. To my horror, they weren't there – in I immediately assumed that I might have left my keys in the lock on my bike, and figured there was a fair likelihood that

$$\text{my bike has been stolen because I left the keys in it.} \qquad (s)$$

But once I recalled that

$$\text{I had lent my bike to a friend,} \qquad (l)$$

...my fear subsided. I would have said that prior to remembering, while I thought it might be possible that my bike was stolen due to my negligence, if I remembered what I had done then I no longer would have entertained that possibility. I would express this observation as:

$$\Diamond s \wedge \boxplus(\Box l \rightarrow \Box\neg s)$$

I consider $\boxminus$ and $\boxplus$ to be inverse modalities of each other, in exactly the same way that *past* and *future* are inverse modalities in temporal logic. This is perhaps a little unusual; it is arguably more natural to think of *forgetting* as the inverse modality of remembering, and there doesn't appear to be an natural inverse operation corresponding to casting into doubt. Following the idea of the *web of belief* due to Quine, as presented in §1.10, I would extend a position asserting that remembering factive data is the same as embracing as much of one's evidence as possible.

In terms of the semantics outlined, $\boxminus$ corresponds to a subsetset relation while $\boxplus$ corresponds to a superset relation. Because of this, I sometimes read $\boxminus\phi$ closer to the formal semantics, as saying something like "for all subsets of the agent's beliefs, ϕ holds" and dually for $\boxplus\phi$. This is admittedly even less natural than the reading of remembering as the opposite of casting into doubt. So be it; I am comfortable with EvIL agents being at best twisted cartoon versions of actual people, who actually have minds and engage in remembering, imagining, and other similar activities. After all, according to the semantics stipulated in §2.1.1, EvIL agents apparently have sets for brains, which makes an EvIL agent a strange effigy for a person indeed – with the possible exception of set theorists, whose brains are typically constructed entirely of sets or urelements.

Furthermore, it is under the set theoretical reading that $\circlearrowleft$ makes the most sense. I read it as asserting something like "the basis for the EvIL agent's beliefs is sound" or "the EvIL agent's arguments only use true premises." It further means that the actual state of affairs is compatible with what the agent believes - reality has not been ruled out by something that the agent is taking as evidence. Moreover, sound premises intuitively exhibit the following property - any subset of them is also sound, since soundness isn't a phenomenon that is subject to synchronicity or other failures of compositionality. A set of premises is sound if and only if all of its subsets are also sound.

2.1.3 Validities

The previous philosophical readings of EvIL immediately suggest certain validities will hold the semantics. For instance, the assertion "A set of premises is sound if and only if all of its subsets are sound." would be expressed as

$$\models\circlearrowleft \leftrightarrow \boxminus\circlearrowleft \qquad (2.1.1)$$

...and indeed, this is a validity of EvIL. There is another, related validity associated with $\circlearrowleft$; namely that if the EvIL agent's assumptions are sound, then anything she concludes from them is

true (employing the reading which naturally arises from Theorem 2.1.7). This is expressed as

$$\Vdash \circlearrowleft\!\!\rightarrow \Box\phi \rightarrow \phi \tag{2.1.2}$$

The formula (2.1.1) expresses that the soundness of one's premises is something *persistent* as the EviL agent carries on casting doubt on assumptions and discarding them. Another thing that is persistent this way is the EviL agent's imagination:

$$\Vdash \Diamond\phi \rightarrow \boxminus\Diamond\phi \tag{2.1.3}$$

I read (2.1.3) as saying something like "If the EviL agent can imagine something, then no matter things she casts into doubt, she can still imagine it." One can also express something like the dual of this, namely

$$\Vdash \Box\phi \rightarrow \boxplus\, \Box\, \phi \tag{2.1.4}$$

. . . which I read as asserting "If th agent can compose an argument then she'll still be able to compose that argument if she remembers more premises she has available." In general, many of the assertions here have an interplay like this – interest in these relationships is taken up in §2.2.1.

Furthermore, for better or for worse the EviL semantics make true the following: if something is achievable by repeatedly casting assumptions into doubt, then it's achievable by casting assumptions into doubt only once:

$$\Vdash \diamondminus^{+}\phi \rightarrow \diamondminus\phi \tag{2.1.5}$$

. . . where $^{+}$ is taken from the syntax for *regular expressions* commonly used in computer science and UNIX programming to mean "one or more" (Friedl, 2006). Similarly, I have assumed that discarding no assumptions is, in a way, vacuously casting assumptions into doubt. In light of this EviL also makes true the following:

$$\Vdash \phi \rightarrow \diamondminus\phi \tag{2.1.6}$$

Furthermore, it is worth mentioning some harder to understand validities of this system. The first one is that when the agent believes something, they believe it regardless of the process of doubting or embracing their beliefs:

$$\Vdash \Box\phi \rightarrow \Box\, \boxminus\, \phi \tag{2.1.7}$$

$$\Vdash \Box\phi \rightarrow \Box\, \boxplus\, \phi \tag{2.1.8}$$

We can observe that this generalizes to multiple agents, as specified in §2.2.3.

Another more challenging validity is the fact that if some proposition ϕ holds , then for any restriction of EviL agent's beliefs (or dually, any extension), if those beliefs are sound, then ϕ must be conceivable. This is expressed as the following two validities:

$$\Vdash \phi \rightarrow \boxminus(\circlearrowleft\!\!\rightarrow \Diamond\phi) \tag{2.1.9}$$

$$\Vdash \phi \rightarrow \boxplus(\circlearrowleft\!\!\rightarrow \Diamond\phi) \tag{2.1.10}$$

Finally, another a peculiarity of EviL is that not all of its validities are *schematic*. For instance, there is a kind of *Cartesian dualism* present in the semantics, where the EviL agent's deliberation on her evidence does not bear on brute matters of fact. For a world pair (a, A), A and a are basically

separate - an EviL agent's mind and the world they live are composed of different substance. This gives rise to the following four validities:

$$\Vdash p \rightarrow \boxminus p \tag{2.1.11}$$

$$\Vdash p \rightarrow \boxplus p \tag{2.1.12}$$

$$\Vdash \neg p \rightarrow \boxminus \neg p \tag{2.1.13}$$

$$\Vdash \neg p \rightarrow \boxplus \neg p \tag{2.1.14}$$

Hence, EviL is not a *normal* logic. On the other hand, the duality between doubting and embracing one's experience, belief and imagination, as well as the soundness and unsoundness give rise to an interplay which I don't believe is present in any *normal* logic. This shall be the subject of study in §2.2.1.

2.2 EviL Basics

2.2.1 Elimination

In section §2.1.3, I presented the structural validities of EviL from a philosophical perspective. That being the case, my manner of presentation followed my intuition, which I admit is altogether unorganized. In this section, I shall give the validities of EviL a more systematic presentation. In doing so, I shall showcase an elimination theorem, that I feel sits at the heart of EviL.

To start, the following lemma summerizes the structural validities that I will be studying in the subsequent discussion:

Lemma 2.2.1. *The following validities hold for all* EviL *models:*

$$\begin{array}{ll}
\Vdash \boxminus p \leftrightarrow p & \Vdash \boxplus p \leftrightarrow p \\
\Vdash \boxminus \neg p \leftrightarrow \neg p & \Vdash \boxplus \neg p \leftrightarrow \neg p \\
\Vdash \boxminus \Diamond \phi \leftrightarrow \Diamond \phi & \Vdash \boxplus \Box \phi \leftrightarrow \Box \phi \\
\Vdash \diamondminus \Box \phi \leftrightarrow \Box \phi & \Vdash \diamondplus \Diamond \phi \leftrightarrow \Diamond \phi \\
\Vdash \boxminus \boxminus \phi \leftrightarrow \boxminus \phi & \Vdash \boxplus \boxplus \phi \leftrightarrow \boxplus \phi \\
\Vdash \boxminus \diamondplus \phi \leftrightarrow \diamondplus \phi & \Vdash \boxplus \diamondminus \phi \leftrightarrow \diamondminus \phi \\
\Vdash \boxminus \circlearrowleft \leftrightarrow \circlearrowleft & \Vdash \boxplus \neg \circlearrowleft \leftrightarrow \neg \circlearrowleft
\end{array}$$

These validities suggest a definite interplay between the modalities of EviL; they are highly suggestive of a general elimination theorem. To see what arises from Lemma 2.2.1, first observe that EviL makes true the usual substitution rule:

Lemma 2.2.2. *If* $\Vdash \phi \leftrightarrow \psi$ *is a validity, then* $\Vdash \chi \leftrightarrow \chi[\phi/\psi]$ *is a validity for any* $\chi \in \mathcal{L}(\Phi)$.

Next, I offer two sublanguages of the main language of EviL:

Definition 2.2.3. *Define the following fragments:*[9]

[9] I was inspired to look at the fragment $\mathcal{L}_A(\Phi)$ by thinking about the continuous fragment of μPML (Fontaine, 2008).

$\mathcal{L}_A(\Phi)$:

$$\phi ::= p \mid \neg p \mid \top \mid \bot \mid \circlearrowleft \mid \phi \wedge \psi \mid \phi \vee \psi \mid \Diamond \phi \mid \boxminus \phi \mid \diamondminus \phi$$

$\mathcal{L}_B(\Phi)$:

$$\phi ::= \neg p \mid p \mid \bot \mid \top \mid \neg \circlearrowleft \mid \phi \vee \psi \mid \phi \wedge \psi \mid \Box \phi \mid \boxminus \phi \mid \boxplus \phi$$

Definition 2.2.4. *Define two dualizing operations* $(\cdot)^A : \mathcal{L}_B(\Phi) \to \mathcal{L}_A(\Phi)$ *and* $(\cdot)^B : \mathcal{L}_A(\Phi) \to \mathcal{L}_B(\Phi)$*, using recursion, such that:*

$\neg p^A := p$	$p^B := \neg p$
$p^A := \neg p$	$\neg p^B := p$
$\bot^A := \top$	$\top^B := \bot$
$\top^A := \bot$	$\bot^B := \top$
$\neg \circlearrowleft^A := \circlearrowleft$	$\circlearrowleft^B := \neg \circlearrowleft$
$(\phi \vee \psi)^A := \phi^A \wedge \psi^A$	$(\phi \wedge \psi)^B := \phi^B \vee \psi^B$
$(\phi \wedge \psi)^A := \phi^A \vee \psi^A$	$(\phi \vee \psi)^B := \phi^B \wedge \psi^B$
$(\Box \psi)^A := \Diamond(\psi^A)$	$(\Diamond \psi)^B := \Box(\psi^B)$
$(\boxminus \psi)^A := \diamondminus(\psi^A)$	$(\diamondminus \psi)^B := \boxminus(\psi^B)$
$(\boxplus \psi)^A := \dotplus(\psi^A)$	$(\dotplus \psi)^B := \boxplus(\psi^B)$

With the above definition in hand, it is straightforward to see the following duality theorem:

Theorem 2.2.5 (Duality)**.** *Observe that for all* $\phi \in \mathcal{L}_A(\Phi)$ *and* $\psi \in \mathcal{L}_B(\Phi)$*,* $(\phi^B)^A = \phi$ *and* $(\psi^A)^B = \psi$*. Moreover, we have the following validities:* $\Vdash \neg(\phi^B) \leftrightarrow \phi$ *and* $\Vdash \neg(\psi^A) \leftrightarrow \psi$*.*

The above duality is convenient, since it can be leveraged to transfer results proven for the fragment $\mathcal{L}_A(\Phi)$ to $\mathcal{L}_B(\Phi)$ and vice versa.

With the above machinery in place, I present what I feel is the natural consequence of the logical equivalence given in Lemma 2.2.1:

Definition 2.2.6. *If* $\phi \in \mathcal{L}(\Phi)$ *then let* ϕ^* *be the same formula, with all instances of* $\boxplus$*,* $\boxminus$*,* $\diamondminus$ *and* $\dotplus$ *eliminated.*

Theorem 2.2.7 (EvIL Elimination)**.** *For all* $\phi \in \mathcal{L}_A(\Phi)$ *or* $\phi \in \mathcal{L}_B(\Phi)$*, we have the following validity:*

$$\Vdash \phi \leftrightarrow \phi^*$$

Proof. The prove proceeds in three steps.

Step 1: First, use induction on $\phi \in \mathcal{L}_A(\Phi)$, and show the following two facts simultaneously:

$$\Vdash \boxminus \phi \leftrightarrow \phi \qquad \Vdash \dotplus \phi \leftrightarrow \phi$$

- Cases p, $\neg p$, $\bot$, $\top$, $\circlearrowleft$: In all of these situations, the result follows directly from the validities illustrated in Lemma 2.2.1.

- Cases $\wedge, \vee$: For $\boxminus$ the connective $\wedge$ is simple, and dually for $\boxplus$ for the connective $\vee$. This is because in each case one may simply use distribution, such as can be done here:

$$\begin{aligned} \Vdash \boxminus(\phi \wedge \psi) &\leftrightarrow \boxminus\phi \wedge \boxminus\psi \\ &\leftrightarrow \phi \wedge \psi \end{aligned}$$

  On the other hand, $\vee$ is more interesting for $\boxminus$, and dually $\wedge$ for $\boxplus$. Using induction, Lemma 2.2.1, and subsitution, and distribution, we have the line of reasoning:

$$\begin{aligned} \Vdash \boxminus(\phi \vee \psi) &\leftrightarrow \boxminus(\boxplus\phi \vee \boxplus\psi) \\ &\leftrightarrow \boxminus\boxplus(\phi \vee \psi) \\ &\leftrightarrow \boxplus(\phi \vee \psi) \\ &\leftrightarrow \boxplus\phi \vee \boxplus\psi \\ &\leftrightarrow \phi \vee \psi \end{aligned}$$

- Case $\Diamond$: Once again, this follows immediately from the validities of Lemma 2.2.1, namely $\Vdash \boxminus\Diamond\phi \leftrightarrow \Diamond\phi$ and $\Vdash \boxplus\Diamond\phi \leftrightarrow \Diamond\phi$

- Cases $\boxminus, \boxplus$: The final step follows from one more application of Lemma 2.2.1, namely by employing the following four validities

$$\begin{array}{ll} \Vdash \boxminus\boxplus\phi \leftrightarrow \boxplus\phi & \Vdash \boxplus\boxplus\phi \leftrightarrow \boxplus\phi \\ \Vdash \boxminus\boxminus\phi \leftrightarrow \boxminus\phi & \Vdash \boxplus\boxminus\phi \leftrightarrow \boxminus\phi \end{array}$$

Step 2: With the above, we can prove for any $\phi \in \mathcal{L}_A(\Phi)$ that $\Vdash \phi \leftrightarrow \phi^*$. Once again, the proof proceeds by induction, the only steps worth noting involve $\boxminus$ and $\boxplus$. In either case, these may be completed using Step 1. For instance, we know that $\Vdash \boxminus\phi \leftrightarrow \phi$, hence $\Vdash \boxminus\phi \leftrightarrow \phi^*$ by induction.

Step 3: With the result for $\mathcal{L}_A(\Phi)$ in hand, just observe that for $\psi \in \mathcal{L}_B(\Phi)$ we have that $(\psi^A)^* = (\psi^*)^A$. With this, substitution, and duality, we have the following chain of reasoning:

$$\begin{aligned} \Vdash \psi &\leftrightarrow \neg(\psi^A) \\ &\leftrightarrow \neg((\psi^A)^*) \\ &\leftrightarrow \neg((\psi^*)^A) \\ &\leftrightarrow \neg(\neg(((\psi^*)^A)^B)) \\ &\leftrightarrow \neg\neg\psi^* \\ &\leftrightarrow \psi^* \end{aligned}$$

QED

Example 2.2.8. *The following validities of* EvidL *are consequences of Theorem 2.2.7:*

$$\Vdash \boxminus\boxplus\Diamond t \vee \boxplus\Box\boxplus\neg t$$

$$\Vdash ((\boxplus\boxplus\Box q \wedge \boxplus\boxplus\Box q) \vee \Box\boxplus\boxplus q) \wedge ((\boxplus\Box\boxplus q \vee \Box\boxplus\boxplus q) \wedge \boxplus\Box\boxplus q) \leftrightarrow \Box q$$

$$\Vdash \quad \Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond$$
$$\Diamond\Diamond\Diamond\Diamond\Diamond \boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\Diamond\Diamond\Diamond\Diamond\Diamond$$
$$\Diamond\Diamond\Diamond\Diamond \boxminus\boxminus\Box\Box\Box\Box\Box\Box\Box\Box\boxminus\boxminus\Diamond\Diamond\Diamond\Diamond$$
$$\Diamond\Diamond \boxminus\boxminus\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\Box\boxminus\boxminus\boxminus\Diamond$$
$$\Diamond \boxminus\Box\Box\Box\Box\Diamond\Diamond \Box\Box\Box\Box\Diamond\Diamond \Box\Box\Box\Box\boxminus\Diamond$$
$$\Diamond \boxminus\Box\Box\Box\Box\Diamond\Diamond \Box\Box\Box\Box\Diamond\Diamond \Box\Box\Box\Box\boxminus\Diamond$$
$$\Diamond \boxminus\Box\Box\Diamond \Box\Box\Box\Box\Box\Box\Box\Box\Diamond \Box\Box\boxminus\Diamond$$
$$\Diamond \boxminus\Box\Box\Box\Diamond \Box\Box\Box\Box\Box\Box\Box\Diamond \Box\Box\Box\boxminus\Diamond$$
$$\Diamond \boxminus\boxminus\boxminus\Box\Box\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond \Box\Box\boxminus\boxminus\boxminus\Diamond$$
$$\Diamond\Diamond\Diamond \boxminus\boxminus\Box\Box\Box\Box\Box\Box\Box\Box\Box\boxminus\boxminus\Diamond\Diamond\Diamond$$
$$\Diamond\Diamond\Diamond\Diamond \boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\boxminus\Diamond\Diamond\Diamond\Diamond$$
$$\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\Diamond\bot \quad \leftrightarrow \Box^{70}\bot$$

The way I read Theorem 2.2.7 is that $\boxminus$ and $\Diamond$ are empty modalities on $\mathcal{L}_A(\Phi)$, and dually for $\mathcal{L}_B(\Phi)$ with $\Diamondblack$ and $\boxplus$. Further, note that $\mathcal{L}_0(\Phi) \subseteq \mathcal{L}_A(\Phi) \cap \mathcal{L}_B(\Phi)$ (up to translation), which means that all four of $\boxplus$, $\boxminus$ along with their duals $\Diamond$ and $\Diamondblack$ vanish on the propositional language. Inspecting the semantics, this is to be expected, since neither $\boxplus$ nor $\boxminus$ interact with propositional truth values.

Finally, I should remark that Theorem 2.2.7 reflects one of the basic themes of EvIL - the interplay between belief, reflected by $\Box$, and imagination, reflected by $\Diamond$. I feel that these two phenomena are just two sides of the same coin - furthermore, one couldn't have more natural opposites. Belief and imagination exemplify what I naturally feel are two warring forces dwelling within any EvIL agent's heart. Evidently soundness $\circlearrowleft$ is aligned with imagination and unsoundness $\neg \circlearrowleft$ is aligned with belief. However, I admit that I do not understand what philosophical light is shed by Theorem 2.2.7, if there is indeed any at all.

2.2.2 Failure of Compactness

In this section I shall demonstrate that EvIL is not compact by giving an example of an infinite set of formulae for which every finite subset is satisfiable while the entirety is not.

Lemma 2.2.9. *Let $\tau : \Phi \to \mathcal{L}$ be defined as follows:*

$$\begin{aligned}\tau(p) := p \wedge \Diamond\top \wedge \\ \Box\ (\neg p \wedge \Diamond\top \wedge \\ \Box\ (p \wedge \Diamond p))\end{aligned}$$

Every finite subset of $\tau[\Phi]$ is satisfiable, but not the entirety, which is infinite.

Proof. That $\tau[\Phi]$ is infinite is immediate, as Φ was stipulated to be infinite.

So let $S \subseteq \tau[\Phi]$ be finite. I shall provide a model that satisfies S. First observe that there is a finite $\Psi \subset \Phi$ such that $S = \tau[\Psi]$. Let $\{q, r, s\} \subseteq \Phi \setminus \Psi$ contain three distinct letters - this can be done, since $\Phi \setminus \Psi$ is infinite. Let $\mathfrak{M} := \{(a, \{q\}), (b, \{r\}), (c, \{s\})\}$, where

$$a := \Psi \cup \{s\}$$
$$b := \{q\}$$
$$c := \Psi \cup \{r\}$$

Using the same visualization convention I introduced in §1.8, $\mathfrak{M}$ can be visualizedin Fig. 6. It is straightforward to check that $\mathfrak{M}, (a, \{q\}) \Vdash \tau(p)$ for all $p \in \Psi$.

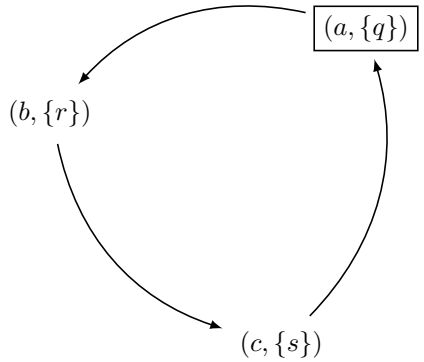


Figure 6: $\mathfrak{M}, (a, \{q\}) \Vdash \tau[\Psi]$

On the other hand, suppose there was some model $\mathfrak{N}$ such that $\mathfrak{N}, (a, A) \Vdash \tau(p)$ for all $p \in \Psi$. This implies that $\mathfrak{N}, (a, A) \Vdash p$ for each $p \in \Phi$. Moreover, let $(b, B) \in \mathfrak{N}$ be such that $b \models A$ (one exists since by hypothesis $\mathfrak{N}, (a, A) \Vdash \Diamond p$). By the semantics of $\tau(p)$, it is evident that $\mathfrak{N}, (b, B) \Vdash \neg p$ and that there's a $(c, C) \in \mathfrak{N}$ such that $c \models B$ and $\mathfrak{N}, (c, C) \Vdash p \wedge \Diamond p$. But then from this it must be that a and c both contain exactly the same sentence letters, so $a = b$ which means that $a \models B$ as well, since $B \subseteq \mathcal{L}_0$ (as per the grammar restriction). However it cannot be that $\mathfrak{N}, (a, A) \Vdash \Diamond p$ since $\mathfrak{N}, (a, A) \Vdash \Box \neg p$, which follows from the assumption that $\mathfrak{N}, (a, A) \Vdash \tau(p)$. Thus it is impossible that $\mathfrak{N}, (a, A) \Vdash \tau[\Phi]$. $\lightning$ QED

The above argument illustrates that while EvIL semantics present themselves as similar to the traditional Kripke Semantics for modal logic, they aren't the same. The similarity particularly manifests itself when thinking about visualizations like Fig. 6. But if for some reason two world-pairs (a, A) and (b, B) make true the same sentence letters, then $a = b$. Moreover, for any $C \subseteq \mathcal{L}_0$, we have that $a \models C$ if and only if $b \models C$ in this case. This is crucial - even though EvIL is modal, it has no accessibility relations; the role of accessibility relations is instead picked up by basic sets of beliefs C. It is necessarily the case that for a basic set of beliefs C that $C \subseteq \mathcal{L}_0$, due to grammar restriction I decided upon to ensure the semantics of EvIL were well-defined. It is from this basic design choice that my unusual failure of compactness manifests itself. I shall return to considering the relationship of Kripke semantics and EvIL models in §2.2.4.

The failure of compactness, while a fairly basic result in the model theory of EvIL, is far reaching. As a consequence, there is no hope of achieving completeness using infinitary Lindenbaum construc-

tions that are typically employed in modal logic, such as is done in (Blackburn et al., 2001, chapter 4), for instance. Hence the completeness theorem for EvIL presented in §2.3 will necessarily have to be finite in nature.

2.2.3 Multiple Agents

In this section I turn to extending the semantics for EvIL from a single agent, as presented in §2.1.1, to accommodate multiple agents. This is primarily of interest since further results in EvIL, namely completeness, can naturally be abstracted beyond the single agent case. But I will freely admit that my EvIL intuitions are principally grounded in the single agent case – I recommend thinking about the multi-agent case as just a generalization of the single agent case.

The following provides the definition of the language of multi-agent EvIL:

Definition 2.2.10. *Define* $\mathcal{L}(\Phi, \mathcal{A})$ *by the following Backus-Naur grammar:*

$$\phi ::= p \in \Phi \mid \phi \to \psi \mid \bot \mid \Box_X \phi \mid \boxminus_X \phi \mid \boxplus_X \phi \mid \circlearrowleft_X$$

...where $X \in \mathcal{A}$ *and* $\mathcal{A}$ *is non-empty.*

As in the single agent case, multi-agent EvIL models are sets $\mathfrak{M} \subseteq \wp\Phi \times ((\wp\mathcal{L}_0(\Phi))^{\mathcal{A}})$ – that is, $\mathfrak{M}$ is a set of pairs of sets of proposition letters, and indexed sets of propositional formulae.

The semantic entailment relation for multi-agent EvIL is

$$(\Vdash) : \wp(\wp\Phi \times (\mathcal{A} \to \wp\mathcal{L}_0(\Phi))) \to \wp\Phi \times (\mathcal{A} \to \wp\mathcal{L}_0(\Phi)) \to \mathcal{L}(\Phi, \mathcal{A}) \to \mathsf{bool}.$$

The input/output behavior of $(\Vdash)$ is just as it was defined before in §2.1.1, the only difference in this setting is that instead of taking a pair as an input, where the second element is a set, it takes an indexed set.

I shall now provide a formal definition of the semantics for the multi-agent $(\Vdash)$: [10]

Definition 2.2.11.

$$\begin{aligned}
\mathfrak{M}, (a, A) \Vdash p &\iff p \in a \\
\mathfrak{M}, (a, A) \Vdash \phi \to \psi &\iff \mathfrak{M}, (a, A) \Vdash \phi \text{ implies } \mathfrak{M}, (a, A) \Vdash \psi \\
\mathfrak{M}, (a, A) \Vdash \bot &\iff False \\
\mathfrak{M}, (a, A) \Vdash \Box_X \phi &\iff \forall (b, B) \in \mathfrak{M}. (\forall \psi \in A_X. b \models \psi) \text{ implies } \mathfrak{M}, (b, B) \Vdash \phi \\
\mathfrak{M}, (a, A) \Vdash \boxminus_X \phi &\iff \forall (b, B) \in \mathfrak{M}. a = b \text{ and } B_X \subseteq A_X \text{ implies } \mathfrak{M}, (b, B) \Vdash \phi \\
\mathfrak{M}, (a, A) \Vdash \boxplus_X \phi &\iff \forall (b, B) \in \mathfrak{M}. a = b \text{ and } B_X \supseteq A_X \text{ implies } \mathfrak{M}, (b, B) \Vdash \phi \\
\mathfrak{M}, (a, A) \Vdash \circlearrowleft_X &\iff \forall \psi \in A_X. a \models \psi
\end{aligned}$$

Just as in §2.1.1, Lemma 2.1.5 and Theorem 2.1.7 can be seen to obtain for the new generalized semantics. Furthermore, all of the validities mentioned in §2.1.3 and §2.2.1 hold, along with Theorem 2.2.7, where $\Box$, $\Diamond$, $\boxminus$, $\boxplus$, $\diamondminus$, $\diamondplus$ and $\circlearrowleft$ are all replaced with $\Box_X$, $\Diamond_X$, $\boxminus_X$, $\boxplus_X$, $\diamondminus_X$, $\diamondplus_X$ and $\circlearrowleft_X$ respectively, for any fixed $X \in \mathcal{A}$. Furthermore, compactness still fails, just as presented in §2.2.2.

[10] Where $X \in \mathcal{A}$, I use A_X to denote $A(X)$ provided that $A : \mathcal{A} \to \wp\mathcal{L}_0(\Phi)$

Finally, there is are two novel validities that these semantics give rise to:

$$\Vdash \Box_X \phi \rightarrow \Box_X \boxminus_Y \phi$$
$$\Vdash \Box_X \phi \rightarrow \Box_X \boxplus_Y \phi$$

This is just to say, that as the EvIL agent's deliberative process was opaque to her beliefs in the single agent case, as expressed by (2.1.7) and (2.1.8) in §2.1.3, in a similar fashion she cannot read anyone else's mind, nor anyone else hers.

I will admit that little use shall be made of EvIL generalized this way, since the single agent case is far more natural to think about for me. However, it will be of interest to the main theorems of EvIL, which I shall present in §2.3, that they be proved in the widest generality.

2.2.4 Kripke Structures

The language of EvIL is evidently modal, and in previous sections the semantics have largely suggested that there are clear connections to conventional Kripke semantics. In this section, I will demonstrate that every EvIL model corresponds to some highly structured Kripke model, with a minor modification on the standard definition. However, it will turn out that this correspondence is one way - the class of Kripke models for which EvIL is strongly complete do not, in general, possess corresponding EvIL models.

To elucidate my intuition for understanding EvIL models as Kripke models, I would like to return to the visualization technique for EvIL models I introduced in §1.8. This involved, roughly, thinking of the EvIL models as *posets* with arrows, as I first presented in Fig. 3. I have given additional examples in Figs. 7(a) and 7(b). In all of these depictions, the implicit relational structure of EvIL models is given visual expression. So it seems only natural to me that this graphically perceived structure could also find formal expression.

Following the modified semantics provided in §2.2.3, the developments this section will assume multiple agents.

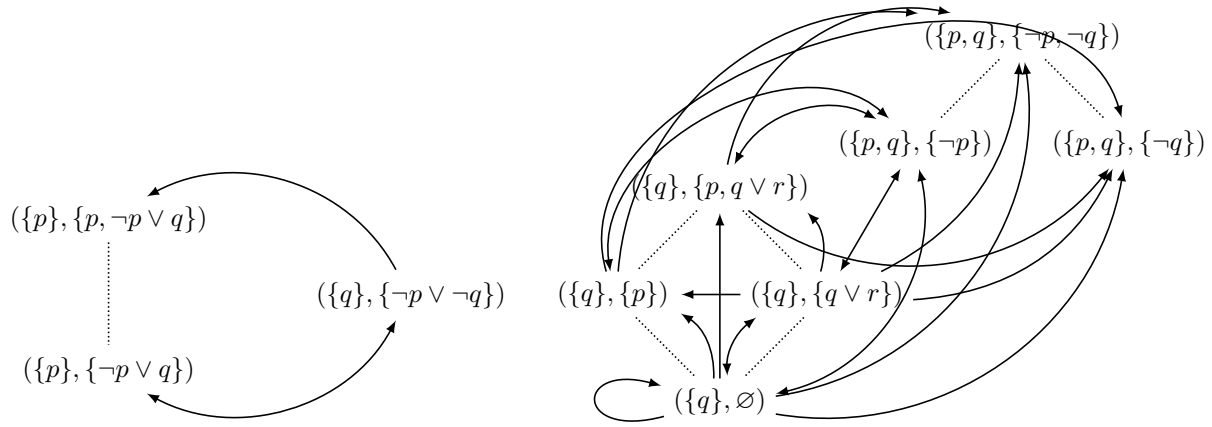
Definition 2.2.12. *Let Φ be a set of letters and let $\mathcal{A}$ be a set of agents. A **Kripke structure** is a state transition system $\mathbb{M} = \langle W^{\mathbb{M}}, R^{\mathbb{M}}, \sqsubseteq^{\mathbb{M}}, \sqsupseteq^{\mathbb{M}}, V^{\mathbb{M}}, P^{\mathbb{M}}_{\circlearrowleft} \rangle$ where[11]:*

- *$W^{\mathbb{M}}$ is a set of worlds*
- *$R^{\mathbb{M}} : \mathcal{A} \rightarrow \wp(W \times W)$, $\sqsubseteq^{\mathbb{M}}$: $\mathcal{A} \rightarrow \wp(W \times W)$, and $\sqsupseteq^{\mathbb{M}}$: $\mathcal{A} \rightarrow \wp(W \times W)$ are $\mathcal{A}$-indexed sets of relations*[12]
- *$V : \Phi \rightarrow \wp(W)$ is a predicate letter valuation*
- *$P_{\circlearrowleft} : \mathcal{A} \rightarrow \wp(W)$ are sets of worlds indexed by agents*

Let $\mathcal{K}_{\Phi,\mathcal{A},I}$ denote the class of Kripke structures for letters Φ, agents $\mathcal{A}$, and where $W \subseteq I$.

[11] Where the context is clear, I shall drop $\mathbb{M}$ from the superscripts I am employing.

[12] I will abbreviate $R(X)$, $\sqsubseteq(X)$ and $\sqsupseteq(X)$ as $R(X)$, $\sqsubseteq(X)$ and $\sqsupseteq(X)$ as R_X, $\sqsubseteq_X$ and $\sqsupseteq_X$ respectively.



(a) A fairly simple example

(b) A more complex example

Figure 7: EvIL model visualizations

Kripke semantics given by $(\Vdash) : \mathcal{K}_{\Phi,\mathcal{A},I} \to I \to \mathsf{bool}$ for these models are defined recursively as usual, granted the exceptional behavior of $P_{\circlearrowleft}$.

Definition 2.2.13. *Let $\mathbb{M}$ be in the class $\mathcal{K}_{\Phi,\mathcal{A},I}$*

$$\begin{aligned}
\mathbb{M}, w \Vdash p &\iff w \in V^{\mathbb{M}}(p) \\
\mathbb{M}, w \Vdash \phi \to \psi &\iff \mathbb{M}, w \Vdash \phi \textit{ implies } \mathbb{M}, w \Vdash \psi \\
\mathbb{M}, w \Vdash \bot &\iff \textit{False} \\
\mathbb{M}, w \Vdash \Box_X \phi &\iff \forall v \in W^{\mathbb{M}}. w R_X^{\mathbb{M}} v \textit{ implies } \mathbb{M}, v \Vdash \phi \\
\mathbb{M}, w \Vdash \boxminus_X \phi &\iff \forall v \in W^{\mathbb{M}}. w \sqsupseteq_X^{\mathbb{M}} v \textit{ implies } \mathbb{M}, v \Vdash \phi \\
\mathbb{M}, w \Vdash \boxplus_X \phi &\iff \forall v \in W^{\mathbb{M}}. w \sqsubseteq_X^{\mathbb{M}} v \textit{ implies } \mathbb{M}, v \Vdash \phi \\
\mathbb{M}, w \Vdash \circlearrowleft_X &\iff w \in P_{\circlearrowleft}^{\mathbb{M}}(X)
\end{aligned}$$

Kripke structures can be observe to have a lot less structure than EvIL models. However, EvIL models can be understood as Kripke structures in disguise. To illustrate this, observe the following lemma:

Definition 2.2.14 ($\mho^{\mathfrak{M}}$ Translation)**.** *Let $\mathfrak{M}$ be an* EvIL *model. Define $\mho^{\mathfrak{M}} := \langle \mathfrak{M}, R^{\mathfrak{M}}, \sqsubseteq^{\mathfrak{M}}, \sqsupseteq^{\mathfrak{M}}, V^{\mathfrak{M}}, P_{\circlearrowleft}^{\mathfrak{M}} \rangle$, where*

- $(a, A) R_X^{\mathfrak{M}} (b, B) \iff \forall \psi \in A_X. b \models \psi$
- $(a, A) \sqsupseteq_X^{\mathfrak{M}} (b, B) \iff a = b$ *and* $A_X \supseteq B_X$
- $(a, A) \sqsubseteq_X^{\mathfrak{M}} (b, B) \iff a = b$ *and* $A_X \subseteq B_X$
- $(a, A) \in P_{\circlearrowleft}^{\mathfrak{M}}(X) \iff \forall \psi \in A_X. a \models \psi$

Lemma 2.2.15. *For all $\mathfrak{M}$ and all $(a, A) \in \mathfrak{M}$, $\mathfrak{M}, (a, A) \Vdash \phi$ if and only if $\mho^{\mathfrak{M}}, (a, A) \Vdash \phi$*

Proof. This follows from a straightforward induction on ϕ. QED

The following summarizes the structural properties of EvIL models, when transformed into Kripke structures:

Proposition 2.2.16. *For any* EvIL *model* $\mathfrak{M}$, $\mho^{\mathfrak{M}}$ *has the following properties*[13]*:*

(I) $\sqsupseteq_X^{\mathfrak{M}}$ *is reflexive*

(II) $\sqsupseteq_X^{\mathfrak{M}}$ *is transitive*

(III) $\sqsupseteq_X^{\mathfrak{M}}$ *is anti-symmetric*

(IV) $w \sqsupseteq_X^{\mathfrak{M}} v$ *if and only if* $v \sqsubseteq_X^{\mathfrak{M}} w$

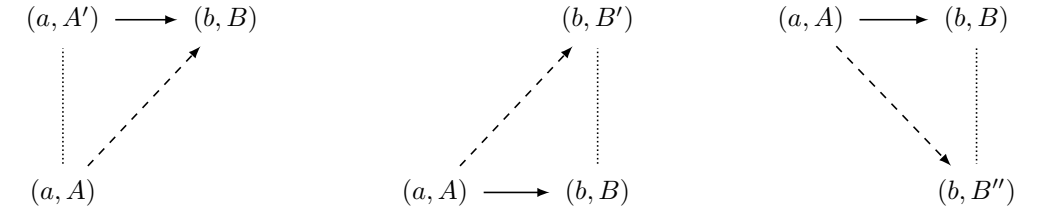
(V) *If* $w \sqsubseteq_X^{\mathfrak{M}} v$ *then* $w \in V(p)$ *if and only if* $v \in V(p)$

(VI) $R_X^{\mathfrak{M}} \circ \sqsubseteq_X^{\mathfrak{M}} \subseteq R_X^{\mathfrak{M}} \subseteq R_X^{\mathfrak{M}} \circ \sqsupseteq_X^{\mathfrak{M}}$

(VII) $\sqsubseteq_Y^{\mathfrak{M}} \circ R_X^{\mathfrak{M}} = R_X^{\mathfrak{M}} = \sqsupseteq_Y^{\mathfrak{M}} \circ R_X^{\mathfrak{M}}$

$(VIII)$ $w \in P_{\circlearrowleft}^{\mathfrak{M}}(X)$ *if and only if* $wR_X^{\mathfrak{M}}w$

... the situation in (VI) *can be visualized in 8(a), while* (VII) *can be split into Figs. 8(b) and 8(c).*



(a) $R_X^{\mathfrak{M}} \circ \sqsubseteq_X^{\mathfrak{M}} \subseteq R_X^{\mathfrak{M}}$ and $R_X^{\mathfrak{M}} \subseteq R_X^{\mathfrak{M}} \circ \sqsupseteq_X^{\mathfrak{M}}$

(b) $\sqsubseteq_Y^{\mathfrak{M}} \circ R_X^{\mathfrak{M}} \subseteq R_X^{\mathfrak{M}}$ and $R_X^{\mathfrak{M}} \subseteq \sqsupseteq_Y^{\mathfrak{M}} \circ R_X^{\mathfrak{M}}$

(c) $\sqsupseteq_Y^{\mathfrak{M}} \circ R_X^{\mathfrak{M}} \subseteq R_X^{\mathfrak{M}}$ and $R_X^{\mathfrak{M}} \subseteq \sqsubseteq_Y^{\mathfrak{M}} \circ R_X^{\mathfrak{M}}$

Figure 8: Visualizations of the relationships in Proposition 2.2.16

Proof. Everything except (VI) follows directly from the definitions - so I shall only demonstrate $R_X^{\mathfrak{M}} \subseteq R_X^{\mathfrak{M}} \circ \sqsupseteq_X^{\mathfrak{M}}$.

Suppose that $(a, A) R_X^{\mathfrak{M}} (b, B)$, then evidently $\forall \psi \in A_X . b \models \psi$. Now assume that $(a, A) \sqsupseteq_X^{\mathfrak{M}} (c, C)$. Then we know that $A_X \supseteq C_X$. But then $\forall \psi \in C_X . b \models \psi$, so $(c, C) R_X^{\mathfrak{M}} (b, B)$, which suffices to show the claim. QED

Definition 2.2.17. *A Kripke structure is called* EvIL *if it makes true the above properties* (I) *through* $(VIII)$ *(with the exception of* (III)*, which is optional).*

These properties are definitive - as we shall demonstrate, EvIL is sound and weakly complete for EvIL models.

[13] Note that in this we have that $\{w, v\} \subseteq \wp\Phi \times \wp\mathcal{L}_0$ in the subsequent discussion

The Kripke semantics also serve to provide proper intuition behind EvIL models. I think of the defined relations given as follows:

- If $xR_X^{\mathfrak{M}}y$, then at world x the agent X can imagine y is true, since y is compatible with what the agent believes
- If $x \sqsubseteq_X^{\mathfrak{M}} y$, then at world x, agent X's assumptions (or the experiences they are taking under consideration) are contained in her evidence at y

Given this perspective, the proof of (VI) can be understood in the following way - if the agent assumes fewer things, more things are imaginable, since it's easier for a world to be incompatible with an agent's evidence.

In fact, in light of Theorem 2.1.7, the above follows from an underlying, order theoretic relationship present in model theory. For a given Kripke structure $\mathbb{M}$, define two operators $Mod^{\mathbb{M}} : \wp\mathcal{L}(\Phi, \mathcal{A}) \to \wp(W^{\mathbb{M}})$ and $Th^{\mathbb{M}} : \wp(W^{\mathbb{M}}) \to \wp\mathcal{L}(\Phi, \mathcal{A})$

$$Mod^{\mathbb{M}}(\Delta) = \{x{\in}W \mid \forall\psi{\in}\Delta.\mathbb{M}, x{\Vdash}\psi\}$$
$$Th^{\mathbb{M}}(\nabla) = \{\psi{\in}\mathcal{L}(\Phi, \mathcal{A}) \mid \forall x{\in}\nabla.\mathbb{M}, x{\Vdash}\psi\}$$

We then have, for any $\Delta \in \wp\mathcal{L}(\Phi, \mathcal{A})$ and $\nabla \in \wp(W^{\mathbb{M}})$:

$$\nabla \subseteq Mod^{\mathbb{M}}(\Delta) \text{ if and only if } \Delta \subseteq Th^{\mathbb{M}}(\nabla)$$

...hence we these two operations form what is refered an *antitone Galois connection*, between the lattice $\wp(W^{\mathbb{M}})$ and the lattice $\wp\mathcal{L}(\Phi, \mathcal{A})$. It follows from the theory of Galois connections (Roman, 2008, chapter 3) that the following two properties:

- If $\nabla \supseteq \nabla'$ then $Th^{\mathbb{M}}(\nabla) \subseteq Th^{\mathbb{M}}(\nabla')$
- If $\Delta \supseteq \Delta'$ then $Mod^{\mathbb{M}}(\Delta) \subseteq Mod^{\mathbb{M}}(\Delta')$

We can see that $((VI))$ follows from the second of these two properties. To see this, assume that $(a, A) \sqsupseteq_X^{\mathfrak{M}} (b, B)$, hence $A_X \supseteq B_X$ and thus $Mod^{\mathfrak{M}}(A_X) \subseteq Mod^{\mathfrak{M}}(B_X)$. But it follows from semantics of EvIL we have that $(c, C) \in Mod^{\mathfrak{M}}(A_X)$ if and only if $(a, A)R_X^{\mathfrak{M}}(c, C)$, and likewise for B_X. Hence if $(a, A)R_X^{\mathfrak{M}}(c, C)$ then $(b, B)R_X^{\mathfrak{M}}(c, C)$.

2.3 EviL Completeness

2.3.1 Axiom Systems

In Table 1, I have provided a Hilbert-style axiom system for EvIL. In addition to giving each axiom, I have also provided my own philosophical reading of what each axiom says. One unusual feature of this logic is that it is not *normal*, that is it is not closed under variable substitution.

This logic makes true a variety of relationships between the various modalities, which are given in the following lemma:

(1)	$\vdash \phi \to \psi \to \phi$	
(2)	$\vdash (\phi \to \psi \to \chi) \to (\phi \to \psi) \to \phi \to \chi$	*Axioms for basic propositional logic*
(3)	$\vdash (\neg\phi \to \neg\psi) \to \psi \to \phi$	
(4)	$\vdash \boxplus_X \phi \to \phi$	*If ϕ holds under any further evidence X considers, then ϕ holds simpliciter, since considering no additional evidence is trivially considering further evidence*
(5)	$\vdash \boxplus_X \phi \to \boxplus_X \boxplus_X \phi$	*If ϕ holds under any further evidence X considers, then ϕ holds whenever X considers even further evidence beyond that*
(6)	$\vdash p \to \boxminus_X p$	*Changing one's mind does not bear on matters of fact*
(7)	$\vdash p \to \boxplus_X p$	
(8)	$\vdash \Diamond_X \phi \to \boxminus_X \Diamond_X \phi$	*The more evidence X discards, the freer her imagination can run*
(9)	$\vdash \Box_X \phi \to \Box_X \boxminus_Y \phi$	*If X believes a proposition, she believes it regardless of what anyone else thinks*
(10)	$\vdash \Box_X \phi \to \Box_X \boxplus_Y \phi$	
(11)	$\vdash \circlearrowleft_X \to \Box_X \phi \to \phi$	*If X's premises are sound, then her logical conclusion are correct*
(12)	$\vdash \circlearrowleft_X \to \boxminus_X \circlearrowleft_X$	*If X's premises are sound then any subset of will be sound as well*
(13)	$\vdash \phi \to \boxminus_X \oplus_X \phi$	*Embracing evidence is the inverse of discarding evidence*
(14)	$\vdash \phi \to \boxplus_X \ominus_X \phi$	
(15)	$\vdash \Box_X(\phi \to \psi) \to \Box_X \phi \to \Box_X \psi$	
(16)	$\vdash \boxminus_X(\phi \to \psi) \to \boxminus_X \phi \to \boxminus_X \psi$	*Variations on axiom K*
(17)	$\vdash \boxplus_X(\phi \to \psi) \to \boxplus_X \phi \to \boxplus_X \psi$	
(I)	$\dfrac{\vdash \phi \to \psi \qquad \vdash \phi}{\vdash \psi}$	*Modus Ponens*
(II)	$\dfrac{\vdash \phi}{\vdash \Box_X \phi}$	*Variations on necessitation*
(III)	$\dfrac{\vdash \phi}{\vdash \boxminus_X \phi}$	
(IV)	$\dfrac{\vdash \phi}{\vdash \boxplus_X \phi}$	

Table 1: A Hilbert style axiom system for EvIL

(1)	$\vdash \phi \to \psi \to \phi$
(2)	$\vdash (\phi \to \psi \to \chi) \to (\phi \to \psi) \to \phi \to \chi$
(3)	$\vdash (\neg\phi \to \neg\psi) \to \psi \to \phi$
(4)	$\vdash \boxminus_X \phi \to \phi$
(5)	$\vdash \boxminus_X \phi \to \boxminus_X \boxminus_X \phi$
(6)	$\vdash p \to \boxminus_X p$
(7)	$\vdash \neg p \to \boxminus_X \neg p$
(8)	$\vdash \Diamond_X \phi \to \boxminus_X \Diamond_X \phi$
(9)	$\vdash \Box_X \phi \to \Box_X \boxminus_Y \phi$
(10)	$\vdash \phi \to \boxminus_X (\circlearrowleft_X \to \Diamond_X \phi)$
(11)	$\vdash \circlearrowleft_X \to \boxminus_X \circlearrowleft_X$
(12)	$\vdash \Box_X(\phi \to \psi) \to \Box_X \phi \to \Box_X \psi$
(13)	$\vdash \boxminus_X(\phi \to \psi) \to \boxminus_X \phi \to \boxminus_X \psi$
(I)	$\dfrac{\vdash \phi \to \psi \qquad \vdash \phi}{\vdash \psi}$
(II)	$\dfrac{\vdash \phi}{\vdash \Box_X \phi}$
(III)	$\dfrac{\vdash \phi}{\vdash \boxminus_X \phi}$

(1)	$\vdash \phi \to \psi \to \phi$
(2)	$\vdash (\phi \to \psi \to \chi) \to (\phi \to \psi) \to \phi \to \chi$
(3)	$\vdash (\neg\phi \to \neg\psi) \to \psi \to \phi$
(4)	$\vdash \boxplus_X \phi \to \phi$
(5)	$\vdash \boxplus_X \phi \to \boxplus_X \boxplus_X \phi$
(6)	$\vdash p \to \boxplus_X p$
(7)	$\vdash \neg p \to \boxplus_X \neg p$
(8)	$\vdash \Box_X \phi \to \boxplus_X \Box_X \phi$
(9)	$\vdash \Box_X \phi \to \Box_X \boxplus_Y \phi$
(10)	$\vdash \phi \to \boxplus_X (\circlearrowleft_X \to \Diamond_X \phi)$
(11)	$\vdash \neg\, \circlearrowleft_X \to \boxplus_X \neg\, \circlearrowleft_X$
(12)	$\vdash \Box_X(\phi \to \psi) \to \Box_X \phi \to \Box_X \psi$
(13)	$\vdash \boxplus_X(\phi \to \psi) \to \boxplus_X \phi \to \boxplus_X \psi$
(I)	$\dfrac{\vdash \phi \to \psi \qquad \vdash \phi}{\vdash \psi}$
(II)	$\dfrac{\vdash \phi}{\vdash \Box_X \phi}$
(III)	$\dfrac{\vdash \phi}{\vdash \boxplus_X \phi}$

Table 2: Axiom system EvIL$^{\boxminus}$ and EvIL$^{\boxplus}$ respectively

Lemma 2.3.1. *We have the following provable equivalences:*

$$\vdash \Box_X \phi \leftrightarrow \boxminus_X \Box_X \phi \qquad \vdash \Box_X \phi \leftrightarrow \Box_X \boxminus_Y \phi \qquad \vdash \Box_X \phi \leftrightarrow \Box_X \boxplus_Y \phi$$
$$\vdash \boxminus_X \phi \leftrightarrow \boxminus_X \boxminus_X \phi \qquad \vdash \boxplus_X \phi \leftrightarrow \boxplus_X \boxplus_X \phi \qquad \vdash \circlearrowleft_X \leftrightarrow \boxminus_X \circlearrowleft_X$$

In addition to the main system presented above, it can be understood to contain two subsystems, corresponding to two fragments of the main grammar:

Definition 2.3.2. *Define* $\mathcal{L}^{\boxminus}(\Phi, \mathcal{A})$ *as the fragment:*

$$\phi ::= p \in \Phi \mid \phi \to \psi \mid \bot \mid \Box_X \phi \mid \boxminus_X \phi \mid \circlearrowleft_X$$

And define $\mathcal{L}^{\boxplus}(\Phi, \mathcal{A})$ *as the fragment:*

$$\phi ::= p \in \Phi \mid \phi \to \psi \mid \bot \mid \Box_X \phi \mid \boxplus_X \phi \mid \circlearrowleft_X$$

Table 2 gives the axioms systems for these two fragments. For now, we shall observe that EvIL extends EvIL$^{\boxminus}$ and EvIL$^{\boxplus}$. In §2.3.5 we shall make this precise.

From the definitions so far the following can be seen to hold:

Lemma 2.3.3 (Soundness)**.** *If* $\vdash \phi$ *then for any model* $\mathfrak{M}$ *and any* $(a, A) \in \mathfrak{M}$ *we have that* $\mathfrak{M}, (a, A) \models \phi$

The proof of the converse, that is *completeness*, proceeds by a three stage construction:

- The first step is to construct a Kripke model $\dagger^{\phi}$ consisting of finite maximally consistent sets of formulae related to ϕ where $\dagger^{\phi}, w \nVdash \phi$ for some world $w \in W^{\dagger^{\phi}}$. This model will be shown to make true nine properties.
- The second step is to construct a model $\lightning^{\dagger^{\phi}}$ which is bisimular to $\dagger^{\phi}$. This model also makes true these nine properties as well as an additional tenth property.
- The final third step is to construct an EvIL model $\bigstar_{\phi}^{\lightning^{\dagger^{\phi}}}$. I shall then show that for each $w \in W^{\lightning^{\dagger^{\phi}}}$ there is a corresponding $(a, A) \in \bigstar_{\phi}^{\lightning^{\dagger^{\phi}}}$ such that $\lightning^{\dagger^{\phi}}, w \Vdash \psi$ if and only if $\bigstar_{\phi}^{\lightning^{\dagger^{\phi}}}, (a, A) \models \psi$ for all subformulae ψ of ϕ.

These three steps together suffice to prove completeness. I shall now proceed to demonstrate these constructions.

2.3.2 Subformula Model Construction

In this section we provide definitions and lemmas related to the subformula construction $\dagger^{\phi}$. I consciously imitate Boolos (1995) in my approach, as well as the "Fischer-Ladner Closure" used in the completeness theorem of PDL (Blackburn et al., 2001).

Definition 2.3.4.

$$\sim \phi := \begin{cases} \psi & \text{if } \phi = \neg\psi \\ \neg\phi & \text{o/w} \end{cases} \qquad \boxdot_X \phi := \begin{cases} \phi & \text{if } \phi = \boxminus_X \psi \\ \boxminus_X \phi & \text{o/w} \end{cases} \qquad \boxtimes_X \phi := \begin{cases} \phi & \text{if } \phi = \boxplus_X \psi \\ \boxplus_X \phi & \text{o/w} \end{cases}$$

Lemma 2.3.5. *By Lemma 2.3.1 we have*

$$\vdash \sim \phi \leftrightarrow \neg\phi \qquad \vdash \boxdot_X \phi \leftrightarrow \boxminus_X \phi \qquad \vdash \boxtimes_X \phi \leftrightarrow \boxplus_X \phi$$

. . . moreover . . .

$$\boxdot_X \phi = \boxdot_X \boxdot_X \phi \qquad \boxtimes_X \phi = \boxtimes_X \boxtimes_X \phi$$

Definition 2.3.6. *Let $\delta(\phi) \subseteq \mathcal{A}$ be the set of agents that occur in ϕ*[14]

Definition 2.3.7. *Define $\Sigma(\Delta, \phi)$ using primitive recursion as follows:*

$$\begin{aligned}
&\Sigma(\Delta, p) := \{p, \neg p, \bot, \neg\bot\} \cup \bigcup\{\{\boxminus_X p, \neg \boxminus_X p, \boxplus_X p, \neg \boxplus_X p\} \mid X \in \Delta\} \\
&\Sigma(\Delta, \bot) := \{\bot, \neg\bot\} \\
&\Sigma(\Delta, \circlearrowleft_X) := \{\circlearrowleft_X, \neg \circlearrowleft_X, \boxminus_X \circlearrowleft_X, \neg\boxminus_X \circlearrowleft_X, \bot, \neg\bot\} \\
&\Sigma(\Delta, \phi \to \psi) := \{\phi \to \psi, \neg(\phi \to \psi)\} \cup \Sigma(\Delta, \phi) \cup \Sigma(\Delta, \psi) \\
&\Sigma(\Delta, \Box_X \phi) := \{\Box_X \phi, \neg \Box_X \phi, \boxplus_X \Box_X \phi, \neg \boxplus_X \Box_X \phi\} \\
&\qquad\qquad \cup \bigcup\{\{\Box_X \boxdot_Y \phi, \neg \Box_X \boxdot_Y \phi, \Box_X \boxtimes_Y \phi, \neg \Box_X \boxtimes_Y \phi, \boxdot_Y \phi, \neg\boxdot_Y \phi, \boxtimes_Y \phi, \neg\boxtimes_Y \phi\} \mid Y \in \Delta\} \\
&\qquad\qquad \cup \Sigma(\Delta, \phi) \\
&\Sigma(\Delta, \boxminus_X \phi) := \{\boxminus_X \phi, \neg \boxminus_X \phi\} \cup \Sigma(\Delta, \phi) \\
&\Sigma(\Delta, \boxplus_X \phi) := \{\boxplus_X \phi, \neg \boxplus_X \phi\} \cup \Sigma(\Delta, \phi)
\end{aligned}$$

[14] In natural language, we read $\delta(\phi)$ as "the dudes mentioned by ϕ."

Lemma 2.3.8. *$\Sigma(\delta(\phi),\phi)$ is finite. Moreover, we have the following:*

- *If $\psi \in \Sigma(\delta(\phi),\phi)$ then $\sim\psi \in \Sigma(\delta(\phi),\phi)$*
- *If $\psi \in \Sigma(\delta(\phi),\phi)$ and χ is a subformula of ψ, then $\chi \in \Sigma(\delta(\phi),\phi)$*
- *If $\boxminus_X\phi \in \Sigma(\delta(\phi),\phi)$ then $\boxdot_X\phi \in \Sigma(\delta(\phi),\phi)$*
- *If $\boxplus_X\phi \in \Sigma(\delta(\phi),\phi)$ then $\boxtimes_X\phi \in \Sigma(\delta(\phi),\phi)$*

Definition 2.3.9. *Let $At(\Psi)$ denote the maximally consistent subsets of Ψ*

Lemma 2.3.10 (Lindenbaum Lemma)**.** *If $\Gamma \nvdash \phi$ and $\Gamma \subseteq \Sigma(\delta(\phi),\phi)$, then there is a $\Gamma' \in At(\Sigma(\delta(\phi),\phi))$ such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash \phi$*

Definition 2.3.11. *Define $\mathsf{t}^\phi := \langle W^{\mathsf{t}^\phi}, V^{\mathsf{t}^\phi}, P_X^{\mathsf{t}^\phi}, R_{\Box_X}^{\mathsf{t}^\phi}, R_{\boxminus_X}^{\mathsf{t}^\phi}, R_{\boxplus_X}^{\mathsf{t}^\phi}\rangle$ where:*

$W^{\mathsf{t}^\phi} := At(\Sigma(\delta(\phi),\phi))$
$V^{\mathsf{t}^\phi}(p) := \{w \in W^{\mathsf{t}^\phi} \mid p \in w\}$
$P_X^{\mathsf{t}^\phi} := \{w \in W^{\mathsf{t}^\phi} \mid \circlearrowleft_X \in w\} \cup \{w \in W^{\mathsf{t}^\phi} \mid X \notin \delta(A)\}$
$R_{\Box_X}^{\mathsf{t}^\phi} := \{(w,v) \in W^{\mathsf{t}^\phi} \times W^{\mathsf{t}^\phi} \mid \{\psi \mid \Box_X\psi \in w\} \subseteq v\}$
$R_{\boxminus_X}^{\mathsf{t}^\phi} := \{(w,v) \in W^{\mathsf{t}^\phi} \times W^{\mathsf{t}^\phi} \mid \bigcup\{\{\psi, \boxdot_X\psi\} \mid \boxdot_X\psi \in w\} \subseteq v \wedge \bigcup\{\{\psi, \boxtimes_X\psi\} \mid \boxtimes_X\psi \in v\} \subseteq w\}$
$R_{\boxplus_X}^{\mathsf{t}^\phi} := \{(v,w) \in W^{\mathsf{t}^\phi} \times W^{\mathsf{t}^\phi} \mid \bigcup\{\{\psi, \boxdot_X\psi\} \mid \boxdot_X\psi \in w\} \subseteq v \wedge \bigcup\{\{\psi, \boxtimes_X\psi\} \mid \boxtimes_X\psi \in v\} \subseteq w\}$

Lemma 2.3.12 (Truth Lemma)**.** *For any subformula $\psi \in \Sigma(\delta(\phi),\phi)$ and any $w \in W^{\mathsf{t}^\phi}$, we have that $\mathsf{t}^\phi, w \Vdash \psi$ if and only if $\psi \in w$*

Proof. The proof proceeds by induction on ψ. Most of the steps are routine, with the exception of the right to left directions for the boxes.

I shall demonstrate the right to left direction for $\boxminus_X$. Assume that $\boxminus_X\psi \notin w$, then $w \nvdash \boxminus_X\psi$. By Lemma 2.3.5 this is true if and only if $w \nvdash \boxdot_X\psi$. Now abbreviate:

$$A := \bigcup\{\{\chi, \boxdot_X\chi\} \mid \boxdot_X\chi \in w\}$$
$$B := \{\sim \boxtimes_X\chi \mid \boxtimes_X\chi \in \Sigma(\delta(\phi),\phi) \wedge \sim\chi \in w\}$$

Now suppose towards a contradiction that $\{\sim\psi\} \cup A \cup B \vdash \bot$. Then $A \cup B \vdash \psi$, and furthermore by Lemma 2.3.5 and rule (III) from the axioms we have that $\boxdot_X A \cup \boxdot_X B \vdash \boxdot_X\psi$.[15] But then let

$$A' := \{\boxdot_X\chi \mid \boxdot_X\chi \in w\}$$
$$B' := \{\sim\chi \mid \sim\chi \in w\}$$

Since $\boxdot_X\boxdot_X\chi = \boxdot_X\chi$ by Lemma 2.3.5, we have $A' = \boxdot_X A$. Moreover, by Lemma 2.3.5, axiom 13, and classical logic we can see that

$$\vdash \sim\chi \to \boxdot_X \sim \boxtimes_X\psi$$

[15] Here $\boxdot_X S$ is shorthand for $\{\boxdot_X\chi \mid \chi \in S\}$.

Thus for every $\beta \in \boxtimes_X B$ we have that $B' \vdash \beta$. Hence by n applications of the Cut rule we can arrive at

$$A' \cup B' \vdash \boxtimes_X \chi$$

However, evidently $A' \cup B' \subseteq w$, hence $w \vdash \boxtimes_X \psi$, which contradicts what has been stipulated. $\lightning$

Hence it must be that $\{\sim \psi\} \cup A \cup B \nvdash \bot$. In addition, from the fact that $w \subseteq \Sigma(\delta(\phi), \phi)$ with Lemma 2.3.8 and the hypothesis we have that $\{\sim \psi\} \cup A \cup B \subseteq \Sigma(\delta(\phi), \phi)$. Hence by the Lindenbaum Lemma we have that there is some $v \in At(\Sigma(\delta(\phi), \phi))$ such that $\{\sim \psi\} \cup A \cup B \subseteq v$. By the inductive hypothesis we have that $\mathsf{t}^\phi, v \nVdash \psi$.

To complete the argument, we have to show that $wR^{\mathsf{t}^\phi}_{\boxminus_X} v$. Since $A \subseteq v$ we just need to check that $\bigcup\{\{\psi, \boxtimes_X \psi\} \mid \boxtimes_X \psi \in v\} \subseteq w$. Suppose that $\boxtimes_X \psi \in v$ but $\psi \notin w$. Since w is maximally consistent we have then that $\notin \psi \in w$. Thus $\sim \boxtimes_X \psi \in v$, which contradicts that v is consistent. $\lightning$ Now suppose that $\boxtimes_X \psi \in v$ but $\boxtimes_X \psi \notin w$, hence $\sim \boxtimes_X \psi \in w$ and thus $\sim \boxtimes_X \boxtimes_X \psi \in v$. However we know from Lemma 2.3.5 that $\boxtimes_X \boxtimes_X \psi = \boxtimes_X \psi$, which once again implies that v is inconsistent. $\lightning$ QED

Lemma 2.3.13 (t^ϕ is Partly EviL)**.** *t^ϕ makes true the following properties:*

(1) $R^{\mathsf{t}^\phi}_{\boxminus_X} \subseteq W^{\mathsf{t}^\phi} \times W^{\mathsf{t}^\phi}$

(2) W^{t^ϕ} *is finite*

(3) For all $w \in W^{\mathsf{t}^\phi}$ *we have* $wR^{\mathsf{t}^\phi}_{\boxminus_X} w$

(4) If $wR^{\mathsf{t}^\phi}_{\boxminus_X} v$ *and* $vR^{\mathsf{t}^\phi}_{\boxminus_X} z$ *then* $vR^{\mathsf{t}^\phi}_{\boxminus_X} z$

(5) $R^{\mathsf{t}^\phi}_{\boxplus_X} = (R^{\mathsf{t}^\phi}_{\boxminus_X})^{-1}$

(6) If $wR^{\mathsf{t}^\phi}_{\boxminus_X} v$ *then* $w \in V^{\mathsf{t}^\phi}(p)$ *if and only if* $v \in V^{\mathsf{t}^\phi}(p)$

(7) If $wR^{\mathsf{t}^\phi}_{\boxminus_X} v$ *and* $vR^{\mathsf{t}^\phi}_{\Box_X} u$ *then* $vR^{\mathsf{t}^\phi}_{\Box_X} u$

(8) If $wR^{\mathsf{t}^\phi}_{\boxminus_X} v$ *then* $uR^{\mathsf{t}^\phi}_{\Box_X} w$ *if and only if* $uR^{\mathsf{t}^\phi}_{\Box_X} v$

(9) If $w \in P^{\mathsf{t}^\phi}_X$ *then* $wR^{\mathsf{t}^\phi}_{\Box_X} w$

... for all $\{X, Y\} \subseteq \mathcal{A}$*. Any model with the same modal similarity type as* t^ϕ *that makes the above true is said to be* ***partly*** **EviL**

Unfortunately, while t^ϕ is nearly what is necessary to derive completeness for my semantics, it is not perfect. Another stage of the construction is necessary.

2.3.3 Bisimulation

I first introduce a Backus-Naur form grammar for the Either type constructor, which may be viewed as a coproduct in category theory (in the category of Sets)[16]:

$$\mathsf{Either}\ a\ b ::= a_l \mid b_r$$

[16] Either is taken from the functional programming language `Haskell`

Definition 2.3.14. *Let $\mathbb{M}$ be a Kripke model, then define $\dashv^{\mathbb{M}}$ as a model*

$$\langle W^{\dashv^{\mathbb{M}}}, V^{\dashv^{\mathbb{M}}}, P_X^{\dashv^{\mathbb{M}}}, R_{\Box_X}^{\dashv^{\mathbb{M}}}, R_{\boxminus_X}^{\dashv^{\mathbb{M}}}, R_{\boxplus_X}^{\dashv^{\mathbb{M}}} \rangle$$

... where ...

$W^{\dashv^{\mathbb{M}}} := \bigcup\{\{w_l, w_r\} \mid w \in W^{\mathbb{M}}\}$
$V^{\dashv^{\mathbb{M}}}(p) := \bigcup\{\{w_l, w_r\} \mid w \in V^{\mathbb{M}}(p)\}$
$P_X^{\dashv^{\mathbb{M}}} := \bigcup\{\{w_l, w_r\} \mid w \in P_X^{\mathbb{M}}\}$
$R_{\Box_X}^{\dashv^{\mathbb{M}}} := \bigcup\{\{(w_l, v_r), (w_r, v_l)\} \mid wR_{\Box_X}^{\mathbb{M}} v \wedge w \notin P_X^{\mathbb{M}}\} \cup \bigcup\{\{w_l, w_r\} \times \{v_l, v_r\} \mid wR_{\Box_X}^{\mathbb{M}} v \wedge w \in P_X^{\mathbb{M}}\}$
$R_{\boxminus_X}^{\dashv^{\mathbb{M}}} := \bigcup\{\{(w_l, v_l), (w_r, v_r)\} \mid wR_{\boxminus_X}^{\mathbb{M}} v\}$
$R_{\boxplus_X}^{\dashv^{\mathbb{M}}} := \bigcup\{\{(w_l, v_l), (w_r, v_r)\} \mid wR_{\boxplus_X}^{\mathbb{M}} v\}$

Lemma 2.3.15. *For any Kripke model $\mathbb{M} = \langle W, V, P_X, R_{\Box_X}, R_{\boxminus_X}, R_{\boxplus_X} \rangle$, we have the following bisimulation Z between $\mathbb{M}$ and $\dashv^{\mathbb{M}}$:*

$$wZw_l \quad \& \quad wZw_r$$

Lemma 2.3.16. *If $\mathbb{M}$ is partly* EviL *then $\dashv^{\mathbb{M}}$ is partly* EviL *as well. It also makes true another, novel property:*

(10) If $wR_{\Box_X}^{\dashv^{\mathbb{M}}} w$ then $w \in P_X^{\dashv^{\mathbb{M}}}$

Any partly EviL *Kripke model that makes true this tenth property is said to be* ***completely* EviL**

2.3.4 Translation

In the subsequent discussion, it will be useful to exploit certain properties of *partly* EviL models. To this end we introduce the concept of a *column*.

Definition 2.3.17. *Let $\mathbb{M}$ be a partly* EviL *Kripke structure. I shall make the following definition:*

$$\lceil\!\lfloor w \rfloor\!\rceil^{\mathbb{M}} := \{v \mid w(R_{\boxminus_X}^{\mathbb{M}} \cup R_{\boxplus_X}^{\mathbb{M}})^* v\}$$

... where R^* is the reflexive transitive closure of R

Lemma 2.3.18 (Column Lemma)**.** *The following hold if $\mathbb{M}$ is partly* EviL*:*

(1) For all w we have $w \in \lceil\!\lfloor w \rfloor\!\rceil^{\mathbb{M}}$
(2) If $w \in \lceil\!\lfloor v \rfloor\!\rceil^{\mathbb{M}}$ then $\lceil\!\lfloor w \rfloor\!\rceil^{\mathbb{M}} = \lceil\!\lfloor v \rfloor\!\rceil^{\mathbb{M}}$
(3) If $wR_{\Box_X}^{\mathbb{M}} v$ then for all $u \in \lceil\!\lfloor v \rfloor\!\rceil^{\mathbb{M}}$ we have $wR_{\Box_X}^{\mathbb{M}} u$
(4) If $w \in \lceil\!\lfloor v \rfloor\!\rceil^{\mathbb{M}}$ then $w \in V^{\mathbb{M}}(p)$ if and only if $v \in V^{\mathbb{M}}(p)$ for all $p \in \Phi$

Definition 2.3.19. *Let $L(\phi) := \{p \in \Phi \mid p$ is a subformula of $\phi\}$*
Let $\Lambda^{\mathbb{M}} := \bigcup\{\{\{w\}, \lceil\!\lfloor w \rfloor\!\rceil^{\mathbb{M}}\} \mid w \in W^{\mathbb{M}}\}$
Let $\rho_\phi^{\mathbb{M}} : \Lambda^{\mathbb{M}} \rightarrowtail \Phi \setminus L(\phi)$ be an injection
Let $\theta_\phi^{\mathbb{M}} : W^{\mathbb{M}} \to \wp\Phi \times \wp(\mathcal{L}|_{\mathsf{Prop}}(\Phi))$ be defined such that:

$$\theta^{\mathbb{M}}_{\phi}(w) := (\{p \in L(\phi) \mid \mathbb{M}, w \Vdash p\} \cup \{\rho^{\mathbb{M}}_{\phi}(\ulcorner w \urcorner^{\mathbb{M}})\}, \lambda X.\{\neg\rho^{\mathbb{M}}_{\phi}(\ulcorner v \urcorner^{\mathbb{M}}) \mid \neg w R^{\mathbb{M}}_{\Box_X} v\} \cup \{\bot \rightarrow \rho^{\mathbb{M}}_{\phi}(\{v\}) \mid w R^{\mathbb{M}}_{\boxminus_X} v\})$$

Let $\maltese^{\mathbb{M}}_{\phi} := \theta^{\mathbb{M}}_{\phi}[W^{\mathbb{M}}]$

Lemma 2.3.20. *Let* $\mathbb{M}$ *be a completely* EvиL *Kripke structure. Then for any subformula* ψ *of* ϕ *and any* $w \in W^{\mathbb{M}}$*, we have* $\mathbb{M}, w \Vdash \psi$ *if and only if* $\maltese^{\mathbb{M}}_{\phi}, \theta^{\mathbb{M}}_{\phi}(w) \models \psi$

Proof. Apply induction. The only challenging cases involve the boxes, so we shall illustrate $\Box_X\psi$.

Assume that $\mathbb{M}, w \nVdash \Box_X\psi$, then there's some $v \in W^{\mathbb{M}}$ such that $wR^{\mathbb{M}}_{\Box_X} v$ $\mathbb{M}, v \nVdash \psi$. Let $(a, A) := \theta^{\mathbb{M}}(w)$ and $(b, B) := \theta^{\mathbb{M}}(v)_{\phi}$. By the inductive hypothesis it suffices to show that $\maltese^{\mathbb{M}}_{\phi}, (b, B) \models A_X$. But the only things in A_X are tautologies or formulae of the form $\neg\rho^{\mathbb{M}}_{\phi}(\ulcorner u \urcorner^{\mathbb{M}})$ where $\neg w R^{\mathbb{M}}_{\Box_X} u$. But then Lemma 2.3.18 it can't be that $\neg\rho^{\mathbb{M}}_{\phi}(\ulcorner v \urcorner^{\mathbb{M}}) \in A_X$, and this suffices.

Now assume that $\maltese^{\mathbb{M}}_{\phi}, (a, A) \nvDash \Box_X\psi$ where $(a, A) = \theta^{\mathbb{M}}(w)$, so there must be some $v \in W^{\mathbb{M}}$ such that $\maltese^{\mathbb{M}}_{\phi}, (b, B) \nvDash \psi$ where $(b, B) = \theta^{\mathbb{M}}(v)$ and $\maltese^{\mathbb{M}}_{\phi}, (b, B) \models A_X$. By the inductive hypothesis it suffices to show that $wR^{\mathbb{M}}_{\Box_X} v$, but this must be the case for otherwise $\neg\rho^{\mathbb{M}}_{\phi}(\ulcorner v \urcorner^{\mathbb{M}}) \in A_X$ and then it couldn't be that $\maltese^{\mathbb{M}}_{\phi}, (b, B) \models A_X$ since $\rho^{\mathbb{M}}_{\phi}(\ulcorner v \urcorner^{\mathbb{M}}) \in B$.

The inductive steps for the other boxes follow by similar reasoning. QED

2.3.5 Completeness

Theorem 2.3.21. *If* $\nvdash \phi$ *then there is some model* $\mathfrak{M}$ *and some* $(a, A) \in \mathfrak{M}$ *such that* $\mathfrak{M}, (a, A) \nvDash \phi$

Proof. Assume $\nvdash \phi$, then by Lemmas 2.3.12 and 2.3.13 we have some partly EvиL Kripke structure and world such that $\mathbb{A}, a \nVdash \phi$. By Lemma 2.3.15 we have that there is a completely EvиL Kripke structure $\mathbb{B}$ such that $\mathbb{A} \underline{\leftrightarrow} \mathbb{B}$, thus there is some world b such that $\mathbb{B}, b \nVdash \phi$. Finally by Lemma 2.3.20 we have that there's a model $\mathfrak{C}$ in EvиL semantics and a pair $(c, C) \in \mathfrak{C}$ such that $\mathfrak{C}, (c, C) \nvDash \phi$. QED

2.3.6 Conservativity, Decidability & Complexity

In this section, we discuss basic computability results for EvиL. I demonstrate that all of the fragments of EvиL are decidable, and establish a lower bound on the computational complexity.

I shall first prove the following lemma:

Lemma 2.3.22. EvиL, EvиL$^{\boxminus}$ *and* EvиL$^{\boxplus}$ *with a single agent are all conservative extensions of the basic modal logic with just axiom K. That is, if* $\nvdash_K \phi$ *then* $\nvdash_{\text{EvиL}} \phi$ *and similarly for the fragments* EvиL$^{\boxminus}$ *and* EvиL$^{\boxplus}$.

EvиL *with* $m > n$ *agents is a conservative extension of* EvиL *with* n *agents, and likewise for the fragments* EvиL$^{\boxminus}$ *and* EvиL$^{\boxplus}$

Proof. Assume that $\nvdash_K \phi$, then we know from modal logic that there's a finite Kripke Structure $\mathbb{M} := \langle W, V, R\rangle$ such and a world $w \in W$ such that $\mathbb{M}, w \nvdash \phi$. Now extend $\mathbb{M}$ to $\mathbb{M}' := \langle W, V, P, R_\Box, R_\boxminus, R_\boxplus\rangle$ where

- $P := \{(v, v) \mid vRv\}$
- $R_\boxminus := R_\boxplus := \{(w, w) \mid w \in W\}$

...this model is trivially completely EvIL. Moreover we know that $\mathbb{M}$ is an elementary submodel of $\mathbb{M}'$, so $\mathbb{M}', w \nvdash \phi$. Hence by the Lemma 2.3.20 we have a model $\mathfrak{M}$ and $(a, A) \in \mathfrak{M}$ such that $\mathfrak{M}, (a, A) \nvDash \phi$; so by soundness for EvIL we have the desired result.

Similarly, if we $\nvdash_{\text{EvIL}_\mathcal{A}} \phi$ then by completeness can find a witnessing $\mathfrak{M}$ and $(a, A) \in \mathfrak{M}$ such that $\mathfrak{M}, (a, A) \nvDash \phi$. But then we can embed $\mathfrak{M}$ into $\mathfrak{M}'$ for agents $\mathcal{B} \supseteq \mathcal{A}$ where $\mathfrak{M}' := \{(a, A') \mid (a, A) \in \mathfrak{M}\}$ and

$$A'_X := \begin{cases} A_X & X \in \mathcal{A} \\ \varnothing & X \notin \mathcal{A} \end{cases}$$

QED

By similar arguments, EvIL is a conservative extension of EvIL$^\boxminus$ and EvIL$^\boxplus$, and that all three of these are conservative extensions of K. This is summerized in the Fig. 9.

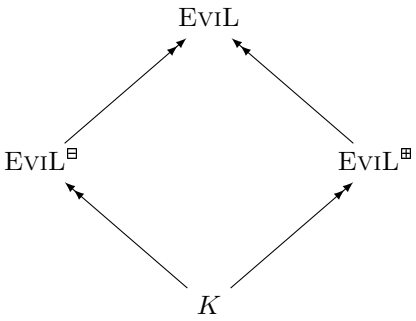


Figure 9: EvIL conservative extensions of K

Lemma 2.3.23. EvIL *is PSPACE hard*

Proof. This follows trivially from the fact that EvIL is a conservative extension of basic modal logic, and the decision problem for basic modal logic is PSPACE complete. QED

3 Applications

3.1 Collapse

3.2 Epistemic Plurality

3.2.1 Different Kinds of Knowledge

3.2.2 Moore's Paradox

3.2.3 Fitch's Paradox

3.3 Intuitionistic Logic

3.3.1 The Gödel Embedding and $\mathcal{L}^{\boxminus}(\Phi)$

3.3.2 Knowledge

3.3.3 Imagination

3.3.4 $ImK_{\Box}$

4 Formal Methods

4.1 LCF Theorem Proving

4.2 Formalizing the EviL Completeness Theorem

5 Epilogue

5.1 Comparison to Other Approaches

5.2 Failures

There are several points of failure of EviL that I feel must be addressed:

(I) EviL is not really a logic, because it is non-normal and non-compact, so it therefore any kind of reasonable algebraic duality is impossible (for details on this, see Blackburn et al., 2001, chapter 5)

(II) EviL is not dynamic and therefore fails to conform to the prevailing paradigm for epistemic logics

(*III*) EvIL is not completely computer verified - only the completeness theorem for the central axiom system for EvIL has been produced; none of the many auxiliary results have been verified

(*IV*) EvIL only partly accommodates irrationality

(*V*) EvIL is inhuman - the assumptions it makes for the nature of knowledge and EvIL agent's cognitive abilities are unrealistic

A Alternate Semantics

In this section, I shall present an alternative work to the framework proposed in §1.3. These semantics are inspired by game semantics for modal logic, such as those in van Benthem (2010, chapter 2).

Consider structures of the form $\langle W, V, \beta, \iota \rangle$ consisting of:

- A set of worlds W
- A propositional valuation function $V : \Phi \to \wp W$
- An belief function $\beta : W \to \wp\mathcal{L}(\Phi)$
- An imagination function $\iota : W \to \wp W$

We shall call these *belief-imagination models*. One can think of a model $\mathfrak{M}$ sort of like a of tuples like in §2; however in this case evidently it would have to be $\mathfrak{M} \subseteq \wp\Phi \times \wp\mathcal{L}(\Phi) \times \wp\mathfrak{M}$, so apparently it would have to be a non-wellfounded set. This is somewhat natural, given a modal logic setting - see for instance (Barwise and Moss, 1996) for an elaboration on these connections.

Definition A.0.1. *Define by recursion the following two truth relations:*

First relation:

$\mathfrak{M}, w \Vdash p \iff p \in V(w)$

$\mathfrak{M}, w \Vdash \phi \wedge \psi \iff$ *both* $\mathfrak{M}, w \Vdash \phi$ *and* $\mathfrak{M}, w \Vdash \psi$

$\mathfrak{M}, w \Vdash \phi \vee \psi \iff$ *either* $\mathfrak{M}, w \Vdash \phi$ *or* $\mathfrak{M}, w \Vdash \psi$

$\mathfrak{M}, w \Vdash \neg\phi \iff \mathfrak{M}, w \nVdash \phi$

$\mathfrak{M}, w \Vdash \Box\phi \iff \beta(w) \vdash^* \phi$

...where $\vdash^*$ *is a sequent that's closed under reflection and resolution:*

$$\frac{\phi \in \Gamma}{\Gamma \vdash^* \phi} \qquad \frac{\Gamma \vdash^* \neg\phi \vee \psi \quad \Delta \vdash^* \phi}{\Gamma \cup \Delta \vdash^* \psi}$$

Second relation:

$\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash p \iff p \notin V(w)$

$\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \phi \wedge \psi \iff$ *either* $\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \phi$ *or* $\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \psi$

$\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \phi \vee \psi \iff$ *both* $\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \phi$ *and* $\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \psi$

$\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \neg\phi \iff \mathfrak{M}, w \nVdash \phi$

$\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \Box\phi \iff$ *there is some* $v \in \iota(w)$ *such that* $\mathfrak{M}, v \Vdash\!\!\!\mid\!\!\!\vdash \phi$

I feel it is necessary to motivate the intuition behind these semantics. Informally, I think of these two truth relations correspond to two players, whom I call the *logician* and the *philosopher*. The logician wields a set beliefs given by β and tries to compose compelling arguments, and the philosopher employs a corpus of thought experiments given by ι to thwart the logician's arguments. Of course, the logician and the philosopher are really just two aspects of a single epistemic agent I am trying to model; I imagine epistemic agents modeled by this system to be embroiled in internal conflict. I feel this sort of dissension between reason and imagination rages on within us all – it's fundamental to human nature.

These semantics are not naturally bivalent; that is it doesn't hold that either $\mathfrak{M}, w \Vdash \phi$ or $\mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \phi$, exclusively. To see this consider a model where $\beta(w) = \iota(w) = \varnothing$; then evidently $\mathfrak{M}, w \nVdash \Box\, p$ and $\mathfrak{M}, w \nVdash\!\!\!\mid\!\!\!\vdash \Box\, p$.

However, bivalence has a convenient semantic characterization:

Proposition A.0.2. *Let* $\mathbb{M}^{\mathfrak{M}} = \langle W^{\mathfrak{M}}, V^{\mathfrak{M}}, R^{\mathfrak{M}} \rangle$ *be a model for basic modal logic model based on a belief/imagination model* $\mathfrak{M}$*, where* $wR^{\mathfrak{M}}v := v \in \iota(w)$*, and let* $\Vdash_{\Box}$ *be the modal truth predicate. We have that* $\Vdash$ *and* $\Vdash\!\!\!\mid\!\!\!\vdash$ *are bivalent if and only if* $\mathfrak{M}, w \Vdash \phi \iff \mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \phi$.

Proof. ($\Longrightarrow$) Assume that $\Vdash$ and $\Vdash\!\!\!\mid\!\!\!\vdash$ are bivalent and consider any $\phi \in \mathcal{L}(\Phi)$. The proof that $\mathfrak{M}, w \Vdash \phi$ is equivalent to $\mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \phi$ proceeds by induction. The case for proposition letters, conjunction and disjunction are straightforward, so we shall only consider negation and modality.

Negation: We have the following chain of equivalences:

$$\begin{aligned}
\mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \neg\phi &\iff \mathbb{M}^{\mathfrak{M}}, w \nVdash_{\Box} \phi \\
&\iff \mathfrak{M}, w \nVdash \phi \quad \text{(inductive step)} \\
&\iff \mathfrak{M}, w \Vdash\!\!\!\mid\!\!\!\vdash \phi \quad \text{(bivalence)} \\
&\iff \mathfrak{M}, w \Vdash \neg\phi
\end{aligned}$$

Modality: We have another chain of equivalences:

$$\begin{aligned}
\mathfrak{M}, w \Vdash \Box\phi &\iff \mathfrak{M}, w \nVdash\!\!\!\mid\!\!\!\vdash \Box\, \phi && \text{(bivalence)} \\
&\iff \forall v \in \iota(w). \mathfrak{M}, w \nVdash\!\!\!\mid\!\!\!\vdash \phi && \text{(definition)} \\
&\iff \forall v \in \iota(w). \mathfrak{M}, w \Vdash \phi && \text{(bivalence)} \\
&\iff \forall v \in \iota(w). \mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \phi && \text{(inductive step)} \\
&\iff \forall v. wR^{\mathfrak{M}}v \Longrightarrow \mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \phi && \text{(definition)} \\
&\iff \mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \Box\phi
\end{aligned}$$

...this completes the induction.

($\Longleftarrow$) Assume that $\mathfrak{M}, w \Vdash \phi$ and $\mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \phi$ are always equivalent. We have:

$$\begin{aligned} \mathfrak{M}, w \Vdash \phi &\iff \mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \phi \\ &\iff \mathfrak{M}, w \nVdash_{\Box} \neg\phi \\ &\iff \mathfrak{M}, w \nVdash \neg\phi \quad \text{(hypothesis)} \\ &\iff \mathfrak{M}, w \Vdash\!\!\!\!/\, \phi \end{aligned}$$

QED

Corollary A.0.3. *If $\Vdash$ and $\Vdash\!\!\!\!/$ are bivalent, then $\beta(w) \vdash^* \phi$ for all $\phi \in \mathrm{Th}_{\Vdash}(\mathfrak{M})$ for all $w \in W^{\mathfrak{M}}$, where $\mathrm{Th}_{\Vdash}(\mathfrak{M}) = \{\phi \in \mathcal{L}(\Phi) \mid \mathfrak{M}, w \Vdash \phi \text{ for all } w \in W^{\mathfrak{M}}\}$.*

Evidently bivalence of $\Vdash$ and $\Vdash\!\!\!\!/$ gives rise to semantics where the agent has a proof for every proposition they believe. Furthermore, we can take any modal logic model $\mathbb{M} := \langle W^{\mathbb{M}}, V^{\mathbb{M}}, R^{\mathbb{M}} \rangle$ and define an equivalent belief/imagination model $\mathfrak{M}^{\mathbb{M}} := \langle W^{\mathbb{M}}, V^{\mathbb{M}}, \beta^{\mathbb{M}}, \iota^{\mathbb{M}} \rangle$ where:

$$\begin{aligned} \beta^{\mathbb{M}}(w) &:= \{\phi \in \mathcal{L}(\Phi) \mid \mathbb{M}, w \Vdash_{\Box} \Box\phi\} \\ \iota^{\mathbb{M}}(w) &:= \{v \in W^{\mathbb{M}} \mid wR^{\mathbb{M}}v\} \end{aligned}$$

We can immediately leverage this to give the a characterization of these semantics:

Proposition A.0.4. *The basic modal logic K is sound and strongly complete for bivalent belief/imagination models.*

Proof. Soundness is trivial given the previous lemma, strong completeness follows by considering the canonical model $\mathbb{K}$ and looking at $\mathfrak{M}^{\mathbb{K}}$. QED

However, reflecting on my remarks in §1.2, I think that it's wrong for agents to be able to have everything they believe in their minds; this is about as bad as the thermometer theory of knowledge in my opinion. However, this is evidently not entirely necessary. Call a belief/imagination model *reasonable* if the following two constraints are satisfied:

- $\beta(w) \vdash^* \phi$ for all $\phi \in \mathrm{Th}_{\Vdash}(\mathfrak{M})$ for all $w \in W^{\mathfrak{M}}$, where $\mathrm{Th}_{\Vdash}(\mathfrak{M}) = \{\phi \in \mathcal{L}(\Phi) \mid \mathfrak{M}, w \Vdash \phi \text{ for all } w \in W^{\mathfrak{M}}\}$
- $\mathrm{Mod}^{\mathfrak{M}}_{\Vdash\!\!\!\!/}(\beta(w)) \subseteq \iota(w)$, where $\mathrm{Mod}^{\mathfrak{M}}_{\Vdash\!\!\!\!/}(\beta(w)) = \{v \in W^{\mathfrak{M}} \mid \mathfrak{M}, v \Vdash\!\!\!\!/\, \phi \text{ for all } \phi \in \beta(w)\}$
- $\beta(w) \setminus \mathrm{Th}_{\Vdash}(\mathfrak{M})$ is finite

Evidently, forcing these requirements suffices to force bivalence:

Proposition A.0.5. *For any reasonable model $\mathfrak{M}$ and any $w \in W^{\mathfrak{M}}$, we have:*

(*i*) *If $\mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \phi$ then $\mathfrak{M}, w \Vdash \phi$*

(*ii*) *If $\mathbb{M}^{\mathfrak{M}}, w \nVdash_{\Box} \phi$ then $\mathfrak{M}, w \Vdash\!\!\!\!/\, \phi$*

... where $\mathbb{M}^{\mathfrak{M}}$ is as defined in Prop. A.0.2. Hence we have $\Vdash$ and $\Vdash\!\!\Vdash$ are bivalent.

Proof. The propositional, disjunctive and conjunctive cases are all straightforward; I shall focus on negation and modality.

Negation: In the case of (i), we know that

$$\begin{aligned} \mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \neg\phi &\iff \mathbb{M}^{\mathfrak{M}}, w \nVdash_{\Box} \phi \\ &\implies \mathfrak{M}, w \Vdash\!\!\Vdash \phi \quad \text{(by the inductive step)} \\ &\iff \mathfrak{M}, w \Vdash \neg\phi \end{aligned}$$

...the proof for (ii) is similar.

Modality: In the case of (i), assume that $\mathbb{M}^{\mathfrak{M}}, w \Vdash_{\Box} \Box\phi$. Using the definition of reasonableness and the inductive step we know for all $v \in W^{\mathfrak{M}}$ that if $\mathfrak{M}, v \nVdash \psi$ for all $\psi \in \beta(w) \setminus \mathrm{Th}(\mathfrak{M})$ then $\mathfrak{M}, v \Vdash \phi$.

From this and the fact that $\mathfrak{M}$ is reasonable we can infer that $\bigvee_{\psi \in \beta(w) \setminus \mathrm{Th}(\mathfrak{M})} \neg\psi \vee \phi \in \mathrm{Th}_{\Vdash}(\mathfrak{M})$. We know further from reasonableness that we have $\mathrm{Th}_{\Vdash}(\mathfrak{M}) \subseteq \beta(w)$. So we can prove by induction that repeatedly applying resolution gets $\beta(w) \vdash^{*} \phi$, which just means that $\mathfrak{M}, w \Vdash \Box\phi$, as desired.

The case of (ii) follows trivially by induction. QED

We may continue to obtain weak completeness for these semantics:

Proposition A.0.6. *$\vdash_K \phi$ if and only if $\mathfrak{M}, w \Vdash \phi$ for all reasonable models $\mathfrak{M}$ and $w \in W^{\mathfrak{M}}$*

Proof. Left to right follows straightforwardly, so we just need to prove right to left.

Assume $\nvdash_K \phi$. As before, let $\mathbb{M} = \langle W^{\mathbb{M}}, V^{\mathbb{M}}, R^{\mathbb{M}} \rangle$ be a finite model and with a world $w \in W^{\mathbb{M}}$ such that $\mathbb{M}, w \nVdash_{\Box} \phi$. Now consider a slightly modified model $\mathbb{M}' := \langle W^{\mathbb{M}}, V', R^{\mathbb{M}} \rangle$ where

$$V'(p) := \begin{cases} \{v\} & p = \rho(v) \\ V(p) & o/w \end{cases}$$

A proof by induction on subformulae ψ of ϕ verifies that $\mathbb{M}, w \Vdash_{\Box} \psi$ if and only if $\mathbb{M}', w \Vdash_{\Box} \psi$.

So now consider $\mathfrak{M} := \langle W^{\mathbb{M}'}, V^{\mathbb{M}'}, \tau, \lambda x. R^{\mathbb{M}'}[x] \rangle$ such that

$$\tau(w) := \mathrm{Th}(\mathbb{M}') \cup \left\{ \bigvee_{v \in R^{\mathbb{M}'}[w]} \rho(v) \right\}$$

...where $\mathrm{Th}(\mathbb{M}') := \{\psi \in \mathcal{L}(\Phi) \mid \mathbb{M}', v \Vdash \psi \text{ for all } v \in W^{\mathbb{M}'}\}$. A proof by induction on ψ shows that $\mathbb{M}', w \Vdash_{\Box} \psi$, $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, v \nVdash \psi$ are equivalent for all $\psi \in \mathcal{L}(\Phi)$. Thus we have that for all $v \in W^{\mathfrak{M}}$ that $\mathfrak{M}, v \nVdash \psi$ for all $\psi \in \mathrm{Th}(\mathbb{M}')$. Moreover, evidently $w R^{\mathbb{M}'} v$ if and only if $\mathbb{M}', v \Vdash_{\Box} \bigvee_{u \in R^{\mathbb{M}'}[w]} \rho(u)$, whence we have that $w R^{\mathbb{M}'} v$ if and only if $\mathfrak{M}, v \nVdash \chi$ for all $\chi \in \tau(w)$. With this we can employ induction and establish that $\mathbb{M}', w \Vdash_{\Box} \psi$ if and only if $\mathfrak{M}, w \models \psi$ for all $\psi \in \mathcal{L}(\Phi)$. Since $\mathbb{M}', w \nVdash_{\Box} \phi$, we have that $\mathfrak{M}, w \nvDash \phi$. Finally, note that in this model we have

that $\mathrm{Mod}^{\mathfrak{M}}_{\not\Vdash}(\beta(w)) = R^{\mathbb{M}'}[w]$. With this and the definition of $\mathfrak{M}$, we can see that $\mathfrak{M}$ is evidently reasonable, and thus we may complete the proof.

QED

Now, while reasonable models attain the goal of modeling agents that have proofs for the things they believe, I do not consider them adequate. These models are only reasonable in the sense that they indeed model agents providing nontrivial proofs for their beliefs. However, they aren't reasonable in the sense that they are simple to reckon with. So while the semantics provided in §2 requires a grammar restriction, which might be considered inelegant, I have settled on it, rather than the formulation given above, precisely because I consider this to be less manageable.

B Isabelle/HOL's Logic

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